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The bialgebra of specified graphs and external structures

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Abstract. We construct a Hopf algebra structure on the space of specified Feynman graphs of a quantum field theory. We introduce a convolution product and a semigroup of characters of this Hopf algebra with values in some suitable commutative algebra taking momenta into account. We then implement the renormalization described by A. Connes and D. Kreimer in [2] and the Birkhoff decomposition for two renormalization schemes: the minimal subtraction scheme and the Taylor expansion scheme.

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Keywords. Bialgebra, Hopf algebra, Feynman graphs, Convolution product, Birkhoff decomposition.

1. Introduction

Hopf algebras of Feynman graphs have been studied by A. Connes and D. Kreimer in [2], [4], [3], and [10] as a powerful tool to explain the combinatorics of renormalization in quantum field theory. In this note we are interested in the Hopf algebras of specified Feynman graphs, an example of which has been studied by A. Connes and D. Kreimer in [2].

In Section 2, we review the simpler case of Hopf algebras of locally one-particle irreducible (1PI) Feynman graphs, neglecting the specification at this stage. First we consider a theory of fields $\mathcal{T}$ (for example φ^3 in [2], φ^4 in [15], QED and QCD in [15], [16], etc.) which gives rise to Feynman graph types determined by $\mathcal{T}$: the type of a vertex is determined by the type of half-edges that are adjacent to it. We then construct a structure of commutative bialgebra $\widetilde{\mathcal{H}}_{\mathcal{T}}$ on the space of locally 1PI graphs of $\mathcal{T}$. The coproduct is given by

$$\Delta(\Gamma) = \sum_{\substack{\gamma\subseteq\Gamma \\ \Gamma/\gamma\in\mathcal{T}}} \gamma\otimes\Gamma/\gamma,$$

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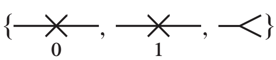
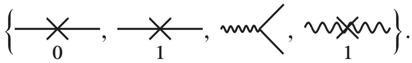
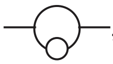
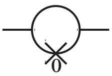
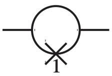
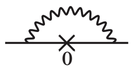
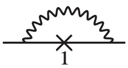
where the sum runs over all locally 1PI covering subgraphs of Γ such that the contracted subgraph Γ/γ is in the theory $\mathcal{T}$ (in other words, locally 1PI superficially divergent subgraphs [1]). The Hopf algebra $\mathcal{H}_\mathcal{T}$ is obtained by taking the quotient of the bialgebra $\widetilde{\mathcal{H}}_\mathcal{T}$ above by the ideal generated by $\mathbf{1} - \Gamma$, where the unit $\mathbf{1}$ is the empty graph and Γ is a 1PI graph without internal edges.

In Section 3, we introduce the specification. We are led by quantum field theory to distinguish between vertices of the same type. For example the list of vertices admitted in φ^3 theory and in QED are respectively

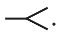
 and .

The contraction of a subgraph on a point rises a problem: For example in φ^3 theory, if we contract the subgraph  inside the graph , shall we get  or  ?

Similarly for QED, does the contraction of  inside  give  or ?

We will remedy to this in a purely combinatorial way, by introducing the specified graph

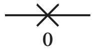
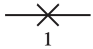
$$\bar{\Gamma} = (\Gamma, \underline{i})$$

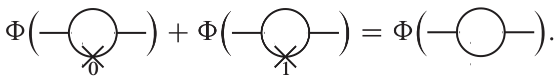
where $\underline{i}$ is a multi-index which identifies the type of residue of Γ, that is to say, the vertices obtained by contracting each connected components onto a point, for example res  = . The formula for the coproduct of specified graphs is then given by:

$$\Delta(\bar{\Gamma}) = \sum_{\substack{\bar{\gamma} \subseteq \bar{\Gamma} \\ \bar{\Gamma}/\bar{\gamma} \in \mathcal{T}}} \bar{\gamma} \otimes \bar{\Gamma}/\bar{\gamma},$$

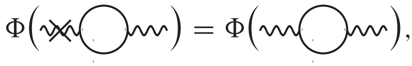
where the sum runs over all locally 1PI specified covering subgraphs $\bar{\gamma} = (\gamma, \underline{j})$ of $\bar{\Gamma} = (\Gamma, \underline{i})$ (see definition 1), such that the contracted subgraph $(\Gamma/(\gamma, \underline{j}), \underline{i})$ is in the theory $\mathcal{T}$. Here $\underline{j}$ is a multi-index that identifies the residue of each of the connected components of γ. The Hopf algebra $\mathcal{H}_\mathcal{T}$ is again obtained by identifying the specified graphs without internal edges with the unit.

In Section 4, we are interested in external structures. Feynman rules associate to each graph a function which depends on momenta associated with each half edge of the graph, with the constraints $p_e + p_{e'} = 0$ for each internal edge (ee') and

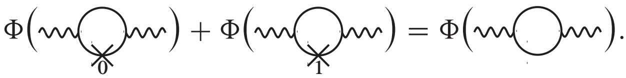
$\sum_{e\in \mathrm{st}(v)} p_e = 0$ for any vertex v, where $\mathrm{st}(v)$ is the set of half-edges adjacent to v. Feynman rules Φ do depend on the refined types of vertices, but do not depend on the overall specification. Refined types of vertices combine themselves in a nice way; for example, in φ^3 theory,  and  combine to an edge , namely [2]:



Similarly in QED we have



and



Considering the relations above we could have chosen only one type of bivalent vertex for φ^3 or for the electron edges in QED, and discard the bivalent vertex for the photon edges: these are the conventions adopted in [16]. We have chosen not to consider this simplification, in order to follow [2] more closely.

We introduce a semi-group G of characters of $\widetilde{\mathcal{H}}_{\mathcal{T}}$ with values in some suitable commutative algebra $\mathcal{B}$, and a convolution product $\circledast$ on G. We then implement in this framework the renormalization described by A. Connes and D. Kreimer in [2] (see also [5], Chapter 1, §5 and §6), replacing $\mathcal{B}$ by

$$\mathcal{A} \stackrel{\text{def}}{=} \mathcal{B}[z^{-1}, z]].$$

We show that each element of G has a unique Birkhoff decomposition for minimal renormalization scheme

$$\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+,$$

where

$$\mathcal{A}_+ \stackrel{\text{def}}{=} \mathcal{B}[[z]] \quad \text{and} \quad \mathcal{A}_- \stackrel{\text{def}}{=} z^{-1}\mathcal{B}[z^{-1}].$$

We also implement the Birkhoff decomposition associated with Taylor expansions in the algebra $\mathcal{B}$ itself, along the lines of [14]. The interest of the construction presented here is the purely combinatorial nature of the bialgebra $\widetilde{\mathcal{H}}_{\mathcal{T}}$ and the Hopf algebra $\mathcal{H}_{\mathcal{T}}$: all the dependence on momenta is removed in the target algebra $\mathcal{B}$ described in Section 4. The Feynman rules, given by the integration of these functions on the internal momenta, will be the subject of a future article.

Acknowledgements. We would likes to thank Kurusch Ebrahimi-Fard for discussions and remarks.

2. Hopf algebras of Feynman graphs

2.1. Basic definitions. A *Feynman graph* is a (non-planar) graph with a finite number of vertices and edges, which can be either internal or external: an internal edge is an edge connected at both ends to a vertex; an external edge is an edge with one open end, the other end being connected to a vertex. The edges are obtained by using half-edges.

More precisely, let us consider two finite sets $\mathcal{V}$ and $\mathcal{E}$. A graph Γ with $\mathcal{V}$ (resp. $\mathcal{E}$) as set of vertices (resp. half-edges) is defined as follows. Let

$$\sigma : \mathcal{E} \longrightarrow \mathcal{E}$$

be an involution and let

$$\partial : \mathcal{E} \longrightarrow \mathcal{V}.$$

For any vertex $v \in \mathcal{V}$ we denote by

$$\mathrm{st}(v) \stackrel{\text{def}}{=} \{e \in \mathcal{E} / \partial(e) = v\}$$

the set of half-edges adjacent to v. The fixed points of σ are the *external edges* and the *internal edges* are given by the pairs

$$\{e, \sigma(e)\}, \quad e \neq \sigma(e).$$

The graph Γ associated to these data is obtained by attaching half-edges $e \in \mathrm{st}(v)$ to any vertex $v \in \mathcal{V}$, and joining the two half-edges e and $\sigma(e)$ if $\sigma(e) \neq e$.

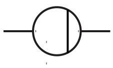
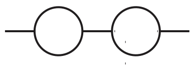
Several types of half-edges will be considered later on: the set $\mathcal{E}$ is partitioned into several pieces $\mathcal{E}_i$. In that case we ask that the involution σ respects the different types of half-edge, i.e. $\sigma(\mathcal{E}_i) \subset \mathcal{E}_i$.

We denote by $\mathcal{I}(\Gamma)$ the set of internal edges and by $\mathrm{Ext}(\Gamma)$ the set of external edges. The loop number of a graph Γ is given by:

$$L(\Gamma) = |\mathcal{I}(\Gamma)| - |\mathcal{V}(\Gamma)| + |\pi_0(\Gamma)|,$$

where $\pi_0(\Gamma)$ is the set of connected components of Γ.

A *one-particle irreducible graph* (in short, 1PI *graph*) is a connected graph which remains connected when we cut any internal edge. A disconnected graph is said to be *locally* 1PI if any of its connected components is 1PI.

 is 1PI and  is not.

A *covering subgraph of* Γ is a Feynman graph γ (not necessarily connected), obtained from Γ by cutting internal edges. In other words:

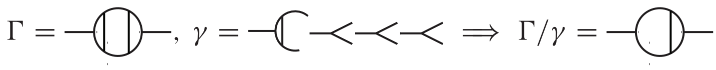
(1) $\mathcal{V}(\gamma) = \mathcal{V}(\Gamma)$;

(2) $\mathcal{E}(\gamma) = \mathcal{E}(\Gamma)$;

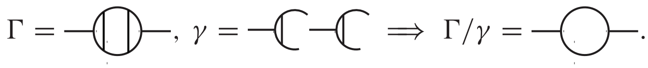
(3) $\sigma_\Gamma(e) = e \implies \sigma_\gamma(e) = e$;

(4) if $\sigma_\gamma(e) \neq \sigma_\Gamma(e)$, then we have $\sigma_\gamma(e) = e$ and $\sigma_\gamma(\sigma_\Gamma(e)) = \sigma_\Gamma(e)$.

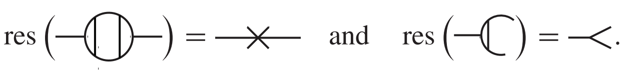
For any covering subgraph γ, the contracted graph Γ/γ is defined by shrinking all connected components of γ inside Γ onto a point, i.e.



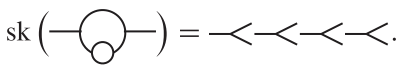
and



The *residue* of the graph Γ, denoted by $\mathrm{res}(\Gamma)$, is the contracted graph Γ/Γ. In other words, it is the only graph with no internal edge and the same external edges than those of Γ.

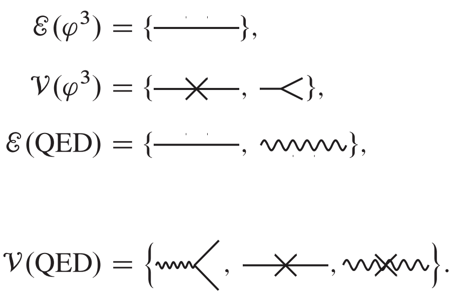


The *skeleton* of a graph Γ denoted by $\mathrm{sk}(\Gamma)$ is a graph obtained by cutting all internal edges, for example:



2.2. The bialgebra $\widetilde{\mathscr{H}}_{\mathcal{T}}$. We will work inside a physical theory $\mathcal{T}$, which involves Feynman graphs of some prescribed type: φ^3, φ^4, QED, QCD, etc. We denote by $\mathcal{E}(\mathcal{T})$ the set of possible types of half-edges and by $\mathcal{V}(\mathcal{T})$ the set of possible types of vertices.

