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A Galois D-groupoid for q-difference equations

A GALOIS D-GROUPOID FOR q-DIFFERENCE EQUATIONS

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ABSTRACT. — We first recall Malgrange's definition of D-groupoid and we define a Galois D-groupoid for q-difference equations. Then, we compute explicitly the Galois D-groupoid of a constant linear q-difference system, and show that it corresponds to the q-difference Galois group. Finally, we establish a conjugation between the Galois D-groupoids of two equivalent constant linear q-difference systems, and define a local Galois D-groupoid for Fuchsian linear q-difference systems by giving its realizations.

RÉSUMÉ. — Nous rappelons d'abord la définition de Malgrange de D-groupoïde et nous définissons un D-groupoïde de Galois pour les équations aux q-différences. Nous calculons ensuite le D-groupoïde de Galois d'un système aux q-différences linéaire à coefficients constants et montrons qu'il correspond au groupe de Galois aux q-différences. Nous établissons enfin une conjugaison entre les D-groupoïdes de Galois de deux systèmes aux q-différences linéaires à coefficients constants équivalents et définissons ainsi, en donnant ses réalisations, un D-groupoïde de Galois local pour les systèmes aux q-différences linéaires fuchsiens.

Introduction

The notion of D-groupoid was introduced and developed by B. Malgrange in [9], about ten years ago, in order to define, for non linear differential equations, an object which generalizes the differential Galois group of linear differential equations. Roughly speaking, a D-groupoid on a complex variety N is a groupoid of germs of local diffeomorphisms of N defined by a system of partial differential equations. In addition, Malgrange's exact definition allows to consider an algebraic and differential envelope of any groupoid of germs of local diffeomorphisms of N. Malgrange then defines the Galois groupoid of a foliation as the D-envelope of the holonomy

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groupoid, and proves in [9] that, for a linear differential equation, the classical differential Galois group can be recovered from the Galois groupoid of the associated foliation.

Moreover, this construction can be adapted to other dynamical systems. For example, it was considered by G. Casale in [1] for a germ of diffeomorphism of $(\mathbb{C}, 0)$. He computed the list of possible D-groupoids on a disk, and establishing conditions, in each case, on the considered germ of diffeomorphism to be a solution of the D-groupoid, he recovered a part of the Martinet-Ramis analytic classification of germs of diffeomorphisms, as presented in [11].

On the other hand, in the last twenty years, several authors have developed a Galois theory for linear q-difference equations, mainly from the Picard-Vessiot point of view (*cf e.g.* [12]), or using Tannakian tools (*cf e.g.* [14]). Malgrange's theory of D-groupoids gives a method to define a Galoisian object for non linear q-difference equations.

In section 1 of this paper, we recall Malgrange's definition of D-groupoid, and how it leads to define the Galois groupoid of a dynamical system. We illustrate this presentation in the q-difference equations's context, and give the definition of the Galois D-groupoid of a q-difference system.

In section 2, we compute explicitly the Galois D-groupoid of a constant linear q-difference system, and therefore obtain its complete description. We observe that it corresponds exactly to J. Sauloy's description in [14] of the q-difference Galois group of such a system.

Finally, in section 3, we establish a conjugation between the Galois D-groupoids of two equivalent constant linear q-difference systems. We explain how this result allows to define a local Galois D-groupoid for Fuchsian q-difference systems by giving its realizations. Moreover, we observe that we recover Sauloy's definition in [14] of the local Galois group of a Fuchsian q-difference system.

1. Malgrange's D-groupoids

This section recalls the definitions and results about D-groupoids needed throughout this text. We follow Malgrange's presentation in [9], and as in [5], we use an example coming from the q-difference equations's case as an illustration.

Since it will be more convenient to introduce notations and consider examples, we specialize the definitions to the particular base space $P^1\mathbb{C} \times \mathbb{C}^n$.

It is the complex variety where the Galois D-groupoid of a q-difference system of size $n \in \mathbb{N}^*$ is defined.

Fix $n \in \mathbb{N}^*$, and denote M the analytic complex variety $P^1\mathbb{C} \times \mathbb{C}^n$. We call *local diffeomorphism of M* any biholomorphism between two open sets of M, and denote $Aut(M)$ the set of germs of local diffeomorphisms of M.

1.1. D-varieties

Let $r \in \mathbb{N}$. Denote $|J_r^*(M, M)|$ the set of r-jets of local diffeomorphisms of M, that is the set of the equivalence classes of germs of local diffeomorphisms of M modulo the equivalence relation "to have the same Taylor expansion of order r at the source point". This set has a natural structure of analytic complex variety, and can be endowed with a system of global coordinates which represent respectively the r-jets's source point in M, their target point, and their partial derivatives of order $\leqslant r$ evaluated at the source point. Depending the need for synthesis or detail, we adopt different types of notation for these global coordinates. They are the following:

- the source:
$$y = (y_0, y_1, \ldots, y_n) = (z, X_1, \ldots, X_n) = (z, X),$$

- the target:
$$\bar{y} = (\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_n) = (\bar{z}, \bar{X}_1, \ldots, \bar{X}_n) = (\bar{z}, \bar{X}),$$

- the first partial derivatives evaluated at the source point:
$$\partial \bar{y} = \frac{\partial \bar{y}}{\partial y} = \left(\frac{\partial \bar{y}_i}{\partial y_j} \right) = \begin{pmatrix} \frac{\partial \bar{z}}{\partial z} & \frac{\partial \bar{z}}{\partial X_1} & \cdots & \frac{\partial \bar{z}}{\partial X_n} \\ \frac{\partial \bar{X}_1}{\partial z} & \frac{\partial \bar{X}_1}{\partial X_1} & \cdots & \frac{\partial \bar{X}_1}{\partial X_n} \\ \vdots & \vdots & \frac{\partial \bar{X}_i}{\partial X_j} & \vdots \\ \frac{\partial \bar{X}_n}{\partial z} & \frac{\partial \bar{X}_n}{\partial X_1} & \cdots & \frac{\partial \bar{X}_n}{\partial X_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial \bar{z}}{\partial z} & \frac{\partial \bar{z}}{\partial X} \\ \frac{\partial \bar{X}}{\partial z} & \frac{\partial \bar{X}}{\partial X} \end{pmatrix},$$

- the partial derivatives of order 2 to r evaluated at the source point:
$$\partial^2 \bar{y} = \frac{\partial^2 \bar{y}}{\partial y^2} = \ldots = \left(\frac{\partial^2 \bar{z}}{\partial z^2}, \frac{\partial^2 \bar{z}}{\partial z \partial X}, \frac{\partial^2 \bar{z}}{\partial X^2}, \frac{\partial^2 \bar{X}}{\partial z^2}, \frac{\partial^2 \bar{X}}{\partial z \partial X}, \frac{\partial^2 \bar{X}}{\partial X^2} \right) = \left(\partial^2 \bar{z}, \partial^2 \bar{X} \right),$$

$$\vdots$$

$$\partial^r \bar{y} = \frac{\partial^r \bar{y}}{\partial y^r} = \ldots = \left(\partial^r \bar{z}, \partial^r \bar{X} \right).$$

We also denote $\delta = \det(\frac{\partial \bar{y}}{\partial y})$ the coordinate which represents the Jacobian of a r-jet evaluated at the source point.

The set $|J_r^*(M,M)|$ of r-jets of local diffeomorphisms of M will be considered as an affine variety on M^2 in the sense of [8] and [10]. This structure is given by the ringed space denoted $J_r^*(M,M) = (M^2, \mathcal{O}_{J_r^*(M,M)})$, where the sheaf $\mathcal{O}_{J_r^*(M,M)}$ on the analytic complex variety M^2 is defined, for an open set Ω in M^2, by

$$\left(\mathcal{O}_{J_r^*(M,M)}\right)_{|\Omega} = (\mathcal{O}_{M^2})_{|\Omega}\left[\frac{\partial \bar{y}}{\partial y}, \delta^{-1}, \frac{\partial^2 \bar{y}}{\partial y^2}, \ldots, \frac{\partial^r \bar{y}}{\partial y^r}\right].$$

Thus, the local sections of the sheaf $\mathcal{O}_{J_r^*(M,M)}$ are functions of the form

$$E(y, \bar{y}, \frac{\partial \bar{y}}{\partial y}, \delta^{-1}, \frac{\partial^2 \bar{y}}{\partial y^2}, \ldots, \frac{\partial^r \bar{y}}{\partial y^r}),$$

depending holomorphically on the coordinates y and $\bar{y}$, and polynomially on the coordinates $\partial \bar{y}, \delta^{-1}, \partial^2 \bar{y}, \ldots, \partial^r \bar{y}$. These local sections are called *equations*, or *partial differential equations*.

