ANNALES DE L'INSTITUT FOURIER

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Examples of deformed G_2-instantons/Donaldson–Thomas connections

EXAMPLES OF DEFORMED
G₂-INSTANTONS/DONALDSON–THOMAS
CONNECTIONS

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ABSTRACT. — In this note, we provide the first non-trivial examples of deformed G₂-instantons, originally called deformed Donaldson–Thomas connections. As a consequence, we see how deformed G₂-instantons can be used to distinguish between nearly parallel G₂-structures and isometric G₂-structures on 3-Sasakian 7-manifolds. Our examples give non-trivial deformed G₂-instantons with obstructed deformation theory and situations where the moduli space of deformed G₂-instantons has components of different dimensions. We finally study the relation between our examples and a Chern–Simons type functional which has deformed G₂-instantons as critical points.

RÉSUMÉ. — Dans cette note, nous fournissons les premiers exemples non triviaux de G₂-instantons déformés, initialement appelés connexions Donaldson–Thomas déformées. En conséquence, on peut utiliser G₂-instantons déformés pour faire la distinction entre des G₂-structures presque parallèles et des G₂-structures isométriques sur des 7-variétés 3-Sasakiennes. Nos exemples donnent des G₂-instantons déformés non triviaux avec une théorie de la déformation obstruée et des situations où l'espace des modules des G₂-instantons déformés a des composantes de dimensions différentes. Nous étudions enfin la relation entre nos exemples et une fonctionnelle de type Chern–Simons qui a les G₂ -instantons déformés en points critiques.

1. Introduction

Gauge theory and calibrated geometry are central topics of study in the context of G₂ geometry, and they are intimately related. Based on

ideas stemming from Mirror Symmetry, particularly SYZ fibrations, and the real Fourier–Mukai transform, the authors of [15] introduced the following gauge-theoretic equation (in the context of complex line bundles over G_2-manifolds) as a proposed mirror to certain calibrated cycles in G_2-manifolds.

DEFINITION 1.1. — *Let (X^7, φ) be a 7-manifold with a coclosed G_2 structure φ, let $\psi = *_\varphi \varphi$ be the dual of φ, and let L be a Hermitian complex line bundle on X. A unitary connection A on L is a* deformed G_2-instanton *if its curvature F_A satisfies*

$$\text{(1.1)} \qquad \frac{1}{6} F_A^3 + F_A \wedge \psi = 0.$$

The definition can obviously be extended to higher rank vector bundles and principal bundles but, based on [15], one is primarily interested in the case of complex line bundles. When (X, φ) is additionally a G_2-manifold, deformed G_2-instantons are, in a certain sense, "mirror" to (co)associative cycles.

Remark 1.2. — In [15], deformed G_2-instantons are called *deformed Donaldson–Thomas connections*. However, since A is a G_2-*instanton* on (X^7, φ) if and only if

$$\text{(1.2)} \qquad\qquad F_A \wedge \psi = 0,$$

the authors feel it is more appropriate that solutions of (1.1) are called deformed G_2-instantons. Moreover, there is a natural relationship between deformed G_2-instantons and deformed Hermitian-Yang–Mills connections, which is parallel to the relationship between G_2-instantons and Hermitian-Yang–Mills connections, that gives further rationale for the nomenclature.

1.1. Main results

The main results of this article are the first constructions of non-trivial solutions to the deformed G_2-instanton equation (1.1). Here, by non-trivial, we mean deformed G_2-instantons that are not flat and do not arise via pullback from lower-dimensional constructions. Our construction takes place on compact 7-manifolds equipped with G_2-structures related to the existence of a 3-Sasakian structure. This is a setting where interesting families of G_2-structures can be found and at the same time symmetries can be used to turn the problem of finding deformed G_2-instantons into a more tractable one.

As an application of our construction we have the following result (Corollary 4.5).

THEOREM 1.3. — *Let X^7 be a compact 3-Sasakian 7-manifold and let L_0 be the trivial complex line bundle on X. Let φ^{ts} be the standard nearly parallel G_2-structure on X inducing the 3-Sasakian Einstein metric g^{ts}, and let φ^{np} be the second (strictly) nearly parallel G_2-structure on X inducing the "squashed" Einstein metric g^{np}.*

- *There is a circle of non-trivial deformed G_2-instantons on L_0 for φ^{ts}.*
- *There is a 2-sphere of non-trivial deformed G_2-instantons on L_0 for φ^{np}.*

Remark 1.4. — There are infinitely many compact 3-Sasakian 7-manifolds X^7 [3,11]. Key examples are giving by the 7-sphere S^7 and the Aloff–Wallach space $SU(3)/U(1)_{1,1}$ (cf. Example 3.8), which is the $SO(3)$-frame bundle of $\Lambda_+^2 \overline{\mathbb{CP}^2}$.

Remark 1.5. — We recall that, if X^7 is a compact 3-Sasakian 7-manifold, then the metric cone on (X^7, g^{ts}) is hyperkähler, and has holonomy $Sp(2)$ if it is not flat, whilst the metric cone on (X^7, g^{np}) has holonomy $Spin(7)$.

Remark 1.6. — Theorem 1.3 shows how non-trivial deformed G_2-instantons distinguish between the two nearly parallel G_2-structures φ^{np} and φ^{ts}. We shall also see that non-trivial deformed G_2-instantons can discriminate between two coclosed G_2-structures inducing the same metric, including the Einstein metrics g^{ts} and g^{np} (cf. Corollary 4.7).

We can also apply our results to non-trivial complex line bundles when X^7 is the 3-Sasakian Aloff–Wallach space (Corollary 4.12).

THEOREM 1.7. — *Let $\pi : X \to \overline{\mathbb{CP}^2}$ be the $SO(3)$-frame bundle of $\Lambda_+^2 \overline{\mathbb{CP}^2}$ and let $k \in \mathbb{Z}$.*

- *There is a circle of non-trivial deformed G_2-instantons on $\pi^* \mathcal{O}(k)$ for φ^{ts}.*
- *There is a 2-sphere of non-trivial deformed G_2-instantons on $\pi^* \mathcal{O}(k)$ for φ^{np}.*

More generally, we give examples of deformed G_2-instantons for families of coclosed G_2-structures $\varphi_{t,\varepsilon}$ on a compact 3-Sasakian 7-manifold X depending on two parameters: $t > 0$ and $\varepsilon \in \{\pm 1\}$. Our ansatz depends on $a_1, a_2, a_3 \in \mathbb{R}$, where $a_1 = a_2 = a_3 = 0$ yields the trivial flat connection in the case of the trivial complex line bundle L_0. Hence, if we let

$r = \sqrt{a_1^2 + a_2^2 + a_3^2}$, which can be viewed as the distance to the trivial connection on L_0, we can represent our main result (Proposition 4.4) in Figure 1.1 below. The overall picture for the non-trivial line bundles on the 3-Sasakian Aloff–Wallach space (Proposition 4.10) is the same.
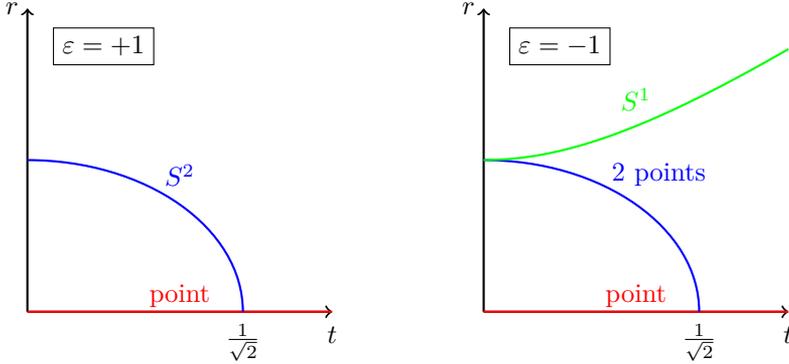

Figure 1.1. The space of deformed G_2-instantons for $\varphi_{t,\varepsilon}$ on L_0.

Remark 1.8. — In particular, we observe in Figure 1.1 how deformed G_2-instantons die as t varies from 0 to $1/\sqrt{2}$ in both the cases $\varepsilon = \pm 1$, but when $\varepsilon = -1$ we have a circle that lives for all $t > 0$. For context, we mention that φ^{np} corresponds to $(t, \varepsilon) = (1/\sqrt{5}, +1)$ and φ^{ts} corresponds to $(t, \varepsilon) = (1, -1)$. We also note that $\varphi_{t,\varepsilon}$ induce the same metric for a fixed value of t; it is an interesting feature resulting from our work that the space of deformed G_2-instantons differs for these isometric G_2-structures.

Remark 1.9. — The parameter t in Figure 1.1 corresponds to the size of the fibres for the canonical SU(2) or SO(3) fibration of the 3-Sasakian 7-manifold X over a 4-orbifold. We see that we have non-trivial deformed G_2-instantons which tend to a non-trivial limit connection as $t \to 0$ (i.e. as the 3-dimensional fibres collapse), even though the norms of their curvatures are blowing up (with respect to the t-dependent metric). This suggests possible links of our study to adiabatic limits and compactness issues for deformed G_2-instantons.

We may also relate our examples to the recently developed moduli space theory for deformed G_2-instantons in [14]. We refer the reader to Definition 4.14 for the formal definition for a deformed G_2-instanton to be obstructed in the sense of deformation theory.

THEOREM 1.10. — *Let X be a compact 3-Sasakian 7-manifold and let L_0 be the trivial complex line bundle on X. The non-trivial deformed G_2-instantons given by Theorem 1.3 are obstructed. Moreover, the moduli spaces of deformed G_2-instantons on L_0 for the nearly parallel G_2-structures φ^{np} and φ^{ts} both contain at least two components of different dimensions.*

A Chern–Simons type functional was introduced in [12] whose critical points are deformed G_2-instantons. We study this functional in our setting and, in particular, deduce the following (cf. Lemma 4.19 and Corollary 5.7), which reflects the fact that non-trivial deformed G_2-instantons coalesce with the trivial connection at $t = 1/\sqrt{2}$.