Example 1. We have



An element of $\mathcal{V}(\mathcal{T})$ can be seen as a function from $\mathcal{E}(\mathcal{T})$ into $\mathbb{N}$ which to each type of half-edge associates the number of half-edges of that type arriving on the vertex in question. Actually the typology of vertices presented here is too coarse, we will return to this point in Section 2, with the introduction of specified graphs.

Let $\widetilde{V}_{\mathcal{T}}$ be the set of 1PI connected graphs Γ with edges in $\mathcal{E}(\mathcal{T})$ and vertices in $\mathcal{V}(\mathcal{T})$ such that $\mathrm{res}(\Gamma)$ is a vertex in $\mathcal{V}_{\mathcal{T}}$ (condition of superficial divergence; see [1], [2], and [10]). Let

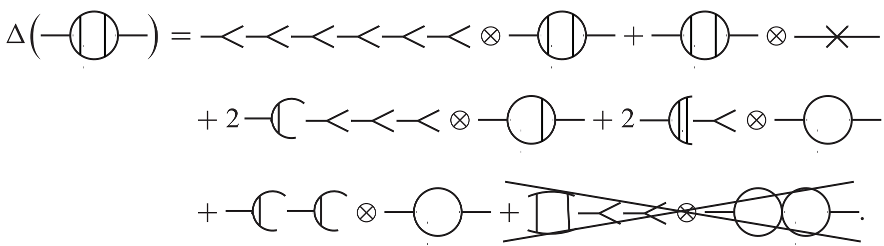
$$\widetilde{\mathcal{H}}_{\mathcal{T}} = S(\widetilde{V}_{\mathcal{T}})$$

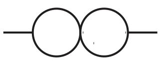
be the vector space generated by superficially divergent, locally 1PI, not necessarily connected Feynman graphs. The product is given by concatenation, the unit **1** is identified to the empty graph, and the coproduct is defined by

$$\Delta(\Gamma) \quad = \sum_{\substack{\gamma \subseteq \Gamma \\ \Gamma/\gamma \in \mathcal{T}}} \gamma \otimes \Gamma/\gamma.$$

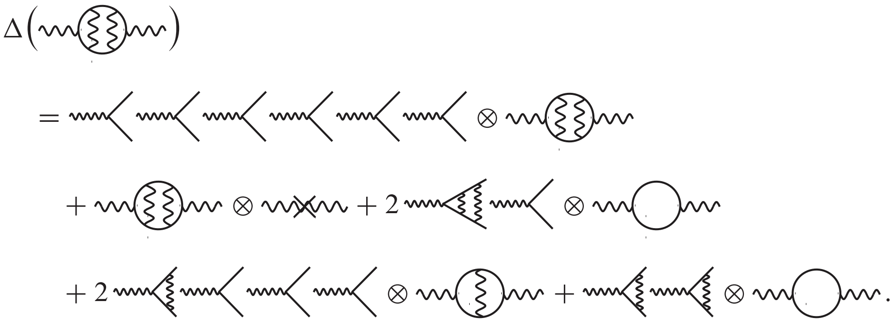
In the above sum, γ runs over all locally 1PI covering subgraphs of Γ such that the contracted subgraph Γ/γ is in the theory $\mathcal{T}$.

Example 2. In φ^3 theory we have



The last term is removed because  $\notin \varphi^3$.

In QED we have



Theorem 1. *Equipped with this coproduct, $\widetilde{\mathcal{H}}_{\mathcal{T}}$ is a bialgebra.*

Proof. Δ is coassociative. Indeed we have

$$(\Delta \otimes \mathrm{id})\Delta(\Gamma) = \sum_{\substack{\gamma\subseteq\Gamma \\ \Gamma/\gamma\in\mathcal{T}}} \Delta(\gamma)\otimes\Gamma/\gamma = \sum_{\substack{\delta\subseteq\gamma\subseteq\Gamma \\ \gamma/\delta;\,\Gamma/\gamma\in\mathcal{T}}} \delta\otimes\gamma/\delta\otimes\Gamma/\gamma$$

and

$$(\mathrm{id}\otimes\Delta)\Delta(\Gamma) = \sum_{\substack{\delta\subseteq\Gamma \\ \Gamma/\delta\in\mathcal{T}}} \delta\otimes\Delta(\Gamma/\delta) = \sum_{\substack{\delta\subseteq\Gamma\,;\tilde{\gamma}\subseteq\Gamma/\delta \\ (\Gamma/\delta)/\tilde{\gamma};\,\Gamma/\delta\in\mathcal{T}}} \delta\otimes\tilde{\gamma}\otimes(\Gamma/\delta)/\tilde{\gamma}.$$

For any covering subgraph δ of Γ such that $\Gamma/\delta\in\mathcal{T}$, there is an obvious bijection

$$\gamma \longmapsto \tilde{\gamma} = \gamma/\delta$$

from covering subgraphs of Γ containing δ such that $\Gamma/\gamma\in\mathcal{T}$ and $\gamma/\delta\in\mathcal{T}$, onto covering subgraphs of Γ/δ such that $(\Gamma/\delta)/\tilde{\gamma}\in\mathcal{T}$, given by shrinking δ; see [11]. For all $\tilde{\gamma}\subseteq\Gamma/\delta$ there exist a unique covering subgraph γ of Γ containing δ such that $\tilde{\gamma}\cong\gamma/\delta$ and we have $(\Gamma/\delta)/\tilde{\gamma}\cong\Gamma/\gamma$. We then obtain

$$(\mathrm{id}\otimes\Delta)\Delta(\Gamma) = \sum_{\substack{\delta\subseteq\gamma\subseteq\Gamma \\ \Gamma/\delta\in\mathcal{T}}} \delta\otimes\Delta(\Gamma/\delta) = \sum_{\substack{\delta\subseteq\gamma\subseteq\Gamma \\ \Gamma/\gamma;\,\gamma/\delta\in\mathcal{T}}} \delta\otimes\gamma/\delta\otimes\Gamma/\gamma.$$

The two expressions coincide, therefore Δ is coassociative. The counit is given by $\varepsilon(\Gamma) = 1$ if Γ has no internal edges, and $\varepsilon(\Gamma) = 0$ for any graph having at least one internal edge. The bialgebra $\widetilde{\mathcal{H}}_{\mathcal{T}}$ is graded and the grading is given by the number L. □

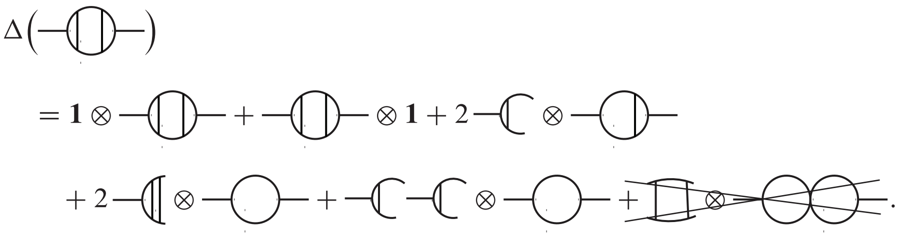
2.3. The Hopf algebra $\mathcal{H}_{\mathcal{T}}$. The Hopf algebra $\mathcal{H}_{\mathcal{T}}$ is given by identifying all elements of degree zero (the residues) to unit **1**:

$$\mathcal{H}_{\mathcal{T}} = \widetilde{\mathcal{H}}_{\mathcal{T}}/\mathcal{J},$$

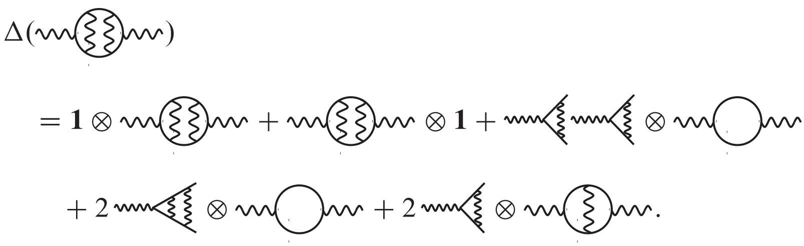
where $\mathcal{J}$ is the ideal generated by the elements $\mathbf{1}-\mathrm{res}(\Gamma)$ where Γ is an 1PI graph. $\mathcal{H}_{\mathcal{T}}$ is a connected graded bialgebra, it is therefore a connected graded Hopf algebra, which can be identified as a commutative algebra with $S(\mathcal{V}_{\mathcal{T}})$, where $\mathcal{V}_{\mathcal{T}}$ is the vector space generated by the 1PI connected Feynman graphs. The coproduct then becomes:

$$\Delta(\Gamma) = \mathbf{1}\otimes\Gamma + \Gamma\otimes\mathbf{1} + \sum_{\substack{\gamma \text{ proper subgraph of } \Gamma \\ \text{loc 1PI. } \Gamma/\gamma\in\mathcal{T}}} \gamma\otimes\Gamma/\gamma.$$

Example 3. In φ^3 theory we have



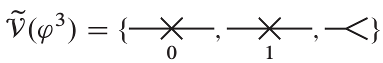
In QED we have



3. The specified Feynman graphs Hopf algebra

3.1. The bialgebra $\widetilde{\mathcal{H}}_{\mathcal{T}}$. In this paragraph, we denote by $\widetilde{\mathcal{V}}(\mathcal{T})$ the set of possible refined types of vertices: for $t \in \mathcal{V}(\mathcal{T})$, you can have a vertices of the same type t but with different refined type. For any refined type $\tilde{t} \in \widetilde{\mathcal{V}}(\mathcal{T})$ we denote by $[\tilde{t}]$ the underlying vertex type. We denote also $\tilde{t} = (t, i)$ where the index i serves to distinguish the refined types of same underlying type.

Example 4. We have



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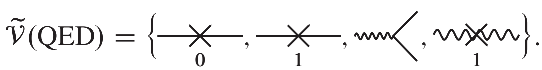


Remark 1. Note that the types of half-edges adjacent to a vertex v are not sufficient to determine its refined type.

Definition 1. A specified graph of theory $\mathcal{T}$ is a couple $(\Gamma, \underline{i})$ where

(1) Γ is a locally 1PI superficially divergent graph with half-edges and vertices of the type prescribed in $\mathcal{T}$;

(2) $\underline{i}\colon \pi_0(\Gamma) \to \mathbb{N}$, the values of $\underline{i}(\gamma)$ being prescribed by the possible types of vertex obtained by contracting the connected component γ on a point.

We will say that $(\gamma, \underline{j})$ is a *specified covering subgraph of* $(\Gamma, \underline{i})$, $((\gamma, \underline{j}) \subset (\Gamma, \underline{i}))$ if

(1) γ is a covering subgraph of Γ and

(2) if γ_0 is a full connected component of γ, i.e if γ_0 is also a full connected component of Γ, then $\underline{j}(\gamma_0) = \underline{i}(\gamma_0)$.

Remark 2. Sometimes we denote by $\bar{\Gamma} = (\Gamma, \underline{i})$ the specified graph, and we will write $\bar{\gamma} \subset \bar{\Gamma}$ for $(\gamma, \underline{j}) \subset (\Gamma, \underline{i})$.

Definition 2. Let be $(\gamma, \underline{j}) \subset (\Gamma, \underline{i})$. The contracted specified subgraph is written

$$\bar{\Gamma}/\bar{\gamma} = (\Gamma/\bar{\gamma}, \underline{i}),$$

where $\Gamma/\bar{\gamma}$ is obtained by contracting each connected component of γ on a point, and specifying the vertex obtained with $\underline{j}$.

Remark 3. The specification $\underline{i}$ is the same for the graph $\bar{\Gamma}$ and the contracted graph $\bar{\Gamma}/\bar{\gamma}$.

Let $\widetilde{\mathcal{H}}_{\mathcal{T}}$ be the vector space generated by the specified superficially divergent Feynman graphs of a field theory $\mathcal{T}$. The product is given by the concatenation, the unit **1** is identified with the empty graph and the coproduct is defined by

$$\Delta(\bar{\Gamma}) = \sum_{\substack{\bar{\gamma} \subseteq \bar{\Gamma} \\ \bar{\Gamma}/\bar{\gamma} \in \mathcal{T}}} \bar{\gamma} \otimes \bar{\Gamma}/\bar{\gamma},$$

where the sum runs over all locally 1PI specified covering subgraphs $\bar{\gamma} = (\gamma, \underline{j})$ of $\bar{\Gamma} = (\Gamma, \underline{i})$, such that the contracted subgraph $(\Gamma/(\gamma, \underline{j}), \underline{i})$ is in the theory $\mathcal{T}$.

