Counting tensor model observables
and branched covers of the 2-sphere

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Abstract. Lattice gauge theories of permutation groups with a simple topological action (henceforth permutation-TFTs) have recently found several applications in the combinatorics of quantum field theories (QFTs). They have been used to solve counting problems of Feynman graphs in QFTs and ribbon graphs of large N, often revealing inter-relations between different counting problems. In another recent development, tensor theories generalizing matrix theories have been actively developed as models of random geometry in three or more dimensions. Here, we apply permutation-TFT methods to count gauge invariants for tensor models, colored as well as non-colored. For the colored case, we exhibit a relationship with the counting of branched covers of the 2-sphere, where the rank d of the tensor gets related to a number of branch points. We give explicit generating functions for the relevant countings and describe algorithms for the enumeration of the invariants. While the counting of colored tensor invariants is related to the classic counting of Hurwitz equivalence classes of branched covers with fixed branch points, collecting these under a further equivalence of permuting the branch points is relevant to the color-symmetrized tensor invariant counting. We also apply the permutation-TFT methods to obtain some formulae for correlators of the tensor model invariants.

Contents

1. Introduction

Motivated by the problem of understanding the precise dictionary between observables in string theory and in gauge theory, in the context of gauge-string duality (see [1] and [2]), permutation group techniques have recently been used to solve a variety of problems in the combinatorics of single- and multi-matrix models; see [3], [5], [6], [7], [8], [9], [10], [11], and [12].

While the matrix models often arise from the study of particular sectors of four dimensional $\mathcal{N} = 4$ super-Yang–Mills theory, or other four dimensional gauge theories, it has been fruitful to revisit, with these permutation techniques, the study of matrix models as mathematical models of gauge-string duality in their own right [13], [14], [15], and [16]. This line of research draws key ideas from the discovery that the large N expansion of two dimensional Yang–Mills (YM) theory, can be reformulated as a string theory, a link where permutations play a crucial role; see [17], [18], [19], [20], and [21]. The exact partition function of $U(N)$ 2dYM on a Riemann surface Σ_g; see [22] can be expanded in $1/N$ and the coefficients in the expansion were recognized as counting holomorphic maps between Riemann surfaces $\Sigma_h \to \Sigma_g$. The power of N is related to the genus of the covering space Σ_h, which is interpreted as the string worldsheet.

There are three main elements to this YM-string connection. The first is the mathematical fact of *Schur–Weyl duality* which relates the world of unitary groups, more generally classical groups, to the world of permutations. The second is *two dimensional topological field theory of permutations* (permutation-TFT), a simple physical construction based on lattice gauge theory, with symmetric groups as gauge groups, where edges variables take values in a symmetric group. The plaquette weight

of the lattice theory is a simple delta function which gives one when the edge variables around the plaquette multiply to one, and gives zero otherwise. The third is the link between *permutations and covering spaces*, a basic fact of algebraic topology. Now in two dimensions, branched covers are equivalently holomorphic maps, leading to deep links between combinatorics and complex geometry in the form of the Riemann existence theorem. Combining these ingredients leads to an interpretation of the permutation sums that appear in the large N expansion of 2dYM in terms of spaces of branched covers, equivalently holomorphic maps, called Hurwitz spaces.

The link between permutations and strings – at a topological level – is of course rather simple, and deep in this simplicity: strings winding around a circle have a winding number. For a fixed total winding number n, multi-string configurations contain a configuration for every partition of n. This has motivated the investigation of Feynman graph counting problems in QFT in terms of permutations, including situations without large N; see [23]. The structure of a graph can be coded using numbers to give labeled structures, in such a way that there is an action of permutation groups (of re-arrangements of the numbers) on the labeled structures, and the counting of the graphs involves modding out by certain permutation equivalences. This leads to the combinatoric description of graphs in terms of double cosets, which was heavily exploited in [23]. Problems of refined graph counting, in this case of graphs embedded in Riemann surfaces, were studied in [24] using further techniques such as graph quotients. The central role of permutation-TFTs continues to persist in these cases. As a unifying description of diverse counting problems, the permutation-TFTs often reveal surprising connections, a notable one being the link between the counting of vacuum graphs in Quantum Electrodynamics and that of ribbon graphs, which are normally encountered in a large N context. The cyclic orientation provided by the electron circulating in loops, can be mapped to a problem of graphs with vertices equipped with cyclic orientation, which are precisely ribbon graphs. While this is a good way to understand the surprising link in retrospect, it is easier to derive it by manipulating some delta functions over symmetric groups.

In this paper, we will undertake some counting problems motivated by tensor models, using the framework of permutation-TFTs, and we will find that this framework continues to be a source of non-trivial links between apparently very different counting problems. Let us review a little more explicitly some concepts from [23], which will set the stage for our current investigations. A Feynman graph can be coded in terms of labeled combinatoric data, by first introducing in the middle of all the existing edges a new type of vertex to get a new graph. We can call the formerly existing vertices - black vertices, and the newly introduced bivalent vertices - white vertices. Now label the edges of the new graph with integers $\{1, 2, \ldots, 2d\}$, where d is the number of edges of the original graph. Next, cut along all these 2d edges. All graphs with a fixed vertex structure can be obtained by re-connecting these cuts. The different reconnections can be parametrized by a permutation $\sigma \in S_{2d}$. This is illustrated in Figure 1 for the case where we have v 4-valent vertices in the original graph and $d = 4v$. Different permutations can give the same graph if they are re-

lated by equations of the form $\sigma' = \gamma_1 \sigma \gamma_2$. The γ_1, γ_2 live in subgroups H_1, H_2 of S_{2d} related to the symmetries of the black vertices and of the white vertices, respectively. This allows us to count Feynman graphs by counting points in double cosets of permutation groups.
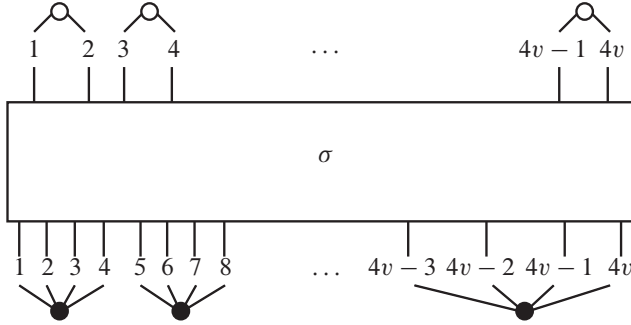

Figure 1. Double coset connection.

Burnside's lemma leads to formulae for counting these equivalence classes as sums of some delta functions over symmetric groups. These can in turn be recognized as the partition functions of permutation-TFT on a cylinder, with S_{2d} gauge group, and with boundary observables related to H_1, H_2. The above framework for relating graph counting to permutation groups and in particular permutation-TFTs is rather general.