THEOREM 1.11. — *Let X be a compact 3-Sasakian 7-manifold with the trivial complex line bundle L_0, and let A_0 be the trivial flat connection on L_0. Then A_0 is unobstructed (and hence rigid and locally isolated) as a deformed G_2-instanton with respect to $\varphi^{np} = \varphi_{1/\sqrt{5},+1}$ and $\varphi^{ts} = \varphi_{1,-1}$, but obstructed as a deformed G_2-instanton with respect to $\varphi_{1/\sqrt{2},\varepsilon}$ for $\varepsilon = \pm 1$.*

1.2. Summary

This article is organized as follows. Section 2 introduces some background on 3-Sasakian geometry which will later be of use in constructing the examples of deformed G_2-instantons. In section 3 we give some simple examples of deformed G_2-instantons which arise from pulling back connections in 6 and 4 dimensions. The non-trivial examples which constitute the main contribution of this article are constructed and presented in section 4. Finally, in section 5, this article concludes with a discussion of the Chern–Simons type functional mentioned above; in particular, the functional is explicitly computed and analyzed in some of the cases developed in this article.

Remark 1.12. — We shall use summation convention over repeated indices throughout the article.

2. 3-Sasakian geometry

We shall now give a short introduction to some aspects of 3-Sasakian geometry and its relation to G_2 geometry. We refer the reader to the survey article [2] and references therein for more information on 3-Sasakian geometry.

DEFINITION 2.1. — *A complete 7-dimensional Riemannian manifold* (X^7, g^{ts}) *is 3-Sasakian if it has three orthonormal Killing vector fields* $\{\xi_i\}_{i=1}^3$ *satisfying the relations* $[\xi_i, \xi_j] = 2\epsilon_{ijk}\xi_k$, *for* ϵ_{ijk} *the sign of the permutation taking* $(1, 2, 3)$ *to* (i, j, k), *such that each* ξ_i *induces a Sasakian structure on* (X^7, g^{ts}).

As a consequence of this definition, for a 3-Sasakian 7-manifold (X^7, g^{ts}), the complete metric g^{ts} is Einstein with positive scalar curvature and its metric cone has holonomy contained in Sp(2). In particular, X^7 is compact by Myers' theorem.

2.1. SU(2) leaf space

For a 3-Sasakian manifold (X^7, g^{ts}) as in Definition 2.1, the Killing vector fields $\{\xi_i\}_{i=1}^3$ generate a locally free action of SU(2). The leaf space of this SU(2) action, denoted Z^4, can be endowed with a metric g_Z so that the canonical projection

$$(2.1) \qquad\qquad \pi : X \to Z$$

is an orbifold Riemannian submersion. This metric is anti-self-dual and Einstein of positive scalar curvature $s > 0$. For convenience, we will scale the metric g^{ts} so that $s = 48$: this fits with the canonical example of S^7 with its constant curvature 1 metric. In the particular case when Z is spin, we shall regard (2.1) as the lift to SU(2) = Spin(3) of an SO(3)-(orbi)bundle of frames of $\Lambda_+^2 Z$, the bundle of self-dual 2-forms on Z.

The Levi-Civita connection of Z induces a connection η on the bundle (2.1) which, seen as a 1-form on X with values in $\mathfrak{su}(2)$, can be written as

$$(2.2) \qquad\qquad \eta = \eta_i \otimes T_i,$$

where the T_i are a standard basis of $\mathfrak{su}(2)$ satisfying $[T_i, T_j] = 2\epsilon_{ijk}T_k$, and the η_i are 1-forms on X. The horizontal space of the connection is $H = \ker(\eta)$. Knowing that Z is anti-self-dual Einstein means that the curvature of the connection η in (2.2) is given by

$$(2.3) \qquad\qquad F_\eta = \mathrm{d}\,\eta + \frac{1}{2}[\eta \wedge \eta] = -2\omega_i \otimes T_i,$$

with the triple of 2-forms $\omega_1, \omega_2, \omega_3$ forming an orthogonal basis of $(\Lambda_+^2 H, g^{ts}|_H)$ with $|\omega_i| = \sqrt{2}$. Notice that we have the relations

$$(2.4) \qquad\qquad \omega_i \wedge \omega_j = 2\delta_{ij}\pi^* \mathrm{vol}_Z,$$

where vol_Z is the Riemannian volume form on (Z, g_Z).

2.2. Metrics and G_2-structures

The 3-Sasakian metric g^{ts} can be written

$$(2.5) \qquad g^{ts} = \eta_i \otimes \eta_i + \pi^* g_Z$$

and it is well-known to be Einstein. In fact, there is a second Einstein metric on X for which π is a Riemannian submersion. This can be obtained from g^{ts} by squashing the fibers of π by a factor which reduces the length of any ξ_i-orbit by $\sqrt{5}$. This yields the metric

$$(2.6) \qquad g^{np} = \frac{1}{5} \eta_i \otimes \eta_i + \pi^* g_Z.$$

The metrics g^{ts} in (2.5) and g^{np} in (2.6) are induced by natural distinguished G_2-structures φ^{ts} and φ^{np} on X^7. (For details on G_2-structures we refer the reader to [13], for example.) Using (2.2) and (2.3), it is convenient to consider two 1-parameter families, depending on $t \in \mathbb{R}^+$ and $\varepsilon \in \{\pm 1\}$, of G_2-structures on X^7 determined by the 3-forms

$$(2.7) \qquad \varphi_{t,\varepsilon} = \varepsilon t^3 \eta_{123} - t(\eta_1 \wedge \omega_1 + \eta_2 \wedge \omega_2 + \varepsilon \eta_3 \wedge \omega_3),$$

where we have used the notation $\eta_{123} = \eta_1 \wedge \eta_2 \wedge \eta_3$ for brevity. Each $\varphi_{t,\varepsilon}$ determines the metric

$$(2.8) \qquad g_t = t^2 \eta_i \otimes \eta_i + \pi^* g_Z,$$

which is *independent* of ε, and the associated 4-forms $\psi_{t,\varepsilon} = *_{g_t} \varphi_{t,\varepsilon}$ are

$$(2.9) \qquad \psi_{t,\varepsilon} = \pi^* \operatorname{vol}_Z - t^2 \left(\varepsilon \eta_{23} \wedge \omega_1 + \varepsilon \eta_{31} \wedge \omega_2 + \eta_{12} \wedge \omega_3 \right),$$

where we write $\eta_{ij} = \eta_i \wedge \eta_j$ for short. Note that if we set

$$(2.10) \qquad \varphi^{ts} = \varphi_{1,-1} \quad \text{and} \quad \varphi^{np} = \varphi_{1/\sqrt{5},+1},$$

then (2.8) shows that φ^{ts} induces g^{ts} in (2.5) and φ^{np} induces g^{np} in (2.6).

Remark 2.2. — Changing the sign of the parameter ϵ corresponds to the change $\eta_3 \mapsto -\eta_3$, which gives a change of orientation on the vertical space in the projection (2.1). However, since we have fixed the structure equations on SU(2), we are not free to change $\eta_3 \to -\eta_3$, and so ε represents a genuine parameter.

We now recall a notable class of G_2-structures in this setting.

DEFINITION 2.3. — *A G_2-structure determined by a 3-form φ, with dual 4-form ψ, is nearly parallel if $\mathrm{d}\,\varphi = \lambda \psi$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. By possibly changing orientation, we can always ensure that $\lambda > 0$.*

Using the equation (2.3) for the curvature of η, we find the following structure equations:

$$(2.11) \qquad\qquad \mathrm{d}\,\eta_i = -2\omega_i - 2\eta_j \wedge \eta_k,$$

$$(2.12) \qquad\qquad \mathrm{d}\,\omega_i = 2\omega_j \wedge \eta_k - 2\eta_j \wedge \omega_k,$$

for (i, j, k) a cyclic permutation of $(1, 2, 3)$. Equations (2.11)-(2.12) immediately show that

$$(2.13) \qquad\qquad \mathrm{d}\,\psi_{t,\varepsilon} = 0$$

for all t, ε. Using (2.11)–(2.12) we compute from (2.7) and (2.9) that the equation

$$(2.14) \qquad\qquad \mathrm{d}\,\varphi_{t,\varepsilon} = \lambda\psi_{t,\varepsilon}$$

only has solutions when

$$(2.15) \qquad (t, \varepsilon, \lambda) = \left(\frac{1}{\sqrt{5}}, +1, \frac{12}{\sqrt{5}} \right) \quad \text{and} \quad (t, \varepsilon, \lambda) = (1, -1, 4)\,.$$

By Definition 2.3, $(t, \varepsilon) = (1/\sqrt{5}, +1)$ and $(t, \varepsilon) = (1, -1)$ are the only values for which $\varphi_{t,\varepsilon}$ is nearly parallel [10]. We see from (2.5), (2.6) and (2.8) that $g_{1/\sqrt{5}} = g^{np}$ and $g_1 = g^{ts}$, and (2.10), (2.14) and (2.15) show that φ^{np} and φ^{ts} are nearly parallel.

We conclude this section with a familiar concrete example.

Example 2.4. — The standard 7-sphere S^7 with its round constant curvature 1 metric g_{S^7} is 3-Sasakian, and its SU(2) leaf space is $Z = S^4$, naturally endowed with its round constant curvature 4 metric g_Z. It is well-known that S^7 admits two homogeneous Einstein metrics: g_{S^7} (which is g^{ts} in (2.5)) and its "squashed" metric (which is g^{np} in (2.6)). Each of these Einstein metrics is induced by a homogeneous nearly parallel G_2-structure, given in (2.10) by φ^{ts} and φ^{np} respectively.

