Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

Efthymios Sofos & Erik Visse-Martindale

The density of fibres with a rational point for a fibration over hypersurfaces of low degree

Tome 71, nº 2 (2021), p. 679-709.

<http://aif.centre-mersenne.org/item/AIF_2021__71_2_679_0>

Ann. Inst. Fourier, Grenoble
71, 2 (2021) 679-709

THE DENSITY OF FIBRES WITH A RATIONAL POINT FOR A FIBRATION OVER HYPERSURFACES OF LOW DEGREE

by Efthymios SOFOS & Erik VISSE-MARTINDALE

Abstract. — We prove asymptotics for the proportion of fibres with a rational point in a conic bundle fibration. The base of the fibration is a general hypersurface of low degree.

Résumé. — Nous établissons une formule asymptotique concernant la proportion de fibres possédant un point rationnel dans le cas d'une fibration en coniques, la base de la fibration étant une hypersurface générique de bas degré.

1. Introduction

Serre's problem [15] regards the density of elements in a family of varieties defined over $\mathbb{Q}$ that have a $\mathbb{Q}$-rational point. Special cases have been considered by Hooley [5, 6] Poonen–Voloch [11], Sofos [17], Browning–Loughran [2], and Loughran–Takloo-Bighash–Tanimoto [9]. The recent investigation of Loughran [7] and Loughran–Smeets [8] provides an appropriate formulation of the problem and proves the conjectured upper bound in considerable generality.

Assume that X is a variety over $\mathbb{Q}$ equipped with a dominant morphism $\phi : X \to \mathbb{P}^n_{\mathbb{Q}}$. Letting H denote the usual Weil height on $\mathbb{P}^n(\mathbb{Q})$, Loughran and Smeets conjectured [8, Conj. 1.6] under suitable assumptions on ϕ, that for all large enough positive t, the cardinality of points $b \in \mathbb{P}^n(\mathbb{Q})$ with height $H(b) \leqslant t$ and such that the fibre $\phi^{-1}(b)$ has a point in $\mathbb{R}$ and $\mathbb{Q}_p$ for every prime p, has order of magnitude

$$\frac{\#\{b \in \mathbb{P}^n(\mathbb{Q}) : H(b) \leqslant t\}}{(\log t)^{\Delta(\phi)}}$$

Keywords: Hardy-Littlewood circle method, Serre's problem, fibres with a rational point.
2020 *Mathematics Subject Classification:* 14G05, 14D06, 11P55, 14D10.

for a non-negative quantity $\Delta(\phi)$ that is defined in [8, (1.3)].

The cardinality of fibres of height at most t and possessing a $\mathbb{Q}$-rational point is bounded by the quantity they considered, while the two quantities coincide if every fibre satisfies the Hasse principle. The problem of obtaining the conjectured lower bound for the number of fibres of bounded height with a $\mathbb{Q}$-rational point when ϕ is general is considered rather hard because there is no general machinery for producing $\mathbb{Q}$-rational points on varieties.

There are only two instances in the literature of the subject where asymptotics have been proved unconditionally:

- the base of the fibration is a toric variety (Loughran [7]),
- the base of the fibration is a wonderful compactification of an adjoint semi-simple algebraic group (Loughran–Takloo-Bighash–Tanimoto [9]).

Our aim in this article is to extend the list above by proving asymptotics in a case of a rather different nature. The base of the fibration of our main theorem will be a generic hypersurface of large dimension compared to its degree.

1.1. The set-up of our results

Let f_1 and f_2 be homogeneous forms in $\mathbb{Z}[t_0,\dots,t_{n-1}]$, of equal and even degree $d>0$ subject to some assumptions which are to follow.

We assume that both the projective varieties defined by $f_1(\mathbf{t})=0$ and $f_2(\mathbf{t})=0$ are smooth and irreducible. Moreover we assume that the variety defined by $f_1(\mathbf{t})=f_2(\mathbf{t})=0$ is a complete intersection. This is satisfied for generic f_1 and f_2 of fixed degree and in a fixed number of variables. The next condition is artificial in nature but its presence allows to adapt the arguments of Birch [1] to our problem. Letting $\sigma(f_1,f_2)$ denote the dimension of the variety given by

$$\mathrm{rk}\left(\frac{\partial f_i}{\partial t_j}\right)_{0\leqslant j\leqslant n-1}^{1\leqslant i\leqslant 2}(\mathbf{t})\leqslant 1$$

when considered as a subvariety in $\mathbb{A}^n_{\mathbb{C}}$, we shall demand the validity of

$$n-\sigma(f_1,f_2)>3(d-1)2^d. \tag{1.1}$$

With more work along the lines of the present article, most of these assumptions may be removed. However, the assumption that $\deg(f_1)$ is even seems necessary and (1.1) is vital for the entire strategy of the proof.

Remark 1.1. — We assume that the varieties defined by $f_i(\mathbf{t}) = 0$ are smooth, so they are also irreducible since smooth hypersurfaces in $\mathbb{P}_{\mathbb{Q}}^{n-1}$ are irreducible if $n \geqslant 3$ holds. In particular we have $n > 12$ by (1.1).

Let $B \subset \mathbb{P}_{\mathbb{Q}}^{n-1}$ be the hypersurface given by $f_2(\mathbf{t}) = 0$. We recall that by the work of Birch [1], B satisfies the Hasse principle, and moreover it satisfies weak approximation by work of Skinner [16]. From now on we also assume $B(\mathbb{Q}) \neq \varnothing$.

For every $i \in \{0, \dots, n-1\}$ consider the subvariety X_i of $\mathbb{P}_{\mathbb{Q}}^2 \times \mathbb{A}_{\mathbb{Q}}^{n-1}$ defined by

$$x_0^2 + x_1^2 = f_1(t_0, \dots, t_{i-1}, 1, t_{i+1}, \dots, t_{n-1}) x_2^2,$$
$$f_2(t_0, \dots, t_{i-1}, 1, t_{i+1}, \dots, t_{n-1}) = 0.$$

The maps $g_i : X_i \to B \subset \mathbb{P}_{\mathbb{Q}}^{n-1}$ sending a pair

$$((x_0 : x_1 : x_2), (t_0, \dots, t_{i-1}, 1, t_{i+1}, \dots, t_{n-1}))$$

to $(t_0 : \dots : t_{i-1} : 1 : t_{i+1} : \dots : t_{n-1})$ glue together, defining a conic bundle X over the base B – this uses that f_1 has even degree. By assumption, f_1 is not a multiple of f_2, so the generic fibre of X is smooth.

If we were interested in counting $\mathbb{Q}$-rational points on X then it would be necessary to make a further study into the equations defining a projective embedding of X (as in [3, §2]). Currently however, we are only interested in counting how many fibres of the conic bundle have a $\mathbb{Q}$-rational point. A *conic bundle* is a dominant morphism all of whose fibres are conics and whose generic fibre is smooth. In this article we consider the conic bundle

$$\phi : X \to B \tag{1.2}$$

defined locally by g_i. We shall estimate asymptotically the probability with which the fibre $\phi^{-1}(b)$ has a $\mathbb{Q}$-point as b ranges over $B(\mathbb{Q})$. For this, we define

$$N(\phi, t) := \#\big\{b \in B(\mathbb{Q}) : H(b) \leqslant t,\ b \in \phi(X(\mathbb{Q}))\big\},\ t \in \mathbb{R}_{>0},$$

where H is the usual naive Weil height on $\mathbb{P}^{n-1}(\mathbb{Q})$.

Remark 1.2. — Since the degree of f_1 is even, the question if for a given $b \in B$ the fibre $\phi^{-1}(b)$ contains a rational point is independent of a chosen representative.

Consider the small quantity

$$\varepsilon_d := \frac{1}{5(d-1)2^{d+5}}. \tag{1.3}$$

Theorem 1.3. — *In the set-up above there exists a constant c_ϕ such that for $t \geqslant 2$ we have*

$$N(\phi, t) = c_\phi \frac{t^{n-d}}{(\log t)^{\frac{1}{2}}} + O\left(\frac{t^{n-d}}{(\log t)^{\frac{1}{2}+\varepsilon_d}} \right).$$

If ϕ has a smooth fibre with a $\mathbb{Q}$-point then c_ϕ is positive. This will be shown in Theorem 5.4, where we shall also provide an interpretation for the leading constant c_ϕ. The proof of Theorem 1.3 will be given in Section 4.3. The main idea is to feed sieve estimates coming from the Rosser–Iwaniec half-dimensional sieve into the major arcs of the Birch circle method.

Theorem 1.3 settles the first case in the literature of an asymptotic for the natural extension of Serre's problem to fibrations over a base that does not have the structure of a toric variety nor a wonderful compactification of an adjoint semi-simple algebraic group. Fibrations that have a base other than the projective space were also studied in the recent work of Browning and Loughran [2, §1.2.2]. In light of the work of Birch [1], our assumptions imply

$$\#\left\{ b \in B(\mathbb{Q}) : H(b) \leqslant t \right\} \asymp t^{n-d}.$$

A very special case of [2, Thm. 1.4] proves $\lim_{t\to\infty} N(\phi, t)/t^{n-d} = 0$, whereas Theorem 1.3 provides asymptotics.

1.2. The logarithmic exponent

The power of $\log t$ occurring in our result is the one expected in the literature. Indeed, in the works of Loughran and Smeets [8, (1.4)], and Browning and Loughran [2, (1.3)], one may find the expected power $\Delta(\phi)$ defined as follows. For any $b \in B$ with residue field $\kappa(b)$, the fibre $X_b = \phi^{-1}(b)$ is called *pseudo-split* if every element of $\mathrm{Gal}(\overline{\kappa(b)}/\kappa(b))$ fixes some multiplicity-one irreducible component of $X_b \times \mathrm{Spec}(\overline{\kappa(b)})$. The fibre X_b is called *split* if it contains a multiplicity-one irreducible component that is also geometrically irreducible. Note that a split fibre is always pseudo-split and further note that for conic bundles these two notions are the same as the singular fibres are either double lines, or two lines intersecting.

Now for every codimension one point $D \in B^{(1)}$ choose a finite group Γ_D through which the action of $\mathrm{Gal}(\overline{\kappa(D)}/\kappa(D))$ on the irreducible components of $X_{\overline{\kappa(D)}}$ factors. Let Γ_D° be the subset of elements of Γ_D which fix some multiplicity one irreducible component. One sets $\delta_D = \#\Gamma_D^\circ / \#\Gamma_D$ and

$$\Delta(\phi) = \sum_{D \in B^{(1)}} \left(1 - \delta_D \right).$$

By considering the possible singular fibres, it is clear that for a conic bundle, δ_D is different from 1 if and only if D is non-split.

In all the cases in the literature so far the power of $(\log t)^{-1}$ turns out to be Δ. Indeed, this is also the case here. The only relevant codimension one point to take into account is $D := Z(f_1, f_2)$; every other fibre is smooth and hence split. Suppose that D is geometrically reducible, then the intersection between any two geometrically irreducible components lies in the singular locus of D, say D^{sing}. Being the intersection between varieties in projective space of codimension at most 2, its codimension is at most 4.

The affine cone above D^{sing} is a subvariety of the affine variety defined by

$$\operatorname{rk}\left(\frac{\partial f_i}{\partial t_j}\right)_{0\leqslant j\leqslant n-1}^{1\leqslant i\leqslant 2}(\mathbf{t}) \leqslant 1.$$

As a subvariety, the affine cone over D^{sing} is at most $\sigma(f_1, f_2)$, so its codimension is at least $n-\sigma(f_1, f_2)$. Hence the codimension of D^{sing} in $\mathbb{P}^n_{\mathbb{Q}}$ is at least $n - \sigma(f_1, f_2) - 1$. Hence we are led to an inequality

$$4 \geqslant n - \sigma(f_1, f_2) - 1 > 3(d-1)2^d - 1 \geqslant 11,$$

violating the combined assumptions (1.1) and $d \geqslant 2$. We conclude that D is geometrically irreducible.

The fibre above D is given by $x_0^2+x_1^2 = 0$ over the function field $\kappa(D)$ and it is split if and only if -1 is a square in $\kappa(D)$. However, it is well known that the function field of a geometrically irreducible variety contains no non-trivial separable algebraic extensions of the base field. Since -1 is not a square in $\mathbb{Q}$, neither is it in $\kappa(D)$. Therefore, under the assumptions of Theorem 1.3 we conclude that $\Delta(\phi) = \delta_D = \frac{1}{2}$.

Alternatively, it was kindly remarked by the referee that one can prove that D is geometrically integral by applying the Lefschetz hyperplane section theorem to the hypersurface $f_1(\mathbf{t}) = 0$. Its divisor D can only be reducible if the variety defined by $f_2(\mathbf{t}) = 0$ is also reducible, which contradicts our assumptions on f_2.

Notation. — The symbol $\mathbb{N}$ will denote the set of strictly positive integers. As usual, we denote the divisor, Euler and Möbius function by τ, φ and μ. We shall make frequent use of the estimates

$$\tau(m) \ll m^{\frac{1}{\log\log m}} \tag{1.4}$$

and

$$\varphi(m) \gg m/\log\log m \tag{1.5}$$

valid for all integers $m \geqslant 3$ and found in [18, Thm. 5.4] and [18, Thm. 5.6] respectively.

