Horn's problem and Harish-Chandra's integrals. Probability density functions

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Abstract. Horn's problem – to find the support of the spectrum of eigenvalues of the sum $C = A + B$ of two n by n Hermitian matrices whose eigenvalues are known – has been solved by Klyachko and by Knutson and Tao. Here the probability distribution function (PDF) of the eigenvalues of C is explicitly computed for low values of n, for A and B uniformly and independently distributed on their orbit, and confronted to numerical experiments. Similar considerations apply to skew-symmetric and symmetric real matrices under the action of the orthogonal group. In the latter case, where no analytic formula is known in general and we rely on numerical experiments, curious patterns of enhancement appear.

Contents

1. Horn's problem for Hermitian matrices

1.1. A short review and summary of results. Let H_n be the n^2-dimensional (real) space of Hermitian matrices of size n. Any matrix $A \in H_n$ may be diagonalized by a unitary transformation $U \in U(n)$

$$A = U \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) U^\dagger. \tag{1}$$

Since permutations of S_n belong to $U(n)$, one may always assume that these (real) eigenvalues have been ordered according to

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n. \tag{2}$$

In the following we are mostly interested in the generic case where all these inequalities are strict, with no pair of equal eigenvalues. We denote by α the multiplets of eigenvalues thus ordered and by $\underset{\sim}{\alpha}$ the diagonal matrix

$$\underset{\sim}{\alpha} = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n).$$

Conversely, given such an α, the set of matrices A with that spectrum of eigenvalues forms the orbit Ω_α of $\underset{\sim}{\alpha}$ under the adjoint action of $U(n)$.

Horn's problem deals with the following question: given two multiplets α and β ordered as in (2), and $A \in \Omega_\alpha$ and $B \in \Omega_\beta$, what can be said about the eigenvalues γ of $C = A + B$? Obviously γ belongs to the hyperplane in $\mathbb{R}^n$ defined by

$$\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} (\alpha_i + \beta_i), \tag{3}$$

expressing that $\operatorname{tr} C = \operatorname{tr} A + \operatorname{tr} B$.

Horn [1] had conjectured the form of a set of necessary and sufficient inequalities to be satisfied by γ to belong to the spectrum of a matrix C. After contributions by several authors, see in particular [2], and [3] for a history of the problem, these conjectures were proved by Knutson and Tao [4, 5], see also [6], through the introduction of combinatorial objects, honeycombs and hives, see examples below.

What makes Horn's problem fascinating are its many facets [2, 3]. The problem has unexpected interpretations and applications in symplectic geometry, Schubert calculus, … and representation theory. In the latter, the above problem has a direct connection with the determination of Littlewood–Richardson (LR) coefficients, *i.e.,* with the computation of multiplicities in the decomposition of the tensor product of two irreducible polynomial representations of $GL(n)$.

In the present work, we show that for two random matrices A and B chosen uniformly on the orbits Ω_α and Ω_β, respectively, (uniformly in the sense of the U(n) Haar measure on these orbits), the probability density function (PDF) $p(\gamma \mid \alpha, \beta)$ of γ may be written in terms of the integral

$$\mathcal{H}(\alpha, \mathrm{i}\,x) = \int_{\mathrm{U}(n)} DU \, \exp(\mathrm{i}\operatorname{tr} \underline{x} U \underline{\alpha} U^\dagger) \tag{4}$$

where $\underline{x} = \operatorname{diag}(x_1, x_2, \ldots, x_n)$, in the general form

$$p(\gamma \mid \alpha, \beta) = \mathrm{const.} \, \Delta(\gamma)^2 \int d^n x \, \Delta(x)^2 \mathcal{H}(\alpha, \mathrm{i}\,x)\mathcal{H}(\beta, \mathrm{i}\,x)\mathcal{H}(\gamma, \mathrm{i}\,x)^*, \tag{5}$$

see Proposition 1 below.

In the present case this integral $\mathcal{H}(\alpha, x)$ is well known and has a simple expression, the so-called HCIZ integral [7, 8]. Then the x integration may be carried out, at least for low values of n, resulting in explicit expressions for the PDF.

The method generalizes to other sets of matrices and their adjoint orbits under appropriate groups. We discuss the case of the real orthogonal group acting on real symmetric or skew-symmetric matrices. Similarities and differences between these cases are pointed out.

Equation (5) is reminiscent of a well-known analogous formula for the determination of LR-coefficients in terms of characters. This is no coincidence, as there exist deep connections between the two problems: Horn's problem may be regarded as a semi-classical limit of the Littlewood–Richardson one, as anticipated by Heckman [9] and made explicit in [4, 5]. We intend to return to these connections in a forthcoming paper [10].

The general formula (5) is an explicit realization of the content of Theorem 4 in [5] and may have been known to many people, see [11, 12, 13, 14] for related work. The main original results of the present paper are the detailed calculations carried out in various cases of low dimension, and their confrontation with numerical "experiments." This work may thus be regarded as an exercise in *concrete and experimental mathematics.…*

1.2. The probability density function (PDF). Let A be a random matrix of H_n chosen uniformly on the orbit Ω_α, *i.e.,* $A = U\underline{\alpha}U^\dagger$, with U uniformly distributed in U(n) in the sense of the normalized Haar measure DU. The characteristic function of the random variable A may be written as

$$\varphi_A(X) := \mathbb{E}(e^{\mathrm{i}\operatorname{tr} XA}) = \int_{\mathrm{U}(n)} DU \, \exp(\mathrm{i}\operatorname{tr} X U \underline{\alpha} U^\dagger) \tag{6}$$

where $X \in H_n$. This is referred to as the Fourier transform of the orbital measure in the literature. For two independent random matrices $A \in \Omega_\alpha$ and $B \in \Omega_\beta$, the characteristic function of the sum $C = A + B$ is the product

$$\mathbb{E}(e^{\mathrm{i\,tr}\,XC}) = \varphi_A(X)\varphi_B(X)$$

from which the PDF of C may be recovered by an inverse Fourier transform

$$p(C \mid \alpha, \beta) = \frac{1}{(2\pi)^{n^2}} \int DX e^{-\mathrm{i\,tr}\,XC} \varphi_A(X)\varphi_B(X), \tag{7}$$

which is, *a priori*, a distribution (in the sense of generalized function).

Here DX stands for the Lebesgue measure on Hermitian matrices. If $X = U_X \underline{x} U_X^\dagger$, that measure may be expressed as $DX = \kappa \prod_i dx_i \Delta(x)^2 DU_X$, where[1]

$$\kappa = (2\pi)^{n(n-1)/2} / \prod_{p=1}^{n} p! \tag{8}$$

and

$$\Delta(x) = \prod_{i<j}(x_i - x_j) \tag{9}$$

is the Vandermonde determinant of the x's. It is clear that $\varphi_A(X)$ and $\varphi_B(X)$ depend only on the eigenvalues α_i, β_i and x_i of A, B and X, namely

$$\varphi_A(X) = \mathcal{H}(\alpha, \mathrm{i}x) \quad \varphi_B(X) = \mathcal{H}(\beta, \mathrm{i}x)$$

in terms of the HCIZ integral introduced above. Also $p(C \mid \alpha, \beta)$ is invariant under conjugation of C by unitary matrices of $\mathrm{U}(n)$ and is thus only a function of the eigenvalues γ_i of C. The PDF of the γ's must incorporate the Jacobian from the measure, hence

$$p(\gamma \mid \alpha, \beta) = \kappa \Delta(\gamma)^2 p(C \mid \alpha, \beta)$$

$$= \frac{\kappa^2}{(2\pi)^{n^2}} \Delta(\gamma)^2 \int_{\mathbb{R}^n} \prod_{i=1}^{n} dx_i \, \Delta(x)^2 \mathcal{H}(\alpha, \mathrm{i}x)\mathcal{H}(\beta, \mathrm{i}x)\mathcal{H}(\gamma, \mathrm{i}x)^* \tag{10}$$

with three copies of the HCIZ integral

$$\mathcal{H}(\alpha, \mathrm{i}x) = \hat{\kappa}\,\mathrm{i}^{-n(n-1)/2} \frac{\det e^{\mathrm{i}x_i\alpha_j}}{\Delta(x)\Delta(\alpha)} \tag{11}$$

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[1] For this and other normalizing constants, see Appendix A.

where[1]

$$\hat{\kappa} = \prod_{p=1}^{n-1} p!. \tag{12}$$

Thus finally

Proposition 1. *The probability distribution function of eigenvalues γ, given α and β, is*

$$p(\gamma \mid \alpha, \beta) = \frac{\kappa^2 \hat{\kappa}^3}{(2\pi)^{n^2}} i^{-n(n-1)/2} \frac{\Delta(\gamma)}{\Delta(\alpha)\Delta(\beta)}$$
$$\int \frac{d^n x}{\Delta(x)} \det e^{i x_i \alpha_j} \det e^{i x_i \beta_j} \det e^{-i x_i \gamma_j}. \tag{13}$$

where κ and $\hat{\kappa}$ are given in (8) *and* (12).

Note that while α and β are ordered as in (2), the integration over the group mixes the order of the γ's and the PDF (13) thus applies to unordered γ's. In particular p is normalized by $\int_{\mathbb{R}^n} d^n \gamma p(\gamma \mid \alpha, \beta) = 1$.

Let's us sketch the way the above integral may be handled. One writes for each determinant

$$\det e^{i x_i \alpha_j} = e^{i \frac{1}{n} \sum_{j=1}^{n} x_j \sum_{k=1}^{n} \alpha_k} \det e^{i(x_i - \frac{1}{n} \sum x_k)\alpha_j}$$

$$= e^{i \frac{1}{n} \sum_{j=1}^{n} x_j \sum_{k=1}^{n} \alpha_k} \sum_{P \in S_n} \varepsilon_P \prod_{j=1}^{n-1} e^{i(x_j - x_{j+1})(\sum_{k=1}^{j} \alpha_{P(k)} - \frac{j}{n} \sum_{k=1}^{n} \alpha_k)}, \tag{14}$$

where ε_P is the signature of permutation P.

