Université Grenoble Alpes

ANNALES DE L'INSTITUT FOURIER

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Tome 71, nº 2 (2021), p. 843-848.
<http://aif.centre-mersenne.org/item/AIF_2021__71_2_843_0>

Ann. Inst. Fourier, Grenoble
71, 2 (2021) 843-848

ON THE UNIVERSAL REGULAR HOMOMORPHISM IN CODIMENSION 2

by Bruno KAHN

Abstract. — We point out a gap in Murre's proof of the existence of a universal regular homomorphism for codimension 2 cycles on a smooth projective variety, and offer two arguments to fill this gap.

Résumé. — On signale une lacune dans la preuve de l'existence d'un homomorphisme régulier universel pour les cycles de codimension 2 sur une variété projective lisse par Murre, et on donne deux arguments différents pour combler cette lacune.

In [11], Jacob Murre shows the existence of a universal regular homomorphism for algebraically trivial cycles of codimension 2 on a smooth projective variety over an algebraically closed field. This theorem has been largely used in the literature, most lately in [1, 7] and [3]; for example, it is essential in [3] for descending the method of Clemens and Griffiths [6] to non-algebraically closed fields, thus allowing Benoist and Wittenberg to obtain new examples of geometrically rational nonrational 3-folds.

Unfortunately its proof contains a gap, but fortunately this gap can be filled, actually by two different methods. This is the purpose of this note, which is a slight modification of a letter to Murre on December 5, 2018.

Recall the set-up, with the notation of [11]: V is a smooth projective variety over an algebraically closed field k and $A^n(V)$ denotes the group of codimension n cycles algebraically equivalent to 0 on V, modulo rational equivalence. Following Samuel, given an abelian k-variety A, a homomorphism

$$\phi : A^n(V) \to A(k)$$

is said to be *regular* if, for any pointed smooth projective k-variety (T, t_0) and any correspondence $Z \in CH^n(T \times V)$, the composition

$$T(k) \xrightarrow{w_Z} A^n(V) \xrightarrow{\phi} A(k) \tag{0.1}$$

Keywords: Algebraic cycles, abelian varieties.
2020 *Mathematics Subject Classification:* 14C25, 14K30.

is induced by a morphism $f : T \to A$; here w_Z is the composition

$$T(k) \to A_0(T) \xrightarrow{Z_*} A^n(V) \tag{0.2}$$

where the first map sends t to $[t] - [t_0]$. (Note that f is then unique, by Zariski density of the rational points in T.)

Using fancy language, regular homomorphisms from $A^n(V)$ form a category and a *universal regular homomorphism* is an initial object of this category, if it exists. This initial object is well-known to exist when $n = 0$, $n = 1$ (the Picard variety) and $n = \dim X$ (the Albanese variety). Murre's theorem is:

THEOREM 0.1 ([11, Theorem 1.9]). — *A universal regular homomorphism ϕ_0 exists when $n = 2$ for any V (of dimension $\geqslant 2$).*

Recall the main steps of his proof. First, given a regular homomorphism ϕ, its image in $A(k)$ is given by the points of some sub-abelian variety $A' \subseteq A$ [11, Lemma 1.6.2 (i)]. From this, one deduces [11, Proposition 2.1] that ϕ_0 exists if and only if $\dim A$ is bounded when ϕ runs through the *surjective* regular homomorphisms. Now, Murre's key idea is to bound $\dim A$ by the torsion of $A^2(V)$, which is controlled by the Merkurjev–Suslin theorem (Bloch's observation).

Let us elaborate a little on this point, to avoid the l-adic argument of loc. cit.: it suffices to prove that ϕ induces a surjection

$$A^2(V)\{l\} \twoheadrightarrow A(k)\{l\} \tag{0.3}$$

for some prime $l \neq \operatorname{char} k$, where $M\{l\}$ denotes the l-primary torsion of an abelian group M: indeed, $\operatorname{corank} A(k)\{l\} = 2 \dim A$. Mainly by Merkurjev–Suslin (Diagram in [11, Proposition 6.1])[(1)],

$$\operatorname{corank} CH^2(V)\{l\} \leqslant \operatorname{corank} H^3_{\text{ét}}(V, \mathbb{Q}_l/\mathbb{Z}_l(2)) \ (= b_3(V))$$

so the same holds *a fortiori* for $\operatorname{corank} A^2(V)\{l\}$.