3. Deformed Hermitian-Yang–Mills connections and ASD instantons

In this section we provide examples of deformed G_2-instantons arising from lower-dimensional geometries: specifically, deformed Hermitian-Yang–Mills connections on Calabi–Yau 3-folds and anti-self-dual instantons on anti-self-dual Einstein 4-orbifolds and hypersymplectic 4-manifolds.

3.1. Deformed Hermitian-Yang–Mills connections

We recall the following definition, which originated in [16, 17].

DEFINITION 3.1. — *Let (Y, ω) be a Kähler n-fold, where ω is the Kähler form, and let L be a Hermitian complex line bundle on Y. A unitary connection A on L is a deformed Hermitian-Yang–Mills connection (with phase $e^{i\alpha}$) if*

$$(3.1) \qquad F_A^{(0,2)} = 0 \quad and \quad \Im(e^{-i\alpha}(\omega + F_A)^n) = 0.$$

When Y is a Calabi–Yau manifold, deformed Hermitian-Yang–Mills connections are, in a sense, "mirror" to special Lagrangian n-folds.

We are interested in the case where (Y, ω, Ω) is a Calabi–Yau 3-fold, with holomorphic volume form Ω, and A is a deformed Hermitian-Yang–Mills connection with phase 1. In this case (3.1) can be rewritten as

$$(3.2) \qquad F_A \wedge \Im\Omega = 0 \quad and \quad \frac{1}{6}F_A^3 + F_A \wedge \frac{1}{2}\omega^2 = 0.$$

The analogy between (1.1) and (3.2) should be clear. We then provide an observation which extends Lemma 5.5 in [14].

LEMMA 3.2. — *Let (Y, ω, Ω) be a Calabi–Yau 3-fold and let L be a Hermitian complex line bundle on Y. Let $\pi : X^7 \to Y$ be an S^1-bundle over Y with a connection 1-form η, which is Hermitian-Yang–Mills, endowed with the standard G_2-structure*

$$(3.3) \qquad \varphi = \eta \wedge \pi^*\omega + \pi^*\Re\Omega \quad and \quad \psi = \frac{1}{2}\pi^*\omega^2 - \eta \wedge \pi^*\Im\Omega.$$

Note that as η is Hermitian-Yang–Mills we have $d\psi = 0$.

Then A is a deformed Hermitian-Yang–Mills connection with phase 1 on L if and only if π^*A is a deformed G_2-instanton on π^*L.

Proof. — The proof follows immediately from (1.1), (3.2) and (3.3), just as in the proof of Lemma 5.5 in [14]. □

Remark 3.3. — Lemma 3.2 has a well-known analogue where deformed Hermitian-Yang–Mills connections and deformed G_2-instantons are replaced by Hermitian-Yang–Mills conections and G_2-instantons. When $X^7 = S^1 \times Y$ with the product G_2-structure and $\eta = d\theta$ for θ the coordinate on S^1, G_2-instantons on π^*L are related via a "broken gauge" with the pullback of Hermitian-Yang–Mills connections over Y [18]. In particular, this implies that π^*A is a G_2-instanton if and only if A is a Hermitian-Yang–Mills connection.

There are now many examples of deformed Hermitian-Yang–Mills connections, in particular provided by the recent relationship between existence of such connections and stability proved in [5]. We may construct a simple example of a G_2-manifold, i.e. X^7 with a torsion-free G_2-structure φ (satisfying $\mathrm{d}\,\varphi = 0$ and $\mathrm{d}\,\psi = 0$), with a non-trivial line bundle admitting a deformed G_2-instanton which is *not* a G_2-instanton as follows.

Example 3.4. — Suppose that (Y, ω, Ω) is a Calabi–Yau 3-fold such that $[\sqrt{3}\omega]$ is an integral class in $H^2(Y)$, and so defines a Hermitian complex line bundle L with a unitary connection A such that $F_A = \mathrm{i}\,\sqrt{3}\omega$. Then (3.2) is satisfied and so A is a deformed Hermitian-Yang–Mills connection with phase 1.

If we let $X = S^1 \times Y$ with the product torsion-free G_2-structure as in (3.3), where $\pi : X \to Y$ is the natural projection and $\eta = \mathrm{d}\,\theta$ for θ the coordinate on S^1, then π^*A is a deformed G_2-instanton on π^*L by Lemma 3.2. Notice that $\pi^*F_A^3 \neq 0$ and so π^*A is *not* a G_2-instanton on $X = S^1 \times Y$.

Remark 3.5. — One can perform a similar construction to Example 3.4 for so-called Calabi–Yau links X^7 in S^9, which are nontrivial S^1-bundles over Calabi–Yau 3-(orbi)folds Y arising as hypersurfaces in $\mathbb{CP}^4$, to obtain deformed G_2-instantons which are not G_2-instantons on X. The study of G_2-instantons on Calabi–Yau links was initiated in [4], using Hermitian-Yang–Mills connections on Y.

3.2. ASD instantons on anti-self-dual Einstein 4-orbifolds

Let X^7 be a 3-Sasakian 7-manifold as in Definition 2.1 and let (Z^4, g_Z) as in (2.1) be the SU(2) leaf space. Recall that (Z^4, g_Z) is an anti-self-dual Einstein 4-orbifold, and recall the G_2-structures on X whose 4-forms are given by $\psi_{t,\varepsilon}$ in (2.9). In particular, recall the forms ω_i in (2.3), which are pullbacks of self-dual 2-forms on Z, used to construct $\psi_{t,\varepsilon}$.

We now have the following simple observation concerning *anti-self-dual* (ASD) instantons on Z, i.e. connections A on Z whose curvature satisfies

$$(3.4) \qquad\qquad\qquad F_A = - * F_A.$$

Since $\pi^*F_A^3 = 0$ automatically for dimension reasons, for any connection A on Z, we see from (1.1) that the notions of deformed G_2-instanton and G_2-instanton coincide for π^*A. We may thus obtain trivial examples of deformed G_2-instantons as follows.

LEMMA 3.6. — *Let X^7 be a 3-Sasakian 7-manifold and let Z be its* SU(2) *leaf space as in (2.1). Let L be a Hermitian complex line bundle on Z, and let A be a unitary connection on L.*

Then π^*A is a (deformed) G_2-instanton on π^*L over X, with respect to some (and hence all) $\psi_{t,\varepsilon}$ in (2.9) if and only if A is an ASD instanton.

Proof. — First, let A be an anti-self-dual (ASD) instanton on Z. Then, since F_A is anti-self-dual and the forms ω_i appearing in (2.9) are self-dual on the horizontal space H for the projection (2.1), we see that $\pi^*F_A \wedge \psi_{t,\varepsilon} = 0$, i.e. that π^*A is a G_2-instanton.

Conversely, if $\pi^*F_A \wedge \psi_{t,\varepsilon} = 0$ for some t and ε, then we must have

$$(3.5) \qquad \pi^*F_A \wedge \omega_i = 0 \quad \text{for all } i.$$

Since the ω_i span the self-dual 2-forms on H, (3.5) implies that (3.4) holds, i.e. A is an ASD instanton. $\square$

Remark 3.7. — Lemma 3.6 has some overlap with Proposition 18 in [1].

We now give an example of using Lemma 3.6 which will be useful later.

Example 3.8. — Let X be the 7-dimensional Aloff–Wallach space $SU(3)/U(1)_{1,1}$ where

$$(3.6) \qquad U(1)_{1,1} = \left\{ \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-2i\theta} \end{pmatrix} \in SU(3) : \theta \in \mathbb{R} \right\}.$$

Then X is a homogeneous 3-Sasakian 7-manifold whose SU(2) leaf space is $Z = \overline{\mathbb{CP}^2}$. Let $L = \mathcal{O}(k)$ on Z for $k \in \mathbb{Z}$. The connection A on L with harmonic curvature will be unitary and have the property that F_A is a multiple of the Fubini–Study form, and so will be an ASD instanton (since $\overline{\mathbb{CP}^2}$ has the opposite orientation). Moreover, A will be non-trivial whenever $k \neq 0$. Lemma 3.6 then gives a deformed G_2-instanton (which is a non-trivial G_2-instanton for $k \neq 0$) on π^*L with respect to both of the homogeneous nearly parallel G_2-structures φ^{ts} and φ^{np} in (2.10) on X.

Remark 3.9. — Gauge theory on $X = SU(3)/U(1)_{1,1}$, in particular G_2-instantons with respect to the two homogeneous nearly parallel G_2-structures, is studied in some detail in [1].

3.3. ASD instantons on hypersymplectic 4-manifolds

We shall now give some simple examples arising from pull-backs of anti-self-dual connections on a hypersymplectic 4-dimensional manifold. As we

shall see, this is directly analogous to the construction arising from anti-self-dual Einstein 4-orbifolds considered in the previous subsection.

DEFINITION 3.10. — *A hypersymplectic* structure *on an oriented 4-manifold Z^4 is a triple of closed 2-forms $(\omega_1, \omega_2, \omega_3)$ on Z so that, for a volume form vol_Z on Z,*

$$(3.7) \qquad\qquad \omega_i \wedge \omega_j = 2Q_{ij}\, \mathrm{vol}_Z,$$

where the matrix Q_{ij} is positive definite. We may choose vol_Z in (3.7) so that the matrix Q_{ij} has determinant one. In this manner, the hypersymplectic structure determines a Riemannian metric g_Z on Z whose bundle of self-dual two forms is spanned by $\{\omega_1, \omega_2, \omega_3\}$ and whose Riemannian volume form is vol_Z.

Example 3.11. — If (Z^4, g_Z) is a K3 surface with $(\omega_1, \omega_2, \omega_3)$ a hyperkähler triple, then (3.7) is satisfied with $Q_{ij} = \delta_{ij}$ and g_Z in Definition 3.10 is the associated Ricci-flat Kähler metric. See [8] for more about hypersymplectic structures and their relation to kyperkähler geometry.