One then defines the D-variety of jets of local diffeomorphisms of M as the ringed space $J^*(M,M) = (M^2, \mathcal{O}_{J^*(M,M)})$, where the structural sheaf is the direct limit $\mathcal{O}_{J^*(M,M)} = \varinjlim \mathcal{O}_{J_r^*(M,M)}$ endowed with the natural derivations with respect to the source coordinates defined by

$$D_{y_i}.E = \frac{\partial E}{\partial y_i} + \sum_{j=0}^{n}\sum_{|\alpha|\geqslant 0} \frac{\partial E}{\partial\left(\frac{\partial^{|\alpha|}\bar{y}_j}{\partial y^\alpha}\right)}\frac{\partial^{|\alpha|+1}\bar{y}_j}{\partial y_i \partial y^\alpha}.$$

A D-subvariety of $J^*(M,M)$ is essentially a ringed space of the form $(M^2, \mathcal{O}_{J^*(M,M)}/\mathcal{I})$, with $\mathcal{I}$ an ideal (*i.e.* a sheaf of ideals) of $\mathcal{O}_{J^*(M,M)}$ such that:

- the ideal $\mathcal{I}$ is pseudo-coherent in the sense of [9] and [10], *i.e.* the ideals $\mathcal{I}_r = \mathcal{I} \cap \mathcal{O}_{J_r^*(M,M)}$ are coherent,

- the ideal $\mathcal{I}$ is differential, *i.e.* stable by the derivations D_{y_i}.

A *solution* of an ideal $\mathcal{I}_r \subset \mathcal{O}_{J_r^*(M,M)}$ (resp. $\mathcal{I} \subset \mathcal{O}_{J^*(M,M)}$) is a germ of a local diffeomorphism $g : (M,a) \to (M,b)$ from a neighbourhood of a in M to a neighbourhood of b in M with $g(a) = b$, such that, for any equation E of the stalk $(\mathcal{I}_r)_{(a,b)}$ (resp. $\mathcal{I}_{(a,b)}$), the function defined by

$$E.g : y \mapsto E(y, g(y), \frac{\partial g}{\partial y}(y), \det(\frac{\partial g}{\partial y}(y))^{-1}, \frac{\partial^2 g}{\partial y^2}(y), \ldots, \frac{\partial^r g}{\partial y^r}(y))$$

is zero in a neighbourhood of a in M. We denote $sol(\mathcal{I}_r)$ (resp. $sol(\mathcal{I})$) the set of solutions of $\mathcal{I}_r$ (resp. $\mathcal{I}$). It is a subset of $Aut(M)$.

Consider for example the coherent sheaf of ideals

$$\mathcal{Q}_2 = \left\langle \frac{\partial \bar{z}}{\partial X}, \frac{\partial \bar{z}}{\partial z} z - \bar{z}, \partial^2 \bar{z} \right\rangle = \left\langle \frac{\partial \bar{z}}{\partial X}, \frac{\partial \bar{z}}{\partial z} z - \bar{z}, \frac{\partial^2 \bar{z}}{\partial z \partial X}, \frac{\partial^2 \bar{z}}{\partial X^2}, \frac{\partial^2 \bar{z}}{\partial z^2} \right\rangle$$

of $\mathcal{O}_{J_2^*(M,M)}$, whose the generators are considered as global sections of $\mathcal{O}_{J_2^*(M,M)}$. It generates, in the differential sheaf $\mathcal{O}_{J^*(M,M)}$, the following differential and pseudo-coherent ideal:

$$\mathcal{Q}' = \left\langle \frac{\partial \bar{z}}{\partial X}, \frac{\partial \bar{z}}{\partial z} z - \bar{z}, \partial^{2+l} \bar{z}; l \geqslant 0 \right\rangle.$$

Since the ideal $\mathcal{Q}'$ is obtained from $\mathcal{Q}_2$ by derivation, these two ideals have the same set of solutions, denoted $sol(\mathcal{Q}_2)$ or $sol(\mathcal{Q}')$, which is the set of germs of local diffeomorphisms of M of the form $(z, X) \mapsto (\alpha z, G(z, X))$, with $\alpha \in \mathbb{C}^*$ and G any local map.

1.2. D-groupoids

The set $Aut(M)$ has a natural groupoid structure, given by the following maps[1]:

- the source projection:

$$s : Aut(M) \to M$$
$$_a[g]_{g(a)} \mapsto a$$

- the target projection:

$$t : Aut(M) \to M$$
$$_a[g]_{g(a)} \mapsto g(a)$$

- the composition, defined on the set $(Aut(M), t) \times_M (Aut(M), s)$ of pairs of germs (g, h) such that $t(g) = s(h)$:

$$c : (Aut(M), t) \times_M (Aut(M), s) \to Aut(M)$$
$$_a[g]_{g(a)}, \quad _b[h]_{h(b)} \mapsto {}_a[h \circ g]_{h(g(a))}$$

- the identity map:

$$e : M \to Aut(M)$$
$$a \mapsto {}_a[id]_a$$

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[1] They are made explicit using the notation $_a[g]_b$ to designate the germ of a local diffeomorphism g of M at a source point a, with target point $b = g(a)$.

- the inversion:

$$i : Aut(M) \to Aut(M)$$

$$_a[g]_{g(a)} \mapsto_{g(a)} [g^{-1}]_a$$

A subgroupoid of $Aut(M)$ is a subset to which these maps can be restricted. More explicitly, it is a subset of $Aut(M)$ stable by composition c of the "compatible" germs, and by inversion i of any germ, and which contains all the germs of the identity map of M, so that the base space M is not restricted. For example, the set $sol(\mathcal{Q}_2) \subset Aut(M)$ of germs of the form

$$(z, X) \mapsto (\alpha z, G(z, X)), \quad \text{with } \alpha \in \mathbb{C}^*,$$

is a subgroupoid of $Aut(M)$.

The groupoid structure of $Aut(M)$ is inherited by projection by the sets $|J_r^*(M, M)|$, and induces an additive natural structure on the structural sheaf of the D-variety $J^*(M, M)$. This might be compared with the Hopf algebra structure naturally defined on the affine algebra of an affine algebraic group. However, in the case of groupoids, we deal with sheaves of equations, and we have to take into account the compatibility condition in the definition of the map c of composition.

The groupoid structure of the affine variety $J_r^*(M, M)$ is given by the following comorphisms[2]:

- the cosource:

$$s^* : \mathcal{O}_M \to s_* \mathcal{O}_{J_r^*(M,M)}$$

$$f \mapsto f \circ s$$

$$f \mapsto f(y)$$

- the cotarget:

$$t^* : \mathcal{O}_M \to t_* \mathcal{O}_{J_r^*(M,M)}$$

$$f \mapsto f \circ t$$

$$f \mapsto f(\bar{y})$$

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[2] Attaching the notation * as an exponent designates an associated comorphism, and attaching the notation $_*$ as a subscript designates the direct image of a sheaf by the considered morphism.

- the cocomposition[(3)]:

$$c^* : \mathcal{O}_{J_r^*(M,M)} \to s_* \mathcal{O}_{J_r^*(M,M)} \widehat{\otimes}_{\mathcal{O}_M(M)} t_* \mathcal{O}_{J_r^*(M,M)}$$

$$E \mapsto E \circ c$$

$$E\left(y, \bar{y}, \frac{\partial \bar{y}}{\partial y}, \delta^{-1}, \frac{\partial^2 \bar{y}}{\partial y^2}, \dots, \frac{\partial^r \bar{y}}{\partial y^r}\right) \mapsto E\left(y, \bar{\bar{y}}, \frac{\partial \bar{\bar{y}}}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial y}, \det\left(\frac{\partial \bar{\bar{y}}}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial y}\right)^{-1}, \dots\right)$$

- the coidentity:

$$e^* : \mathcal{O}_{J_r^*(M,M)} \to e_* \mathcal{O}_M$$

$$E \mapsto E \circ e$$

$$E\left(y, \bar{y}, \frac{\partial \bar{y}}{\partial y}, \delta^{-1}, \frac{\partial^2 \bar{y}}{\partial y^2}, \dots, \frac{\partial^r \bar{y}}{\partial y^r}\right) \mapsto E\left(y, y, Id_{n+1}, 1, 0, \dots, 0\right)$$

- the coinversion:

$$i^* : \mathcal{O}_{J_r^*(M,M)} \to i_* \mathcal{O}_{J_r^*(M,M)}$$

$$E \mapsto E \circ i$$

$$E\left(y, \bar{y}, \frac{\partial \bar{y}}{\partial y}, \delta^{-1}, \frac{\partial^2 \bar{y}}{\partial y^2}, \dots, \frac{\partial^r \bar{y}}{\partial y^r}\right) \mapsto E\left(\bar{y}, y, \left(\frac{\partial \bar{y}}{\partial y}\right)^{-1}, \delta, \dots\right)$$

A subgroupoid of $J_r^*(M, M)$ is essentially a ringed space of the form $(M^2, \mathcal{O}_{J_r^*(M,M)}/\mathcal{G}_r)$, with $\mathcal{G}_r$ a coherent ideal of $\mathcal{O}_{J_r^*(M,M)}$ which satisfies the following three conditions:

(i)

$$c^* \mathcal{G}_r \subset (s_* \mathcal{G}_r) \widehat{\otimes}_{\mathcal{O}_M(M)} \left(t_* \mathcal{O}_{J_r^*(M,M)}\right) + \left(s_* \mathcal{O}_{J_r^*(M,M)}\right) \widehat{\otimes}_{\mathcal{O}_M(M)} \left(t_* \mathcal{G}_r\right)$$

which will also be (improperly) written:

$$c^* \mathcal{G}_r \subset \mathcal{G}_r \widehat{\otimes} \mathcal{O}_{J_r^*(M,M)} + \mathcal{O}_{J_r^*(M,M)} \widehat{\otimes} \mathcal{G}_r,$$

(ii)

$$\mathcal{G}_r \subset \ker e^*,$$

(iii)

$$\iota^* \mathcal{G}_r \subset \mathcal{G}_r,$$

so that the quotient sheaf $\mathcal{O}_{J_r^*(M,M)}/\mathcal{G}_r$ is endowed by the comorphisms induced by $s^*, t^*, c^*, e^*,$ and i^*. The conditions *(i), (ii)*, and *(iii)* force the solutions $sol(\mathcal{G}_r)$ to be a subgroupoid of $Aut(M)$. For example, we check in Appendix A, going into details, that the coherent ideal $\mathcal{Q}_2 \subset \mathcal{O}_{J_2^*(M,M)}$ satisfies them, and defines therefore

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[(3)] We use the symbol $\widehat{\otimes}_{\mathcal{O}_M(M)}$ to designate the total tensor product, as defined in [4], of two sheaves on the ring $\mathcal{O}_M(M)$.

a subgroupoid of $J_2^*(M, M)$. In addition, it has already been mentioned that the solutions $sol(\mathcal{Q}_2)$ form a subgroupoid of $Aut(M)$.