In the product of the three determinants, the prefactor

$$e^{i \sum_{j=1}^{n} x_j \sum_{k=1}^{n} (\alpha_k + \beta_k - \gamma_k)/n}$$

yields, upon integration over $\frac{1}{n} \sum x_j$, 2π times a Dirac delta of $\sum_k (\alpha_k + \beta_k - \gamma_k)$, expressing the conservation of the trace in Horn's problem. One is left with an integration over $(n-1)$ variables[2] $u_j := x_j - x_{j+1}$ of $(n!)^3$ terms of the form

$$\int_{\mathbb{R}^{n-1}} \frac{d u_j}{\widetilde{\Delta}(u)} \prod_j e^{i u_j A_j (P, P', P'')},$$

where

$$\widetilde{\Delta}(u) := \prod_{1 \leq i < j \leq n} (u_i + u_{i+1} + \cdots + u_{j-1}) \tag{15}$$

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[2] The Jacobian from $(x_1, \ldots, x_n)$ to $(\frac{1}{n} \sum x_j, u_1, \ldots, u_{n-1})$ is $(-1)^{n-1}$.

and

$$A_j(P, P', P'') = \sum_{k=1}^{j}(\alpha_{P(k)} + \beta_{P'(k)} - \gamma_{P''(k)}) - \frac{j}{n}\sum_{k=1}^{n}(\alpha_k + \beta_k - \gamma_k). \quad (16)$$

It is also easy to see that one may absorb P'' through a redefinition of the x's by $P'': x_j \mapsto x_{P''(j)}$ (which introduces a welcome sign $\varepsilon_{P''}$ from the Vandermonde $\Delta(x)$) and a change of P and P' into $P''P$ and $P''P'$. Thus P'' may be taken to be the trivial permutation I in the above, with an overall factor $n!$. Hence

$$p(\gamma \mid \alpha, \beta) = \frac{\kappa^2 \hat{\kappa}^3 n!}{(2\pi)^{n(n-1)}}\delta\Big(\sum_k(\alpha_k + \beta_k - \gamma_k)\Big)\frac{\Delta(\gamma)}{\Delta(\alpha)\Delta(\beta)}\mathcal{J}_n(\alpha, \beta; \gamma) \quad (17)$$

$$\mathcal{J}_n(\alpha, \beta; \gamma) = \frac{\mathrm{i}^{-n(n-1)/2}}{2^{n-1}\pi^{n-1}}\sum_{P,P'\in S_n}\varepsilon_P \varepsilon_{P'}\int \frac{d^{n-1}u}{\widetilde{\Delta}(u)}\prod_{j=1}^{n-1}e^{\mathrm{i}u_j A_j(P,P',I)}. \quad (18)$$

This is the expression that we are going to study in more detail for $n = 2$, $n = 3$ and (to a lesser extent) $n = 4$, $n = 5$. The constant in front of (17) reads

$$\frac{\kappa^2\hat{\kappa}^3 n!}{(2\pi)^{n(n-1)}} = \frac{\prod_1^{n-1}p!}{n!},$$

which is equal to $\frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{12}{5}, \dots$ for $n = 2, 3, 4, 5, \dots$

Remarks. 1. Note that in that computation of p, the last term in the right hand side of (16) drops out, because of the relation (3) embodied in the Dirac delta. The merit of that term is to make explicit the invariance of A_j under a simultaneous translation of all γ's: for all i, $\gamma_i \to \gamma_i + c$, expressing the fact that the PDF of eigenvalues of $C = A + B$ takes the same values as that of $A + B + cI$, on a shifted support.

2. Convergence of $\mathcal{J}_n$. $\mathcal{J}_n$ in (18) is a double sum over the symmetric group S_n of the Fourier transform of $\widetilde{\Delta}(u)^{-1}$ evaluated at $A_j(P, P', I)$. Each of these integrals is absolutely convergent at infinity for $n > 2$, and is only semi-convergent for $n = 2$. Each one exhibits poles for vanishing partial sums

$$(u_i + u_{i+1} + \cdots + u_{j-1}),$$

(*i.e.*, $x_i = x_j$), but the sum is regular at these points, as a result of the (x_i, x_j) anti-symmetry of the determinant in (14). This enables us to introduce a Cauchy principal value prescription at each of these points, including infinity, and to compute each integral on the right hand side of (18) by repeated contour integrals (generalized Dirichlet integrals), see below. The resulting function of γ is a piecewise polynomial of degree $(n-1)(n-2)/2$, a "box spline" as defined in [15].

3. In accordance with Theorem 4 of [5], the interpretation of $\mathcal{J}_n$ is that it gives the volume of the polytope in honeycomb space. This will be discussed in more detail in [10].

4. The normalization of $\mathcal{J}_n$ follows from that of p

$$n! \int_{\substack{\gamma_n \leq \gamma_{n-1} \leq \cdots \leq \gamma_1 \\ \sum_i \gamma_i = \sum_i \alpha_i + \sum_i \beta_i}} d^{n-1}\gamma\, p(\gamma \mid \alpha, \beta) = 1$$

hence

$$\int_{\substack{\gamma_n \leq \gamma_{n-1} \leq \cdots \leq \gamma_1 \\ \sum_i \gamma_i = \sum_i \alpha_i + \sum_i \beta_i}} d^{n-1}\gamma\, \frac{\Delta(\gamma)}{\Delta(\alpha)\Delta(\beta)} \mathcal{J}_n(\alpha, \beta; \gamma) = \frac{1}{\prod_1^{n-1} p!} \tag{19}$$

which equals $1, \frac{1}{2}, \frac{1}{12}, \frac{1}{288}, \ldots$ for $n = 2, 3, 4, 5$.

1.3. The case $n = 2$

1.3.1. Direct calculation. For $n = 2$, the averaging of $B = \mathrm{diag}(\beta_1, \beta_2)$ over the U(2) unitary group may be worked out directly, since in $UBU^\dagger$, one may take simply $U = \exp{-i\sigma_2\psi}$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ the Pauli matrix, ψ an Euler angle between 0 and π with the measure $\frac{1}{2}\sin\psi\,d\psi$. The (unordered) eigenvalues of $A + UBU^\dagger$ are then

$$\gamma_{1,2} = \frac{1}{2}\left[\alpha_1 + \alpha_2 + \beta_1 + \beta_2 \pm \sqrt{\alpha_{12}^2 + \beta_{12}^2 + 2\alpha_{12}\beta_{12}\cos\psi}\right] \tag{20}$$

(here and below, $\alpha_{12} := \alpha_1 - \alpha_2$ etc.) whence

$$\gamma_{12} = \pm\sqrt{\alpha_{12}^2 + \beta_{12}^2 + 2\alpha_{12}\beta_{12}\cos\psi} \tag{21}$$

whose density is

$$\rho(\gamma_{12}) = -\frac{1}{4}\sin\psi\,\frac{d\psi}{d\gamma_{12}} = \frac{1}{2}\frac{|\gamma_{12}|}{\alpha_{12}\beta_{12}}, \tag{22}$$

on its support

$$|\alpha_{12} - \beta_{12}| \leq \gamma_{12} \leq \alpha_{12} + \beta_{12} \ \cup\ -(\alpha_{12} + \beta_{12}) \leq \gamma_{12} \leq -|\alpha_{12} - \beta_{12}|, \tag{23}$$

in agreement with Horn's inequalities. Indeed if we now choose $\gamma_2 \leq \gamma_1$, the latter read

$$\max(\alpha_1 + \beta_2, \alpha_2 + \beta_1) \leq \gamma_1 \leq \alpha_1 + \beta_1 \quad \alpha_2 + \beta_2 \leq \gamma_2 \leq \min(\alpha_1 + \beta_2, \alpha_2 + \beta_1)$$

whence

$$|\alpha_{12} - \beta_{12}| \leq \gamma_{12} \leq \alpha_{12} + \beta_{12},$$

a triangular inequality familiar from the "rules of addition of angular momenta," also know as *Littlewood–Richardson coefficients* for SU(2).

1.3.2. Applying equations ([17](#)) and ([18](#)). According to ([18](#)), for $n = 2$,

$$\mathcal{J}_2(\alpha, \beta; \gamma) = \frac{1}{2\pi \mathrm{i}} \sum_{P, P' \in S_2} \varepsilon_P \varepsilon_{P'} \int_{\mathbb{R}} \frac{du}{u} e^{\mathrm{i} u A(P, P', I)}$$

with

$$A(P, P', I) = \frac{1}{2}(\alpha_{P(12)} + \beta_{P'(12)} - \gamma_{12}) = \frac{1}{2}(\varepsilon_P \alpha_{12} + \varepsilon_{P'} \beta_{12} - \gamma_{12}).$$

Recall that $\alpha_{12}, \beta_{12} \geq 0$ by convention, while γ_{12} is unconstrained at this stage. As explained above, the u integral, not absolutely convergent at infinity and with a pole at 0, is to be interpreted as a Cauchy principal value and then computed by a standard contour integral (Dirichlet integral)

$$P \int_{\mathbb{R}} \frac{du}{u} e^{\mathrm{i} u A} = \mathrm{i}\pi\epsilon(A), \quad \text{if } A \neq 0, \tag{24}$$

with ϵ the sign function. Thus

$$\mathcal{J}_2(\alpha, \beta; \gamma) = \frac{1}{4} \sum_{P, P', P'' \in S_2} \varepsilon_P \varepsilon_{P'} \epsilon(A(P, P', I)),$$

if all $A(P, P', I) \neq 0$, which turns out to be expressible in terms of the characteristic (indicator) functions $\mathbf{1}_I$ and $\mathbf{1}_{-I}$ of the intervals $I = (|\alpha_{12} - \beta_{12}|, \alpha_{12} + \beta_{12})$ and $-I$

$$\mathcal{J}_2(\alpha, \beta; \gamma) = \frac{1}{2}(\epsilon(\gamma_{12} - \alpha_{12} + \beta_{12}) + \epsilon(\gamma_{12} + \alpha_{12} - \beta_{12})$$
$$- \epsilon(\gamma_{12} - \alpha_{12} - \beta_{12}) - \epsilon(\gamma_{12} + \alpha_{12} + \beta_{12}))$$
$$= (\mathbf{1}_I(\gamma_{12}) - \mathbf{1}_{-I}(\gamma_{12})).$$

If one of the arguments of the sign functions $\epsilon(\gamma_{12} \pm \alpha_{12} \pm \beta_{12})$ vanishes, *i.e.,* if γ_{12} stands at one of the end points of one of the intervals I or $-I$, one may see, returning to the original integral, that one must take the corresponding $\epsilon(0) = 0$, or equivalently the characteristic function $\mathbf{1}$ takes the value $\frac{1}{2}$ at the end points of its support.

Our final result for the $n = 2$ PDF thus reads

$$p(\gamma \mid \alpha, \beta) = \frac{(\gamma_1 - \gamma_2)}{2(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)}(\mathbf{1}_I(\gamma_1 - \gamma_2) - \mathbf{1}_{-I}(\gamma_1 - \gamma_2))$$
$$\delta(\gamma_1 + \gamma_2 - \alpha_1 - \alpha_2 - \beta_1 - \beta_2) \tag{25}$$

which does integrate to 1 over $\mathbb{R}^2$, as it should. In that case, the density is a discontinuous, piece-wise linear function over its support. This is in full agreement with the results ([20](#)), ([22](#)), and ([23](#)).