Graph counting has also come into the centre of attention from a completely different perspective, namely random tensor models. Graphs are related (see for example [25]) to the counting of tensor invariants a problem with classical origins; see [26] and [27]. Tensor models have been proposed as a way to understand higher dimensional random geometry (see [28], [29], [30], and [31]), generalizing the powerful results connecting matrix models to two dimensional quantum gravity from the eighties/nineties; see [32]. A tensor model is defined via a field which is a rank d tensor over an abstract multi-dimensional representation space. In a dual space, a rank d tensor is viewed as a $(d-1)$-simplex. The interaction in such models is dually described by a d-simplex and is formed by the gluing of $(d-1)$-basic simplices along their $(d-2)$-boundary simplex. For example, if $d = 2$, the field can be a real matrix M representing a 1-simplex or a segment; the simplest interaction is of the form of an invariant $\text{Tr}[M^3]$ and represent a triangle formed by the gluing of 1-simplices along their 0-simplex boundaries. This is the simplest non trivial matrix model. The simplest higher rank extension of this model, is a rank 3 tensor model. Here, the field is a rank 3 tensor representing a 2-simplex or triangle. The interaction is obtained by a specific contraction of tensor fields and represents a 3-simplex or tetrahedron formed by the gluing of triangles along their 1-simplex or boundary segments. Generally, in a rank d model, a Feynman graph corresponds to a simplicial complex obtained from the gluing of d-simplexes along their $(d-1)$-boundary.

Recent works have focused on colored tensor models; see [33] and [34], where the $1/N$ expansion has been developed in [35], [36], [37], and [38]. This has triggered a plethora of new results on higher dimensional statistical mechanics and renormalizability of tensor models; see [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], and [49]. "Melonic graphs," which can be counted by mapping to a tree counting problem, with counting functions given by generalized Catalan numbers have played a central role, notably in connection with solving Schwinger–Dyson equations; see [43]. Specific types of melonic tensor invariants have been used in QFT to determine renormalizable actions; see [47] and [49].

In this paper, we will consider tensor models where the basic fields are $\Phi_{i_1,...,i_d}$ and $\bar{\Phi}_{j_1,...j_d}$. The indices $i_1, \ldots i_d$ transform as $\otimes_{a=1}^{n} V_a$ of $U(N_a)^{\times d}$, while $j_1 \ldots j_d$ transform as $\bar{V}_a^{\otimes n}$, with V_a being the fundamental of $U(N_a)$ and $\bar{V}_a$ the anti-fundamental. The emphasis will be on the complete enumeration of tensor invariants, given specified gauge invariance constraints. We will concentrate our attention on the case $d = 3, 4$, and take n to be the number of Φ's, which has to be equal to the number of $\bar{\Phi}$'s. Based on the expectations from [23] and [24], we find that these counting problems can be expressed neatly in terms of permutation-TFTs. And we find that these problems, for any d, can be mapped to the counting of branched covers of the 2-sphere. The parameter d appears as the number of branch points on the 2-sphere.

These formulations in terms of TFTs and branched covers allow the expression of the counting in terms of extracting coefficients of certain multi-variable generating functions. These expressions can be evaluated to high orders with the help of Mathematica, where the enumeration of the tensor invariants by hand becomes hopeless. Another useful piece of software is GAP [50], which gives not only the numbers of invariants, but can also store the detailed information about the structure of the invariant in the form of some permutation data, once the correct permutation formulation of the tensor counting problem has been found.

The plan of the paper is as follows. The next section reviews the definition of unitary tensor invariants (sometimes referred to, simply as tensor invariants or even trace invariants) and tensor models. Section 3 deals with the counting of invariants that can be built from d-index tensors $\Phi_{i_1,...,i_d}, \bar{\Phi}_{j_1,...,j_d}$, when we have n copies of Φ and $\bar{\Phi}$. We first formulate the problem in terms of counting invariants of an action of $U(N)^{\times d}$ on a certain symmetrized tensor product of fundamental representations. This is mapped to a counting of a d-tuple of permutations subject to certain constraints, which are themselves given by the action of two permutations. These two permutations correspond to the symmetries of re-ordering the Φ's and $\bar{\Phi}$'s respectively. This problem is expressed in terms of sums over delta functions over symmetric groups, which are then simplified to yield a problem of counting a sequence of just $(d - 1)$ permutations, subject to an equivalence given by one permutation. This leads to a solution of the counting in the form of sums over partitions, weighted by powers of the symmetry factors of the partitions (see eq. (11)). We distinguish connected and disconnected invariant countings, which are related by the plethystic logarithm function. Section 4 interprets the symmetric group delta functions arising

in the solution above in terms of topological lattice gauge theory on a certain complex. The simplification is shown to be related to a coarsening of the complex, which leaves the answer invariant because of the topological invariance of the lattice theory. The final permutation problem involving $(d - 1)$ permutations with one conjugation constraint is explained, using the classic Riemann existence theorem, to be related to the counting of branched covers of the sphere, equivalently to holomorphic maps from Riemann surfaces to the sphere.

Section 5 describes a *color symmetrized* version of the counting problem of Section 3. This is based on the fact that the counting of colored tensor invariants for rank d admits an S_d permutation symmetry of renaming the colors, so it is natural to count equivalence classes under this symmetry. This problem is expressed in precise form in terms of $U(N)^{\times d}$ invariants in an appropriate vector space. Again, since $U(N)$ invariants are generated by products of δ_{ij}, we can count them by parametrizing the possible ways the i's go with the j's, which is given by permutations σ_a (for a going from 1 to d). The color-symmetrized counting involves imposing a further equivalence of permuting the σ_a. We solve this using the permutation group algebra techniques, the upshot being simple elegant formulae in terms of delta functions over symmetric groups, leading to generating functions involving multiple variables. There is a subtlety in the relation between connected and disconnected case, so that the connected counting is no longer given by taking a plethystic log. This subtlety is explained.

Section 6 turns to the counting of general tensor invariants where contraction between the different i-indices on Φ can occur with j indices on $\bar{\Phi}$, irrespective of the positions of these indices. This distinguishes the counting from the "colored-case" where the different slots along the Φ and $\bar{\Phi}$ are distinguished, so we may call this "non-colored counting." This is a problem in invariants of $U(N)$ acting on an appropriate vector space, rather than $U(N)^{\times d}$. A variation where the tensors Φ and $\bar{\Phi}$ are symmetric is also solved. Section 7 gives some formulae for correlators related to the counting of Section 3.