A *D-groupoid on M* is, according to Malgrange's definition in [9], a reduced, differential, and pseudo-coherent ideal $\mathcal{G}$ of $\mathcal{O}_{J^*(M,M)}$ such that:

(i') for any relatively compact open set U of M, there exists a close complex analytic subvariety Z of U of codimension $\geqslant 1$, and a positive integer $r_0 \in \mathbb{N}$ such that, above $(U \setminus Z)^2$, and for all $r \geqslant r_0$, one has:

$$c^* \mathcal{G}_{r|(U\setminus Z)^2} \subset \mathcal{G}_{r|(U\setminus Z)^2} \, \widehat{\otimes} \, \mathcal{O}_{J_r^*(U\setminus Z, U\setminus Z)} + \mathcal{O}_{J_r^*(U\setminus Z, U\setminus Z)} \, \widehat{\otimes} \, \mathcal{G}_{r|(U\setminus Z)^2}$$

(ii') for all $r \in \mathbb{N}$, one has:

$$\mathcal{G}_r \subset \ker e^*$$

(iii') for all $r \in \mathbb{N}$, one has:

$$i^* \mathcal{G}_r \subset \mathcal{G}_r$$

These conditions, combined with the analytic continuation theorem, force the solutions $sol(\mathcal{G})$ to be a subgroupoid of $Aut(M)$. We insist on the fact that the terminology of *D-groupoid* designates an ideal of equations, whereas this object has to be interpreted from the point of view of solutions.

The flexibility introduced by Malgrange in his definition of *D-groupoid* is made, on one hand, to take into account the natural situation described by the following theorem:

THEOREM 1.1 (Malgrange, [9]). — *Let $\mathcal{G}_r$ be a coherent ideal of $\mathcal{O}_{J_r^*(M,M)}$ which defines a subgroupoid of $J_r^*(M, M)$, i.e. which satisfies the conditions (i), (ii), and (iii). Then, the reduced differential ideal of $\mathcal{O}_{J^*(M,M)}$ generated by $\mathcal{G}_r$, which we denote by $\sqrt{\mathcal{G}'}$, is a D-groupoid on M.*

This theorem allows to produce examples of *D-groupoids*. Actually, it implies that the radical $\sqrt{\mathcal{Q}'}$ of the differential ideal $\mathcal{Q}' \subset \mathcal{O}_{J^*(M,M)}$ is a *D-groupoid* on M. Moreover, Malgrange also proves in [9] that, generically, any *D-groupoid* on M comes from such a situation.

1.3. *D*-envelopes

Malgrange's definition of *D-groupoid* is also especially designed so that the following theorem holds:

THEOREM 1.2 (Malgrange, [9]). — *Let J be any set. For a family of D-groupoids $\mathcal{G}^j \subset \mathcal{O}_{J^*(M,M)}$, with $j \in J$, the reduced differential ideal $\sqrt{\sum_{j \in J} \mathcal{G}^j} \subset \mathcal{O}_{J^*(M,M)}$ is also a D-groupoid on M.*

Such a D-groupoid is called, according to [9], the *intersection* of the D-groupoids $\mathcal{G}^j$. This terminology has to be understood from the point of view of solutions because of the equality $sol(\sqrt{\sum \mathcal{G}^j}) = (\cap sol(\mathcal{G}^j))$.

Then, this theorem allows us to define the D-envelope of any subgroupoid H of $Aut(M)$: it is the intersection of the D-groupoids on M whose solutions contain H. Hence, denoting $\mathcal{G}^j$ the D-groupoids on M such that $H \subset sol(\mathcal{G}^j)$, the D-envelope of H is the D-groupoid $\sqrt{\sum \mathcal{G}^j}$; and because of the inclusions $H \subset sol(\sqrt{\sum \mathcal{G}^j}) \subset sol(\mathcal{G}^j)$, one also says that the D-envelope of H is the *least* D-groupoid on M whose solutions contain H.

The first fundamental example, which was Malgrange's motivation to develop this D-groupoid notion, is the definition of the Galois groupoid of a foliation. According to [9], the Galois groupoid of a differential system is defined as the D-envelope of the dynamics of the system, that is, more precisely, the intersection of the D-groupoids on the phase space whose solutions contain the holonomy transformations.

Two discrete examples were then considered by Casale. He studies the Galois groupoid of a germ of local diffeomorphism of $(\mathbb{C}, 0)$ in [1], and the Galois groupoid of a rational map of $P^1\mathbb{C}$ in [2]. The two constructions can be summarized by a more general situation : let $\Phi : N \mapsto N$ be a generically locally invertible analytic map on a complex variety N, then, the Galois groupoid of Φ is the D-envelope of the subgroupoid of $Aut(N)$ generated by the germs of Φ at any point of N where it is well defined and invertible.

1.4. A Galois D-groupoid for q-difference systems

Inspired by the above examples, we shall now define a Galois D-groupoid for q-difference systems. Fix $q \in \mathbb{C}^*$, consider $X(qz) = F(z, X(z))$ a rational q-difference system of size n, *i.e.* with $F(z, X) \in \mathbb{C}(z, X)^n$, and remember that $M = P^1\mathbb{C} \times \mathbb{C}^n$.

Define $Dyn(F)$ as the subgroupoid of $Aut(M)$ generated by the germs of $(z, X) \mapsto (qz, F(z, X))$ at any point of M where this map is well defined and invertible. We note that the local diffeomorphisms of $Dyn(F)$ are the natural maps which preserve the graph in M of each solution of the q-difference system $X(qz) = F(z, X(z))$.

DEFINITION 1.3. — *We define the Galois D-groupoid $\mathcal{G}al(F)$ of the q-difference system $X(qz) = F(z, X(z))$ as the D-envelope on M of the dynamics $Dyn(F)$.*

Therefore, the Galois D-groupoid $\mathcal{G}al(F)$ is the intersection of the D-groupoids on M whose solutions contain $Dyn(F)$. It is also the least D-groupoid on M whose solutions contain $Dyn(F)$. A family of examples will be computed in the next section.

To finish, explain the significance, in the context of q-difference equations, of the running example we used to illustrate Malgrange's definition of D-groupoid. On the one hand, the dynamics $Dyn(F)$ of any q-difference system $X(qz) = F(z, X(z))$ is the subgroupoid of $Aut(M)$ generated the germs of $(z, X) \mapsto (qz, F(z, X))$. Therefore, all its elements are necessarily of the form $(z, X) \mapsto (q^k z, \star)$, with $k \in \mathbb{Z}$. On the other hand, we recall that the solutions of the D-groupoid $\sqrt{\mathcal{Q}'}$, defined in subsection 1.2, are the germs of local diffeomorphisms of the form $(z, X) \mapsto (\alpha z, G(z, X))$, with $\alpha \in \mathbb{C}^*$, and G any local map. Thus, we have the inclusion $Dyn(F) \subset sol(\sqrt{\mathcal{Q}'})$. By Definition 1.3 of the Galois D-groupoid $\mathcal{G}al(F)$, one necessary gets the inclusions $\sqrt{\mathcal{Q}'} \subset \mathcal{G}al(F)$, and $Dyn(F) \subset sol(\mathcal{G}al(F)) \subset sol(\sqrt{\mathcal{Q}'})$. Considering the point of view of solutions, we say that the D-groupoid $\sqrt{\mathcal{Q}'}$ is an *upper* bound of the Galois D-groupoid $\mathcal{G}al(F)$.

The aim of the two following sections of this text is to provide justifications for Definition 1.3, proving that, for some classes of linear q-difference systems, the q-difference Galois group can be recovered from the Galois D-groupoid.

2. The Galois D-groupoid of a constant linear q-difference system

Fix $q \in \mathbb{C}^*$, consider $X(qz) = AX(z)$ a constant linear q-difference system of rank n defined by an invertible constant matrix $A \in GL_n(\mathbb{C})$, and recall that $M = P^1\mathbb{C} \times \mathbb{C}^n$. We are going to compute explicitly the Galois D-groupoid $\mathcal{G}al(A)$ of the q-difference system $X(qz) = AX(z)$, and to show that for such a system, the Galois D-groupoid and the q-difference Galois group correspond exactly.