Consider a 3-torus bundle $\pi : X \to Z$ over such a hypersymplectic 4-manifold Z with an anti-self-dual connection η. We regard the connection $\eta = (\eta_1, \eta_2, \eta_3) \in \Omega^1(X, \mathbb{R}^3)$ as three 1-forms on the total space whose curvatures $\mathrm{d}\,\eta_i$ are the pullback of anti-self-dual 2-forms on (Z, g_Z). Then, we consider G_2-structures on X whose corresponding 4-forms are

$$(3.8) \qquad \psi_t = \pi^* \mathrm{vol}_Z - t^2 \left(\eta_{23} \wedge \pi^*\omega_1 + \eta_{31} \wedge \pi^*\omega_2 + \eta_{12} \wedge \pi^*\omega_3 \right),$$

for some constant $t > 0$. (For more details on the relation between G_2-structures and hypersymplectic structures see [9].) We see that $\mathrm{d}\,\psi_t = 0$ as the $\mathrm{d}\,\eta_i$ are assumed to be the pull-back of anti-self-dual 2-forms and thus $\mathrm{d}\,\eta_i \wedge \omega_j = 0$ for all i, j. This is the only point for which the condition that η is anti-self-dual is used.

We now construct some simple deformed G_2-instantons with respect to ψ_t in (3.8). We regard these as being trivial examples since, as in Lemma 3.6 above, they are G_2-instantons for which the cubic term in the curvature vanishes. The proof is almost identical to Lemma 3.6 so we omit it.

LEMMA 3.12. — *Let Z be a hypersymplectic 4-manifold, let L be a Hermitian complex line bundle on Z and let A be a unitary connection on L. Let $\pi : X \to Z$ be a T^3-bundle over Z with anti-self-dual connection η and let ψ_t be as in (3.8).*

Then π^*A is a (deformed) G_2-instanton on π^*L with respect to some (and hence all) ψ_t if and only if A is an ASD instanton.

Example 3.13. — Let Z be a K3 surface and $\omega_1, \omega_2, \omega_3$ a hyperkähler triple, and let $X = T^3 \times Z$ so that η is the trivial flat connection on $\pi : X \to Z$. Note that in this case X is a G_2-manifold with the product G_2-structure whose 4-form is $\psi = \psi_1$ in (3.8).

Denote by $\mathcal{H}_+ = \langle \omega_1, \omega_2, \omega_3 \rangle$ the space of self-dual harmonic 2-forms and by $\mathcal{H}_-$ that of anti-self-dual ones. Then $H^2(X, \mathbb{R}) \cong \mathcal{H}_+ \oplus \mathcal{H}_-$ and suppose that $\mathcal{H}_- \cap H^2(X, \mathbb{Z}) \neq 0$. Then, for any complex line bundle so that $c_1(L) \in \mathcal{H}_- \cap H^2(X, \mathbb{Z})$, Lemma 3.12 applies and we obtain a connection A which is both a G_2-instanton and a deformed G_2-instanton for ψ on the G_2-manifold $X = T^3 \times Z$.

Remark 3.14. — Similar constructions of G_2-instantons to Example 3.13, also performed in the higher rank case, can be found in [6].

4. Examples

We now turn to the main goal of this article, which is to construct the first non-trivial examples of deformed G_2-instantons. At the end of the section, we shall also discuss implications of this construction for the deformation theory of deformed G_2-instantons.

In this section, unless we state otherwise, we let X^7 be a 3-Sasakian 7-manifold as in Definition 2.1, and we shall use the notation introduced in section 2. In particular we let Z^4 be the leaf space of the $\mathrm{SU}(2)$ action on X given in (2.1). We also let L_0 denote the trivial complex line bundle over X.

4.1. G_2-instantons

We consider connections A on the trivial complex line bundle L_0 over X, given by

$$(4.1) \qquad A = \mathrm{i}(a_1 \eta_1 + a_2 \eta_2 + a_3 \eta_3),$$

for $a_1, a_2, a_3 \in \mathbb{R}$, where the η_i are given in (2.2). (Here, we identify the Lie algebra $\mathfrak{u}(1)$ with $\mathrm{i}\,\mathbb{R}$.) The curvature of A is given by

$$(4.2) \qquad F_A = -2\,\mathrm{i}\,a_1\,(\omega_1 + \eta_{23}) - 2\,\mathrm{i}\,a_2\,(\omega_2 + \eta_{31}) - 2\,\mathrm{i}\,a_3\,(\omega_3 + \eta_{12}),$$

where the ω_i are given in (2.3), and we recall that $\eta_{ij} = \eta_i \wedge \eta_j$. Using (2.4), (2.9) and (4.2), we compute

$$(4.3) \quad F_A \wedge \psi_{t,\varepsilon} = -2\,\mathrm{i}\left((1 - 2\varepsilon t^2)\,(a_1 \eta_{23} + a_2 \eta_{31})\right.$$
$$\left. + (1 - 2t^2)a_3 \eta_{12}\right) \wedge \pi^* \mathrm{vol}_Z.$$

From (1.2) and (4.3) we are led to the following conclusion.

PROPOSITION 4.1. — *Suppose that the connection A in (4.1) is a G_2-instanton with respect to $\psi_{t,\varepsilon}$ in (2.9). Then*

- *$a_1 = a_2 = a_3 = 0$, in which case A is the trivial flat connection, or*
- *$t = 1/\sqrt{2}$, $\varepsilon = -1$, $a_1 = a_2 = 0$ and $a_3 \neq 0$, so $A = i\,a_3\eta_3$ is a G_2-instanton with respect to $\psi_{1/\sqrt{2},-1}$, or*
- *$t = 1/\sqrt{2}$ and $\varepsilon = +1$, in which case all A in (4.1) are G_2-instantons with respect to $\psi_{1/\sqrt{2},+1}$.*

Remark 4.2. — The G_2-structures with $(t, \varepsilon) = (1/\sqrt{2}, +1)$ are special, as they support a real 3-parameter family of G_2-instantons on the trivial complex line bundle L_0 on X by Proposition 4.1. This extends observations made in Proposition 17 of [1]. Moreover, the G_2-structures given by $(t, \varepsilon) = (1/\sqrt{2}, -1)$ admit a real 1-parameter family of G_2-instantons on L_0 on X.

Remark 4.3. — There are examples of G_2-instantons on higher rank bundles on X, such as Example 2 in [7], which describes an irreducible G_2-instanton with gauge group $SO(3)$ in this setting.

4.2. Deformed G_2-instantons

We now analyze solutions to the deformed G_2-instanton (or, equivalently, deformed Donaldson–Thomas connection) equation with respect to $\psi_{t,\varepsilon}$:

$$(4.4) \qquad \frac{1}{6}F_A^3 + F_A \wedge \psi_{t,\varepsilon} = 0,$$

for A a connection on the trivial complex line bundle L_0 on X. For A as in (4.1) we compute

$$(4.5) \qquad \frac{1}{6}F_A^3 = 8\,i\left(a_1^2 + a_2^2 + a_3^2\right)(a_1\eta_{23} + a_2\eta_{31} + a_3\eta_{12}) \wedge \pi^* \mathrm{vol}_Z,$$

Using (4.3) and (4.5), we see that A solves (4.4) if and only if

$$(4.6) \quad \left(4(a_1^2 + a_2^2 + a_3^2) - (1 - 2\varepsilon t^2)\right)(a_1\eta_{23} + a_2\eta_{31})$$
$$+ \left(4(a_1^2 + a_2^2 + a_3^2) - (1 - 2t^2)\right)a_3\eta_{12} = 0.$$

We see that (4.6) always has the solution $a_1 = a_2 = a_3 = 0$, which corresponds to the flat connection. Otherwise, if $\varepsilon = +1$ then we must have

$$(4.7) \qquad a_1^2 + a_2^2 + a_3^2 = \frac{1}{4}\left(1 - 2t^2\right).$$

We immediately see that (4.7) can be solved for a non-flat connection A if and only if $2t^2 < 1$, in which case there is a whole 2-sphere of solutions. If, instead, $\varepsilon = -1$ then if $a_1^2 + a_2^2 \neq 0$ we must have

$$(4.8) \qquad a_1^2 + a_2^2 = \frac{1}{4}(1 + 2t^2) \quad \text{and} \quad a_3 = 0,$$

and if $a_3 \neq 0$ we must have

$$(4.9) \qquad a_1^2 + a_2^2 = 0 \quad \text{and} \quad a_3^2 = \frac{1}{4}(1 - 2t^2).$$

Clearly, (4.8) gives a circle of solutions for any $t > 0$, whereas (4.9) gives non-trivial solutions if and only if $2t^2 < 1$, in which case there are two solutions.

We state these findings as follows.

PROPOSITION 4.4. — *Let $a_1, a_2, a_3 \in \mathbb{R}$ and let $A = \mathrm{i}\, a_j \eta_j$ as in (4.1) be a connection on the trivial complex line bundle L_0 on X. Then A is a deformed G_2-instanton with respect to $\psi_{t,\varepsilon}$ in (2.9) if and only if either $a_1 = a_2 = a_3 = 0$, so A is the trivial flat connection, or one of the following holds:*

- *$t \in (0, 1/\sqrt{2})$, $\varepsilon = +1$ and a_1, a_2, a_3 satisfy (4.7) (so there is a 2-sphere of solutions);*
- *$t \in (0, 1/\sqrt{2})$, $\varepsilon = -1$ and a_1, a_2, a_3 satisfy (4.8) or (4.9) (so the solutions consist of a circle and two points);*
- *$t \geqslant 1/\sqrt{2}$, $\varepsilon = -1$, and a_1, a_2, a_3 satisfy (4.8) (so there is a circle of solutions).*

By (2.10), Proposition 4.4 immediately gives the following two results.