Theorem 2. *Equipped with the coproduct* Δ, $\widetilde{\mathcal{H}}_{\mathcal{T}}$ *is a bialgebra.*

Proof. Δ is coassociative. Indeed we have

$$(\Delta \otimes \mathrm{id})\Delta(\bar{\Gamma}) = \sum_{\substack{\bar{\gamma} \subseteq \bar{\Gamma} \\ \bar{\Gamma}/\bar{\gamma} \in \mathcal{T}}} \Delta(\bar{\gamma}) \otimes \bar{\Gamma}/\bar{\gamma} = \sum_{\substack{\bar{\delta} \subseteq \bar{\gamma} \subseteq \bar{\Gamma} \\ \bar{\gamma}/\bar{\delta};\ \bar{\Gamma}/\bar{\gamma} \in \mathcal{T}}} \bar{\delta} \otimes \bar{\gamma}/\bar{\delta} \otimes \bar{\Gamma}/\bar{\gamma}.$$

and

$$(\mathrm{id} \otimes \Delta)\Delta(\bar{\Gamma}) = \sum_{\substack{\bar{\delta} \subseteq \bar{\Gamma} \\ \bar{\Gamma}/\bar{\delta} \in \mathcal{T}}} \bar{\delta} \otimes \Delta(\bar{\Gamma}/\bar{\delta}) = \sum_{\substack{\bar{\delta} \subseteq \bar{\Gamma};\ \bar{\alpha} \subseteq \bar{\Gamma}/\bar{\delta} \\ (\bar{\Gamma}/\bar{\delta})/\bar{\alpha};\ \bar{\Gamma}/\bar{\delta} \in \mathcal{T}}} \bar{\delta} \otimes \bar{\alpha} \otimes (\bar{\Gamma}/\bar{\delta})/\bar{\alpha}.$$

For any specified covering subgraph $\bar{\delta}$ of Γ such that $\bar{\Gamma}/\bar{\delta} \in \mathcal{T}$, there is an obvious bijection

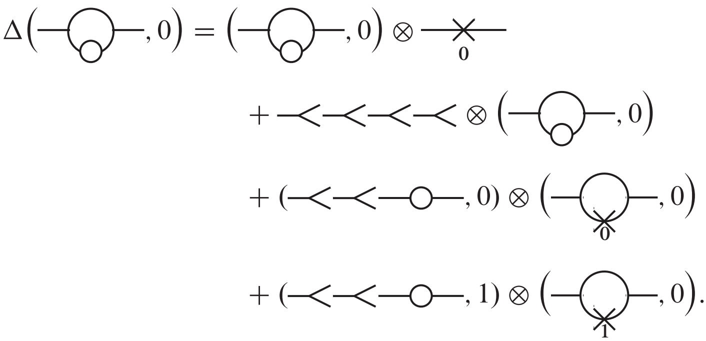
$$\bar{\gamma} \longmapsto \bar{\alpha} = \bar{\gamma}/\bar{\delta}$$

from specified covering subgraphs of $\bar{\Gamma}$ containing $\bar{\delta}$ such that $(\bar{\Gamma}/\bar{\delta})/\bar{\alpha}$, $\bar{\Gamma}/\bar{\delta} \in \mathcal{T}$, onto specified covering subgraphs of $\bar{\Gamma}/\bar{\delta}$ such that $\bar{\gamma}/\bar{\delta}$, $\bar{\Gamma}/\bar{\gamma} \in \mathcal{T}$, given by shrinking $\bar{\delta}$. Then for any specified covering subgraph $\bar{\alpha} = (\alpha, \underline{j})$ of $\Gamma/\bar{\delta}$ there exists a unique specified covering subgraph $\bar{\gamma} = (\gamma, \underline{j})$ of $\bar{\Gamma}$ such that $\delta \subseteq \gamma$ and $\alpha \cong \gamma/\bar{\delta}$, we have $\bar{\alpha} \cong \bar{\gamma}/\bar{\delta}$ and $(\bar{\Gamma}/\bar{\delta})/\bar{\alpha} \cong \bar{\Gamma}/\bar{\gamma}$. We obtain

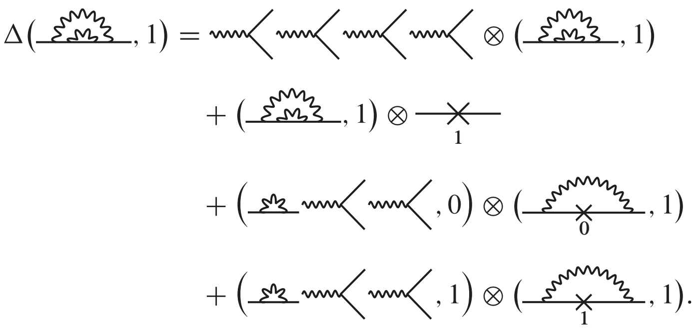
$$(\mathrm{id} \otimes \Delta)\Delta(\bar{\Gamma}) = \sum_{\substack{\bar{\delta} \subseteq \bar{\gamma} \subseteq \bar{\Gamma} \\ \bar{\gamma}/\bar{\delta};\ \bar{\Gamma}/\bar{\gamma} \in \mathcal{T}}} \bar{\delta} \otimes \bar{\gamma}/\bar{\delta} \otimes \bar{\Gamma}/\bar{\gamma}.$$

Then Δ is coassociative, $\widetilde{\mathcal{H}}_{\mathcal{T}}$ is a bialgebra where the counit is given by $\varepsilon(\bar{\Gamma}) = 1$ if $\bar{\Gamma}$ has no internal edges, and $\varepsilon(\bar{\Gamma}) = 0$ for any graph $\bar{\Gamma}$ having at least one internal edge. □

Example 5. In φ^3 theory we have



In QED we have



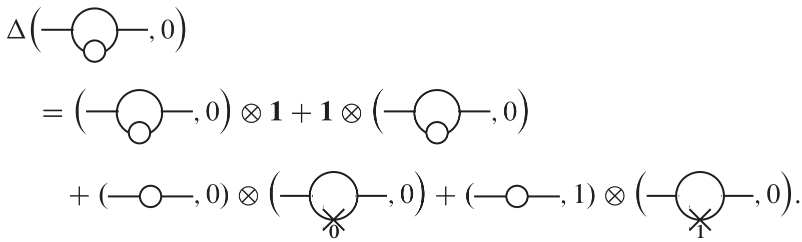
3.2. The Hopf algebra $\mathcal{H}_{\mathcal{T}}$. The Hopf algebra $\mathcal{H}_{\mathcal{T}}$ is given by identifying all elements of degree zero (the residues) to unit **1**:

$$\mathcal{H}_{\mathcal{T}} = \widetilde{\mathcal{H}}_{\mathcal{T}}/\mathcal{J},$$

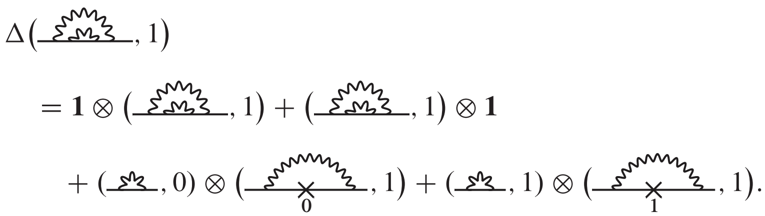
where $\mathcal{J}$ is the ideal generated by the elements $\mathbf{1} - \operatorname{res}(\bar{\Gamma})$, where $\bar{\Gamma}$ is an 1PI specified graph. $\mathcal{H}_{\mathcal{T}}$ is a connected graded bialgebra. It is therefore a connected graded Hopf algebra. The coproduct then becomes

$$\Delta(\bar{\Gamma}) = \mathbf{1} \otimes \bar{\Gamma} + \bar{\Gamma} \otimes \mathbf{1} + \sum_{\substack{\bar{\gamma} \text{ proper subgraph of } \bar{\Gamma} \\ \text{loc 1PI. } \bar{\Gamma}/\bar{\gamma} \in \mathcal{T}}} \bar{\gamma} \otimes \bar{\Gamma}/\bar{\gamma}.$$

Example 6. Taking the same graphs as in Example 5, we obtain in φ^3 theory (see [2])



In QED we have (see [16])



4. External structures

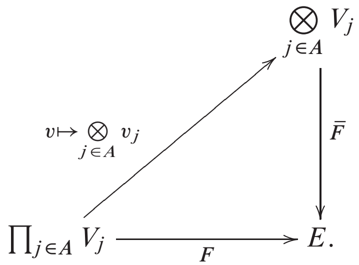
4.1. The unordered tensor product. Let A be a finite set, and let V_j be a vector space for any $j \in A$. The product $\prod_{j \in A} V_j$ is defined by

$$\prod_{j \in A} V_j \stackrel{\text{def}}{=} \Big\{ v : A \longrightarrow \coprod_{j \in A} V_j : v(i) \in V_i,\ i \in A \Big\}.$$

The space

$$V \stackrel{\text{def}}{=} \bigotimes_{j \in A} V_j$$

is then defined by the following universal property: for any vector space E and for any multilinear map $F\colon \prod_{j\in A} V_j \longrightarrow E$, there exists a unique linear map $\overline{F}$ such that the following diagram is commutative:



Remark 4. Let $(e_\lambda)_{\lambda\in\Lambda_j}$ be a basis of V_j. A basis of $\bigotimes_{j\in A} V_j$ is given by

$$\Big(f_\mu = \bigotimes_{j\in A} e_{\mu(j)}\Big)_{\mu\in\Lambda},$$

where

$$\Lambda = \prod_{j\in A}\Lambda_j = \Big\{\mu\colon A \longrightarrow \coprod_{j\in A}\Lambda_j : \mu(j)\in\Lambda_j\Big\}.$$

4.2. An algebra of $\mathcal{C}^\infty$ functions. Let D be an integer ≥ 1 (the dimension). For any half-edge e of Γ we denote by $p_e \in \mathbb{R}^D$ the corresponding moment. More precisely the moment space of graph Γ is defined by

$$\begin{aligned} W_\Gamma = \Big\{p\colon \mathcal{E}(\Gamma) \longrightarrow \mathbb{R}^D, \sum_{e\in \mathrm{st}(v)} p_e = 0\colon v\in\mathcal{V}(\Gamma),\\ p_e + p_{\sigma(e)} = 0,\\ e\in\mathcal{V}(\Gamma),\ e\neq\sigma(e)\Big\}. \end{aligned}$$

In particular, we have

$$W_{\mathrm{res}(\Gamma)} = \Big\{(p_1,\dots,p_{|\mathrm{Ext}(\Gamma)|})\colon p_j\in\mathbb{R}^D, \sum_{j=1}^{|\mathrm{Ext}(\Gamma)|} p_j = 0\Big\}.$$

We introduce then

$$V_\Gamma \stackrel{\text{def}}{=} \mathcal{C}^\infty(W_\Gamma, \mathbb{C}), \quad \Gamma \text{ connected},$$

$$V \stackrel{\text{def}}{=} \prod_{\Gamma \text{ connected}} V_\Gamma,$$

and, finally,

$$\mathcal{B} \stackrel{\text{def}}{=} \prod_{\Gamma \text{ connected or not}} V_\Gamma, \tag{1}$$

where the space

$$V_\Gamma \stackrel{\text{def}}{=} \bigotimes_{j\in A} V_{\Gamma_j}$$

is the unordered tensor product of the V_{Γ_j}'s and where the Γ_j's are the connected components of Γ. The space V_Γ is naturally identified with a subspace of $\mathcal{C}^\infty(W_\Gamma, \mathbb{C})$ via

$$\bigotimes_{j\in A} v_i(p) \stackrel{\text{def}}{=} \prod_{j\in A} v_j(p_j)$$

with

$$p_j \stackrel{\text{def}}{=} p_{|\mathcal{E}(\Gamma_j)}.$$

We equip also $\mathcal{B}$ with the unordered concatenation product denoted by "$\bullet$": for

$$v = \bigotimes_{j\in A} v_j \in V_\Gamma \quad \text{and} \quad v' = \bigotimes_{j\in B} v_j \in V_{\Gamma'}$$

(with $A \cap B = \emptyset$), the product $v \bullet v' \in V_{\Gamma\Gamma'}$ is defined by

$$v \bullet v' = \bigotimes_{j\in A \coprod B} v_j.$$

The product $\bullet$ is commutative by definition. This definition extends naturally to a bilinear product $\mathcal{B} \times \mathcal{B} \to \mathcal{B}$.