2.1. Calculation

Recall first that the dynamics $Dyn(A)$ of the system $X(qz) = AX(z)$ is the subgroupoid of $Aut(M)$ generated by the germs of the global diffeomorphism $(z, X) \mapsto (qz, AX)$ of M. Therefore, the elements of $Dyn(A)$ are the germs at any point of M of the diffeomorphisms $(z, X) \mapsto (q^k z, A^k X)$, for all $k \in \mathbb{Z}$.

Step 1. — Note that the elements of $Dyn(A)$ have a particular shape which can be expressed in terms of partial differential equations: their first (resp. second) component is independent on the variables X (resp. the variable z) and depends linearly on the variable z (resp. the variables X). Hence, consider the ideal:

$$\mathcal{C}onst_2 = \left\langle \frac{\partial \bar{z}}{\partial X}, \partial^2 \bar{z}, \frac{\partial \bar{z}}{\partial z}z - \bar{z}, \frac{\partial \bar{X}}{\partial z}, \partial^2 \bar{X}, \frac{\partial \bar{X}}{\partial X}X - \bar{X} \right\rangle \subset \mathcal{O}_{J_2^*(M,M)}.$$

One can check, extending exactly what is done in Appendix A for the ideal $\mathcal{Q}_2$, that this coherent ideal defines a subgroupoid of $J_2^*(M, M)$, *i.e.* satisfies the conditions *(i)*, *(ii)*, and *(iii)* of 1.2. Thus, by Theorem 1.1, it generates a D-groupoid $\sqrt{\mathcal{C}onst'}$. The solutions are given by:

$$sol(\sqrt{\mathcal{C}onst'}) = \{(z, X) \mapsto (\alpha z, \beta X) \, ; \, \alpha \in \mathbb{C}^*, \beta \in GL_n(\mathbb{C})\}.$$

They obviously contain the elements of $Dyn(A)$. Consequently, by Definition 1.3 of $\mathcal{G}al(A)$, one gets the inclusion $\sqrt{\mathcal{C}onst'} \subset \mathcal{G}al(A)$, and, from the point of view of solutions, the inclusions

$$Dyn(A) \subset sol(\mathcal{G}al(A)) \subset sol(\sqrt{\mathcal{C}onst'}).$$

The D-groupoid $\sqrt{\mathcal{C}onst'}$ is an upper bound of the Galois D-groupoid $\mathcal{G}al(A)$.

Step 2. — Comparing the solutions $sol(\sqrt{\mathcal{C}onst'})$ with the elements of the dynamics $Dyn(A)$, we remark that if the Galois D-groupoid $\mathcal{G}al(A)$ contained more equations than the bound $\sqrt{\mathcal{C}onst'}$, the additional equations would give constraints on the first partial derivatives. Hence, consider the algebraic subgroup $Jac \subset GL_{n+1}(\mathbb{C})$ generated by the Jacobian matrices $diag(q^k, A^k) = \begin{pmatrix} q^k & 0 \\ 0 & A^k \end{pmatrix}$ of the elements of $Dyn(A)$, denote $I(Jac)$ the

ideal of the Hopf algebra $\mathbb{C}[T_{i,j}, \Delta^{-1}]$ of $GL_{n+1}(\mathbb{C})$ which defines the algebraic subgroup Jac, and consider the following morphism τ of $\mathbb{C}$-algebras:

$$\mathbb{C}[T_{i,j}, \Delta^{-1}] \to \Gamma(M^2, \mathcal{O}_{J_2^*(M,M)})$$

$$\begin{pmatrix} T_{0,0} & T_{0,1} & \cdots & T_{0,n} \\ T_{1,0} & & & \\ \vdots & & (T_{i,j})_{i,j} & \\ T_{n,0} & & & \end{pmatrix} \mapsto \begin{pmatrix} \frac{\partial \bar{z}}{\partial z} & \frac{\partial \bar{z}}{\partial X_1} & \cdots & \frac{\partial \bar{z}}{\partial X_n} \\ \frac{\partial \bar{X}_1}{\partial z} & & & \\ \vdots & & (\frac{\partial \bar{X}_i}{\partial X_j})_{i,j} & \\ \frac{\partial \bar{X}_n}{\partial z} & & & \end{pmatrix},$$

between the polynomial equations on $GL_{n+1}(\mathbb{C})$, and the global partial differential equations of $\mathcal{O}_{J_2^*(M,M)}$. The equations of the image $\tau(I(Jac))$ are satisfied by the elements of $Dyn(A)$. Hence, consider the ideal:

$$\mathcal{G}(A)_2 = \left\langle \frac{\partial \bar{z}}{\partial X}, \partial^2 \bar{z}, \frac{\partial \bar{z}}{\partial z} z - \bar{z}, \frac{\partial \bar{X}}{\partial z}, \partial^2 \bar{X}, \frac{\partial \bar{X}}{\partial X} X - \bar{X}, \tau(I(Jac)) \right\rangle$$

of $\mathcal{O}_{J_2^*(M,M)}$. It defines a subgroupoid of $J_2^*(M,M)$ because the ideal $Const_2$ also does, and the ideal $I(Jac)$ defines a subgroup of $GL_{n+1}(\mathbb{C})$. Thus, it generates a D-groupoid $\sqrt{\mathcal{G}(A)'}$ whose solutions are given by:

$$sol(\sqrt{\mathcal{G}(A)'}) = \{(z,X) \mapsto (\alpha z, \beta X) \,;\, diag(\alpha, \beta) \in Jac\}.$$

They obviously contain the elements of $Dyn(A)$. Therefore, one has the inclusions of ideals $\sqrt{Const'} \subset \sqrt{\mathcal{G}(A)'} \subset \mathcal{G}al(A)$, and, from the point of view of solutions, the inclusions $Dyn(A) \subset sol(\mathcal{G}al(A)) \subset sol(\sqrt{\mathcal{G}(A)'}) \subset sol(\sqrt{Const'})$. It means that the D-groupoid $\sqrt{\mathcal{G}(A)'}$ is a better upper bound than $\sqrt{Const'}$ of the Galois D-groupoid $\mathcal{G}al(A)$.

Step 3. — We are going to prove the equality $\sqrt{\mathcal{G}(A)'} = \mathcal{G}al(A)$. Fix $(a,b) \in M^2$, and let E be an equation of the fiber $\mathcal{G}al(A)_{(a,b)}$. We can assume, up to multiply the equation E by a positive power of the invertible equation δ, that the equation E, of order denoted r, is of the form:

$$E\left(z, X, \bar{z}, \bar{X}, \frac{\partial \bar{z}}{\partial z}, \frac{\partial \bar{z}}{\partial X}, \frac{\partial \bar{X}}{\partial z}, \frac{\partial \bar{X}}{\partial X}, \partial^2 \bar{z}, \partial^2 \bar{X}, \ldots, \partial^r \bar{z}, \partial^r \bar{X}\right),$$

and depends holomorphically on the source and target variables, and polynomially on the partial derivatives variables.

Perform successively the Euclidean divisions of E, and then of its successive remainders, considered as polynomials in the partial derivatives variables, by the monic polynomials $\frac{\partial \bar{z}}{\partial X}, \frac{\partial \bar{X}}{\partial z}, \partial^2 \bar{z}, \partial^2 \bar{X}, \ldots, \partial^r \bar{z}, \partial^r \bar{X}$, which are equations of the differential ideal $\sqrt{\mathcal{G}(A)'}$. We get, as a remainder, an equation $E_1(z, X, \bar{z}, \bar{X}, \frac{\partial \bar{z}}{\partial z}, \frac{\partial \bar{X}}{\partial X}) \in \mathcal{G}al(A)_{(a,b)}$ which depends only on the variables $z, X, \bar{z}, \bar{X}, \frac{\partial \bar{z}}{\partial z}, \frac{\partial \bar{X}}{\partial X}$, and such that $E_1 \equiv E \mod \sqrt{\mathcal{G}(A)'}_{(a,b)}$. Then,

using the Weierstrass preparation theorem, *e.g.* as stated in [6], perform the successive divisions of E_1, and of its remainders, by the equations $\bar{z} - \frac{\partial \bar{z}}{\partial z} z$ and $\bar{X} - \frac{\partial \bar{X}}{\partial X} X$. We get, as a remainder, an equation $E_2(z, X, \frac{\partial \bar{z}}{\partial z}, \frac{\partial \bar{X}}{\partial X}) \in \mathcal{G}al(A)_{(a,b)}$ such that $E_2 \equiv E_1 \mod \sqrt{\mathcal{G}(A)'}_{(a,b)}$.

The equation E_2 is satisfied by the elements of $Dyn(A)$. So, for all point (z_0, X_0) in a neighbourhood of a in M, and for all $k \in \mathbb{Z}$, one has: $E_2(z_0, X_0, q^k, A^k) = 0$, and so $E_2(z_0, X_0, T_{0,0}, T_{i,j}) \in I(Jac)$. By Lemma 3.6, stated and proved in Appendix B, we deduce that

$$E_2 \left(z, X, \frac{\partial \bar{z}}{\partial z}, \frac{\partial \bar{X}}{\partial X} \right) \in (\mathcal{G}(A)_2)_{(a,b)},$$

and so $E \in \sqrt{\mathcal{G}(A)'}_{(a,b)}$.

