Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

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Tome 71, nº 6 (2021), p. 2471-2517.

https://doi.org/10.5802/aif.3451

Les *Annales de l'Institut Fourier* sont membres du Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org e-ISSN : 1777-5310

Ann. Inst. Fourier, Grenoble
71, 6 (2021) 2471-2517

A BORG–LEVINSON THEOREM FOR MAGNETIC SCHRÖDINGER OPERATORS ON A RIEMANNIAN MANIFOLD

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Abstract. — We establish uniqueness and stability results for the inverse spectral problem of recovering the magnetic field and the electric potential in a Riemannian manifold from the knowledge of boundary spectral data of the corresponding magnetic Schrödinger operator with Dirichlet boundary condition. The spectral data consist in the knowledge of asymptotic properties, that we specify hereafter, of the sequence of eigenvalues and Neumann traces of the corresponding sequence of eigenfunctions. We also prove similar results for Schrödinger operators with Neumann boundary conditions. To our knowledge our results are the first ones involving such weak boundary spectral data.

Résumé. — Nous établissons des résultats d'unicité et de stabilité pour le problème qui consiste à reconstruire, à partir de données spectrales au bord, le champ magnétique et le potentiel électrique, qui apparaissent dans une équation de Schrödinger magnétique sur une variété riemannienne compacte, avec une condition aux limites de Dirichlet. Les données spectrales consistent en la connaissance du comportement asymptotique, dans un sens que nous préciserons, de la suite des valeurs propres, de l'opérateur de Schrödinger magnétique avec une condition aux limites de Dirichlet, et des traces des dérivées normales des fonctions propres associées. Nous démontrons également des résultats similaires pour un opérateur de Schrödinger magnétique avec une condition aux limites de Neumann. A notre connaissance nos résultats sont les premiers concernant les problèmes spectraux inverses avec des données spectrales au bord aussi faibles.

1. Introduction and main results

Keywords: Borg–Levinson type theorem, magnetic Schrödinger operator, simple Riemannian manifold, uniqueness, stability estimate.
2010 *Mathematics Subject Classification:* 35R30, 35J10, 35P99.
(*) MC and YK are supported by the grant ANR-17-CE40-0029 of the French National Research Agency ANR (project MultiOnde).
DDSF is partly supported by ANR grant *iproblems*.
PS partly supported by NSF Grant DMS-1600327.

1.1. Statement of the problem

Let $\mathrm{M} = (\mathrm{M}, \mathrm{g})$ be a smooth and compact Riemannian manifold of dimension $n \geqslant 2$ and with boundary $\partial\mathrm{M}$. We denote the Laplace–Beltrami operator associated to the Riemannian metric g by Δ. In local coordinates, the metric reads $\mathrm{g} = (\mathrm{g}_{jk})$, and the Laplace–Beltrami operator Δ is given by

$$\Delta = \frac{1}{\sqrt{|\mathrm{g}|}} \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} \left(\sqrt{|\mathrm{g}|}\, \mathrm{g}^{jk} \frac{\partial}{\partial x_k} \right).$$

Here (g^{jk}) is the inverse of the metric g and $|\mathrm{g}| = \det(\mathrm{g}_{jk})$.

Given a couple of magnetic and electric potentials $B = (A, q)$, where $q \in L^\infty(\mathrm{M})$ is real-valued, and $A = a_j \mathrm{d}x^j$ is a covector field (1-form) with real-valued coefficients, $a_j \in W^{1,\infty}(\mathrm{M})$, we consider the magnetic Laplacian

$$\begin{aligned} \mathcal{H}_B &= \frac{1}{\sqrt{|\mathrm{g}|}} \sum_{j,k=1}^{n} \left(\frac{1}{i} \frac{\partial}{\partial x_j} + a_j \right) \sqrt{|\mathrm{g}|}\, \mathrm{g}^{jk} \left(\frac{1}{i} \frac{\partial}{\partial x_k} + a_k \right) + q \\ &= -\Delta - 2i\, A \cdot \nabla - i\, \delta A + |A|^2 + q \\ &:= -\Delta_A + q. \end{aligned} \tag{1.1}$$

Here, the dot product is in the metric with A and ∇ considered as covectors, δ is the coderivative (codifferential) operator, corresponding to the divergence with identifying vectors and covectors, which sends 1-forms to functions by the formula

$$\delta A = \frac{1}{\sqrt{|\mathrm{g}|}} \sum_{j,k=1}^{n} \frac{\partial}{\partial x^j} \left(\mathrm{g}^{jk} \sqrt{|\mathrm{g}|} a_k \right),$$

and we recall that, for $A = a_j \mathrm{d}x^j$, we have $|A|^2 = \mathrm{g}^{jk} a_j a_k$.

For $B = (A, q)$ with $q \in L^\infty(\mathrm{M})$ and $A = a_j \mathrm{d}x^j$, $a_j \in W^{1,\infty}(\mathrm{M})$, define on $L^2(\mathrm{M})$ the unbounded self-adjoint operator H_B as follows

$$H_B u = \mathcal{H}_B u \tag{1.2}$$

and

$$\mathscr{D}(H_B) = \left\{ u \in H_0^1(\mathrm{M}),\ -\Delta_A u + qu \in L^2(\mathrm{M}) \right\}. \tag{1.3}$$

Here, for $k \in \mathbb{N}$, $H^k(\mathrm{M})$, denotes the standard definition of the Sobolev spaces.

The operator H_B is self adjoint and has compact resolvent, therefore its spectrum $\sigma(H_B)$ consists in a sequence $\boldsymbol{\lambda}_B = (\lambda_{B,k})$ of real eigenvalues, counted according to their multiplicities, so that

$$-\infty < \lambda_{B,1} \leqslant \lambda_{B,2} \leqslant \ldots \leqslant \lambda_{B,k} \to +\infty \quad \text{as } k \to \infty. \tag{1.4}$$

In the sequel $\boldsymbol{\phi}_B = (\phi_{B,k})$ denotes an orthonormal basis of $L^2(\mathrm{M})$ consisting in eigenfunctions with $\phi_{B,k}$ associated to $\lambda_{B,k}$, for each k. In the rest of this text, we often use the following notation, where $k \geqslant 1$,

$$\psi_{B,k} = (\partial_\nu + iA(\nu))\,\phi_{B,k}, \quad \text{on } \partial\mathrm{M}$$

and $\boldsymbol{\psi}_B = (\psi_{B,k})$, where ν the outward unit normal vector field on ∂M with respect to the metric g.

We address the question of whether one can recover, in some suitable sense, the magnetic field A and the potential q from some asymptotic knowledge of the boundary spectral data $(\boldsymbol{\lambda}_B, \boldsymbol{\psi}_B)$ with $B = (A, q)$. As for most inverse problems, the main issues are uniqueness and stability.

1.2. Obstruction to uniqueness

We recall that there is an obstruction to the recovery of the electromagnetic potential B from the boundary spectral data $(\boldsymbol{\lambda}_B, \boldsymbol{\psi}_B)$. Indeed, let $B = (A, q)$, and let $V \in \mathcal{C}^1(\mathrm{M})$ be such that $V_{|\partial\mathrm{M}} = 0$ and set $\breve{B} = (A + \mathrm{d}V, q)$. Then it is straightforward to check that

$$e^{-iV}\mathcal{H}_B e^{iV} = \mathcal{H}_{\breve{B}}, \quad (\boldsymbol{\lambda}_B, \boldsymbol{\psi}_B) = (\boldsymbol{\lambda}_{\breve{B}}, \boldsymbol{\psi}_{\breve{B}}). \tag{1.5}$$

Therefore, the magnetic potential A cannot be uniquely determined by the boundary spectral data $(\boldsymbol{\lambda}_B, \boldsymbol{\psi}_B)$ and our inverse problem needs to be stated differently.

According to [47], for every covector $A \in H^k(\mathrm{M}, T^*\mathrm{M})$, there exist uniquely determined $A^s \in H^k(\mathrm{M}, T^*\mathrm{M})$ and $V \in H^{k+1}(\mathrm{M})$ such that

$$A = A^s + \mathrm{d}V, \quad \delta A^s = 0, \quad V|_{\partial\mathrm{M}} = 0. \tag{1.6}$$

Following the well established terminology, A^s and $\mathrm{d}V$ are called respectively the solenoidal and potential parts of the covector A. In view of the obstruction described above, the best one can expect is the simultaneous recovery of A^s and q from some knowledge of the boundary spectral data $(\boldsymbol{\lambda}_B, \boldsymbol{\psi}_B)$. From now on, we focus our attention on this problem.

1.3. Known results

There is a vast literature devoted to inverse spectral problems in one dimension. We refer for instance to the pioneer works by Ambartsumian [2], Borg [11], Levinson [40], Gel'fand and Levitan [20]. The first multidimensional uniqueness result of this type is due to Nachman, Sylvester and

Uhlmann [42] for the operator $-\Delta + q$ with g Euclidean. They showed that q is uniquely determined by the Dirichlet eigenvalues and the traces of the normal derivatives of the corresponding eigenfunctions. Later, Isozaki [21] proved that if finitely many eigenvalues and eigenfunctions are omitted, we still have uniqueness. In [50], Sun studied, in this context, the recovery of magnetic Schrödinger operator from boundary measurements. The result of [50] requires an assumption of smallness of the magnetic field. This assumption was removed by Nakamura, Sun and Uhlmann in [43] as a consequence of their result on the Calderón's problem for such operators. These results have been extended to bounded electromagnetic potentials by [37] and to unbounded domains by [28, 29]. We mention also the work of [9, 35, 36] dealing with the sable recovery of an electromagnetic potential appearing in a dynamical Schrödinger equation. Developing further Isozaki's approach, Choulli and Stefanov [15] gave a generalization of Isozaki's uniqueness result together with a Hölder stability estimate with respect to appropriate metrics for the spectral data. We mention that, following a remark of Isozaki which goes back to [21], the uniqueness and stability results of [15] were stated with only some asymptotic closeness of the boundary spectral data. We mention also the work of [12, 13], dealing with recovery of general non-smooth coefficients from the full boundary spectral data, the work [26] who have considered a similar inverse spectral problem for Schrödinger operators in an infinite cylindrical waveguide and the work of [44] devoted to the extension of the approach of [21] to the recovery of non-smooth coefficients from partial boundary spectral data. We refer also to the works [12, 25, 30, 32, 34] for applications of inverse spectral problems to other class of inverse problems.

Another approach for getting uniqueness in the spectral inverse problem for the Laplace–Beltrami operator was introduced by Belishev [4] and Belishev and Kurylev [5]. This approach consists in reducing the inverse spectral problem under consideration into an inverse hyperbolic problem for which one can apply the so called boundary control method. This method allows to consider the trace of the normal derivative of eigenfunctions only in a part of the boundary. We refer to [5, 23, 24, 31, 38, 39] and [33] in the case of non-smooth coefficients. We mention that none of these papers considered this problem with observations corresponding to some asymptotic knowledge of the boundary spectral data. Actually, to our best knowledge, beside the present paper, there is no other results dealing with inverse spectral problem on non flat manifolds with data similar to the one considered by [15, 26].

One of the first stability estimate for inverse spectral problems was established by Alessandrini and Sylvester [1]. This result was reformulated by the second author in a more precise way in [14]. A similar result in the case of the Laplace–Beltrami operator was proved by the first and the third authors in [8] using the idea introduced in [1]. With the help of a result quantifying the uniqueness of continuation for a Cauchy problem with data on a part of the boundary for a wave equation, the first two authors and Yamamoto [7] proved a double logarithmic stability estimate under the assumption that the potential is known near the boundary. In [15], the second and the last authors provided one of the first Hölder type stability estimate for the multi-dimensional Borg–Levinson theorem of determining the potential from some asymptotic knowledge of the boundary spectral data of the associated Schrödinger operator. In [26], the fourth author, Kavian and Soccorsi proved a similar result for an inverse spectral problem in an infinite cylindrical waveguide.

1.4. Preliminaries

We briefly recall some notations and known results in Riemannian geometry. We refer for instance to [22] for more details. By Riemannian manifold with boundary, we mean a C^∞-smooth manifold with boundary in the usual sense, endowed with a metric g.

As before M denotes a compact Riemannian manifold of dimension $n \geqslant 2$. Fix a local coordinate system $x = \left(x^1, \dots, x^n\right)$ and let $(\partial_1, \dots, \partial_n)$ be the corresponding tangent vector fields. For $x \in \mathrm{M}$, the inner product and the norm on the tangent space $T_x\mathrm{M}$ are given by

$$\mathrm{g}(X,Y) = \langle X, Y\rangle = \sum_{j,k=1}^{n} \mathrm{g}_{jk} X^j Y^k,$$

$$|X| = \langle X, X\rangle^{1/2}, \qquad X = \sum_{i=1}^{n} X^i \partial_i, \qquad Y = \sum_{i=1}^{n} Y^i \partial_i.$$

The cotangent space $T_x^*\mathrm{M}$ is the dual of $T_x\mathrm{M}$. Its elements are called covectors or one-forms. The disjoint union of the tangent spaces

$$T\mathrm{M} = \bigcup_{x\in\mathrm{M}} T_x\mathrm{M}$$

is called the tangent bundle of M. Similarly, the cotangent bundle $T^*\mathrm{M}$ is the disjoint union of the spaces $T_x^*\mathrm{M}$, $x \in \mathrm{M}$. A 1-form A on the manifold M is a function that assigns to each point $x \in \mathrm{M}$ a covector $A(x) \in T_x^*\mathrm{M}$.

An example of a 1-form is the differential of a function $f \in \mathcal{C}^\infty(\mathrm{M})$, which is defined by

$$\mathrm{d}f_x(X) = \sum_{j=1}^n X^j \frac{\partial f}{\partial x_j}, \quad X = \sum_{j=1}^n X^j \partial_j.$$

Hence f defines the mapping $\mathrm{d}f : T\mathrm{M} \to \mathbb{R}$, which is called the differential of f given by

$$\mathrm{d}f(x, X) = \mathrm{d}f_x(X).$$

In local coordinates,

$$\mathrm{d}f = \sum_{j=1}^n \partial_j f \mathrm{d}x^j.$$

where $(\mathrm{d}x^1, \dots, \mathrm{d}x^n)$ is the basis in the space $T_x^*\mathrm{M}$, dual to the basis $(\partial_1, \dots, \partial_n)$.

The Riemannian metric g induces a natural isomorphism $\imath : T_x\mathrm{M} \to T_x^*\mathrm{M}$ given by $\iota(X) = \langle X, \cdot \rangle$. For $X \in T_x\mathrm{M}$ denote $X^\flat = \imath(X)$, and similarly for $A \in T_x^*\mathrm{M}$ we denote $A^\sharp = \imath^{-1}(A)$, $\imath$ and $\imath^{-1}$ are called musical isomorphisms. The *sharp* operator is given by

$$T_x^*\mathrm{M} \longrightarrow T_x\mathrm{M}, \quad A \longmapsto A^\sharp, \tag{1.7}$$

given in local coordinates by

$$(a_j \mathrm{d}x^j)^\sharp = a^j \partial_j, \quad a^j = \sum_{k=1}^n \mathrm{g}^{jk} a_k. \tag{1.8}$$

Define the inner product of 1-forms in $T_x^*\mathrm{M}$ by

$$\langle A, B \rangle = \langle A^\sharp, B^\sharp \rangle = \sum_{j,k=1}^n \mathrm{g}^{jk} a_j b_k = \sum_{j,k=1}^n \mathrm{g}_{jk} a^j b^k. \tag{1.9}$$

The metric tensor g induces the Riemannian volume

$$\mathrm{dv}^n = |\mathrm{g}|^{1/2} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n.$$

We denote by $L^2(\mathrm{M})$ the completion of $\mathcal{C}^\infty(\mathrm{M})$ endowed with the usual inner product

$$(f_1, f_2) = \int_\mathrm{M} f_1(x) \overline{f_2(x)} \, \mathrm{dv}^n, \qquad f_1, f_2 \in \mathcal{C}^\infty(\mathrm{M}).$$

A section of a vector bundle E over the Riemannian manifold M is a C^∞ map $\mathfrak{s} : \mathrm{M} \to E$ such that for each $x \in \mathrm{M}$, $\mathfrak{s}(x)$ belongs to the fiber over x. We denote by $\mathcal{C}^\infty(\mathrm{M}, E)$ the space of smooth sections of the vector bundle E. According to this definition, $\mathcal{C}^\infty(\mathrm{M}, T\mathrm{M})$ denotes the space of vector fields on M and $\mathcal{C}^\infty(\mathrm{M}, T^*\mathrm{M})$ denotes the space of 1-forms on M. Similarly, we may define the spaces $L^2(\mathrm{M}, T^*\mathrm{M})$ (resp. $L^2(\mathrm{M}, T\mathrm{M})$) of square integrable 1-forms (resp. vectors) by using the inner product

$$(A, B) = \int_{\mathrm{M}} \langle A, \overline{B} \rangle \,\mathrm{dv}^n, \quad A, B \in L^2(\mathrm{M}, T^*\mathrm{M}). \tag{1.10}$$

For $k \in \mathbb{N}$, we define the Sobolev space $H^k(\mathrm{M})$ as the completion of $\mathcal{C}^\infty(\mathrm{M})$ with respect to the norm

$$\|f\|^2_{H^k(\mathrm{M})} = \|f\|^2_{L^2(\mathrm{M})} + \sum_{k=1}^{n} \|\nabla^k f\|^2_{L^2(\mathrm{M}, T^k\mathrm{M})},$$

where ∇^k is the covariant differential of f in the metric g. Moreover, following [24, p. 59], [41, p. 40] and [41, Theorem 9.6, Chapter 1], for $s \in [0, +\infty) \setminus \mathbb{N}$, we denote by $H^s(\mathrm{M})$ the space of interpolation of order $1 - (s - [s])$ between $H^{[s]+1}(\mathrm{M})$ and $H^{[s]}(\mathrm{M})$. Here $[s]$ denotes the integer part of s and we refer to [41, Definition 2.1, Chapter 1] for the definition of space of interpolation. Since, for $k \in \mathbb{N}$, $H^k(\mathrm{M}) = H^k(\mathrm{M}_{\mathrm{int}})$, where $\mathrm{M}_{\mathrm{int}}$ denotes the interior of M, this definition of the space $H^s(\mathrm{M})$, $s \in [0, +\infty)$, coincides with the space $H^s(\mathrm{M}_{\mathrm{int}})$ defined by interpolation in [41, p. 40]. Then, applying [41, Theorems 9.1, 9.2, Chapter 1] (see also the proof of [41, Theorem 9.4, Chapter 1]), one can check that the definition of the space $H^s(\mathrm{M}_{\mathrm{int}})$ by interpolation coincides with its definition by local coordinates. Therefore, our definition of the space $H^s(\mathrm{M})$ coincides with the definition of $H^s(\mathrm{M}_{\mathrm{int}})$ by local coordinates. We denote also by $H^s_0(\mathrm{M})$ the closure of $\mathcal{C}^\infty_0(\mathrm{M}_{\mathrm{int}})$ in $H^s(\mathrm{M})$ and, in view of [41, Theorem 11.6, Chapter 1], we recall that for $s \notin \frac{1}{2} + \mathbb{N}$, $H^s_0(\mathrm{M})$ coincide with the space of interpolation of order $1 - (s - [s])$ between $H_0^{[s]+1}(\mathrm{M})$ and $H_0^{[s]}(\mathrm{M})$.

If f is a $\mathcal{C}^\infty$ function on M, then ∇f is the vector field defined by

$$X(f) = \langle \nabla f, X \rangle,$$

for every vector field X on M. In the local coordinates system, the last identity can be rewritten in the form

$$\nabla f = \sum_{i,j=1}^{n} \mathrm{g}^{ij} \frac{\partial f}{\partial x_i} \partial_j = (\mathrm{d}f)^\sharp. \tag{1.11}$$

The normal derivative of a function u is given by the formula

$$\partial_\nu u := \langle \nabla u, \nu \rangle = \sum_{j,k=1}^{n} \mathrm{g}^{jk} \nu_j \frac{\partial u}{\partial x_k}, \tag{1.12}$$

where ν is the unit outward vector field to $\partial\mathrm{M}$.