Section 8 gives a summary of our results and avenues for future research. The discussion includes, as well, some observations on the relations between color-symmetrized counting of invariants and the counting of braid orbits of branched covers, a subject that is studied from completely different motivations by pure group theorists. Appendix A gives a short review on group actions, including Burnside's lemma and some key facts about the symmetric group. Appendix B proves some formulae stated in the main text. Appendix C provides details about derivations of formulae for correlators in Gaussian tensor models given in Section 7. Appendix D contains some GAP and Mathematica codes used to obtain the explicit counting sequences.[1] Some of these are identified with known ones in OEIS (see [51]); others are not in OEIS.

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[1]These can certainly be improved in efficiency but are included for illustrations.

2. Tensor model invariants: a review

In this section, we review the construction of unitary tensor invariants and their graphical representation. The results presented here are largely based on [42]. We also discuss the simplest way to introduce tensor models and their Feynman graphs.

2.1. Tensor invariants. Let $V_1, V_2, \ldots, V_d$ be some complex vector spaces of respective dimensions $N_1, N_2, \ldots N_d$. Consider rank $d \geq 2$ covariant tensors Φ with components $\Phi_{i_1,\ldots,i_d}$ transforming as $\otimes_{a=1}^d V_a$ with $i_a \in \{1, \ldots, N_a\}$, $a = 1, 2, \ldots, d$, with no symmetry assumed under permutation of their indices. These tensors transform under the action of the tensor product of fundamental representations of unitary groups $\otimes_{a=1}^d U(N_a)$ where each unitary group $U(N_a)$ acts on a tensor index i_a independently. The complex conjugate of $\Phi_{i_1 i_2 \ldots i_d}$ is a contravariant tensor of the same rank and is given by $\bar{\Phi}_{i_1 i_2 \ldots i_d}$. We have the following transformation:

$$\Phi_{i_1 i_2 \ldots i_d} = \sum_{j_1, \ldots, j_d} U^{(1)}_{i_1 j_1} U^{(2)}_{i_2 j_2} \ldots U^{(d)}_{i_d j_d} \Phi_{j_1 j_2 \ldots j_d} \tag{1a}$$

and

$$\bar{\Phi}_{i_1 i_2 \ldots i_d} = \sum_{j_1, \ldots, j_d} \bar{U}^{(1)}_{i_1 j_1} \bar{U}^{(2)}_{i_2 j_2} \ldots \bar{U}^{(d)}_{i_d j_d} \bar{\Phi}_{j_1 j_2 \ldots j_d}, \tag{1b}$$

where $U^{(a)} \in U(N_a)$ and may be very well all distinct. In the next discussion, we will be primarily interested in $d \geq 3$.

Invariants with respect to the unitary action (1) built on tensors can be obtained by contracting, in all possible ways, pairs of covariant and contravariants tensors. It turns out that these contractions are in bijection with closed d–colored graphs that we must now introduce.

A *bi-partite closed d-colored graph* is a graph

$$\mathcal{B} = (\mathcal{V}(\mathcal{B}), \mathcal{E}(\mathcal{B}))$$

that is a collection $\mathcal{V}(\mathcal{B})$ of vertices with fixed valence (or degree or coordination) d and set $\mathcal{E}(\mathcal{B})$ of edges, with incidence relation between edges and vertices, such that

- $\mathcal{V}(\mathcal{B})$ can be partitioned into two disjoint sets $\mathcal{V}^+$ and $\mathcal{V}^-$, of equal size, such that each edge e is may only connect a vertex $v^+ \in \mathcal{V}^+$ and a vertex $v^- \in \mathcal{V}^-$ (this is the bi-partite property);

- the graph has a d-line coloring α, that is an assignment of a color to each edge,

$$\alpha \colon \mathcal{E}(\mathcal{B}) \longrightarrow \{1, 2, \ldots, d\},$$

such that two adjacent edges cannot have the same color (two edges are called adjacent if they are incident to a same vertex). Note that $\alpha^{-1}(i)$ is the subset of lines of color i.

The fact that the graph is closed simply implies that the number of edges in the graph fully saturates the valence of the vertices $2|\mathcal{E}(\mathcal{B})| = d|\mathcal{V}(\mathcal{B})|$.

One can construct the graph associated with a tensor invariant built from the contraction of some tensors in the following way. Consider $\Phi_{i_1 \ldots i_d}$ (respectively, $\bar{\Phi}_{i_1 \ldots i_d}$) and assign it to a vertex $v^+ \in \mathcal{V}^+$ (respectively, to a vertex $v^- \in \mathcal{V}^-$). The position of an index in the tensor becomes a color: i_a has the color a. The contraction of an index i_a of some $\Phi_{\ldots i_a \ldots}$ with and index j_a of some $\bar{\Phi}_{\ldots j'_a \ldots}$ is represented by a line of color a between a vertex v^+ associated with Φ and a vertex v^- associated with $\bar{\Phi}$. Some examples are provided in Figure 2. The trace invariant associated with $\mathcal{B}$ is given by

$$\mathrm{Tr}_{\mathcal{B}}(\Phi, \bar{\Phi}) = \sum_{i,j} \delta^{\mathcal{B}}_{i,j} \prod_{v,v' \in \mathcal{V}(\mathcal{B})} \Phi_{i_v} \bar{\Phi}_{j_v}, \tag{2a}$$

and

$$\delta^{\mathcal{B}}_{i,j} = \prod_{a=1}^{d} \prod_{l^a \in \alpha^{-1}(a)} \delta_{i^a_{v^+(l^a)}, j^a_{v^-(l^a)}}, \tag{2b}$$
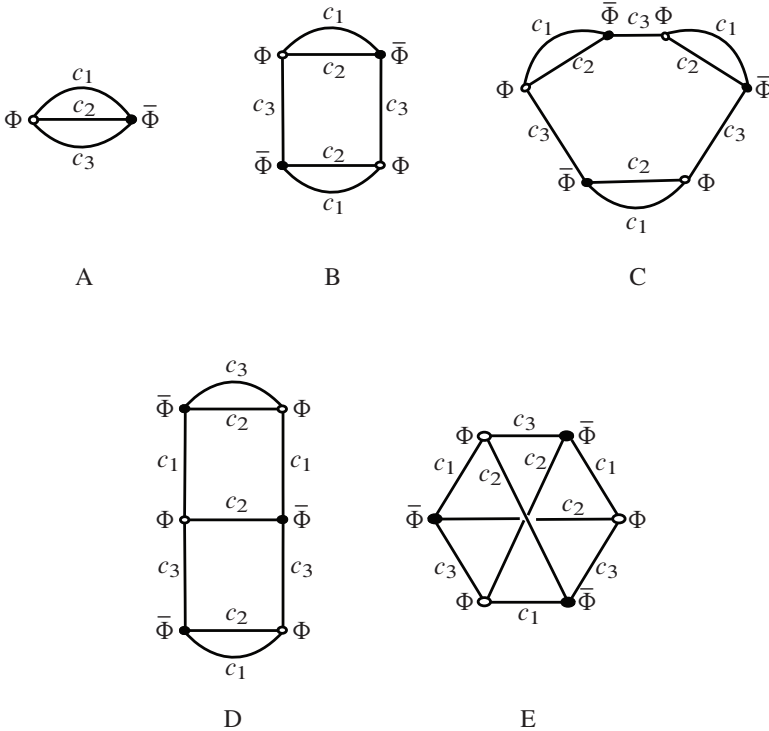