We have proved the following theorem:

THEOREM 2.1. — *The Galois D-groupoid $\mathcal{G}al(A)$ of the constant linear q-difference system $X(qz) = AX(z)$ is the D-groupoid on $P^1\mathbb{C} \times \mathbb{C}^n$ generated by the following coherent ideal of $\mathcal{O}_{J_2^*(M,M)}$:*

$$\left\langle \frac{\partial \bar{z}}{\partial X}, \partial^2 \bar{z}, \frac{\partial \bar{z}}{\partial z} z - \bar{z}, \frac{\partial \bar{X}}{\partial X}, \partial^2 \bar{X}, \frac{\partial \bar{X}}{\partial X} X - \bar{X}, \tau(I(Jac)) \right\rangle,$$

which defines a subgroupoid of $J_2^*(M, M)$. Its solutions $sol(\mathcal{G}al(A))$ are the germs of the following global diffeomorphisms:

$$\{(z, X) \mapsto (\alpha z, \beta X) \, ; \, \alpha \in \mathbb{C}^*, \beta \in GL_n(\mathbb{C}), diag(\alpha, \beta) \in Jac\} .$$

2.2. Correspondence with the q-difference Galois group

Suppose now, and for all the following, that $|q| > 1$. To compare the Galois D-groupoid and the q-difference Galois group of the constant linear q-difference system $X(qz) = AX(z)$, we are going to use a complete description of these two objects, and to observe that they correspond exactly.

On the one hand, by Theorem 2.1 above, the Galois D-groupoid $\mathcal{G}al(A)$ of $X(qz) = AX(z)$ is essentially determined by the algebraic group Jac. This group is generated by only one matrix, and therefore, its description is known and given by the following proposition:

PROPOSITION 2.2. — *The algebraic group Jac is completely described by:*

$$Jac = \left\{ \begin{pmatrix} \gamma(q) & 0 \\ 0 & \gamma(A_s) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A_u^\lambda \end{pmatrix} ; (\gamma, \lambda) \in Hom_{gp}(\mathbb{C}^*, \mathbb{C}^*) \times \mathbb{C} \right\},$$

where A_s and A_u are respectively the semi-simple and unipotent parts of A in its multiplicative Jordan decomposition, and the matrix $\gamma(A_s)$ is defined by $\gamma(A_s) = P^{-1}\gamma(D)P$ for any diagonalization $A_s = P^{-1}DP$.

COROLLARY 2.3. — *The solutions $sol(\mathcal{G}al(A))$ of the Galois D-groupoid $\mathcal{G}al(A)$ are the germs of global diffeomorphisms of M of the form:*

$$(z, X) \mapsto (\gamma(q)z, \gamma(A_s)A_u^\lambda X),$$

with $(\gamma, \lambda) \in Hom_{gp}(\mathbb{C}^*, \mathbb{C}^*) \times \mathbb{C}$.

On the other hand, Sauloy defines in [14], using Tannakian tools, a Galois group for constant linear q-difference systems. He endows the neutral Tannakian category over $\mathbb{C}$ of these systems with a family of fiber functors indexed by $\mathbb{C}^*$. By Tannaka duality, this gives rise to a Galois set groupoid, indexed by $\mathbb{C}^*$, associated to the constant linear q-difference system $X(qz) = AX(z)$. It is computed in [14]. The set of its arrows between two base points $z_0, z_1 \in \mathbb{C}^*$ is:

$$Gal(A)(z_0, z_1) = \left\{\gamma(A_s)A_u^\lambda \,;\, (\gamma, \lambda) \in Hom_{gp}(\mathbb{C}^*, \mathbb{C}^*) \times \mathbb{C}, \gamma(q)z_0 = z_1\right\},$$

and, in particular, the Galois group is given by $Gal(A)(z_0, z_0)$, for any point $z_0 \in \mathbb{C}^*$, that is:

$$Gal(A) = \left\{\gamma(A_s)A_u^\lambda \,;\, (\gamma, \lambda) \in Hom_{gp}(\mathbb{C}^*, \mathbb{C}^*) \times \mathbb{C}, \gamma(q) = 1\right\}.$$

Comparing Sauloy's description with the statement of Corollary 2.3, we consider first, for any $z_0, z_1 \in \mathbb{C}^*$, the solutions of $\mathcal{G}al(A)$ which transform the generic transversal line $\{z_0\} \times \mathbb{C}^n$ into $\{z_1\} \times \mathbb{C}^n$, and denote them:

$$sol(\mathcal{G}al(A))_{z_0, z_1} = \left\{(z, X) \mapsto (\frac{z_1}{z_0}z, \gamma(A_s)A_u^\lambda X) \,;\, \gamma(q)z_0 = z_1\right\}.$$

We consider then the solutions of $\mathcal{G}al(A)$ which preserve the transversal lines $\{z\} \times \mathbb{C}^n$, for all $z \in P^1\mathbb{C}$, and we denote them:

$$sol(\mathcal{G}al(A))_z = \left\{(z, X) \mapsto (z, \gamma(A_s)A_u^\lambda X) \,;\, \gamma(q) = 1\right\}.$$

We obtain the following theorem:

THEOREM 2.4. — *The correspondence between the Galois D-groupoid $\mathcal{G}al(A)$, and the q-difference Galois group $Gal(A)$ of the constant linear q-difference system $X(qz) = AX(z)$ is given by the map:*

$$\left[(z, X) \mapsto (\gamma(q)z, \gamma(A_s)A_u^\lambda X)\right] \mapsto \gamma(A_s)A_u^\lambda,$$

which gives the following bijections:

$$sol(\mathcal{G}al(A))_{z_0, z_1} \simeq Gal(A)(z_0, z_1),$$
$$sol(\mathcal{G}al(A))_z \simeq Gal(A).$$

3. A local Galois D-groupoid for Fuchsian q-difference equations

A Fuchsian linear q-difference system is locally equivalent to a constant linear one, for which we know the Galois D-groupoid. We are going to establish a conjugation between the Galois D-groupoids of two equivalent constant linear q-difference systems. This will allow us to define a local Galois D-groupoid for Fuchsian q-difference systems by giving as its realizations the Galois D-groupoids of its locally equivalent constant q-difference systems.

3.1. Conjugation

Consider two equivalent constant linear q-difference systems $X(qz) = AX(z)$ and $X(qz) = BX(z)$, defined by two constant matrices $A, B \in GL_n(\mathbb{C})$. Thus, according to [14], there exists a gauge transform $F(z) \in GL_n(\mathbb{C}[z, z^{-1}])$ such that $F(qz)AF(z)^{-1} = B$.

This last functional equation extends to $F(q^k z)A^k F(z)^{-1} = B^k$, for all $k \in \mathbb{Z}$. This can be translated into a conjugation between the dynamics $Dyn(A)$ and $Dyn(B)$, that we are going to state precisely.

Denote $M^* = \mathbb{C}^* \times \mathbb{C}^n \subset P^1\mathbb{C} \times \mathbb{C}^n = M$, and consider, first, the global diffeomorphism of M^*:

$$f : M^* \to M^*$$
$$(z, X) \mapsto (z, F(z)X).$$

Denote then $Aut(M^*)$ the groupoid of germs of local diffeomorphisms of M^*, and consider the conjugation:

$$\varphi : Aut(M^*) \to Aut(M^*)$$
$$_a[g]_{g(a)} \mapsto \, _{f(a)}[f \circ g \circ f^{-1}]_{f(g(a))}.$$

This map preserves the groupoid structure of $Aut(M^*)$. Actually, the maps f and φ on one side, and, on the other side, the maps which define the groupoid structure of $Aut(M^*)$, as in 1.2, satisfy the commutativity relations:

$$s \circ \varphi = f \circ s, t \circ \varphi = f \circ t, c \circ (\varphi \times_M \varphi) = \varphi \circ c, e = \varphi \circ e, \text{ and } i \circ \varphi = \varphi \circ i.$$

PROPOSITION 3.1. — *The dynamics $Dyn(A)$ and $Dyn(B)$ are conjugated over M^* by:*

$$\varphi \left[Dyn(A) \cap Aut(M^*) \right] = Dyn(B) \cap Aut(M^*).$$

Proof. — Recall that $Dyn(A) = \{(z, X) \mapsto (q^k z, A^k X)\}$ and $Dyn(B) = \{(z, X) \mapsto (q^k z, B^k X)\}$. Then, using the relation $F(q^k z)A^k F(z)^{-1} = B^k$, we check that for all $k \in \mathbb{Z}$, one has:

$$f \circ [(z, X) \mapsto (q^k z, A^k X)] \circ f^{-1} = [(z, X) \mapsto (q^k z, F(q^k z)A^k F(z)^{-1} X)]$$
$$= [(z, X) \mapsto (q^k z, B^k X)].$$

$\square$

The conjugation $\varphi : Aut(M^*) \mapsto Aut(M^*)$ can be projected on the sets $|J_r^*(M^*, M^*)|$, and induces a comorphism on the D-variety $J^*(M^*, M^*)$ which respects its groupoid structure.