Likewise, we say that a 1-form $A = a_j \mathrm{d}x^j$ belongs to $H^k(\mathrm{M}, T^*\mathrm{M})$ if each $a_j \in H^k(\mathrm{M})$. The space $H^k(\mathrm{M}, T^*\mathrm{M})$ is a Hilbert space when it is endowed with the norm

$$\|A\|_{H^k(\mathrm{M},T^*\mathrm{M})} = \left(\sum_{j=1}^{n} \|a_j\|^2_{H^k(\mathrm{M})}\right)^{\frac{1}{2}}.$$

As usual, the vector space of smooth 2-forms on M is denoted by $\Omega^2(\mathrm{M})$. In local coordinates, a 2-form ω is represented as

$$\omega = \sum_{j,k=1}^{n} \omega_{jk} \mathrm{d}x^j \wedge \mathrm{d}x^k,$$

where ω_{jk} are real-valued functions on M. Similarly as before, ω is in $H^s(\mathrm{M}, \Omega^2(\mathrm{M}))$, $s \in \mathbb{R}$, if $\omega_{jk} \in H^s(\mathrm{M})$ for each j, k. Additionally, $H^s(\mathrm{M}, \Omega^2(\mathrm{M}))$ is a Hilbert space for the norm

$$\|\omega\|_{H^s(\mathrm{M},\Omega^2(\mathrm{M}))} = \left(\sum_{j,k} \|\omega_{jk}\|^2_{H^s(\mathrm{M})}\right)^{\frac{1}{2}}.$$

In the rest of this text, the scalar product of $L^2(\partial\mathrm{M})$ is also denoted by $\langle \cdot, \cdot \rangle$:

$$\langle f_1, f_2 \rangle = \int_{\partial\mathrm{M}} f_1(x)\, \overline{f_2(x)}\, \mathrm{d}\sigma^{n-1} \tag{1.13}$$

where $\mathrm{d}\sigma^{n-1}$ is the volume form of $\partial\mathrm{M}$.

1.5. Main results

Prior to the statement of our main results, we introduce the notion of simple manifolds [48]. We say that the boundary $\partial\mathrm{M}$ is strictly convex if the second fundamental form is positive-definite for any $x \in \partial\mathrm{M}$.

Definition 1.1. — *A manifold* M *is simple if* $\partial\mathrm{M}$ *is strictly convex and, for any* $x \in \mathrm{M}$, *the exponential map* $\exp_x : \exp_x^{-1}(\mathrm{M}) \to \mathrm{M}$ *is a diffeomorphism.*

Note that if M is simple, then it is diffeomorphic to a ball, and every two points can be connected by a unique minimizing geodesic depending smoothly on its endpoints. Also, one can extend it to a simple manifold M_1 such that $\mathrm{M}_1^{\mathrm{int}} \supset \mathrm{M}$.

We now introduce the admissible sets of magnetic potentials A and electric potentials q. Set

$$\mathscr{B} = W^{2,\infty}(\mathrm{M}, T^*\mathrm{M}) \oplus L^\infty(\mathrm{M}).$$

We endow $\mathscr{B}$ with its natural norm

$$\|B\|_{\mathscr{B}} = \|A\|_{W^{2,\infty}(\mathrm{M},T^*\mathrm{M})} + \|q\|_{L^\infty(\mathrm{M})}.$$

For $r > 0$, set

$$\mathscr{B}_r = \{B = (A, q) \in \mathscr{B},\ \|B\|_{\mathscr{B}} \leqslant r\}. \tag{1.14}$$

Let $B_\ell \in \mathscr{B}_r$, $\ell = 1, 2$, we denote by $(\lambda_{\ell,k}, \phi_{\ell,k})$, $k \geqslant 1$, the eigenvalues and normalized eigenfunctions of the operator H_{B_ℓ}.

For $\ell = 1$ or $\ell = 2$, let

$$\psi_{\ell,k} = (\partial_\nu + iA_\ell(\nu))\,\phi_{\ell,k}, \quad k \geqslant 1. \tag{1.15}$$

At this point we remark that when $A_1 = A_2$ it is clear that $H_{B_1} - H_{B_2} = q_1 - q_2$ whence by the min-max principle,

$$\sup_{k\geqslant 1} |\lambda_{1,k} - \lambda_{2,k}| \leqslant \|q_1 - q_2\|_{L^\infty(M)} < \infty.$$

Assume now that $A_1 \neq A_2$ and $\delta A_1 = \delta A_2$. Then we have

$$H_{B_1} - H_{B_2} = -2i(A_1 - A_2)\nabla + |A_1|^2 - |A_2|^2 + q_1 - q_2.$$

Thus, $H_{B_1} - H_{B_2} \notin \mathcal{B}(L^2(M))$. Therefore, we can reasonably expect that

$$\sup_{k\geqslant 1} |\lambda_{1,k} - \lambda_{2,k}| = +\infty.$$

Keeping in mind this property and the obstruction described in Section 1.2, it seems natural to expect the recovery of the solenoidal part of the magnetic potential from a rate of growth of the eigenvalues. Our first result gives a positive answer to this issue together with the recovery of the electric potential.

THEOREM 1.2. — *Assume that* M *is simple. Let* $B_\ell = (A_\ell, q_\ell) \in \mathscr{B}_r$, $\ell = 1, 2$, *such that*

$$\partial_x^\alpha A_1(x) = \partial_x^\alpha A_2(x), \quad x \in \partial M,\ |\alpha| \leqslant 1. \tag{1.16}$$

Furthermore, assume that there exists $t \in [0, 1/2)$ *so that*

$$\sup_{k\geqslant 1} k^{-t/n}|\lambda_{1,k} - \lambda_{2,k}| + \sum_{k\geqslant 1} k^{-2t/n}\|\psi_{1,k} - \psi_{2,k}\|^2_{L^2(\partial\mathrm{M})} < \infty. \tag{1.17}$$

Then $A_1^s = A_2^s$. *Moreover, under the additional conditions*

$$\lim_{k\to+\infty} |\lambda_{1,k} - \lambda_{2,k}| = 0, \quad \textit{and} \quad \sum_{k\geqslant 1} \|\psi_{1,k} - \psi_{2,k}\|^2_{L^2(\partial\mathrm{M})} < \infty, \tag{1.18}$$

we have $q_1 = q_2$.

In the spirit of [15, 26], we consider also the stability issue for this problem stated as follows.

Theorem 1.3. — *Assume that* M *is simple. Let* $B_\ell = (A_\ell, q_\ell) \in \mathscr{B}_r$, $\ell = 1, 2$, *such that* A_1 *and* A_2 *satisfies* (1.16) *and* $q_1 - q_2 \in H_0^1(\mathrm{M})$ *satisfies*

$$\|q_1 - q_2\|_{H_0^1(\mathrm{M})} \leqslant r.$$

Furthermore, assume that there exists $t \in (0, 1/2)$ *so that*

$$\sup_{k\geqslant 1} k^{-t/n}|\lambda_{1,k} - \lambda_{2,k}| + \sum_{k\geqslant 1} \|\psi_{1,k} - \psi_{2,k}\|^2_{L^2(\partial\mathrm{M})} < \infty. \tag{1.19}$$

Then $A_1^s = A_2^s$ *and*

$$\|q_1 - q_2\|_{L^2(\mathrm{M})} \leqslant C \left(\limsup_{k\to\infty}|\lambda_{1,k} - \lambda_{2,k}|\right)^{\frac{1}{2}} < \infty, \tag{1.20}$$

the constant C *only depends on* r *and* M.

To our best knowledge Theorems 1.2 and 1.3 are the first results dealing with inverse spectral problems for Schrödinger operators, with nonconstant leading coefficients, from asymptotic knowledge of boundary spectral data similar to the one considered by [15, 26]. Note also that Theorem 1.3 seems to be the first stability result of recovering the electric potential from partial boundary spectral data in such general context (the only other similar results can be found in [7, 15, 26] where stable recovery of Schrödinger operators on a bounded domain, with an Euclidean metric and without magnetic potential, have been considered).

We recall that multi-dimensional Borg–Levinson type theorems for magnetic Schrödinger operators have been already considered in [23, 27, 46]. Among them, only [27] considered the uniqueness issue from boundary spectral data similar to (1.18). The results in the present work can be seen as an improvement of that in [27] in four directions. First of all, we prove for the first time the extension of such results to a general simple Riemanian manifold by proving the connection between our problem and the injectivity of the so called geodesic ray transform borrowed from [3, 19, 47, 48]. In addition, by using some results of [48], we establish stability estimates for this problem where [27] treated only the uniqueness. In contrast to [27], we do not require the knowledge of the magnetic potentials on the neighborhood of the boundary. This condition is relaxed, by considering only some knowledge of the magnetic potentials at the boundary given by (1.16). Finally, we show, for what seems to be the first time, that even a rate of growth of the difference of eigenvalues like (1.17), (1.19) can determine the magnetic potential appearing in a magnetic Schrödinger operator.

The main ingredient in our analysis is a suitable representation formula that involves the magnetic potential A and the electric potential q in terms of the Dirichlet-to-Neumann map associated to the equations $\mathcal{H}_B u - \lambda u = 0$, for a well chosen set of complex λ's. In [21, 27], the authors considered such a representation for a bounded domain with flat metric. Using a construction inspired by [6, 8, 16, 17, 18, 49] we show how one can extend such approach to more general manifolds. Note that this construction differs from the one considered by [16, 17, 18] for recovering the magnetic Schrödinger operators from boundary measurements. Actually, our results hold for a general simple manifold even in the case $n \geqslant 3$, whereas the determination of Schrödinger operators from boundary measurements in the same context is still an open problem (see [17, 18]).

In this paper we treat also the problem of determining the Neumann realization of magnetic Schrödinger operator. For simplicity and in order to avoid any confusion between the results for the different operators, we give the statement of the result for the Neumann realization of magnetic Schrödinger operator in Theorem 6.1. The result of Theorem 6.1 is stated with an optimal growth of the difference of eigenvalues (see the discussion just after Theorem 6.1).

We believe that following the idea of [9, 27, 37, 45], one can relax the regularity condition imposed to the magnetic potentials as well as condition (1.16). This approach requires the construction of ansatz depending on an approximation of the magnetic potential instead of the magnetic potential itself. In order to avoid the inadequate expense of the size of the paper, we do not consider this issue.

1.6. Outline

The outline of the paper is as follows. We review in Section 2 the geodesic ray transform for 1-one forms and functions on a manifold. Section 3 is devoted to an asymptotic spectral analysis. We construct in Section 4 geometrical optics solutions for magnetic Schrödinger equations. We particularly focus our attention on the solvability of the eikonal and the transport equations which are essential in the construction of geometric optic solutions. Additionally, we provide a representation formula. The proof of Theorems 1.2 and 1.3 are given in Section 5. The Neumann case is briefly discussed in Section 6. Finally, we prove some uniform estimates related to the Weyl's formula for the magnetic Schrödinger operator in the Appendix.

2. A short review on the geodesic ray transform on a simple manifold

We collect in this section some known results on the geodesic ray transform for functions and 1-forms on a smooth simple Riemannian manifold (M,g). These results will be used later in this text.

Denote by $\operatorname{div} X$ the divergence of a vector field $X\in H^1(\mathrm{M},T\mathrm{M})$ on M, i.e. in local coordinates ([24, p. 42]),

$$\operatorname{div} X=\frac{1}{\sqrt{|\mathrm{g}|}}\sum_{i=1}^n\partial_i\Big(\sqrt{|\mathrm{g}|}\,X^i\Big),\quad X=\sum_{i=1}^n X^i\partial_i. \tag{2.1}$$

Using the inner product of a 1-form, we can define the coderivative operator δ as the adjoint of the exterior derivative via the relation

$$(\delta A,v)=(A,\mathrm{d}v)\,,\quad A\in\mathcal{C}^\infty(M,T^*M),\ v\in\mathcal{C}^\infty(\mathrm{M}). \tag{2.2}$$

Then δA is related to the divergence of vector fields by $\delta A=\operatorname{div}(A^\sharp)$, where the divergence is given by (2.1). If $X\in H^1(\mathrm{M},T\mathrm{M})$ the divergence formula reads

$$\int_{\mathrm{M}}\operatorname{div} X\,\mathrm{dv}^n=\int_{\partial\mathrm{M}}\langle X,\nu\rangle\,\mathrm{d}\sigma^{n-1}. \tag{2.3}$$

For $f\in H^1(\mathrm{M})$, we have the following Green formula

$$\int_{\mathrm{M}}\operatorname{div} X\,f\,\mathrm{dv}^n=-\int_{\mathrm{M}}\langle X,\nabla f\rangle\,\mathrm{dv}^n+\int_{\partial\mathrm{M}}\langle X,\nu\rangle f\,\mathrm{d}\sigma^{n-1}. \tag{2.4}$$

Therefore, for $u,w\in H^2(\mathrm{M})$, the following identity holds

$$\begin{aligned}\int_{\mathrm{M}}\Delta_A u\overline{w}\,\mathrm{dv}^n &=-\int_{\mathrm{M}}\langle\nabla_A u,\overline{\nabla_A w}\rangle\,\mathrm{dv}^n+\int_{\partial\mathrm{M}}(\partial_\nu u+iA(\nu)u)\overline{w}\,\mathrm{d}\sigma^{n-1}\\ &=\int_{\mathrm{M}}u\overline{\Delta_A w}\,\mathrm{dv}^n\\ &\quad+\int_{\partial\mathrm{M}}\Big((\partial_\nu u+iA(\nu)u)\overline{w}-u(\overline{\partial_\nu w+iA(\nu)w})\Big)\,\mathrm{d}\sigma^{n-1},\end{aligned} \tag{2.5}$$

where $\nabla_A u=\nabla u+iuA^\sharp$. For $x\in\mathrm{M}$ and $\theta\in T_x\mathrm{M}$, denote by $\gamma_{x,\theta}$ the unique geodesic starting from x and directed by θ.

Recall that the sphere bundle and co-sphere bundle of M are respectively given by

$$S\mathrm{M}=\{(x,\theta)\in T\mathrm{M};\ |\theta|=1\}\,,\quad S^*\mathrm{M}=\{(x,p)\in T^*\mathrm{M};\ |p|=1\}\,,$$

The exponential map $\exp_x : T_x\mathrm{M} \to \mathrm{M}$ is defined as follows

$$\exp_x(v) = \gamma_{x,\theta}(|v|), \quad \theta = \frac{v}{|v|}. \tag{2.6}$$

We assume in the rest of this section that M is simple and we point out that any arbitrary pair of points in M can be joined by an unique geodesic of finite length.

Given $(x,\theta) \in S\mathrm{M}$ and denote by $\gamma_{x,\theta}$ the unique geodesic $\gamma_{x,\theta}$ satisfying the initial conditions $\gamma_{x,\theta}(0) = x$ and $\dot\gamma_{x,\theta}(0) = \theta$, which is defined on the maximal interval $[\ell_-(x,\theta), \ell_+(x,\theta)]$, with $\gamma_{x,\theta}(\ell_\pm(x,\theta)) \in \partial\mathrm{M}$. Define the geodesic flow φ_t by

$$\varphi_t : S\mathrm{M} \longrightarrow S\mathrm{M}, \quad \varphi_t(x,\theta) = (\gamma_{x,\theta}(t), \dot\gamma_{x,\theta}(t)), \quad t \in [\ell_-(x,\theta), \ell_+(x,\theta)], \tag{2.7}$$

and observe that $\varphi_t \circ \varphi_s = \varphi_{t+s}$.

Introduce now the submanifolds of inner and outer vectors of $S\mathrm{M}$

$$\partial_\pm S\mathrm{M} = \left\{(x,\theta) \in S\mathrm{M},\ x \in \partial\mathrm{M},\ \pm\langle\theta, \nu(x)\rangle < 0\right\}, \tag{2.8}$$

where ν is the unit outer normal vector field on $\partial\mathrm{M}$.

Note that $\partial_+ S\mathrm{M}$ and $\partial_- S\mathrm{M}$ are compact manifolds with the same boundary $S(\partial\mathrm{M})$ and

$$\partial S\mathrm{M} = \partial_+ S\mathrm{M} \cup S\partial\mathrm{M} \cup \partial_- S\mathrm{M}.$$

It is straightforward to check that $\ell_\pm : S\mathrm{M} \to \mathbb{R}$ satisfy

$$\begin{gathered}
\ell_-(x,\theta) \leqslant 0, \quad \ell_+(x,\theta) \geqslant 0, \quad \ell_+(x,\theta) = -\ell_-(x,-\theta), \\
\ell_-(x,\theta) = 0, \quad (x,\theta) \in \partial_+ S\mathrm{M}, \\
\ell_-(\varphi_t(x,\theta)) = \ell_-(x,\theta) - t, \quad \ell_+(\varphi_t(x,\theta)) = \ell_+(x,\theta) + t.
\end{gathered}$$

To each 1-form $A \in \mathcal{C}^\infty(\mathrm{M}, T^*\mathrm{M})$, with $A = a_j \mathrm{d}x^j$, associate the smooth symbol $\sigma_A \in \mathcal{C}^\infty(S\mathrm{M})$ given by

$$\sigma_A(x,\theta) = \sum_{j=1}^n a_j(x)\theta^j = \langle A^\sharp(x), \theta\rangle, \quad (x,\theta) \in S\mathrm{M}. \tag{2.9}$$

Recall that the Riemannian scalar product on $T_x\mathrm{M}$ induces the volume form on $S_x\mathrm{M}$ given by

$$\mathrm{d}\omega_x(\theta) = \sqrt{|\mathrm{g}|}\, \sum_{k=1}^n (-1)^k \theta^k \mathrm{d}\theta^1 \wedge \cdots \wedge \widehat{\mathrm{d}\theta^k} \wedge \cdots \wedge \mathrm{d}\theta^n.$$

As usual, the notation $\widehat{\cdot}$ means that the corresponding factor has been dropped.

We also consider the volume form dv^{2n-1} on the manifold $S\mathrm{M}$ defined as follows

$$\mathrm{dv}^{2n-1}(x,\theta) = \mathrm{d}\omega_x(\theta) \wedge \mathrm{dv}^n,$$

where dv^n is the Riemannnian volume form on M.

By Liouville's theorem, the form dv^{2n-1} is preserved by the geodesic flow. The corresponding volume form on the boundary

$$\partial S\mathrm{M} = \{(x,\theta) \in S\mathrm{M},\, x \in \partial \mathrm{M}\}$$

is given by

$$\mathrm{d}\sigma^{2n-2} = \mathrm{d}\omega_x(\theta) \wedge \mathrm{d}\sigma^{n-1},$$

where $\mathrm{d}\sigma^{n-1}$ is the volume form of $\partial\mathrm{M}$.

Santaló's formula will be useful in the sequel:

$$\begin{aligned}(2.10)\qquad \int_{S\mathrm{M}} F(x,\theta)\,\mathrm{dv}^{2n-1}(x,\theta) \\ = \int_{\partial_+ S\mathrm{M}} \left(\int_0^{\ell_+(x,\theta)} F(\varphi_t(x,\theta))\,\mathrm{d}t \right) \mu(x,\theta)\,\mathrm{d}\sigma^{2n-2},\end{aligned}$$

for any $F \in \mathcal{C}(S\mathrm{M})$, where we set $\mu(x,\theta) = |\langle \theta, \nu(x)\rangle|$.

For the sake of simplicity $L^2(\partial_+ S\mathrm{M}, \mu\,\mathrm{d}\sigma^{2n-2})$ is denoted by $L^2_\mu(\partial_+ S\mathrm{M})$.