Proposition 1. *Let Γ_1 and Γ_2 be two (not necessarily connected), graphs and let $\Gamma = \Gamma_1\Gamma_2$. For any $v_1, v_1' \in V_{\Gamma_1}$ and $v_2, v_2' \in V_{\Gamma_2}$ we have the equality in V_Γ*

$$(v_1v_1') \bullet (v_2v_2') = (v_1 \bullet v_2)(v_1' \bullet v_2').$$

Proof. For $p_1 \in W_{\Gamma_1}$ and $p_2 \in W_{\Gamma_2}$ we have

$$\begin{aligned}
(v_1v_1') \bullet (v_2v_2')(p_1, p_2) &= v_1v_1'(p_1)v_2v_2'(p_2) \\
&= v_1(p_1)v_1'(p_1)v_2(p_2)v_2'(p_2) \\
&= v_1(p_1)v_2(p_2)v_1'(p_1)v_2'(p_2) \\
&= (v_1 \bullet v_2)(p_1, p_2)(v_1' \bullet v_2')(p_1, p_2) \\
&= (v_1 \bullet v_2)(v_1' \bullet v_2')(p_1, p_2).
\end{aligned}$$

□

4.3. The convolution product $\circledast$. Let Γ be a graph and γ a covering subgraph of Γ. We denote by

$$i_{\Gamma,\gamma} \colon V_{\Gamma/\gamma} \hookrightarrow V_\Gamma$$

and

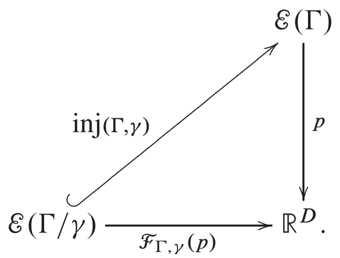
$$\pi_{\gamma,\Gamma} \colon V_\gamma \twoheadrightarrow V_\Gamma$$

two morphisms of algebras which are defined as follows.

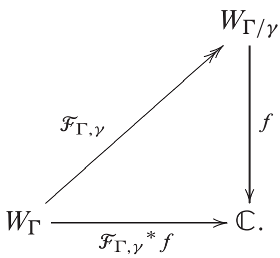
Let

$$\mathcal{F}_{\Gamma,\gamma} \colon W_\Gamma \longrightarrow W_{\Gamma/\gamma}$$

be the projection of W_Γ onto $W_{\Gamma/\gamma}$ by neglecting the internal moments of γ, that we can still be defined by the following commutative diagram:



where $\mathrm{inj}(\Gamma,\gamma)$ is the natural injection and $\mathcal{F}_{\Gamma,\gamma} = \mathrm{inj}(\Gamma,\gamma)^*$. We now consider the following commutative diagram:



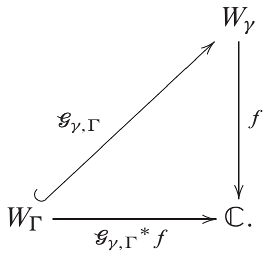
We define the injection $i_{\Gamma,\gamma}$ by

$$i_{\Gamma,\gamma} = {\mathcal{F}_{\Gamma,\gamma}}^*.$$

We denote by

$$\mathcal{G}_{\gamma,\Gamma} \colon W_\Gamma \hookrightarrow W_\gamma$$

the natural inclusion of W_Γ in W_γ and we consider the following commutative diagram:



We define the surjection $\pi_{\gamma,\Gamma}$ by

$$\pi_{\gamma,\Gamma} \stackrel{\text{def}}{=} \mathcal{G}_{\gamma,\Gamma}{}^* \colon V_\gamma \longrightarrow\!\!\!\!\rightarrow V_\Gamma.$$

Let $\widetilde{\mathcal{H}}_{\mathcal{T}}$ be the specified Feynman graphs bialgebra associated with a theory $\mathcal{T}$. We denote by $\mathcal{L}(\widetilde{\mathcal{H}}_{\mathcal{T}}, \mathcal{B})$ the space of $\mathbb{C}$-linear maps

$$\chi \colon \widetilde{\mathcal{H}}_{\mathcal{T}} \longrightarrow \mathcal{B}$$

and by $\widetilde{\mathcal{L}}(\widetilde{\mathcal{H}}_{\mathcal{T}}, \mathcal{B})$ the subspace of $\mathcal{L}(\widetilde{\mathcal{H}}_{\mathcal{T}}, \mathcal{B})$ of χ such that

(1) χ does not depend on the specification of Γ, in other words: $\chi(\Gamma, \underline{i}) = \chi(\Gamma)$;

(2) $\chi(\Gamma) \in V_\Gamma$ for any graph Γ, i.e. the projection of $\chi(\Gamma)$ on $V_{\Gamma'}$ vanishes for any graph $\Gamma' \neq \Gamma$;

(3) $\chi(\Gamma) = \mathbf{1}_{V_\Gamma}$ if Γ has no internal edges, where $\mathbf{1}_{V_\Gamma}$ denotes the constant function equal to $\mathbf{1}$ on W_Γ.

Then we define a convolution product $\circledast$ for all $\chi, \eta \in \widetilde{\mathcal{L}}(\widetilde{\mathcal{H}}_{\mathcal{T}}, \mathcal{B})$ and for all specified graphs $(\Gamma, \underline{i})$ by

$$\begin{aligned}(\chi \circledast \eta)(\Gamma, \underline{i}) &= (\chi \circledast \eta)(\Gamma) \\ &= \sum_{\substack{(\gamma,\underline{j})\subset(\Gamma,\underline{i}) \\ (\Gamma,\underline{i})/(\gamma,\underline{j})\in\mathcal{T}}} \pi_{\gamma,\Gamma}[\chi(\gamma)] i_{\Gamma,\gamma}[\eta(\Gamma/(\gamma,\underline{j}))]. \end{aligned} \tag{2}$$

The product used in the right hand side is the pointwise product in V_Γ.

Theorem 3. *The product $\circledast$ is associative.*

Proof. Let χ, η and ξ be three elements of $\widetilde{\mathcal{L}}(\widetilde{\mathcal{H}}_{\mathcal{T}}, \mathcal{B})$ and $(\Gamma, \underline{i})$ a specified graph. We denote indifferently $\overline{\Gamma} = (\Gamma, \underline{i})$, $\bar{\gamma} = (\gamma, \underline{j})$, $\bar{\delta} = (\delta, \underline{k})$ and $\overline{\Gamma}/\bar{\gamma} = (\Gamma/(\gamma, \underline{j}), \underline{i})$.

First, we have

$$\begin{aligned}
&\chi \circledast (\eta \circledast \xi)(\Gamma, \underline{i})\\
&\quad = \chi \circledast (\eta \circledast \xi)(\Gamma)\\
&\quad = \sum_{\substack{\bar\delta \subset \bar\Gamma \\ \bar\Gamma/\bar\delta \in \mathcal{T}}} \pi_{\delta,\Gamma}[\chi(\delta)] i_{\Gamma,\delta}[(\eta \circledast \xi)(\bar\Gamma/\bar\delta)]\\
&\quad = \sum_{\substack{\bar\delta \subset \bar\Gamma \\ \bar\Gamma/\bar\delta \in \mathcal{T}}} \pi_{\delta,\Gamma}[\chi(\delta)] i_{\Gamma,\delta}\Big[\sum_{\substack{\bar\alpha \subset \bar\Gamma/\delta \\ (\bar\Gamma/\bar\delta)/\bar\alpha \in \mathcal{T}}} \pi_{\alpha,\Gamma/\delta}[\eta(\alpha)] i_{\Gamma/\delta,\alpha}[\xi((\bar\Gamma/\bar\delta)/\bar\alpha)]\Big].
\end{aligned}$$

By identifying $\bar\alpha$ with $\bar\gamma/\bar\delta$ where $\bar\gamma$ is a subgraph of $\bar\Gamma$ containing $\bar\delta$, and $(\bar\Gamma/\bar\delta)/\bar\alpha$ with $\bar\Gamma/\bar\gamma$ we obtain

$$\begin{aligned}
&\chi \circledast (\eta \circledast \xi)(\Gamma, \underline{i})\\
&\quad = \chi \circledast (\eta \circledast \xi)(\Gamma)\\
&\quad = \sum_{\substack{\bar\delta \subset \bar\gamma \subset \bar\Gamma \\ \bar\gamma/\bar\delta;\ \bar\Gamma/\bar\gamma \in \mathcal{T}}} \pi_{\delta,\Gamma}[\chi(\delta)] i_{\Gamma,\delta}[\pi_{\gamma/\delta,\Gamma/\delta}[\eta(\bar\gamma/\bar\delta)] i_{\Gamma/\delta,\gamma/\delta}[\xi(\bar\Gamma/\bar\gamma)]]\\
&\quad = \sum_{\substack{\bar\delta \subset \bar\gamma \subset \bar\Gamma \\ \bar\gamma/\bar\delta;\ \bar\Gamma/\bar\gamma \in \mathcal{T}}} \pi_{\delta,\Gamma}[\chi(\delta)] i_{\Gamma,\delta} \pi_{\gamma/\delta,\Gamma/\delta}[\eta(\bar\gamma/\bar\delta)] i_{\Gamma,\delta} i_{\Gamma/\delta,\gamma/\delta}[\xi(\bar\Gamma/\bar\gamma)].
\end{aligned}$$

Secondly we have

$$\begin{aligned}
(\chi \circledast \eta) \circledast \xi(\Gamma, \underline{i}) &= (\chi \circledast \eta) \circledast \xi(\Gamma)\\
&= \sum_{\substack{\bar\gamma \subset \bar\Gamma \\ \bar\Gamma/\bar\gamma \in \mathcal{T}}} \pi_{\gamma,\Gamma}[\chi \circledast \xi(\gamma)] i_{\Gamma,\gamma}[\xi(\bar\Gamma/\bar\gamma)]\\
&= \sum_{\substack{\bar\delta \subset \bar\gamma \subset \bar\Gamma \\ \bar\gamma/\bar\delta;\ \bar\Gamma/\bar\gamma \in \mathcal{T}}} \pi_{\gamma,\Gamma}[\pi_{\delta,\gamma}\chi(\delta) i_{\delta,\gamma}[\eta(\bar\gamma/\bar\delta)]] i_{\Gamma,\gamma}[\xi(\bar\Gamma/\bar\gamma)]\\
&= \sum_{\substack{\bar\delta \subset \bar\gamma \subset \bar\Gamma \\ \bar\gamma/\bar\delta;\ \bar\Gamma/\bar\gamma \in \mathcal{T}}} \pi_{\gamma,\Gamma}\pi_{\delta,\gamma}[\chi(\delta)] \pi_{\gamma,\Gamma} i_{\delta,\gamma}[\eta(\bar\gamma/\bar\delta)] i_{\Gamma,\gamma}[\xi(\bar\Gamma/\bar\gamma)].
\end{aligned}$$

The two following diagrams commute:

$$\begin{CD}
W_\Gamma @>{\mathscr{F}_{\Gamma,\delta}}>> W_{\Gamma/\delta} \\
@V{\mathscr{G}_{\gamma,\Gamma}}VV @VV{\mathscr{G}_{\gamma/\delta,\Gamma/\delta}}V \\
W_\gamma @>>{\mathscr{F}_{\gamma,\delta}}> W_{\gamma/\delta}
\end{CD}$$

and

$$\begin{array}{ccc}
 & & W_{\Gamma/\delta} \\
 & \overset{\mathscr{F}_{\Gamma,\delta}}{\nearrow} & \Big\downarrow{\scriptstyle \mathscr{F}_{\Gamma/\delta,\gamma/\delta}} \\
W_\Gamma & \xrightarrow[\mathscr{F}_{\Gamma,\gamma}]{} & W_{\Gamma/\gamma}.
\end{array}$$