Now, in [11, Lemma 1.6.2 (ii)], Murre constructs an abelian variety B (pointed at 0) and a correspondence $Z \in CH^2(B \times V)$ such that (0.1) is surjective for $T = B$. Since this map is induced by a morphism of abelian varieties sending 0 to 0 (hence a homomorphism), it restricts to a surjection

$$B\{l\} \twoheadrightarrow A\{l\}. \tag{0.4}$$

This allows me to explain

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(1) One could replace this diagram by the injection $CH^2(V) \hookrightarrow H^4_{\text{ét}}(V, \Gamma(2))$ of [9, Theorem 2.13 (c)], together with the surjection $H^3_{\text{ét}}(V, \mathbb{Q}_l/\mathbb{Z}_l(2)) \twoheadrightarrow H^4_{\text{ét}}(V, \Gamma(2))\{l\}$, cf. loc. cit., proof of Theorem 2.15; here, $\Gamma(2)$ is Lichtenbaum's complex.

The gap

A priori (0.4) does not imply (0.3), because w_Z is in general only a set-theoretic map, not a group homomorphism (see e.g. [4, Theorem 3.1 (a)]).

We now fix a surjective regular homomorphism ϕ as above. We shall give two ways to fill this gap:

(A) construct (B, Z) such that w_Z is a homomorphism;

(B) prove that w_Z always sends torsion to torsion.

(A) was my initial idea, and (B) was inspired by a discussion with Murre.

1. Explanation of (A)

We have

LEMMA 1.1. — *Take (T, t_0, z) with T of dimension 1 and $z \in CH^2(T \times V)$. Let $J = J(T)$ be the jacobian of T. Then the homomorphism $z_* : A_0(T) = J(k) \to A^2(V)$ is of the form w_α for some correspondence $\alpha \in CH^2(J \times V)$ (using $0 \in J(k)$ as base point).*

Proof. — Let g be the genus of T. Recall from [10, Example 3.12] the universal relative Cartier divisor D_{can} on $T \times T^{(g)}/T^{(g)}$, parametrising the effective divisors of degree g on T. It defines a correspondence $D_{\text{can}} : T^{(g)} \to T$. Composing with the graph of the birational map $J \dashrightarrow T^{(g)}$ inverse to $(t_1, \ldots, t_g) \mapsto \sum t_i - gt_0$, we find a (Chow) correspondence $D : J \to T$. I claim that $\alpha = z \circ D$ answers the question. Indeed, one checks immediately that the homomorphism

$$D_* : A_0(J) \to A_0(T)$$

is the Albanese morphism for J; hence the composition

$$J(k) \to A_0(J) \xrightarrow{D_*} A_0(T)$$

is the identity. □

Remark 1.2. — On the other hand, the morphism $T \to A$ given by the regularity of ϕ factors through a homomorphism

$$J(T) \to A. \qquad (1.1)$$

This homomorphism coincides with the one underlying $\phi \circ z_*$ in view of Lemma 1.1. Indeed, by uniqueness, it suffices to see that (1.1) induces $\phi \circ z_*$ on k-points; this is clear since $T(k)$ generates $J(T)(k)$ as an abelian group.

Consider all triples (T, t_0, z) with $\dim T = 1$. The homomorphism $\bigoplus A_0(T) \xrightarrow{(z_*)} A^2(V)$ is surjective, hence so is $\bigoplus A_0(T) \xrightarrow{(z_*)} A^2(V) \twoheadrightarrow A(k)$. As in Remark 1.2, each summand of this homomorphism is induced by a homomorphism $\rho_{T,\,t_0,\,z} : J(T) \to A$, so

$$B := \prod_{(T,\,t_0,\,z)\,\in\, S} J(T) \xrightarrow{(\rho_{T,\,t_0,\,z})} A$$

is surjective (faithfully flat) for a suitable finite set S. For each (T, t_0, z), let $\alpha = \alpha_z$ be a correspondence given by Lemma 1.1. Write $\pi_{T,\,t_0,z} : B \to J(T)$ for the canonical projection, viewed as an algebraic correspondence. The pair given by B and $Z = \sum_{(T,\,t_0,z)} \alpha_z \circ \pi_{T,\,t_0,z}$ yields (A).