Note that $L^2_\mu(\partial_+ S\mathrm{M})$ is a Hilbert space when it is endowed with the scalar product

$$(2.11)\qquad (u,v)_\mu = \int_{\partial_+ S\mathrm{M}} u(x,\theta)\bar{v}(x,\theta)\mu(x,\theta)\,\mathrm{d}\sigma^{2n-2}.$$

Until the end of this section, we assume that M is simple.

2.1. Geodesic ray transform of 1-forms

The ray transform of 1-forms on M is defined as the linear operator

$$\mathcal{I}_1 : \mathcal{C}^\infty(\mathrm{M}, T^*\mathrm{M}) \longrightarrow \mathcal{C}^\infty(\partial_+ S\mathrm{M})$$

acting as follows

$$\begin{aligned}\mathcal{I}_1(A)(x,\theta) = \int_{\gamma_{x,\theta}} A = \sum_{j=1}^n \int_0^{\ell_+(x,\theta)} a_j(\gamma_{x,\theta}(t))\dot{\gamma}^j_{x,\theta}(t)\,\mathrm{d}t \\ = \int_0^{\ell_+(x,\theta)} \sigma_A(\varphi_t(x,\theta))\,\mathrm{d}t.\end{aligned}$$

It is easy to check that $\mathcal{I}_1(\mathrm{d}\varphi) = 0$ for any $\varphi \in \mathcal{C}^\infty(\mathrm{M})$ satisfying $\varphi_{|\partial\mathrm{M}} = 0$. On the other hand, it is known that $\mathcal{I}_1$ is injective on the space of solenoidal 1-forms satisfying $\delta A = 0$. Therefore, if $A \in H^1(\mathrm{M}, T^*\mathrm{M})$ is so that $\mathcal{I}_1(A) = 0$, then $A^s = 0$. Whence, there exists $\varphi \in H_0^1(M) \cap H^2(M)$ such that $A = \mathrm{d}\varphi$. As a consequence of this observation, we have

$$|\mathcal{I}_1(A)(x,\theta)| = |\mathcal{I}_1(A^s)(x,\theta)| \leqslant C\|A^s\|_{\mathcal{C}^0}, \quad A \in \mathcal{C}^0(\mathrm{M}, T^*\mathrm{M}). \tag{2.12}$$

With reference to [47], we recall that $\mathcal{I}_1^* : L^2_\mu(\partial_+ S\mathrm{M}) \to L^2(\mathrm{M}, T^*\mathrm{M})$ is given by

$$\left(\mathcal{I}_1^*\Psi(x)\right)_j = \int_{S_x\mathrm{M}} \theta^j \breve{\Psi}(x,\theta)\,\mathrm{d}\omega_x(\theta). \tag{2.13}$$

Here $\breve{\Psi}$ is the extension of Ψ from $\partial_+ S\mathrm{M}$ to $S\mathrm{M}$, which is constant on every orbit of the geodesic flow. That is

$$\breve{\Psi}(x,\theta) = \Psi\big(\gamma_{x,\theta}(\ell_-(x,\theta)), \dot{\gamma}_{x,\theta}(\ell_-(x,\theta))\big) = \Psi(\varphi_{\ell_-(x,\theta)}(x,\theta)), \ \ (x,\theta) \in S\mathrm{M}.$$

One can check [47] that $\mathcal{I}_1$ has a bounded extension, still denoted by $\mathcal{I}_1$,

$$\mathcal{I}_1 : H^k(\mathrm{M}, T^*\mathrm{M}) \longrightarrow H^k(\partial_+ S\mathrm{M}).$$

We complete this subsection by results borrowed from [48]. We extend (M, g) to a smooth Riemannian manifold (M_1, g) such that $\mathrm{M}_1^{\mathrm{int}} \supset \mathrm{M}$ and we consider the normal operator $N_1 = \mathcal{I}_1^*\mathcal{I}_1$. Then there exist $C_1 > 0$, $C_2 > 0$ such that

$$C_1\|A^s\|_{L^2(\mathrm{M})} \leqslant \|N_1(A)\|_{H^1(\mathrm{M}_1)} \leqslant C_2\|A^s\|_{L^2(\mathrm{M})}, \tag{2.14}$$

for any $A \in L^2(\mathrm{M}, T^*\mathrm{M})$. If $\mathcal{O}$ is an open set of M_1, N_1 is an elliptic pseudo-differential operator of order -1 on $\mathcal{O}$ having as principal symbol $\varrho(x,\xi) = (\varrho_{jk}(x,\xi))_{1\leqslant j,k\leqslant n}$, where

$$\varrho_{j,k}(x,\xi) = \frac{c_n}{|\xi|}\left(\mathrm{g}_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right).$$

Therefore, for each integer $k \geqslant 0$, there exists a constant $C_k > 0$ such that, for any $A \in H^k(\mathrm{M}, T^*\mathrm{M})$ compactly supported in $\mathcal{O}$, we have

$$\|N_1(A)\|_{H^{k+1}(\mathrm{M}_1)} \leqslant C_k\|A^s\|_{H^k(\mathcal{O})}. \tag{2.15}$$

2.2. Geodesic ray transform of functions

Following [47, Lemma 4.1.1], the ray transform of functions is the linear operator

$$\mathcal{I}_0 : \mathcal{C}^\infty(\mathrm{M}) \longrightarrow \mathcal{C}^\infty(\partial_+ S\mathrm{M}) \tag{2.16}$$

acting as follows

$$\mathcal{I}_0 f(x,\theta) = \int_0^{\ell_+(x,\theta)} f(\gamma_{x,\theta}(t))\,\mathrm{d}t. \tag{2.17}$$

Similarly to $\mathcal{I}_1$, $\mathcal{I}_0$ has an extension, still denoted by $\mathcal{I}_0$:

$$\mathcal{I}_0 : H^k(\mathrm{M}) \longrightarrow H^k(\partial_+ S\mathrm{M}) \tag{2.18}$$

for every integer $k \geqslant 0$. We refer to [47, Theorem 4.2.1] for details.

Considering $\mathcal{I}_0$ as a bounded operator from $L^2(\mathrm{M})$ into $L^2_\mu(\partial_+ S\mathrm{M})$, we can compute its adjoint $\mathcal{I}_0^* : L^2_\mu(\partial_+ S\mathrm{M}) \to L^2(\mathrm{M})$

$$\mathcal{I}_0^* \Psi(x) = \int_{S_x\mathrm{M}} \check{\Psi}(x,\theta)\,\mathrm{d}\omega_x(\theta), \tag{2.19}$$

where $\check{\Psi}$ is the extension of Ψ from $\partial_+ S\mathrm{M}$ to $S\mathrm{M}$ which is constant on every orbit of the geodesic flow:

$$\check{\Psi}(x,\theta) = \Psi(\gamma_{x,\theta}(\ell_-(x,\theta))).$$

Let M_1 be a simple manifold so that $\mathrm{M}_1^{\mathrm{int}} \supset \mathrm{M}$ and consider the normal operator $N_0 = \mathcal{I}_0^* \mathcal{I}_0$. Then there exist two constants $C_1 > 0$, $C_2 > 0$ such that

$$C_1 \|f\|_{L^2(\mathrm{M})} \leqslant \|N_0(f)\|_{H^1(\mathrm{M}_1)} \leqslant C_2 \|f\|_{L^2(\mathrm{M})} \tag{2.20}$$

for any $f \in L^2(\mathrm{M})$, see [48].

If $\mathcal{O}$ is an open set of M_1, N_0 is an elliptic pseudo-differential operator of order -1 on Ω, whose principal symbol is a multiple of $|\xi|^{-1}$, see [48]. Therefore there exists a constant $C_k > 0$ such that, for all $f \in H^k(\mathcal{O})$ compactly supported in $\mathcal{O}$,

$$\|N_0(f)\|_{H^{k+1}(\mathrm{M}_1)} \leqslant C_k \|f\|_{H^k(\mathcal{O})}. \tag{2.21}$$

3. Asymptotic spectral analysis

We fix in all of this section $B_\ell = (A_\ell, q_\ell) \in \mathscr{B}_r$, $\ell = 1, 2$, satisfying the assumptions of Theorem 1.2. As in Section 1, H_{B_ℓ}, $\ell = 1, 2$, is the operator defined by (1.2) and (1.3) when $B = B_\ell$. Furthermore, for $\lambda \in \rho(H_{B_\ell})$, denote by $R_{B_\ell}(\lambda)$ the resolvent of H_{B_ℓ} and recall the following resolvent estimate.

LEMMA 3.1. — *Let $B_\ell = (A_\ell, q_\ell) \in \mathscr{B}_r$, $\ell = 1,2$ and consider the set $S := \{z \in \mathbb{C} \setminus \mathbb{R} : \ |z| \geqslant 1\}$. Then for all $s \in [0,1]$ there exists $C_s > 0$, depending on B_1, B_2, s and Ω, such that*

$$\|R_{B_\ell}(\lambda)\|_{\mathscr{L}(L^2(\mathrm{M});H^{2s}(\mathrm{M}))} \leqslant \frac{C_s(1+|\lambda|)^s}{|\Im\lambda|}, \quad \ell = 1,2, \ \lambda \in S. \tag{3.1}$$

Proof. — For $s = 0$, (3.1) follows from the classical resolvent estimate

$$\|R_{B_\ell}(\lambda)\|_{\mathscr{L}(L^2(\mathrm{M}))} \leqslant \frac{1}{\mathrm{dist}(\lambda, \sigma(H_{B_\ell}))} = \frac{1}{|\Im\lambda|}, \quad \lambda \in S.$$

Now let us fix $h \in L^2(\mathrm{M})$, $\lambda \in S$ and consider $\|R_{B_\ell}(\lambda)h\|_{D(H_{B_\ell})}$. We have

$$\begin{aligned}
&\|R_{B_\ell}(\lambda)h\|^2_{D(H_{B_\ell})} \\
&\leqslant \sum_{\ell=1}^{+\infty} \left(\frac{1+|\lambda_k|}{|\lambda-\lambda_k|}\right)^2 |\langle h, \phi_{B_\ell,k}\rangle|^2 \\
&\leqslant \sum_{|\lambda_k|\leqslant 2|\lambda|} \left(\frac{1+|\lambda_k|}{|\lambda-\lambda_k|}\right)^2 |\langle h, \phi_{B_\ell,k}\rangle|^2 + \sum_{|\lambda_k|\geqslant 2|\lambda|} \left(\frac{1+|\lambda_k|}{|\lambda-\lambda_k|}\right)^2 |\langle h, \phi_{B_\ell,k}\rangle|^2 \\
&\leqslant \left[\left(\frac{1+2|\lambda|}{|\Im\lambda|}\right)^2 + \sup_{k\geqslant 1}\left(\frac{4(1+|\lambda_k|)}{1+|\lambda_k|}\right)^2\right] \sum_{\ell=1}^{+\infty} |\langle h, \phi_{B_\ell,k}\rangle|^2 \\
&\leqslant C\left(\frac{1+|\lambda|}{|\Im\lambda|}\right)^2 \|h\|^2_{L^2(\mathrm{M})},
\end{aligned}$$

with $C > 0$ a constant. Combining this with the fact that $D(H_{B_\ell})$ is embedded continuously into $H^2(\mathrm{M})$, thanks to the elliptic regularity of the operator H_{B_ℓ}, we deduce that (3.1) is true for $s = 1$. Then, by interpolation, we deduce (3.1) for all $s \in [0,1]$. □

For $f \in H^{3/2}(\partial\mathrm{M})$ and $\lambda \in \rho(H_{B_\ell})$, $\ell = 1,2$, consider the Dirichlet problem

$$\begin{cases} (\mathcal{H}_{B_\ell} - \lambda)u = 0 & \text{in M}, \\ u = f & \text{on } \partial\mathrm{M}. \end{cases} \tag{3.2}$$

Let κ be a boundary defining function, that is a smooth function $\kappa : \overline{\mathrm{M}} \to \mathbb{R}_+$ such that

- $\kappa(x) > 0$ for all $x \in \mathrm{M}^{\mathrm{int}}$,
- $\kappa|_{\partial\mathrm{M}} = 0$ and $d\kappa(x) \neq 0$, for all $x \in \partial M$.

We recall that one can construct such a function by combining local coordinates with boundary distance functions or by considering the first eigenvalue of the Dirichlet Laplacian. We can now state the following result.

Lemma 3.2. — *If $f \in H^{3/2}(\partial \mathrm{M})$ and $\lambda \in \rho(H_{B_\ell})$, then the BVP (3.2) has a unique solution $u_\ell(\lambda) = u_\ell^f(\lambda) \in H^2(\mathrm{M})$ given by the series*

$$u_\ell(\lambda) = \sum_{k \geqslant 1} \frac{\langle f, \psi_{\ell,k}\rangle}{\lambda - \lambda_{\ell,k}}\, \phi_{\ell,k}, \tag{3.3}$$

the convergence takes place in $H^1(\mathrm{M})$. Moreover, for any neighborhood $\mathcal{V}$ of $\partial \mathrm{M}$ in M, we have

$$\lim_{\lambda \to -\infty} \left(\|u_\ell(\lambda)\|_{L^2(\mathrm{M})} + \|\kappa \nabla u_\ell(\lambda)\|_{L^2(\mathrm{M})} \right) = 0. \tag{3.4}$$

Proof. — The proof of (3.3) and

$$\lim_{\lambda \to -\infty} \|u_\ell(\lambda)\|^2_{L^2(\mathrm{M})} = 0$$

is quite similar to that of [27, Lemma 2.1].
The proof of (3.4) is then completed by establishing the following Cacciop-poli's type inequality, where $\lambda < 0$:

$$\|\kappa \mathrm{d} u_\ell(\lambda)\|_{L^2(\mathrm{M})} \leqslant C \|u_\ell(\lambda)\|_{L^2(\mathrm{M})}, \tag{3.5}$$

the constant C only depends on r and M.

For the sake of simplicity, we omit the subscript ℓ in $u_\ell(\lambda)$ and B_ℓ. Multiplying the first equation of (3.2) by $\kappa^2 \overline{u}(\lambda)$, using the fact that $\kappa_{|\partial \mathrm{M}} = 0$ and applying Green's formula, we obtain

$$\begin{aligned} 0 &= -\int_{\mathrm{M}} \Delta_A u(\lambda) \kappa^2 \overline{u}(\lambda)\, \mathrm{dv}^n + \int_{\mathrm{M}} (q - \lambda) \kappa^2 |u(\lambda)|^2 \,\mathrm{dv}^n \\ &= \int_{\mathrm{M}} |\kappa \mathrm{d} u(\lambda)|^2 \,\mathrm{dv}^n + 2 \int_{\mathrm{M}} \langle \kappa \mathrm{d} u(\lambda), \overline{u}(\lambda) \mathrm{d}\kappa \rangle \,\mathrm{dv}^n \\ &\quad + 2\Im \int_{\mathrm{M}} \langle \kappa \overline{u}(\lambda) A, \kappa \mathrm{d} u(\lambda)\rangle \,\mathrm{dv}^n \\ &\quad + \int_{\mathrm{M}} \left(2i \langle A, \kappa \mathrm{d}\kappa\rangle + (|A|^2 + q - \lambda)\kappa^2\right) |u(\lambda)|^2 \,\mathrm{dv}^n. \end{aligned} \tag{3.6}$$

An application of Cauchy–Schwarz's inequality yields

$$\begin{aligned} \|\kappa \mathrm{d} u(\lambda)\|^2_{L^2(\mathrm{M})} - \lambda \|\kappa u(\lambda)\|^2_{L^2(\mathrm{M})} &\leqslant C \|u(\lambda)\|_{L^2(\mathrm{M})} \|\kappa \mathrm{d} u(\lambda)\|_{L^2(\mathrm{M})} \\ &\qquad + C \|u(\lambda)\|^2_{L^2(\mathrm{M})} \\ &\leqslant C' \|u(\lambda)\|^2_{L^2(\mathrm{M})} + \frac{1}{2} \|\kappa \mathrm{d} u(\lambda)\|^2_{L^2(\mathrm{M})}. \end{aligned}$$

Then, it follows

$$\frac{1}{2} \|\kappa \mathrm{d} u(\lambda)\|^2_{L^2(\mathrm{M})} - \lambda \|\kappa u(\lambda)\|^2_{L^2(\mathrm{M})} \leqslant C \|u(\lambda)\|^2_{L^2(\mathrm{M})} \tag{3.7}$$

and since $-\lambda > 0$, we get

$$\|\kappa \mathrm{d}u(\lambda)\|^2_{L^2(\mathrm{M})} \leqslant C\|u(\lambda)\|^2_{L^2(\mathrm{M})}, \tag{3.8}$$

implying Caccioppoli's inequality (3.5). □

Lemma 3.3. — *Let $f \in H^{3/2}(\partial\mathrm{M})$ and, for $\mu \in \rho(H_{B_1}) \cap \rho(H_{B_2})$, set*

$$w_{1,2}(\mu) = u_1(\mu) - u_2(\mu) \in H^2(\mathrm{M}),$$

where $u_\ell(\mu)$ is the corresponding solution to (3.2) with B_ℓ and λ are substituted by B_ℓ and μ. Then we have that $w_{1,2}(\mu)$ converges to 0 in $H^2(\mathrm{M})$ as $\mu \to -\infty$. In particular, $\partial_\nu w_{1,2}(\mu) \to 0$ in $L^2(\partial\mathrm{M})$ as $\mu \to -\infty$.