Denote φ_r the projection of φ on $|J_r^*(M^*, M^*)|$. For example, the conjugation φ_0 is given on $(M^*)^2$, in coordinates, by: $\varphi_0(z, X, \bar{z}, \bar{X}) = (z, F(z)X, \bar{z}, F(\bar{z})\bar{X})$. Hence, the comorphism associated to φ is defined by:

$$\varphi^* : \mathcal{O}_{J^*(M^*, M^*)} \to (\varphi_0)_* \mathcal{O}_{J^*(M^*, M^*)}$$
$$E \mapsto E \circ \varphi_r$$

with r the order of the equation E. One can check that this comorphism preserves the groupoid structure of $J^*(M^*, M^*)$, and that we have the following result:

PROPOSITION 3.2. — *Let $\mathcal{G} \subset \mathcal{O}_{J^*(M^*, M^*)}$ be a D-groupoid on M^*. Then, the image $(\varphi_0^{-1})_* (\varphi^*\mathcal{G})$ is a D-groupoid on M^* whose solutions are conjugated to the solutions of $\mathcal{G}$ by the following relation:*

$$\varphi \left[sol \left((\varphi_0^{-1})_* (\varphi^*\mathcal{G}) \right) \right] = sol(\mathcal{G}).$$

The conjugation φ between the dynamics $Dyn(A)$ and $Dyn(B)$ extends to their D-envelopes in the following sense:

THEOREM 3.3. — *The restrictions to M^* of the Galois D-groupoids $\mathcal{G}al(A)$ and $\mathcal{G}al(B)$ are conjugated by:*

$$(\varphi_0^{-1})_* \left(\varphi^*\mathcal{G}al(B)_{|(M^*)^2} \right) = \mathcal{G}al(A)_{|(M^*)^2},$$

and consequently, their solutions are conjugated by:

$$\varphi \left[sol(\mathcal{G}al(A)) \cap Aut(M^*) \right] = sol(\mathcal{G}al(B)) \cap Aut(M^*).$$

Proof. — The restriction $\mathcal{G}al(B)_{|(M^*)^2}$ is a D-groupoid on M^*. Thus, by Proposition 3.2, the image $(\varphi_0^{-1})_* \left(\varphi^*\mathcal{G}al(B)_{|(M^*)^2} \right)$ is a D-groupoid on M^* whose solutions are $\varphi^{-1} \left[sol(\mathcal{G}al(B)_{|(M^*)^2}) \right]$. By Proposition 3.1, they consequently contain $Dyn(A) \cap Aut(M^*) = \varphi^{-1} \left[Dyn(B) \cap Aut(M^*) \right]$.

On the other hand, calculating the D-envelope of $Dyn(A) \cap Aut(M^*)$ on M^*, following what was done in 2.1 on M, we obtain that this D-envelope is exactly the restriction $\mathcal{G}al(A)_{|(M^*)^2}$. Thus, the D-groupoid $\mathcal{G}al(A)_{|(M^*)^2}$ is the least D-groupoid on M^* whose solutions contain $Dyn(A) \cap Aut(M^*)$, and we consequently have the inclusion:

$$(\varphi_0^{-1})_* \left(\varphi^* \mathcal{G}al(B)_{|(M^*)^2} \right) \subset \mathcal{G}al(A)_{|(M^*)^2}.$$

The reverse inclusion can be obtained by the same method, but considering the conjugation φ^{-1} instead of φ. □

3.2. A local Galois D-groupoid for Fuchsian q-difference equations

Let $X(qz) = A(z)X(z)$ be a rational linear q-difference system, with $A(z) \in GL_n(\mathbb{C}(z))$, and assume it is Fuchsian at 0. Then, according to [12] and [14], there exists a local gauge transform $F^{(0)}(z) \in GL_n(\mathbb{C}(\{z\}))$, and a constant matrix $A^{(0)} \in GL_n(\mathbb{C})$ such that $F^{(0)}(qz)A(z)F^{(0)}(z)^{-1} = A^{(0)}$.

We propose here a definition of a local Galois D-groupoid for Fuchsian q-difference systems, that we are going to legitimate just after.

DEFINITION 3.4. — *The realizations of the local Galois D-groupoid* $\mathcal{G}al^{(0)}(A(z))$ *of* $X(qz) = A(z)X(z)$ *are the Galois D-groupoids* $\mathcal{G}al(A^{(0)})$ *of the constant linear q-difference systems* $X(qz) = A^{(0)}X(z)$, *with* $A^{(0)} \in GL_n(\mathbb{C})$, *locally equivalent to* $X(qz) = A(z)X(z)$.

Consider respectively two constant linear q-difference systems, defined by two constant matrices $A_1^{(0)}, A_2^{(0)} \in GL_n(\mathbb{C})$, locally equivalent to $X(qz) = A(z)X(z)$, *via* two local gauge transforms $F_1^{(0)}(z), F_2^{(0)}(z) \in GL_n(\mathbb{C}(\{z\}))$ which satisfy

$$F_1^{(0)}(qz)A(z)F_1^{(0)}(z)^{-1} = A_1^{(0)}$$

and

$$F_2^{(0)}(qz)A(z)F_2^{(0)}(z)^{-1} = A_2^{(0)}.$$

Then, the matrix $F(z) = F_2^{(0)}(z)F_1^{(0)}(z)^{-1} \in GL_n(\mathbb{C}(\{z\}))$ is a local gauge transform such that $F(qz)A_1^{(0)}F(z)^{-1} = A_2^{(0)}$. According to [14], this last functional equation necessarily implies that $F(z) \in GL_n(\mathbb{C}[z, z^{-1}])$, that is $F(z)$ is global gauge transform between the two constant linear q-difference systems $X(qz) = A_1^{(0)}X(z)$ and $X(qz) = A_2^{(0)}X(z)$. Hence, by Theorem 3.3, the Galois D-groupoids of those two equivalent constant linear q-difference systems are conjugated. This legitimates Definition 3.4.

Moreover, from the point of view of solutions, one has the following conjugation:

$$\varphi\left[sol(\mathcal{G}al(A_1^{(0)}))\cap Aut(M^*)\right] = sol(\mathcal{G}al(A_2^{(0)}))\cap Aut(M^*).$$

Considering, in this equality, solutions which preserve the transversal lines $\{z\}\times\mathbb{C}^n$, for all $z\in P^1\mathbb{C}$, and using the correspondence given by Theorem 2.4, we get the following corollary:

COROLLARY 3.5. — *The Galois groups $Gal(A_1^{(0)})$ and $Gal(A_2^{(0)})$ (as defined in [14], and recalled in 2.2) are conjugated by:*

$$F(z_0)Gal(A_1^{(0)})F(z_0)^{-1} = Gal(A_2^{(0)}),$$

for all $z_0\in\mathbb{C}^*$.

This is exactly the relation used in [14], and obtained from the Tannakian machinery, to legitimate the definition of the local Galois group $Gal^{(0)}(A(z))$ of the Fuchsian q-difference system $X(qz)=A(z)X(z)$, whose realizations are the Galois groups $Gal(A^{(0)})$ of the constant linear q-difference systems $X(qz)=A^{(0)}X(z)$ locally equivalent to $X(qz)=A(z)X(z)$. Therefore, we recover from Definition 3.4 Sauloy's definition in [14] of the local Galois group for such systems.

We could also define directly a local Galois D-groupoid for the Fuchsian q-difference system $X(qz)=A(z)X(z)$ as the D-envelope on $M^{(0)}=\mathbb{C}\times\mathbb{C}^n\subset P^1\mathbb{C}\times\mathbb{C}^n=M$ of the dynamics $Dyn(A(z))\cap Aut(M^{(0)})$. Actually, the q-invariant open set $\mathbb{C}\subset P^1\mathbb{C}$ is, in the q-difference equations context, the privileged open neighbourhood of 0 in $P^1\mathbb{C}$. To legitimate this explicit definition, one would then have to describe a conjugation, as in Theorem 3.3, between this explicit local Galois D-groupoid on $M^{(0)}$, and the local Galois D-groupoid of a locally equivalent constant linear q-difference system $X(qz)=A^{(0)}X(z)$, which is the restriction $\mathcal{G}al(A^{(0)})_{|(M^{(0)})^2}$.

Appendix A

We check here that the coherent ideal $\mathcal{Q}_2\subset\mathcal{O}_{J_2^*(M,M)}$, defined by

$$\mathcal{Q}_2 = \left\langle\frac{\partial\bar{z}}{\partial X}, \frac{\partial\bar{z}}{\partial z}z - \bar{z}, \frac{\partial^2\bar{z}}{\partial z\partial X}, \frac{\partial^2\bar{z}}{\partial X^2}, \frac{\partial^2\bar{z}}{\partial z^2}\right\rangle,$$

defines a subgroupoid of $J_2^*(M,M)$, *i.e.*, it satisfies the conditions *(i)*, *(ii)*, and *(iii)* defined in subsection 1.2. It is enough to check these stability conditions on the generators of $\mathcal{Q}_2$.

This necessitates to explicit the morphisms c^*, e^*, and i^* which define the groupoid structure of the affine variety $J_2^*(M, M)$. To perform that, we need to remember that for any two $\mathcal{C}^2$-maps $f, g : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, one has the two following formulas:

$$\frac{\partial(g \circ f)_l}{\partial x_i} = \sum_{j=0}^{n} \frac{\partial f_j}{\partial x_i}\left(\frac{\partial g_l}{\partial y_j} \circ f\right),$$

$$\frac{\partial^2(g \circ f)_l}{\partial x_k \partial x_i} = \sum_{j=0}^{n} \frac{\partial^2 f_j}{\partial x_k \partial x_i}\left(\frac{\partial g_l}{\partial y_j} \circ f\right) + \sum_{j=0}^{n} \frac{\partial f_j}{\partial x_i} \sum_{p=0}^{n} \frac{\partial f_p}{\partial x_k}\left(\frac{\partial^2 g_l}{\partial y_p \partial y_j} \circ f\right).$$

Condition *(i)*: We recall that, in coordinates, the map c is defined by

$$c\left(\left(y, \bar{y}, \frac{\partial \bar{y}}{\partial y}, \frac{\partial^2 \bar{y}}{\partial y^2}\right), \left(\bar{y}, \bar{\bar{y}}, \frac{\partial \bar{\bar{y}}}{\partial \bar{y}}, \frac{\partial^2 \bar{\bar{y}}}{\partial \bar{y}^2}\right)\right) = \left(y, \bar{\bar{y}}, \frac{\partial \bar{\bar{y}}}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial y}, *\right),$$

where the $*$ has to be expanded using the second preliminary formula.