We consider the forms f_1 and f_2 constant throughout our paper, thus the implied constants in the Vinogradov/Landau notation $\ll, O(\,\cdot\,)$ are allowed to depend on ϕ, f_1, f_2, n and d without further mention. Any dependence of the implied constants on other parameters will be explicitly recorded by the appropriate use of a subscript. For $z \in \mathbb{C}$ we let

$$\mathrm{e}(z) := \exp(2\pi i z).$$

The symbol $v_p(m)$ will refer to the standard p-adic valuation of an integer m. Lastly, we shall use the Ramanujan sum, defined for $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ as

$$c_q(a) := \sum_{x \in (\mathbb{Z}/q\mathbb{Z})^*} \mathrm{e}(ax/q). \tag{1.6}$$

Denoting the indicator function of a set A by $\mathbb{1}_A$, we have the following equality,

$$c_{p^m}(a) = p^{m-1}\big(p\mathbb{1}_{\{v_p(a) \geqslant m\}} - \mathbb{1}_{\{v_p(a) \geqslant m-1\}}\big), \qquad (p \text{ prime}, a \in \mathbb{Z}, m \geqslant 1). \tag{1.7}$$

Lastly, we shall make frequent use of the constant

$$\mathscr{C}_0 := \prod_{\substack{p \text{ prime} \\ p \equiv 3 (\mathrm{mod}\ 4)}} \Big(1 - \frac{1}{p^2}\Big)^{1/2}. \tag{1.8}$$

Acknowledgements

This work started while Efthymios Sofos had a position at Leiden University. It was completed while Erik Visse-Martindale was visiting the Max Planck Institute in Bonn, the hospitality of which is greatly acknowledged. The authors are very grateful to Daniel Loughran for useful comments that helped improve the introduction and the end of Section 5.

2. Using the Hardy–Littlewood circle method for Serre's problem

We begin by estimating the main quantity in Theorem 1.3 by averages of an arithmetic function over a thin subset of integer vectors. Let us first define $\vartheta_{\mathbb{Q}} : \mathbb{Z} \to \{0,1\}$ as the indicator function of those integers m such that the curve $x_0^2 + x_1^2 = m x_2^2$ has a point over $\mathbb{Q}$. For $P \in \mathbb{R}_{>0}$ we let

$$\Theta_{\mathbb{Q}}(P) := \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \cap P[-1,1]^n \\ f_1(\mathbf{x}) \neq 0, f_2(\mathbf{x}) = 0}} \vartheta_{\mathbb{Q}}(f_1(\mathbf{x})). \tag{2.1}$$

In order to go from $\mathbb{Q}$-solutions to coprime $\mathbb{Z}$-solutions, we perform a standard Möbius transformation, where we cut off the range of summation at the price of an error term. This is the content of the following lemma.

Lemma 2.1. — *Under the assumptions of Theorem 1.3 we have for* $t \geqslant 1$,

$$N(B, \phi, t) = \frac{1}{2} \sum_{l \in \mathbb{N} \cap [1, \log(2t)]} \mu(l) \Theta_{\mathbb{Q}}(t/l) + O(t^{n-d} (\log 2t)^{-1}).$$

Proof. — For any $b \in \mathbb{P}^{n-1}(\mathbb{Q})$ there exists a unique, up to sign, $\mathbf{y} \in \mathbb{Z}^n$ with $\gcd(y_0, \ldots, y_{n-1}) = 1$ and $b = [\pm \mathbf{y}]$. Recalling that the degree of f_1 is even, allows to infer that the fibre $\phi^{-1}(b)$ has a rational point if and only if $\vartheta_{\mathbb{Q}}(f_1(\mathbf{y})) = 1$, hence

$$N(B, \phi, t) = \frac{1}{2} \# \left\{ \mathbf{y} \in \mathbb{Z}^n \cap t[-1,1]^n : \begin{array}{c} \gcd(y_0, \ldots, y_{n-1}) = 1, \\ f_2(\mathbf{y}) = 0, \vartheta_{\mathbb{Q}}(f_1(\mathbf{y})) = 1 \end{array} \right\}.$$

If $f_1(\mathbf{y}) = 0$ then $\vartheta_{\mathbb{Q}}(f_1(\mathbf{y})) = 1$ (since $(0:0:1)$ is a point in $\phi^{-1}([\mathbf{y}])$) and, therefore, the quantity above is

$$\frac{1}{2} \sum_{\substack{\mathbf{y} \in \mathbb{Z}^n \cap t[-1,1]^n \\ \gcd(y_0, \ldots, y_{n-1}) = 1 \\ f_2(\mathbf{y}) = 0, f_1(\mathbf{y}) \neq 0}} \vartheta_{\mathbb{Q}}(f_1(\mathbf{y})) + O(\#\{\mathbf{y} \in \mathbb{Z}^n \cap [-t,t]^n : f_1(\mathbf{y}) = f_2(\mathbf{y}) = 0\}).$$

The assumption (1.1) allows to apply [1, Thm. 1, p. 260] with $R = 2$ to immediately obtain

$$\#\{\mathbf{y} \in \mathbb{Z}^n \cap t[-1,1]^n : f_1(\mathbf{y}) = f_2(\mathbf{y}) = 0\} \ll t^{n-2d}, \quad (t \geqslant 1).$$

Thus we obtain equality with

$$\frac{1}{2} \sum_{\substack{\mathbf{y} \in \mathbb{Z}^n \cap t[-1,1]^n \\ \gcd(y_0, \ldots, y_{n-1}) = 1 \\ f_1(\mathbf{y}) \neq 0, f_2(\mathbf{y}) = 0}} \vartheta_{\mathbb{Q}}(f_1(\mathbf{y})) + O(t^{n-2d}).$$

Using Möbius inversion and letting $\mathbf{y} = l\mathbf{x}$ we see that the sum over $\mathbf{y}$ equals

$$\sum_{\substack{\mathbf{y} \in \mathbb{Z}^n \cap t[-1,1]^n \\ f_1(\mathbf{y}) \neq 0, f_2(\mathbf{y}) = 0}} \vartheta_{\mathbb{Q}}(f_1(\mathbf{y})) \sum_{\substack{l \in \mathbb{N} \\ l \mid \mathbf{y}}} \mu(l) = \sum_{l \leqslant t} \mu(l) \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \cap \frac{t}{l}[-1,1]^n \\ f_1(\mathbf{x}) \neq 0, f_2(\mathbf{x}) = 0}} \vartheta_{\mathbb{Q}}(f_1(\mathbf{x})),$$

because $\vartheta_{\mathbb{Q}}(f_1(\mathbf{y})) = \vartheta_{\mathbb{Q}}(f_1(\mathbf{x}))$ holds due to $\deg(f_1)$ being even. Hence

$$N(B, \phi, t) = \frac{1}{2} \sum_{l \in \mathbb{N} \cap [1,t]} \mu(l) \Theta_{\mathbb{Q}}(t/l) + O(t^{n-2d}),$$

and now, using that both f_1 and f_2 are smooth, (1.1) and [1, Thm. 1, p. 260] for $R = 1$ yields

$$|\Theta_{\mathbb{Q}}(t)| \leqslant \#\{\mathbf{y} \in \mathbb{Z}^n \cap t[-1,1]^n : f_2(\mathbf{y}) = 0\} \ll t^{n-d},$$

which shows that the collective contribution from large l is

$$\left| \sum_{l \in \mathbb{N} \cap ((\log 2t), t]} \mu(l) \Theta_{\mathbb{Q}}(t/l) \right| \ll \sum_{l > \log(2t)} (t/l)^{n-d} \ll t^{n-d} \sum_{l > \log(2t)} l^{-2} \ll t^{n-d} (\log 2t)^{-1},$$

where we used that $n - d \geqslant 2$ holds due to (1.1). □

For $m < 0$ the curve $x_0^2 + x_1^2 = m x_2^2$ has no $\mathbb{R}$-point, and therefore no $\mathbb{Q}$-point, hence $\vartheta_{\mathbb{Q}}(m) = 0$. Thus, denoting $\max\{f_1([-1,1]^n)\} := \max\{f_1(\mathbf{t}) : \mathbf{t} \in [-1,1]^n\}$, it is evident that we have the equality

$$\Theta_{\mathbb{Q}}(P) = \sum_{\substack{m \in \mathbb{N} \\ m \leqslant \max\{f_1([-1,1]^n)\} P^d}} \vartheta_{\mathbb{Q}}(m) \sum_{\substack{\mathbf{x} \in \mathbb{Z}^n \cap P[-1,1]^n \\ f_1(\mathbf{x}) = m, f_2(\mathbf{x}) = 0}} 1.$$

Writing $\mathrm{d}\boldsymbol{\alpha}$ for $\mathrm{d}\alpha_1 \mathrm{d}\alpha_2$ and using the identity

$$\int_{\boldsymbol{\alpha} \in [0,1)^2} \mathrm{e}(\alpha_1(f_1(\mathbf{x}) - m) + \alpha_2 f_2(\mathbf{x})) \mathrm{d}\boldsymbol{\alpha} = \begin{cases} 1, & \text{if } f_1(\mathbf{x}) = m \text{ and } f_2(\mathbf{x}) = 0, \\ 0, & \text{otherwise,} \end{cases}$$

shows the validity of

$$\Theta_{\mathbb{Q}}(P) = \int_{\boldsymbol{\alpha} \in [0,1)^2} S(\boldsymbol{\alpha}) \overline{E_{\mathbb{Q}}(\alpha_1)} \mathrm{d}\boldsymbol{\alpha}, \tag{2.2}$$

where one uses the notation

$$S(\boldsymbol{\alpha}) := \sum_{\mathbf{x} \in \mathbb{Z}^n \cap P[-1,1]^n} \mathrm{e}(\alpha_1 f_1(\mathbf{x}) + \alpha_2 f_2(\mathbf{x})) \tag{2.3}$$

and

$$E_{\mathbb{Q}}(\alpha_1) := \sum_{\substack{m \in \mathbb{N} \\ m \leqslant \max\{f_1([-1,1]^n)\} P^d}} \vartheta_{\mathbb{Q}}(m) \mathrm{e}(\alpha_1 m). \tag{2.4}$$

One has the obvious bound $E_{\mathbb{Q}}(\alpha_1) \ll P^d$ from the triangle inequality. Recall the notation [1, p. 251, (4)–(7)], that we repeat here for the convenience of the reader. For each a_1, a_2, q, the interval $\mathscr{M}_{(a_1,a_2),q}(\theta)$ consists of those $\alpha \in [0,1]^2$ satisfying

$$2|q\alpha_i - a_i| \leqslant P^{-d+2(d-1)\theta}$$

for all $i = 1, 2$. For each $0 < \theta \leqslant 1$ denote the set of "major arcs" by

$$\mathscr{M}(\theta) = \bigcup_{1 \leqslant q \leqslant P^{2(d-1)\theta}} \bigcup_{a} \mathscr{M}_{(a_1,a_2),q}(\theta)$$

where the second union is over those a_1, a_2 satisfying both $\gcd(a_1, a_2, q) = 1$ and $0 \leqslant a_i < q$ for all $i = 1, 2$.

Let us now deal with the complement of $\mathscr{M}(\theta)$ that is usually referred to as the "minor arcs". In our case the number of equations, denoted by R in [1], satisfies $R = 2$. For small positive θ_0 and δ as in [1, p. 251, (10)–(11)], that is $1 > \delta + 16\theta_0$ and $\frac{n-\sigma}{2^{d-1}} - 6(d-1) > 2\delta\theta_0^{-1}$ we have

$$\int_{\boldsymbol{\alpha} \notin \mathscr{M}(\theta_0)} \left|S(\boldsymbol{\alpha})\overline{E_{\mathbb{Q}}(\alpha_1)}\right| \mathrm{d}\boldsymbol{\alpha} \leqslant \left(\int_{\boldsymbol{\alpha} \notin \mathscr{M}(\theta_0)} |S(\boldsymbol{\alpha})| \mathrm{d}\boldsymbol{\alpha}\right) \max_{\alpha_1 \in [0,1)} |E_{\mathbb{Q}}(\alpha_1)|,$$

hence, applying the result of [1, Lem. 4.4] on the first factor, and using the trivial bound $E_{\mathbb{Q}}(\alpha_1) \ll P^d$ leads to the following bound on the integral away from $\mathscr{M}(\theta_0)$:

$$\int_{\boldsymbol{\alpha} \notin \mathscr{M}(\theta_0)} \left|S(\boldsymbol{\alpha})\overline{E_{\mathbb{Q}}(\alpha_1)}\right| \mathrm{d}\boldsymbol{\alpha} \ll P^{n-d-\delta}.$$

By (2.2) this shows

$$\Theta_{\mathbb{Q}}(P) = \int_{\boldsymbol{\alpha} \in \mathscr{M}(\theta_0)} S(\boldsymbol{\alpha})\overline{E_{\mathbb{Q}}(\alpha_1)} \mathrm{d}\boldsymbol{\alpha} + O(P^{n-d-\delta}).$$

Consistently modifying the setup, the following lemma is analogous to [1, Lem. 4.5] and its proof is the same, using the notation introduced above.