1.4. The case $n = 3$

1.4.1. The inequalities and the polygon for $n = 3$.
Assuming the inequalities (2) satisfied by α, β *and* γ

$$\alpha_3 \leq \alpha_2 \leq \alpha_1, \tag{26}$$

$$\beta_3 \leq \beta_2 \leq \beta_1, \tag{27}$$

$$\gamma_3 \leq \gamma_2 \leq \gamma_1 \tag{28}$$

as well as (3), the Horn inequalities read

$$\gamma_{3\,\min} := \alpha_3 + \beta_3$$
$$\leq \gamma_3 \tag{29a}$$
$$\leq \min(\alpha_1 + \beta_3, \alpha_2 + \beta_2, \alpha_3 + \beta_1) =: \gamma_{3\,\max}$$

$$\gamma_{2\,\min} := \max(\alpha_2 + \beta_3, \alpha_3 + \beta_2)$$
$$\leq \gamma_2 \tag{29b}$$
$$\leq \min(\alpha_1 + \beta_2, \alpha_2 + \beta_1) =: \gamma_{2\,\max}$$

$$\gamma_{1\,\min} := \max(\alpha_1 + \beta_3, \alpha_2 + \beta_2, \alpha_3 + \beta_1)$$
$$\leq \gamma_1 \tag{29c}$$
$$\leq \alpha_1 + \beta_1 =: \gamma_{1\,\max}.$$

These inequalities follow from Knutson-Tao's inequalities on the honeycomb ξ variable of Figure 1

$$\max(\alpha_1 - \gamma_1 + \gamma_2, \gamma_3 - \beta_3, \alpha_2, -\beta_2 + \gamma_2,$$
$$\alpha_1 + \alpha_3 + \beta_1 - \gamma_1, \alpha_1 + \alpha_2 + \beta_2 - \gamma_1) \tag{30}$$
$$\leq \xi \leq \min(\alpha_1, -\beta_3 + \gamma_2, \alpha_1 + \alpha_2 + \beta_1 - \gamma_1)$$

Inequalities (29) are the necessary and sufficient conditions for γ to belong to the polygon in the plane γ_1, γ_2 (with γ_3 given by (3)). See [4] for a detailed discussion and proof. This polygon is at most an octagon, see Figure 2. The red lines are AB: $\gamma_3 = \gamma_{3\,\min}$, *i.e.*, $\gamma_1 + \gamma_2 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2$ and DE: $\gamma_3 = \gamma_{3\,\max}$; and by (28), we retain only the part of the polygon below the diagonal $\gamma_1 = \gamma_2$ (broken line IJ) and above HG: $\gamma_3 = \gamma_2$ hence $\gamma_1 + 2\gamma_2 = \sum \alpha_i + \beta_i$ (the blue line). Some of these lines may not cross the quadrangle CC′ FF′, see figures below.
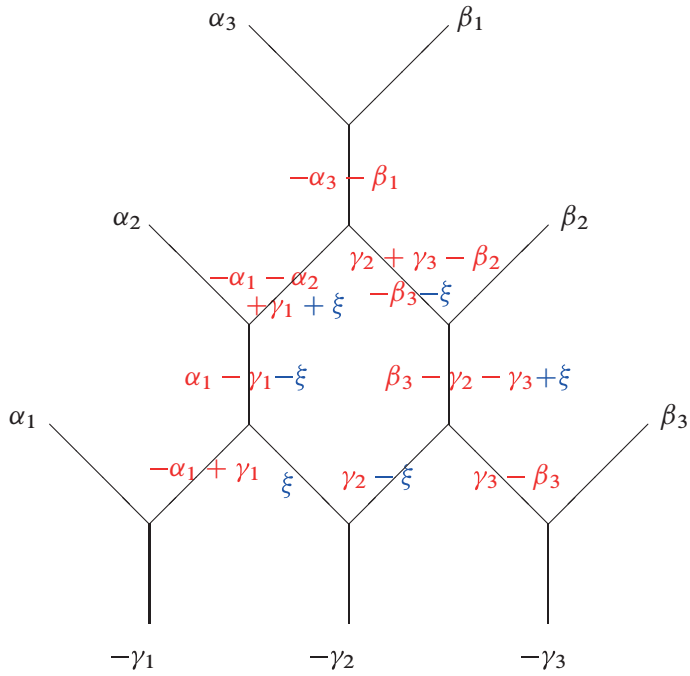
Figure 1. Knutson–Tao's honeycomb for $n = 3$: the inner edges.
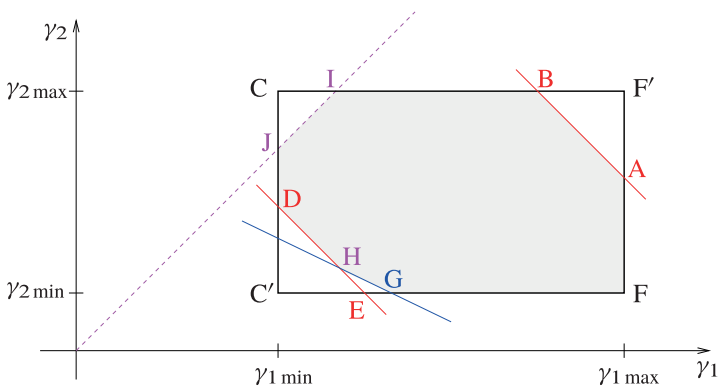

Figure 2. The polygon ABIJDHGF for $n = 3$.

1.4.2. The PDF for $n = 3$. According to (13)–(18), we may write for $n = 3$

$$p(\gamma \mid \alpha, \beta) = \frac{1}{3}\delta\Big(\sum \gamma_i - \alpha_i - \beta_i\Big)\frac{\Delta(\gamma)}{\Delta(\alpha)\Delta(\beta)}\mathcal{J}_3(\alpha, \beta; \gamma), \tag{31}$$

$$\mathcal{J}_3(\alpha, \beta; \gamma) = \frac{\mathrm{i}}{4\pi^2}\int_{\mathbb{R}^2}\frac{du_1 du_2}{u_1 u_2(u_1 + u_2)}\sum_{P, P' \in S_3}\varepsilon_P \varepsilon_{P'}e^{\mathrm{i}(u_1 A_1 + u_2 A_2)}, \tag{32}$$

$$A_1 = \alpha_{P(1)} + \beta_{P'(1)} - \gamma_1 \quad A_2 = -\alpha_{P(3)} - \beta_{P'(3)} + \gamma_3, \tag{33}$$

where use has been made of (3). Integrating once again term by term by principal value and contour integrals, we find

$$\mathcal{J}_3(\alpha, \beta; \gamma) = \frac{1}{4}\sum_{P, P' \in S_3}\varepsilon_P \varepsilon_{P'}\epsilon(A_1)(|A_2| - |A_2 - A_1|). \tag{34}$$

Note that in that expression, the vanishing of A_1 yields a vanishing result. The somewhat ambiguous value of the sign function at 0 is thus irrelevant. In the domain $\gamma_3 \leq \gamma_2 \leq \gamma_1$, the corresponding sum of $2 \times 6^2 = 72$ contributions vanishes if the set of Horn's inequalities (29) is not satisfied, but conversely it is fairly difficult to read these inequalities off expression (34). When (3), (28), and (29) are satisfied, it may be shown that this sum reduces to a sum of 4 terms

$$\mathcal{J}_3(\alpha, \beta; \gamma) = \frac{1}{6}(\alpha_1 - \alpha_3 + \beta_1 - \beta_3 + \gamma_1 - \gamma_3) - \frac{1}{2}|\alpha_2 + \beta_2 - \gamma_2| - \frac{1}{3}\psi_{\alpha\beta}(\gamma) - \frac{1}{3}\psi_{\beta\alpha}(\gamma) \tag{35}$$

where

$$\psi_{\alpha\beta}(\gamma) = \begin{cases} (\gamma_2 - \alpha_3 - \beta_1) - (\gamma_1 - \alpha_1 - \beta_2) \\ \quad \text{if } \gamma_2 - \alpha_3 - \beta_1 \geq 0 \text{ and } \gamma_1 - \alpha_1 - \beta_2 < 0, \\ (\gamma_3 - \alpha_2 - \beta_3) - (\gamma_2 - \alpha_3 - \beta_1) \\ \quad \text{if } \gamma_3 - \alpha_2 - \beta_3 \geq 0 \text{ and } \gamma_2 - \alpha_3 - \beta_1 < 0, \\ (\gamma_1 - \alpha_1 - \beta_2) - (\gamma_3 - \alpha_2 - \beta_3) \\ \quad \text{if } \gamma_1 - \alpha_1 - \beta_2 \geq 0 \text{ and } \gamma_3 - \alpha_2 - \beta_3 < 0. \end{cases} \tag{36}$$

In Figure 3, the three sectors in the (γ_1, γ_2) plane where $\psi_{\alpha\beta}$ takes one of three values of (36) are depicted. It is manifest that $\psi_{\alpha\beta}$ is a continuous function of γ, thanks to (3).
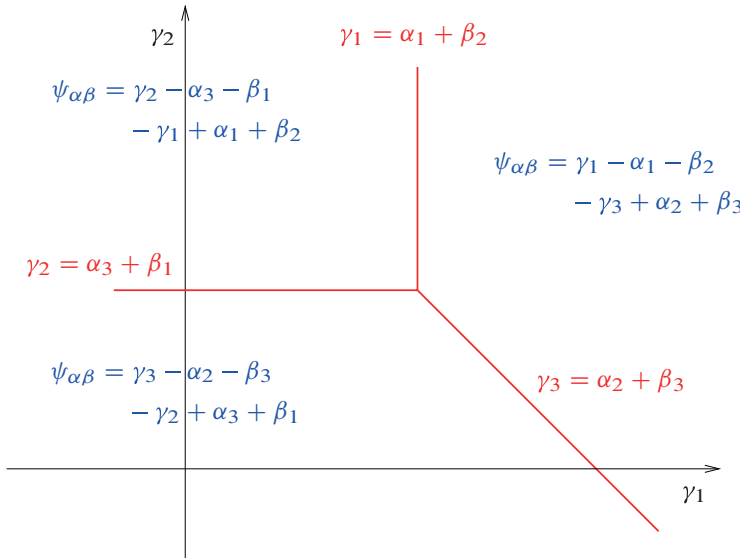
Figure 3. The three sectors defining $\psi_{\alpha\beta}(\gamma)$.

We recall that we have assumed that all α_i's on the one hand, and all β_j's on the other, are distinct.[3] Then the function $\mathcal{J}_3$ is a piece-wise linear continuous function of the γ's, making $p(\gamma \mid \alpha, \beta)$ a "piece-wise degree 4 polynomial" continuous function of those variables. The lines along which $\mathcal{J}_3$ is not differentiable are the segments of the three half-lines depicted on Figure 3 that lie inside the polygon, those obtained when α and β are swapped, and the inside segment of the line $\gamma_2 = \alpha_2 + \beta_2$. These singular lines appear on some of the figures below.