Figure 2. Some rank $d = 3$ tensor invariants.

where in the formula, the sum is performed over all indices of the tensors, the function $\delta^{\mathscr{B}}_{i,j}$ implements the d-line coloring or contraction between tensor indices, such that, given a line l^a of color a incident to vertices $v^+(l^a)$ and $v^-(l^a)$, the indices $i^a_{v^+(l^a)}$ must be equal to $j^a_{v^-(l^a)}$. One can check the formal expression

$$\mathrm{Tr}_{\mathscr{B}}(\Phi^U, \bar{\Phi}^{\bar{U}}) = \mathrm{Tr}_{\mathscr{B}}(\Phi, \bar{\Phi}),$$

where Φ^U stands for the transformed of Φ with respect to the unitary action (1). The trace invariant may factorize over the connected components of $\mathscr{B}$. For instance in Figure 2, combining graphs A and B generates a new rank 3 disconnected invariant made with six tensors.

Finally, we must emphasize that colored graphs of this kind are dual to d-dimensional abstract simplicial pseudo-manifolds [33]. Such a feature is important in the framework of tensor models. In the same way that the study of matrix models provides the statistical sum of random triangulations of Riemannian surfaces and turned out to be important to solve 2D quantum gravity, tensor models generate random triangulations of higher dimensional objects and address gravity in dimension higher than 2. The colored tensor model introduced in [33] yields a first step towards a clearer understanding of the type of "regular" triangulations that can be generated by the partition function using colored tensors. The next section formally introduces the generic type of tensor models.

2.2. Tensor models. The simplest form of rank d tensor models are described by an action with complex tensor field $\Phi_{i_1 \ldots i_d}$ with kinetic term

$$S^{\mathrm{kin}} = \sum_{\{i_a\}} \bar{\Phi}_{i_1 \ldots i_d} \, \Phi_{i_1 \ldots i_d}.$$

In the specific instance S^{kin} corresponds to a mass term. Certainly, more elaborate kinetic terms can be constructed.

A typical $(\bar{\Phi}\Phi)^p$ interaction in such a model may be written as

$$S^{\mathrm{inter}} = \lambda_V \sum_{\{i_a^{(p)}, j_a^{(p)}\}} V(\{i_a^{(p)}, j_a^{(p)}\}) \prod_{p=1}^{n} \bar{\Phi}_{i_1^{(p)} \ldots i_d^{(p)}} \Phi_{j_1^{(p)} \ldots j_d^{(p)}},$$

where λ is a coupling constant and V is constructed from Kronecker delta's and determines the precise form of the interaction. In 1-matrix theory, interaction terms are, say at order 3, of the form $\mathrm{tr}(M^3)$, $[\mathrm{tr}(M^2)](\mathrm{tr}M)$, $(\mathrm{tr}M)^3$: at order n there are $p(n)$ possible interaction terms (number of partitions of n). The enumeration of tensor invariants we give in subsequent sections allows a group theoretic characterization of the interaction terms at each order for tensor models and gives a number $Z_d(n)$ which replaces $p(n)$ when we go from matrix models to tensor models. Particular forms of V might lead to models with different properties. For instance, discussing

perturbative renormalizability, the type of contractions implemented by V should be of form of trace invariants of the melonic kind; see [49].

The partition function associated with the type of tensor model can be written

$$Z = \int d\Phi d\bar\Phi \, e^{-S^{\mathrm{kin}} - S^{\mathrm{inter}}(\bar\Phi, \Phi)}.$$

Either at the Gaussian limit $\lambda = 0$ or, in the perturbative picture, by perturbing around the Gaussian measure, the several types of countings that we will discuss in the following are useful for the understanding of the $2P$ correlation function issued from tensor models as

$$\langle \bar\Phi_{I_1} \Phi_{I_1'} \ldots \bar\Phi_{I_P} \Phi_{I_P'} \rangle = \int d\Phi d\bar\Phi \;\; \bar\Phi_{I_1} \Phi_{I_1'} \ldots \bar\Phi_{I_P} \Phi_{I_P'} \, e^{-S^{\mathrm{kin}} - S^{\mathrm{inter}}(\bar\Phi, \Phi)},$$

where I_i are multi-indices. The external data $(I_1, I_2, \ldots, I_P)$ and $(I_1', I_2', \ldots, I_P')$ are associated with external boundary topological data of the simplex corresponding to the collection of fields $(\bar\Phi_{I_1}, \Phi_{I_1'}, \ldots, \bar\Phi_{I_P}, \Phi_{I_P'})$. Referring to the renormalizable tensor models, it has been proved that, for a primitively divergent correlation function, these momentum data should match with melonic tensor invariant contractions of the same form of the vertex in the initial action. Once again, unitary invariants play a central role in this context.

3. Counting invariants in colored tensor models

For simplicity, we start the discussion by the rank $d = 3$ case, the general situation $d \geq 3$ can be easily inferred from this point.

3.1. Tensor invariants as $U(N)^d$ group action invariants.
Consider a colored tensor Φ of rank d, where the indices are colored. Making the indices explicit, we have $\Phi_{i_1 \ldots i_d}$. We want to know the number $Z_3(n)$ of invariants in the form (2) that one can build from n copies of Φ and n copies of $\bar\Phi$.

This can be formulated as a problem in invariant theory. Given a $U(N)$ representation V, there is a one-dimensional space of linear maps from $V \otimes \bar V$ to $\mathbb{C}$ such that

$$\delta \mid e_i \rangle \otimes \mid \bar e_j \rangle = \delta_{ij},$$

which are invariant in the sense that

$$\delta(U \otimes U) = \delta, \quad U \in U(N).$$

This follows since

$$\delta(U \otimes U) \mid e_i \rangle \otimes \mid \bar e_j \rangle = U_{ki}(U^*)_{lj} \, \delta(\mid e_k \rangle \otimes \mid \bar e_l \rangle) = U_{ki}(U^*)_{kj} = \delta_{ij}.$$

Given $\otimes_{a=1}^{d} V_a$ which is a representation of $U(N)^{\times d}$ of dimension N^d (note that we could equally well work with $U(N_1) \times U(N_2) \times \cdots \times U(N_d)$, in which case we have dimension $N_1 N_2 \ldots N_d$), consider

$$W = \mathrm{Sym}(V_1 \otimes V_2 \otimes \cdots \otimes V_d)^{\otimes n}$$

and

$$\overline{W} = \mathrm{Sym}(\overline{V}_1 \otimes \overline{V}_2 \otimes \cdots \otimes \overline{V}_d)^{\otimes n},$$

where Sym indicates that we are symmetrizing the n copies (in other words these define indistinguishable copies). The first counting problem we solve is to find the dimension of the space of invariants in $W \otimes \overline{W}$. We will assume $N > n$, otherwise there are finite N corrections which we leave for future investigation (see more comments on this in the discussion section).