Thus, we first have $c^*(z, X) = (z, X)$ and $c^*(\bar{z}, \bar{X}) = (\bar{\bar{z}}, \bar{\bar{X}})$, and

$$c^*\left(\begin{pmatrix} \frac{\partial \bar{z}}{\partial z} & \frac{\partial \bar{z}}{\partial X} \\ \frac{\partial \bar{X}}{\partial z} & \frac{\partial \bar{X}}{\partial X} \end{pmatrix}\right) = \begin{pmatrix} \frac{\partial \bar{\bar{z}}}{\partial \bar{z}} & \frac{\partial \bar{\bar{z}}}{\partial \bar{X}} \\ \frac{\partial \bar{\bar{X}}}{\partial \bar{z}} & \frac{\partial \bar{\bar{X}}}{\partial \bar{X}} \end{pmatrix} \begin{pmatrix} \frac{\partial \bar{z}}{\partial z} & \frac{\partial \bar{z}}{\partial X} \\ \frac{\partial \bar{X}}{\partial z} & \frac{\partial \bar{X}}{\partial X} \end{pmatrix}.$$

This gives

$$c^*\left(\frac{\partial \bar{z}}{\partial X}\right) = \frac{\partial \bar{\bar{z}}}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial X} + \frac{\partial \bar{\bar{z}}}{\partial \bar{X}} \frac{\partial \bar{X}}{\partial X} \in \mathcal{O}_{J_2^*(M,M)} \widehat{\otimes} \mathcal{Q}_2 + \mathcal{Q}_2 \widehat{\otimes} \mathcal{O}_{J_2^*(M,M)}$$

$$c^*\left(\frac{\partial \bar{z}}{\partial z} z - \bar{z}\right) = \left(\frac{\partial \bar{\bar{z}}}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial z} + \frac{\partial \bar{\bar{z}}}{\partial \bar{X}} \frac{\partial \bar{X}}{\partial z}\right) z - \bar{\bar{z}}$$

$$= \frac{\partial \bar{\bar{z}}}{\partial \bar{z}}\left(\frac{\partial \bar{z}}{\partial z} z - \bar{z}\right) + \frac{\partial \bar{\bar{z}}}{\partial \bar{z}} \bar{z} - \bar{\bar{z}} + \frac{\partial \bar{\bar{z}}}{\partial \bar{X}} \frac{\partial \bar{X}}{\partial z} z$$

$$\in \mathcal{O}_{J_2^*(M,M)} \widehat{\otimes} \mathcal{Q}_2 + \mathcal{Q}_2 \widehat{\otimes} \mathcal{O}_{J_2^*(M,M)} + \mathcal{Q}_2 \widehat{\otimes} \mathcal{O}_{J_2^*(M,M)}$$

We deduce then from the second preliminary formula that

$$c^*\left(\frac{\partial^2 \bar{z}}{\partial z^2}\right) = \frac{\partial^2 z}{\partial z^2} \frac{\partial \bar{\bar{z}}}{\partial \bar{z}} + \frac{\partial z}{\partial z}\left(\frac{\partial \bar{z}}{\partial z} \frac{\partial^2 \bar{\bar{z}}}{\partial \bar{z}^2} + \sum_{p=1}^{n} \frac{\partial \bar{X}_p}{\partial z} \frac{\partial^2 \bar{\bar{z}}}{\partial \bar{X}_p \partial \bar{z}}\right) +$$

$$+ \sum_{j=1}^{n} \left[\frac{\partial^2 \bar{X}_j}{\partial z^2} \frac{\partial \bar{\bar{z}}}{\partial \bar{X}_j} + \frac{\partial \bar{X}_j}{\partial z}\left(\frac{\partial \bar{z}}{\partial z} \frac{\partial^2 \bar{\bar{z}}}{\partial \bar{z} \partial \bar{X}_j} + \sum_{p=1}^{n} \frac{\partial \bar{X}_p}{\partial z} \frac{\partial^2 \bar{\bar{z}}}{\partial \bar{X}_p \partial \bar{X}_j}\right)\right].$$

The global equations

$$\frac{\partial^2 \bar{z}}{\partial z^2}, \frac{\partial^2 \bar{\bar{z}}}{\partial \bar{z}^2}, \frac{\partial \bar{\bar{z}}}{\partial \bar{X}_j}, \frac{\partial^2 \bar{\bar{z}}}{\partial \bar{X}_p \partial \bar{z}}, \frac{\partial^2 \bar{\bar{z}}}{\partial \bar{z} \partial \bar{X}_j}, \frac{\partial^2 \bar{\bar{z}}}{\partial \bar{X}_p \partial \bar{X}_j}$$

belong to $\mathcal{Q}_2$, so any term of the above last sum belong to $\mathcal{Q}_2 \widehat{\otimes} \mathcal{O}_{J_2^*(M,M)} + \mathcal{O}_{J_2^*(M,M)} \widehat{\otimes} \mathcal{Q}_2$.

Similar computations give

$$c^* \left(\frac{\partial^2 \bar{z}}{\partial z \partial X} \right), \, c^* \left(\frac{\partial^2 \bar{z}}{\partial X^2} \right) \in \mathcal{Q}_2 \widehat{\otimes} \mathcal{O}_{J_2^*(M,M)} + \mathcal{O}_{J_2^*(M,M)} \widehat{\otimes} \mathcal{Q}_2,$$

and consequently, $c^*(\mathcal{Q}_2) \subset \mathcal{Q}_2 \widehat{\otimes} \mathcal{O}_{J_2^*(M,M)} + \mathcal{O}_{J_2^*(M,M)} \widehat{\otimes} \mathcal{Q}_2$.

Condition (ii). — We recall that, in coordinates, the map e is defined by

$$e \left(y, \bar{y}, \frac{\partial \bar{y}}{\partial y}, \frac{\partial^2 \bar{y}}{\partial y^2} \right) = (y, y, Id_{n+1}, 0) \,.$$

This gives

$$e^* \left(\frac{\partial \bar{z}}{\partial X} \right) = 0 \,, \, e^* \left(\frac{\partial^2 \bar{z}}{\partial z \partial X} \right) = 0 \,, \, e^* \left(\frac{\partial^2 \bar{z}}{\partial X^2} \right) = 0,$$

$$e^* \left(\frac{\partial^2 \bar{z}}{\partial z^2} \right) = 0 \,, \, e^* \left(z \frac{\partial \bar{z}}{\partial z} - \bar{z} \right) = z.1 - z = 0,$$

and consequently, $e^*(\mathcal{Q}_2) = 0$.

Condition (iii). — We recall that, in coordinates, the map i is defined by

$$i \left(y, \bar{y}, \frac{\partial \bar{y}}{\partial y}, \frac{\partial^2 \bar{y}}{\partial y^2} \right) = \left(\bar{y}, y, (\frac{\partial \bar{y}}{\partial y})^{-1}, ? \right),$$

where the ? has to be computed using the second preliminary formula.

Thus, we first have $i^*(z, X) = (\bar{z}, \bar{X})$ and $i^*(\bar{z}, \bar{X}) = (z, X)$, and

$$\begin{pmatrix} \frac{\partial \bar{z}}{\partial z} & \frac{\partial \bar{z}}{\partial X_1} & \cdots & \frac{\partial \bar{z}}{\partial X_n} \\ \frac{\partial \bar{X}_1}{\partial z} & & & \\ \vdots & & (\frac{\partial \bar{X}_i}{\partial X_j})_{i,j} & \\ \frac{\partial \bar{X}_n}{\partial z} & & & \end{pmatrix}^{-1} = \delta^{-1} \begin{pmatrix} \det(\frac{\partial \bar{X}_i}{\partial X_j})_{i,j} & *_1 & \cdots & *_n \\ \flat_1 & & & \\ \vdots & & B & \\ \flat_n & & & \end{pmatrix},$$

where the $*_j$ represent linear combinations of $\frac{\partial \bar{z}}{\partial X_1}, \dots, \frac{\partial \bar{z}}{\partial X_n}$, and the matrix B satisfy

$$\delta^{-1} (\flat_i \frac{\partial \bar{z}}{\partial X_j})_{i,j} + \delta^{-1} B (\frac{\partial \bar{X}_i}{\partial X_j})_{i,j} = Id_n.$$

This gives

$$i^* (\frac{\partial \bar{z}}{\partial X_j}) = \delta^{-1} *_j \in \mathcal{Q}_2,$$

and

$$i^* (\frac{\partial \bar{z}}{\partial z} z - \bar{z}) = \delta^{-1} \det(\frac{\partial \bar{X}_i}{\partial X_j})_{i,j} \bar{z} - z.$$

