ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

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Tome XXXI, nº 4 (2022), p. 1213–1221.

https://doi.org/10.5802/afst.1716

CENTRE MERSENNE

Publication membre du centre Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/
e-ISSN : 2258-7519

Annales de la faculté des sciences de Toulouse Volume XXXI, n° 4, 2022
pp. 1213-1221

Asymptotic of the largest Floquet multiplier for cooperative matrices (*)

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Abstract. — The aim of this note is to give a link between the spectral radius of the monodromy matrix of a linear differential equation with periodic coefficients $\frac{\mathrm{d}x}{\mathrm{d}t}(t) = A(t)\,x(t)$, with $A(t)$ a cooperative irreducible matrix, and the mean spectral abscissa $\int_0^1 s(A(u))\,\mathrm{d}u$.

Résumé. — Le but de cet article est d'établir un lien entre le rayon spectral de la matrice de monodromie d'une équation différentielle linéaire à coefficients périodiques $\frac{\mathrm{d}x}{\mathrm{d}t}(t) = A(t)\,x(t)$, avec $A(t)$ une matrice coopérative irréductible, avec la moyenne de l'abcisse spectrale $\int_0^1 s(A(u))\,\mathrm{d}u$.

1. Introduction

The motivation of our main result, Theorem 2.1 comes from the study of the spread of infectious diseases for periodic systems in populations whose individuals can be divided into a finite number of distinct groups.

The most famous threshold quantity used is the *basic reproduction number* R_0 which has the following property: if $R_0 > 1$, then the introduction of one infectious individual (of any type) has a positive probability of inducing a major epidemic; if $R_0 < 1$, then the introduction of one infectious

(*) Reçu le 14 mai 2020, accepté le 25 août 2020.

Keywords: Floquet's theorem, spectral radius, spectral abscissa, ordinary differential equation, non negative matrices.

2020 *Mathematics Subject Classification:* 15A18, 34D08, 15A42, 15B46, 60J80.

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The author thanks the Centre Henri Lebesgue ANR-11-LABX-0020-01 for creating an attractive mathematical environment. This project has received financial support from the CNRS through the 80|Prime program.

Article proposé par Laurent Miclo.

individual (of any type) has a null probability of inducing a major epidemic (see [1, 5, 11]).

The value of R_0 is determined by the study of a linear system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x(t)\,, \tag{1.1}$$

where $A(t)$ is a $d \times d$ continuous matrix function.

Let us consider an example in a periodic case: a vector host model of the African Horse Sickness (see [7, Section 3]). There, $x_1(t)$ (resp. $x_2(t)$) denotes the population of infected hosts (resp. vectors) at time t, and

$$A(t) = \begin{pmatrix} -r & b/s(t) \\ \beta\, s(t) & -\mu \end{pmatrix}, \tag{1.2}$$

with r the removal rate of hosts, μ the death rate of vectors, b (resp. β) the rate constant for infection of hosts by vectors (resp. vectors by hosts), and $s(t)$ the population of susceptible vectors taken to be

$$s(t) = s_0 e^{\mu\delta \sin(2\pi t)}\,. \tag{1.3}$$

By construction,, the matrices involved in these compartmental models are *cooperative*: $A_{ij}(t) \geqslant 0$ if $i \neq j$.

Then R_0 is defined to be the spectral radius of the next generation operator named $\mathcal{K}$ in [6, Equation (4.6)] and L in [13, Equation (2.6)]. In the periodic case, not only do the authors of [6, 13] establish the stability properties of R_0, but they also link it to another threshold quantity λ_d, whose precise definition we give now.

Assume $A(t)$ continuous periodic with period T, and let ϕ_A be the fundamental matrix, that is the solution of

$$\frac{\mathrm{d}\phi_A}{\mathrm{d}t} = A(t)\phi_A(t)\,, \qquad \phi_A(0) = I_d\,. \tag{1.4}$$

Let $\lambda_d = \rho(\phi_A(T))$ be the spectral radius of the monodromy matrix $\phi_A(T)$. We have

$$R_0 > 1 \iff \lambda_d > 1\,, \qquad R_0 < 1 \iff \lambda_d < 1\,, \tag{1.5}$$

(see [6, Theorem 1] or [13, Theorem 2]).