Proof. — For the sake of simplicity, we use in this proof $w(\mu)$ instead of $w_{1,2}(\mu)$. Since the trace map $v \mapsto \partial_\nu v$ is continuous from $H^2(\mathrm{M})$ into $L^2(\partial\mathrm{M})$, it is enough to show that $\|w(\mu)\|_{H^2(\mathrm{M})} \to 0$ when $\mu \to -\infty$. Let $\mu < \mu_* < -2\|q\|_\infty$, for some fixed $\mu_* < 0$. It is straightforward to check that $w(\mu)$ is the solution of the boundary value problem

$$\begin{cases} (\mathcal{H}_{B_1} - \mu)\, w(\mu) = h(\mu) & \text{in M,} \\ w(\mu) = 0 & \text{on } \partial\mathrm{M}. \end{cases} \tag{3.9}$$

Here $h(\mu)$ is given by

$$h(\mu) = -2i\langle A_2 - A_1, \mathrm{d}u_2(\mu)\rangle + (V_2 - V_1)\, u_2(\mu) \tag{3.10}$$

with

$$V_j = -i\delta A_j + |A_j|^2 + q_j, \quad j = 1, 2.$$

Multiplying the first equation of (3.9) by $\overline{w}(\mu)$, we apply Green's formula (2.5) in order to obtain

$$\begin{aligned} \int_\mathrm{M} h(\mu)\overline{w}(\mu)\,\mathrm{dv}^n &= \int_\mathrm{M} \mathcal{H}_{B_1} w(\mu)\overline{w}(\mu)\,\mathrm{dv}^n - \int_\mathrm{M} \mu|w(\mu)|^2\,\mathrm{dv}^n \\ &= \int_\mathrm{M} |\nabla_{A_1} w|^2\,\mathrm{dv}^n + \int_\mathrm{M} (q-\mu)|w|^2\,\mathrm{dv}^n. \end{aligned}$$

We deduce that, for $-\mu$ sufficiently large,

$$\left(-\|q\|_\infty - \frac{\mu}{2}\right)\|w(\mu)\|^2_{L^2(\mathrm{M})} + \frac{|\mu|}{4}\|w(\mu)\|^2_{L^2(\mathrm{M})} \leqslant C\|h(\mu)\|^2_{L^2(\mathrm{M})},$$

for some positive constant C, not dependent on μ, and then we conclude that

$$|\mu|\|w(\mu)\|^2_{L^2(\mathrm{M})} \leqslant C\|h(\mu)\|^2_{L^2(\mathrm{M})}. \tag{3.11}$$

Moreover we have

$$\begin{cases} (H_{B_1} - \mu_*)\, w(\mu) = h(\mu) + (\mu - \mu_*)w(\mu) & \text{in M,} \\ w(\mu) = 0 & \text{on } \partial\mathrm{M}. \end{cases} \tag{3.12}$$

Using that $(H_{B_1} - \mu^*)^{-1}$ is an isomorphism from $L^2(\mathrm{M})$ onto $H^2(\mathrm{M})$, there exists a constant C, depending on M and B_1, so that

$$\begin{aligned}\|w(\mu)\|_{H^2(\mathrm{M})} &\leqslant C\|h(\mu) + (\mu - \mu_*)w(\mu))\|_{L^2(\mathrm{M})} \\ &\leqslant C\left(\|h(\mu)\|_{L^2(\mathrm{M})} + |\mu - \mu_*|\|w(\mu)\|_{L^2(\mathrm{M})}\right) \\ &\leqslant C\left(\|h(\mu)\|_{L^2(\mathrm{M})} + 2|\mu|\|w(\mu)\|_{L^2(\mathrm{M})}\right),\end{aligned} \tag{3.13}$$

where the positive constant C is not dependent on μ. Using now the estimate (3.11), we obtain

$$\|w(\mu)\|_{H^2(\mathrm{M})} \leqslant 4C\|h(\mu)\|_{L^2(\mathrm{M})}. \tag{3.14}$$

On the other hand, in view of (1.16) there exists $C > 0$ such that

$$|A_1(x) - A_2(x)| \leqslant C\kappa(x), \quad x \in \mathrm{M}. \tag{3.15}$$

Applying (3.15), we obtain

$$\|h(\mu)\|_{L^2(\mathrm{M})} \leqslant C''\left(\|\kappa \mathrm{d}u_2(\mu)\|_{L^2(\mathrm{M})} + \|u_2(\mu)\|_{L^2(\mathrm{M})}\right) \tag{3.16}$$

for some constant C'' independent of μ. Then, according to (3.4) in Lemma 3.2, we get

$$\limsup_{\mu\to-\infty} \|h(\mu)\|_{L^2(\mathrm{M})} = 0, \tag{3.17}$$

entailing by (3.14)

$$\limsup_{\mu\to-\infty} \|w(\mu)\|_{H^2(\mathrm{M})} = 0. \tag{3.18}$$

This completes the proof of the lemma. □

The following lemma will be useful in the sequel. We omit its proof since it is quite similar to that in [26, 27].

Lemma 3.4. — *Let $f \in H^{3/2}(\partial\mathrm{M})$ and, for $\mu, \lambda \in \rho(H_{B_\ell})$, set $w_\ell(\lambda,\mu) = u_\ell(\lambda) - u_\ell(\mu)$, where $u_\ell(\mu)$ is the solution of* (3.2) *when λ is substituted by μ. Then we have*

$$(\partial_\nu + iA_\ell(\nu))\, w_\ell(\lambda,\mu) = \sum_{k\geqslant 1} \frac{(\mu-\lambda)\langle f, \psi_{\ell,k}\rangle}{(\lambda-\lambda_{\ell,k})(\mu-\lambda_{\ell,k})}\, \psi_{\ell,k}, \tag{3.19}$$

the convergence takes place in $H^{1/2}(\partial\mathrm{M})$.

4. Isozaki's representation formula

In the present section we provide a version of Isozaki's approach [21], based on the so-called *Born approximation* method. The usual ansatz used to solve the problem of determining the coefficients of a magnetic Laplace–Beltrami operator, from the corresponding Dirichlet-to-Neumann map will be useful in our analysis. Let us describe briefly this method.

In all of this section $B_\ell = (A_\ell, q_\ell) \in \mathscr{B}_r$, $\ell = 1, 2$, with A_ℓ satisfying (1.16). We extend the covector A_1 to a $W^{2,\infty}$ covector on M_1 supported in the interior of M_1 and still denoted by A_1. Then, we consider the extension of A_2 to M_1, still denoted by A_2, defined by

$$A_1(x) = A_2(x), \quad x \in \mathrm{M}_1 \setminus \mathrm{M}. \tag{4.1}$$

Then, (1.16) implies that $A_2 \in W^{2,\infty}(\mathrm{M}_1; T^*\mathrm{M}_1)$. We fix also $A = A_1 - A_2$.

4.1. Representation formula

If $u_\ell(\lambda)$, $\lambda \in \rho(H_{B_1}) \cap \rho(H_{B_2})$, is the solution of (3.2) when $B = B_\ell$, define the Dirichlet-to-Neumann map by

$$\Lambda_{B_\ell}(\lambda) : f \in H^{3/2}(\partial\mathrm{M}) \longmapsto (\partial_\nu + iA_\ell(\nu))\, u_\ell(\lambda)_{|\partial\mathrm{M}}, \quad \ell = 1, 2. \tag{4.2}$$

We fix $\psi \in \mathcal{C}^2(\mathrm{M})$ a function satisfying the eikonal equation

$$|\mathrm{d}\psi|^2 = \sum_{i,j=1}^{n} \mathrm{g}^{ij} \frac{\partial\psi}{\partial x_i} \frac{\partial\psi}{\partial x_j} = 1. \tag{4.3}$$

We also set two functions $\alpha_\ell \in H^2(\mathrm{M})$ solving the transport equations

$$\langle \mathrm{d}\psi, \mathrm{d}\alpha_\ell \rangle + \frac{1}{2}(\Delta\psi)\alpha_\ell = 0, \quad \ell = 1, 2. \tag{4.4}$$

These functions will be given in Section 4.2. Consider also two functions $\beta_{A_\ell} \in H^2(\mathrm{M})$, $\ell = 1, 2$, solutions of the transport equations

$$\langle \mathrm{d}\psi, \mathrm{d}\beta_{A_\ell} \rangle + i\langle A_\ell, \mathrm{d}\psi \rangle \beta_{A_\ell} = 0, \quad \forall\ x \in \mathrm{M}, \quad \ell = 1, 2. \tag{4.5}$$

Henceforth $\tau > 1$ and $\lambda_\tau = \tau + i$. Let

$$\begin{aligned} \varphi^*_{1,\tau}(x) &= e^{i\lambda_\tau \psi(x)} \alpha_1 \beta_{A_1}(x) := e^{i\lambda_\tau \psi(x)} \beta^*_1(x), \\ \varphi^*_{2,\tau}(x) &= e^{i\overline{\lambda_\tau} \psi(x)} \alpha_2 \beta_{A_2}(x) := e^{i\overline{\lambda_\tau} \psi(x)} \beta^*_2(x), \end{aligned} \tag{4.6}$$

where, for $\ell = 1, 2$, α_ℓ is a solution of (4.4) and β_{A_ℓ} is a solution of (4.5).

Define

$$S_{B_\ell}(\tau) = \langle \Lambda_{B_\ell}(\lambda_\tau^2)\varphi^*_{1,\tau}, \varphi^*_{2,\tau} \rangle = \int_{\partial\mathrm{M}} \Lambda_{B_\ell}(\lambda_\tau^2)\varphi^*_{1,\tau} \overline{\varphi^*_{2,\tau}}\, \mathrm{d}\sigma^{n-1}, \quad \ell = 1, 2. \tag{4.7}$$

Lemma 4.1. — *We have*

$$S_{B_1}(\tau) = \int_{\partial \mathrm{M}} \beta_1^* \left(\partial_\nu \overline{\beta_2^*} - iA_1(\nu)\overline{\beta_2^*} - i\lambda_\tau \overline{\beta_2^*}\partial_\nu \psi\right) \mathrm{d}\sigma^{n-1} + \int_{\mathrm{M}} \beta_1^* \overline{\mathcal{H}_{B_1}(\beta_2^*)}\,\mathrm{dv}^n - 2\lambda_\tau \int_{\mathrm{M}} \beta_1^*\overline{\beta_2^*}\langle A, \mathrm{d}\psi\rangle\,\mathrm{dv}^n - \int_{\mathrm{M}} R_{B_1}(\lambda_\tau^2)\left(e^{i\lambda_\tau\psi}\mathcal{H}_{B_1}\left(\beta_1^*\right)\right) \times \left(e^{-i\lambda_\tau\psi}\overline{\mathcal{H}_{B_1}(\beta_2^*)} - 2\lambda_\tau e^{-i\lambda_\tau\psi}\overline{\beta_2^*}\langle A, \mathrm{d}\psi\rangle\right)\mathrm{dv}^n. \tag{4.8}$$

and

$$S_{B_2}(\tau) = \int_{\partial \mathrm{M}} \beta_1^* \left(\partial_\nu \overline{\beta_2^*} - iA_2(\nu)\overline{\beta_2^*} - i\lambda_\tau \overline{\beta_2^*}\partial_\nu \psi\right) \mathrm{d}\sigma^{n-1} + \int_{M} \beta_1^* \overline{\mathcal{H}_{B_2}(\beta_2^*)}\,\mathrm{dv}^n - \int_{\mathrm{M}} R_{B_2}(\lambda_\tau^2)\left(e^{i\lambda_\tau\psi}\left(\mathcal{H}_{B_2}\left(\beta_1^*\right) - 2\lambda_\tau\langle A, \mathrm{d}\psi\rangle\beta_1^*\right)\right) \times \left(e^{-i\lambda_\tau\psi}\overline{\mathcal{H}_{B_2}(\beta_2^*)}\right)\mathrm{dv}^n. \tag{4.9}$$

Here $R_{B_\ell}(\lambda_\tau^2)$ is the resolvent of H_{B_ℓ}.

Proof. — Direct computations yield

$$\left(\mathcal{H}_{B_1} - \lambda_\tau^2\right)\varphi_{1,\tau}^* = e^{i\lambda_\tau\psi}\mathcal{H}_{B_1}\left(\beta_1^*\right) + e^{i\lambda_\tau\psi}\Bigg(\lambda_\tau^2\beta_1^*\left(|\mathrm{d}\psi|^2 - 1\right) - 2i\lambda_\tau\beta_{A_1}\Big(\langle \mathrm{d}\psi, \mathrm{d}\alpha_1\rangle + \frac{\alpha_1}{2}\Delta\psi\Big) - 2i\lambda_\tau\alpha_1\left(\langle \mathrm{d}\psi, \mathrm{d}\beta_{A_1}\rangle + i\langle A_1, \mathrm{d}\psi\rangle\beta_{A_1}\right)\Bigg). \tag{4.10}$$

Taking into account (4.3), (4.4) and (4.5), with $\ell = 1$, the right-hand side of (4.10) becomes

$$\left(\mathcal{H}_{B_1} - \lambda_\tau^2\right)\varphi_{1,\tau}^* = e^{i\lambda_\tau\psi}\mathcal{H}_{B_1}\left(\beta_1^*\right) \equiv e^{i\lambda_\tau\psi}k_1. \tag{4.11}$$

Denote by u_1 the solution of the BVP

$$\begin{cases} \left(\mathcal{H}_{B_1} - \lambda_\tau^2\right)u_1 = 0 & \text{in M},\\ u_1 = \varphi_{1,\tau}^* & \text{on } \partial\mathrm{M}. \end{cases}$$

We split u_1 into two terms, $u_1 = \varphi_{1,\tau}^* + v_1$, where v_1 is the solution of the boundary value problem

$$\begin{cases} \left(\mathcal{H}_{B_1} - \lambda_\tau^2\right)v_1 = -e^{i\lambda_\tau\psi}k_1 & \text{in M},\\ v_1 = 0 & \text{on } \partial\mathrm{M}. \end{cases}$$

Therefore

$$u_1 = \varphi^*_{1,\tau} - (H_{B_1} - \lambda^2_\tau)^{-1}(e^{i\lambda_\tau\psi}k_1) = \varphi^*_{1,\tau} - R_{B_1}(\lambda^2_\tau)\big(e^{i\lambda_\tau\psi}k_1\big)\,. \tag{4.12}$$

As

$$S_{B_1}(\tau) = \int_{\partial \mathrm{M}} (\partial_\nu u_1 + iA_1(\nu)u_1)\,\overline{\varphi^*_{2,\tau}}\,\mathrm{d}\sigma^{n-1}, \tag{4.13}$$

we get by applying formula (2.5)

$$\begin{aligned} S_{B_1}(\tau) = \int_{\mathrm{M}} \Delta_{A_1} u_1 \overline{\varphi^*_{2,\tau}}\,\mathrm{dv}^n - \int_{\mathrm{M}} u_1 \overline{\Delta_{A_1}\varphi^*_{2,\tau}}\,\mathrm{dv}^n \\ + \int_{\partial\mathrm{M}} \varphi^*_{1,\tau}\Big(\overline{\partial_\nu\varphi^*_{2,\tau} + iA_1(\nu)\varphi^*_{2,\tau}}\Big)\,\mathrm{d}\sigma^{n-1}. \end{aligned} \tag{4.14}$$

On the other hand, by a simple computation and using (4.3), (4.4) and (4.5), we get

$$\begin{aligned} \Delta_{A_1}\varphi^*_{2,\tau} &= \Delta_{A_1}(e^{i\bar\lambda_\tau\psi}\beta^*_2) \\ &= -\overline{\lambda}^2_\tau\varphi^*_{2,\tau} + e^{i\bar\lambda_\tau\psi}\Delta_{A_1}(\beta^*_2) \\ &\quad - 2i\overline{\lambda}_\tau e^{i\bar\lambda_\tau\psi}\alpha_2\left(\langle \mathrm{d}\psi, \mathrm{d}\beta_2\rangle + i\langle A_1, \mathrm{d}\psi\rangle\beta_2\right) \\ &\quad + 2i\overline{\lambda}_\tau\beta_2 e^{i\bar\lambda_\tau\psi}\left(\langle \mathrm{d}\psi, \mathrm{d}\alpha_2\rangle + \frac{\alpha_2}{2}\Delta\psi\right) \\ &= -\overline{\lambda}^2_\tau\varphi^*_{2,\tau} + e^{i\bar\lambda_\tau\psi}\Delta_{A_1}(\beta^*_2) \\ &\quad - 2i\overline{\lambda}_\tau e^{i\bar\lambda_\tau\psi}\alpha_2\left(-i\langle A_2, \mathrm{d}\psi\rangle\beta_2 + i\langle A_1, \mathrm{d}\psi\rangle\beta_2\right) \\ &= -\overline{\lambda}^2_\tau\varphi^*_{2,\tau} + e^{i\bar\lambda_\tau\psi}\Delta_{A_1}(\beta^*_2) + 2\overline{\lambda}_\tau e^{i\bar\lambda_\tau\psi}\beta^*_2\langle A, \mathrm{d}\psi\rangle. \end{aligned}$$

Whence, in light of (4.12), we find

$$\begin{aligned} &\int_{\mathrm{M}} u_1\overline{\Delta_{A_1}\varphi^*_{2,\tau}}\,\mathrm{dv}^n \\ &\qquad = \int_{\mathrm{M}} \left(\varphi^*_{1,\tau} - R_{B_1}(\lambda^2_\tau)\left(e^{i\lambda_\tau\psi}k_1\right)\right) \\ &\qquad\qquad \times \left(-\lambda^2_\tau\overline{\varphi^*_{2,\tau}} + e^{-i\lambda_\tau\psi}\overline{\Delta_{A_1}(\beta^*_2)} + 2\lambda_\tau e^{-i\lambda_\tau\psi}\overline{\beta^*_2}\langle A, \mathrm{d}\psi\rangle\right)\,\mathrm{dv}^n, \end{aligned}$$

and, using again (4.12), we get

$$\begin{aligned} \int_{\mathrm{M}} \Delta_{A_1} u_1\overline{\varphi^*_{2,\tau}}\,\mathrm{dv}^n &= -\int_{\mathrm{M}} \mathcal{H}_{B_1} u_1\overline{\varphi^*_{2,\tau}}\,\mathrm{dv}^n + \int_{\mathrm{M}} q_1 u_1\overline{\varphi^*_{2,\tau}}\,\mathrm{dv}^n \\ &= \int_{\mathrm{M}} \big(\varphi^*_{1,\tau} - R_{B_1}(\lambda^2_\tau)(e^{i\lambda_\tau\psi}k_1)\big)\big(-\lambda^2_\tau\overline{\varphi^*_{2,\tau}} + q_1\overline{\varphi^*_{2,\tau}}\big)\,\mathrm{dv}^n. \end{aligned}$$

We deduce that

$$
\begin{aligned}
(4.15)\quad & \int_{\mathrm{M}} \Delta_{A_1} u_1 \overline{\varphi^*_{2,\tau}}\, \mathrm{dv}^n - \int_{\mathrm{M}} u_1 \overline{\Delta_{A_1}\varphi^*_{2,\tau}}\, \mathrm{dv}^n \\
&= \int_{\mathrm{M}} \left(\varphi^*_{1,\tau} - R_{B_1}(\lambda_\tau^2)(e^{i\lambda_\tau\psi}k_1)\right) \\
&\qquad \times \left(e^{-i\lambda_\tau\psi}\overline{\mathcal{H}_{B_1}(\beta_2^*)} - 2\lambda_\tau e^{-i\lambda_\tau\psi}\overline{\beta_2^*}\langle A, \mathrm{d}\psi\rangle\right) \mathrm{dv}^n \\
&= \int_{\mathrm{M}} \beta_1^* \overline{\mathcal{H}_{B_1}(\beta_2^*)}\, \mathrm{dv}^n - 2\lambda_\tau \int_{\mathrm{M}} \beta_1^*\overline{\beta_2^*}\langle A, \mathrm{d}\psi\rangle\, \mathrm{dv}^n \\
&\quad - \int_{\mathrm{M}} R_{B_1}(\lambda_\tau^2)(e^{i\lambda_\tau\psi}k_1)\left(e^{-i\lambda_\tau\psi}\overline{\mathcal{H}_{B_1}(\beta_2^*)} - 2\lambda_\tau e^{-i\lambda_\tau\psi}\overline{\beta_2^*}\langle A, \mathrm{d}\psi\rangle\right) \mathrm{dv}^n.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
(4.16)\quad & \int_{\partial\mathrm{M}} \varphi^*_{1,\tau}\left(\overline{\partial_\nu\varphi^*_{2,\tau} + iA_1(\nu)\varphi^*_{2,\tau}}\right) \mathrm{d}\sigma^{n-1} \\
&= \int_{\partial\mathrm{M}} \beta_1^* \left(\partial_\nu\overline{\beta_2^*} - iA_1(\nu)\overline{\beta_2^*} - i\lambda_\tau\overline{\beta_2^*}\partial_\nu\psi\right) \mathrm{d}\sigma^{n-1}.
\end{aligned}
$$

Finally, we get (4.8) by combining (4.14), (4.15) and (4.16).