From the two preceding diagrams we obtain the following two commutative diagrams:

$$\begin{CD}
V_{\gamma/\delta} @>{i_{\gamma,\delta}}>> V_\gamma \\
@V{\pi_{\gamma/\delta,\Gamma/\delta}}VV @VV{\pi_{\gamma,\Gamma}}V \\
V_{\Gamma/\delta} @>>{i_{\Gamma,\delta}}> V_\Gamma
\end{CD}$$

and

$$\begin{array}{ccc}
 & & V_{\Gamma/\delta} \\
 & \overset{i_{\Gamma/\delta,\gamma/\delta}}{\nearrow} & \Big\downarrow{\scriptstyle \pi_{\Gamma,\delta}} \\
V_{\Gamma/\gamma} & \xrightarrow[i_{\Gamma,\gamma}]{} & V_\Gamma.
\end{array}$$

Hence we can write

$$(\chi \circledast \eta) \circledast \xi(\Gamma, \underline{i}) = \sum_{\substack{\bar\delta \subset \bar\gamma \subset \bar\Gamma \\ \bar\gamma/\bar\delta;\, \bar\Gamma/\bar\gamma \in \mathcal{T}}} \pi_{\delta,\Gamma}[\chi(\delta)] i_{\Gamma,\delta} \pi_{\gamma/\delta,\Gamma/\delta}[\eta(\bar\gamma/\bar\delta)] i_{\Gamma,\delta} i_{\Gamma/\delta,\gamma/\delta}[\xi(\bar\Gamma/\bar\gamma)].$$

Consequently, as $(\chi \circledast \eta) \circledast \xi(\Gamma, \underline{i}) = \chi \circledast (\eta \circledast \xi)(\Gamma, \underline{i})$ for any specified graph $(\Gamma, \underline{i})$, the product $\circledast$ is associative. ☐

Theorem 4. *Let*

$$G = \{\varphi \in \widetilde{\mathcal{L}}(\widetilde{\mathcal{H}}_{\mathcal{T}}, \mathcal{B}) \colon \varphi(\gamma\gamma') = \varphi(\gamma) \bullet \varphi(\gamma'),\ \varphi(\mathbf{1}) = \mathbf{1}_{\mathcal{B}}\}.$$

Equipped with the product $\circledast$, the set G is a subgroup of the semigroup of characters of $\widetilde{\mathcal{H}}_{\mathcal{T}}$ with values in $\mathcal{B}$.

Proof. Let φ, ψ be two elements of G and $\bar\Gamma = (\Gamma, i)$, $\bar\Gamma' = (\Gamma', i')$ two specified graphs. It is clear that, by definition, $\varphi \circledast \psi \in \widetilde{\mathcal{L}}(\widetilde{\mathcal{H}}_{\mathcal{T}}, \mathcal{B})$. Using Proposition 1 we have then

$$\begin{aligned}
(\varphi \circledast \psi)(\bar\Gamma\bar\Gamma') &= (\varphi \circledast \psi)(\Gamma\Gamma') \\
&= \sum_{\substack{\bar\gamma\bar\gamma' \subset \bar\Gamma\bar\Gamma' \\ \bar\Gamma\bar\Gamma'/\bar\gamma\bar\gamma' \in \mathcal{T}}} \pi_{\gamma\gamma',\Gamma\Gamma'}[\varphi(\gamma\gamma')] i_{\Gamma\Gamma',\gamma\gamma'}[\psi(\bar\Gamma\bar\Gamma'/\bar\gamma\bar\gamma')] \\
&= \sum_{\substack{\bar\gamma \subset \bar\Gamma,\, \bar\gamma' \subset \bar\Gamma' \\ \bar\Gamma/\bar\gamma;\, \bar\Gamma'/\bar\gamma' \in \mathcal{T}}} (\pi_{\gamma,\Gamma}[\varphi(\gamma)] \bullet \pi_{\gamma',\Gamma'}[\varphi(\gamma')]) \\
&\qquad\qquad (i_{\Gamma,\gamma}[\psi(\bar\Gamma/\bar\gamma)] \bullet i_{\Gamma',\gamma'}[\psi(\bar\Gamma'/\bar\gamma')]) \\
&= \sum_{\substack{\bar\gamma \subset \bar\Gamma,\, \bar\gamma' \subset \bar\Gamma' \\ \bar\Gamma/\bar\gamma;\, \bar\Gamma'/\bar\gamma' \in \mathcal{T}}} (\pi_{\gamma,\Gamma}[\varphi(\gamma)] i_{\Gamma,\gamma}[\psi(\bar\Gamma/\bar\gamma)]) \bullet (\pi_{\gamma',\Gamma'}[\varphi(\gamma')] i_{\Gamma',\gamma'}[\psi(\bar\Gamma'/\bar\gamma')]) \\
&= (\varphi \circledast \psi)(\bar\Gamma)(\varphi \circledast \psi)(\bar\Gamma').
\end{aligned}$$

The identity element e is defined by $e(\bar\Gamma) = \mathbf{1}_{V_\Gamma}$ if $\bar\Gamma$ is a specified graph of degree zero and $e(\bar\Gamma) = 0$ if it is not. Indeed, for any $\varphi \in \widetilde{\mathcal{L}}(\widetilde{\mathcal{H}}_{\mathcal{T}}, \mathcal{B})$, if $\bar\Gamma$ is of degree zero, we have

$$(e \circledast \varphi)(\bar\Gamma) = e(\bar\Gamma)\varphi(\bar\Gamma) = \varphi(\Gamma)$$

and, similarly,

$$(\varphi \circledast e)(\overline{\Gamma}) = \varphi(\overline{\Gamma})e(\overline{\Gamma}) = \varphi(\Gamma),$$

while, if $\overline{\Gamma}$ of degree ≥ 1, we have

$$\begin{aligned}(e \circledast \varphi)(\overline{\Gamma}) &= \sum_{(\overline{\Gamma})} \pi_{\gamma,\Gamma}[e(\bar{\gamma})] i_{\Gamma,\gamma}[\varphi(\overline{\Gamma}/\bar{\gamma})] \\ &= \pi_{\mathrm{sk}(\Gamma),\Gamma}[\mathbf{1}_{V_{\mathrm{sk}(\Gamma)}}] i_{\Gamma,\mathrm{sk}(\Gamma)}[\varphi(\Gamma)] \\ &= \varphi(\Gamma)\end{aligned}$$

and

$$\begin{aligned}(\varphi \circledast e)(\overline{\Gamma}) &= \sum_{(\Gamma)} \pi_{\gamma,\Gamma}[\varphi(\gamma)] i_{\Gamma,\gamma}[e(\overline{\Gamma}/\bar{\gamma})] \\ &= \pi_{\Gamma,\Gamma}[\varphi(\Gamma)] i_{\Gamma,\Gamma}[e(\mathrm{sk}(\overline{\Gamma}))] \\ &= \varphi(\Gamma).\end{aligned}$$

The inverse of an element φ of G is given by

$$\begin{aligned}\varphi^{\circledast -1}(\overline{\Gamma}) &= (e - (e - \varphi))^{\circledast -1}(\overline{\Gamma}) \\ &= \sum_n (e - \varphi)^{\circledast n}(\overline{\Gamma}).\end{aligned}$$

This sum is well defined: it stops at $n = q$ for specified graph $\overline{\Gamma}$ of degree q. Then we have:

$$\varphi^{\circledast -1} \circledast \varphi = \varphi \circledast \varphi^{\circledast -1} = e. \qquad \square$$

4.4. The Birkhoff decomposition. In this section we will explain how to renormalize a character φ of the specified graphs graded bialgebra $\widetilde{\mathcal{H}}_{\mathcal{T}}$. Let φ be a character with values in the unitary commutative algebra

$$\mathcal{A} \stackrel{\text{def}}{=} \mathcal{B}[z^{-1}, z]]$$

equipped with the minimal subtraction scheme

$$\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+,$$

where

$$\mathcal{A}_+ \stackrel{\text{def}}{=} \mathcal{B}[[z]], \quad \text{and} \quad \mathcal{A}_- \stackrel{\text{def}}{=} z^{-1}\mathcal{B}[z^{-1}].$$

Both $\mathcal{A}_-$ and $\mathcal{A}_+$ are two subalgebras of $\mathcal{A}$, with $\mathbf{1}_{\mathcal{A}} \in \mathcal{A}_+$. We denote by P the projection on $\mathcal{A}_-$ parallel to $\mathcal{A}_+$. The space of linear maps of $\widetilde{\mathcal{H}}_{\mathcal{T}}$ to $\mathcal{A}$ is equipped with the convolution product $\circledast$ defined by (2). We have verified in the previous paragraph that the space of characters $\widetilde{\mathcal{H}}_{\mathcal{T}}$ with values in $\mathcal{A}$ is a group for the convolution product $\circledast$.

Theorem 5. (1) *Any character $\varphi \in G$ has a unique Birkhoff decomposition in G*

$$\varphi = \varphi_-^{\circledast -1} \circledast \varphi_+$$

compatible with the renormalization scheme chosen, in other words, such that φ_+ takes its values in $\mathcal{A}_+$ and such that $\varphi_-(\overline{\Gamma}) \in \mathcal{A}_-$ for any specified graph $(\Gamma, \underline{i})$ of degree ≥ 1. The components φ_+ and φ_- are given by simple recursive formulas. For any $\overline{\Gamma}$ of degree zero (i.e. without internal edges) we put

$$\varphi_-(\overline{\Gamma}) = \varphi_+(\overline{\Gamma}) = \varphi(\overline{\Gamma}) = \mathbf{1}_{V_\Gamma}.$$

If we assume that $\varphi_-(\overline{\Gamma})$ and $\varphi_+(\overline{\Gamma})$ are known for $\overline{\Gamma}$ of degree $k \leq n-1$, we have then for any specified graph $\overline{\Gamma}$ of degree n

$$\varphi_-(\overline{\Gamma}) = \varphi_-(\Gamma) = -P\Big(\varphi(\Gamma) + \sum_{\substack{\bar{\gamma} \subsetneq \overline{\Gamma} \\ \overline{\Gamma}/\bar{\gamma} \in \mathcal{T}}} \pi_{\gamma,\Gamma}[\varphi_-(\gamma)] i_{\Gamma,\gamma}[\varphi(\Gamma/(\gamma, \underline{j}))]\Big)$$

and

$$\varphi_+(\overline{\Gamma}) = \varphi_+(\Gamma) = (I-P)\Big(\varphi(\Gamma) + \sum_{\substack{\bar{\gamma} \subsetneq \overline{\Gamma} \\ \overline{\Gamma}/\bar{\gamma} \in \mathcal{T}}} \pi_{\gamma,\Gamma}[\varphi_-(\gamma)] i_{\Gamma,\gamma}[\varphi(\Gamma/(\gamma, \underline{j}))]\Big).$$

(2) *Both φ_+ and φ_- are characters. We will call φ_+ the* ***renormalized character*** *and φ_- the* ***character of the counterterms.***

Proof. (1) The fact that φ_+ takes its values in $\mathcal{A}_+$ and that $\varphi_-(\Gamma) \in \mathcal{A}_-$ is immediate by definition of P, and we can verify by a simple calculation that $\varphi_+ = \varphi_- \circledast \varphi$:

$$\begin{aligned}\varphi_+(\Gamma) &= (I-P)\Big(\varphi(\Gamma) + \sum_{\substack{\bar{\gamma} \subsetneq \overline{\Gamma} \\ \overline{\Gamma}/\bar{\gamma} \in \mathcal{T}}} \pi_{\gamma,\Gamma}[\varphi_-(\gamma)] i_{\Gamma,\gamma}[\varphi(\Gamma/(\gamma, \underline{j}))]\Big) \\ &= \varphi(\Gamma) + \varphi_-(\Gamma) + \sum_{\substack{\bar{\gamma} \subsetneq \overline{\Gamma} \\ \overline{\Gamma}/\bar{\gamma} \in \mathcal{T}}} \pi_{\gamma,\Gamma}[\varphi_-(\gamma)] i_{\Gamma,\gamma}[\varphi(\Gamma/(\gamma, \underline{j}))].\end{aligned}$$