Lemma 2.2. — *For any θ_0, δ satisfying* [1, p. 251, (10)–(11)] *and under the assumptions of Theorem 1.3 we have*

$$\Theta_{\mathbb{Q}}(P) = \sum_{q \leqslant P^{2(d-1)\theta_0}} \sum_{\substack{\mathbf{a} \in (\mathbb{Z} \cap [0,q))^2 \\ \gcd(a_1,a_2,q)=1}} \int_{\mathscr{M}'_{\mathbf{a},q}(\theta_0)} S(\boldsymbol{\alpha})\overline{E_{\mathbb{Q}}(\alpha_1)} \mathrm{d}\boldsymbol{\alpha} + O(P^{n-d-\delta}),$$

where the modified set $\mathscr{M}'_{\mathbf{a},q}(\theta_0)$ is defined in [1, p. 253] *and consists of those $\alpha \in [0,1]^2$ satisfying $|q\alpha_i - a_i| \leqslant qP^{-d+2(d-1)\theta_0}$.*

For $\mathbf{a} \in (\mathbb{Z} \cap [0,q))^2$, write

$$S_{\mathbf{a},q} := \sum_{\mathbf{x} \in (\mathbb{Z} \cap [0,q))^n} \mathrm{e}\left(\frac{a_1 f_1(\mathbf{x}) + a_2 f_2(\mathbf{x})}{q}\right) \tag{2.5}$$

and for $\mathbf{\Gamma} \in \mathbb{R}^2$ define

$$I(\mathbf{\Gamma}) := \int_{\boldsymbol{\zeta}\in[-1,1]^n} \mathrm{e}(\Gamma_1 f_1(\boldsymbol{\zeta}) + \Gamma_2 f_2(\boldsymbol{\zeta}))\mathrm{d}\boldsymbol{\zeta}. \tag{2.6}$$

Recalling the notation $\eta = 2(d-1)\theta_0$ of [1, p. 254, (2)], we now employ [1, Lem. 5.1] with $\boldsymbol{\nu} = \mathbf{0}$ to evaluate $S(\alpha)$ and to see that under the assumptions of Lemma 2.2 we have

$$\begin{aligned}\Theta_{\mathbb{Q}}(P) - P^n &\sum_{q\leqslant P^{2(d-1)\theta_0}} q^{-n} \\ &\times \sum_{\substack{\mathbf{a}\in(\mathbb{Z}\cap[0,q))^2 \\ \gcd(a_1,a_2,q)=1}} S_{\mathbf{a},q} \int_{|\boldsymbol{\beta}|\leqslant P^{-d+\eta}} I(P^d\boldsymbol{\beta})\overline{E_{\mathbb{Q}}(\beta_1 + a_1/q)}\mathrm{d}\boldsymbol{\beta} \\ \ll P^{n-d-\delta} + P^{n-1+2\eta} &\sum_{q\leqslant P^\eta} \sum_{\substack{\mathbf{a}\in(\mathbb{Z}\cap[0,q))^2 \\ \gcd(a_1,a_2,q)=1}} \int_{|\boldsymbol{\beta}|\leqslant P^{-d+\eta}} |E_{\mathbb{Q}}(\beta_1 + a_1/q)|\mathrm{d}\boldsymbol{\beta}.\end{aligned}$$

By using $E_{\mathbb{Q}}(\alpha) \ll P^d$ once more we infer that the sum over q in the error term above is

$$\ll \sum_{q\leqslant P^\eta} q^2 P^{2(-d+\eta)} P^d \ll P^{-d+5\eta},$$

hence we have proved the following lemma.

Lemma 2.3. — *Under the assumptions of Lemma 2.2 the quantity* $\Theta_{\mathbb{Q}}(P)P^{-n+d}$ *equals*

$$\sum_{q\leqslant P^\eta} q^{-n} \sum_{\substack{\mathbf{a}\in(\mathbb{Z}\cap[0,q))^2 \\ \gcd(a_1,a_2,q)=1}} S_{\mathbf{a},q} \int_{|\boldsymbol{\beta}|\leqslant P^{-d+\eta}} P^d I(P^d\boldsymbol{\beta})\overline{E_{\mathbb{Q}}(\beta_1 + a_1/q)}\mathrm{d}\boldsymbol{\beta} \\ + O(P^{-\delta} + P^{-1+7\eta}).$$

3. Exponential sums with terms detecting the existence of rational points

As made clear by Lemma 2.3, to verify Theorem 1.3 we need to asymptotically estimate

$$E_{\mathbb{Q}}\Big(\frac{a_1}{q} + \beta_1\Big) = \sum_{\substack{m\in\mathbb{N}\cap[1,T] \\ x_0^2+x_1^2=mx_2^2 \text{ has a } \mathbb{Q}\text{-point}}} \mathrm{e}^{2\pi i(\frac{a_1}{q}+\beta_1)m},$$

for integers a_1, q and $\beta_1 \in \mathbb{R}$ and $T = \max\{f_1([-1,1]^n)\}P^d$. It suffices to first study the case $\beta_1 = 0$, and then apply Lemma 3.6 at the end of this section. To study $E_{\mathbb{Q}}(a_1/q)$ we shall rephrase the condition on m in a way that it only regards the prime factorisation of m and then use the Rosser–Iwaniec sieve.

We begin by alluding to the formulas regarding Hilbert symbols in [14, Ch. III, Thm. 1], which show that for strictly positive integers m one has

$$\vartheta_{\mathbb{Q}}(m)=\begin{cases}1, & \text{if } p\equiv 3 \pmod 4 \Rightarrow v_p(m)\equiv 0 \pmod 2,\\ 0, & \text{otherwise.}\end{cases} \tag{3.1}$$

Indeed, for $m\in\mathbb{Z}_{>0}$, the curve $x_0^2+x_1^2=mx_2^2$ defines a smooth conic in $\mathbb{P}^2$ with an $\mathbb{R}$-point and the Hasse principle combined with Hilbert's product formula [14, Ch. III, Thm. 3] proves (3.1). The function in (3.1) is the characteristic function of those integers m that are sums of two integral squares, see [18, §4.8]. Landau [18, (4.90)] proved the following asymptotic:

$$\sum_{1\leqslant m\leqslant x}\vartheta_{\mathbb{Q}}(m)=\frac{1}{2^{1/2}\mathscr{C}_0}\frac{x}{(\log x)^{1/2}}+O\left(\frac{x}{(\log x)^{3/2}}\right), x\in\mathbb{R}_{>1}, \tag{3.2}$$

but this is not sufficient for us, since we will need a similar result restricted to those m in an arithmetic progression. Observe that the following holds due to periodicity,

$$E_{\mathbb{Q}}\left(\frac{a_1}{q}\right)=\sum_{\substack{m\in\mathbb{Z}\cap[1,T]\\ x_0^2+x_1^2=mx_2^2 \text{ has a } \mathbb{Q}\text{-point}}}\mathrm{e}^{2\pi i\frac{a_1}{q}m}=\sum_{\ell\in\mathbb{Z}\cap[0,q)}\mathrm{e}(a_1\ell/q)\sum_{\substack{1\leqslant m\leqslant T\\ m\equiv\ell \pmod q}}\vartheta_{\mathbb{Q}}(m).$$

The work of Rieger [12, Satz 1] could now be invoked to study the sum over $m\equiv\ell \pmod q$ when $\gcd(\ell,q)=1$. One could attempt to use this to get asymptotic formulas for the cases with $\gcd(\ell,q)>1$, however, we found it more straightforward to work instead with the function ϖ in place of $\vartheta_{\mathbb{Q}}$. This function $\varpi:\mathbb{Z}_{>0}\to\{0,1\}$ is defined as

$$\varpi(m):=\begin{cases}1, & \text{if } p\mid m\Rightarrow p\equiv 1 \pmod 4,\\ 0, & \text{otherwise.}\end{cases} \tag{3.3}$$

It is obvious that for all $m,k\in\mathbb{Z}_{>0}$ we have

$$\varpi(mk)=\varpi(m)\varpi(k), \tag{3.4}$$

while a similar property does not hold for $\vartheta_{\mathbb{Q}}$ (to see this take $m=k=p$, where p is any prime which is $3 \pmod 4$). This is the reason for choosing to work with ϖ rather than $\vartheta_{\mathbb{Q}}$. Our next lemma shows how one can replace $\vartheta_{\mathbb{Q}}$ by ϖ, while simultaneously restricting the summation at the price of an error term.

Lemma 3.1. — *For $x, u \in \mathbb{R}_{\geqslant 1}, q \in \mathbb{Z}_{>0}, a_1 \in \mathbb{Z} \cap [0, q)$ we have*

$$\sum_{1 \leqslant m \leqslant x} \vartheta_{\mathbb{Q}}(m) \mathrm{e}(a_1 m/q) = \sum_{\substack{(k,t) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geqslant 0} \\ 2^t k^2 \leqslant u \\ p \mid k \Rightarrow p \equiv 3 (\mathrm{mod}\ 4)}} \sum_{\ell \in \mathbb{Z} \cap [0,q)} \mathrm{e}(a_1 \ell/q) \sum_{\substack{r \in \mathbb{Z}_{>0} \\ 2^t k^2 r \equiv \ell (\mathrm{mod}\ q) \\ 1 \leqslant r \leqslant x 2^{-t} k^{-2}}} \varpi(r) + O\left(\frac{x}{\sqrt{u}}\right),$$

with an absolute implied constant.

Proof. — It is easy to see that for positive m one has $\vartheta_{\mathbb{Q}}(m) = 1$ if and only if $m = 2^t k^2 r$ for $t \in \mathbb{Z}_{\geqslant 0}$, k a positive integer all of whose primes are $3 \,(\mathrm{mod}\ 4)$ and r is such that $\varpi(r) = 1$. This shows that the sum over m is

$$\sum_{\substack{(k,t) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\geqslant 0} \\ p \mid k \Rightarrow p \equiv 3 (\mathrm{mod}\ 4)}} \sum_{\substack{r \in \mathbb{Z}_{>0} \\ r \leqslant x 2^{-t} k^{-2}}} \varpi(r) \mathrm{e}(a_1 2^t k^2 r/q).$$

The contribution of the pairs (k, t) with $2^t k^2 > u$ is at most

$$\sum_{t \geqslant 0} \sum_{k > \sqrt{u 2^{-t}}} x 2^{-t} k^{-2} \ll x \sum_{t \geqslant 0} \frac{2^{-t}}{\sqrt{u 2^{-t}}} \ll \frac{x}{\sqrt{u}}.$$

Noting that $\mathrm{e}(a_1 2^t k^2 r/q)$ as a function of r is periodic modulo q allows to partition all r in congruences $\ell \in \mathbb{Z}/q\mathbb{Z}$, thus concluding the proof. □

The terms in the sum involving ϖ in Lemma 3.1 are in an arithmetic progression that is not necessarily primitive. We next show that we can reduce the evaluation of these sums to similar expressions where the summation is over an arithmetic progression that is primitive. The property (3.4) will be necessary for this.