Now W has an action of S_d of permuting $V_1 \otimes V_2 \otimes \cdots \otimes V_d$, likewise for $\overline{W}$. We can define a linear operator for each $\alpha \in S_d$, denoted $\rho_W(\alpha)$ acting on W and a linear operator $\rho_{\overline{W}}(\alpha)$ acting on $\overline{W}$. Consider the S_d-symmetrizer acting on $W \otimes \overline{W}$ given by

$$\frac{1}{d!} \sum_{\alpha \in S_d} \rho_W(\alpha) \otimes \rho_{\overline{W}}(\alpha).$$

The second problem in invariant theory is to count the dimension of the space of $U(N)^{\times d}$ invariants in the image of the above symmetrizer. This is the color-symmetrized counting we address in Section 5.

3.2. Tensor invariants for $d = 3$ and permutation double coset.

At this point we will, for concreteness, specialize the discussion to $d = 3$, although it will be clear how the steps generalize to general d. Returning to the first problem, the invariants are generated by the different ways of contracting the different copies of V_a in W with the copies of $\overline{V}_a$ in $\overline{W}$. Diagrammatically, one may think about all the possible contractions between n tensors simply as the possible parings in the way given in Figure 3. In other words, the determination of possible graph amounts to count triples

$$(\sigma_1, \sigma_2, \sigma_3) \in (S_n \times S_n \times S_n)$$

with equivalence

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_1 \sigma_1 \gamma_2, \gamma_1 \sigma_2 \gamma_2, \gamma_1 \sigma_3 \gamma_2), \tag{3}$$

where $\gamma_i \in S_n$. Thus, we are counting points in the double coset

$$\mathrm{Diag}(S_n) \backslash (S_n \times S_n \times S_n) / \mathrm{Diag}(S_n).$$

We denote the number of points in this double coset as $Z_3(n)$. For general subgroups $H_1 \subset G$, $H_2 \subset G$, the cardinality of this double coset is given by

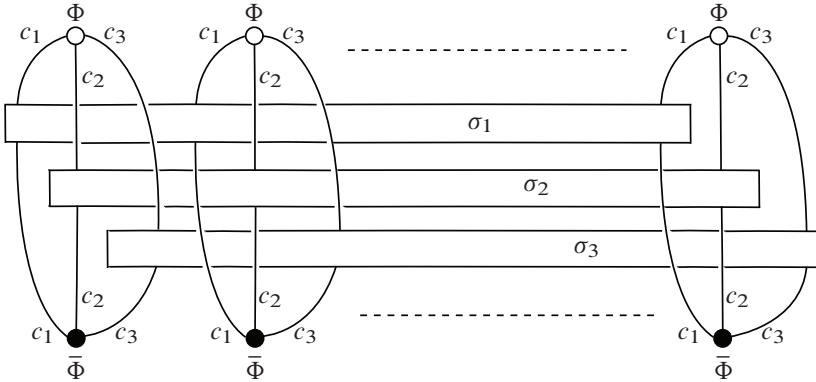$$|H_1 \backslash G / H_2| = \frac{1}{|H_1||H_2|} \sum_C Z_C^{H_1 \to G} Z_C^{H_2 \to G} \mathrm{Sym}(C). \tag{4}$$

Figure 3. Diagrammatic tensor contraction defining $(\sigma_1, \sigma_2, \sigma_3)$.

The sum is over conjugacy classes of G, and $Z_C^{H \to G}$ is the number of elements of H in the conjugacy class C of G. This formula appears in the context of graph counting in [52] and is used for a variety of Feynman graph problems in [23].

Let us explain the proof of this formula using Burnside's lemma reviewed in Appendix A. We think of the double coset as the number of orbits of the $H_1 \times H_2$ action on G. The fixed-point counting formula for the number of orbits becomes

$$|H_1 \backslash G / H_2| = \frac{1}{|H_1||H_2|} \sum_{h_1 \in H_1} \sum_{h_2 \in H_2} \sum_{g \in G} \delta(h_1 g h_2 g^{-1}), \qquad (5)$$

where δ is the delta function over the group G, equal to 1 if its argument is the identity element and 0 otherwise. This means that h_1 and h_2 have to be in the same conjugacy class of G. Now organize the sums according to conjugacy classes C of G. The number of elements in conjugacy class C from H_1 and H_2 are denoted $Z_C^{H_1}$ and $Z_C^{H_2}$, respectively. So the counting has a factor $Z_C^{H_1} Z_C^{H_2}$ from the h_1 and h_2 sums. For each such pair, there are $\mathrm{Sym}(C)$ possible g's. Hence we get the above formula.

The conjugacy classes of $S_n \times S_n \times S_n$ are entirely determined by a triple (p_1, p_2, p_3) where each p_i is a partition of n (see Proposition 1 in Appendix A). This correspondence holds because each conjugacy class is determined by a cycle structure. Now, the diagonal subgroup produces conjugacy classes (p, p, p). So applying (4), we get

$$Z_3(n) = \frac{1}{(n!)^2} \sum_{p \vdash n} \left(\frac{n!}{\mathrm{Sym}(p)} \right)^2 (\mathrm{Sym}(p))^3 = \sum_{p \vdash n} \mathrm{Sym}(p), \qquad (6)$$

where

$$\mathrm{Sym}(p) \stackrel{\mathrm{def}}{=} \prod_{i=1}^{n} (i^{p_i})(p_i!)$$

and the sum over $p = \{p_1, p_2, \ldots, p_n\}$ is performed over all partitions of $n = \sum_i i\, p_i$. The cardinality of a conjugacy class T_p of S_n with cycle structure determined by a partition p is given by $|T_p| = n!/\mathrm{Sym}(p)$; see Proposition 3 in the same appendix.

We can generate this sequence (using a GAP or Mathematica program, see GAP code and Mathematica code 1 in Appendix D) and get (from $n = 1$ up to $n = 10$)

$$1, 4, 11, 43, 161, 901, 5579, 43206, 378360, 3742738, \ldots. \tag{7}$$

This series is recognized in the OEIS website as A110143. The same sequence also matches with the counting of n-fold coverings of a graph [53]. This link will be clarified through the discussion vis permutation-TFT in Section 4.