Upon integration over γ_1, γ_2, the function p of (31) sums to $1/6$ in the domain defined by (3), (28), and (29), hence to 1 on the 3! sectors obtained by relaxing (28).

Remark. There is an alternative expression of $\mathcal{J}_3$ that follows from its identification with the "volume" of the polytope of honeycombs, here simply the length of the ξ-interval (30). This will be discussed in more detail in [10]. Thus we may also write, again when (3), (28), and (29) are satisfied

$$\mathcal{J}_3(\alpha, \beta; \gamma) = \min(\alpha_1, -\beta_3 + \gamma_2, \alpha_1 + \alpha_2 + \beta_1 - \gamma_1)$$
$$- \max(\alpha_1 - \gamma_1 + \gamma_2, \gamma_3 - \beta_3, \alpha_2, -\beta_2 + \gamma_2, \qquad (37)$$
$$\alpha_1 + \alpha_3 + \beta_1 - \gamma_1, \alpha_1 + \alpha_2 + \beta_2 - \gamma_1).$$

The non-differentiability of $\mathcal{J}_3$ occurs along lines where two arguments of the min or of the max functions coincide, but the detailed pattern is more difficult to grasp

---

[3] Otherwise, $\mathcal{J}_n$ vanishes, by antisymmetry of the determinant in (14).

than on expression (35,36).

1.4.3. Examples. Take for example $\alpha = \beta = (1, 0, -1)$. Then (γ_1, γ_2) subject to inequality (28) is restricted to a quadrangular domain ABDF with corners at $(2, 0)$, $(1, 1)$, $(0, 0)$, $(2, -1)$. A typical plot of eigenvalues in that domain and their histogram obtained with samples of respectively 10,000 and 10^6 random unitary matrices U in $\mathrm{diag}(\alpha) + U \,\mathrm{diag}(\beta)U^\dagger$ is displayed in Figure 4.a and 4.b, while the plot of the function $p(\gamma \mid \alpha, \beta)$ is in Figure 4.c. Finally Figure 4.d gives the full distribution when inequality (28) is relaxed.
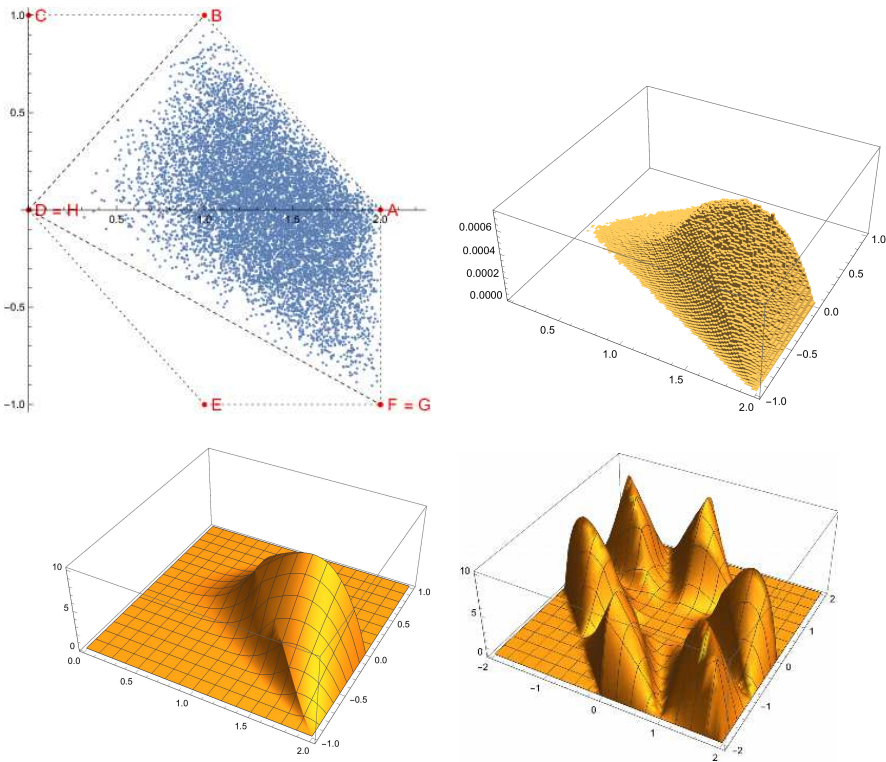

Figure 4. Example of $\alpha = \beta = (1, 0, -1)$. TOP, LEFT. distribution of 10,000 eigenvalues in the γ_1, γ_2 plane. RIGHT. histogram of 10^6 eigenvalues. BELOW, LEFT. plot of the PDF of (35) for $\gamma_1 \geq \gamma_2 \geq \gamma_3$. RIGHT. the full γ_1, γ_2 plane.

Other examples are displayed in Figure 5, exhibiting the lines of non-differentiability, as well as the sharp features of the PDF as two (or more) of the eigenvalues α or β coalesce. All these plots, histograms and figures have been computed in Mathematica[16], making use in particular of the

    RandomVariate[CircularUnitaryMatrixDistribution[n]]

(resp. RandomVariate[CircularRealMatrixDistribution[n]] in Sections 2 and 3 below) to generate unitary, resp. real orthogonal matrices, uniformly distributed according to the Haar measure of SU(n), resp. O(n) or SO(n).

Our result (35) is in excellent agreement with these numerical experiments, as seen on the figures.

1.5. The cases $n = 4$ and $n = 5$. The cases $n = 4$ and $n = 5$ have also been worked out, see Appendix B for some indications.

2. The probability density function (PDF) for real symmetric matrices

One may also consider Horn's problem for real symmetric matrices of size n. Given two n-plets of real eigenvalues α and β, ordered as in (2), what is the range of eigenvalues γ of $\text{diag}(\alpha) + O\,\text{diag}(\beta)O^T$ where now $O \in$ O(n), the group of real orthogonal matrices ? According to Fulton [3], the ordered γ's still live in a convex domain given by the same conditions as in the Hermitian case. What about their PDF? It turns out it looks quite different from the Hermitian case.

For $n = 2$, we have the sum rule $\gamma_1 + \gamma_2 = \alpha_1 + \alpha_2 + \beta_1 + \beta_2$. The difference $\gamma_{12} := \gamma_1 - \gamma_2$, taken to be non negative by convention, depends only on $\alpha_{12} := \alpha_1 - \alpha_2 \geq 0$ and $\beta_{12} \geq 0$, namely $\gamma_{12} = \sqrt{\alpha_{12}^2 + \beta_{12}^2 + 2\alpha_{12}\beta_{12}\cos(2\theta)}$, with $0 \leq \theta \leq 2\pi$ the angle of the relative O(2) rotation O between A and B, whence a density $\rho(\gamma_{12}) = -\frac{2}{\pi}\frac{d\theta}{d\gamma_{12}}$, equal to

$$\rho(\gamma) = \begin{cases} \dfrac{2}{\pi}\dfrac{\gamma}{\sqrt{(\gamma_{12\,\text{max}}^2 - \gamma^2)(\gamma^2 - \gamma_{12\,\text{min}}^2)}} & \text{if}\,\gamma_{12\,\text{min}} \leq \gamma \leq \gamma_{12\,\text{max}}, \\ 0 & \text{otherwise}, \end{cases} \tag{38}$$

with $\gamma_{12\,\text{min}} = |\alpha_{12} - \beta_{12}|$, $\gamma_{12\,\text{max}} = \alpha_{12} + \beta_{12}$. This function is singular (but integrable) at the edges $\gamma_{12\,\text{min}}$ and $\gamma_{12\,\text{max}}$ of the support if $\gamma_{12\,\text{min}} \neq 0$, and only at $\gamma_{12\,\text{max}}$ if $\gamma_{12\,\text{min}} = 0$, see Figure 6.
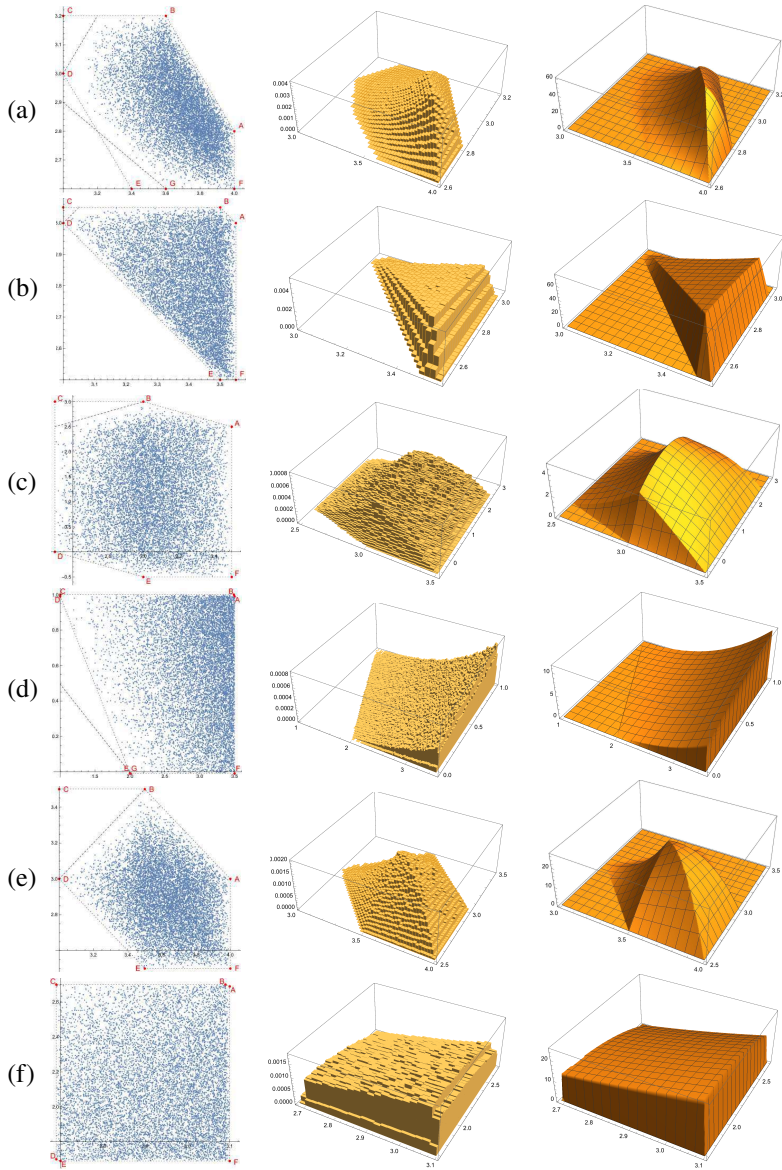
Figure 5. 2-D plot of 10^4 eigenvalues γ_1, γ_2, histogram of 10^6 eigenvalues, and PDF of (35) for a sample of α's and β's. From top to bottom, (a) $\alpha = (2, 1.2, 1)$, $\beta = (2, 1.6, 1)$; (b) $\alpha = (1.55, 1.5, 1)$, $\beta = (2, 1.5, -3.5)$; (c) $\alpha = (1.5, 1, -2)$, $\beta = (2, 1.5, -3.5)$; (d) $\alpha = (2, 1.99, -0.5)$, $\beta = (1.5, -1, -2)$; (e) $\alpha = (2, 1.5, 1)$, $\beta = (2, 1.5, -4)$; (f) $\alpha = (1.5, 1.49, -3)$, $\beta = (1.6, 1.2, 0.2)$.
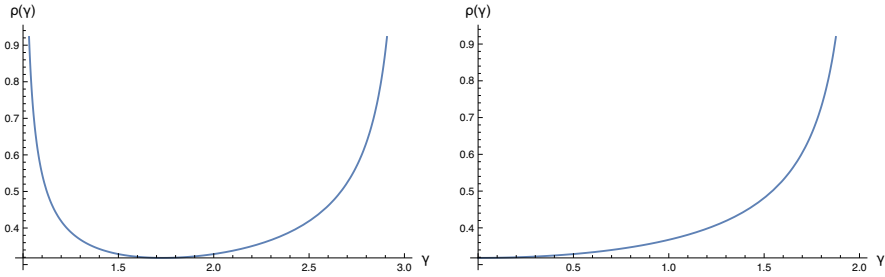
Figure 6. The density ρ. LEFT. For $\alpha_{12} = 1, \beta_{12} = 2$. RIGHT. $\alpha_{12} = \beta_{12} = 1$.