The easiest quantity to compute numerically is λ_d by simulating the ODE system on one period, even if efficient computation methods of R_0 have been devised (see e.g. [3]).

However estimating the influence of a small periodic perturbation on a constant system may prove very difficult even when using the threshold λ_d (see [4]).

We introduce in this paper a new quantity, the *mean spectral abscissa*, that is also a threshold quantity for large periods: this is our main theorem. Perturbation theory is much easier to apply to the mean spectral abscissa (see [4]).

2. Statement of the main result

Let us consider the linear system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = A(t)\, x(t) \tag{2.1}$$

where $A : \mathbb{R} \to \mathcal{M}_{dxd}$ is a continuous matrix function. The fundamental matrix is the matrix $\phi = \phi_A(t)$ solution of

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = A(t)\phi(t)\,, \qquad \phi(0) = I_d\,. \tag{2.2}$$

Assume that A is T periodic, that is

$$A(T+t) = A(t)\,. \tag{2.3}$$

Floquet's theorem, see e.g. [12, Theorem 3.15, Chapter 3.6], establishes the existence of a possibly complex matrix B and a continuous T-periodic matrix function P such that $P(0) = I_d$ and

$$\phi(t) = P(t)e^{tB}\,. \tag{2.4}$$

For a matrix M with spectrum $\sigma(M)$ we let $s(M)$ and $\rho(M)$ be its *spectral abscissa* and its *spectral radius*:

$$s(M) = \sup\{\Re(z), z \in \sigma(M)\}\,, \qquad \rho(M) = \sup\{|z|, z \in \sigma(M)\}\,. \tag{2.5}$$

The *characteristic multipliers* are the eigenvalues of the *monodromy matrix* $\phi(T) = e^{TB}$. The eigenvalues of B are called the *characteristic exponents.*

If the spectral radius of $\phi(T) = e^{TB}$ is also an eigenvalue, then the largest (in modulus) characteristic exponent is the spectral abscissa of B, also an eigenvalue of B, and we have the relation:

$$\frac{1}{T}\ln\rho(\phi(T)) = s(B)\,. \tag{2.6}$$

Therefore when A is constant, and $\rho(e^{tA})$ is an eigenvalue of e^{tA}, we have $\frac{1}{T}\ln\rho(\phi(T)) = s(A)$ and our goal is to find a similar equation valid for periodic matrices A, in the limit $T \to +\infty$.

We say that a matrix M is non negative, $M \geqslant 0$, if for all i, j, $M_{ij} \geqslant 0$ and we say that M is positive, $M > 0$ if for all i, j, $M_{ij} > 0$.

A matrix M is *irreducible* if for any $i \neq j$ there exists an integer $k \geqslant 1$ and $i_0 = i, i_1, \dots, i_k = j$ such that $M_{i_p, i_{p+1}} \neq 0$ for $0 \leqslant p \leqslant k-1$.[1]

We say that a matrix M is *cooperative* if $M_{ij} \geqslant 0$ for $i \neq j$.

Let $A : \mathbb{R} \to \mathcal{M}_{d \times d}$ be a continuous 1-periodic matrix function, with fundamental matrix $\phi(t)$. We let $\phi^{(T)}(t)$ be the fundamental matrix of the T periodic matrix $t \to A(t/T)$.

Theorem 2.1. — *Assume that the matrix function $t \to A(t)$ is Lipschitz and 1 periodic. Assume that for every $t \geqslant 0$ the matrix $A(t)$ is cooperative and irreducible. Then*

$$\lim_{T \to +\infty} \frac{1}{T} \ln \rho(\phi^{(T)}(T)) = \int_0^1 s(A(u))\, \mathrm{d}u\,. \tag{2.7}$$

Remark 2.2. — The assumption that $A(t)$ is irreducible is important. One easily sees, by taking a diagonal 2×2 matrix

$$A(t) = \begin{pmatrix} u(t) & 0 \\ 0 & v(t) \end{pmatrix}, \tag{2.8}$$

that the quantities

$$\frac{1}{T} \ln \rho(\phi^{(T)}(T)) = \max\left(\int_0^1 u(s)\, \mathrm{d}s, \int_0^1 v(s)\, \mathrm{d}s \right),$$
$$\int_0^1 s(A(r))\, \mathrm{d}r = \int_0^1 \max(u(s), v(s))\, \mathrm{d}s\,,$$

are in general different.