The number $Z_3(n)$ actually includes disconnected invariants. One can easily give a graphical representation to the first order terms:

- $Z_3(1) = 1$ consists in a single connected mass term (see Figure 2 A) of the form

$$\sum_{i,j,k} \overline{\Phi}_{ijk} \Phi_{ijk};$$

- $Z_3(2) = 4$ consists in 3 connected invariants (see Figure 2 B), one of these is given by

$$\sum_{i,i'} \overline{\Phi}_{i_1 i_2 i_3} \Phi_{i'_1 i_2 i_3} \overline{\Phi}_{i'_1 i'_2 i'_3} \Phi_{i_1 i'_2 i'_3}$$

and the two others are obtained by simple color permutation $1 \to 2 \to 3$, plus one disconnected invariant of the form

$$\left(\sum_{i,j,k} \overline{\Phi}_{ijk} \Phi_{ijk} \right)^2$$

This term is nothing but twice a mass term (same as in Figure 2 A above). Such disconnected invariant terms in higher rank Tensorial Group Field Theory framework should be interesting since they appear as "anomalous" terms generated by the Renormalization Group flow; see [47].

3.3. Connected invariants. To get the connected invariants, we can use the so-called plethystic logarithm (Plog) function (for recent applications of this function in supersymmetric gauge theory and further references, see [54]). This can be achieved in the following manner. Define the generating function of the disconnected invariants as

$$Z_3(x) = \sum_{n=0}^{\infty} Z_3(n) x^n. \tag{8}$$

The Plog of $Z_3(x)$ is the function

$$\text{Plog}[Z_3(x)] = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log[Z_3[x^k]] \tag{9}$$

where

$$\mu(k) = \begin{cases} 0 & \text{if } k \text{ has repeated prime factors,} \\ 1 & \text{if } k = 1, \\ (-1)^n & \text{if } k \text{ is a product of } n \text{ distinct primes,} \end{cases}$$

is the so-called *Möbius function.* This series can be expanded at finite order by Mathematica (see Mathematica code 1 in Appendix D). We get the following expansion at the lowest order (up to $n = 10$):

$$x + 3x^2 + 7x^3 + 26x^4 + 97x^5 + 624x^6 + 4163x^7$$
$$+ 34470x^8 + 314493x^9 + 3202839x^{10} + O[x^{11}], \tag{10}$$

where a_n the coefficient of x^n gives now the number of connected diagrams with n black vertices (corresponding to $\bar{\Phi}$) and n white vertices (for Φ). This is again recognized in the OEIS as the series A057005 giving the number of conjugacy classes of subgroups of index n in the free group of rank 2. The first orders $Z_{3;\text{connected}}(1) = 1$, $Z_{3;\text{connected}}(2) = 3$ and $Z_{3;\text{connected}}(3) = 7$ are represented graphically in Figure 4.

Returning to a previous sequence, $Z_3(n)$, see eq. (7), includes disconnected invariants. For the first orders, $Z_3(2) = 4$ includes B1, B2 and B3 as connected objects plus a disconnected graph given by twice A; $Z_3(3) = 11$, contains all seven connected graphs with six external legs which are C1, C2,..., E, drawn in Figure 4, plus four other disconnected graphs given by the following combinations of connected pieces: (A, A, A), (A, B1), (A, B2) and (A, B3) (we will keep that notation for disconnected components graphs).

3.4. Generalized rank d case.

For rank d tensors, using $d-$tuples of permutations $(\sigma_1, \ldots, \sigma_d) \in (S_n)^{\times d}$ equivalent under the diagonal action $\text{Diag}(S_n)$ such that

$$(\sigma_1, \ldots, \sigma_d) \sim (\gamma_1 \sigma_1 \gamma_2, \ldots, \gamma_1 \sigma_d \gamma_2),$$

following the same procedure and in adapted notations, it is direct to obtain the number of tensor invariants made with $2n$ fields as

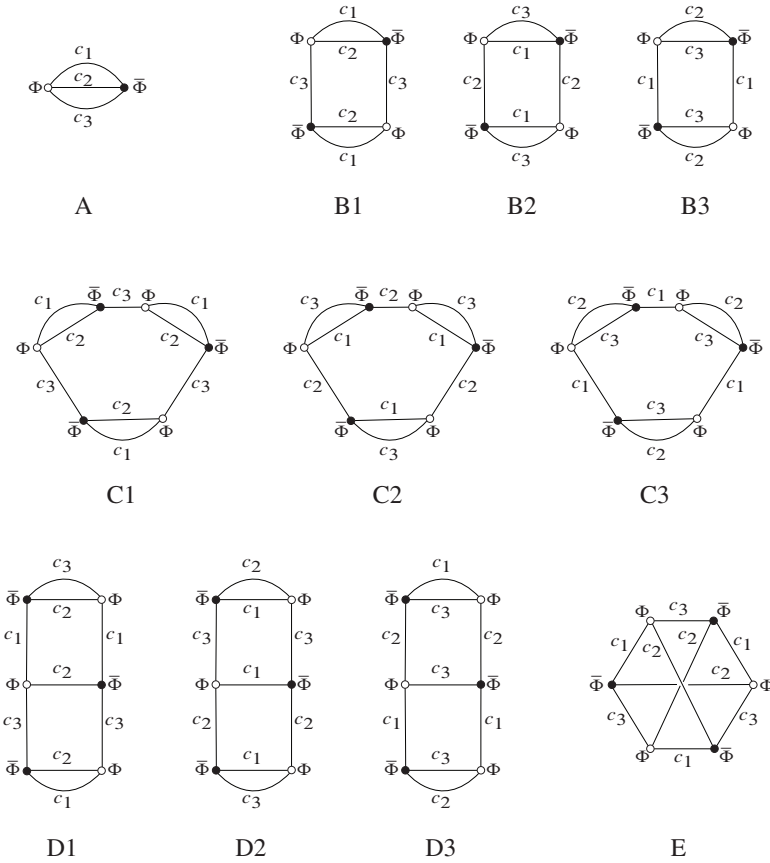$$Z_d(n) = \sum_{p \vdash n} (\text{Sym}(p))^{d-2}. \tag{11}$$

Figure 4. The colored graphs associated with $Z_{3;\text{connected}}(n)$: $n = 1$, A; $n = 2$, B1, B2, and B3; $n = 3$, C1, C2, C3, D1, D2, D3, and E.