COROLLARY 4.5. — *Recall the nearly parallel G_2-structures φ^{np} and φ^{ts} on X given in (2.10).*

- *There is a 2-sphere of non-trivial deformed G_2-instantons on L_0 over X with respect to φ^{np} arising from (4.1).*
- *There is a circle of non-trivial deformed G_2-instantons on L_0 over X with respect to φ^{ts} arising from (4.1).*

Remark 4.6. — Corollary 4.5 demonstrates how Proposition 4.4 can be used to show that deformed G_2-instantons can discriminate between G_2-structures on X; in particular, between the two natural nearly parallel G_2-structures on X. We also see that the family of deformed G_2-structures for these two nearly parallel G_2-structures has different dimensions, and there is no obvious relation between them.

Corollary 4.7. — *Recall the two Einstein metrics g^{ts} and g^{np} on X given in (2.5) and (2.6), and recall that $\varphi_{1,\varepsilon}$ induces g^{ts} and $\varphi_{1/\sqrt{5},\varepsilon}$ induces g^{np} for $\varepsilon \in \{\pm 1\}$.*

- *Using the ansatz (4.1), there is a circle of non-trivial deformed G_2-instantons with respect to $\varphi_{1,-1}$, whereas there are no non-trivial deformed G_2-instantons with respect to $\varphi_{1,+1}$.*
- *Using the ansatz (4.1), there is a circle plus two isolated examples of non-trivial deformed G_2-instantons with respect to $\varphi_{1/\sqrt{5},-1}$, whereas there is a 2-sphere of non-trivial deformed G_2-instantons with respect to $\varphi_{1/\sqrt{5},+1}$.*

Remark 4.8. — Corollary 4.7 indicates how Proposition 4.4 shows that deformed G_2-instantons can be used to distinguish between isometric G_2-structures on X; in particular, between the two natural Einstein metrics on X. However, we observe that for these two Einstein metrics, whilst the spaces of deformed G_2-instantons having the form (4.1) are very different for the two isometric G_2-structures, their Euler characteristics are the same. This is pertinent since one might hope to use the Euler characteristic of the moduli space as a possible enumerative invariant for deformed G_2-instantons.

We give a concrete example of the construction.

Example 4.9. — Take the 7-sphere S^7 as in Example 2.4 and let L_0 be the trivial complex line bundle over S^7. Corollary 4.5 gives a 2-sphere of deformed G_2-instantons on L_0 over (S^7, φ^{np}), and a circle of deformed G_2-instanton on L_0 over (S^7, φ^{ts}). On the other hand, Corollary 4.7 shows that we have a family of deformed G_2-instantons on L_0 consisting of a circle plus two further points for another G_2-structure inducing the squashed metric g^{np}, and we have no known non-trivial deformed G_2-instantons on L_0 for another G_2-structure inducing the round metric g^{ts}.

In this way, deformed G_2-instantons on the trivial complex line bundle can be used to distinguish between the two homogeneous nearly parallel G_2-structures on S^7, and between isometric G_2-structures for the two homogeneous Einstein metrics on S^7.

4.3. Non-trivial line bundles

One may ask whether there are non-trivial examples of deformed G_2-instantons on non-trivial line bundles. The natural approach is to use a

non-trivial bundle L over Z equipped with an anti-self-dual connection A_0, using the ideas in subsection 3.2. We shall give a particular example, which may clearly be generalized to other 3-Sasakian 7-manifolds, of a existence result for non-trivial deformed G_2-instantons on non-trivial line bundles.

Consider the setting of Example 3.8, where X is the Aloff–Wallach space $\mathrm{SU}(3)/\mathrm{U}(1)_{1,1}$, where $\mathrm{U}(1)_{1,1}$ is given in (3.6), and recall that $Z = \overline{\mathbb{CP}^2}$. Let $L = \mathcal{O}_{\overline{\mathbb{CP}^2}}(k)$ for some $k \in \mathbb{Z}$, and let A_0 be the connection on L with harmonic curvature (which must be an ASD instanton as observed in Example 3.8).

Recalling the η_i in (2.2), we may consider a connection A on π^*L over X given by

$$(4.10) \qquad A = \pi^*A_0 + \mathrm{i}\,a, \quad \text{for} \quad a = a_j\eta_j,$$

where $a_1, a_2, a_3 \in \mathbb{R}$. The curvature of A is $F_A = F_{A_0} + \mathrm{i}\,\mathrm{d}\,a$ and as A_0 is anti-self-dual we have $\pi^*F_{A_0} \wedge \psi_{t,\varepsilon} = 0$ by Lemma 3.6. Hence,

$$(4.11)$$
$$F_A \wedge \psi_{t,\varepsilon} = \mathrm{i}\,\mathrm{d}\,a \wedge \psi_{t,\varepsilon}$$
$$= -2\mathrm{i}\left((1 - 2\varepsilon t^2)(a_1\eta_{23} + a_2\eta_{31}) + (1 - 2t^2)a_3\eta_{12}\right) \wedge \pi^*\,\mathrm{vol}_Z\,.$$

by (4.3). As for the cubic term in the curvature we find that

$$(4.12) \qquad F_A^3 = 3\,\mathrm{i}\,\pi^*F_{A_0}^2 \wedge \mathrm{d}\,a - 3\pi^*F_{A_0} \wedge (\mathrm{d}\,a)^2 - \mathrm{i}(\mathrm{d}\,a)^3$$

as $F_{A_0}^3 = 0$ for dimensional reasons. By inspection, we see that $(\mathrm{d}\,a)^2 = \beta_i \wedge \omega_i$ for some 2-forms β_i. As the ω_i are self-dual and F_{A_0} is anti-self-dual we find that $F_{A_0} \wedge (\mathrm{d}\,a)^2 = 0$. Hence, (4.12) becomes

$$(4.13) \qquad F_A^3 = -\mathrm{i}(\mathrm{d}\,a)^3 + 3\,\mathrm{i}\,\pi^*F_{A_0}^2 \wedge \mathrm{d}\,a.$$

The first term in (4.13) is given by the right-hand side of (4.5) (multiplied by 6). As for the second term, we find that $F_{A_0}^2 = |F_{A_0}|^2\,\mathrm{vol}_Z$, with $|\cdot|$ denoting the norm with respect to g_Z, since F_{A_0} is anti-self-dual (recalling that F_{A_0} is imaginary-valued). Thus, using (4.2), we see that

$$(4.14) \quad 3\,\mathrm{i}\,\pi^*F_{A_0}^2 \wedge \mathrm{d}\,a = -6\,\mathrm{i}\,\pi^*|F_{A_0}|^2\,(a_1\eta_{23} + a_2\eta_{31} + a_3\eta_{12}) \wedge \pi^*\,\mathrm{vol}_Z\,.$$

Inserting (4.5) and (4.14) in (4.13) shows that the deformed G_2-instanton equation (4.4) for $\psi_{t,\varepsilon}$ is equivalent to

$$(4.15) \quad \left(8(a_1^2 + a_2^2 + a_3^2) - \pi^*|F_{A_0}|^2 - 2(1 - 2\varepsilon t^2)\right)(a_1\eta_{23} + a_2\eta_{31})$$
$$+ \left(8(a_1^2 + a_2^2 + a_3^2) - \pi^*|F_{A_0}|^2 - 2(1 - 2t^2)\right)a_3\eta_{12} = 0.$$

At this point, we use the fact that $F_{A_0} = \mathrm{i}\,k\omega$, where ω is the Fubini–Study form on $\overline{\mathbb{CP}^2}$, and thus $|F_{A_0}|^2 = 2k^2$. (Note that we need $|F_{A_0}|^2$ to be

constant in order for (4.15) to have a solution for constant a_1, a_2, a_3 which are not all zero.) Inserting this into (4.15) for $\varepsilon = +1$ gives that non-trivial solutions must satisfy

$$(4.16) \qquad a_1^2 + a_2^2 + a_3^2 = \frac{1}{4}(1 - 2t^2 + k^2).$$

We see that (4.16) has non-trivial solutions for a_1, a_2, a_3 if and only if $2t^2 < 1 + k^2$. We therefore obtain deformed G_2-instantons with respect to $\psi_{t,+1}$ on $\pi^*\mathcal{O}(k)$ for $k \neq 0$ arising from the ansatz (4.10) which are *not* given by the pullback of the ASD instanton on $\mathcal{O}(k)$ (which is a deformed G_2-instanton by Lemma 3.6) for these values of t.

If we instead look at (4.15) for $\epsilon = -1$ then for non-trivial solutions we either have

$$(4.17) \qquad a_1^2 + a_2^2 = \frac{1}{4}(1 + 2t^2 + k^2) \quad \text{and} \quad a_3 = 0, \text{ or}$$

$$(4.18) \qquad a_1^2 + a_2^2 = 0 \quad \text{and} \quad a_3^2 = \frac{1}{4}(1 - 2t^2 + k^2).$$

We see that (4.17) admits non-trivial solutions for all t, whereas (4.18) admits non-trivial solutions if and only if $2t^2 < 1 + k^2$, just as for (4.16).

Overall, we have the following proposition, which generalizes Proposition 4.4 for the case of the Aloff–Wallach space $\mathrm{SU}(3)/\mathrm{U}(1)_{1,1}$.

PROPOSITION 4.10. — *Let X be the Aloff–Wallach space $\mathrm{SU}(3)/\mathrm{U}(1)_{1,1}$ as in Example 3.8, with projection $\pi : X \to Z = \overline{\mathbb{CP}^2}$. Let $L = \mathcal{O}(k)$ on Z for $k \in \mathbb{Z}$, let A_0 be the unitary ASD instanton on L and let A be a connection on π^*L as in (4.10) which is a deformed G_2-instanton with respect to $\psi_{t,\varepsilon}$ given in (2.9). Then either $a_1 = a_2 = a_3 = 0$, and so $A = \pi^*A_0$ (and thus is a G_2-instanton), or one of the following holds:*

- *$t \in (0, \sqrt{(1+k^2)/2})$, $\varepsilon = +1$ and a_1, a_2, a_3 satisfy (4.16);*
- *$t \in (0, \sqrt{(1+k^2)/2})$, $\varepsilon = -1$ and a_1, a_2, a_3 satisfy (4.17) or (4.18);*
- *$t \geqslant \sqrt{(1+k^2)/2}$, $\varepsilon = -1$ and a_1, a_2, a_3 satisfy (4.17).*

Remark 4.11. — We see that (4.18) has non-trivial solutions for $t = 1$ if and only if $|k| > 1$. In particular, Proposition 4.10 gives non-trivial deformed G_2-instantons on $\pi^*\mathcal{O}(k)$ over X with respect to the G_2-structure $\varphi_{1,+1}$, which induces the 3-Sasakian metric g^{ts}, if and only if $|k| > 1$. Moreover, when $|k| > 1$, Proposition 4.10 gives a 2-sphere of non-trivial deformed G_2-instantons on $\pi^*\mathcal{O}(k)$ with respect to $\varphi_{1,+1}$, whereas it gives a family of deformed G_2-instantons consisting of a circle and two points with respect to φ^{ts}.