By using the fact that $\varphi_-(\Gamma) = \varphi(\Gamma) = \mathbf{1}_{V_\Gamma}$, for any graph Γ of degree zero we have

$$\varphi_- \circledast \varphi(\Gamma) = \sum_{\substack{\bar{\gamma} \subset \overline{\Gamma} \\ \overline{\Gamma}/\bar{\gamma} \in \mathcal{T}}} \pi_{\gamma,\Gamma}[\varphi_-(\gamma)] i_{\Gamma,\gamma}[\varphi(\Gamma/(\gamma, \underline{j}))]$$

$$= \pi_{\mathrm{sk}(\Gamma),\Gamma}[\varphi_-(\mathrm{sk}(\Gamma))]i_{\Gamma,\mathrm{sk}(\Gamma)}[\varphi(\Gamma)] + \pi_{\Gamma,\Gamma}[\varphi_-(\Gamma)]i_{\Gamma,\Gamma}[\varphi(\mathrm{res}(\Gamma))]$$

$$+ \sum_{\substack{\bar{\gamma}\subsetneq\bar{\Gamma} \\ \bar{\Gamma}/\bar{\gamma}\in\mathcal{T}}} \pi_{\gamma,\Gamma}[\varphi_-(\gamma)]i_{\Gamma,\gamma}[\varphi(\Gamma/(\gamma,\underline{j}))]$$

$$= \varphi(\Gamma) + \varphi_-(\Gamma) + \sum_{\substack{\bar{\gamma}\subsetneq\bar{\Gamma} \\ \bar{\Gamma}/\bar{\gamma}\in\mathcal{T}}} \pi_{\gamma,\Gamma}[\varphi_-(\gamma)]i_{\Gamma,\gamma}[\varphi(\Gamma/(\gamma,\underline{j}))].$$

Hence, $\varphi_+ = \varphi_- \circledast \varphi$ is equivalent to saying that $\varphi = \varphi_-^{\circledast-1} \circledast \varphi_+$.

We now assume that $\varphi = \varphi_-^{\circledast-1} \circledast \varphi_+ = \psi_-^{\circledast-1} \circledast \psi_+$. Thus we obtain

$$\varphi_+ \circledast \psi_+^{\circledast-1} = \varphi_- \circledast \psi_-^{\circledast-1}.$$

The right-hand side of the equality sends any specified graph of degree ≥ 1 in $\mathcal{A}_+$ but the left-hand side sends it in $\mathcal{A}_-$. Hence for any graph $\bar{\Gamma}$ of degree ≥ 1 we have

$$\varphi_+ \circledast \psi_+^{\circledast-1}(\Gamma) = \varphi_- \circledast \psi_-^{\circledast-1}(\Gamma) = 0.$$

Then we observe $\varphi_+ \circledast \psi_+^{\circledast-1} = \varphi_- \circledast \psi_-^{\circledast-1} = e$, which proves the uniqueness of the Birkhoff decomposition.

(2) We will just prove that φ_- is a character. Then $\varphi_+ = \varphi_- \circledast \varphi$ is also a character. The idea follows from the fact that the projection P satisfies the Rota-Baxter equality

$$P(a)P(b) = P(-ab + P(a)b + P(b)a). \tag{3}$$

Let φ be an element of G. The proof is obtained by induction on the degree of the graph $\Gamma\Gamma'$. For $\bar{\Gamma}\bar{\Gamma}'$ of degree zero we have $\mathbf{1}_{V_\Gamma} \bullet \mathbf{1}_{V'_\Gamma} = \mathbf{1}_{V_{\Gamma\Gamma'}}$. We assume that $\varphi_-(\Gamma\Gamma') = \varphi_-(\Gamma) \bullet \varphi_-(\Gamma')$ for any $\bar{\Gamma},\bar{\Gamma}' \in \widetilde{\mathcal{H}}_{\mathcal{T}}$ such that: $|\bar{\Gamma}| + |\bar{\Gamma}'| \leq d-1$ and we show the equality for $\bar{\Gamma}, \bar{\Gamma}' \in \widetilde{\mathcal{H}}_{\mathcal{T}}$ such that $|\bar{\Gamma}| + |\bar{\Gamma}'| = d$, where $|\bar{\Gamma}|$ denotes the degree of $\bar{\Gamma}$. We have

$$\varphi_-(\Gamma) \bullet \varphi_-(\Gamma') = P(X) \bullet P(Y),$$

where

$$X = \varphi(\Gamma) + \sum_{\substack{\bar{\gamma}\subsetneq\bar{\Gamma} \\ \bar{\Gamma}/\bar{\gamma}\in\mathcal{T}}} \pi_{\gamma,\Gamma}[\varphi_-(\gamma)]i_{\Gamma,\gamma}[\varphi(\Gamma/(\gamma,\underline{j}))]$$

and

$$Y = \varphi(\Gamma') + \sum_{\substack{\bar{\gamma}'\subsetneq\bar{\Gamma}' \\ \bar{\Gamma}'/\bar{\gamma}'\in\mathcal{T}}} \pi_{\gamma',\Gamma'}[\varphi_-(\gamma')]i_{\Gamma',\gamma'}[\varphi(\Gamma'/(\gamma',\underline{j}))].$$

We have

$$\varphi_-(\Gamma)\bullet\varphi_-(\Gamma') = P(X)\bullet P(Y) = P(-X\bullet Y + P(X)\bullet Y + X\bullet P(Y)).$$

Since $P(X) = -\varphi_-(\Gamma)$ and $P(Y) = -\varphi_-(\Gamma')$, we obtain

$$\varphi_-(\Gamma)\bullet\varphi_-(\Gamma') = -P(X\bullet Y + \varphi_-(\Gamma)\bullet Y + X\bullet\varphi_-(\Gamma')).$$

Therefore we obtain

$$\begin{aligned}
&\varphi_-(\Gamma)\bullet\varphi_-(\Gamma')\\
&\quad = -P\Big[\varphi(\Gamma)\bullet\varphi(\Gamma') + \varphi_-(\Gamma)\bullet\varphi(\Gamma') + \varphi(\Gamma)\bullet\varphi_-(\Gamma')\\
&\qquad\quad + \sum_{\substack{\bar\gamma\subsetneq\bar\Gamma\\ \bar\Gamma/\bar\gamma\in\mathcal{T}}} (\pi_{\gamma,\Gamma}[\varphi_-(\gamma)]i_{\Gamma,\gamma}[\varphi(\bar\Gamma/\bar\gamma)])\bullet(\varphi_-(\Gamma')+\varphi(\Gamma'))\\
&\qquad\quad + \sum_{\substack{\bar\gamma'\subsetneq\bar\Gamma'\\ \bar\Gamma'/\bar\gamma'\in\mathcal{T}}} (\pi_{\gamma',\Gamma'}[\varphi_-(\gamma')]i_{\Gamma',\gamma'}[\varphi(\bar\Gamma'/\bar\gamma')])\bullet(\varphi_-(\Gamma)+\varphi(\Gamma))\\
&\qquad\quad + \sum_{\substack{\bar\gamma\subsetneq\bar\Gamma,\,\bar\gamma'\subsetneq\bar\Gamma'\\ \bar\Gamma/\bar\gamma;\,\bar\Gamma'/\bar\gamma'\in\mathcal{T}}} (\pi_{\gamma,\Gamma}[\varphi_-(\gamma)]i_{\Gamma,\gamma}[\varphi(\bar\Gamma/\bar\gamma)])\\
&\qquad\qquad\qquad\qquad \bullet(\pi_{\gamma',\Gamma'}[\varphi_-(\gamma')]i_{\Gamma',\gamma'}[\varphi(\bar\Gamma'/\bar\gamma')])\Big].
\end{aligned}$$

The coproduct $\Delta(\bar\Gamma\bar\Gamma')$ is given by

$$\begin{aligned}
\Delta(\bar\Gamma\bar\Gamma') &= \bar\Gamma\bar\Gamma'\otimes \operatorname{res}(\bar\Gamma)\operatorname{res}(\bar\Gamma') + \operatorname{sk}(\bar\Gamma)\operatorname{sk}(\bar\Gamma')\otimes\bar\Gamma\bar\Gamma'\\
&\quad + \bar\Gamma\operatorname{sk}(\bar\Gamma')\otimes\bar\Gamma'\operatorname{res}(\bar\Gamma) + \bar\Gamma'\operatorname{sk}(\bar\Gamma)\otimes\bar\Gamma\operatorname{res}(\bar\Gamma')\\
&\quad + \sum_{\substack{\bar\gamma\subsetneq\bar\Gamma\\ \bar\Gamma/\bar\gamma\in\mathcal{T}}} \bar\gamma\bar\Gamma'\otimes(\bar\Gamma/\bar\gamma)\operatorname{res}(\bar\Gamma') + \bar\gamma\operatorname{sk}(\bar\Gamma')\otimes(\bar\Gamma/\bar\gamma)\bar\Gamma'\\
&\quad + \sum_{\substack{\bar\gamma'\subsetneq\bar\Gamma'\\ \bar\Gamma'/\bar\gamma'\in\mathcal{T}}} \bar\Gamma\bar\gamma'\otimes(\bar\Gamma'/\bar\gamma')\operatorname{res}(\bar\Gamma) + \bar\gamma'\operatorname{sk}(\bar\Gamma)\otimes\bar\Gamma(\bar\Gamma'/\bar\gamma')\\
&\quad + \sum_{\substack{\bar\gamma\subsetneq\bar\Gamma;\,\bar\gamma'\subsetneq\bar\Gamma'\\ \bar\Gamma/\bar\gamma;\,\bar\Gamma'/\bar\gamma'\in\mathcal{T}}} \bar\gamma\bar\gamma'\otimes(\bar\Gamma/\bar\gamma)(\bar\Gamma'/\bar\gamma').
\end{aligned}$$

Since $\varphi_-(\Gamma\Gamma') = -P(\varphi_- \circledast \varphi(\Gamma\Gamma') - \varphi_-(\Gamma\Gamma'))$, we have

$$
\begin{aligned}
&\varphi_-(\Gamma\Gamma')\\
&\quad = -P\Big[\pi_{\Gamma\Gamma',\,\Gamma\Gamma'}[\varphi_-(\Gamma\Gamma')]i_{\Gamma\Gamma',\,\Gamma\Gamma'}[\varphi(\operatorname{res}(\bar\Gamma)\operatorname{res}(\bar\Gamma'))]\\
&\qquad\qquad + \pi_{\operatorname{sk}(\Gamma)\operatorname{sk}(\Gamma'),\Gamma\Gamma'}[\varphi_-(\operatorname{sk}(\bar\Gamma)\operatorname{sk}(\bar\Gamma'))]i_{\Gamma\Gamma',\,\operatorname{sk}(\Gamma)\operatorname{sk}(\Gamma')}[\varphi(\Gamma\Gamma')]\\
&\qquad\qquad + \pi_{\Gamma\operatorname{sk}(\Gamma'),\,\Gamma\Gamma'}[\varphi_-(\Gamma\operatorname{sk}(\bar\Gamma'))]i_{\Gamma\Gamma',\,\Gamma\operatorname{sk}(\Gamma')}[\varphi(\Gamma'\operatorname{res}(\bar\Gamma))]\\
&\qquad\qquad + \pi_{\Gamma'\operatorname{sk}(\Gamma),\,\Gamma\Gamma'}[\varphi_-(\Gamma'\operatorname{sk}(\bar\Gamma))]i_{\Gamma\Gamma',\,\Gamma'\operatorname{sk}(\Gamma)}[\varphi(\Gamma\operatorname{res}(\bar\Gamma'))]\\
&\qquad\qquad + \sum_{\substack{\bar\gamma\subsetneq\bar\Gamma\\ \bar\Gamma/\bar\gamma\in\mathcal{T}}} \pi_{\gamma\gamma',\,\Gamma\Gamma'}[\varphi_-(\gamma\Gamma')]i_{\Gamma\Gamma',\,\gamma\gamma'}[\varphi(\bar\Gamma/\bar\gamma\operatorname{res}(\bar\Gamma'))]\\
&\qquad\qquad\qquad\qquad + \pi_{\gamma\gamma',\,\Gamma\Gamma'}[\varphi_-(\gamma\operatorname{sk}(\bar\Gamma'))]i_{\Gamma\Gamma',\,\gamma\gamma'}[\varphi(\bar\Gamma/\bar\gamma\Gamma')]\\
&\qquad\qquad + \sum_{\substack{\bar\gamma'\subsetneq\bar\Gamma'\\ \bar\Gamma'/\bar\gamma'\in\mathcal{T}}} \pi_{\gamma\gamma',\,\Gamma\Gamma'}[\varphi_-(\gamma'\Gamma)]i_{\Gamma\Gamma',\,\gamma\gamma'}[\varphi(\bar\Gamma'/\bar\gamma'\operatorname{res}(\bar\Gamma))]\\
&\qquad\qquad\qquad\qquad + \pi_{\gamma\gamma',\,\Gamma\Gamma'}[\varphi_-(\gamma'\operatorname{sk}(\bar\Gamma))]i_{\Gamma\Gamma',\,\gamma\gamma'}[\varphi(\bar\Gamma'/\bar\gamma'\Gamma)]\\
&\qquad\qquad + \sum_{\substack{\bar\gamma\subsetneq\bar\Gamma;\ \bar\gamma'\subsetneq\bar\Gamma'\\ \bar\Gamma/\bar\gamma;\ \bar\Gamma'/\bar\gamma'\in\mathcal{T}}} \pi_{\gamma\gamma',\,\Gamma\Gamma'}[\varphi_-(\gamma\gamma')]i_{\Gamma\Gamma',\,\gamma\gamma'}[\varphi(\bar\Gamma/\bar\gamma\bar\Gamma'/\bar\gamma')] - \varphi_-(\Gamma\Gamma')\Big].
\end{aligned}
$$