However,

$$\frac{\partial \bar{z}}{\partial z} z - \bar{z} \in \mathcal{Q}_2, \text{ so } \delta^{-1} \det(\frac{\partial \bar{X}_i}{\partial X_j})_{i,j} \frac{\partial \bar{z}}{\partial z} z - \delta^{-1} \det(\frac{\partial \bar{X}_i}{\partial X_j})_{i,j} \bar{z} \in \mathcal{Q}_2,$$

which consequently gives

$$i^*(\frac{\partial \bar{z}}{\partial z} z - \bar{z}) \equiv (\delta^{-1} \det(\frac{\partial \bar{X}_i}{\partial X_j})_{i,j} \frac{\partial \bar{z}}{\partial z} - 1) z$$

$$\equiv -\delta^{-1}(\sum_{k=1}^{n} \frac{\partial \bar{z}}{\partial X_k} \flat_k) \equiv 0 \mod \mathcal{Q}_2.$$

Consider $g = f^{-1}$ in the second preliminary formula. We get:

$$0 = \sum_{j=0}^{n} \frac{\partial^2 f_j}{\partial x_k \partial x_i}(\frac{\partial f_l^{-1}}{\partial x_j} \circ f) + \sum_{j=0}^{n} \frac{\partial f_j}{\partial x_i} \sum_{p=0}^{n} \frac{\partial f_p}{\partial x_k}(\frac{\partial^2 f_l^{-1}}{\partial x_p \partial x_j} \circ f).$$

It can be also written:

$$\begin{pmatrix} \frac{\partial f_0}{\partial x_i} & \cdots & \frac{\partial f_n}{\partial x_i} \end{pmatrix} \left(H(f_l^{-1}) \circ f\right) \begin{pmatrix} \frac{\partial f_0}{\partial x_k} \\ \vdots \\ \frac{\partial f_n}{\partial x_k} \end{pmatrix}$$

$$= -\begin{pmatrix} \frac{\partial f_l^{-1}}{\partial x_0} \circ f & \cdots & \frac{\partial f_l^{-1}}{\partial x_n} \circ f \end{pmatrix} \begin{pmatrix} \frac{\partial^2 f_0}{\partial x_i \partial x_k} \\ \vdots \\ \frac{\partial^2 f_n}{\partial x_i \partial x_k} \end{pmatrix} := h_{i,k}^l$$

where $H(f_l^{-1})$ is the Hessian matrix of the component f_l^{-1}, whose coefficients are $\frac{\partial^2 f_l^{-1}}{\partial x_p \partial x_j}$. Hence, the coefficients $h_{i,k}^l$ are the values of the symmetric bilinear form represented by the Hessian matrix $H(f_l^{-1})$ with respect to the base made with column vectors of the Jacobian matrix $Jf = \left(\frac{\partial f_j}{\partial x_k}\right)_{j,k}$.
Performing a base change, we get:

$$\left(H(f_l^{-1}) \circ f\right) = {}^t (Jf)^{-1} \left(h_{i,k}^l\right)_{i,k} (Jf)^{-1}.$$

We deduce from this formula that $-i^*\left(H\bar{z}\right)$ is the product of the three following matrices:

$$ {}^t i^* \begin{pmatrix} \frac{\partial \bar{z}}{\partial z} & \frac{\partial \bar{z}}{\partial X} \\ \frac{\partial \bar{X}}{\partial z} & \frac{\partial \bar{X}}{\partial X} \end{pmatrix}, $$

$$ \begin{pmatrix} i^* \begin{pmatrix} \frac{\partial \bar{z}}{\partial z} & \frac{\partial \bar{z}}{\partial X} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \bar{z}}{\partial z^2} \\ \frac{\partial^2 \bar{X}}{\partial z^2} \end{pmatrix} & i^* \begin{pmatrix} \frac{\partial \bar{z}}{\partial z} & \frac{\partial \bar{z}}{\partial X} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \bar{z}}{\partial z \partial X} \\ \frac{\partial^2 \bar{X}}{\partial z \partial X} \end{pmatrix} \\ i^* \begin{pmatrix} \frac{\partial \bar{z}}{\partial z} & \frac{\partial \bar{z}}{\partial X} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \bar{z}}{\partial z \partial X} \\ \frac{\partial^2 \bar{X}}{\partial z \partial X} \end{pmatrix} & i^* \begin{pmatrix} \frac{\partial \bar{z}}{\partial z} & \frac{\partial \bar{z}}{\partial X} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \bar{z}}{\partial X^2} \\ \frac{\partial^2 \bar{X}}{\partial X^2} \end{pmatrix} \end{pmatrix}, $$

$$ i^* \begin{pmatrix} \frac{\partial \bar{z}}{\partial z} & \frac{\partial \bar{z}}{\partial X} \\ \frac{\partial \bar{X}}{\partial z} & \frac{\partial \bar{X}}{\partial X} \end{pmatrix}. $$

Since

$$ i^* \left(\frac{\partial \bar{z}}{\partial X} \right), \frac{\partial^2 \bar{z}}{\partial z^2}, \frac{\partial^2 \bar{z}}{\partial z \partial X}, \frac{\partial^2 \bar{z}}{\partial X^2} \in \mathcal{Q}_2, $$

the coefficients of second matrix belong to $\mathcal{Q}_2$, and so $i^*\left(H\bar{z}\right) \in \mathcal{Q}_2$. Consequently, $i^*(\mathcal{Q}_2) \subset \mathcal{Q}_2$.

Appendix B

We state here the lemma needed in the step 3 of Calculation 2.1, and give a proof of it based on Groebner bases (*cf e.g.* [3]) and an adapted version of a division algorithm.

Let first $u = (u_0, \ldots, u_l)$ and $v = (v_0, \ldots, v_m)$ denote two sets of independent indeterminates, with $l, m \in \mathbb{N}$. Then, consider the ring $\mathbb{C}[v]$ of polynomials over $\mathbb{C}$ with v as indeterminates, and the ring $\mathbb{C}\{u\}[v]$ of polynomials also with v as indeterminates, but over the ring of convergent series over $\mathbb{C}$ with u as indeterminates.

LEMMA 3.6. — *Let I be an ideal of $\mathbb{C}[v]$, and let $E(u, v)$ be an element of $\mathbb{C}\{u\}[v]$ such that, for all point u^0 in a neighbourhood of 0 in $\mathbb{C}^{n+1}$, the evaluation $E(u^0, v)$ is an element of $I \subset \mathbb{C}[v]$. Then, the polynomial $E(u, v)$ is an element of the ideal $\langle I \rangle$ of $\mathbb{C}\{u\}[v]$ generated by I.*

Proof. — Let $(g_1(v), \ldots, g_d(v))$ be a Groebner base of the ideal $I \subset \mathbb{C}[v]$. The ring $\mathbb{C}\{u\}$ is an integral domain, and the polynomials $g_1(v), \ldots, g_d(v)$ are monic polynomials in the polynomial ring $\mathbb{C}\{u\}[v]$. Following the division algorithm presented in [3] in a polynomial ring over a commutative field, we observe that each step remains possible if one considers polynomials with coefficients in an integral domain, and monic polynomials

as divisors. Thus, perform the division of $E(u,v)$ by the Groebner base $(g_1(v), \ldots, g_d(v))$. We get some polynomials

$$a_1(u,v), \ldots, a_d(u,v), R(u,v) \in \mathbb{C}\{u\}[v]$$

such that:

$$E(u,v) = a_1(u,v)g_1(v) + \cdots + a_d(u,v)g_d(v) + R(u,v),$$

with $R = 0$, or with R such that any of its terms, with respect to the indeterminates v, is divisible by any leading term of the $g_i(v)$.

For all u^0 in a neighbourhood of 0 in $\mathbb{C}^{n+1}$, the evaluation given by $u = u^0$ in this last expression gives:

$$E(u^0, v) = a_1(u^0, v)g_1(v) + \cdots + a_d(u^0, v)g_d(v) + R(u^0, v).$$

According to the hypothesis, the polynomial $E(u^0, v)$ belongs to I, and the Groebner base $(g_1(v), \ldots, g_d(v))$ is also included in I. Therefore, we get $R(u^0, v) \in I$ for all u^0 in a neighbourhood of 0.

Assume that there exists u^0 close enough to 0 such that $R(u^0, v) \neq 0$. Then, on the one hand, because the monomials of $R(u^0, v)$ are also monomials of $R(u,v)$ (which are not divisible by the leading term of any $g_i(v)$), any term of $R(u^0, v)$ is divisible by the leading term of any $g_i(v)$. On the other hand, the polynomial $R(u^0, v)$ belongs to I, and $(g_1(v), \ldots, g_d(v))$ is a Groebner base of I. Therefore, by definition of Groebner bases, the leading term of $R(u^0, v)$ is divisible by the leading term of one of the $g_i(v)$. This gives rise to a contradiction.

Consequently, for all point u^0 in a neighbourhood of 0, we have that $R(u^0, v) = 0$, which implies that $R = 0$. There remains:

$$E(u,v) = a_1(u,v)g_1(v) + \cdots + a_d(u,v)g_d(v),$$

which means that the polynomial E belongs to the ideal $\langle I \rangle$ generated by I in $\mathbb{C}\{u\}[v]$. $\qquad\square$

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