2. Explanation of (B)

It suffices to show that the map

$$f : B(k) \to A_0(B)$$

sends l-primary torsion to l-primary torsion. Let $d = \dim B$. By Bloch's theorem [4, Theorem (0.1)], we have $A_0(B)^{*(d+1)} = 0$, where $*$ denotes Pontrjagin product. In other words, f has "degree $\leqslant d$" in the sense that its $(d+1)^{\text{st}}$ deviation [8, Section 8] is identically 0. It remains to show:

Lemma 2.1. — *Let $f : M \to N$ be a map of degree $\leqslant d$ between two abelian groups, such that $f(0) = 0$. Let $m_0 \in M$ be an element such that $am_0 = 0$ for some integer $a > 0$. Then*

$$a^{\binom{d+1}{2}} f(m_0) = 0.$$

Proof. — Induction on d. The case $d = 1$ is trivial. Assume $d > 1$. By hypothesis, the d^{th} deviation of f is multilinear, which implies that the map

$$g_a(m) = f(am) - a^d f(m)$$

is of degree $\leqslant d-1$. By induction, $a^{\binom{d}{2}} g_a(m_0) = 0$, hence the conclusion. □

Remark 2.2. — Of course, either argument proves more generally the following: the map $\phi : A^n(V)\{l\} \to A(k)\{l\}$ is surjective for any integer n, any surjective regular homomorphism $\phi : A^n(V) \to A(k)$ and any prime $l \neq \operatorname{char} k$.

Remark 2.3. — In [2, Section 6, Lemma and Proposition 11], Beauville gives a different proof that f sends torsion to torsion. Moreover, he observes that Roĭtman's theorem [13] then implies that the restriction of f to torsion *is actually an isomorphism, hence a homomorphism.*

If we apply Roĭtman's theorem together with Lemma 2.1, we obtain the following stronger result: *if $m, m_0 \in B(k)$ and m_0 is torsion, then* $f(m+m_0) = f(m) + f(m_0)$. (Fixing m, the map $f_m : m' \mapsto f(m+m') - f(m) - f(m')$ is of degree $< d$, hence $a^{\binom{d}{2}} f_m(m_0) = 0$ if $am_0 = 0$ by Lemma 2.1, and therefore $f_m(m_0) = 0$ by Roĭtman's theorem.)

3. Some expectation

The landmark work of Bloch and Esnault [5] yields the existence of 4-folds V over fields k of characteristic 0 such that the l-torsion of $A^3(V)$ is infinite (hence its l-primary torsion has infinite corank). One example, used by Rosenschon–Srinivas [14] and Totaro [16] and relying on Nori's theorem [12] and Schoen's results [15], is the following: start from the generic abelian 3-fold A, whose field of constants k_0 is finitely generated over $\mathbb{Q}$; choose an elliptic curve $E/k_0(t)$, not isotrivial with respect to k_0, and take $V = A_{k_0(t)} \times E$, $k =$ algebraic closure of $k_0(t)$.

Conjecture 3.1. — *For this V, a universal regular homomorphism on $A^3(V)$ does not exist.*

Ackowledgements

I am indebted to Jacob Murre for discussions around this problem, and for his encouragement to publish this note. I am also indebted to the referee for a careful reading, pointing out an incorrect earlier formulation of Lemma 1.1, as well as the reference to [2]. (The referee credits in turn Charles Vial for this reference.)

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Manuscrit reçu le 21 juin 2019,
révisé le 7 janvier 2020,
accepté le 13 mai 2020.

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