We notice that the first and last terms in the right side cancel each other. Since φ is a character, $\varphi(\operatorname{sk}(\bar\Gamma)) = \varphi_-(\operatorname{sk}(\bar\Gamma)) = \mathbf{1}_{V_\Gamma}$ and by the induction hypothesis we obtain

$$
\begin{aligned}
&\varphi_-(\Gamma\Gamma')\\
&\quad = -P\Big[\varphi(\Gamma)\bullet\varphi(\Gamma')\\
&\qquad\qquad + (\pi_{\Gamma\operatorname{sk}(\Gamma'),\,\Gamma\Gamma'}[\varphi_-(\Gamma)]\bullet\pi_{\Gamma\operatorname{sk}(\Gamma'),\,\Gamma\Gamma'}[\varphi_-(\operatorname{sk}(\bar\Gamma'))])\\
&\qquad\qquad\qquad \times (i_{\Gamma\Gamma',\,\Gamma\operatorname{sk}(\Gamma')}[\varphi(\Gamma')]\bullet i_{\Gamma\Gamma',\,\Gamma\operatorname{sk}(\Gamma')}[\varphi(\operatorname{res}(\bar\Gamma))])\\
&\qquad\qquad + (\pi_{\Gamma'\operatorname{sk}(\Gamma),\,\Gamma\Gamma'}[\varphi_-(\Gamma')]\bullet\pi_{\Gamma'\operatorname{sk}(\Gamma),\,\Gamma\Gamma'}[\varphi_-(\operatorname{sk}(\bar\Gamma))])\\
&\qquad\qquad\qquad \times (i_{\Gamma\Gamma',\,\Gamma'\operatorname{sk}(\Gamma)}[\varphi(\Gamma)]\bullet i_{\Gamma\Gamma',\,\Gamma'\operatorname{sk}(\Gamma)}[\varphi(\operatorname{res}(\bar\Gamma'))])\\
&\qquad\qquad + \sum_{\substack{\bar\gamma\subsetneq\bar\Gamma\\ \bar\Gamma/\bar\gamma\in\mathcal{T}}} (\pi_{\gamma,\,\Gamma}[\varphi_-(\gamma)]i_{\Gamma,\,\gamma}[\varphi(\bar\Gamma/\bar\gamma)])\bullet(\varphi_-(\Gamma')+\varphi(\Gamma'))\\
&\qquad\qquad + \sum_{\substack{\bar\gamma'\subsetneq\bar\Gamma'\\ \bar\Gamma'/\bar\gamma'\in\mathcal{T}}} (\pi_{\gamma',\,\Gamma'}[\varphi_-(\gamma')]i_{\Gamma',\,\gamma'}[\varphi(\bar\Gamma'/\bar\gamma')])\bullet(\varphi_-(\Gamma)+\varphi(\Gamma))
\end{aligned}
$$

$$+ \sum_{\substack{\bar\gamma \subsetneq \bar\Gamma;\ \bar\gamma' \subsetneq \bar\Gamma' \\ \bar\Gamma/\bar\gamma;\ \bar\Gamma'/\bar\gamma' \in \mathcal{T}}} (\pi_{\gamma,\Gamma}[\varphi_-(\gamma)] \bullet \pi_{\gamma',\Gamma'}[\varphi_-(\gamma')]) \times (i_{\Gamma,\gamma}[\varphi(\bar\Gamma/\bar\gamma)] \bullet i_{\Gamma',\gamma'}[\varphi(\bar\Gamma'/\bar\gamma')])\Big].$$

By Proposition 1 we can write

$$\begin{aligned}
\varphi_-(\Gamma\Gamma') = -P\Big[&\varphi(\Gamma) \bullet \varphi(\Gamma') + \varphi_-(\Gamma) \bullet \varphi(\Gamma') + \varphi(\Gamma) \bullet \varphi_-(\Gamma') \\
&+ \sum_{\substack{\bar\gamma \subsetneq \bar\Gamma \\ \bar\Gamma/\bar\gamma \in \mathcal{T}}} (\pi_{\gamma,\Gamma}[\varphi_-(\gamma)] i_{\Gamma,\gamma}[\varphi(\bar\Gamma/\bar\gamma)]) \bullet (\varphi_-(\Gamma') + \varphi(\Gamma')) \\
&+ \sum_{\substack{\bar\gamma' \subsetneq \bar\Gamma' \\ \bar\Gamma'/\bar\gamma' \in \mathcal{T}}} (\pi_{\gamma',\Gamma'}[\varphi_-(\gamma')] i_{\Gamma',\gamma'}[\varphi(\bar\Gamma'/\bar\gamma')]) \bullet (\varphi_-(\Gamma) + \varphi(\Gamma)) \\
&+ \sum_{\substack{\bar\gamma \subsetneq \bar\Gamma, \bar\gamma' \subsetneq \bar\Gamma' \\ \bar\Gamma/\bar\gamma;\ \bar\Gamma'/\bar\gamma' \in \mathcal{T}}} (\pi_{\gamma,\Gamma}[\varphi_-(\gamma)] i_{\Gamma,\gamma}[\varphi(\bar\Gamma/\bar\gamma)]) \\
&\qquad \bullet (\pi_{\gamma',\Gamma'}[\varphi_-(\gamma')] i_{\Gamma',\gamma'}[\varphi(\bar\Gamma'/\bar\gamma')])\Big] \\
= \varphi_-(\Gamma) \bullet \varphi_-(\Gamma'),&
\end{aligned}$$

which shows that φ_- is a character. □

4.5. Taylor expansions. We adapt here a construction from [7], §9, also used by [14], §3.7, (see also [8] and [9]).

Definition 3. Let $\mathcal{B}$ be the commutative algebra defined by (1). For $m \in \mathbb{N}$' the order m Taylor expansion operator is

$$P_m \in \operatorname{End}(\mathcal{B}), \quad P_m f(v) \stackrel{\text{def}}{=} \sum_{|\beta| \le m} \frac{v^\beta}{\beta!} \partial_0^\beta f, \tag{4}$$

where $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ with the usual notations, $\beta \le \alpha$ if and only if $\beta_i \le \alpha_i$ for all i,

$$|\beta| \stackrel{\text{def}}{=} \beta_1 + \cdots + \beta_n$$

and

$$v^\beta \stackrel{\text{def}}{=} \prod_{1 \le k \le n} v_k^{\beta_k}, \quad \beta! \stackrel{\text{def}}{=} \prod_{1 \le k \le n} \beta_k!, \quad \partial_0^\beta \stackrel{\text{def}}{=} \prod_{1 \le k \le n} \frac{\partial^{\beta_k}}{\partial v_k^{\beta_k}}_{|v_k=0}.$$

We can now implement the general momentum scheme using these projections P_m. Let $\widetilde{\mathcal{H}}_{\mathcal{T}} = \bigoplus_n \widetilde{\mathcal{H}}_{\mathcal{T},n}$ be the specified Feynman graphs graded bialgebra. We define a Birkhoff decomposition

$$\varphi = \varphi_-^{\circledast -1} \circledast \varphi_+.$$

The components φ_+ and φ_- are given by simple recursive formulas. For any $\overline{\Gamma}$ of degree zero (i.e without internal edges) we put

$$\varphi_-(\overline{\Gamma}) = \varphi_+(\overline{\Gamma}) = \varphi(\overline{\Gamma}) = \mathbf{1}_{V_\Gamma}.$$

If we assume that $\varphi_-(\overline{\Gamma})$ and that $\varphi_+(\overline{\Gamma})$ are known for $\overline{\Gamma}$ of degree $k \le m-1$, we have then for any specified graph $\overline{\Gamma}$ of degree m

$$\varphi_-(\overline{\Gamma}) = -P_m\Big(\varphi(\Gamma) + \sum_{\substack{\bar{\gamma}\subsetneq\overline{\Gamma} \\ \overline{\Gamma}/\bar{\gamma}\in\mathcal{T}}} \pi_{\gamma,\Gamma}[\varphi_-(\gamma)]i_{\Gamma,\gamma}[\varphi(\Gamma/(\gamma,\underline{j}))]\Big) \tag{5}$$

and

$$\varphi_+(\overline{\Gamma}) = (I - P_m)\Big(\varphi(\Gamma) + \sum_{\substack{\bar{\gamma}\subsetneq\overline{\Gamma} \\ \overline{\Gamma}/\bar{\gamma}\in\mathcal{T}}} \pi_{\gamma,\Gamma}[\varphi_-(\gamma)]i_{\Gamma,\gamma}[\varphi(\Gamma/(\gamma,\underline{j}))]\Big). \tag{6}$$

The operators P_m form a Rota–Baxter family in the sense of K. Ebrahimi-Fard, J. Gracia-Bondia, and F. Patras [7], Proposition 9.1 and Proposition 9.2. The analogue of the Rota-Baxter equality defined by the formula (3) is given by following theorem; see [7], and [14].