For $n \geq 3$, we have no analytic formula, but numerical experiments reveal curious enhanced regions and ridges in the density of points or histogram, see Figures 7. Empirically,[4] for $n = 3$, these enhancements take place along the same half-lines that appeared in the discussion of equations (35) and (36), namely

$$(\gamma_1 = \alpha_1 + \beta_2, \gamma_2 \geq \alpha_3 + \beta_1),$$

$$(\gamma_2 = \alpha_3 + \beta_1, \gamma_1 \leq \alpha_1 + \beta_2),$$

$$(\gamma_1 + \gamma_2 = \alpha_1 + \alpha_3 + \beta_1 + \beta_2, \gamma_1 \geq \alpha_1 + \beta_2),$$

restricted to their segments inside the polygon; the same with α and β swapped; and the segment of the line $\gamma_2 = \alpha_2 + \beta_2$ inside the polygon. Similar features also occur for higher n. The nature of these enhancements, presumably a weak integrable singularity, or even better, an analytic expression for the PDF, remain to be found.

3. The probability density function (PDF) for real skew-symmetric matrices

The same Horn's problem may again be posed about real skew-symmetric matrices of size n with the adjoint action of the group $O(n)$ or $SO(n)$. Such matrices may always be block-diagonalized in the form

$$A = \begin{cases} \mathrm{diag}\left(\begin{pmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{pmatrix}_{i=1,\dots,m} \right) & \text{for even } n = 2m, \\[2em] \mathrm{diag}\left(\begin{pmatrix} 0 & \alpha_i \\ -\alpha_i & 0 \end{pmatrix}_{i=1,\dots,m}, 0 \right) & \text{for odd } n = 2m + 1. \end{cases} \tag{39}$$

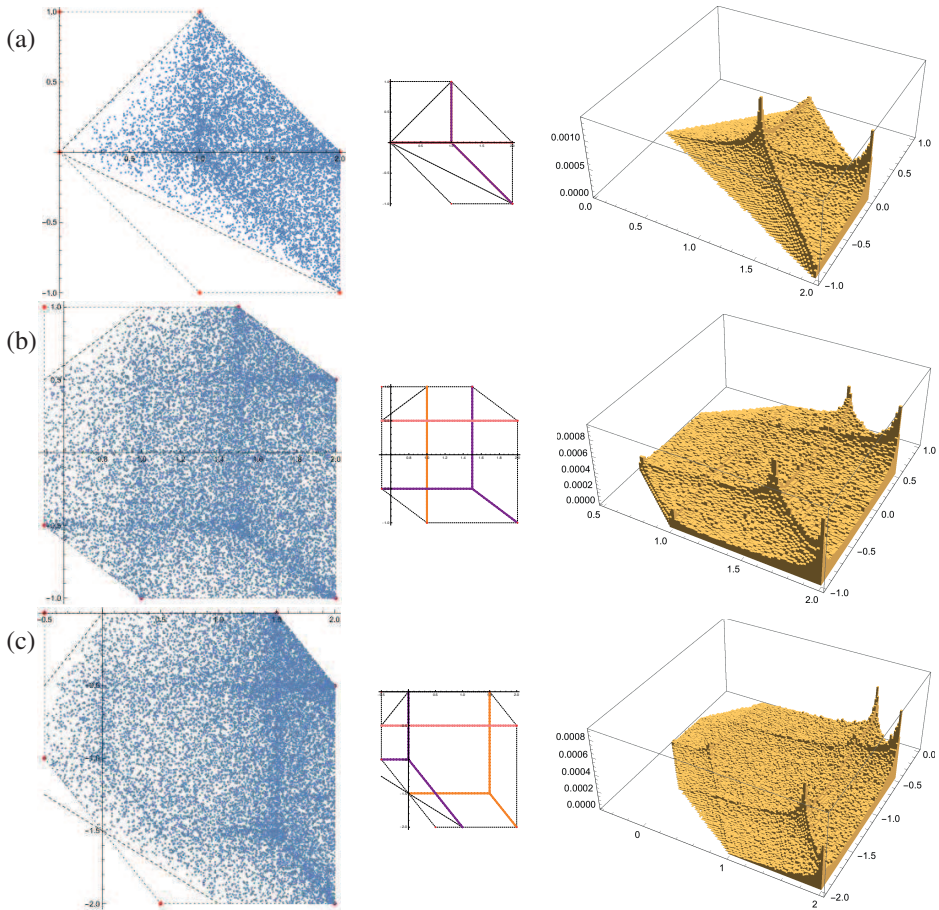[4] M. Vergne (private communication) has shown that this is indeed the case.

Figure 7. (a) Plot (left) and histogram (right) of respectively 10^4 and 10^6 eigenvalues γ_1, γ_2 for the sum of 3×3 symmetric matrices of eigenvalues $\alpha = \beta = (1, 0, -1)$. The density appears to be enhanced along the lines (middle) $\gamma_1 = 1, \gamma_2 = 0$ and $\gamma_3 = -\gamma_1 - \gamma_2 = -1$. Same with (b) $\alpha = (1, 0.5, -2.5), \beta = (1, 0, -1.5)$ and (c) $\alpha = (1, -1, -2.5), \beta = (1, 0.5, -2)$. (Michèle Vergne had obtained the same plot (a) in a prior work [17].)

We refer to such α's as the "eigenvalues" of A. (The actual eigenvalues are in fact the $\pm i\alpha_j$, $j = 1, \ldots, m$, together with 0 if $n = 2m + 1$.) In the case of O(n) or SO$(2m + 1)$, one may again order the α's as in (2) and choose them to be non negative. For the group SO$(2m)$, however, the matrix that swaps the sign of any α_i or β_i is of determinant -1: only an even number of sign changes are allowed but we may still impose

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{m-1} \geq |\alpha_m| \geq 0$$

and likewise[5] for the β_j's. As elsewhere in the present work, we focus on the case where the inequalities are strict.

Given two skew-symmetric matrices A and B and their eigenvalues α and β, what is the range and density of the eigenvalues γ of $A + OBO^T$ when O runs over the real orthogonal group O(n) or SO(n)?

In that case we have a Harish-Chandra integral at our disposal

$$
\int_G DO \exp \operatorname{tr} AOBO^T
$$

$$
= \begin{cases}
\hat{\kappa}_m \dfrac{\det(\cosh(2\alpha_i\beta_j))_{1\le i,j\le m}}{\Delta_O(\alpha)\Delta_O(\beta)} & \text{if } G = \text{O}(2m), \\[2ex]
\dfrac{\hat{\kappa}_m}{2^{m-1}} \dfrac{\displaystyle\sum_{\varepsilon_j=\pm1}{}' \det(e^{-2\varepsilon_i\alpha_i\beta_j})_{1\le i,j\le m}}{\Delta_O(\alpha)\Delta_O(\beta)}, & \text{if } G = \text{SO}(2m), \\[2ex]
\hat{\kappa}'_m \dfrac{\det(\sinh(2\alpha_i\beta_j))_{1\le i,j\le m}}{\Delta_O(\alpha)\Delta_O(\beta)}, & \text{if } G = \text{O}(2m+1), \text{ or } \text{SO}(2m+1),
\end{cases}
$$

(40)

where on the second line, the primed sum $\sum'$ runs over an even number of minus signs. In the denominator, Δ_O stands for

$$
\Delta_O(\alpha) = \begin{cases}
\displaystyle\prod_{1\le i<j\le m}(\alpha_i^2 - \alpha_j^2) & \text{for } n = 2m, \\[2ex]
\displaystyle\prod_{1\le i<j\le m}(\alpha_i^2 - \alpha_j^2)\prod_i \alpha_i & \text{for } n = 2m+1,
\end{cases}
$$

(41)

if $m > 1$, while for $m = 1$, by convention $\prod_{1\le i<j\le m}(\alpha_i^2 - \alpha_j^2) \equiv 1$. Finally the constants are (see Appendix A)

$$
\hat{\kappa}_m = \frac{(m-1)!\displaystyle\prod_{p=1}^{m-1}(2p-1)!}{2^{(m-1)^2}}, \quad \hat{\kappa}'_m = \frac{\displaystyle\prod_{p=1}^{m}(2p-1)!}{2^{m^2}},
$$

(the numerators of which may also be regarded as the products $\prod_i m_i!$ of factorials of the Coxeter exponents of the Lie algebra $D_m = so(2m)$ (for $m \ge 4$), resp. of $B_m = so(2m+1)$ (for $m \ge 2$)).

---

[5] This reflects the structure of the Weyl group of type B_m or D_m.