Given d and n, this number can be evaluated by a GAP or Mathematica program (see GAP and Mathematica code 1, in Appendix D). This counting function $Z_d(n)$ has also been studied in connection with counting of n-fold coverings of the one-vertex graph with $(d - 1)$ edges (which we denote by F_{d-1}, the flower graph with $d - 1$ legs) which is equivalent to counting n-fold branched covers of the sphere with d branch points [53]. The link between the tensor-invariant counting, which we related to the double coset $\text{Diag}(S_n) \backslash S_n^{\times d} / \text{Diag}(S_n)$, and the counting of covers will become clearer when we develop the permutation-TFT description in the next section.

For the $d = 4$ case, the counting of invariants yields the sequence

$$1, 8, 49, 681, 14721, 524137, 25471105,$$
$$1628116890, 131789656610, 13174980291658, \ldots, \tag{12}$$

for $n = 1, 2, \ldots, 10$, respectively. For the case of connected invariants we use the

Plog function to get, for $n = 1, 2, 3, \ldots, 10$,

$$1, 7, 41, 604, 13753, 504243, 24824785,$$
$$1598346352, 129958211233, 13030565312011, \ldots. \tag{13}$$

This is recognized as the A057006 sequence by OEIS or as the number of conjugacy classes of subgroups of index n in the free group of rank 3. This sequence is also discussed the context of connected covers of F_{d-1} in [53].

4. Tensor model invariants and permutation-TFTs

In Section 3, the counting of tensor invariants was related to the number of points in a double coset. To calculate this we used a sum over group elements weighted by a delta function over the group (5) to arrive at formula (4). Such delta functions arise in a very simple physical construction, namely topological lattice gauge theory, where permutation groups play the role of gauge groups. We give a brief review of this construction, and refer the reader to a more detailed review in Section 5.1 of [11] and the original literature [55] and [56]. Then we will show that the topological invariance of this lattice construction illuminates the link between the counting of tensor invariants and the counting of branched covers of the 2-dimensional sphere.

4.1. Permutation TFTs – lightning review. On any cellular complex X, one can define a partition function for a finite group G by assigning a group element g_e to each edge e and to each plaquette P a weight $w(g_P)$, where

$$g_P = \prod_{e \in P} g_e.$$

A most simple and natural choice independent of the plaquette size is given by

$$w(g_P) = \delta(g_P) = \begin{cases} 1 & \text{if } g_P = \text{id}, \\ 0 & \text{otherwise.} \end{cases}$$

The partition function in the model is given by

$$Z[X; \, G] = \frac{1}{|G|^V} \sum_{g_e} \prod_P w_P \Big(\prod_{e \in P} g_e \Big), \tag{14}$$

where V is the number of vertices in the cell decomposition. This theory is topological in the sense that it is invariant under refinement of the cellular decomposition. We will be interested in cases where G is taken to be the symmetric group S_n, of all permutations of n objects. This simple topological field theory construction, with n arbitrary, has a variety of applications in QFT combinatorics; see [23], [24], and [11].

Take the torus realized as a rectangle, with opposite sides identified (Figure 5). This is a cell decomposition with a single 0-cell, two 1-cells a, c and a single 2-cell. Assign to each 1-cell a group element in G:

$$a \longrightarrow \sigma \quad \text{and} \quad c \longrightarrow \gamma. \tag{15}$$

Thus the plaquette weight for the single 2-cell (plaquette) is

$$w(g_P) = \delta(\gamma \sigma \gamma^{-1} \sigma^{-1}) \tag{16}$$

and the partition function is

$$Z(T^2; S_n) = \frac{1}{n!} \sum_{\sigma, \gamma \in S_n} \delta(\gamma \sigma \gamma^{-1} \sigma^{-1}). \tag{17}$$

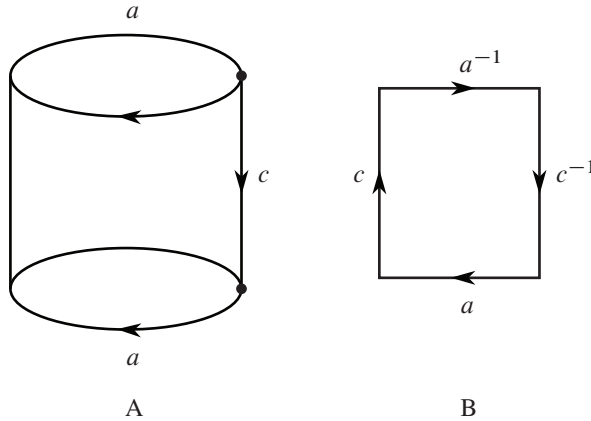

Figure 5. Periodic torus and the plaquette action.

This partition function, for a topological space X, counts equivalence classes of homomorphisms from $\pi_1(X)$ to S_n (weighted by the number of elements of S_n which fix the homomorphism under conjugation). By a standard theorem of algebraic topology, this is the same to count equivalence classes of covering spaces of X of degree n (see e.g. [57]), counted with weight equal to inverse of the order of the automorphism group of the cover). The partition function (17) thus counts n-fold covers of the torus and plays a role in the string theory interpretation of two-dimensional YM theory [17], [18], and [20]. Given a cover, we can pick a generic point on the target space, label the inverse images $\{1, \ldots, n\}$ and obtain permutations $\sigma, \gamma \in S_n$ as we follow the inverse images of the 1-cells a, c on the torus. The combination $aca^{-1}c^{-1}$ is a contractible path (shrinkable on the rectangle of Figure 5), so must give a trivial permutation of the sheets, which is enforced by the delta function.

4.2. Topological invariance of permutation-TFT: double coset to conjugation equivalence. We first start with the rank 3 case and then generalize the ideas to any rank d. The partition function $Z_3(n)$ can be written by applying Burnside's lemma (see Appendix A, Proposition 2) as

$$Z_3(n)$$
$$= \frac{1}{n!^2} \sum_{\gamma_1,\gamma_2 \in S_n} \sum_{\sigma_1,\sigma_2,\sigma_3 \in S_n} \delta(\gamma_1\sigma_1\gamma_2\sigma_1^{-1})\delta(\gamma_1\sigma_2\gamma_2\sigma_2^{-1})\delta(\gamma_1\sigma_3\gamma_2\sigma_3^{-1}).$$

Having seen the connection between sums over group delta functions and lattice TFTs, the natural question is: what topological space has a permutation-TFT partition function given by $Z_3(n)$? This allows us to see an emergence of geometry (more precisely topology at this stage, but see comment on holomorphic maps later) directly from the structure of the counting problem.

Consider the graph G_3 in Figure 6, which has two vertices and three edges. Next consider $G_3 \times S^1$, which can be visualized as being obtained by evolving G_3 along a vertical time direction and then compactifying the time, which amounts to identifying the graph at the base of the Figure 6 with the one at the top. The three 2-cells of this cell-complex are shaded.