Proposition 4.10 has the following immediate corollary.

COROLLARY 4.12. — *Let X^7 be the Aloff–Wallach space $\mathrm{SU}(3)/\mathrm{U}(1)_{1,1}$ as in Example 3.8, with projection $\pi : X \to \overline{\mathbb{CP}^2}$, and recall the nearly parallel G$_2$-structures φ^{np} and φ^{ts} on X given in (2.10).*

- *For every $k \in \mathbb{Z}$, there is a 2-sphere of non-trivial deformed G$_2$-instantons on $\pi^* \mathcal{O}(k)$ with respect to φ^{np}.*
- *If $k \in \{0, \pm 1\}$, there is a circle of non-trivial deformed G$_2$-instantons on $\pi^* \mathcal{O}(k)$ with respect to φ^{ts}, and if $k \in \mathbb{Z} \backslash \{0, \pm 1\}$ there is a circle and two further examples of non-trivial deformed G$_2$-instantons on $\pi^* \mathcal{O}(k)$ with respect to φ^{ts}.*

Remark 4.13. — Proposition 4.10 continues to demonstrate how deformed G$_2$-instantons can distinguish between nearly parallel G$_2$-structures and isometric G$_2$-structures. Moreover, for isometric G$_2$ structures, the observed equality between the Euler characteristics of the families of deformed G$_2$-instantons, having the form (4.1), continues to hold in this setting.

4.4. Moduli spaces

We now make some observations on the families of deformed G$_2$-instantons we have constructed, and their relation to the deformation theory of deformed G$_2$-instantons developed in [14]. To state the deformation theory result we recall the following definition.

DEFINITION 4.14. — *For a 7-manifold X with a coclosed G$_2$-structure φ and a Hermitian complex line bundle L on X, we let $\mathcal{M}(X, \varphi, L)$ denote the moduli space of deformed G$_2$-instantons on L with respect to φ. Let $A \in \mathcal{M}(X, \varphi, L)$ and consider the complex*

$$(4.19) \qquad 0 \longrightarrow \Omega^0(X) \xrightarrow{\mathrm{d}} \Omega^1(X) \xrightarrow{(\frac{1}{2}F_A^2 + *\varphi) \wedge \mathrm{d}} \mathrm{d}\,\Omega^5(X) \longrightarrow 0.$$

Then A is unobstructed *if $H^2 = 0$ for (4.19), i.e. if the linearisation of the deformed G$_2$-instanton condition $(\frac{1}{2}F_A^2 + *\varphi) \wedge \mathrm{d} : \Omega^1(X) \to \mathrm{d}\,\Omega^5(X)$ is surjective; otherwise A is* obstructed.

We now state a deformation theory result that follows from [14], which motivates the definition of unobstructed in Definition 4.14.

THEOREM 4.15. — *Let (X^7, φ) be a compact 7-manifold with a coclosed G$_2$-structure, and let L be a Hermitian complex line bundle on X. Then $\mathcal{M}(X, \varphi, L)$ has expected dimension $b^1(X)$. Therefore, if $A \in \mathcal{M}(X, \varphi, L)$ is unobstructed, then $\mathcal{M}(X, \varphi, L)$ is a smooth manifold near A of dimension $b^1(X)$.*

Moreover, if $A \in \mathcal{M}(X, \varphi, L)$ and

- $\mathrm{d}\,\varphi = \lambda *_\varphi \varphi$ *for some* $\lambda \in \mathbb{R}$, *or*
- $F_A^3 \neq 0$ *everywhere*,

then for generic coclosed G_2-*structures* φ' *sufficiently near* φ *so that* $[*_{\varphi'}\varphi'] = [*_\varphi\varphi] \in H^4(X)$, *the subset of* $\mathcal{M}(X, \varphi', L)$ *of connections sufficiently near* A *is a smooth manifold of dimension* $b^1(X)$ *(if it is non-empty).*

Theorem 4.15 has the following consequence.

COROLLARY 4.16. — *Let* (X^7, φ) *be a compact 7-manifold with a coclosed* G_2-*structure, and suppose that* X^7 *admits a nearly parallel* G_2-*structure. Let* L *be a Hermitian complex line bundle on* X. *Then the expected dimension of* $\mathcal{M}(X, \varphi, L)$ *is* 0. *In particular, if* $A \in \mathcal{M}(X, \varphi, L)$ *is unobstructed then* A *is rigid and locally isolated in* $\mathcal{M}(X, \varphi, L)$.

Moreover, given $A \in \mathcal{M}(X, \varphi, L)$ *such that* $F_A^3 \neq 0$ *everywhere, for generic coclosed* G_2-*structures* φ' *near* φ *with* $[*_{\varphi'}\varphi'] = [*_\varphi\varphi]$, *the subset of* $\mathcal{M}(X, \varphi', L)$ *of connections sufficiently near* A *is a discrete collection of points (if it is non-empty).*

Proof. — The result follows from Theorem 4.15 and the fact that X^7 must have finite fundamental group by Myers theorem, since the induced metric from a nearly parallel G_2-structure is Einstein with positive scalar curvature. □

Since the non-trivial deformed G_2-instantons given by Corollary 4.5 are not rigid for the nearly parallel G_2-structures φ^{np} and φ^{ts} on the 3-Sasakian X^7, Corollary 4.16 yields the following result.

PROPOSITION 4.17. — *All of the non-trivial deformed* G_2-*instantons from Propositions 4.4 and 4.10 that exist in positive-dimensional families are obstructed.*

In particular, the non-trivial deformed G_2-*instantons on the trivial line bundle* L_0 *over* (X^7, φ^{ts}) *and* (X^7, φ^{np}) *from Proposition 4.4 are obstructed.*

Remark 4.18. — We see from (4.5) and Propositions 4.4 and 4.10 that all of the non-trivial deformed G_2-instantons A with respect to $\varphi_{t,\varepsilon}$ we have constructed have the property that $F_A^3 \neq 0$ everywhere, and thus Corollary 4.16 applies. Moreover, when $\varepsilon = +1$ we see that $[\psi_{t,+1}] = [\psi_{t',+1}]$ for all t, t'. However, $\mathcal{M}(X, \varphi_{t',+1}, L_0)$ still contains an S^2 family of deformed G_2-instantons near A for all t' near t. We conclude that the family $\varphi_{t,+1}$ is not sufficiently generic to enable us to perturb A to become locally isolated.

We now make an elementary observation, which follows from Remark 5.13 in [14].

LEMMA 4.19. — *Let (X^7, φ) be a compact 7-manifold with a nearly parallel G_2-structure, and let L_0 be the trivial complex line bundle on X. Then the trivial flat connection is unobstructed as a deformed G_2-instanton, thus rigid and locally isolated in $\mathcal{M}(X, \varphi, L_0)$.*

Lemma 4.19 yields the following interesting result.

PROPOSITION 4.20. — *For any 3-Sasakian 7-manifold X, the moduli spaces $\mathcal{M}(X, \varphi^{np}, L_0)$ and $\mathcal{M}(X, \varphi^{ts}, L_0)$ have at least two components of different dimensions.*

Proof. — Since the trivial flat connection A_0 lies in the moduli spaces $\mathcal{M}(X, \varphi^{np}, L_0)$ and $\mathcal{M}(X, \varphi^{ts}, L_0)$ and is locally isolated, it must define a component of the moduli space in each case. Therefore, in each case the positive-dimensional family of non-trivial deformed G_2-instantons from Proposition 4.4 must lie in a different component of the moduli space to the trivial flat connection. □

5. A Chern–Simons type functional

In this section, we study a functional of Chern–Simons type, introduced in [12], which has deformed G_2-instantons as critical points, in the setting of our examples.

5.1. The functional

Let X^7 be a compact 7-manifold with a coclosed G_2-structure φ. (The assumption of compactness is to ensure that integrals of various quantities over X are guaranteed to be well-defined.) Let L be a Hermitian complex line bundle over X and fix a unitary connection A_0 on L. We let t denote a coordinate on $[0, 1]$ and we shall pullback various quantities from X to $X \times [0, 1]$ such as $\psi = *_\varphi \varphi$ and L; for ease of notation, we shall omit the pullback symbol.

Then, for any other unitary connection A on L we consider a unitary connection $\mathbf{A}$ on the pullback of L to $[0, 1] \times X$, given by

$$(5.1) \qquad \mathbf{A} = A_0 + t(A - A_0).$$

We let $A_t = \mathbf{A}|_{\{t\} \times X}$ and let F_t be the curvature of A_t, which is

$$(5.2) \qquad F_t = F_{A_0} + t(F_A - F_{A_0}).$$

Hence, the curvature of $\mathbf{A}$ in (5.1) can be written

$$(5.3) \qquad\qquad \mathbf{F} = \mathrm{d}\,t \wedge (A - A_0) + F_t.$$

With this notation, we can make the following definition.