Lemma 3.2. — *Let $t \in \mathbb{Z}_{\geqslant 0}$, $q \in \mathbb{Z}_{>0}$, $\ell \in \mathbb{Z} \cap [0, q)$ and $k \in \mathbb{Z}_{>0}$ such that every prime divisor of k is $3 \,(\mathrm{mod}\ 4)$. For $y \in \mathbb{R}_{>0}$ consider the sum*

$$\sum_{\substack{r \in \mathbb{Z}_{>0} \cap [1,y] \\ 2^t k^2 r \equiv \ell (\mathrm{mod}\ q)}} \varpi(r).$$

The sum vanishes if $\gcd(2^t k^2, q) \nmid \ell$, and it otherwise equals

$$\varpi\left(\frac{\gcd(\ell, q)}{\gcd(2^t k^2, q)}\right) \sum_{\substack{s \in \mathbb{Z}_{>0} \cap [1, y \gcd(2^t k^2, q) \gcd(\ell, q)^{-1}] \\ \frac{2^t k^2}{\gcd(2^t k^2, q)} s \equiv \frac{\ell}{\gcd(\ell, q)} \left(\mathrm{mod}\ \frac{q}{\gcd(\ell, q)}\right)}} \varpi(s).$$

Proof. — If $\gcd(2^t k^2, q) \nmid \ell$ then the congruence $2^t k^2 r \equiv \ell \pmod q$ does not have a solution r, in which case the sum over r vanishes. If it holds then we see that the congruence for r can be written equivalently as

$$\frac{2^t k^2}{\gcd(2^t k^2, q)} r \equiv \frac{\ell}{\gcd(2^t k^2, q)} \left(\bmod \frac{q}{\gcd(2^t k^2, q)}\right).$$

Note that any solution r of this must necessarily satisfy

$$\gcd\left(\frac{\ell}{\gcd(2^t k^2, q)}, \frac{q}{\gcd(2^t k^2, q)}\right) \Big| \frac{2^t k^2}{\gcd(2^t k^2, q)} r$$

and the fact of

$$\gcd\left(\frac{\gcd(\ell, q)}{\gcd(2^t k^2, q)}, \frac{2^t k^2}{\gcd(2^t k^2, q)}\right) = 1$$

shows that r must be divisible by $\gcd(\ell, q) \gcd(2^t k^2, q)^{-1}$. Therefore there exists an $s \in \mathbb{Z}_{>0}$ with

$$r = \frac{\gcd(\ell, q)}{\gcd(2^t k^2, q)} s$$

and substituting this into the sum over r in our lemma concludes the proof because

$$\varpi(r) = \varpi\left(\frac{\gcd(\ell, q)}{\gcd(2^t k^2, q)}\right) \varpi(s)$$

holds due to the complete multiplicativity seen in (3.4). □

We are now in a position to apply [4, Thm. 14.7], which is a result on the distribution of the function ϖ along primitive arithmetic progressions and which we include as a proposition for the convenience of the reader. We first introduce the following notation for $Q \in \mathbb{Z}_{>0}$,

$$\dot{Q} := \prod_{p \equiv 1 \pmod 4} p^{v_p(Q)} \quad \text{and} \quad \ddot{Q} := \prod_{p \equiv 3 \pmod 4} p^{v_p(Q)}. \tag{3.5}$$

Proposition 3.3 ([4, Thm. 14.7]). — *Assume that Q is a positive integer that is a multiple of 4, that a is an integer satisfying $\gcd(a, Q) = 1, a \equiv 1 \pmod 4$ and let z be any real number with $z \geqslant Q$. Then*

$$\sum_{\substack{r \in \mathbb{Z}_{>0} \cap [1, z] \\ r \equiv a \pmod Q}} \varpi(r) = 2^{1/2} \mathscr{C}_0 \frac{\ddot{Q}}{\varphi(\ddot{Q})} \frac{z}{Q\sqrt{\log z}} \left\{1 + O\left(\left(\frac{\log Q}{\log z}\right)^{1/7}\right)\right\},$$

where the implied constant is absolute.

Remark 3.4. — This result was proved using the semi-linear Rosser–Iwaniec sieve. We should remark that there is a typo in the reference, namely [4, (14.22)] should instead read

$$V(D)=\prod_{2<p<D}\left(1-\frac{1}{p}\right)^{\frac12}\prod_{p<D}\left(1-\frac{\chi(p)}{p}\right)^{-\frac12}\prod_{\substack{2<p<D\\ p\equiv 3(\bmod\ 4)}}\left(1-\frac{1}{p^2}\right)^{\frac12},$$

and as a result, [4, (14.39)] must be replaced by the asymptotic in Proposition 3.3. After fixing this typo, one can show, as in the proof of [4, (14.24)], that for $D\geqslant 2$, we have

$$\prod_{\substack{p<D\\ p\equiv 3(\bmod\ 4)}}\left(1-\frac{1}{p}\right)=\frac{\sqrt{\pi}}{\sqrt{2\mathrm{e}^{\gamma}}}\mathscr{C}_0\frac{1}{\sqrt{\log D}}+O\left(\frac{1}{(\log D)^{3/2}}\right). \tag{3.6}$$

There is a further typo in [4, (14.26)], namely, $c\sqrt{2}$ should be replaced by $2^{1/2}\mathscr{C}_0/4$.

We will now proceed to the application of Proposition 3.3. For $q\in\mathbb{Z}_{>0}, a_1\in\mathbb{Z}\cap[0,q)$ define

$$\begin{aligned}\mathfrak{F}(a_1,q):=&\sum_{\substack{(k,t)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{\geqslant 0}\\ p\mid k\Rightarrow p\equiv 3(\bmod\ 4)}}\frac{\gcd(2^tk^2,q)}{2^tk^2}\\ &\times\sum_{\substack{\ell\in\mathbb{Z}\cap[0,q)\\ \gcd(2^tk^2,q)\mid\ell,(3.8)}}\frac{\varpi\left(\frac{\gcd(\ell,q)}{\gcd(2^tk^2,q)}\right)\mathrm{e}\left(\frac{a_1\ell}{q}\right)}{\gcd(\ell,q)\operatorname{lcm}\left(4,\frac{q}{\gcd(\ell,q)}\right)}\prod_{\substack{p\equiv 3(\bmod\ 4)\\ v_p(q)>v_p(\ell)}}\left(1-\frac{1}{p}\right)^{-1},\end{aligned} \tag{3.7}$$

where ℓ in the summation satisfies

$$\frac{2^tk^2}{\gcd(2^tk^2,q)}\equiv\frac{\ell}{\gcd(\ell,q)}\left(\bmod\ \gcd\left(4,\frac{q}{\gcd(\ell,q)}\right)\right). \tag{3.8}$$

The result of the following lemma aims to separate out the dependence on x from the apparent pandemonium that is hidden in $\mathfrak{F}(a_1,q)$.

Lemma 3.5. — *For $x\in\mathbb{R}_{\geqslant 1}, A\in\mathbb{R}_{>0}, q\in\mathbb{Z}_{>0}, a_1\in\mathbb{Z}\cap[0,q)$ with $q\leqslant(\log x)^A$ we have*

$$\sum_{\substack{m\in\mathbb{Z}\cap[1,x]\\ x_0^2+x_1^2=mx_2^2\text{ has a }\mathbb{Q}\text{-point}}}\mathrm{e}^{2\pi i a_1\frac{m}{q}}=2^{1/2}\mathscr{C}_0\mathfrak{F}(a_1,q)\frac{x}{(\log x)^{1/2}}+O\left(\frac{q^3x}{(\log x)^{1/2+1/7}}\right),$$

where the implied constant depends at most on A.

Proof. — Combining Lemma 3.1 with $u = (\log x)^{100}$ and Lemma 3.2 shows that, up to an error term which is $\ll x(\log x)^{-50}$, the sum over m in our lemma equals

$$\sum_{\substack{(k,t)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{\geqslant 0}\\ 2^t k^2\leqslant(\log x)^{100}\\ p\mid k\Rightarrow p\equiv 3 \pmod 4}}\ \sum_{\substack{\ell\in\mathbb{Z}\cap[0,q)\\ \gcd(2^tk^2,q)\mid\ell}} \varpi\Big(\frac{\gcd(\ell,q)}{\gcd(2^tk^2,q)}\Big)\mathrm{e}(a_1\ell/q)$$
$$\times \sum_{\substack{s\in\mathbb{Z}_{>0}\cap[1,x2^{-t}k^{-2}\gcd(2^tk^2,q)\gcd(\ell,q)^{-1}]\\ \frac{2^tk^2}{\gcd(2^tk^2,q)}s\equiv\frac{\ell}{\gcd(\ell,q)}\left(\bmod \frac{q}{\gcd(\ell,q)}\right)}} \varpi(s).$$

We note that $\varpi(s)$ vanishes unless $s \equiv 1 \pmod 4$. This means that we can add the condition $s \equiv 1 \pmod 4$ in the last sum over s, thus resulting with the double congruence

$$s \equiv 1 \pmod 4,\ \frac{2^tk^2}{\gcd(2^tk^2,q)}s \equiv \frac{\ell}{\gcd(\ell,q)} \left(\bmod \frac{q}{\gcd(\ell,q)}\right).$$

By the Chinese remainder theorem this has a solution if and only if (3.8) is satisfied. Assuming that this happens, the solution is unique modulo

$$Q := \operatorname{lcm}\Big(4, \frac{q}{\gcd(\ell,q)}\Big),$$

hence by Proposition 3.3 we get that the sum over m in our lemma equals

$$\mathrm{MT} := 2^{1/2}\mathscr{C}_0 \sum_{\substack{(k,t)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{\geqslant 0}\\ 2^t k^2\leqslant(\log x)^{100}\\ p\mid k\Rightarrow p\equiv 3 \pmod 4}}\ \sum_{\substack{\ell\in\mathbb{Z}\cap[0,q),(3.8)\\ \gcd(2^tk^2,q)\mid\ell}} \varpi\Big(\frac{\gcd(\ell,q)}{\gcd(2^tk^2,q)}\Big)\mathrm{e}(a_1\ell/q)$$
$$\times \frac{\ddot{Q}}{\varphi(\ddot{Q})}\frac{1}{\operatorname{lcm}(4,q/\gcd(\ell,q))}\frac{x2^{-t}k^{-2}\gcd(2^tk^2,q)\gcd(\ell,q)^{-1}}{\sqrt{\log(x2^{-t}k^{-2}\gcd(2^tk^2,q)\gcd(\ell,q)^{-1})}}$$

up to an error term which is

$$\ll \frac{x}{(\log x)^{50}} + \sum_{\substack{(k,t)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{\geqslant 0}\\ 2^t k^2\leqslant(\log x)^{100}\\ p\mid k\Rightarrow p\equiv 3 \pmod 4}}$$
$$\times \sum_{\substack{\ell\in\mathbb{Z}\cap[0,q),(3.8)\\ \gcd(2^tk^2,q)\mid\ell}} (\log\log\ddot{Q})\frac{x2^{-t}k^{-2}\gcd(2^tk^2,q)}{\gcd(\ell,q)\sqrt{\log x}}\left(\frac{\log Q}{\log x}\right)^{1/7} \tag{3.9}$$

owing to (1.5), which gives $\ddot{Q}/\varphi(\ddot{Q}) \ll \log\log \ddot{Q} \leqslant \log\log Q$. Note that we have made use of

$$\log(x2^{-t}k^{-2}\gcd(2^tk^2,q)\gcd(\ell,q)^{-1}) = \log x + O_A(\log\log x), \tag{3.10}$$

which follows from

$$\frac{x}{(\log x)^{100+A}} \leqslant \frac{x}{2^tk^2q} \leqslant x2^{-t}k^{-2}\gcd(2^tk^2,q)\gcd(\ell,q)^{-1} \leqslant xq \leqslant x(\log x)^A.$$

The bound $\ddot{Q} \leqslant Q \leqslant 4q$ shows that the sum over t,k in (3.9) is

$$\begin{aligned}
&\ll (\log\log q)(\log q)^{1/7}\frac{x}{(\log x)^{1/2+1/7}}\sum_{(k,t)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{\geqslant 0}}\ \sum_{\ell\in\mathbb{Z}\cap[0,q)} 2^{-t}k^{-2}\gcd(2^tk^2,q)\\
&\ll (\log\log q)(\log q)^{1/7}\frac{x}{(\log x)^{1/2+1/7}}q^2\sum_{(k,t)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{\geqslant 0}} 2^{-t}k^{-2}\\
&\ll q^3\frac{x}{(\log x)^{1/2+1/7}},
\end{aligned}$$

which is satisfactory. To conclude the proof, it remains to show that the quantity MT gives rise to the main term as stated in our lemma. By (3.10) we see that

$$\frac{1}{\sqrt{\log(x2^{-t}k^{-2}\gcd(2^tk^2,q)\gcd(\ell,q)^{-1})}} = \frac{1}{\sqrt{\log x}} + O\left(\frac{\log\log x}{(\log x)^{3/2}}\right),$$

hence $\mathrm{MT} = \mathrm{M}' + \mathrm{R}$, where M' is defined by

$$\frac{x2^{1/2}\mathscr{C}_0}{(\log x)^{1/2}}\sum_{\substack{(k,t)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{\geqslant 0}\\ 2^tk^2\leqslant(\log x)^{100}\\ p\mid k\Rightarrow p\equiv 3 \,(\mathrm{mod}\ 4)}} \frac{\gcd(2^tk^2,q)}{2^tk^2} \times \sum_{\substack{\ell\in\mathbb{Z}\cap[0,q)\\ \gcd(2^tk^2,q)\mid\ell,(3.8)}} \frac{\varpi(\gcd(\ell,q)/\gcd(2^tk^2,q))\mathrm{e}(a_1\ell/q)\ddot{Q}}{\gcd(\ell,q)\,\mathrm{lcm}(4,q/\gcd(\ell,q))\varphi(\ddot{Q})}$$

and R is a quantity that satisfies

$$\mathrm{R} \ll \sum_{(k,t)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{\geqslant 0}}\ \sum_{\ell\in\mathbb{Z}\cap[0,q)} \frac{\ddot{Q}}{\varphi(\ddot{Q})}\frac{x2^{-t}k^{-2}\gcd(2^tk^2,q)}{(\log\log x)^{-1}(\log x)^{3/2}} \ll q^3\frac{x\log\log x}{(\log x)^{3/2}}.$$

To complete the summation over t,k in M' we use the bound

$$\sum_{\substack{(k,t)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{\geqslant 0}\\ 2^tk^2>(\log x)^{100}}} \frac{\gcd(2^tk^2,q)}{2^tk^2}\sum_{\substack{\ell\in\mathbb{Z}\cap[0,q),(3.8)\\ \gcd(2^tk^2,q)\mid\ell}} \frac{\ddot{Q}}{\varphi(\ddot{Q})} \ll q^3 \sum_{\substack{(k,t)\in\mathbb{N}\times\mathbb{Z}_{\geqslant 0}\\ 2^tk^2>(\log x)^{100}}} \frac{1}{2^tk^2} \ll \frac{q^3}{(\log x)^{50}},$$

while the observation

$$\frac{\ddot{Q}}{\varphi(\ddot{Q})} = \prod_{\substack{p\equiv 3 (\mathrm{mod}\ 4) \\ p \mid q(\gcd(\ell,q))^{-1}}} \left(1-\frac{1}{p}\right)^{-1} = \prod_{\substack{p\equiv 3 (\mathrm{mod}\ 4) \\ v_p(q)>v_p(\ell)}} \left(1-\frac{1}{p}\right)^{-1}$$

allows to remove $\ddot{Q}$ from M'. □

We note that one immediate corollary of the last lemma is the bound

$$\mathfrak{F}(a_1, q) \ll 1, \tag{3.11}$$

with an absolute implied constant. Indeed, this can be shown by taking $A = 1/100$ in Lemma 3.5, dividing throughout by $x/\sqrt{\log x}$ in the asymptotic it provides and alluding to (3.2) to obtain

$$\begin{aligned} 2^{1/2}\mathscr{C}_0\mathfrak{F}(a_1,q) &\ll \frac{(\log x)^{1/2}}{x}\Big|\sum_{1\leqslant m\leqslant x}\vartheta_{\mathbb{Q}}(m)\mathrm{e}(a_1 m/q)\Big| + \frac{q^3}{(\log x)^{1/7}} \\ &\ll 1 + \frac{(\log x)^{3/100}}{(\log x)^{1/7}}. \end{aligned}$$

As announced at the beginning of this section, studying $E_{\mathbb{Q}}\big(\frac{a_1}{q}+\beta_1\big)$ is first done in the case $\beta_1 = 0$ as above. The following lemma shows that this is sufficient, up to introducing an extra factor.