3.1. Case of even $n = 2m$. A calculation similar to that of Section 1.2 then leads to

$$p(\gamma \mid \alpha, \beta) = \frac{\prod\limits_{p=1}^{m-1}(2p-1)!}{2^{(m-1)^2}\pi^m m^2} \frac{\Delta_O(\gamma)}{\Delta_O(\alpha)\Delta_O(\beta)} \mathcal{I}_m$$

$$\mathcal{I}_m = \begin{cases} (-1)^{m(m-1)/2} \displaystyle\int_{\mathbb{R}^m} \frac{d^m x}{\Delta_O(x)} \det(\cos(2x_i\alpha_j)) \\ \qquad \det(\cos(2x_i\beta_j))\det(\cos(2x_i\gamma_j)) \qquad \text{for } O(2m), \quad (42) \\ \dfrac{(-1)^{m(m-1)/2}}{2^{3(m-1)}} \displaystyle\int_{\mathbb{R}^m} \frac{d^m x}{\Delta_O(x)} \sideset{}{'}\sum_{\varepsilon,\varepsilon',\varepsilon''} \\ \det(\exp(2\,\mathrm{i}\,\varepsilon_j x_i \alpha_j))\det(\exp(2\,\mathrm{i}\,\varepsilon'_j x_i \beta_j)) \\ \qquad \det(\exp(-2\,\mathrm{i}\,\varepsilon''_j x_i \gamma_j)) \qquad\quad \text{for } SO(2m), \end{cases}$$

with as before, an even number of minus signs for ε, and likewise for $\varepsilon', \varepsilon''$.

For $m = 1$, Horn's problem is trivial: any skew-symmetric matrix $B = \left(\begin{smallmatrix} 0 & \beta \\ -\beta & 0 \end{smallmatrix}\right)$ commutes with an SO(2) rotation matrix while for the permutation $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ that belongs to O(2) but not to SO(2), $P.B.P = -B$. When $O \in \text{O}(2)$, resp. $\in \text{SO}(2)$, the "eigenvalues" of $A + O.B.O^T$ are $\pm\alpha \pm \beta$ with two independent signs, resp. simply $\alpha + \beta$, which is precisely what is given by (42) when the x integration is worked out:

$$p(\gamma \mid \alpha, \beta) = \begin{cases} \dfrac{1}{4}(\delta(\gamma + \alpha + \beta) + \delta(\gamma + \alpha - \beta) \\ \qquad +\delta(\gamma - \alpha + \beta) + \delta(\gamma - \alpha - \beta)) \quad \text{for } O(2), \quad (43) \\ \delta(\gamma - \alpha - \beta) \qquad\qquad\qquad\qquad\quad \text{for } SO(2). \end{cases}$$

For $m = 2$ (4×4 skew-symmetric matrices), using variables $s = (x_1 + x_2)$ and $t = (x_1 - x_2)$, we write in the SO(4) case

$$\mathcal{I}_2 = 2^2 \int \frac{ds}{s} \int \frac{dt}{t} [\sin(s(\alpha_1 + \alpha_2))\sin(t(\alpha_1 - \alpha_2))][\text{same with } \beta][\text{same with } \gamma]$$

while in the O(4) case, each square bracket is replaced by

$$\frac{1}{2}\left[\sin s(\alpha_1 + \alpha_2)\sin t(\alpha_1 - \alpha_2) + \sin s(\alpha_1 - \alpha_2)\sin t(\alpha_1 + \alpha_2)\right].$$

After expansion and use of the formula

$$\sin as \sin bs \sin cs = \frac{1}{4}(\sin(-a + b + c)s + \sin(a - b + c)s + \sin(a + b - c)s - \sin(a + b + c)s)$$

one finds for SO(4)

$$p(\gamma \mid \alpha, \beta) = \frac{1}{2^3} \frac{\Delta_O(\gamma)}{\Delta_O(\alpha)\Delta_O(\beta)} \begin{pmatrix} (\mathbf{1}_I(\gamma_1 + \gamma_2) - \mathbf{1}_{-I}(\gamma_1 + \gamma_2)) \\ (\mathbf{1}_{I'}(\gamma_1 - \gamma_2) - \mathbf{1}_{-I'}(\gamma_1 - \gamma_2))) \end{pmatrix} \tag{44}$$

with the indicator functions of the intervals

$$I = (|(\alpha_1 + \alpha_2) - (\beta_1 + \beta_2)|, (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2)), \tag{45a}$$

$$I' = (|(\alpha_1 - \alpha_2) - (\beta_1 - \beta_2)|, (\alpha_1 - \alpha_2) + (\beta_1 - \beta_2)). \tag{45b}$$

In the O(4) case, the result would be similar, with the big bracket in (44) replaced by

$$\left(\frac{1}{4} \sum_{\varepsilon, \varepsilon'} \begin{pmatrix} (\mathbf{1}_{I(\varepsilon,\varepsilon')}(\gamma_1 + \gamma_2) - \mathbf{1}_{-I(\varepsilon,\varepsilon')}(\gamma_1 + \gamma_2)) \\ (\mathbf{1}_{I'(\varepsilon,\varepsilon')}(\gamma_1 - \gamma_2) - \mathbf{1}_{-I'(\varepsilon,\varepsilon')}(\gamma_1 - \gamma_2)) \end{pmatrix} \right) \tag{44'}$$

and a sum over intervals

$$I(\varepsilon, \varepsilon') = (|(\alpha_1 + \varepsilon\alpha_2) - (\beta_1 + \varepsilon'\beta_2)|, (\alpha_1 + \varepsilon\alpha_2) + (\beta_1 + \varepsilon'\beta_2)), \tag{46a}$$

$$I'(\varepsilon, \varepsilon') = (|(\alpha_1 - \varepsilon\alpha_2) - (\beta_1 - \varepsilon'\beta_2)|, (\alpha_1 - \varepsilon\alpha_2) + (\beta_1 - \varepsilon'\beta_2)), \tag{46b}$$

where $\varepsilon, \varepsilon'$ are two independent signs.

It is an easy exercise to check that p integrates to 1 over the whole γ-plane.

The resulting PDF is much more irregular than in the $n = 4$ Hermitian case, with discontinuities across some lines. Its support is clearly convex in the SO(4) case, in accordance with general theorems. In the O(4) case, the support may be non convex, as apparent on Figure 8. This is a consequence of the *non connectivity* of the group. When the contributions of the two connected parts SO(4) and O(4)\ SO(4) are computed separately, one sees clearly that convexity of the support is restored for each.[6]

---

[6] My thanks to Allen Knutson and Michèle Vergne for emphasizing the rôle of connectivity of the group in the convexity theorem.

3.2. Case of odd $n = 2m + 1$. We now write

$$p(\gamma \mid \alpha, \beta) = \frac{(-1)^{m(m-1)/2} \prod_{p=1}^{m}(2p-1)!}{2^{m^2}\pi^m m!^2}$$
$$\frac{\Delta_O(\gamma)}{\Delta_O(\alpha)\Delta_O(\beta)} \int_{\mathbb{R}^m} \frac{d^m x}{\Delta_O(x)} \det(\sin(2x_i\alpha_j)) \qquad (47)$$
$$\det(\sin(2x_i\beta_j)) \det(\sin(2x_i\gamma_j)).$$

For $m = 1$, *i.e., $n = 3$*, the calculation is essentially identical to that of Section 1.3.2.[7]

$$p(\gamma \mid \alpha, \beta) = \frac{1}{2\pi}\frac{\gamma}{\alpha\beta}\int_{\mathbb{R}}\frac{ds}{s}\sin(2\alpha x)\sin(2\beta x)\sin(2\gamma x)$$

$$= \frac{1}{4}\frac{\gamma}{\alpha\beta}\begin{cases} 1 & \text{if } |\alpha-\beta| \leq \gamma \leq \alpha+\beta, \\ -1 & \text{if } -(\alpha+\beta) \leq \gamma \leq -|\alpha-\beta| \\ 0 & \text{otherwise,} \end{cases} \qquad (48)$$

thus a piece-wise linear and discontinuous function of γ.

For $n = 5$, $m = 2$, we have

$$p(\gamma|\alpha, \beta) = -\frac{3}{32\pi^2}\frac{\Delta_O(\gamma)}{\Delta_O(\alpha)\Delta_O(\beta)}$$
$$\underbrace{\int \frac{d^2 x}{\Delta_O(x)} \det(\sin(2x_i\alpha_j)) \det(\sin(2x_i\beta_j)) \det(\sin(2x_i\gamma_j))}_{\mathfrak{I}}.$$
$$(49)$$

We then make use as above of variables $s = (x_1 + x_2)$ and $t = (x_1 - x_2)$ and of the identity

$$\det(\sin(2x_i\alpha_j)) = \sin(s(\alpha_1 + \alpha_2))\sin(t(\alpha_1 - \alpha_2))$$
$$- \sin(t(\alpha_1 + \alpha_2))\sin(s(\alpha_1 - \alpha_2)),$$

and the x-integral in (49) reduces to

$$\mathfrak{I} = \frac{1}{2}\int\frac{ds\,dt}{st(s^2 - t^2)}[\sin s(\alpha_1 + \alpha_2)\sin t(\alpha_1 - \alpha_2)$$
$$- \sin s(\alpha_1 - \alpha_2)\sin t(\alpha_1 + \alpha_2)]$$
$$[\text{same with }\beta]\,[\text{same with }\gamma].$$