3. Proof of the main theorem

We shall need the following Lemmas, whose proof is postponed.

Lemma 3.1. — *Assume that A and B are continuous matrix functions on $I = [0, a]$ such that:*

(1) *$B(0)$ is irreducible and for all t, $B(t)$ is cooperative.*
(2) *For any $t \geqslant 0$, $B(t) \geqslant A(t) \geqslant 0$ and for $t > 0$, $B(t) - A(t) \neq 0$.*

[1] in other words the directed graph with vertices $1, \dots, d$ and directed edges (i, j) for $M_{ij} \neq 0$ is connected by directed paths.

Let ϕ_A and ϕ_B be the fundamental matrices associated with A and B. Then, for any $t \in I$,

$$\phi_A(t) \leqslant \phi_B(t)\,. \tag{3.1}$$

The following Lemma results from an application of Perron Frobenius Theorem to e^A.

LEMMA 3.2. — *Let A be a cooperative irreducible matrix. Then the spectral abscissa $s(A)$ is an isolated eigenvalue of A with a positive eigenvector. Consequently, $s(A) = \rho(A)$ is also A's spectral radius.*

If $x, y > 0$ are positive vectors then we consider the vector $\left(\frac{x}{y}\right)_i = \frac{x_i}{y_i}$ and we have obviously $x \leqslant \left\|\frac{x}{y}\right\|_\infty y$. A non negative matrix is *primitive* if there exists an integer $k \geqslant 1$ such that $M^k > 0$.

Observe that if a matrix is primitive, then it is irreducible and cooperative.

Let us state a simple, but useful, bound on the spectral radius of the product of primitive matrices that has been established by [8]. For sake of completeness we supply a proof later.

LEMMA 3.3. — *Let $A_1, A_2, \ldots, A_n$ be primitive non negative matrices with associated eigenvectors $u_i > 0$: $A_i u_i = \rho(A_i) u_i$. Then*

$$\rho(A_1 A_2 \ldots A_n) \leqslant \rho(A_1) \cdots \rho(A_n) \alpha(u_1, u_2) \ldots \alpha(u_{n-1}, u_n) \tag{3.2}$$

with $\alpha(x, y) = \left\|\frac{x}{y}\right\|_\infty \left\|\frac{y}{x}\right\|_\infty \geqslant 1$. Similarly we have the lower bound

$$\rho(A_1 A_2 \ldots A_n) \geqslant \rho(A_1) \cdots \rho(A_n) (\alpha(u_1, u_2) \ldots \alpha(u_{n-1}, u_n))^{-1}\,. \tag{3.3}$$

Proof of the upper bound of the main theorem. — Since $t \to A(t)$ is continuous, and $A(t)$ is cooperative irreducible, by Lemma 3.2, for every t, $s(A(t))$ is an isolated eigenvalue associated to a positive eigenvector. By the continuity of the isolated eigenvalue and the corresponding eigenvector (see e.g. [9, 10]) there exists a continuous function $t \to u(t)$ such that $u(t) > 0$ satisfies

$$A(t)u(t) = s(A(t))\, u(t)\,. \tag{3.4}$$

Let us fix $\gamma > 0$. Let J be the matrix with all coefficients equal to 1. Again, by continuity of both the isolated eigenvalue and the corresponding eigenvector there exists $\eta_0 = \eta_0(\gamma)$ such that for $\eta \in [0, \eta_0)$ there exists a positive vector $u_\eta^+(t)$ such that

$$(A(t) + \eta J) u_\eta^+(t) = s(A(t) + \eta J) u_\eta^+(t)\,, \quad |s(A(t) + \eta J) - s(A(t))| \leqslant \gamma\,, \tag{3.5}$$

$$\left\|u_\eta^+(t) - u(t)\right\| \leqslant \gamma\,. \tag{3.6}$$

Let C be a Lipschitz constant such that

$$|A_{ij}(s) - A_{ij}(t)| \leqslant C|s-t| \qquad (\forall\, s,t,i,j). \tag{3.7}$$

Let N be a large integer. For any integer k, with $\eta = \frac{2C}{N}$

$$A(t) \leqslant A(k/N) + \frac{C}{N} J < A(k/N) + \eta J \qquad \text{for } t \in \left[\frac{k}{N}, \frac{k+1}{N}\right]. \tag{3.8}$$