The proof of (4.9) is quite similar to that of (4.8). But, for the reader's convenience, we detail the proof of (4.9). By a simple computation we find

$$
\begin{aligned}
(4.17)\quad & \left(\mathcal{H}_{B_2} - \lambda_\tau^2\right)\varphi^*_{1,\tau} = e^{i\lambda_\tau\psi}\mathcal{H}_{B_2}\left(\beta_1^*\right) \\
&+ e^{i\lambda_\tau\psi}\Bigg(\lambda_\tau^2\beta_1^*\left(|\mathrm{d}\psi|^2 - 1\right) - 2i\lambda_\tau\beta_{A_1}\left(\langle \mathrm{d}\psi, \mathrm{d}\alpha_1\rangle + \frac{\alpha_1}{2}\Delta\psi\right) \\
&\qquad - 2i\lambda_\tau\alpha_1(\langle \mathrm{d}\psi, \mathrm{d}\beta_{A_1}\rangle + i\langle A_2, \mathrm{d}\psi\rangle\beta_{A_1})\Bigg).
\end{aligned}
$$

Taking into account (4.3), (4.4) and (4.5), the right-hand side of (4.17) takes the form

$$
(4.18)\quad (\mathcal{H}_{B_2} - \lambda_\tau^2)\varphi^*_{1,\tau} = e^{i\lambda_\tau\psi(x)}(\mathcal{H}_{B_2}\left(\beta_1^*\right) - 2\lambda_\tau\langle A, \mathrm{d}\psi\rangle\beta_1^*) \equiv e^{i\lambda_\tau\psi(x)}k_2.
$$

Let u_2 be the solution of the BVP

$$
\begin{cases}
\left(\mathcal{H}_{B_2} - \lambda_\tau^2\right) u_2 = 0 & \text{in M,} \\
u_2 = \varphi^*_{1,\tau} & \text{on } \partial\mathrm{M}.
\end{cases}
$$

As for u_1, we split u_2 into two terms, $u_2 = \varphi^*_{1,\tau} + v_2$, where v_2 is the solution of the BVP

$$
\begin{cases}
\left(\mathcal{H}_{B_2} - \lambda_\tau^2\right) v_2 = -e^{i\lambda_\tau\psi}k_2 & \text{in M} \\
v_2 = 0 & \text{on } \partial\mathrm{M}.
\end{cases}
$$

Therefore

$$u_2 = \varphi^*_{1,\tau} - \left(H_{B_2} - \lambda^2_\tau\right)^{-1} (e^{i\lambda_\tau\psi} k_2) = \varphi^*_{1,\tau} - R_{B_2}(\lambda^2_\tau)(e^{i\lambda_\tau\psi} k_2). \tag{4.19}$$

Since

$$S_{B_2}(\tau) = \int_{\partial \mathrm{M}} (\partial_\nu u_2 + iA_2(\nu)u_2)\, \overline{\varphi^*_{2,\tau}}\, \mathrm{d}\sigma^{n-1}, \tag{4.20}$$

we obtain, by applying formula (2.5),

$$\begin{aligned} S_{B_2}(\tau) = \int_{\mathrm{M}} \Delta_{A_2} u_2 \overline{\varphi^*_{2,\tau}}\, \mathrm{dv}^n - \int_{\mathrm{M}} u_2 \overline{\Delta_{A_2}\varphi^*_{2,\tau}}\, \mathrm{dv}^n \\ + \int_{\partial \mathrm{M}} \varphi^*_{1,\tau}(\overline{\partial_\nu \varphi^*_{2,\tau} + iA_2(\nu)\varphi^*_{2,\tau}})\, \mathrm{d}\sigma^{n-1}. \end{aligned} \tag{4.21}$$

On the other hand, by using (4.3), (4.4) and (4.5), we find

$$\Delta_{A_2}\varphi^*_{2,\tau} = \Delta_{A_2}(e^{i\bar{\lambda}_\tau\psi}\beta^*_2) = -\overline{\lambda}^2_\tau \varphi^*_{2,\tau} + e^{i\bar{\lambda}_\tau\psi}\Delta_{A_2}(\beta^*_2). \tag{4.22}$$

Whence

$$\begin{aligned} \int_{\mathrm{M}} u_2 \overline{\Delta_{A_2}\varphi^*_{2,\tau}}\, \mathrm{dv}^n = \int_{\mathrm{M}} (\varphi^*_{1,\tau} - R_{B_2}(\lambda^2_\tau)(e^{i\lambda_\tau\psi}k_2)) \\ \times \left(-\lambda^2_\tau \overline{\varphi^*_{2,\tau}} + e^{-i\lambda_\tau\psi}\overline{\Delta_{A_2}(\beta^*_2)}\right) \mathrm{dv}^n \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} \int_{\mathrm{M}} \Delta_{A_2} u_2 \overline{\varphi^*_{2,\tau}}\, \mathrm{dv}^n &= -\int_{\mathrm{M}} \mathcal{H}_{B_2} u_2 \overline{\varphi^*_{2,\tau}}\, \mathrm{dv}^n + \int_{\mathrm{M}} q_2 u_2 \overline{\varphi^*_{2,\tau}}\, \mathrm{dv}^n \\ &= \int_M \left(\varphi^*_{1,\tau} - R_{B_2}(\lambda^2_\tau)(e^{i\lambda_\tau\psi}k_2)\right)\left(-\lambda^2_\tau\overline{\varphi^*_{2,\tau}} + q_2\overline{\varphi^*_{2,\tau}}\right) \mathrm{dv}^n. \end{aligned}$$

Thus,

$$\begin{aligned} &\int_{\mathrm{M}} \Delta_{A_2} u_2 \overline{\varphi^*_{2,\tau}}\, \mathrm{dv}^n - \int_{\mathrm{M}} u_2 \overline{\Delta_{A_2}\varphi^*_{2,\tau}}\, \mathrm{dv}^n \\ &= \int_{\mathrm{M}} \left(\varphi^*_{1,\tau} - R_{B_2}(\lambda^2_\tau)(e^{i\lambda_\tau\psi}k_2)\right) \left(e^{-i\lambda_\tau\psi}\overline{\mathcal{H}_{\mathcal{B}_\in}(\beta^*_\in)}\right) \mathrm{dv}^n \\ &= \int_M \beta^*_1 \overline{\mathcal{H}_{B_2}(\beta^*_2)}\, \mathrm{dv}^n - \int_{\mathrm{M}} R_{B_2}(\lambda^2_\tau)\left(e^{i\lambda_\tau\psi}k_2\right)\left(e^{-i\lambda_\tau\psi}\overline{\mathcal{H}_{B_1}(\beta^*_2)}\right) \mathrm{dv}^n \\ &= \int_M \beta^*_1 \overline{\mathcal{H}_{B_2}(\beta^*_2)}\, \mathrm{dv}^n \\ &\quad - \int_{\mathrm{M}} R_{B_2}(\lambda^2_\tau)\left(e^{i\lambda_\tau\psi}(\mathcal{H}_{B_2}(\beta^*_1) - 2\lambda_\tau\langle A, \mathrm{d}\psi\rangle\beta^*_1)\right)\left(e^{-i\lambda_\tau\psi}\overline{\mathcal{H}_{B_1}(\beta^*_2)}\right) \mathrm{dv}^n. \end{aligned} \tag{4.24}$$

Moreover, we have

$$(4.25)\quad \int_{\partial M} \varphi_{1,\tau}^{*}\big(\overline{\partial_\nu \varphi_{2,\tau}^{*} + iA_2(\nu)\varphi_{2,\tau}^{*}}\big)\,\mathrm{d}\sigma^{n-1} = \int_{\partial M} \beta_1^{*}\left(\partial_\nu \overline{\beta_2^{*}} - iA_2(\nu)\overline{\beta_2^{*}} - i\lambda_\tau \overline{\beta_2^{*}}\partial_\nu\psi\right)\,\mathrm{d}\sigma^{n-1}.$$

Inserting (4.25) and (4.24) in (4.21), we obtain

$$(4.26)\quad S_{B_2}(\tau) = \int_{\partial M} \beta_1^{*}\left(\partial_\nu \overline{\beta_2^{*}} - iA_2(\nu)\overline{\beta_2^{*}} - i\lambda_\tau \overline{\beta_2^{*}}\partial_\nu\psi\right)\,\mathrm{d}\sigma^{n-1} + \int_{M} \beta_1^{*}\overline{\mathcal{H}_{B_2}(\beta_2^{*})}\,\mathrm{dv}^n - \int_M R_{B_2}(\lambda_\tau^2)\big(e^{i\lambda_\tau\psi}(\mathcal{H}_{B_2}(\beta_1^{*}) - 2\lambda_\tau\langle A, \mathrm{d}\psi\rangle\beta_1^{*})\big)\Big(e^{-i\lambda_\tau\psi}\overline{\mathcal{H}_{B_2}(\beta_2^{*})}\Big)\mathrm{dv}^n.$$

This completes the proof of the Lemma. □

Subtracting side by side (4.8) and (4.9), and using the fact that $A_1 = A_2$ on ∂M, we obtain the following identity, that we will use later in the text.

$$(4.27)\quad S_{B_1}(\tau) - S_{B_2}(\tau) = -2\lambda_\tau \int_M \beta_1^{*}\overline{\beta_2^{*}}\langle A, \mathrm{d}\psi\rangle\,\mathrm{dv}^n + \int_M \beta_1^{*}\overline{(\mathcal{H}_{B_1} - \mathcal{H}_{B_2})(\beta_2^{*})}\,\mathrm{dv}^n - \int_M R_{B_1}(\lambda_\tau^2)\big(e^{i\lambda_\tau\psi}\mathcal{H}_{B_1}(\beta_1^{*})\big)\Big(e^{-i\lambda_\tau\psi}\Big(\overline{\mathcal{H}_{B_1}(\beta_2^{*})} - 2\lambda_\tau\overline{\beta_2^{*}}\langle A, \mathrm{d}\psi\rangle\Big)\Big)\mathrm{dv}^n + \int_M R_{B_2}(\lambda_\tau^2)\big(e^{i\lambda_\tau\psi}\,(\mathcal{H}_{B_2}(\beta_1^{*}) - 2\lambda_\tau\langle A, \mathrm{d}\psi\rangle\beta_1^{*})\big)\Big(e^{-i\lambda_\tau\psi}\overline{\mathcal{H}_{B_2}(\beta_2^{*})}\Big)\mathrm{dv}^n.$$

4.2. Solving the eikonal and transport equations

We construct the phase function ψ solution to the eikonal equation (4.3) and the amplitudes α_ℓ and β_ℓ, $\ell = 1, 2$, solutions to the transport equations (4.4) and (4.5).

Let $y \in \partial M_1$. Denote points in M_1 by (r, θ) where (r, θ) are polar normal coordinates in M_1 with center y. That is, $x = \exp_y(r\theta)$, where $r > 0$ and

$$\theta \in S_y^{+}M_1 = \{\theta \in T_yM_1,\ |\theta| = 1,\ \langle\theta, \nu\rangle < 0\}.$$

In these coordinates (depending on the choice of y) the metric has the form

$$\widetilde{\mathrm{g}}(r, \theta) = \mathrm{d}r^2 + \mathrm{g}_0(r, \theta).$$

If u is a function in M, set, for $r > 0$ and $\theta \in S_yM_1$,

$$\widetilde{u}(r, \theta) = u(\exp_y(r\theta)),$$

If u is compactly supported, $\widetilde{u}$ is naturally extended by 0 outside M.

The geodesic distance to y provide an explicit solution of the eikonal equation (4.3):

$$\psi(x) = d_{\mathrm{g}}(x, y). \tag{4.28}$$

Since $y \in \mathrm{M}_1 \backslash \overline{\mathrm{M}}$, we have $\psi \in \mathcal{C}^\infty(\mathrm{M})$ and

$$\widetilde{\psi}(r, \theta) = r = d_{\mathrm{g}}(x, y). \tag{4.29}$$

We now solve the transport equation (4.4). To this and, recall that if $f(r)$ is any function of the geodesic distance r, then

$$\Delta_{\widetilde{\mathrm{g}}} f(r) = f''(r) + \frac{\varrho^{-1}}{2}\frac{\partial \varrho}{\partial r} f'(r). \tag{4.30}$$

Here $\varrho = \varrho(r, \theta)$ denotes the square of the volume element in geodesic polar coordinates. In the new coordinates system, equation (4.4) takes the form

$$\frac{\partial \widetilde{\psi}}{\partial r}\frac{\partial \widetilde{\alpha}}{\partial r} + \frac{1}{4}\widetilde{\alpha}\varrho^{-1}\frac{\partial \varrho}{\partial r}\frac{\partial \widetilde{\psi}}{\partial r} = 0. \tag{4.31}$$

Thus $\widetilde{\alpha}$ satisfies

$$\frac{\partial \widetilde{\alpha}}{\partial r} + \frac{1}{4}\widetilde{\alpha}\varrho^{-1}\frac{\partial \varrho}{\partial r} = 0. \tag{4.32}$$

For $\eta \in H^2(S_y^+\mathrm{M})$, we seek $\widetilde{\alpha}$ in the form

$$\widetilde{\alpha}(r, \theta) = \varrho^{-1/4}\eta(y, \theta). \tag{4.33}$$

Direct computations yield

$$\frac{\partial \widetilde{\alpha}}{\partial r}(r, \theta) = -\frac{1}{4}\varrho^{-5/4}\frac{\partial \varrho}{\partial r}\eta(y, \theta). \tag{4.34}$$

Finally, (4.33) and (4.34) entail

$$\frac{\partial \widetilde{\alpha}}{\partial r}(r, \theta) = -\frac{1}{4}\varrho^{-1}\widetilde{\alpha}(r, \theta)\frac{\partial \varrho}{\partial r}. \tag{4.35}$$

In the rest of this subsection we are concerned with transport equation (4.5). Using that, in polar coordinates, $\nabla\psi(x)$ can be expressed in term of $\dot{\gamma}_{y,\theta}(r)$ (see for instance [6, Appendix C]), we have

$$\langle \widetilde{A}_\ell(r, y, \theta), \mathrm{d}\psi\rangle = \langle \widetilde{A}_\ell^\sharp(r, y, \theta), \nabla\psi\rangle = \sigma_{A_\ell}(\varphi_r(y, \theta)) = \widetilde{\sigma}_{A_\ell}(r, y, \theta).$$

Consequently, in polar coordinates system, (4.5) has the form

$$\frac{\partial \widetilde{\psi}}{\partial r}\frac{\partial \widetilde{\beta}}{\partial r} + i\widetilde{\sigma}_{A_\ell}(r, y, \theta)\widetilde{\beta} = 0, \tag{4.36}$$

where $\widetilde{\sigma}_{A_\ell}(r, y, \theta) := \sigma_{A_\ell}(\Phi_r(y, \theta)) = \langle \dot{\gamma}_{y,\theta}(r), A_\ell^\sharp(\gamma_{y,\theta}(r))\rangle$. Thus $\widetilde{\beta}$ satisfies

$$\frac{\partial \widetilde{\beta}}{\partial r} + i\widetilde{\sigma}_{A_\ell}(r, y, \theta)\widetilde{\beta} = 0. \tag{4.37}$$

Thus, we can choose $\widetilde{\beta}$ defined as follows

$$\widetilde{\beta}(y,r,\theta) = \exp\left(i\int_0^{\ell_+(y,\theta)} \widetilde{\sigma}_{A_\ell}(r+s,y,\theta)\,\mathrm{d}s\right).$$

In other words, we solved (3.4). Note that here, since the support of A_ℓ is contained into the interior of M_1, the support of $\widetilde{\sigma}_{A_\ell}$ is contained into the interior of M_1 and the function $r \mapsto \widetilde{\sigma}_{A_\ell}(r,y,\theta)$ is supported on $[0,\ell_+(y,\theta)]$. Therefore, in the previous integral, for $r,s \in [0,\ell_+(y,\theta)]$ if $r+s \geqslant \ell_+(y,\theta)$ we have $\widetilde{\sigma}_{A_\ell}(r+s,y,\theta) = 0$ which makes sense.

In the remainder of this paper we use the following notations:

$$\widetilde{\beta}_{A_\ell}(y,r,\theta) = \exp\left(i\int_0^{\ell_+(y,\theta)} \widetilde{\sigma}_{A_\ell}(r+s,y,\theta)\,\mathrm{d}s\right), \quad \ell = 1,2, \tag{4.38}$$

and

$$\widetilde{\alpha}_1(r,\theta) = \varrho^{-1/4}\eta(y,\theta), \quad \widetilde{\alpha}_2(r,\theta) = \varrho^{-1/4}. \tag{4.39}$$

4.3. Asymptotic behavior of the boundary representation formula

We discuss in this subsection the asymptotic behavior of $S_{B_1}(\tau) - S_{B_2}(\tau)$, as well as the asymptotic behavior of $[S_{B_1}(\tau) - S_{B_2}(\tau)]/\tau$, as $\tau \to \infty$.

As before, $B_\ell = (A_\ell, q_\ell) \in \mathscr{B}_r$, $\ell = 1,2$ are such that A_ℓ's satisfy (1.16). Set

$$A(x) = (A_1 - A_2)(x), \quad q(x) = (q_1 - q_2)(x).$$

Note that A, extended by 0 outside M, belongs to $\mathcal{C}^0(\mathrm{M}_1, T^*\mathrm{M}_1)$. We also extend q by 0 outside M. This extension, still denoted by q, is an element of $L^\infty(\mathrm{M}_1)$.

Lemma 4.2. — *For any $\eta \in H^2(S_y^+\mathrm{M}_1)$, we have*

$$\lim_{\tau \to +\infty} \frac{S_{B_1}(\tau) - S_{B_2}(\tau)}{\tau} = 2i\int_{S_y^+\mathrm{M}_1} \left(e^{i\mathcal{I}_1 A(y,\theta)} - 1\right)\eta(y,\theta)\,\mathrm{d}\omega_y(\theta). \tag{4.40}$$

Proof. — By the resolvent estimate, we have

$$\|R_{B_\ell}(\lambda_\tau^2)\|_{\mathscr{L}(L^2(\mathrm{M}))} \leqslant \frac{1}{|\Im(\lambda_\tau^2)|} = \frac{1}{2\tau}, \quad \ell = 1,2. \tag{4.41}$$

Inequalities (4.41) and (4.27) yield in a straightforward manner

$$\begin{aligned}(4.42)\qquad \lim_{\tau\to+\infty}\frac{S_{B_1}(\tau)-S_{B_2}(\tau)}{\tau} &= 2\int_{\mathrm{M}}\beta_1^*\overline{\beta_2^*}\langle A,\mathrm{d}\psi\rangle\,\mathrm{dv}^n\\ &= 2\int_{\mathrm{M}}\alpha_1\alpha_2\beta_{A_1}\overline{\beta_{A_2}}\langle A,\mathrm{d}\psi\rangle\,\mathrm{dv}^n.\end{aligned}$$

Applying (4.1) and making the change of variable $x=\exp_y(r\theta)$, with $r>0$ and $\theta\in S_y\mathrm{M}_1$, we get

$$\begin{aligned}(4.43)\quad & 2\int_{\mathrm{M}}\langle A,\mathrm{d}\psi\rangle(\alpha_1\alpha_2)(x)(\beta_{A_1}\overline{\beta}_{A_2})(x)\,\mathrm{dv}^n\\ &= 2\int_{S_y^+\mathrm{M}_1}\int_0^{\ell_+(y,\theta)}\widetilde{\sigma}_A(r,y,\theta)(\widetilde{\alpha}_1\widetilde{\alpha}_2)(r,\theta)(\widetilde{\beta}_{A_1}\overline{\widetilde{\beta}}_{A_2})(r,\theta)\varrho^{1/2}\,\mathrm{d}r\,\mathrm{d}\omega_y(\theta)\\ &= 2\int_{S_y^+\mathrm{M}_1}\int_0^{\ell_+(y,\theta)}\widetilde{\sigma}_A(r,y,\theta)\widetilde{\beta}_{A_1}(r,\theta)\overline{\widetilde{\beta}}_{A_2}(r,\theta)\eta(y,\theta)\mathrm{d}r\,\mathrm{d}\omega_y(\theta)\\ &= \int_{S_y^+\mathrm{M}_1}\int_0^{\ell_+(y,\theta)}\widetilde{\sigma}_A(r,y,\theta)\exp\left(i\int_0^{\ell_+(y,\theta)}\widetilde{\sigma}_A(r+s,y,\theta)\,\mathrm{d}s\right)\eta(y,\theta)\mathrm{d}r\,\mathrm{d}\omega_y(\theta).\end{aligned}$$