Lemma 3.6. — *For $\Gamma_1 \in \mathbb{R}, A \in \mathbb{R}_{>0}, q \in \mathbb{Z}_{>0}, a_1 \in \mathbb{Z} \cap [0,q)$ with $q \leqslant (\log P)^A$ we have*

$$E_{\mathbb{Q}}\left(\frac{a_1}{q}+\frac{\Gamma_1}{P^d}\right) = 2^{1/2}\mathscr{C}_0\mathfrak{F}(a_1,q)\left(\int_2^{\max\{f_1([-1,1]^n)\}P^d}\frac{\mathrm{e}(\Gamma_1 P^{-d}t)}{\sqrt{\log t}}\mathrm{d}t\right) + O_A\left(\frac{q^3(1+|\Gamma_1|)P^d}{(\log P)^{1/2+1/7}}\right),$$

with an implied constant depending at most on A.

Proof. — To ease the notation we temporarily put $c := 2^{1/2}\mathscr{C}_0\mathfrak{F}(a_1,q)$. Fix $\beta \in \mathbb{R}$. By partial summation $\sum_{m\leqslant x}\vartheta_{\mathbb{Q}}(m)\mathrm{e}(m(\beta+a_1/q))$ equals

$$\left(\sum_{m\leqslant x}\vartheta_{\mathbb{Q}}(m)\mathrm{e}(a_1 m/q)\right)\mathrm{e}(x\beta) - \int_0^x \mathrm{e}(\beta t)'\left(\sum_{m\leqslant t}\vartheta_{\mathbb{Q}}(m)\mathrm{e}(a_1 m/q)\right)\mathrm{d}t.$$

If $q \leqslant (\log x)^A$ then Lemma 3.5 shows that this equals

$$c\left(\frac{x}{\sqrt{\log x}}\mathrm{e}(x\beta) - \int_2^x \frac{t}{\sqrt{\log t}}\mathrm{e}(\beta t)'\mathrm{d}t\right) + O\left(\frac{q^3 x(1+|\beta|x)}{(\log x)^{1/2+1/7}}\right),$$

with an implied constant depending at most on A. Using partial integration this is plainly

$$c\left(\int_2^x \Big(\frac{t}{\sqrt{\log t}}\Big)' \mathrm{e}(\beta t)\mathrm{d}t\right) + O\left(\frac{q^3(1+|\beta|x)x}{(\log x)^{1/2+1/7}}\right),$$

and using $(t(\log t)^{-1/2})' = (\log t)^{-1/2} - 2^{-1}(\log t)^{-3/2}$ shows that the last integral can be evaluated as $\int_2^x \mathrm{e}(\beta t)(\log t)^{-1/2}\mathrm{d}t + O(x(\log x)^{-3/2})$. Invoking the bound $c \ll 1$ (that is implied by (3.11)) we obtain

$$\sum_{m\leqslant x} \vartheta_{\mathbb{Q}}(m)\mathrm{e}(m(\beta + a_1/q)) = c\left(\int_2^x \frac{\mathrm{e}(\beta t)}{\sqrt{\log t}}\mathrm{d}t\right) + O\bigg(\frac{q^3(1+|\beta|x)x}{(\log x)^{1/2+1/7}}\bigg).$$

Using this for $x = \max\{f_1([-1,1]^n)\}P^d$ and putting $\beta = \Gamma_1 P^{-d}$ concludes the proof. □

4. Proof of the asymptotic

We are ready to prove the asymptotic in Theorem 1.3. We shall do so with different leading constants than those given in Theorem 1.3; showing equality of the constants is delayed until Section 5.

4.1. Restricting the range in the major arcs

The first reasonable step for the proof of the asymptotics would be to inject Lemma 3.6 into Lemma 2.3. However, this would give poor results because the error term in Lemma 3.6 is only powerful when Γ_1 is close to zero and q is small in comparison to P. For this reason we restrict the sum over q and the integration over $\boldsymbol{\beta}$ in Lemma 2.3. For its proof we shall need certain bounds. First, by (3.2), one has

$$E_{\mathbb{Q}}(\alpha_1) \ll P^d(\log P)^{-1/2}. \tag{4.1}$$

Next, letting $K := (n - \sigma(f_1, f_2))2^{-d+1}$, we use [1, Lem. 5.2, Lem. 5.4] to obtain the following bounds for every $\varepsilon > 0$, $\mathbf{\Gamma} \in \mathbb{R}^2$ and $\mathbf{a} \in \mathbb{Z}^2, q \in \mathbb{N}$ satisfying $\gcd(a_1, a_2, q) = 1$:

$$I(\mathbf{\Gamma}) \ll_\varepsilon \min\{1, |\mathbf{\Gamma}|^{-K/(2(d-1))+\varepsilon}\} \text{ and } S_{\mathbf{a},q} \ll_\varepsilon q^{n-K/(2(d-1))+\varepsilon}.$$

By our assumption (1.1), we have

$$I(\mathbf{\Gamma}) \ll \min\{1, |\mathbf{\Gamma}|^{-5/2}\}, \tag{4.2}$$

and, furthermore, that for all $0 < \lambda < 2^{-d}(d-1)^{-1}$ we have

$$S_{\mathbf{a},q} \ll_\lambda q^{n-3-\lambda}. \tag{4.3}$$

LEMMA 4.1. — *Keep the assumptions of Lemma 2.2 and let $Q_1, Q_2 \in \mathbb{R}_{\geqslant 1}$ with $Q_1, Q_2 \leqslant P^\eta$. Then for any λ satisfying*

$$0 < \lambda < \min\Big\{1, \frac{1}{2}\Big(\frac{n-\sigma(f_1,f_2)}{2^d(d-1)} - 3\Big)\Big\}, \tag{4.4}$$

we have

$$\begin{aligned}\sum_{q\leqslant P^\eta} q^{-n} \sum_{\substack{\mathbf{a}\in(\mathbb{Z}\cap[0,q))^2\\ \gcd(a_1,a_2,q)=1}} S_{\mathbf{a},q} \int_{|\boldsymbol{\beta}|\leqslant P^{-d+\eta}} P^d I(P^d\boldsymbol{\beta})\overline{E_{\mathbb{Q}}(\beta_1+a_1/q)}\mathrm{d}\boldsymbol{\beta}\\ = \sum_{q\leqslant Q_1} q^{-n} \sum_{\substack{\mathbf{a}\in(\mathbb{Z}\cap[0,q))^2\\ \gcd(a_1,a_2,q)=1}} S_{\mathbf{a},q} \int_{|\boldsymbol{\Gamma}|\leqslant Q_2} \frac{I(\boldsymbol{\Gamma})}{P^d}\overline{E_{\mathbb{Q}}(\Gamma_1 P^{-d}+a_1/q)}\mathrm{d}\boldsymbol{\Gamma}\\ + O_{\delta,\lambda,\theta_0}\big((\log P)^{-1/2}\min\big\{Q_1^{-\lambda}, Q_2^{-1/2}\big\}\big).\end{aligned}$$

Proof. — Using the change of variables $P^d\boldsymbol{\beta} = \boldsymbol{\Gamma}$ we obtain

$$\begin{aligned}\int_{P^{-d}Q_2<|\boldsymbol{\beta}|\leqslant P^{-d+\eta}} P^d I(P^d\boldsymbol{\beta})\overline{E_{\mathbb{Q}}(\beta_1+a_1/q)}\mathrm{d}\boldsymbol{\beta}\\ = P^{-d}\int_{Q_2<|\boldsymbol{\Gamma}|\leqslant P^\eta} I(\boldsymbol{\Gamma})\overline{E_{\mathbb{Q}}(\Gamma_1 P^{-d}+a_1/q)}\mathrm{d}\boldsymbol{\Gamma}\end{aligned}$$

and combining (4.1) with (4.2) shows that

$$\int_{P^{-d}Q_2<|\boldsymbol{\beta}|\leqslant P^{-d+\eta}} P^d I(P^d\boldsymbol{\beta})\overline{E_{\mathbb{Q}}(\beta_1+a_1/q)}\mathrm{d}\boldsymbol{\beta} \ll \frac{1}{\sqrt{Q_2\log P}}. \tag{4.5}$$

This shows that the sum over $q \leqslant P^\eta$ in the statement of our lemma equals

$$\sum_{q\leqslant P^\eta} q^{-n} \sum_{\substack{\mathbf{a}\in(\mathbb{Z}\cap[0,q))^2\\ \gcd(a_1,a_2,q)=1}} S_{\mathbf{a},q} \int_{|\boldsymbol{\Gamma}|\leqslant Q_2} \frac{I(\boldsymbol{\Gamma})}{P^d}\overline{E_{\mathbb{Q}}(\Gamma_1 P^{-d}+a_1/q)}\mathrm{d}\boldsymbol{\Gamma},$$

up to a term that is

$$\ll \frac{1}{\sqrt{Q_2\log P}} \sum_{q\leqslant P^\eta} \sum_{\substack{\mathbf{a}\in(\mathbb{Z}\cap[0,q))^2\\ \gcd(a_1,a_2,q)=1}} \frac{|S_{\mathbf{a},q}|}{q^n} \ll \frac{\sum_{q\leqslant P^\eta} q^{-1-\lambda}}{\sqrt{Q_2\log P}} \ll \frac{1}{\sqrt{Q_2\log P}},$$

where (4.3) has been utilised. Note that the bound $\int_{\mathbb{R}^2} |I(\boldsymbol{\Gamma})|\mathrm{d}\boldsymbol{\Gamma} < \infty$ is a consequence of (4.2). Using this with (4.1) shows that

$$\sum_{q>Q_1} \sum_{\substack{\mathbf{a}\in(\mathbb{Z}\cap[0,q))^2 \\ \gcd(a_1,a_2,q)=1}} \frac{S_{\mathbf{a},q}}{q^n} \int_{|\boldsymbol{\beta}|\leqslant P^{-d}Q_2} P^d I(P^d\boldsymbol{\beta})\overline{E_{\mathbb{Q}}(\beta_1+a_1/q)}\mathrm{d}\boldsymbol{\beta} \ll \frac{\sum_{q>Q_1} q^{-1-\lambda}}{\sqrt{\log P}} \ll \frac{Q_1^{-\lambda}}{\sqrt{\log P}},$$

where we have alluded to (4.3). This concludes the proof of the lemma. □

Lemma 4.2. — *Keep the assumptions of Lemma 2.2, fix any two positive A_1, A_2 and let*

$$\lambda_0 := \frac{1}{2}\min\left\{1, \frac{1}{2}\left(\frac{n-\sigma(f_1,f_2)}{2^d(d-1)} - 3\right)\right\}. \tag{4.6}$$

Then for all sufficiently large P the quantity $\Theta_{\mathbb{Q}}(P)P^{-n+d}$ equals

$$\sum_{q\leqslant(\log P)^{A_1}} \sum_{\substack{\mathbf{a}\in(\mathbb{Z}\cap[0,q))^2 \\ \gcd(a_1,a_2,q)=1}} \frac{S_{\mathbf{a},q}}{q^n} \int_{|\boldsymbol{\Gamma}|\leqslant(\log P)^{A_2}} \frac{I(\boldsymbol{\Gamma})}{P^d}\overline{E_{\mathbb{Q}}\left(\frac{a_1}{q}+\frac{\Gamma_1}{P^d}\right)}\mathrm{d}\boldsymbol{\Gamma} + O_{A_1,A_2}\left((\log P)^{-1/2-\min\{A_1\lambda_0,A_2/2\}}\right).$$