Therefore if $\tau = \frac{T}{N}$, then

$$A(t/T) < A(k/N) + \eta J \qquad \text{for } t \in [k\tau, (k+1)\tau]. \tag{3.9}$$

Applying Lemma 3.1 to $t \to A(\frac{t}{T} + k/N)$ and to the constant irreducible cooperative matrix $A(k/N) + \eta J$ yields then

$$\phi^{(T)}((k+1)\tau)\, \phi^{(T)}(k\tau)^{-1} \leqslant e^{\tau(A(k/N)+\eta J)}. \tag{3.10}$$

Since all the matrices are non negative, we can combine the inequalities to infer

$$\phi^{(T)}(T) \leqslant e^{\tau(A((N-1)/N)+\eta J)} \cdots e^{\tau(A(1/N)+\eta J)}\, e^{\tau(A(0/N)+\eta J)}. \tag{3.11}$$

We now apply Lemma 3.3 to obtain

$$\rho(\phi^{(T)}(T)) \leqslant \exp\left(\tau \sum_{k=0}^{N-1} s(A(k/N)+\eta J)\right) \times \prod_{k=0}^{N-2} \alpha(u_\eta^+(k/N), u_\eta^+((k+1)/N)). \tag{3.12}$$

By continuity of the functions α and u and the bounds (3.5) we have, for $N \geqslant N_0(\gamma)$ that ensures $\eta \leqslant \eta_0(\gamma)$,

$$C_1 = C_1(\gamma) = \sup_t \alpha(u_\eta^+(t), u_\eta^+(t+1/N)) < +\infty. \tag{3.13}$$

Combining with the bounds (3.5), we get

$$\frac{1}{T}\ln(\rho(\phi^{(T)}(T))) \leqslant \gamma + \frac{1}{N}\sum_{k=0}^{N-1} s(A(k/N)) + \frac{N-1}{T}\ln(C_1). \tag{3.14}$$

Therefore,

$$\limsup_{T\to+\infty} \frac{1}{T}\ln(\rho(\phi^{(T)}(T))) \leqslant \gamma + \frac{1}{N}\sum_{k=0}^{N-1} s(A(k/N)). \tag{3.15}$$

Hence, by the convergence of Riemann sums, letting $N \to +\infty$,

$$\limsup_{T\to+\infty} \frac{1}{T}\ln(\rho(\phi^{(T)}(T))) \leqslant \gamma + \int_0^1 s(A(u))\,\mathrm{d}u. \tag{3.16}$$

We conclude by letting $\gamma \to 0$. □

Proof of the lower bound. — The proof follows the same steps except that we need to be sure to have cooperative matrices. We define

$$\psi_\eta^-(M)_{ij} = (M_{ii} - \eta)\, \mathbb{1}_{(i=j)} + (M_{ij} - \eta)^+\, \mathbb{1}_{(i\neq j)}\,. \tag{3.17}$$

For $\tau = T/N$ we have

$$A(t/T) \geqslant \psi_\eta^-(A(k/N)) \qquad \text{for } t \in [k\tau, (k+1)\tau]\,. \tag{3.18}$$

We apply Lemma 3.1 and get

$$\phi^{(T)}((k+1)\tau)\phi^{(T)}(k\tau)^{-1} \geqslant e^{\tau\psi_\eta^-(A(k/N))}\,. \tag{3.19}$$

We know that $e^{\tau\psi_\eta^-(A(k/N))} \geqslant 0$, since $\psi_\eta^-(A(t))$ is cooperative for all $t \in [0,1]$. Then we resume the proof as before, mutatis mutandis. □

4. Postponed proofs of Lemmas

Proof of Lemma 3.1. —

First Step. — Assume that x_0 and y_0 are vectors such that $x_0 \geqslant 0$ and $y_0 > x_0$. If $x(t) = \phi_A(t)x_0$ and $y(t) = \phi_B(t)y_0$ then these are continuous functions such that $y(0) = y_0 > x(0) = x_0$. We let $z(t) = y(t) - x(t)$ and assume that there exists $t > 0$ such that $z(t) > 0$ is false. We let

$$\tau = \inf\{t > 0 : z(t) > 0 \text{ is false}\} \tag{4.1}$$

be the first time this happens. We have $0 < \tau < +\infty$.