Also

$$\begin{aligned}(4.44)\quad \int_0^{\ell_+(y,\theta)}\widetilde{\sigma}_A(r,y,\theta)\exp\left(i\int_0^{\ell_+(y,\theta)}\widetilde{\sigma}_A(r+s,y,\theta)\mathrm{d}s\right)\mathrm{d}r &\\ = -i\int_0^{\ell_+(y,\theta)}\partial_r\left[\exp\left(i\int_0^{\ell_+(y,\theta)}\widetilde{\sigma}_A(r+s,y,\theta)\mathrm{d}s\right)\right]\mathrm{d}r &\\ = i\left[\exp\left(i\int_0^{\ell_+(y,\theta)}\widetilde{\sigma}_A(s,y,\theta)\mathrm{d}s\right)-1\right], &\end{aligned}$$

entailing

$$\begin{aligned} & 2\int_{\mathrm{M}}\langle A,\mathrm{d}\psi\rangle(\alpha_1\alpha_2)(x)(\beta_{A_1}\overline{\beta}_{A_2})(x)\,\mathrm{dv}^n\\ &\qquad = 2i\int_{S_y^+\mathrm{M}_1}\left(\exp\left(i\mathcal{I}_1A(y,\theta)\right)-1\right)\eta(y,\theta)\mathrm{d}\omega_y(\theta).\end{aligned}$$

This in (4.42) gives the expected inequality. □

Lemma 4.3. — *Assume that $A_1=A_2$. Then, for any $\eta\in H^2(S_y^+\mathrm{M}_1)$, we have*

$$(4.45)\qquad \lim_{\tau\to+\infty}\left(S_{B_1}(\tau)-S_{B_2}(\tau)\right)=\int_{S_y^+\mathrm{M}_1}\mathcal{I}_0(q)(y,\theta)\eta(y,\theta)\mathrm{d}\omega_y(\theta).$$

Proof. — Since $A_1 = A_2$, (4.27) is reduced to the following formula

$$S_{B_1}(\tau) - S_{B_2}(\tau) = \int_{\mathrm{M}} q(x)\beta_1^*(x)\overline{\beta_2^*}(x)\,\mathrm{dv}^n - \int_{\mathrm{M}} R_{B_1}(\lambda_\tau^2)\big(e^{i\lambda_\tau\psi}\mathcal{H}_{B_1}(\beta_1^*)\big)\Big(e^{-i\lambda_\tau\psi}\overline{\mathcal{H}_{B_1}(\beta_2^*)}\Big)\,\mathrm{dv}^n + \int_{\mathrm{M}} R_{B_2}(\lambda_\tau^2)\big(e^{i\lambda_\tau\psi}\mathcal{H}_{B_2}(\beta_1^*)\big)\Big(e^{-i\lambda_\tau\psi}\overline{\mathcal{H}_{B_2}(\beta_2^*)}\Big)\,\mathrm{dv}^n. \tag{4.46}$$

Once again the resolvent estimate enables us to get

$$\lim_{\tau\to+\infty}\left(S_{B_1}(\tau) - S_{B_2}(\tau)\right) = \int_{\mathrm{M}} q(x)(\alpha_1\alpha_2)(x)\,\mathrm{dv}^n. \tag{4.47}$$

We complete the proof by mimicking the end of the previous proof in order to obtain

$$\int_{\mathrm{M}} q(x)(\alpha_1\alpha_2)(x)\,\mathrm{dv}^n = \int_{S_y^+\mathrm{M}_1} \mathcal{I}_0(q)(y,\theta)\eta(y,\theta)\,\mathrm{d}\omega_y(\theta). \tag{4.48}$$

This completes the proof. □

5. Proof of the main results

5.1. Asymptotic behavior of the spectral data

Prior to the completion of the proofs of Theorems 1.2 and 1.3, we establish some technical lemmas. Assumptions and notations are the same as in the preceding one.

Lemma 5.1. — *For $t \in [0, 1/2)$ and $\ell = 1, 2$, we have*

$$\sum_{k\geqslant 1} k^{2t/n}\left|\frac{\langle \varphi_{1,\tau}^*, \psi_{\ell,k}\rangle}{\lambda_{\ell,k} - \lambda_\tau^2}\right|^2 \leqslant C_\ell \tau^{2t}\|\eta\|^2_{H^2(S_y^+\mathrm{M}_1)} \tag{5.1}$$

and

$$\sum_{k\geqslant 1} k^{2t/n}\left|\frac{\langle \varphi_{2,\tau}^*, \psi_{2,k}\rangle}{\lambda_{\ell,k} - \lambda_\tau^2}\right|^2 \leqslant C_\ell \tau^{2t}, \tag{5.2}$$

the constant C_ℓ depends on t, M, r and B_ℓ if $t > 0$, and it is independent on B_ℓ when $t = 0$.

Proof. — By Lemma 3.2 the solution of the boundary value problem (3.2), with $f = \varphi^*_{1,\tau}$, $\lambda = \lambda_\tau$ and $B = B_1$, is given by the series

$$u_1(\lambda_\tau) = \sum_{k \geqslant 1} \frac{\langle \varphi^*_{1,\tau}, \psi_{1,k} \rangle}{\lambda_\tau^2 - \lambda_{1,k}} \phi_{1,k}. \tag{5.3}$$

If $\mu = 2r+1$, then the operator $H_{B_\ell} + \mu$ is positive. Indeed, for $u \in H^1_0(\mathrm{M})$, we have

$$\int_{\mathrm{M}} (H_{B_\ell} + \mu)\, u\bar{u} \,\mathrm{dv}^n = \int_{\mathrm{M}} |\nabla_{A_\ell} u|^2 \,\mathrm{dv}^n + \int_{\mathrm{M}} (q_\ell + \mu)|u|^2 \,\mathrm{dv}^n$$
$$\geqslant \int_{\mathrm{M}} |\mathrm{d}u|^2 \,\mathrm{dv}^n + (\mu - \|q_\ell\|_\infty - 2\|A_\ell\|_\infty) \int_{\mathrm{M}} |u|^2 \,\mathrm{dv}^n.$$

Since $\mathscr{D}((H_{B_\ell} + \mu)^{\frac{1}{2}}) = H^1_0(\mathrm{M})$ we have, by interpolation, $\mathscr{D}((H_{B_\ell} + \mu)^{\frac{t}{2}}) = H^t_0(\mathrm{M}) = H^t(\mathrm{M})$ (e.g. [41, Chapter 1, Theorems 11.1 and 11.6]). Whence, for $w \in H^t(\mathrm{M})$, we have

$$\sum_{k \geqslant 1} (1 + |\lambda_{\ell,k}|)^t |(w, \phi_{\ell,k})|^2 \leqslant C_\ell \|w\|^2_{H^t(\mathrm{M})}, \quad \ell = 1, 2, \tag{5.4}$$

the constant C_ℓ only depends on t, r and M and B_ℓ.

On the other hand, we get from (4.12) that

$$\|u_1(\lambda_\tau)\|_{H^t(\mathrm{M})} = \|\varphi^*_{1,\tau} - R_{B_1}(\lambda_\tau^2)(e^{i\lambda_\tau \psi} \mathcal{H}_{B_1} \beta^*_1)\|_{H^t(\mathrm{M})}$$
$$\leqslant \|\varphi^*_{1,\tau}\|_{H^t(\mathrm{M})} + \|R_{B_1}(\lambda_\tau^2)(e^{i\lambda_\tau \psi} \mathcal{H}_{B_1} \beta^*_1)\|_{H^t(\mathrm{M})}.$$

Combining this with (3.1) and the fact that $\Im \lambda_\tau^2 = 2\tau$, we get

$$\|u_1(\lambda_\tau)\|_{H^t(\mathrm{M})} \leqslant C \left(\tau^t + \frac{(1 + |\lambda_\tau|)^t}{2\tau} \right) \|\eta\|_{H^2(S^+_y \mathrm{M}_1)} \tag{5.5}$$
$$\leqslant C \tau^t \|\eta\|_{H^2(S^+_y \mathrm{M}_1)},$$

with $C > 0$ a constant independent of τ and η. Here again the constant C only depends on t, r, M and B_1, where we used that $\exp_y^{-1}(M) \subset \{r\theta : r > 0,\ \theta \in S^+_y(M_1)\}$ in order to restrict the norm of η to $S^+_y \mathrm{M}_1$.
This estimate and (5.4) with $w = u_1(\lambda_\tau)$ and $\ell = 1$ entails

$$\sum_{k \geqslant 1} (1 + |\lambda_{1,k}|)^t |(u_1(\lambda_\tau), \phi_{1,k})|^2 \leqslant C_1 \tau^{2t} \|\eta\|^2_{H^2(S^+_y \mathrm{M}_1)}. \tag{5.6}$$

We get the first estimate (5.1) for $\ell = 1$, by using (A.1) in Appendix A and the identity

$$(u_1(\lambda_\tau), \phi_{1,k}) = \frac{\langle \varphi^*_{1,\tau}, \psi_{1,k} \rangle}{\lambda_\tau^2 - \lambda_{1,k}}. \tag{5.7}$$

To prove the first inequality (5.1) for $\ell = 2$, we consider $u_2(\lambda_\tau)$, the solution of the BVP (3.2) when $\lambda = \lambda_\tau$, $f = \varphi^*_{1,\tau}$ and $B = B_2$. By Lemma 3.2, this solution is given by the series

$$u_2(\lambda_\tau) = \sum_{k\geqslant 1} \frac{\langle \varphi^*_{1,\tau}, \psi_{2,k}\rangle}{\lambda_\tau^2 - \lambda_{2,k}} \phi_{2,k}. \tag{5.8}$$

On the other hand, we get from (4.19) and (3.1)

$$\begin{aligned} &\|u_2(\lambda_\tau)\|_{H^t(\mathrm{M})} \\ &\leqslant \|\varphi^*_{1,\tau}\|_{H^t(\mathrm{M})} + \|R_{B_2}(\lambda_\tau^2)(e^{i\lambda_\tau\psi}(\mathcal{H}_{B_2}(\beta_1^*) - 2\lambda_\tau\langle A, \mathrm{d}\psi\rangle\beta_2^*)\|_{H^t(\mathrm{M})} \\ &\leqslant C\left(\tau^t + \frac{|\lambda_\tau|^{1+t}}{\tau}\right)\|\eta\|_{H^2(S_y^+\mathrm{M}_1)} \leqslant C\tau^t\|\eta\|_{H^2(S_y^+\mathrm{M}_1)}. \end{aligned} \tag{5.9}$$

Applying again (5.4) with $w = u_2(\lambda_\tau)$ and $\ell = 2$ entails

$$\sum_{k\geqslant 1}(1 + |\lambda_{2,k}|)^t |(u_2(\lambda_\tau), \phi_{2,k})|^2 \leqslant C_2\tau^{2t}\|\eta\|^2_{H^2(S_y^+\mathrm{M}_1)}. \tag{5.10}$$

Since

$$(u_2(\lambda_\tau), \phi_{2,k}) = \frac{\langle \varphi^*_{1,\tau}, \psi_{2,k}\rangle}{\lambda_\tau^2 - \lambda_{2,k}}, \tag{5.11}$$

we obtain (5.1) with $\ell = 2$.

The second inequality of (5.2) is proved similarly. □

Let us recall some notations that we introduced in Section 3. For $f \in H^{3/2}(\partial\mathrm{M})$ fixed and $\lambda, \mu \in \rho(H_{B_1}) \cap \rho(H_{B_2})$, if $u_\ell(\lambda)$ (resp. $u_\ell(\mu)$) is the solution of the boundary value problem (3.2) for $B = B_\ell$ (resp. $B = B_\ell$ and $\lambda = \mu$), $\ell = 1, 2$, we have posed

$$w_\ell(\lambda, \mu) = u_\ell(\lambda) - u_\ell(\mu), \tag{5.12}$$

$$w_{1,2}(\mu) = u_1(\mu) - u_2(\mu). \tag{5.13}$$

Let

$$\mathcal{K}(\tau, \mu, f) = (\partial_\nu + iA_1(\nu))\, w_1(\lambda_\tau^2, \mu) - (\partial_\nu + iA_2(\nu))\, w_2(\lambda_\tau^2, \mu) \quad \text{on } \partial\mathrm{M}. \tag{5.14}$$

Then, by (3.19), we obtain

$$\begin{aligned} &\mathcal{K}(\tau, \mu, f) \\ &= \sum_{k\geqslant 1}\left[\frac{(\mu - \lambda_\tau^2)\langle f, \psi_{1,k}\rangle}{(\lambda_\tau^2 - \lambda_{1,k})(\mu - \lambda_{1,k})}\psi_{1,k} - \frac{(\mu - \lambda_\tau^2)\langle f, \psi_{2,k}\rangle}{(\lambda_\tau^2 - \lambda_{2,k})(\mu - \lambda_{2,k})}\psi_{2,k}\right]. \end{aligned} \tag{5.15}$$

We define

$$\mathcal{L}(\tau,\mu)=\langle \mathcal{K}(\tau,\mu,\varphi_{1,\tau}^*),\varphi_{2,\tau}^*\rangle. \tag{5.16}$$

From (5.15), we get

$$\begin{aligned}\mathcal{L}(\tau,\mu)& \\ &=\sum_{k\geqslant 1}(\mu-\lambda_\tau^2)\left[\frac{\langle\varphi_{1,\tau}^*,\psi_{1,k}\rangle\langle\psi_{1,k},\varphi_{2,\tau}^*\rangle}{(\lambda_\tau^2-\lambda_{1,k})(\mu-\lambda_{1,k})}-\frac{\langle\varphi_{1,\tau}^*,\psi_{2,k}\rangle\langle\psi_{2,k},\varphi_{2,\tau}^*\rangle}{(\lambda_\tau^2-\lambda_{2,k})(\mu-\lambda_{2,k})}\right].\end{aligned} \tag{5.17}$$

Define

$$\mathcal{L}^*(\tau)=\sum_{k\geqslant 1}\mathcal{L}_{1,k}^*(\tau)+\sum_{k\geqslant 1}\mathcal{L}_{2,k}^*(\tau)+\sum_{k\geqslant 1}\mathcal{L}_{3,k}^*(\tau), \tag{5.18}$$

with

$$\begin{aligned}\mathcal{L}_{1,k}^*(\tau)&=\frac{\left\langle\varphi_{1,\tau}^*,\psi_{1,k}-\psi_{2,k}\right\rangle\left\langle\psi_{1,k},\varphi_{2,\tau}^*\right\rangle}{\lambda_\tau^2-\lambda_{1,k}},\\ \mathcal{L}_{2,k}^*(\tau)&=\frac{\left\langle\varphi_{1,\tau}^*,\psi_{2,k}\right\rangle\left\langle\psi_{1,k}-\psi_{2,k},\varphi_{2,\tau}^*\right\rangle}{\lambda_\tau^2-\lambda_{1,k}},\\ \mathcal{L}_{3,k}^*(\tau)&=\left\langle\varphi_{1,\tau}^*,\psi_{2,k}\right\rangle\left\langle\psi_{2,k},\varphi_{2,\tau}^*\right\rangle\left(\frac{1}{(\lambda_\tau^2-\lambda_{1,k})}-\frac{1}{(\lambda_\tau^2-\lambda_{2,k})}\right).\end{aligned}$$

LEMMA 5.2. — *Under assumption* (1.17), $\mathcal{L}(\tau,\mu)$ *converges to* $\mathcal{L}^*(\tau)$ *as* $\mu\to-\infty$ *and, for* $t\in[0,1/2)$, *we have*

$$\limsup_{\tau\to\infty}\tau^{-t}|\mathcal{L}^*(\tau)|\leqslant C\|\eta\|_{H^2(S_y^+\mathrm{M}_1)}\limsup_{k\to\infty}k^{-t/n}|\lambda_{1,k}-\lambda_{2,k}|. \tag{5.19}$$

Proof. — We split $\mathcal{L}(\tau,\mu)$ into three series

$$\mathcal{L}(\tau,\mu)=\sum_{k\geqslant 1}\mathcal{L}_{1,k}(\mu,\tau)+\sum_{k\geqslant 1}\mathcal{L}_{2,k}(\mu,\tau)+\sum_{k\geqslant 1}\mathcal{L}_{3,k}(\mu,\tau),$$

with

$$\begin{aligned}\mathcal{L}_{1,k}(\tau,\mu)&=(\mu-\lambda_\tau^2)\frac{\langle\varphi_{1,\tau}^*,\psi_{1,k}-\psi_{2,k}\rangle\langle\psi_{1,k},\varphi_{2,\tau}^*\rangle}{(\lambda_\tau^2-\lambda_{1,k})(\mu-\lambda_{1,k})},\\ \mathcal{L}_{2,k}(\tau,\mu)&=(\mu-\lambda_\tau^2)\frac{\langle\varphi_{1,\tau}^*,\psi_{2,k}\rangle\langle\psi_{1,k}-\psi_{2,k},\varphi_{2,\tau}^*\rangle}{(\lambda_\tau^2-\lambda_{1,k})(\mu-\lambda_{1,k})},\\ \mathcal{L}_{3,k}(\tau,\mu)&=(\mu-\lambda_\tau^2)\langle\varphi_{1,\tau}^*,\psi_{2,k}\rangle\langle\psi_{2,k},\varphi_{2,\tau}^*\rangle\\ &\qquad\times\left(\frac{1}{(\lambda_\tau^2-\lambda_{1,k})(\mu-\lambda_{1,k})}-\frac{1}{(\lambda_\tau^2-\lambda_{2,k})(\mu-\lambda_{2,k})}\right).\end{aligned}$$

Under assumption (1.17) and in light of (5.1), we can see that the series in $\mathcal{L}_{1,k}(\tau,\mu)$, $\mathcal{L}_{2,k}(\tau,\mu)$ and $\mathcal{L}_{3,k}(\tau,\mu)$ converge uniformly with respect to $\mu\ll-1$. Therefore, $\mathcal{L}(\tau,\mu)$ converges to $\mathcal{L}^*(\tau)$ as $\mu\to-\infty$.