Proof. — The proof follows immediately by using Lemma 4.1 with $Q_i = (\log P)^{A_i}$ and Lemma 2.3 with some fixed η and θ_0 satisfying [1, p. 251, (10)–(11)] and $\eta < 1/7$. □

4.2. Injecting the sieve estimates into the restricted major arcs

We are now in position to inject Lemma 3.6 into Lemma 4.2. It will be convenient to start by studying the archimedean density. Recall (2.6) and define for $P > 3/\min\{f_1([-1,1]^n)\}$,

$$\mathfrak{I}_\Phi(P) := \int_{\boldsymbol{\Gamma}\in\mathbb{R}^2} \frac{I(\boldsymbol{\Gamma})}{P^d}\left(\int_3^{\max\{f_1([-1,1]^n)\}P^d} \frac{\mathrm{e}(-\Gamma_1 P^{-d}t)}{\sqrt{\log t}}\mathrm{d}t\right)\mathrm{d}\boldsymbol{\Gamma}. \tag{4.7}$$

The assumptions of Theorem 1.3 ensure that the integral converges absolutely, since by (4.2) we have

$$\int_{\boldsymbol{\Gamma}\in\mathbb{R}^2} \frac{|I(\boldsymbol{\Gamma})|}{P^d} \int_2^{\max\{f_1([-1,1]^n)\}P^d} \frac{\mathrm{d}t}{\sqrt{\log t}}\mathrm{d}\boldsymbol{\Gamma} \ll \int_{\boldsymbol{\Gamma}\in\mathbb{R}^2} \frac{\min\{1,|\boldsymbol{\Gamma}|^{-5/2}\}}{P^d}\frac{P^d}{\sqrt{\log P}} \ll \frac{1}{\sqrt{\log P}}.$$

Lemma 4.3. — *Under the assumptions of Theorem 1.3 we have*

$$\mathfrak{J}_\Phi(P)=\frac{1}{\sqrt{\log(P^d)}}\int_{\mathbf{\Gamma}\in\mathbb{R}^2}I(\mathbf{\Gamma})\bigg(\int_0^{\max\{f_1([-1,1]^n)\}}\mathrm{e}(-\Gamma_1\mu)\mathrm{d}\mu\bigg)\mathrm{d}\mathbf{\Gamma}\\ +O((\log P)^{-3/2}).$$

Proof. — Observe that the change of variables $\mu=P^{-d}t$ in (4.7) shows that

$$\mathfrak{J}_\Phi(P)=\int_{\mathbf{\Gamma}\in\mathbb{R}^2}I(\mathbf{\Gamma})\bigg(\int_{3P^{-d}}^{\max\{f_1([-1,1]^n)\}}\frac{\mathrm{e}(-\Gamma_1\mu)}{\sqrt{\log(\mu P^d)}}\mathrm{d}\mu\bigg)\mathrm{d}\mathbf{\Gamma}.$$

It is easy to verify that $(1+x)^{-1/2}=1+O(x)$ for $|x|<1$, hence for μ and P in the range $0<\mu<P^d$ we have

$$\big(\log(\mu P^d)\big)^{-1/2}=\big(\log(P^d)\big)^{-1/2}\bigg(1+\frac{\log\mu}{\log(P^d)}\bigg)^{-1/2}\\ =\big(\log(P^d)\big)^{-1/2}+O\bigg(\frac{\log\mu}{(\log P)^{3/2}}\bigg).$$

Using this for $0<\mu\leqslant\max\{f_1([-1,1]^n)\}$, we infer the following estimate for all sufficiently large P,

$$\mathfrak{J}_\Phi(P)-\frac{1}{\sqrt{\log(P^d)}}\int_{\mathbf{\Gamma}\in\mathbb{R}^2}I(\mathbf{\Gamma})\int_{3P^{-d}}^{\max\{f_1([-1,1]^n)\}}\mathrm{e}(-\Gamma_1\mu)\mathrm{d}\mu\mathrm{d}\mathbf{\Gamma}\\ \ll\frac{1}{(\log P)^{3/2}}\int_{\mathbf{\Gamma}\in\mathbb{R}^2}|I(\mathbf{\Gamma})|\mathrm{d}\mathbf{\Gamma},$$

which is $\ll(\log P)^{-3/2}$ due to (4.2). □

Define

$$\mathfrak{J}:=\int_{\Gamma\in\mathbb{R}}\int_{\{\mathbf{t}\in[-1,1]^n:x_0^2+x_1^2=f_1(\mathbf{t})x_2^2\text{ has an }\mathbb{R}\text{-point}\}}\mathrm{e}(\Gamma f_2(\mathbf{t}))\mathrm{d}\mathbf{t}\mathrm{d}\Gamma \tag{4.8}$$

and note that the integral converges absolutely owing to the smoothness of f_1 and f_2, (1.1) and [1, Lem. 5.2] with $R=1$. The arguments in [1, §6] show that if there is a non-singular real point of $f_2=0$ contained in the set $\{\mathbf{t}\in[-1,1]^n:f_1(\mathbf{t})\geqslant 0\}$ then $\mathfrak{J}>0$. In the situation of Theorem 1.3 this condition holds, because its assumptions include that $B(\mathbb{Q})\neq\varnothing$ and that f_2 is non-singular.

Lemma 4.4. — *Under the assumptions of Theorem 1.3 we have*

$$\int_{\mathbf{\Gamma}\in\mathbb{R}^2}I(\mathbf{\Gamma})\bigg(\int_0^{\max\{f_1([-1,1]^n)\}}\mathrm{e}(-\Gamma_1\mu)\mathrm{d}\mu\bigg)\mathrm{d}\mathbf{\Gamma}=\mathfrak{J}.$$

Proof. — Define for $m \in \mathbb{N}$ the function $\varphi_m : \mathbb{R} \to \mathbb{R}$ through $\varphi_m(x) := \pi^{-1/2} m \exp(-m^2 x^2)$. First one may show

$$\lim_{m\to+\infty} \int_{\mathbf{\Gamma}\in\mathbb{R}^2} \frac{I(\mathbf{\Gamma})}{\mathrm{e}^{\pi^2\Gamma_1^2 m^{-2}}} \int_0^{\max\{f_1([-1,1]^n)\}} \mathrm{e}(-\Gamma_1\mu)\mathrm{d}\mu\mathrm{d}\mathbf{\Gamma} = \int_{\mathbf{\Gamma}\in\mathbb{R}^2} I(\mathbf{\Gamma}) \int_0^{\max\{f_1([-1,1]^n)\}} \mathrm{e}(-\Gamma_1\mu)\mathrm{d}\mu\mathrm{d}\mathbf{\Gamma},$$

for example by considering the difference between the right-hand side of this equality and each individual member of the limit on the left-hand side. Then one shows that this difference is $o(1)$ independently of m, by splitting the integral over Γ_1 up into the ranges $0 < |\Gamma_1| < \log m$ and $\log m < |\Gamma_1|$ and showing that the two resulting integrals are both $o(1)$. One will need (4.2) for this.

Recalling (2.6) and using the following formula with $x = f_1(\mathbf{t}) - \mu$,

$$\varphi_m(x) = \int_{\mathbb{R}} \mathrm{e}^{-\pi^2\Gamma_1^2 m^{-2}} \mathrm{e}(x\Gamma_1)\mathrm{d}\Gamma_1,$$

that can be established by Fourier's inversion formula, allows us to rewrite the integral inside the limit as

$$\int_{\substack{\mathbf{t}\in[-1,1]^n: f_1(\mathbf{t})\neq 0\\ f_1(\mathbf{t})\neq\max\{f_1([-1,1]^n)\}}} \left(\int_0 \varphi_m(f_1(\mathbf{t})-\mu)\mathrm{d}\mu\right)\left(\int_{\Gamma_2\in\mathbb{R}} \mathrm{e}(\Gamma_2 f_2(\mathbf{t}))\mathrm{d}\Gamma_2\right)\mathrm{d}\mathbf{t}.$$

Note that we used (2.6) with $[-1,1]^n$ replaced by the range of integration for $\mathbf{t}$ in the expression above; this is clearly allowable as it only removes a set of measure zero from the integration in (2.6). It is now easy to see that the limit

$$\lim_{m\to+\infty} \int_{c_1}^{c_2} \varphi_m(\mu)\mathrm{d}\mu$$

equals 1 if $c_1 < 0 < c_2$ and that it vanishes when $c_1 > 0$. This proves that if $\mathbf{t} \in [-1,1]^n$ satisfies $f_1(\mathbf{t}) > 0$, then the limit

$$\lim_{m\to+\infty} \int_0^{\max\{f_1([-1,1]^n)\}} \varphi_m(f_1(\mathbf{t})-\mu)\mathrm{d}\mu$$

equals 1, while, if $f_1(\mathbf{t}) < 0$ then the limit vanishes. The dominated convergence theorem then gives

$$\int_{\mathbf{\Gamma}\in\mathbb{R}^2} I(\mathbf{\Gamma}) \int_0^{\max\{f_1([-1,1]^n)\}} \mathrm{e}(-\Gamma_1\mu)\mathrm{d}\mu\mathrm{d}\mathbf{\Gamma} = \int_{\substack{\mathbf{t}\in[-1,1]^n: f_1(\mathbf{t})\neq 0\\ f_1(\mathbf{t})\neq\max\{f_1([-1,1]^n)\}}} \left(\int_{\Gamma_2\in\mathbb{R}} \mathrm{e}(\Gamma_2 f_2(\mathbf{t}))\mathrm{d}\Gamma_2\right)\mathrm{d}\mathbf{t},$$

which concludes the proof. □

Having dealt with the integral part of Lemma 2.3, we now turn our attention to the summation. Recall the definition of $S_{\mathbf{a},q}$ and $\mathfrak{F}(a_1,q)$ respectively in (2.5) and (3.7) and let

$$\mathbb{L}_\phi := \sum_{q\in\mathbb{N}} q^{-n} \sum_{\substack{\mathbf{a}\in(\mathbb{Z}\cap[0,q))^2\\ \gcd(a_1,a_2,q)=1}} S_{\mathbf{a},q}\overline{\mathfrak{F}(a_1,q)}. \tag{4.9}$$

Under the assumptions of Theorem 1.3 the sum $\mathbb{L}_\phi$ converges absolutely, since by (3.11) and (4.3) we have for all $x>1$,

$$\begin{aligned}\sum_{\substack{q\in\mathbb{N}\\ q>x}} q^{-n} &\sum_{\substack{\mathbf{a}\in(\mathbb{Z}\cap[0,q))^2\\ \gcd(a_1,a_2,q)=1}} \left|S_{\mathbf{a},q}\overline{\mathfrak{F}(a_1,q)}\right| \\ &\ll \sum_{\substack{q\in\mathbb{N}\\ q>x}} q^{-n} \sum_{\substack{\mathbf{a}\in(\mathbb{Z}\cap[0,q))^2\\ \gcd(a_1,a_2,q)=1}} q^{n-3-\lambda_0} \leqslant \sum_{\substack{q\in\mathbb{N}\\ q>x}} q^{-1-\lambda_0} \ll x^{-\lambda_0}.\end{aligned} \tag{4.10}$$

Lemma 4.5. — *Under the assumptions of Theorem 1.3 we have for all $P\geqslant 2$,*

$$\Theta_{\mathbb{Q}}(P) = \mathscr{C}_0\mathfrak{I}\frac{\mathbb{L}_\phi\sqrt{2}}{d^{1/2}}\frac{P^{n-d}}{(\log P)^{1/2}} + O\left((\log P)^{-\frac{1}{40}\frac{1}{(d-1)2^{d+2}}}\frac{P^{n-d}}{(\log P)^{1/2}}\right).$$

Proof. — Combining Lemmas 3.6 and 4.2 shows that

$$\frac{\Theta_{\mathbb{Q}}(P)}{P^{n-d}} = 2^{1/2}\mathscr{C}_0\mathscr{R}_1\mathscr{R}_2 + \mathscr{R}_3 + O\big((\log P)^{-1/2-\min\{A_1\lambda_0,A_2/2\}}\big), \tag{4.11}$$

where

$$\mathscr{R}_1 := \sum_{q\leqslant(\log P)^{A_1}} q^{-n} \sum_{\substack{\mathbf{a}\in(\mathbb{Z}\cap[0,q))^2\\ \gcd(a_1,a_2,q)=1}} S_{\mathbf{a},q}\overline{\mathfrak{F}(a_1,q)},$$

$$\mathscr{R}_2 := \int_{|\mathbf{\Gamma}|\leqslant(\log P)^{A_2}} \frac{I(\mathbf{\Gamma})}{P^d}\left(\int_2^{\max\{f_1([-1,1]^n)\}P^d} \frac{\mathrm{e}(-\Gamma_1P^{-d}t)}{\sqrt{\log t}}\mathrm{d}t\right)\mathrm{d}\mathbf{\Gamma},$$

and $\mathscr{R}_3$ is a quantity that satisfies

$$\mathscr{R}_3 \ll \sum_{q\leqslant(\log P)^{A_1}} \sum_{\substack{\mathbf{a}\in(\mathbb{Z}\cap[0,q))^2\\ \gcd(a_1,a_2,q)=1}} \frac{|S_{\mathbf{a},q}|}{q^n}\int_{|\mathbf{\Gamma}|\leqslant(\log P)^{A_2}} \frac{|I(\mathbf{\Gamma})|}{P^d}\frac{q^3(1+|\Gamma_1|)P^d}{(\log P)^{1/2+1/7}}\mathrm{d}\mathbf{\Gamma}.$$