Then on $[0,\tau)$ the function $z(t) > 0$ and there exists an index i such that $z_i(\tau) = 0$. Since z is C^1, we have $z_i'(\tau) \leqslant 0$. Therefore, since $A(\tau) \geqslant 0$ and $z(\tau) \geqslant 0$,

$$\begin{aligned} 0 &\geqslant z_i'(\tau) = y_i'(\tau) - x_i'(\tau) \\ &= (B(\tau)y(\tau) - A(\tau)x(\tau))_i = ((B-A)(\tau)y(\tau))_i + (A(\tau)z(\tau))_i \\ &\geqslant ((B-A)(\tau)y(\tau))_i\,. \end{aligned}$$

Since $B(0)$ is irreducible and for all t, $B(t)$ cooperative, we know from [2, Lemma 2] that, since $\tau > 0$, $y(\tau) > 0$. Moreover, $(B-A)(\tau) \geqslant 0$ and $(B-A)(\tau) \neq 0$ therefore $(B-A)(\tau)y(\tau) > 0$ and we obtain a contradiction. We have thus established that for all $t \geqslant 0$, $y(t) > x(t)$.

Second Step. — Consider a sequence $(y_0(n))_{n\in\mathbb{N}}$ of vectors such $y_0(n) > x_0$ and $y_0(n) \to x_0$, then, by the first step, for all $t > 0$,

$$\phi_B(t)y_0(n) > \phi_A(t)x_0 \,.$$

By continuity of the flow, letting $n \to +\infty$ we get that for all t, for all $x_0 \geqslant 0$,

$$\phi_B(t)x_0 \geqslant \phi_A(t)x_0 \,. \qquad \square$$

Proof of Lemma 3.2. — From [2, Lemma 2] we deduce that $e^A > 0$. Therefore, by Perron Frobenius, its spectral radius $\rho > 0$ is an isolated eigenvalue with a positive eigenvector u: $e^A u = \rho u$. Since

$$e^A Au = Ae^A u = \rho Au \tag{4.2}$$

and the eigenspace of ρ is of dimension 1, we obtain that for a constant c, $Au = cu$. Thus, $e^A u = e^c u$, that is $e^c = \rho$. We conclude by observing the functional relation between the spectra

$$\sigma(e^A) = \{e^z : z \in \sigma(A)\}\,, \tag{4.3}$$

that entails that $c = s(A)$ and that $s(A)$ is an isolated eigenvalue of A. $\square$

Proof of Lemma 3.3. — The proof is quite elementary and relies on the fact that $A \geqslant 0$ and $x \leqslant y$ implies $Ax \leqslant Ay$. Indeed,

$$\begin{aligned} A_1A_2\dots A_nu_n &= \rho(A_n)A_1A_2A_{n-1}u_n \\ &\leqslant \rho(A_n)\left\|\frac{u_n}{u_{n-1}}\right\|_\infty A_1A_2A_{n-1}u_{n-1} \\ &= \rho(A_n)\left\|\frac{u_n}{u_{n-1}}\right\|_\infty \rho(A_{n-1})A_1\dots A_{n-2}u_{n-1} \\ &\leqslant \dots \leqslant \prod_{k=1}^{n}\rho(A_k)\prod_{k=0}^{n-2}\left\|\frac{u_{n-k}}{u_{n-k-1}}\right\|_\infty u_1 \\ &\leqslant \prod_{k=1}^{n}\rho(A_k)\prod_{k=0}^{n-2}\left\|\frac{u_{n-k}}{u_{n-k-1}}\right\|_\infty \left\|\frac{u_1}{u_2}\right\|_\infty u_2 \\ &\leqslant \dots \leqslant \prod_{k=1}^{n}\rho(A_k)\prod_{k=0}^{n-2}\alpha(u_{n-k},u_{n-k-1})\ u_n\,. \end{aligned}$$

By Perron Frobenius theorem applied to the matrix $M = A_1\cdots A_n$, we have a vector $u_n > 0$ and a constant $r \geqslant 0$ such that $Mu_n \leqslant ru_n$, and therefore $\rho(M) \leqslant r$. The proof of the lower bound is similar. $\square$

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