We have

$$(5.20)\quad |\mathcal{L}^*_{1,k}(\tau)| \leqslant \|\varphi^*_{1,\tau}\|_{L^2(\partial \mathrm{M})}\|\psi_{1,k}-\psi_{2,k}\|_{L^2(\partial \mathrm{M})}\left|\frac{\langle \psi_{1,k},\varphi^*_{2,\tau}\rangle}{\lambda_\tau^2-\lambda_{1,k}}\right|,$$

$$(5.21)\quad |\mathcal{L}^*_{2,k}(\tau)| \leqslant \frac{\|\varphi^*_{1,\tau}\|_{L^2(\partial \mathrm{M})}\|\varphi^*_{2,\tau}\|_{L^2(\partial \mathrm{M})}\|\psi_{1,k}-\psi_{2,k}\|^2_{L^2(\partial \mathrm{M})}}{|\lambda_\tau^2-\lambda_{1,k}|} + \|\varphi^*_{2,\tau}\|_{L^2(\partial \mathrm{M})}\|\psi_{1,k}-\psi_{2,k}\|_{L^2(\partial \mathrm{M})}\left|\frac{\langle \varphi^*_{1,\tau},\psi_{1,k}\rangle}{\lambda_\tau^2-\lambda_{1,k}}\right|,$$

$$(5.22)\quad |\mathcal{L}^*_{3,k}(\tau)| \leqslant \|\varphi^*_{1,\tau}\|_{L^2(\partial \mathrm{M})}\|\psi_{1,k}-\psi_{2,k}\|_{L^2(\partial \mathrm{M})}\frac{|\lambda_{2,k}-\lambda_{1,k}|}{|\lambda_\tau^2-\lambda_{2,k}|}\left|\frac{\langle \psi_{2,k},\varphi^*_{2,\tau}\rangle}{\lambda_\tau^2-\lambda_{2,k}}\right| + |\lambda_{2,k}-\lambda_{1,k}|\left|\frac{\langle \varphi^*_{1,\tau},\psi_{1,k}\rangle}{\lambda_\tau^2-\lambda_{1,k}}\right|\left|\frac{\langle \psi_{2,k},\varphi^*_{2,\tau}\rangle}{\lambda_\tau^2-\lambda_{2,k}}\right|.$$

But

$$(5.23)\quad \sup_{\tau>1}\|\varphi^*_{1,\tau}\|_{L^2(\partial \mathrm{M})} \leqslant \|\beta_1^*\|_{L^2(\partial \mathrm{M})} \leqslant C\|\eta\|_{H^2(S_y\mathrm{M}_1)}$$

and

$$(5.24)\quad \sup_{\tau>1}\|\varphi^*_{2,\tau}\|_{L^2(\partial \mathrm{M})} \leqslant \|\beta_1^*\|_{L^2(\partial \mathrm{M})} \leqslant C,$$

the constant C only depends on M. This estimate entails in particular that

$$\limsup_{\tau\to+\infty} \tau^{-t}|\mathcal{L}^*_{1,k}(\tau)| = 0, \quad k \geqslant 1.$$

Thus, for an arbitrary positive integer n_1, we get

$$\limsup_{\tau\to+\infty} \tau^{-t}\sum_{k=1}^{\infty}|\mathcal{L}^*_{1,k}(\tau)| = \limsup_{\tau\to+\infty} \tau^{-t}\sum_{k=n_1}^{\infty}|\mathcal{L}^*_{1,k}(\tau)|.$$

This estimate together with (5.1), (5.20), (5.23) and (5.24) imply

$$\tau^{-t}\sum_{k=n_1}^{\infty}|\mathcal{L}^*_{1,k}(\tau)| \leqslant C\left(\sup_{\tau>1}\tau^{-2t}\sum_{k=1}^{\infty}k^{2t/n}\left|\frac{\langle \psi_{1,k},\varphi^*_{2,\tau}\rangle}{\lambda_\tau^2-\lambda_{1,k}}\right|^2\right)^{1/2} \times \left(\sum_{k=n_1}^{\infty}k^{-2t/n}\|\psi_{1,k}-\psi_{2,k}\|^2_{L^2(\partial \mathrm{M})}\right)^{1/2} \leqslant C\left(\sum_{k=n_1}^{\infty}k^{-2t/n}\|\psi_{1,k}-\psi_{2,k}\|^2_{L^2(\partial \mathrm{M})}\right)^{1/2},$$

the constant C is independent on τ. Since the last term goes to zero as n_1 tends to ∞ by (1.18), we easily get

$$\limsup_{\tau\to+\infty} \tau^{-t}\sum_{k=1}^{\infty}|\mathcal{L}^*_{1,k}(\tau)| = 0. \tag{5.25}$$

In the sequel, we use the following useful observation: for $r > 1$ the map $\tau \mapsto |\lambda_\tau^2 - r|$ reach its minimum at $\tau = \sqrt{r-1}$. Hence

$$|\lambda_\tau^2 - r| \geqslant 2\sqrt{r-1}, \quad \tau > 0.$$

This observation together with (5.1), (5.21) and (A.1) in Appendix A yield

$$\begin{aligned}
&\limsup_{\tau\to+\infty} \tau^{-t}\sum_{k=1}^{\infty}|\mathcal{L}^*_{2,k}(\tau)| = \limsup_{\tau\to+\infty} \tau^{-t}\sum_{k=n_1}^{\infty}|\mathcal{L}^*_{2,k}(\tau)| \\
&\leqslant C\sum_{k=n_1}^{\infty} k^{-1/n}\|\psi_{1,k}-\psi_{2,k}\|^2_{L^2(\partial\mathrm{M})} \\
&\quad + C\left(\sup_{\tau>1}\tau^{-2t}\sum_{k=1}^{\infty}k^{2t/n}\left|\frac{\langle\psi_{1,k},\varphi^*_{1,\tau}\rangle}{\lambda_\tau^2-\lambda_{1,k}}\right|^2\right)^{\frac12}\left(\sum_{k=n_1}^{\infty}k^{-2/n}\|\psi_{1,k}-\psi_{2,k}\|^2_{L^2(\partial\mathrm{M})}\right)^{\frac12} \\
&\leqslant C\sum_{k=n_1}^{\infty} k^{-2t/n}\|\psi_{1,k}-\psi_{2,k}\|^2_{L^2(\partial\mathrm{M})} + C\left(\sum_{k=n_1}^{\infty}k^{-2t/n}\|\psi_{1,k}-\psi_{2,k}\|^2_{L^2(\partial\mathrm{M})}\right)^{\frac12}.
\end{aligned}$$

Then, using again the fact that n_1 is arbitrary and (1.17), we find

$$\limsup_{\tau\to+\infty} \tau^{-t}\sum_{k=1}^{\infty}|\mathcal{L}^*_{2,k}(\tau)| = 0. \tag{5.26}$$

The same argument as before enables us to obtain

$$\limsup_{\tau\to+\infty} \tau^{-t}\sum_{k=1}^{\infty}|\mathcal{L}^*_{3,k}(\tau)| \leqslant C\|\eta\|_{H^2(S_y^+\mathrm{M}_1)}\limsup_{k\to+\infty} k^{-t/n}|\lambda_{1,k}-\lambda_{2,k}|. \tag{5.27}$$

The expected result follows from (5.25), (5.26) and (5.27). □

5.2. End of the proof of the main results

We are now ready to complete the proof of Theorems 1.2 and 1.3.

Proof of Theorem 1.2. — Since A_ℓ, $\ell = 1, 2$, satisfy (1.16) and $w_{1,2}(\mu) = 0$ on $\partial\mathrm{M}$, we easily obtain the following identity, useful in the sequel,

$$\begin{aligned}
\mathcal{K}(\tau,\mu,\varphi^*_{1,\tau}) &= (\partial_\nu + iA_1(\nu))u_1(\lambda) \\
&\quad - (\partial_\nu + iA_2(\nu))u_2(\lambda) - \partial_\nu w_{1,2}(\mu) \quad \text{on } \partial\mathrm{M}.
\end{aligned} \tag{5.28}$$

By formula (5.16) we get

$$\begin{aligned}(5.29)\quad \mathcal{L}(\tau,\mu) &= \int_{\partial \mathrm{M}} \mathcal{K}(\tau,\mu,\varphi_{1,\tau}^*)\overline{\varphi_{2,\tau}^*}\,\mathrm{d}\sigma^{n-1}\\ &= \int_{\partial \mathrm{M}} (\partial_\nu + iA_1(\nu))\,u_1(\lambda)\overline{\varphi_{2,\tau}^*}\,\mathrm{d}\sigma^{n-1}\\ &\quad - \int_{\partial \mathrm{M}} (\partial_\nu + iA_2(\nu))\,u_2(\lambda)\overline{\varphi_{2,\tau}^*}\,\mathrm{d}\sigma^{n-1} - \int_{\partial \mathrm{M}} \partial_\nu w_{1,2}(\mu)\overline{\varphi_{2,\tau}^*}\,\mathrm{d}\sigma^{n-1}\\ &= \int_{\partial \mathrm{M}} \Lambda_{B_1}(\lambda_\tau^2)\varphi_{1,\tau}^*\overline{\varphi_{2,\tau}^*}\,\mathrm{d}\sigma^{n-1}\\ &\quad - \int_{\partial \mathrm{M}} \Lambda_{B_2}(\lambda_\tau^2)\varphi_{1,\tau}^*\overline{\varphi_{2,\tau}^*}\,\mathrm{d}\sigma^{n-1} - \int_{\partial \mathrm{M}} \partial_\nu w_{1,2}(\mu)\overline{\varphi_{2,\tau}^*}\,\mathrm{d}\sigma^{n-1}\\ &= S_{B_1}(\tau) - S_{B_2}(\tau) - \int_{\partial \mathrm{M}} \partial_\nu w_{1,2}(\mu)\overline{\varphi_{2,\tau}^*}\,\mathrm{d}\sigma^{n-1}.\end{aligned}$$

According to Lemmas 3.3 and 5.2, formula (5.29) and passing to the limit as μ goes to $-\infty$, we get

$$S_{B_1}(\tau) - S_{B_2}(\tau) = \mathcal{L}^*(\tau). \tag{5.30}$$

Furthermore, from (5.19) we have $\tau^{-t}\left(S_{B_1}(\tau) - S_{B_2}(\tau)\right)$ is bounded for $\tau > 1$ and $t \in [0, 1/2)$. Then $\tau^{-1}\left(S_{B_1}(\tau) - S_{B_2}(\tau)\right)$ goes to zero as τ tends to ∞. This in (4.40) yields,

$$\int_{S_y^+\mathrm{M}_1} \left(e^{i\mathcal{I}_1 A(y,\theta)} - 1\right)\eta(y,\theta)\,\mathrm{d}\omega_y(\theta) = 0. \tag{5.31}$$

Since η is arbitrary in $H^2(S_y\mathrm{M})$, we obtain that $\mathcal{I}_1 A(y,\theta) \in 2\pi\mathbb{Z}$ for any $\theta \in S_y^+\mathrm{M}_1$. On the other hand, since ∂M_1 is strictly convex, $S_y^+\mathrm{M}_1 \ni \theta \mapsto \ell_+(y,\theta)$ is continuous, and letting θ tend to a tangent direction $\theta_0 \in S_y\partial M_1$ we get

$$\lim_{\theta\to\theta_0} \ell_+(y,\theta) = 0$$

hence

$$2\pi m = \lim_{\theta\to\theta_0} \mathcal{I}_1 A(y,\theta) = 0$$

and therefore

$$\mathcal{I}_1 A(y,\theta) = 0, \quad \theta \in S_y^+\mathrm{M}_1 \tag{5.32}$$

which implies that $\mathcal{I}_1 A = 0$, because $y \in \partial\mathrm{M}_1$ is arbitrary. From (2.14), we deduce that the solenoidal part A^s in the Hodge decomposition of the 1-form A is equal to zero. This completes the proof of the first part of Theorem 1.2.

Now let us consider the second part of the theorem. For this purpose, we assume that condition (1.18) is fulfilled and we would like to show that

$q_1 = q_2$. Note first that the condition $A^s = 0$ implies $\mathrm{d}A = 0$ and, since M_1 is simply connected, there exists $\varphi \in W^{3,\infty}(\mathrm{M}_1)$ such that $\mathrm{d}\varphi = A$. Since $A = 0$ on $M_1 \setminus M$ by eventually extracting a constant to φ we may assume that $\varphi = 0$ on $M_1 \setminus M$. In particular we have $\varphi_{|\partial\mathrm{M}} = \partial_\nu \varphi_{|\partial\mathrm{M}} = 0$. Let $B_3 = (A_1, q_2)$. Applying (1.5), we deduce that

$$e^{-i\varphi} H_{B_2} e^{i\varphi} = H_{B_3}.$$

In particular, for $\lambda_{3,k}$, $k \geqslant 1$, the non-decreasing sequence of eigenvalues of H_{B_3} we have $\lambda_{3,k} = \lambda_{2,k}$ and $\phi_{3,k} = e^{-i\varphi}\phi_{2,k}$ corresponds to an orthonormal basis of eigenfunctions of H_{B_3}. Moreover, fixing $\psi_{3,k} = (\partial_\nu + iA_2(\nu))\,\phi_{3,k}$, we deduce that

$$\begin{aligned}\psi_{3,k}(x) &= (\partial_\nu + iA_1(\nu))\, e^{-i\varphi}\phi_{2,k}(x) \\ &= e^{-i\varphi}\,(\partial_\nu + iA_1(\nu) - i\partial_\nu\varphi)\,\phi_{2,k}(x) \\ &= (\partial_\nu + iA_2(\nu))\,\phi_{2,k}(x) = \psi_{2,k}(x),\ \ x \in \partial\mathrm{M}.\end{aligned}$$

Combining this with (1.18), we deduce that

$$\lim_{k\to+\infty} |\lambda_{1,k} - \lambda_{3,k}| = 0, \quad \text{and} \quad \sum_{k\geqslant 1} \|\psi_{1,k} - \psi_{3,k}\|^2_{L^2(\partial\mathrm{M})} < \infty.$$

In view of this gauge invariance property, from now on, without loss of generality, we may assume that $A_1 = A_2$. According to (1.18), with $t = 0$, the right hand side of (5.19) is equal to zero. □

Proof of Theorem 1.3. — We already proved that $\mathrm{d}A_1 = \mathrm{d}A_2$ in Theorem 1.2 and according to the gauge invariance property of the boundary spectral data, without loss of generality, we may assume that $A_1 = A_2$. Then a straightforward application of the min-max principle yields

$$|\lambda_{1,k} - \lambda_{2,k}| \leqslant \|q_1 - q_2\|_{L^\infty(\mathrm{M})}. \tag{5.33}$$

In that case (1.19) is reduced to

$$\sum_{k\geqslant 1} \|\psi_{1,k} - \psi_{2,k}\|^2_{L^2(\partial\mathrm{M})} < \infty. \tag{5.34}$$

Combining this with (4.45), (5.19) for $t = 0$ (which is valid in the present case) and taking into account that

$$\limsup_{\tau\to+\infty} |S_{B_1}(\tau) - S_{B_2}(\tau)| = \limsup_{\tau\to+\infty} |\mathcal{L}^*(\tau)|, \tag{5.35}$$

we obtain, for any $\eta \in H^2(S_y^+\mathrm{M}_1)$ real valued, that

$$(5.36)\quad \left|\int_{S_y^+\mathrm{M}_1} \mathcal{I}_0(q)(y,\theta)\eta(y,\theta)\,\mathrm{d}\sigma^{2n-2}\right| \leqslant C\|\eta\|_{H^2(S_y^+\mathrm{M}_1)} \limsup_{k\to+\infty} |\lambda_{1,k}-\lambda_{B_2^k}|.$$

Since $q \in H^1(\mathrm{M}_1)$, by the smoothing effect of the normal operator $N_0 = \mathcal{I}_0^*\mathcal{I}_0$ (see (2.21)), $N_0 q \in H^2(M)$ and

$$(5.37)\quad \|N_0(q)\|_{H^2(\mathrm{M}_1)} \leqslant C\|q\|_{H^1(\mathrm{M})} \leqslant Cr'.$$

Since $\mathcal{I}_0 :\ H^2(\mathrm{M}_1) \to H^2(\partial_+S\mathrm{M}_1)$ is bounded, we can take $\eta = \mathcal{I}_0 N_0(q)$. We integrate with respect to $y \in \partial\mathrm{M}_1$ the left hand side (5.36) in order to get

$$\int_{\partial_+S\mathrm{M}_1} \mathcal{I}_0(q)(y,\theta)\eta(y,\theta)\,\mathrm{d}\sigma^{2n-2} = \int_{\mathrm{M}_1} |N_0(q)|^2\,\mathrm{dv}^n = \|N_0(q)\|^2_{L^2(\mathrm{M}_1)}.$$

Combined with (5.36), this inequality entails

$$(5.38)\quad \|N_0(q)\|^2_{L^2(\mathrm{M}_1)} \leqslant C\|\mathcal{I}_0 N_0(q)\|_{H^2(\partial_+S\mathrm{M}_1)} \limsup_{k\to+\infty} |\lambda_{1,k}-\lambda_{2,k}|.$$

On the other hand, it follows from (5.37)

$$(5.39)\quad \|\mathcal{I}_0 N_0(q)\|_{H^2(\partial_+S\mathrm{M}_1)} \leqslant C\|N_0(q)\|_{H^2(\mathrm{M}_1)} \leqslant C',$$

the constants C and C' only depend on M and r. Thus, (5.38) and (5.39), give

$$(5.40)\quad \|N_0(q)\|^2_{L^2(\mathrm{M}_1)} \leqslant C \limsup_{k\to+\infty} |\lambda_{1,k}-\lambda_{2,k}|,$$

the constant C only depends on M and r'. We complete the proof by using the interpolation inequality

$$\|N_0(q)\|_{H^1(\mathrm{M}_1)} \leqslant C\|N_0(q)\|^{\frac{1}{2}}_{L^2(\mathrm{M}_1)}\|N_0(q)\|^{\frac{1}{2}}_{H^2(\mathrm{M}_1)} \leqslant C'\|N_0(q)\|^{\frac{1}{2}}_{L^2(\mathrm{M}_1)},$$

the constants C and C' only depend on M, r. We then apply (2.20) to get (1.20). □

6. Extension to the Neumann case

We explain in this section how to adapt the preceding analysis to obtain an uniqueness result for an inverse spectral problem fo the Schrödinger operator under Neumann boundary condition.

For $B = (A, q) \in \mathscr{B}$, define the unbounded self-adjoint operator $\mathscr{H}_B$, acting in $L^2(\mathrm{M})$ as follows

$$\mathscr{H}_B u = \mathcal{H}_B u = -\Delta_A u + qu, \quad u \in \mathscr{D}(\mathscr{H}_B), \tag{6.1}$$

with domain

$$\mathscr{D}(\mathscr{H}_B) = \left\{u \in H^1(\mathrm{M}),\ -\Delta_A u + qu \in L^2(\mathrm{M}),\ (\partial_\nu + iA(\nu))u_{|\partial M} = 0\right\}. \tag{6.2}$$

Fix $B_\ell \in \mathscr{B}_r$, $\ell = 1, 2$ and denote by $(\mu_{\ell,k}, \chi_{\ell,k})$, $k \geqslant 1$, the eigenvalues and normalized eigenfunctions of $\mathscr{H}_{B_\ell}$.

We aim in this section to prove the following uniqueness result.

Theorem 6.1. — *Assume that* (1.16) *and the conditions*

$$\sum_{k=1}^{+\infty} \|\chi_{1,k} - \chi_{2,k}\|^2_{L^2(\partial M)} < \infty, \tag{6.3}$$

$$\lim_{k\to+\infty} k^{-\frac{1}{n}} |\mu_{1,k} - \mu_{2,k}| = 0, \tag{6.4}$$

are fulfilled. Then $A_1^s = A_2^s$.

Note that, according to Weyl's formula in [10, p. 114], we have that

$$\lim_{k\to+\infty} k^{-\frac{1}{n}} |\mu_{1,k} - \mu_{2,k}| < \infty.$$

Therefore, condition (6.4) seems to be the optimal rate of growth of the difference of eigenvalues that guaranty the uniqueness of the magnetic potential.

6.1. Boundary representation formulae for the Neumann problem

For $g \in H^{1/2}(\partial\mathrm{M})$ and $\rho(\mathscr{H}_B)$, consider the BVP

$$\begin{cases} (\mathcal{H}_B - \lambda)v = 0 & \text{in M}, \\ (\partial_\nu + iA\nu)v = g & \text{on } \partial\text{M}. \end{cases} \tag{6.5}$$

Similarly to the Dirichlet case, for $\ell = 1, 2$, define the N-to-D map

$$\mathcal{N}_{\ell,\lambda} : g \in H^{\frac{1}{2}}(\partial\mathrm{M}) \longmapsto v_j(\lambda)_{|\partial\mathrm{M}},$$

where $v_j(\lambda) \in H^2(\mathrm{M})$ is the solution of the BVP (6.5).

Define, For $\ell = 1, 2$,

$$\begin{aligned} Q_j(\tau) &= \left\langle \mathcal{N}_{j,\lambda_\tau^2}(\partial_\nu + iA_j\nu)\varphi^*_{1,\tau}, (\partial_\nu + iA_j\nu)\varphi^*_{2,\tau}\right\rangle \\ &= \int_{\partial\mathrm{M}} (\partial_\nu - iA_j\nu)\overline{\varphi^*_{2,\tau}} \mathcal{N}_{j,\lambda_\tau^2}(\partial_\nu + iA_j\nu)\varphi^*_{1,\tau}\, \mathrm{d}\sigma^{n-1}, \end{aligned} \tag{6.6}$$

with $\varphi_{j,\tau}^*$, $j=1,2$, given in (4.6).