DEFINITION 5.1. — *Let (X^7, φ) be a compact 7-manifold with a co-closed G_2-structure and let L be a Hermitian complex line bundle on X. Given a unitary connection A_0 on L we define the functional $\mathcal{F}$ on unitary connections A on L over X by the formula:*

$$(5.4) \qquad\qquad \mathcal{F}(A) = \int_{X \times [0,1]} e^{\mathbf{F} + \psi},$$

where $\psi = *_\varphi \varphi$ and $\mathbf{F}$ is given in (5.3). It is shown in [12] that the deformed G_2-instanton equation (1.1) arises as the critical point equation for the functional $\mathcal{F}$.

Since X is 7-dimensional, it is convenient to expand and rewrite $\mathcal{F}$ in (5.4) as follows:

$$(5.5) \qquad \mathcal{F}(A) = \int_{X \times [0,1]} \frac{1}{3!} (\mathbf{F} + \psi)^3 + \frac{1}{4!} (\mathbf{F} + \psi)^4$$

$$= \frac{1}{2} \int_{X \times [0,1]} \mathbf{F}^2 \wedge \psi + \frac{1}{12} \mathbf{F}^4.$$

On the other hand, we have $\mathbf{F}^k = F_{A_0}^k + \mathrm{d}\,(cs_k(A_0, \mathbf{A}))$ on $[0,1] \times X$, where $\mathbf{F}$ is given in (5.3) and $cs_k(A_0, \mathbf{A})$ is the kth transgression form. This can be explicitly written using the curvature $\mathbf{F}_s$ of the connections $\mathbf{A}_s = A_0 + s(\mathbf{A} - A_0)$ for s a real parameter. Indeed, we have

$$(5.6) \qquad\qquad cs_k(A_0, \mathbf{A}) = k \int_0^1 (\mathbf{A} - A_0) \wedge \mathbf{F}_s^{k-1} \,\mathrm{d}\,s.$$

Thus, from Stokes' theorem and the fact that $F_{A_0}^2 \wedge \psi = 0 = F_{A_0}^4$ by dimensional reasons, we may write (5.5) as:

$$(5.7) \qquad \mathcal{F}(A) = \frac{1}{2} \int_X cs_2(A_0, A) \wedge \psi + \frac{1}{12} cs_4(A_0, A).$$

In particular, we have the following observation.

LEMMA 5.2. — *In the notation of Definition 5.1, let $L = L_0$ be the trivial complex line bundle and let A_0 be the trivial flat connection. Then the functional $\mathcal{F}$ in (5.4) is given by*

$$(5.8) \qquad \mathcal{F}(A) = \frac{1}{2} \int_X (A - A_0) \wedge \left(F_A \wedge \psi + \frac{1}{12} F_A^3 \right).$$

Proof. — The result follows from (5.7) and the observation that $cs_k(A_0, A) = (A - A_0) \wedge F_A^{k-1}$ by (5.6) as $F_{A_0} = 0$. $\square$

5.2. Examples

For connections as in (4.1) on 3-Sasakian 7-manifolds we find, after a lengthy but straightforward computation, the following using (5.8).

PROPOSITION 5.3. — *Let X^7 be a 3-Sasakian 7-manifold with the co-closed G_2-structure $\varphi_{t,\varepsilon}$ on (2.7), and recall the 3-Sasakian metric g^{ts} in (2.5). For connections $A = \mathrm{i} a_j \eta_j$ as in (4.1) on the trivial complex line bundle L_0 on X, we have that the functional $\mathcal{F}$ in (5.4) is given by*

$$(5.9) \qquad \mathcal{F}(A) = -c\left[(x^2 + y^2)(2(x^2 + y^2) - 1) + 2t^2(x^2 + \varepsilon y^2)\right],$$

where $x = a_3$, $y^2 = a_1^2 + a_2^2$ and

$$(5.10) \qquad c = \int_X \eta_{123} \wedge \pi^* \mathrm{vol}_Z = \mathrm{Vol}(X, g^{ts}) > 0.$$

Remark 5.4. — Notice that $\mathcal{F}$ in (5.9) is bounded above and that, as can be seen from (5.10), the 3-Sasakian metric g^{ts} may be rescaled so that $c = 1$, which we will now do for convenience.

Remark 5.5. — The critical points of the functional $\mathcal{F}$ in (5.9), restricted to connections given by the ansatz (4.1), are given by the vanishing of

$$(5.11) \qquad \frac{\partial \mathcal{F}}{\partial x} = -2x\left(4(x^2 + y^2) + 2t^2 - 1\right),$$

$$(5.12) \qquad \frac{\partial \mathcal{F}}{\partial y} = -2y\left(4(x^2 + y^2) + 2\epsilon t^2 - 1\right).$$

We see that we have solutions to (5.11)–(5.12) given by $x = 0 = y$, corresponding to the flat connection, and

$$(5.13) \qquad x^2 + y^2 = \frac{1}{4}(1 - 2t^2) \text{ when } \epsilon = 1 \text{ and } 2t^2 < 1,$$

or $\epsilon = -1$ and either

$$(5.14) \quad x = 0 \text{ and } y^2 = \frac{1}{4}(1 + 2t^2), \quad \text{or}$$

$$y = 0 \text{ and } x^2 = \frac{1}{4}(1 - 2t^2) \text{ when } 2t^2 < 1.$$

Equations (5.13)–(5.14) coincide with the conditions (4.7)–(4.9) we derived earlier that gave our non-trivial deformed G_2-instantons in Proposition 4.4.

Using Proposition 5.3 we can examine the relationship between the trivial flat connection and the functional $\mathcal{F}$. In particular, we see that the nature of the trivial connection as a critical point for $\mathcal{F}$ depends on the choice of G_2-structure.

LEMMA 5.6. — *Recall the notation of Proposition 5.3, in particular the functional $\mathcal{F}$ in (5.9). Let A_0 be the trivial flat connection on the trivial complex line bundle L_0 over X. Then the Hessian of $\mathcal{F}$ is nondegenerate at A_0 if and only if $t \neq 1/\sqrt{2}$. Moreover:*

- *if $t \in (0, 1/\sqrt{2})$ then A_0 is a local minimum of $\mathcal{F}$;*
- *if $t \geqslant 1/\sqrt{2}$ and $\varepsilon = +1$ then A_0 is a local maximum of $\mathcal{F}$;*
- *if $t \geqslant 1/\sqrt{2}$ and $\varepsilon = -1$ then A_0 is a saddle point of $\mathcal{F}$.*

Proof. — By direct computation, one can calculate the Hessian of the functional $\mathcal{F}$ in (5.9). At the flat connection A_0, when $(x, y) = (0, 0)$, the Hessian of $\mathcal{F}$ has eigenvalues

$$(5.15) \qquad\qquad 2(1 - 2t^2) \quad \text{and} \quad 2(1 - 2\varepsilon t^2).$$

We see immediately that the Hessian of $\mathcal{F}$ is degenerate if and only if $2t^2 = 1$ as claimed. Moreover, as long as $2t^2 \neq 1$, the critical point is characterised by the signs of the eigenvalues in (5.15). When $2t^2 = 1$ we see that

$$(5.16) \qquad\qquad \mathcal{F}(A) = -2(x^2 + y^2)^2 + (1 - \varepsilon)y^2.$$

When $\varepsilon = +1$, $\mathcal{F} \leqslant 0$ and equals 0 if and only if $x = y = 0$. When $\varepsilon = -1$, we instead see that inserting $x = 0$ in (5.16) gives a function of y with a local minimum at $y = 0$, and for $y = 0$ in (5.16) we obtain a local maximum at $x = 0$. The result then follows. $\qquad\square$

We already observed the significance of the value $t = 1/\sqrt{2}$ in Proposition 4.4. Lemma 5.6 now leads to the following additional observation concerning this value of t.

COROLLARY 5.7. — *The trivial flat connection on the trivial complex line bundle over X is obstructed as a deformed G_2-instanton for the G_2-structures $\varphi_{1/\sqrt{2}, \varepsilon}$ in (2.7) for $\varepsilon = \pm 1$.*

Proof. — When $t = 1/\sqrt{2}$, Lemma 5.6 shows that the Hessian of $\mathcal{F}$ in (5.9) is degenerate at the trivial flat connection A_0. Thus, there exist non-trivial infinitesimal deformations of A_0 as a deformed G_2-instanton within the ansatz (4.1), i.e. H^1 of the complex (4.19) is non-zero.

However, we know from Proposition 4.4 that A_0 is locally isolated as a deformed G_2-instanton for $t = 1/\sqrt{2}$ amongst those given by (4.1), and so

the non-trivial infinitesimal deformation must be obstructed, i.e. H^2 of the complex (4.19) must also be non-zero. □

Remark 5.8. — The fact that the flat connection is obstructed at $t = 1/\sqrt{2}$ was to be expected, as it is at this point that the transition occurs when the 2-sphere or two points consisting of non-flat deformed G_2-instantons "shrinks" and merges with the flat connection (see Figure 1.1). Nevertheless, this observation contrasts with the case of nearly parallel G_2-structures for which the flat connection is always unobstructed (see Lemma 4.19).

To conclude, we compare the functional $\mathcal{F}$ in (5.9) for the pairs of isometric G_2-structures $\varphi_{1/\sqrt{5},\epsilon}$ and $\varphi_{1,\epsilon}$, for $\epsilon \in \{\pm1\}$, which induce the Einstein metrics g^{np} and g^{ts} respectively.