---

[7] Indeed, the action of U(2) on Hermitian matrices $\left(\begin{smallmatrix}\alpha & 0 \\ 0 & -\alpha\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\beta & 0 \\ 0 & -\beta\end{smallmatrix}\right)$ resembles that of O(2) on skew-symmetric matrices $\left(\begin{smallmatrix}0 & \alpha \\ -\alpha & 0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0 & \beta \\ -\beta & 0\end{smallmatrix}\right)$....
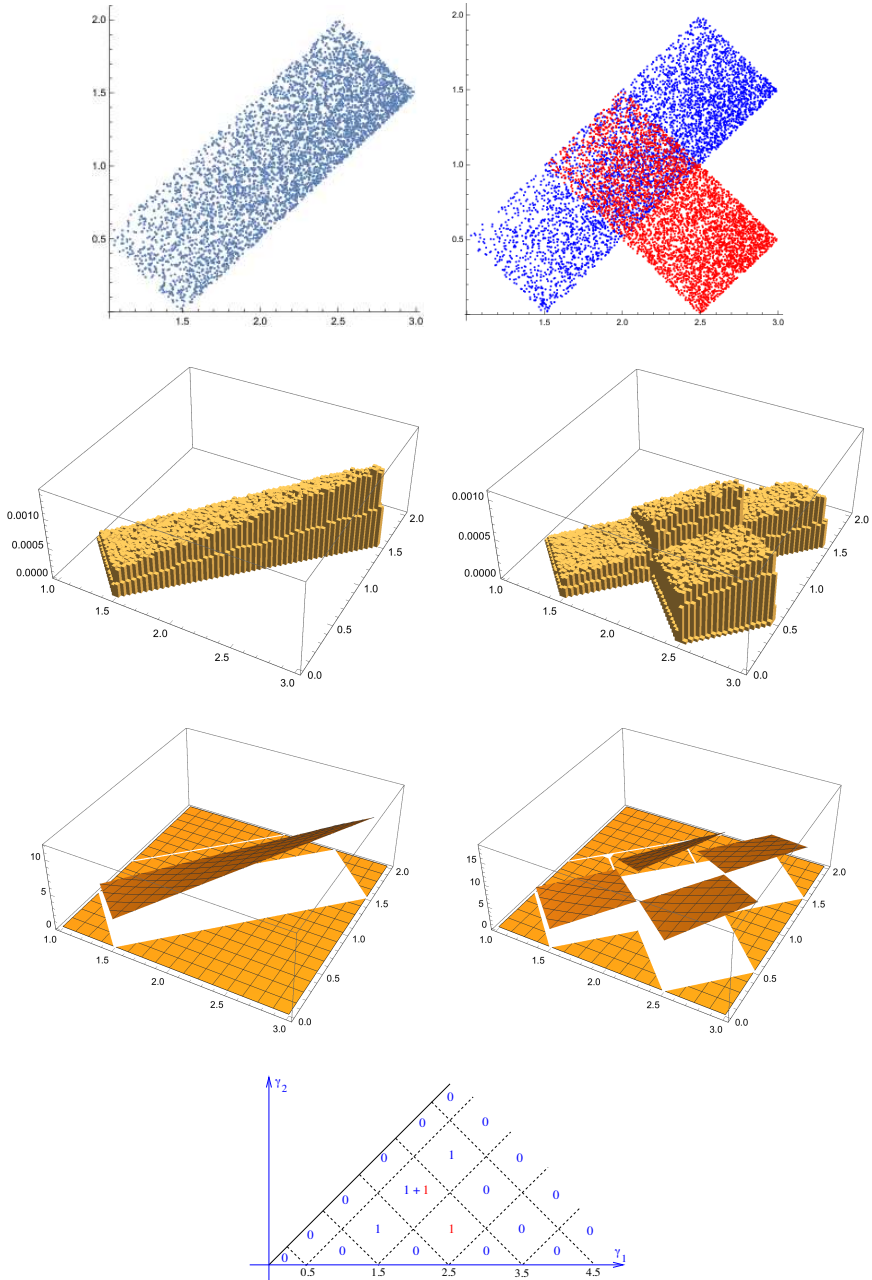
Figure 8. Plot and histogram of eigenvalues γ_1, γ_2 for the sum of 4×4 skew-symmetric matrices of eigenvalues $\alpha = (2,1)$, $\beta = (1, \frac{1}{2})$, 10^4 points in the plots, 10^6 in the histograms. Below, the density $p(\gamma \mid \alpha, \beta)$ as given in eq. (44), with the values of the bracket according to (45) and (46), in the sector $0 \leq \gamma_2 \leq \gamma_1$. Left. Action of SO(4). Right. Action of O(4). Bottom. the values of the bracket of (44) and (44'), in the sector $\gamma_2 \leq \gamma_1$.

We refrain from giving the full expression of $\mathfrak{I}$ (a sum of 2^7 terms ...), which is a continuous and piecewise quadratic function of the γ's, and just display a sample of results for explicit examples, see Figure 9.

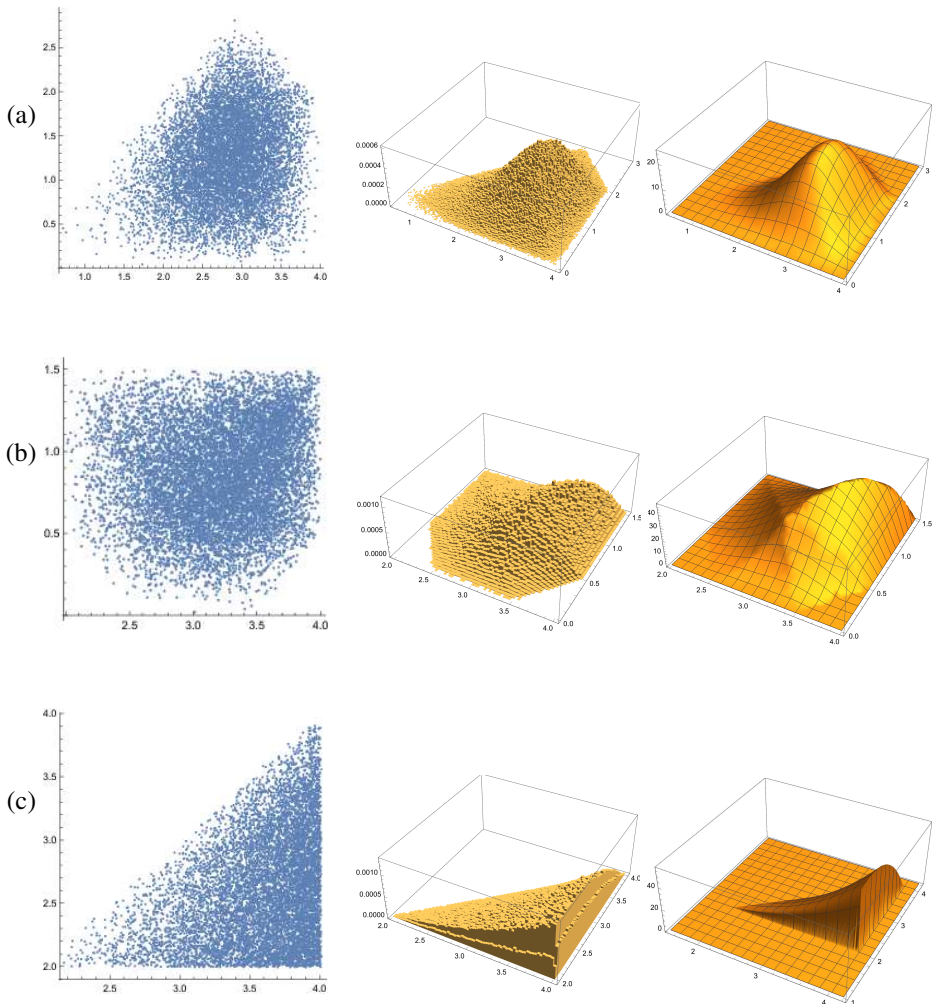In general, the inequalities determining the support have been written by Belkale and Kumar [18].

Figure 9. Plot and histogram of eigenvalues γ_1, γ_2 for the sum of 5 by 5 skew-symmetric matrices and the density $p(\gamma \mid \alpha, \beta)$ as given in eq. (49): (a) $\alpha = \beta = (2, 1)$; (b) $\alpha = (1.01, 1)$ and $\beta = (3, \frac{1}{2})$; (c) $\alpha = (1.01, 1)$, $\beta = (3, 3.005)$.

4. Discussion

The same calculation could be carried out for quaternionic anti-selfdual matrices and their orbits under the action of the group $\mathrm{Sp}(2m)$, where again a Harish-Chandra formula is available. To keep this paper in a reasonable size, we refrain from discussing that case.

Both in the Hermitian/unitary and the skew-symmetric/orthogonal cases, we observe the same feature: the PDF tends to become more and more regular as n increases: a sum of Dirac masses for the lowest values, ($n = 1$, resp. $n = 2$), then a discontinuous function for $n = 2$, resp. $n = 3, 4$, and finally a continuous function of class C^{n-3} for $n \geq 3$, resp. C^p with $p = \lfloor \frac{1}{2}(n-5) \rfloor$ for $n \geq 5$. By Riemann-Lebesgue theorem, this is just a reflection of the increasingly fast decay of its Fourier transform at large x.

We recall that our discussion has left aside the case where two or more eigenvalues coincide...

Appendices

A. Normalization constants

Consider the set $\mathfrak{X}_n$ of Hermitian, resp. real skew-symmetric, n by n matrices.

For $A \in \mathfrak{X}_n$, with eigenvalues α_i (in the sense of (39) in the skew-symmetric case), write the Lebesgue measure on A as $DA = \kappa \Delta(\alpha)^2 \prod_{i=1}^r d\alpha_i \, DU_A$, with $U_A \in \mathrm{U}(n)$, resp. $\in \mathrm{O}(n)$.

The constant κ and the Harish-Chandra integral

$$\mathcal{H}_G(\alpha, \beta) = \int_G Dg \, e^{\mathrm{tr}\, AgBg^{-1}}$$

are given by the following Table 1.

The constant κ may be determined by carrying out the calculation of a Gaussian integral in two different ways, integrating either over the original matrix elements, or over the eigenvalues. The constant $\hat{\kappa}$ may be determined by considering the limit where all α_i are scaled to zero.

Table 1

$\mathcal{X}_n$	$\Delta(\alpha)$	κ	$\mathcal{H}_G(\alpha,\beta)$	$\hat\kappa$
Hermitian H_n	$\displaystyle\prod_{1\le i<j\le n}(\alpha_i-\alpha_j)$	$\dfrac{(2\pi)^{n(n-1)/2}}{\displaystyle\prod_{p=1}^{n}p!}$	$\hat\kappa\,\dfrac{(\det e^{\alpha_i\beta_j})_{i,j=1,\dots,n}}{\Delta(\alpha)\Delta(\beta)}$	$\displaystyle\prod_{p=1}^{n-1}p!$
skew-symmetric A_{2m}	$\displaystyle\prod_{1\le i<j\le m}(\alpha_i^2-\alpha_j^2)$	$\dfrac{2^{2m^2-\frac{3}{2}m}\pi^{m(m-1)}}{m!\displaystyle\prod_{p=1}^{m-1}(2p)!}$	$\hat\kappa\,\dfrac{(\det\cos 2\alpha_i\beta_j)_{i,j=1,\dots,m}}{\Delta(\alpha)\Delta(\beta)}$	$\dfrac{(m-1)!\displaystyle\prod_{p=1}^{m-1}(2p-1)!}{2^{(m-1)^2}}$
skew-symmetric A_{2m+1}	$\displaystyle\prod_i \alpha_i \prod_{1\le i<j\le m}(\alpha_i^2-\alpha_j^2)$	$\dfrac{2^{2m^2+\frac{1}{2}m}\pi^{m^2}}{m!\displaystyle\prod_{p=1}^{m}(2p)!}$	$\hat\kappa\,\dfrac{(\det\sin 2\alpha_i\beta_j)_{i,j=1,\dots,m}}{\Delta(\alpha)\Delta(\beta)}$	$\dfrac{\displaystyle\prod_{p=1}^{m}(2p-1)!}{2^{m^2}}$