Proposition 6.2. — *We have*

$$Q_1(\tau)=\int_{\partial M}(\mathrm{i}\lambda_\tau)\partial_\nu\psi\beta_1^*+\mathrm{i}(A_1\nu)\beta_1^*+\partial_\nu\beta_1^*)\overline{\beta_2^*(x)}\,\mathrm{d}\sigma^{n-1}(x) \tag{6.7}$$
$$-2\lambda_\tau\int_{S_y(M_1)}\int_0^{\ell_+(y,\theta)}\widetilde{\sigma}_A(r,y,\theta)\widetilde{\beta}_1^*\overline{\widetilde{\beta}_2^*}\varrho^{1/2}\,\mathrm{d}r\,\mathrm{d}\omega_y(\theta)-\int_M\beta_1^*\overline{\mathcal{H}_{B_1}(\beta_2^*)}\,\mathrm{dv}^n$$
$$+\int_M\big[(\mathscr{H}_{B_1}-\lambda_\tau^2)^{-1}(e^{\mathrm{i}\lambda_\tau\psi}\mathcal{H}_{A_1,q_1}\beta_1^*)\big]e^{-\mathrm{i}\lambda_\tau\psi}\big[2\lambda_\tau(A\nabla\psi)\overline{\beta_2^*}+\overline{\mathcal{H}_{B_1}\beta_2^*}\big]\,\mathrm{dv}^n$$

and

$$Q_2(\tau) \tag{6.8}$$
$$=\int_{\partial M}(\mathrm{i}\lambda_\tau)\partial_\nu\psi\beta_1^*+\mathrm{i}(A_1\nu)\beta_1^*+\partial_\nu\beta_1^*)\overline{\beta_2^*(x)}\,\mathrm{d}\sigma^{n-1}(x)-\int_M\beta_1^*\overline{\mathcal{H}_{B_2}(\beta_2^*)}\,\mathrm{dv}^n$$
$$+\int_M\big[(\mathscr{H}_{B_2}-\lambda_\tau^2)^{-1}e^{\mathrm{i}\lambda_\tau\psi}(2\lambda_\tau(-A\nabla\psi)\beta_1^*+\mathcal{H}_{B_2}\beta_1^*)\big]\big(e^{-\mathrm{i}\lambda_\tau\psi}\overline{\mathcal{H}_{B_2}\beta_2^*}\big)\,\mathrm{dv}^n.$$

Proof. — Applying Green's formula, we get

$$\begin{aligned}Q_1(\tau)&=\int_M\mathrm{div}(v_1(\lambda_\tau^2)\overline{\nabla_{A_1}\varphi_{2,\tau}^*})\,\mathrm{dv}^n\\&=\int_M\big\langle\nabla_{A_1}v_1(\lambda_\tau^2),\overline{\nabla_{A_1}\varphi_{2,\tau}^*}\big\rangle_g\,\mathrm{dv}^n+\int_M v_1(\lambda_\tau^2)\overline{\Delta_{A_1}\varphi_{2,\tau}^*}\,\mathrm{dv}^n\\&=-\int_M\Delta_{A_1}v_1(\lambda_\tau^2)\overline{\varphi_{2,\tau}^*}\,\mathrm{dv}^n\\&\qquad+\int_{\partial M}(\partial_\nu+\mathrm{i}A_1\nu)v_1(\lambda_\tau^2)\overline{\varphi_{2,\tau}^*}\mathrm{d}\sigma_g+\int_M v_1(\lambda_\tau^2)\overline{\Delta_{A_1}\varphi_{2,\tau}^*}\,\mathrm{dv}^n,\end{aligned}$$

where $v_1(\lambda_\tau^2)$ the solution of the BVP (6.5), with $g=(\partial_\nu+\mathrm{i}A_1\nu)\varphi_{1,\tau}^*$, $\lambda=\lambda_\tau^2$, $A=A_1$, $q=q_1$. Using the fact that

$$(\partial_\nu+\mathrm{i}A_1\nu)v_1(\lambda_\tau^2)(x)=g(x)=(\partial_\nu+\mathrm{i}A_1\nu)\varphi_{1,\tau}^*(x),\quad x\in\partial M,$$

we deduce that

$$\begin{aligned}Q_1(\tau)=\int_{\partial M}&(\mathrm{i}\lambda_\tau)\partial_\nu\psi\beta_1^*+\mathrm{i}(A_1\nu)\beta_1^*+\partial_\nu\beta_1^*)\overline{\beta_2^*(x)}\,\mathrm{d}\sigma^{n-1}(x)\\&-\int_M\Delta_{A_1}v_1(\lambda_\tau^2)\overline{\varphi_{2,\tau}^*}\,\mathrm{dv}^n+\int_M v_1(\lambda_\tau^2)\overline{\Delta_{A_1}\varphi_{2,\tau}^*}\,\mathrm{dv}^n.\end{aligned}$$

This identity at hand, we proceed as in Lemma 4.1 to get (6.7). Similar arguments allow us to derive (6.8). □

As for the derivation of (4.40), we obtain from (6.7) and (6.8) the following identity

$$
\begin{aligned}
(6.9)\quad & Q_2(\tau) - Q_1(\tau) \\
&= 2\lambda_\tau \int_{S_y(\mathrm{M}_1)} \int_0^{\ell_+(y,\theta)} \widetilde{\sigma}_A(r,y,\theta)\widetilde{\beta}_1^* \overline{\widetilde{\beta}_2^*} \varrho^{1/2}\, \mathrm{d}r\, \mathrm{d}\omega_y(\theta) \\
&\quad + \int_{\mathrm{M}} (q_1 - q_2)\beta_1^*\beta_2^*\, \mathrm{dv}^n(x) - \int_{\mathrm{M}} \beta_1^* \overline{(\Delta_{A_1}\beta_2^* - \Delta_{A_2}\beta_2^*)}\, \mathrm{dv}^n \\
&\quad - \int_{\mathrm{M}} \left[(\mathscr{H}_{B_1} - \lambda_\tau^2)^{-1}(e^{\mathrm{i}\lambda_\tau\psi}\mathcal{H}_{B_1}\beta_1^*)\right] e^{-\mathrm{i}\lambda_\tau\psi}\left[2\lambda_\tau(A\nabla\psi)\overline{\beta_2^*} + \overline{\mathcal{H}_{B_1}\beta_2^*}\,\right] \mathrm{dv}^n \\
&\quad + \int_{\mathrm{M}} \left[(\mathscr{H}_{B_2} - \lambda_\tau^2)^{-1} e^{\mathrm{i}\lambda_\tau\psi}(2\lambda_\tau(-A\nabla\psi)\beta_1^* + \mathcal{H}_{B_2}\beta_1^*)\right]\left(e^{-\mathrm{i}\lambda_\tau\psi}\overline{\mathcal{H}_{B_2}\beta_2^*}\right) \mathrm{dv}^n,
\end{aligned}
$$

from which we deduce that, for all $y \in \partial\mathrm{M}_1$ and all $\eta \in H^2(S_y^+\mathrm{M}_1)$,

$$
(6.10)\quad 2i \int_{S_y^+(\mathrm{M}_1)} \left(e^{iI_1 A(y,\theta)} - 1\right) \eta(y,\theta)\, \mathrm{d}\omega_y(\theta) = \lim_{\tau\to+\infty} \frac{Q_2(\tau) - Q_1(\tau)}{\tau}.
$$

The following lemma is needed in the proof of Theorem 6.1.

Lemma 6.3. — *For $\ell = 1, 2$, consider $\varphi_{j,\tau}^*$, $j = 1, 2$, given by* (4.6)*. Then, we have*

$$
(6.11)\quad
\begin{aligned}
&\sum_{k=1}^\infty k^{\frac{2}{n}} \left| \frac{\langle \varphi_{1,\tau}^*, \chi_{\ell,k}\rangle}{\mu_{\ell,k} - \lambda_\tau^2} \right|^2 < C\|\eta\|_{H^2(S_y^+(\mathrm{M}_1))}^2 \tau^2, \\
&\sum_{k=1}^\infty k^{\frac{2}{n}} \left| \frac{\langle \overline{\varphi_{2,\tau}^*}, \chi_{\ell,k}\rangle}{\mu_{\ell,k} - \lambda_\tau^2} \right|^2 \leqslant C\tau^2, \quad \ell = 1, 2,
\end{aligned}
$$

with $C > 0$ independent of τ.

Proof. — Let $\tau = \|q_1\|_{L^\infty(M)} + \|q_2\|_{L^\infty(M)} + 1$ and note that $D((\mathscr{H}_{B_\ell} + \tau)^{1/2}) = H^1(M)$ since it coincides with the domain of the form associated to the operator $\mathscr{H}_{B_\ell} + \tau$. Whence, for any $w \in H^1(M)$, we have

$$
\sum_{k=1}^\infty (1 + |\mu_{\ell,k}|)|(w, \chi_{\ell,k})_{L^2(M)}|^2 \leqslant C\|w\|_{H^1(M)}^2,
$$

the constant C only depends on τ, A_ℓ, q_ℓ and M. Combining this estimate with a Weyl's formula for Neumann magnetic operators, similar to that in Lemma A.1, we get (6.11). □

6.2. End of the proof of Theorem 6.1.

The following lemma is useful in the sequel

Lemma 6.4. — *Let $g \in H^{1/2}(\partial\mathrm{M})$, $B \in \mathscr{B}$, $\lambda \in \rho(\mathscr{H}_B)$ and denote by $v(\lambda)$ the solution of the BVP* (6.5)*. Then*

$$v(\lambda)_{|\partial\mathrm{M}} = \sum_{k\geqslant 1} \frac{\langle g, \chi_k\rangle}{\lambda - \mu_k}\,\chi_k, \tag{6.12}$$

the convergence takes place in $H^{1/2}(\partial\mathrm{M})$.

In light of this lemma, we have

$$Q_2(\tau)-Q_1(\tau) = \sum_{k=1}^{\infty} \frac{\langle(\partial_\nu+iA_1\nu)\varphi_{1,\tau}^*, \chi_{2,k}\rangle\langle\chi_{2,k}, (\partial_\nu+iA_1\nu)\varphi_{2,\tau}^*\rangle}{\lambda_\tau^2-\mu_{2,k}} - \frac{\langle(\partial_\nu+iA_1\nu)\varphi_{1,\tau}^*, \chi_{1,k}\rangle\langle\chi_{1,k}, (\partial_\nu+iA_1\nu)\varphi_{2,\tau}^*\rangle}{\lambda_\tau^2-\mu_{1,k}}. \tag{6.13}$$

Observe that, according to (1.16), A_1 can be substituted by A_2 in the identity above.

On the other hand, we have from (4.6)

$$\begin{aligned} &\|(\partial_\nu+iA_1\nu)\varphi_{j,\tau}^*\|_{L^2(\partial\mathrm{M})} \\ &\qquad\leqslant |\lambda_\tau|\|\partial_\nu\psi\beta_j^*\|_{L^2(\partial\mathrm{M})} + \|(\partial_\nu+iA_1\nu)\beta_j^*\|_{L^2(\partial\mathrm{M})} \\ &\qquad\leqslant C\tau(1+\|\eta\|_{H^2(S_y\mathrm{M}_1)}), \end{aligned} \tag{6.14}$$

the constant C being independent of τ. Thus,

$$|Q_2(\tau)-Q_1(\tau)| \leqslant \sum_{k=1}^{\infty}\mathcal{E}_k(\tau) + \sum_{k=1}^{\infty}\mathcal{F}_k(\tau) + \sum_{k=1}^{\infty}\mathcal{G}_k(\tau),$$

with

$$\begin{aligned} \mathcal{E}_k(\tau) &= \|(\partial_\nu+iA_1\nu)\varphi_{1,\tau}^*\|_{L^2(\partial\mathrm{M})}\|\chi_{2,k}-\chi_{1,k}\|_{L^2(\partial\mathrm{M})}\frac{|\langle\chi_{2,k},(\partial_\nu+iA_1\nu)\varphi_{2,\tau}^*\rangle|}{|\lambda_\tau^2-\mu_{2,k}|} \\ &\leqslant C\|\chi_{2,k}-\chi_{1,k}\|_{L^2(\partial\mathrm{M})}\tau\frac{|\langle\chi_{2,k},(\partial_\nu+iA_1\nu)\varphi_{2,\tau}^*\rangle|}{|\lambda_\tau^2-\mu_{2,k}|}, \end{aligned}$$

$$\begin{aligned} \mathcal{F}_k(\tau) &= \|(\partial_\nu+iA_1\nu)\varphi_{2,\tau}^*\|_{L^2(\partial\mathrm{M})}\|\chi_{2,k}-\chi_{1,k}\|_{L^2(\partial\mathrm{M})}\frac{|\langle(\partial_\nu+iA_1\nu)\varphi_{1,\tau}^*,\chi_{1,k}\rangle|}{|\lambda_\tau^2-\mu_{2,k}|} \\ &\leqslant C\|\chi_{2,k}-\chi_{1,k}\|_{L^2(\partial\mathrm{M})}\tau\frac{|\langle(\partial_\nu+iA_1\nu)\varphi_{1,\tau}^*,\chi_{2,k}\rangle|}{|\lambda_\tau^2-\mu_{2,k}|} \\ &\qquad + C\tau^2\|\chi_{1,k}-\chi_{2,k}\|_{L^2(\partial\mathrm{M})}^2 \end{aligned}$$

and

$$\begin{aligned}&\mathcal{G}_k(\tau)\\&=\frac{|\langle(\partial_\nu+iA_1\nu)\varphi_{1,\tau}^*,\chi_{1,k}\rangle||\langle\chi_{1,k},(\partial_\nu+iA_1\nu)\varphi_{2,\tau}^*\rangle||\mu_{2,k}-\mu_{1,k}|}{|\lambda_\tau^2-\mu_{2,k}||\lambda_\tau^2-\mu_{1,k}|}\\&\leqslant Ck^{-\frac{1}{n}}|\mu_{2,k}-\mu_{1,k}|\|\chi_{1,k}-\chi_{2,k}\|_{L^2(\partial\mathrm{M})}k^{\frac{1}{n}}\frac{|\langle(\partial_\nu+iA_1\nu)\varphi_{1,\tau}^*,\chi_{1,k}\rangle|}{|\lambda_\tau^2-\mu_{1,k}|}\\&\quad+k^{-\frac{1}{n}}|\mu_{2,k}-\mu_{1,k}|k^{\frac{1}{n}}\frac{|\langle(\partial_\nu+iA_1\nu)\varphi_{1,\tau}^*,\chi_{1,k}\rangle|}{|\lambda_\tau^2-\mu_{1,k}|}\frac{|\langle\chi_{2,k},(\partial_\nu+iA_1\nu)\overline{\varphi_{2,\tau}^*}\rangle|}{|\lambda_\tau^2-\mu_{2,k}|},\end{aligned}$$

the constant $C>0$ being independent on τ and k.

Noting that

$$\sup_{\tau>1}\frac{\tau^2}{|\lambda_\tau^2-\mu_{\ell,k}|}<\infty,\quad \ell=1,2,\ k\geqslant 1,$$

we deduce that we have, for all $k\geqslant 1$,

$$\limsup_{\tau\to+\infty}\tau^{-1}\mathcal{E}_k(\tau)=\limsup_{\tau\to+\infty}\tau^{-1}\mathcal{F}_k(\tau)=\limsup_{\tau\to+\infty}\tau^{-1}\mathcal{G}_k(\tau)=0.$$

Then, for any arbitrary integer $n_1\geqslant 1$, we get

$$\begin{aligned}\limsup_{\tau\to+\infty}\tau^{-1}\sum_{k=1}^{\infty}\mathcal{E}_k(\tau)&\leqslant\limsup_{\tau\to+\infty}\tau^{-1}\sum_{k=n_1}^{\infty}\mathcal{E}_k(\tau),\\\limsup_{\tau\to+\infty}\tau^{-1}\sum_{k=1}^{\infty}\mathcal{F}_k(\tau)&\leqslant\limsup_{\tau\to+\infty}\tau^{-1}\sum_{k=n_1}^{\infty}\mathcal{F}_k(\tau),\\\limsup_{\tau\to+\infty}\tau^{-1}\sum_{k=1}^{\infty}\mathcal{G}_k(\tau)&\leqslant\limsup_{\tau\to+\infty}\tau^{-1}\sum_{k=n_1}^{\infty}\mathcal{G}_k(\tau).\end{aligned}$$

We combine these inequalities, estimates (6.11) and Weyl's formula in order to get, by repeating the arguments used to prove Lemma 5.2, that

$$\begin{aligned}&\limsup_{\tau\to+\infty}\left|\frac{Q_2(\tau)-Q_1(\tau)}{\tau}\right|\\&\qquad\leqslant C\big(1+\|\eta\|_{H^2(S_y^+(\mathrm{M}_1))}\big)^2\left(\limsup_{k\to+\infty}k^{-\frac{1}{n}}|\mu_{2,k}-\mu_{1,k}|\right).\end{aligned}$$

Then, from (6.4) and (6.10) we deduce that $\mathcal{I}_1A\in 2\pi\mathbb{Z}$. We proceed similarly to the proof of Theorem 1.2 to get that $A_1^s=A_2^s$. This completes the proof of Theorem 6.1.

Appendix A. Weyl's formula

We establish some uniform estimates related to the Weyl's formula of magnetic Schrödinger operators. Our estimates, which are also valid for the Neuman realization of magnetic Schrödinger operators, can be stated as follows.

LEMMA A.1. — *Let $B = (A,q) \in \mathscr{B}$. Then there exists a constant $C > 1$, only depending on* M *and* $r \geqslant \|A\|^2_{L^\infty(\mathrm{M},T^*\mathrm{M})} + \|q\|_{L^\infty(\mathrm{M})}$ *such that*

$$C^{-1}k^{2/n} \leqslant 1 + |\lambda_B^k| \leqslant Ck^{2/n}, \quad k \geqslant 1. \tag{A.1}$$

Proof. — Let (λ_k) be the sequence of eigenvalues, counted according to their multiplicities, of the Laplace–Beltrami operator under Dirichlet boundary condition. By Weyl's asymptotic formula [10, p. 114]

$$\lambda_k = \mathcal{O}\big(k^{\frac{2}{n}}\big), \quad k \geqslant 1. \tag{A.2}$$

The sesquilinear form associated to H_B is given by

$$\mathbf{a}(u,v) = \int_{\mathrm{M}} \langle \nabla_A u, \overline{\nabla_A v}\rangle \,\mathrm{dv}^n + \int_{\mathrm{M}} qu\bar{v}\,\mathrm{dv}^n, \quad u,v \in H_0^1(\mathrm{M}).$$

Then it is not hard to check that

$$\begin{aligned}\mathbf{a}(u,u) &\leqslant \|\nabla u\|^2_{L^2(\mathrm{M})} + 2\sqrt{r}\|u\|_{L^2(\mathrm{M})}\|\nabla u\|_{L^2(\mathrm{M})} + r\|u\|^2_{L^2(\mathrm{M})} \\ &\leqslant \frac{3}{2}\|\nabla u\|^2_{L^2(\mathrm{M})} + r\|u\|^2_{L^2(\mathrm{M})}\end{aligned}$$

and

$$\begin{aligned}\mathbf{a}(u,u) &\geqslant \|\nabla u\|^2_{L^2(\mathrm{M})} - 2\sqrt{r}\|u\|_{L^2(\mathrm{M})}\|\nabla u\|_{L^2(\mathrm{M})} - r\|u\|^2_{L^2(\mathrm{M})} \\ &\geqslant \frac{1}{2}\|\nabla u\|^2_{L^2(M)} - 3r\|u\|^2_{L^2(\mathrm{M})}.\end{aligned}$$

We get the expected two-sided inequalities (A.1) by using (A.2) and the minmax principle. □

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Manuscrit reçu le 3 août 2018,
révisé le 23 septembre 2019,
accepté le 13 novembre 2020.

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