Example 5.9. — The coclosed G_2-structures $\varphi_{1/\sqrt{5},+1} = \varphi^{np}$ and $\varphi_{1/\sqrt{5},-1}$ both induce the strictly nearly parallel metric g^{np} on the 3-Sasakian 7-manifold X, which has the property that the metric cone on (X, g^{np}) has holonomy Spin(7). These G_2-structures determine rather different functionals $\mathcal{F}$ by Proposition 4.4. Figure 5.1 plots the functional $\mathcal{F}$, restricted to connections given by the ansatz (4.1), as in (5.9) for these two G_2-structures.
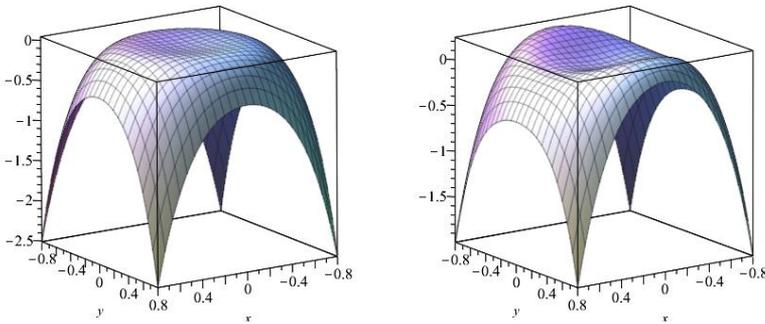

Figure 5.1. The functional $\mathcal{F}$ for $(t, \varepsilon) = (1/\sqrt{5}, +1)$ and $(t, \varepsilon) = (1/\sqrt{5}, -1)$.

One can just discern the local minimum at the origin in the left-hand plot in Figure 5.1, predicted by Lemma 5.6. We also illustrate the difference between these cases via their levels sets in Figure 5.2. One can see the circle of critical points (which are local maxima) for $\varepsilon = +1$ which gives the 2-sphere of non-trivial deformed G_2-instantons in Proposition 4.4, in

contrast to the two pairs of critical points for $\varepsilon = -1$ which give the circle (given by local maxima) and two further examples (which are saddle points) of non-trivial deformed G_2-instantons in Proposition 4.4.
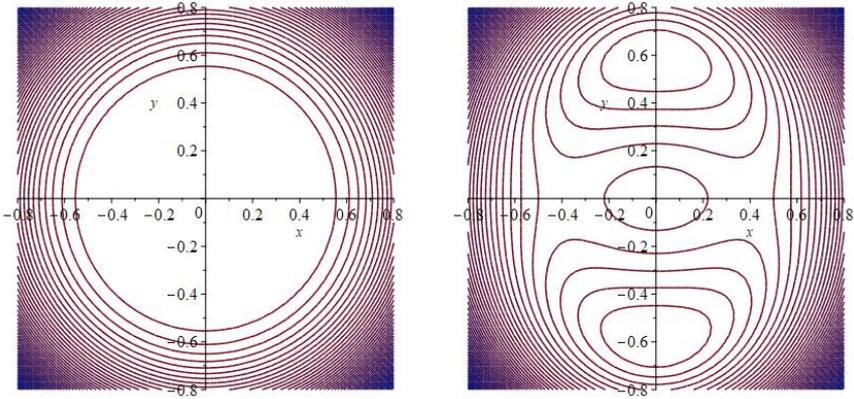

Figure 5.2. Level sets of $\mathcal{F}$ for $(t, \varepsilon) = (1/\sqrt{5}, +1)$ and $(t, \varepsilon) = (1/\sqrt{5}, -1)$.

Example 5.10. — We now focus on the coclosed G_2-structures $\varphi_{1,\epsilon}$, for $\epsilon = \pm 1$, recalling that $\varphi^{ts} = \varphi_{1,-1}$. These G_2-structures induce the 3-Sasakian metric g^{ts} on X, which is that whose metric cone is hyperkähler. In this case, we again already know the functionals $\mathcal{F}$ for these two G_2-structures are very different by Proposition 4.4, and further evidence is provided by the following plots of the functional $\mathcal{F}$ in (5.9) in Figure 5.3.
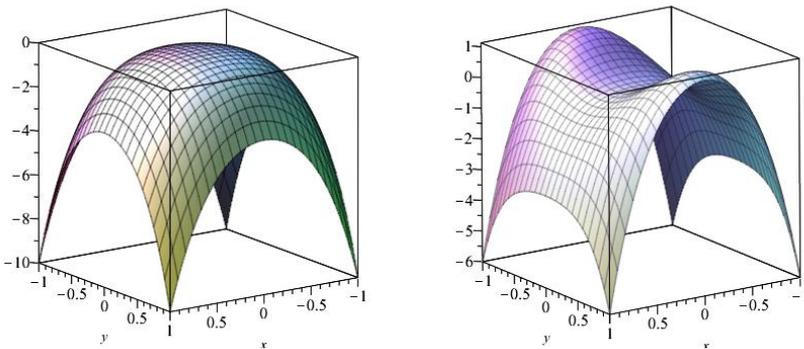

Figure 5.3. The functional $\mathcal{F}$ for $(t, \varepsilon) = (1, +1)$ and $(t, \varepsilon) = (1, -1)$.

As for the level sets of the functional $\mathcal{F}$, these are plotted in Figure 5.4 for each case. For $\varepsilon = +1$ we see that the only critical point is the origin, giving the trivial flat connection, and instead we have a pair of critical points (which are local maxima) for $\varepsilon = -1$ which define a circle of deformed G$_2$-instantons from Proposition 4.4.
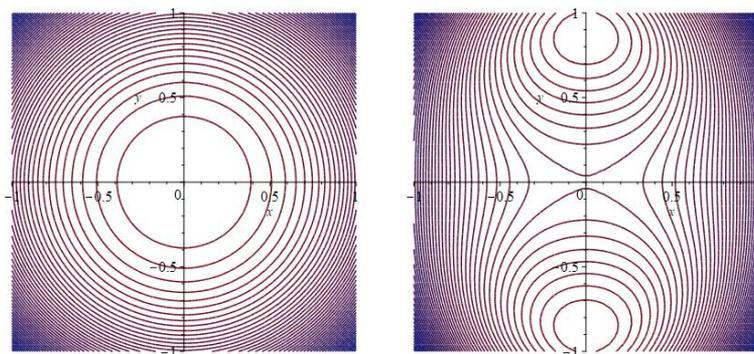

Figure 5.4. Level sets of $\mathcal{F}$ for $(t, \varepsilon) = (1, +1)$ and $(t, \varepsilon) = (1, -1)$.

BIBLIOGRAPHY

[1] G. Ball & G. Oliveira, "Gauge theory on Aloff–Wallach spaces", *Geom. Topol.* **23** (2019), no. 2, p. 685-743.

[2] C. Boyer & K. Galicki, "3-Sasaki manifolds", in *Surveys in Differential Geometry, Vol. 6; Essays on Einstein Manifolds*, vol. 6, International Press, 2001.

[3] C. P. Boyer, K. Galicki & B. M. Mann, "The geometry and topology of 3-Sasakian manifolds", *J. Reine Angew. Math.* **455** (1994), p. 183-220.

[4] O. Calvo-Andrade, L. O. Rodríguez Díaz & H. N. Sá Earp, "Gauge theory and G$_2$-geometry on Calabi–Yau links", *Rev. Mat. Iberoam.* **36** (2020), no. 6, p. 1753-1778.

[5] G. Chen, "Supercritical deformed Hermitian-Yang–Mills equation", `https://arxiv.org/abs/2005.12202`, 2020.

[6] A. Clarke, M. Garcia-Fernandez & C. Tipler, "T-Dual solutions and infinitesimal moduli of the G$_2$-Strominger system", `https://arxiv.org/abs/2005.09977`, 2020.

[7] A. Clarke & G. Oliveira, "Spin(7)-instantons from evolution equations", *J. Geom. Anal.* **31** (2021), no. 4, p. 4328-4355.

[8] S. K. Donaldson, "Two-forms on four-manifolds and elliptic equations", in *Inspired by S. S. Chern*, Nankai Tracts Math., vol. 11, World Sci. Publ., Hackensack, NJ, 2006, p. 153-172.

[9] J. Fine & C. Yao, "Hypersymplectic 4-manifolds, the G$_2$-Laplacian flow, and extension assuming bounded scalar curvature", *Duke Math. J.* **167** (2018), no. 18, p. 3533-3589.

[10] T. FRIEDRICH, I. KATH, A. MOROIANU & U. SEMMELMANN, "On nearly parallel G_2-structures", *J. Geom. Phys.* **23** (1997), no. 3-4, p. 259-286.

[11] K. GROVE, B. WILKING & W. ZILLER, "Positively curved cohomogeneity one manifolds and 3-Sasakian geometry", *J. Differential Geom.* **78** (2008), no. 1, p. 33-111.

[12] S. KARIGIANNIS & N. C. LEUNG, "Hodge theory for G_2-manifolds: intermediate Jacobians and Abel–Jacobi maps", *Proc. Lond. Math. Soc. (3)* **99** (2009), no. 2, p. 297-325.

[13] S. KARIGIANNIS, N. C. LEUNG & J. D. LOTAY (eds.), *Lectures and surveys on* G_2-*manifolds and related topics*, Fields Institute Communications, vol. 84, Springer, New York, 2020, Extended papers from the Minischool and Workshop held as part of the Major Thematic Program on Geometric Analysis at the Fields Institute, Toronto, August 19–25, 2017, xxii+382 pages.

[14] K. KAWAI & H. YAMAMOTO, "Deformation theory of deformed Hermitian Yang–Mills connections and deformed Donaldson–Thomas connections", `arxiv.org/abs/2004.00532`, 2020.

[15] J.-H. LEE & N. C. LEUNG, "Geometric structures on G_2 and Spin(7)-manifolds", *Adv. Theor. Math. Phys.* **13** (2009), no. 1, p. 1-31.

[16] N. C. LEUNG, S.-T. YAU & E. ZASLOW, "From special Lagrangian to Hermitian-Yang–Mills via Fourier–Mukai transform", *Adv. Theor. Math. Phys.* **4** (2000), no. 6, p. 1319-1341.

[17] M. MARIÑO, R. MINASIAN, G. MOORE & A. STROMINGER, "Nonlinear instantons from supersymmetric p-branes", *J. High Energy Phys.* (2000), no. 1, article no. 5 (32 pages).

[18] Y. WANG, "Moduli spaces of G_2 and *Spin*(7)-instantons on product manifolds", *Ann. Henri Poincaré* **21** (2020), no. 9, p. 2997-3033.

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