We can easily see that

$$\mathscr{R}_3 \ll_{A_2} \frac{(\log P)^{3A_1+A_2}}{(\log P)^{1/2+1/7}} \sum_{q\leqslant(\log P)^{A_1}} \sum_{\substack{\mathbf{a}\in(\mathbb{Z}\cap[0,q))^2\\ \gcd(a_1,a_2,q)=1}} \frac{|S_{\mathbf{a},q}|}{q^n}\int_{|\mathbf{\Gamma}|\leqslant(\log P)^{A_2}}|I(\mathbf{\Gamma})|\mathrm{d}\mathbf{\Gamma}.$$

By (4.2) and (4.3) the sum over q is convergent, and so is the integral over $\mathbf{\Gamma}$, therefore

$$\mathscr{R}_3 \ll_{A_2} (\log P)^{3A_1+A_2-1/2-1/7}. \tag{4.12}$$

Using (4.2) we infer that

$$\begin{aligned}\int_{|\mathbf{\Gamma}|>(\log P)^{A_2}} \frac{|I(\mathbf{\Gamma})|}{P^d}\left(\int_2^{\max\{f_1([-1,1]^n)\}P^d} \frac{\mathrm{e}(-\Gamma_1 P^{-d}t)}{\sqrt{\log t}}\mathrm{d}t\right)\mathrm{d}\mathbf{\Gamma}\\ \ll \int_{|\mathbf{\Gamma}|>(\log P)^{A_2}} |I(\mathbf{\Gamma})|\frac{1}{\sqrt{\log P}}\mathrm{d}\mathbf{\Gamma} \ll_{A_2} (\log P)^{-1/2-A_2/2},\end{aligned}$$

therefore

$$\mathscr{R}_2 = \mathfrak{J}_\phi(P) + O_{A_2}((\log P)^{-1/2-A_2/2}). \tag{4.13}$$

Furthermore, by (4.10) we deduce

$$\mathscr{R}_1 = \mathbb{L}_\phi + O_{A_1}((\log P)^{-A_1\lambda_0}). \tag{4.14}$$

By Lemmas 4.3 and 4.4 we have $\mathfrak{J}_\phi(P) \ll (\log P)^{-1/2}$, thus injecting (4.12), (4.13) and (4.14) into (4.11) provides us with

$$\frac{\Theta_{\mathbb{Q}}(P)}{P^{n-d}} = 2^{1/2}\mathscr{C}_0\mathfrak{J}_\phi(P)\mathbb{L}_\phi + O((\log P)^{-1/2-\beta}),$$

where $\beta := \min\{A_1\lambda_0, A_2/2, -3A_1-A_2+1/7\}$. A moment's thought affirms that assumption (1.1) ensures the validity of $\lambda_0 \geqslant (d-1)^{-1}2^{-d-2}$ and choosing $A_1 = \frac{1}{40} = A_2/2$ gives $\beta \geqslant (40(d-1)2^{d+2})^{-1}$. Finally, using Lemmas 4.3 and 4.4 concludes the proof. □

4.3. Proof of Theorem 1.3

Define

$$c_\phi := \frac{\mathfrak{J}}{d^{1/2}}\frac{2^{1/2}}{\zeta(n-d)}\frac{\mathbb{L}_\phi}{2}\mathscr{C}_0. \tag{4.15}$$

By Lemmas 2.1 and 4.5 the quantity $N(B,\phi,t)$ equals

$$\frac{\sqrt{2}}{2}\mathscr{C}_0\mathfrak{J}\mathbb{L}_\phi\frac{t^{n-d}}{d^{1/2}}\sum_{l\leqslant \log t}\frac{\mu(l)}{l^{n-d}(\log(t/l))^{1/2}}$$

up to an error term that is

$$\ll \frac{t^{n-d}}{\log t} + \sum_{l\leqslant\log t}\frac{(t/l)^{n-d}}{(\log(t/l))^{\frac12+\frac1{40}\frac{1}{(d-1)2^{d+2}}}} \ll \frac{t^{n-d}}{(\log t)^{\frac12+\frac1{40}\frac{1}{(d-1)2^{d+2}}}}.$$

Note that for $l \leqslant \log t$ we have $(\log(t/l))^{-1/2} = (\log t)^{-1/2} + O((\log t)^{-1})$, hence

$$\sum_{l \leqslant \log t} \frac{\mu(l)}{l^{n-d}(\log(t/l))^{1/2}} = (\log t)^{-1/2}\left(\sum_{l \leqslant \log t} \frac{\mu(l)}{l^{n-d}}\right) + O((\log t)^{-1}).$$

Assumption (1.1) implies $n-d \geqslant 2$. Denoting the Riemann zeta function by ζ, we use the standard estimate

$$\sum_{l \leqslant \log t} \frac{\mu(l)}{l^{n-d}} = \zeta(n-d)^{-1} + O\left(\frac{1}{(\log t)^{n-d-1}}\right)$$

to obtain

$$\sum_{l \leqslant \log t} \frac{\mu(l)}{l^{n-d}(\log(t/l))^{1/2}} = \zeta(n-d)^{-1}(\log t)^{-1/2} + O((\log t)^{-1}).$$

Thus,

$$\frac{N(B,\phi,t)}{t^{n-d}(\log t)^{-1/2}} - \frac{\mathfrak{J}\mathbb{L}_\phi \mathscr{C}_0}{\zeta(n-d)\sqrt{2d}} \ll \frac{1}{(\log t)^{\varepsilon_d}}, \tag{4.16}$$

which concludes our proof. □

5. The leading constant

The circle method and the half-dimensional sieve allowed us to obtain a proof of the asymptotic, however, this came at a cost because the leading constant c_ϕ in (4.15) is complicated. In this section we shall simplify c_ϕ by relating it to a product of p-adic densities.

We begin by factorising $\mathbb{L}_\phi$. One can use a version of the Chinese Remainder Theorem to show that complete exponential sums form a multiplicative function of the modulus. In the context of the circle method this is very standard and it occurs when one factorises the singular series, see [1, (2), §7], for example. Before stating the factorisation of $\mathbb{L}_\phi$ we introduce the necessary notation for the p-adic factors. For a prime p define

$$\tau_{f_2}(p) := \lim_{N\to+\infty} \frac{\#\left\{\mathbf{t} \in (\mathbb{Z}\cap[0,p^N))^n : f_2(\mathbf{t}) \equiv 0 \left(\bmod p^N\right)\right\}}{p^{N(n-1)}}.$$

For $a \in \mathbb{Z}_{\geqslant 0}$ and $q,k \in \mathbb{N}$ we let

$$\mathscr{W}_{a,q}(k) := \sum_{\substack{\ell \in \mathbb{Z}\cap[0,q) \\ \gcd(\ell,q)=\gcd(k^2,q)}} \mathrm{e}(-a\ell/q) \prod_{\substack{p \text{ prime} \\ v_p(q)>v_p(\ell)}} \left(1-\frac{1}{p}\right)^{-1}$$

and for $p \equiv 3 \pmod 4$ we define

$$E_\Phi(p) := \sum_{\kappa, m \in \mathbb{Z}_{\geqslant 0}} \frac{\gcd(p^{2\kappa}, p^m)}{p^{2\kappa + m(n+1)}} \sum_{\substack{\mathbf{a} \in (\mathbb{Z} \cap [0, p^m))^2 \\ \gcd(a_1, a_2, p^m) = 1}} S_{\mathbf{a}, p^m} \mathscr{W}_{a_1, p^m}(p^\kappa).$$

We furthermore define

$$E_\Phi(2) := \frac{1}{4} \sum_{t, \varrho \in \mathbb{Z}_{\geqslant 0}} \frac{1}{2^{t + \varrho n}} \sum_{\substack{\mathbf{b} \in (\mathbb{Z} \cap [0, 2^\varrho))^2 \\ \gcd(b_1, b_2, 2^\varrho) = 1}} S_{\mathbf{b}, 2^\varrho} \mathrm{e}(-b_1 2^{t - \varrho}) \mathbb{1}_{\{v_2(b_1) \geqslant \varrho - t - 2\}}.$$

Lemma 5.1. — *Keep the assumptions of Theorem 1.3. Then*

$$\mathbb{L}_\Phi = E_\Phi(2) \left(\prod_{p \equiv 1 \pmod 4} \tau_{f_2}(p) \right) \left(\prod_{p \equiv 3 \pmod 4} E_\Phi(p) \right),$$

where both infinite products over p converge absolutely.

The proof of Lemma 5.1 is based on the repeated use of explicit expressions for Ramanujan sums. It is relatively straightforward but tedious and we thus omit the details. The complete proof is given in the Ph.D. thesis of the second named author [19, §3.5.1].

We next relate the exponential sums modulo prime powers that the circle method gives to limits of counting functions related to p-adic solubility.

Proposition 5.2. — *Let p be a prime number with $p \equiv 3 \pmod 4$. Under the assumptions of Theorem 1.3 the following limit exists,*

$$\ell_p := \lim_{N \to +\infty} \frac{\# \left\{ \mathbf{t} \in (\mathbb{Z} \cap [0, p^N))^n : \begin{array}{l} f_2(\mathbf{t}) \equiv 0 \pmod{p^N}, \\ x_0^2 + x_1^2 = f_1(\mathbf{t}) x_2^2 \textit{ has a } \mathbb{Q}_p\textit{-point} \end{array} \right\}}{p^{N(n-1)}}.$$

Furthermore, we have $E_\Phi(p) = \left(1 - 1/p\right)^{-1} \ell_p$.

Proposition 5.3. — *Under the assumptions of Theorem 1.3, the following limit exists,*

$$\ell_2 := \lim_{N \to +\infty} \frac{\# \left\{ \mathbf{t} \in (\mathbb{Z} \cap [0, 2^N))^n : \begin{array}{l} f_2(\mathbf{t}) \equiv 0 \pmod{2^N}, \\ x_0^2 + x_1^2 = f_1(\mathbf{t}) x_2^2 \textit{ has a } \mathbb{Q}_2\textit{-point} \end{array} \right\}}{2^{N(n-1)}}.$$

Furthermore, we have $E_\Phi(2) = \ell_2$.

The proofs of Propositions 5.2-5.3 are straightforward in the context of the circle method and are not given here. Full details can be found in [19, §3.5.2–3.5.3].

For every prime p we define the number

$$\tau_p := \frac{(1-\frac{1}{p^{n-d}})}{(1-\frac{1}{p})} \lim_{N\to+\infty} \frac{\#\left\{\mathbf{t}\in(\mathbb{Z}\cap[0,p^N))^n : \begin{array}{l} p^N \mid f_2(\mathbf{t}), \\ x_0^2+x_1^2=f_1(\mathbf{t})x_2^2 \text{ has a } \mathbb{Q}_p\text{-point}\end{array}\right\}}{p^{N(n-1)}}.$$

This is well-defined because for $p \equiv 1 \,(\mathrm{mod}\ 4)$ the limit coincides with $\tau_{f_2}(p)$ and for $p \not\equiv 1 \,(\mathrm{mod}\ 4)$ the limit coincides with ℓ_p and ℓ_2. The definition of τ_p is motivated by the construction of the Tamagawa measure by Loughran in [7, §5.7.2]. It is useful to recall that if one was counting $\mathbb{Q}$-rational points on the hypersurface $f_2 = 0$ then the corresponding Peyre constant would involve a p-adic density that is the same as the number τ_p except for the condition on $\mathbb{Q}_p$-solubility, see [10, Cor. 3.5]. For $s \in \mathbb{C}$ with $\Re(s) > 1$ let

$$L(s) := \sqrt{\zeta(s)}, \tag{5.1}$$

denote the p-adic factor of $L(s)$ by $L_p(s)$ and write λ_p for $L_p(1)$, i.e.,

$$\lambda_p := \left(1 - \frac{1}{p}\right)^{-1/2}.$$

Recall the definition of the real density $\mathfrak{J}$ in (4.8) and that d denotes the degrees of f_1 and f_2 (which are equal by the assumption of Theorem 1.3).