B. The cases of SU(4) and SU(5)

B.1. Horn's inequalities for 4 × 4 Hermitian matrices. We have

$$\max(\alpha_1 + \beta_4, \alpha_2 + \beta_3, \alpha_3 + \beta_2, \alpha_4 + \beta_1) \le \gamma_1 \le \alpha_1 + \beta_1 \tag{50a}$$

$$\max(\alpha_2 + \beta_4, \alpha_3 + \beta_3, \alpha_4 + \beta_2) \le \gamma_2 \le \min(\alpha_1 + \beta_2, \alpha_2 + \beta_1), \tag{50b}$$

$$\max(\alpha_3 + \beta_4, \alpha_4 + \beta_3) \le \gamma_3 \le \min(\alpha_1 + \beta_3, \alpha_2 + \beta_2, \alpha_3 + \beta_1), \tag{50c}$$

$$\alpha_4 + \beta_4 \le \gamma_4 \le \min(\alpha_1 + \beta_4, \alpha_2 + \beta_3, \alpha_3 + \beta_2, \alpha_4 + \beta_1), \tag{50d}$$

and

$$
\begin{aligned}
\max(&\alpha_1 + \alpha_2 + \beta_3 + \beta_4, \\
&\alpha_1 + \alpha_3 + \beta_2 + \beta_4, \\
&\alpha_2 + \alpha_3 + \beta_2 + \beta_3, \\
&\alpha_1 + \alpha_4 + \beta_1 + \beta_4, \\
&\alpha_2 + \alpha_4 + \beta_1 + \beta_3, \\
&\alpha_3 + \alpha_4 + \beta_1 + \beta_2) \\
&\le \gamma_1 + \gamma_2 \le \alpha_1 + \alpha_2 + \beta_1 + \beta_2,
\end{aligned}
\tag{51a}
$$

$$
\begin{aligned}
\max(&\alpha_1 + \alpha_3 + \beta_3 + \beta_4, \\
&\alpha_1 + \alpha_4 + \beta_2 + \beta_4, \\
&\alpha_2 + \alpha_3 + \beta_2 + \beta_4, \\
&\alpha_3 + \alpha_4 + \beta_1 + \beta_3, \\
&\alpha_2 + \alpha_4 + \beta_1 + \beta_4, \\
&\alpha_2 + \alpha_4 + \beta_2 + \beta_3) \\
&\le \gamma_1 + \gamma_3 \\
&\le \min(\alpha_1 + \alpha_2 + \beta_1 + \beta_3, \\
&\qquad \alpha_1 + \alpha_3 + \beta_1 + \beta_2),
\end{aligned}
\tag{51b}
$$

$$
\begin{aligned}
\max(&\alpha_1 + \alpha_4 + \beta_3 + \beta_4, \\
&\alpha_2 + \alpha_4 + \beta_2 + \beta_4, \\
&\alpha_3 + \alpha_4 + \beta_1 + \beta_4) \\
&\le \gamma_1 + \gamma_4 \\
&\le \min(\alpha_1 + \alpha_2 + \beta_1 + \beta_4, \\
&\qquad \alpha_1 + \alpha_3 + \beta_1 + \beta_3, \\
&\qquad \alpha_1 + \alpha_4 + \beta_1 + \beta_2),
\end{aligned}
\tag{51c}
$$

following from the 41 so-called $(*IJK)$ *inequalities* [3]

$$
\begin{pmatrix}
\gamma_1 \leq \alpha_1 + \beta_1 & \gamma_2 \leq \alpha_1 + \beta_2 \\
\gamma_3 \leq \alpha_1 + \beta_3 & \gamma_4 \leq \alpha_1 + \beta_4 \\
\gamma_2 \leq \alpha_2 + \beta_1 & \gamma_3 \leq \alpha_2 + \beta_2 \\
\gamma_4 \leq \alpha_2 + \beta_3 & \gamma_3 \leq \alpha_3 + \beta_1 \\
\gamma_4 \leq \alpha_3 + \beta_2 & \gamma_4 \leq \alpha_4 + \beta_1 \\
\gamma_1 + \gamma_2 \leq \alpha_1 + \alpha_2 + \beta_1 + \beta_2 & \gamma_1 + \gamma_3 \leq \alpha_1 + \alpha_2 + \beta_1 + \beta_3 \\
\gamma_1 + \gamma_4 \leq \alpha_1 + \alpha_2 + \beta_1 + \beta_4 & \gamma_2 + \gamma_3 \leq \alpha_1 + \alpha_2 + \beta_2 + \beta_3 \\
\gamma_2 + \gamma_4 \leq \alpha_1 + \alpha_2 + \beta_2 + \beta_4 & \gamma_3 + \gamma_4 \leq \alpha_1 + \alpha_2 + \beta_3 + \beta_4 \\
\gamma_1 + \gamma_3 \leq \alpha_1 + \alpha_3 + \beta_1 + \beta_2 & \gamma_1 + \gamma_4 \leq \alpha_1 + \alpha_3 + \beta_1 + \beta_3 \\
\gamma_2 + \gamma_3 \leq \alpha_1 + \alpha_3 + \beta_1 + \beta_3 & \gamma_2 + \gamma_4 \leq \alpha_1 + \alpha_3 + \beta_1 + \beta_4 \\
\gamma_2 + \gamma_4 \leq \alpha_1 + \alpha_3 + \beta_2 + \beta_3 & \gamma_3 + \gamma_4 \leq \alpha_1 + \alpha_3 + \beta_2 + \beta_4 \\
\gamma_1 + \gamma_4 \leq \alpha_1 + \alpha_4 + \beta_1 + \beta_2 & \gamma_2 + \gamma_4 \leq \alpha_1 + \alpha_4 + \beta_1 + \beta_3 \\
\gamma_3 + \gamma_4 \leq \alpha_1 + \alpha_4 + \beta_1 + \beta_4 & \gamma_2 + \gamma_3 \leq \alpha_2 + \alpha_3 + \beta_1 + \beta_2 \\
\gamma_2 + \gamma_4 \leq \alpha_2 + \alpha_3 + \beta_1 + \beta_3 & \gamma_3 + \gamma_4 \leq \alpha_2 + \alpha_3 + \beta_2 + \beta_3 \\
\gamma_2 + \gamma_4 \leq \alpha_2 + \alpha_4 + \beta_1 + \beta_2 & \gamma_3 + \gamma_4 \leq \alpha_2 + \alpha_4 + \beta_1 + \beta_3 \\
\gamma_3 + \gamma_4 \leq \alpha_3 + \alpha_4 + \beta_1 + \beta_2 & \gamma_1 + \gamma_2 + \gamma_3 \leq \alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 \\
\gamma_1 + \gamma_2 + \gamma_4 \leq \alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_4 & \gamma_1 + \gamma_3 + \gamma_4 \leq \alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_3 + \beta_4 \\
\gamma_2 + \gamma_3 + \gamma_4 \leq \alpha_1 + \alpha_2 + \alpha_3 + \beta_2 + \beta_3 + \beta_4 & \gamma_1 + \gamma_2 + \gamma_4 \leq \alpha_1 + \alpha_2 + \alpha_4 + \beta_1 + \beta_2 + \beta_3 \\
\gamma_1 + \gamma_3 + \gamma_4 \leq \alpha_1 + \alpha_2 + \alpha_4 + \beta_1 + \beta_2 + \beta_4 & \gamma_2 + \gamma_3 + \gamma_4 \leq \alpha_1 + \alpha_2 + \alpha_4 + \beta_1 + \beta_3 + \beta_4 \\
\gamma_1 + \gamma_3 + \gamma_4 \leq \alpha_1 + \alpha_3 + \alpha_4 + \beta_1 + \beta_2 + \beta_3 & \gamma_2 + \gamma_3 + \gamma_4 \leq \alpha_1 + \alpha_3 + \alpha_4 + \beta_1 + \beta_2 + \beta_4 \\
\gamma_2 + \gamma_3 + \gamma_4 \leq \alpha_2 + \alpha_3 + \alpha_4 + \beta_1 + \beta_2 + \beta_3 &
\end{pmatrix}
$$

B.2 The PDF for $n = 4$. We have

$$
p(\gamma \mid \alpha, \beta) = \frac{1}{2} \delta \left(\sum \gamma - \alpha - \beta \right) \frac{\Delta(\gamma)}{\Delta(\alpha)\Delta(\beta)} \mathcal{J}_4
$$

and

$$
\begin{aligned}
\mathcal{J}_4 = \frac{1}{8} \sum_{PP' \in S_4} \varepsilon_P \varepsilon_{P'} \epsilon(A_1) \Big[& \frac{1}{3!} \epsilon(A_2 - A_1)(|A_3 - A_1|^3 - |A_3 - A_2 + A_1|^3 \\
& \qquad\qquad - |A_3 - A_2|^3 + |A_3|^3) \\
& - \frac{1}{3} \epsilon(A_2)(|A_3|^3 - |A_3 - A_2|^3) \\
& - \frac{1}{2}(|A_2 - A_1| - |A_2|) \\
& \qquad\quad (|A_3 - A_2|(A_3 - A_2) + |A_3|A_3) \Big]
\end{aligned}
$$

with A_j is a shorthand notation for $A_j(P, P', I)$ given in (16).

For $\gamma_4 \leq \gamma_3 \leq \gamma_2 \leq \gamma_1$, this sum vanishes if the inequalities (50) and (51) are not satisfied. $\mathcal{J}_4$ is normalized according to (19), *i.e.,*

$$\int_{\substack{\text{sector} \\ \gamma_4 \leq \gamma_3 \leq \gamma_2 \leq \gamma_1}} d^3\gamma \frac{\Delta(\gamma)}{\Delta(\alpha)\Delta(\beta)} \mathcal{J}_4 = \frac{1}{12}.$$

Note that the above expression of $\mathcal{J}_4$ has the property that the two sign functions $\epsilon(A_1)$ and $\epsilon(A_2 - A_1)$ are in front of expressions that vanish when A_1, resp. $A_2 - A_1$, vanishes. The somewhat ambiguous value of the sign function at 0 is thus irrelevant.

B.3 A few words about $n = 5$. For $n = 5$, Horn's inequalities and the expression of $\mathcal{J}_5$ are too cumbersome to be given here – it is a spline function made of 628 terms of degree 6...–, but may be found on the web site

http://www.lpthe.jussieu.fr/~zuber/Z_Unpub.html

We have checked a certain number of consistency relations, its vanishing when Horn's inequalities are not satisfied, and the normalization condition (19), namely

$$\int_{\substack{\text{sector} \\ \gamma_5 \leq \gamma_4 \leq \gamma_3 \leq \gamma_2 \leq \gamma_1}} d^4\gamma \frac{\Delta(\gamma)}{\Delta(\alpha)\Delta(\beta)} \mathcal{J}_5 = \frac{1}{288}.$$

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