Theorem 6. *Let* Γ *be a graph, and let* $f, g \in V_\Gamma$*. The Taylor expansion operators fulfill for any* $s, t \in \mathbb{N}$

$$(P_s f)(P_t g) = P_{s+t}[(P_s f)g + f(P_t g) - fg]. \tag{7}$$

Proof. Denote by $\mu(f \otimes g) = fg$ the pointwise product on V_Γ. Using the Leibniz rule,

$$\partial \circ \mu = \mu \circ (\partial \otimes Id + Id \otimes \partial), \tag{8}$$

and the formula

$$\partial_0^\alpha P_s = \partial_0^\alpha \sum_{|\beta|\le s} \frac{v \longmapsto v^\beta}{\beta!}\partial_0^\beta = \sum_{|\beta|\le s} \frac{\partial_0^\alpha(v \longmapsto v^\beta)}{\beta!}\partial_0^\beta = \begin{cases} \partial_0^\alpha & \text{if } |\alpha| \le s, \\ 0 & \text{ontherwise,} \end{cases} \tag{9}$$

by (4) it suffices to check for any multi-index $|\alpha| \le s+t$ that

$$\begin{aligned}
&\partial_0^\alpha[(P_s f)g + f(P_t g) - fg] \\
&= \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \mu \circ (\partial_0^\beta \otimes \partial_0^{\alpha-\beta})[(P_s f) \otimes g + f \otimes (P_t g) - f \otimes g] \\
&= \sum_{\beta\le\alpha} \binom{\alpha}{\beta} [(\partial_0^\beta P_s f)(\partial_0^{\alpha-\beta} g) + (\partial_0^\beta f)(\partial_0^{\alpha-\beta} P_t g) - (\partial_0^\beta f)(\partial_0^{\alpha-\beta} g)]
\end{aligned}$$

$$= \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (\partial_0^\beta P_s f)(\partial_0^{\alpha-\beta} P_t g)$$

$$= \partial_0^\alpha [(P_s f).P_t g)].$$

Here we used that in the middle line, by formula (9) the contributions with $|\beta| > s$ or $|\alpha - \beta| > t$ give zero. For example, if $|\alpha - \beta| > t$ then $|\alpha| - |\beta| > t \implies |\beta| < |\alpha| - t$, since $|\alpha| \le s + t$ then $|\beta| < s$ such that:

$$\partial_0^\beta P_s = \partial_0^\beta, \quad \partial_0^{\alpha-\beta} P_t = 0, \quad \partial_0^{\alpha-\beta} P_t = 0,$$

then

$$(\partial_0^\beta P_s f)(\partial_0^{\alpha-\beta} g) + (\partial_0^\beta f)(\partial_0^{\alpha-\beta} P_t g) - (\partial_0^\beta f)(\partial_0^{\alpha-\beta} g) = 0,$$

and

$$(\partial_0^\beta P_s f)(\partial_0^{\alpha-\beta} P_t g) = 0.$$

Hence only terms with $|\beta| \le s$ and $|\alpha - \beta| \le t$ remain. We obtain

$$\partial_0^\beta P_s = \partial_0^\beta, \quad \partial_0^{\alpha-\beta} P_t = \partial_0^{\alpha-\beta}, \quad \partial_0^\beta P_s = \partial_0^\beta, \quad \partial_0^{\alpha-\beta} P_t = \partial_0^\beta,$$

and then we have

$$\begin{aligned}
&(\partial_0^\beta P_s f)(\partial_0^{\alpha-\beta} g) + (\partial_0^\beta f)(\partial_0^{\alpha-\beta} P_t g) - (\partial_0^\beta f)(\partial_0^{\alpha-\beta} g) \\
&= (\partial_0^\beta f)(\partial_0^{\alpha-\beta} g) \\
&= (\partial_0^\beta P_s f)(\partial_0^{\alpha-\beta} P_t g).
\end{aligned}$$

□

Theorem 7. *Let $\widetilde{\mathcal{H}}_\mathcal{T}$ be the specified graphs graded bialgebra and φ be a character with values in the unitary commutative algebra $\mathcal{B}$. Further let*

$$P_. : \mathbb{N} \longrightarrow \mathrm{End}(\mathcal{B})$$

be an indexed renormalization scheme, that is a family $(P_t)_{t\in\mathbb{N}}$ of endomorphisms such that

$$\mu \circ (P_s \otimes P_t) = P_{s+t} \circ \mu \circ [P_s \otimes \mathrm{Id} + \mathrm{Id} \otimes P_t - \mathrm{Id} \otimes \mathrm{Id}], \qquad (10)$$

for all $s, t \in \mathbb{N}$. Then the two maps φ_- and φ_+ defined by (5) and (6) are two characters.

Proof. We will just prove that φ_- is a character. Then $\varphi_+ = \varphi_- \circledast \varphi$ is also a character. For $\bar\Gamma, \bar\Gamma' \in \ker \varepsilon$, we write $\varphi_-(\bar\Gamma) = -P_{|\Gamma|}(\bar\varphi(\bar\Gamma))$, where

$$\bar\varphi(\bar\Gamma) = \varphi(\Gamma) + \sum_{\substack{\bar\gamma \subsetneq \bar\Gamma \\ \bar\Gamma/\bar\gamma \in \mathcal{T}}} \pi_{\gamma,\Gamma}[\varphi_-(\gamma)] i_{\Gamma,\gamma}[\varphi(\Gamma/(\gamma, \underline{j}))].$$

For proving this theorem we use formulas (5) and (10):

$$
\begin{aligned}
&\varphi_-(\bar{\Gamma}\bar{\Gamma}')\\
&= -P_{|\Gamma\Gamma'|}\Big[\varphi(\Gamma)\bullet\varphi(\Gamma')+\varphi_-(\Gamma)\bullet\varphi(\Gamma')+\varphi(\Gamma)\bullet\varphi_-(\Gamma')\\
&\qquad + \sum_{\substack{\bar{\gamma}\subsetneq\bar{\Gamma}\\ \bar{\Gamma}/\bar{\gamma}\in\mathcal{T}}} (\pi_{\gamma,\Gamma}[\varphi_-(\gamma)]i_{\Gamma,\gamma}[\varphi(\bar{\Gamma}/\bar{\gamma})])\bullet(\varphi_-(\Gamma')+\varphi(\Gamma'))\\
&\qquad + \sum_{\substack{\bar{\gamma}'\subsetneq\bar{\Gamma}'\\ \bar{\Gamma}'/\bar{\gamma}'\in\mathcal{T}}} (\pi_{\gamma',\Gamma'}[\varphi_-(\gamma')]i_{\Gamma',\gamma'}[\varphi(\bar{\Gamma}'/\bar{\gamma}')])\bullet(\varphi_-(\Gamma)+\varphi(\Gamma))\\
&\qquad + \sum_{\substack{\bar{\gamma}\subsetneq\bar{\Gamma},\bar{\gamma}'\subsetneq\bar{\Gamma}'\\ \bar{\Gamma}/\bar{\gamma};\,\bar{\Gamma}'/\bar{\gamma}'\in\mathcal{T}}} (\pi_{\gamma,\Gamma}[\varphi_-(\gamma)]i_{\Gamma,\gamma}[\varphi(\bar{\Gamma}/\bar{\gamma})])\\
&\qquad\qquad\qquad \bullet(\pi_{\gamma',\Gamma'}[\varphi_-(\gamma')]\; i_{\Gamma',\gamma'}[\varphi(\bar{\Gamma}'/\bar{\gamma}')])\Big]\\
&= -P_{|\Gamma|+|\Gamma'|}\Big[\Big(\varphi(\Gamma)+\sum_{\substack{\bar{\gamma}\subsetneq\bar{\Gamma}\\ \bar{\Gamma}/\bar{\gamma}\in\mathcal{T}}} (\pi_{\gamma,\Gamma}[\varphi_-(\gamma)]i_{\Gamma,\gamma}[\varphi(\bar{\Gamma}/\bar{\gamma})])\Big)\\
&\qquad\qquad \bullet\Big(\varphi(\Gamma')+\sum_{\substack{\bar{\gamma}'\subsetneq\bar{\Gamma}'\\ \bar{\Gamma}'/\bar{\gamma}'\in\mathcal{T}}} (\pi_{\gamma',\Gamma'}[\varphi_-(\gamma')]i_{\Gamma',\gamma'}[\varphi(\bar{\Gamma}'/\bar{\gamma}')])\Big)\\
&\qquad + \varphi_-(\Gamma)\bullet\Big(\varphi(\Gamma')+\sum_{\substack{\bar{\gamma}'\subsetneq\bar{\Gamma}'\\ \bar{\Gamma}'/\bar{\gamma}'\in\mathcal{T}}} (\pi_{\gamma',\Gamma'}[\varphi_-(\gamma')]i_{\Gamma',\gamma'}[\varphi(\bar{\Gamma}'/\bar{\gamma}')])\Big)\\
&\qquad + \varphi_-(\Gamma')\bullet\Big(\varphi(\Gamma)+\sum_{\substack{\bar{\gamma}\subsetneq\bar{\Gamma}\\ \bar{\Gamma}/\bar{\gamma}\in\mathcal{T}}} (\pi_{\gamma,\Gamma}[\varphi_-(\gamma)]i_{\Gamma,\gamma}[\varphi(\bar{\Gamma}/\bar{\gamma})])\Big)\Big]\\
&= -P_{|\Gamma|+|\Gamma'|}[\bar{\varphi}(\bar{\Gamma})\bullet\bar{\varphi}(\bar{\Gamma}') - P_{|\Gamma|}(\bar{\varphi}(\bar{\Gamma}))\bullet\bar{\varphi}(\bar{\Gamma}') - P_{|\Gamma'|}(\bar{\varphi}(\bar{\Gamma}'))\bullet\bar{\varphi}(\bar{\Gamma})]\\
&= P_{|\Gamma|+|\Gamma'|}[P_{|\Gamma'|}(\bar{\varphi}(\bar{\Gamma}'))\bullet\bar{\varphi}(\bar{\Gamma}) + P_{|\Gamma|}(\bar{\varphi}(\bar{\Gamma}))\bullet\bar{\varphi}(\bar{\Gamma}') - \bar{\varphi}(\bar{\Gamma})\bullet\bar{\varphi}(\bar{\Gamma}')]\\
&= (P_{|\Gamma|}(\bar{\varphi}(\bar{\Gamma})))\bullet(P_{|\Gamma'|}(\bar{\varphi}(\bar{\Gamma}')))\\
&= \varphi_-(\bar{\Gamma})\bullet\varphi_-(\bar{\Gamma}'). \qquad \square
\end{aligned}
$$

References

[1] S. Agarwala, *The geometry of renormalisation.* Ph.D. thesis. Johns Hopkins University, Baltimore, MD, 2009.

[2] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann–Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem. *Comm. Math. Phys.* **210** (2000), 249–273. MR 1748177 Zbl 1032.81026

[3] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann–Hilbert problem. II. The β-function, diffeomorphisms and the renormalization group. *Comm. in Math. Phys.* **216** (2001), 215–241. MR 1810779 Zbl 1042.81059

[4] A. Connes and D. Kreimer, Hopf algebras, renormalization and noncommutative geometry. *Comm. in Math. Phys.* **199** (1998), 203–242. MR 1660199 Zbl 0932.16038

[5] A. Connes and M. Marcolli, *Noncommutative geometry, quantum fields and motives.* American Mathematical Society Colloquium Publications 55. American Mathematical Society/Hindustan Book Agency, Providence, RI, 2008; Texts and Readings in Mathematics 47. Hindustan Book Agency, New Delhi, 2008. MR 2371808 Zbl 1209.58007 Zbl 1159.58004

[6] K. Ebrahimi-Fard, L. Guo, and D. Manchon, Birkhoff type decompositions and the Baker–Campbell–Hausdorff recursion. *Comm. Math. Phys.* **267** (2006), 821–845. MR 2249793 Zbl 1188.17020

[7] K. Ebrahimi-Fard, J. Gracia-Bondia, and F. Patras, A Lie theoretic approach to renormalization. *Comm. Math. Phys.* **276** (2007), 519–549. MR 2346399 Zbl 1136.81395

[8] K. Ebrahimi-Fard and F. Patras, Exponential renormalization. *Ann. Henri Poincaré* **11** (2010), 943–971. MR 2736528 Zbl 1208.81152

[9] K. Ebrahimi-Fard and F. Patras, Exponential renormalization II. Bogoliubov's *R*-operation and momentum subtraction schemes. *J. Math. Phys.* **53** (2012), Article Id. 083505.

[10] D. Kreimer, On the Hopf algebra structure of perturbative quantum field theories. *Adv. Theor. Math. Phys.* **2** (1998), 303–334. MR 1633004

[11] D. Manchon, On bialgebra and Hopf algebra of oriented graphs. *Confluentes Math.* **4** (2012).

[12] D. Manchon, Hopf algebras and renormalisation. In M. Hazewinkel (ed.), *Handbook of algebra.* Vol 5. Elsevier/North-Holland, Amsterdam, 2008, 365–427. MR 2523455 Zbl 1182.00008 (collection)

[13] M. E. Sweedler, *Hopf algebras.* W. A. Benjamin, New York, 1969. MR 0252485 Zbl 0194.32901

[14] E. Panzer, Hopf algebraic renormalization of Kreimer's toy model. Preprint 2012. arXiv:1202.3552

[15] W. D. van Suijlekom, The structure of renormalization Hopf algebras for gauge theories I. Representing Feynman graphs on BV-algebras. *Comm. Math. Phys.* **290** (2009), 291–319. MR 2520515 Zbl 1207.81083

[16] W. D. van Suijlekom, The Hopf algebra of Feynman graphs in quantum electrodynamics. *Lett. Math. Phys.* **77** (2006), 265–281. MR 2260373 Zbl 1160.81432

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