Theorem 5.4. — *Keep the assumptions of Theorem 1.3.*

(1) *If ϕ has a smooth fibre with a $\mathbb{Q}$-point then the constant c_ϕ in Theorem 1.3 is strictly positive.*
(2) *The infinite product $\prod_p \frac{\tau_p}{\lambda_p}$ taken over all non-archimedean places converges.*
(3) *The constant c_ϕ in Theorem 1.3 satisfies*

$$c_\phi = \frac{\frac{1}{\sqrt{d}}\mathfrak{J}\prod_p \frac{\tau_p}{\lambda_p}}{\sqrt{\pi}}.$$

Remark 5.5. — Recalling that $\sqrt{\pi}$ is the value of the Euler Gamma function at $1/2$ and noting that

$$1 = \lim_{s\to 1_+} (s-1)^{1/2} L(s)$$

allows for a comparison of Theorem 5.4 with the case of [7, Thm. 5.15] that corresponds to

$$\rho_{\mathscr{B}}(X) = \frac{1}{2}.$$

Proof of Theorem 5.4. — To prove (1) observe that due to (4.15), it suffices to show that if ϕ has a smooth fibre with a $\mathbb{Q}$-point then

$$\mathfrak{J} > 0 \quad \text{and} \quad \mathbb{L}_\phi > 0.$$

For the former part, we recall that it is standard that if $\mathscr{B} \subset [-1,1]^n$ is a box with sides parallel to the coordinate axes and the hypersurface $f_2 = 0$ has a non-singular real point inside $\mathscr{B}$ then the corresponding singular integral that is given by

$$\int_{\Gamma\in\mathbb{R}} \int_{\mathbf{t}\in\mathscr{B}} \mathrm{e}(\Gamma f_2(\mathbf{t}))\mathrm{d}\mathbf{t}\mathrm{d}\Gamma$$

is strictly positive. This is proved in [1, §6], for example, but see also [13, §4]. Here, the fact that ϕ has a smooth fibre with a $\mathbb{Q}$-point implies that there exists $b \in \mathbb{P}^n(\mathbb{Q})$ such that $f_2(b) = 0$ and the curve $x_0^2 + x_1^2 = f_1(\mathbf{t})x_2^2$ is smooth and has a $\mathbb{Q}$-point, hence in particular, an $\mathbb{R}$-point. Picking $\mathbf{t}_0 \in \mathbb{Z}^n_{\mathrm{prim}}$ with $b = [\mathbf{t}_0]$ we get that there exists $\mathbf{t}_0 \in \mathbb{R}^n$ with $f_2(\mathbf{t}_0) = 0$ and $f_1(\mathbf{t}_0) > 0$. Note that f_2 is smooth at $\mathbf{t}_0$ due to the assumptions of Theorem 1.3. Thus, by the Implicit Function Theorem there is a non-empty box $\mathscr{B}$ with sides parallel to the axes such that every $\mathbf{t}$ with $f_2(\mathbf{t}) = 0$ and in the interior of $\mathscr{B}$ satisfies $f_1(\mathbf{t}) > 0$. From this, one infers that $\mathfrak{J} > 0$ upon recalling the definition of $\mathfrak{J}$ in (4.8).

To prove that $\mathbb{L}_\phi > 0$, we invoke Lemma 5.1 to see that it is enough to show

$$E_\phi(2) > 0, \quad p \equiv 1 \pmod 4 \Rightarrow \tau_{f_2}(p) > 0 \quad \text{and} \quad p \equiv 3 \pmod 4 \Rightarrow E_\phi(p) > 0. \tag{5.2}$$

For this, note that for every prime p the point $\mathbf{t}_0$ can be viewed as a smooth $\mathbb{Q}_p$-point on the hypersurface $f_2 = 0$ and such that the curve $x_0^2 + x_1^2 = f_1(\mathbf{t}_0)x_2^2$ has a point $\mathbb{Q}_p$-point. If $p \equiv 1 \pmod 4$ this forces no condition on $f_1(\mathbf{t}_0)$, thus $\tau_{f_2}(p) > 0$ because, as mentioned in [1, §7], one can use Hensel's lemma to prove that if $f_2 = 0$ has a smooth $\mathbb{Q}_p$-point then the analogous p-adic density is strictly positive. If $p \equiv 3 \pmod 4$ or if $p = 2$ then the existence of such a $\mathbf{t}_0$ can be used with Hensel's lemma to prove that the quantities ℓ_2 and ℓ_p are strictly positive. The equalities $E_\phi(p) = \ell_p/(1-1/p)$ and $E_\phi(2) = \ell_2$ (proved in Propositions 5.2–5.3) then show the validity of (5.2), which concludes the proof of (1).

Let us now commence the proof of (2). Denoting the limit in the definition of τ_p by ℓ_p we see that

$$\begin{aligned}\lim_{t\to+\infty}\prod_{p\leqslant t}\frac{\tau_p}{\lambda_p} &= \lim_{t\to+\infty}\prod_{p\leqslant t}\frac{(1-\frac{1}{p^{n-d}})}{(1-\frac{1}{p})}\ell_p\left(1-\frac{1}{p}\right)^{1/2}\\ &= \frac{\ell_2 2^{1/2}}{\zeta(n-d)}\lim_{t\to+\infty}\prod_{p\leqslant t}\frac{\ell_p}{(1-\frac{\mathbb{1}_{p\equiv 3(\mathrm{mod}\ 4)}}{p})}\left(\frac{(1-\frac{\mathbb{1}_{p\equiv 3(\mathrm{mod}\ 4)}}{p})}{(1-\frac{\mathbb{1}_{p\equiv 1(\mathrm{mod}\ 4)}}{p})}\right)^{1/2}.\end{aligned}$$

We now let χ stand for the non-trivial Dirichlet character (mod 4) to obtain that

$$\prod_{p\leqslant t}\frac{(1-\frac{\mathbb{1}_{p\equiv 3(\mathrm{mod}\ 4)}}{p})}{(1-\frac{\mathbb{1}_{p\equiv 1(\mathrm{mod}\ 4)}}{p})} = \left(\prod_{p\leqslant t}\frac{1}{1-\frac{\chi(p)}{p}}\right)\prod_{\substack{p\leqslant t\\ p\equiv 3(\mathrm{mod}\ 4)}}\left(1-\frac{1}{p^2}\right)$$

and therefore, alluding to the well-known fact that the Euler product for the Dirichlet series $L(s,\chi)$ of χ converges to $\pi/4$ for $s=1$, we get via Definition (1.8) that

$$\lim_{t\to+\infty}\prod_{p\leqslant t}\left(\frac{(1-\frac{\mathbb{1}_{p\equiv 3(\mathrm{mod}\ 4)}}{p})}{(1-\frac{\mathbb{1}_{p\equiv 1(\mathrm{mod}\ 4)}}{p})}\right)^{1/2} = \frac{\pi^{1/2}}{2}\mathscr{C}_0.$$

We have so far shown that

$$\lim_{t\to+\infty}\prod_{p\leqslant t}\frac{\tau_p}{\lambda_p} = \frac{\ell_2 2^{1/2}}{\zeta(n-d)}\left(\lim_{t\to+\infty}\prod_{p\leqslant t}\frac{\ell_p}{(1-\frac{\mathbb{1}_{p\equiv 3(\mathrm{mod}\ 4)}}{p})}\right)\frac{\pi^{1/2}}{2}\mathscr{C}_0.$$

It is clear that if $p\equiv 1\,(\mathrm{mod}\ 4)$ then $\ell_p=\tau_{f_2}(p)$, and thus,

$$\lim_{t\to+\infty}\prod_{\substack{p\equiv 1(\mathrm{mod}\ 4)\\ p\leqslant t}}\ell_p = \prod_{p\equiv 1(\mathrm{mod}\ 4)}\tau_{f_2}(p).$$

By Proposition 5.2 one gets

$$\prod_{\substack{p\equiv 3(\mathrm{mod}\ 4)\\ p\leqslant t}}\frac{\ell_p}{(1-\frac{1}{p})} = \prod_{\substack{p\equiv 3(\mathrm{mod}\ 4)\\ p\leqslant t}}E_\Phi(p).$$

It is now clear from Lemma 5.1 that the last product converges as $t\to+\infty$, therefore the product $\prod_p \tau_p/\lambda_p$ is convergent, which proves (2).

For the proof of (3) we note that the arguments at the end of the proof of (2) provided us with the equality

$$\prod_p\frac{\tau_p}{\lambda_p} = \frac{\ell_2 2^{1/2}}{\zeta(n-d)}\left(\prod_{p\equiv 1(\mathrm{mod}\ 4)}\tau_{f_2}(p)\right)\left(\prod_{p\equiv 3(\mathrm{mod}\ 4)}E_\Phi(p)\right)\frac{\pi^{1/2}}{2}\mathscr{C}_0.$$

We have $E_\Phi(2) = \ell_2$ due to Proposition 5.3, and alluding to Lemma 5.1 we get

$$\prod_p \frac{\tau_p}{\lambda_p} = \frac{2^{1/2}}{\zeta(n-d)} \mathbb{L}_\Phi \frac{\pi^{1/2}}{2} \mathscr{C}_0.$$

A comparison with (4.15) makes the proof of (3) immediately apparent. □

Let us remark that the arguments in the present section can be easily rearranged to show that $\prod_{p \leqslant t} \tau_p$ diverges and therefore, the numbers λ_p can be viewed as "convergence factors". We are very grateful to Daniel Loughran for suggesting this choice for λ_p, as well as for the L-function in (5.1).

BIBLIOGRAPHY

[1] B. J. Birch, "Forms in many variables", *Proc. R. Soc. Lond., Ser. A* **265** (1962), p. 245-263.

[2] T. Browning & D. Loughran, "Sieving rational points on varieties", *Trans. Am. Math. Soc.* **371** (2019), no. 8, p. 5757-5785.

[3] C. Frei, D. Loughran & E. Sofos, "Rational points of bounded height on general conic bundle surfaces", *Proc. Lond. Math. Soc.* **117** (2018), no. 2, p. 407-440.

[4] J. Friedlander & H. Iwaniec, *Opera de cribro*, Colloquium Publications, vol. 57, American Mathematical Society, 2010, xx+527 pages.

[5] C. Hooley, "On ternary quadratic forms that represent zero", *Glasg. Math. J.* **35** (1993), no. 1, p. 13-23.

[6] ———, "On ternary quadratic forms that represent zero. II", *J. Reine Angew. Math.* **602** (2007), p. 179-225.

[7] D. Loughran, "The number of varieties in a family which contain a rational point", *J. Eur. Math. Soc.* **20** (2018), no. 10, p. 2539-2588.

[8] D. Loughran & A. Smeets, "Fibrations with few rational points", *Geom. Funct. Anal.* **26** (2016), no. 5, p. 1449-1482.

[9] D. Loughran, R. Takloo-Bighash & S. Tanimoto, "Zero-loci of Brauer group elements on semi-simple algebraic groups", *J. Inst. Math. Jussieu* **19** (2020), no. 5, p. 1467-1507.

[10] E. Peyre & Y. Tschinkel, "Tamagawa numbers of diagonal cubic surfaces, numerical evidence", *Math. Comput.* **70** (2001), no. 233, p. 367-387.

[11] B. Poonen & J. F. Voloch, "Random Diophantine equations", in *Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002)*, Progress in Mathematics, vol. 226, Birkhäuser, 2004, With appendices by Jean-Louis Colliot-Thélène and Nicholas M. Katz, p. 175-184.

[12] G. J. Rieger, "Über die Anzahl der als Summe von zwei Quadraten darstellbaren und in einer primen Restklasse gelegenen Zahlen unterhalb einer positiven Schranke. II", *J. Reine Angew. Math.* **217** (1965), p. 200-216.

[13] W. M. Schmidt, "Simultaneous rational zeros of quadratic forms", in *Seminar on Number Theory (Paris, 1980/1981)*, Progress in Mathematics, vol. 22, Birkhäuser, 1982, p. 281-307.

[14] J.-P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics, vol. 7, Springer, 1973, Translated from the French, viii+115 pages.

[15] ———, "Spécialisation des éléments de $\mathrm{Br}_2(\mathbf{Q}(T_1, \ldots, T_n))$", *C. R. Math. Acad. Sci. Paris* **311** (1990), no. 7, p. 397-402.

[16] C. M. Skinner, "Forms over number fields and weak approximation", *Compos. Math.* **106** (1997), no. 1, p. 11-29.

[17] E. Sofos, "Serre's problem on the density of isotropic fibres in conic bundles", *Proc. Lond. Math. Soc.* **113** (2016), no. 2, p. 261-288.

[18] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, third ed., Graduate Studies in Mathematics, vol. 163, American Mathematical Society, 2015, Translated from the 2008 French edition by Patrick D. F. Ion, xxiv+629 pages.

[19] E. Visse-Martindale, "Counting points on K3 surfaces and other arithmetic-geometric objects", PhD Thesis, Leiden University (Netherlands), 2019, `https://openaccess.leidenuniv.nl/handle/1887/67532`.

Manuscrit reçu le 26 mars 2019,
révisé le 10 octobre 2019,
accepté le 20 décembre 2019.

Efthymios SOFOS
The Mathematics and Statistics Building
University of Glasgow
University Place
Glasgow, G12 8QQ (Scotland)
efthymios.sofos@glasgow.ac.uk

Erik VISSE-MARTINDALE
Universiteit Leiden
Mathematisch Instituut
Niels Bohrweg 1, Leiden
2333 CA (Netherlands)
h.d.visse@math.leidenuniv.nl