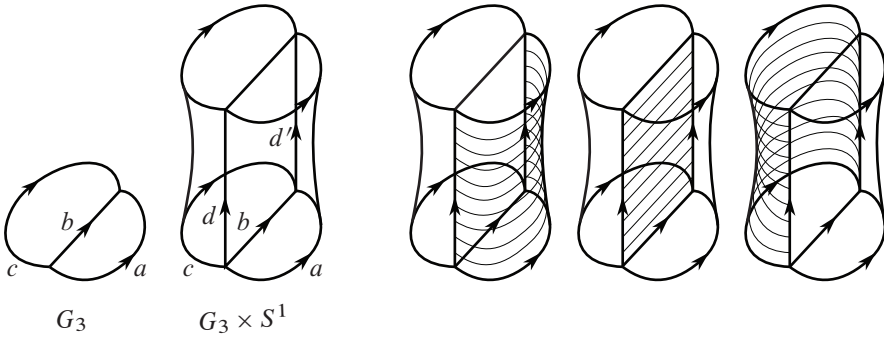Figure 6. G_3, $G_3 \times S^1$, and its different plaquettes (shaded).

To do S_n permutation-TFT on this complex, we assign

$$a \longrightarrow \sigma_1, \quad b \longrightarrow \sigma_2, \qquad c \longrightarrow \sigma_3,$$

where $\sigma_i \in S_n$, and we have two extra edges d and d' to which we assign

$$d \longrightarrow \gamma_1, \quad \text{and} \quad d' \longrightarrow \gamma_2,$$

with $\gamma_i \in S_n$. The partition function of this complex computed according to (14) as

$$Z(G_3 \times S^1; S_n)$$

$$= \frac{1}{n!^2} \sum_{\gamma_1,\gamma_2 \in S_n} \sum_{\sigma_1,\sigma_2,\sigma_3 \in S_n} \delta(\gamma_1\sigma_1\gamma_2\sigma_1^{-1})\delta(\gamma_1\sigma_2\gamma_2\sigma_2^{-1})\delta(\gamma_1\sigma_3\gamma_2\sigma_3^{-1}).$$

We thus recognize that *the counting function for 3-index colored tensor invariants is the permutation-TFT partition function on $G_3 \times S^1$*:

$$\boxed{Z(G_3 \times S^1; S_n) = Z_3(n).}$$

As observed in the lightning review, we can interpret this as counting covering spaces of $G_3 \times S^1$, and this is counting the covering spaces, with weight equal to inverse symmetry factor; see for example [18], and [20] for explanation of this fact.

The power of the permutation-TFT approach is that, not only, it exposes the geometry behind counting problems, but it also allows easy manipulations of the delta functions, which often reveal connections to other geometrical interpretations of the same counting problem. In this case, we can use one the delta functions to solve for γ_1:

$$Z_3(n) = \frac{1}{n!^2} \sum_{\gamma_2} \sum_{\sigma_1,\sigma_2,\sigma_3 \in S_n} \delta(\sigma_1\gamma_2^{-1}\sigma_1^{-1}\sigma_2\gamma_2\sigma_2^{-1})\delta(\sigma_1\gamma_2^{-1}\sigma_1^{-1}\sigma_3\gamma_2\sigma_3^{-1})$$

$$= \frac{1}{n!} \sum_{\gamma \in S_n} \sum_{\tau_1,\tau_2 \in S_n} \delta(\gamma\tau_1\gamma^{-1}\tau_1^{-1})\delta(\gamma\tau_2\gamma^{-1}\tau_2^{-1}). \tag{18}$$

In the last line, we have defined $\tau_1 = \sigma_1^{-1}\sigma_2$, $\tau_2 = \sigma_1^{-1}\sigma_3$, used the invariance of the σ_2, σ_3 sums under this redefinition. We also renamed $\gamma_2 \rightarrow \gamma$. Now recalling Burnside's lemma again, we see that this is counting pairs (τ_1, τ_2) subject to the equivalence

$$(\tau_1, \tau_2) \sim (\gamma\tau_1\gamma^{-1}, \gamma\tau_2\gamma^{-1}). \tag{19}$$

Physically, these manipulations amount to starting from the equivalences

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_1\sigma_1\gamma_2, \gamma_1\sigma_2\gamma_2, \gamma_1\sigma_3\gamma_2),$$

fixing γ_1 gauge symmetry by setting $\sigma_1 = 1$

$$(\sigma_1, \sigma_2, \sigma_3) \rightarrow (1, \tau_1 \equiv \sigma_1^{-1}\sigma_2, \tau_2 \equiv \sigma_1^{-1}\sigma_3).$$

Then we use the remaining γ_2 gauge symmetry, which becomes (19) after renaming $\gamma_2 \rightarrow \gamma$.

Now let us interpret the outcome geometrically. We observe that the expression (18) coincides with a permutation-TFT partition function for a simpler cell-complex. This is $F_2 \times S^1$, where F_2 is the Flower graph, with a single vertex and two edges illustrated in Figure 7.
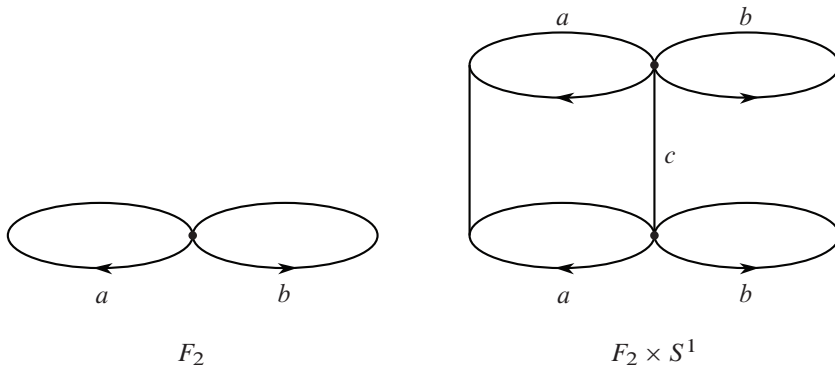

Figure 7. The flower F_2 and its periodic lattice $F_2 \times S^1$.

The flower F_2 has a fundamental group made of two generators without any relations. Consider the periodic flower $F_2 \times S^1$ as given in Figure 7. Opening $F_2 \times S^1$, we get, in the similar way as (15), the following assignments

$$a \longrightarrow \sigma_1, \quad c \longrightarrow \gamma, \quad b \longrightarrow \sigma_2,$$

and to the two different plaquettes present in the theory we assign a weight analogous to (16) as

$$w(g_{P_a}) = \delta(\gamma \sigma_1 \gamma^{-1} \sigma_1^{-1}) \quad \text{and} \quad w(g_{P_b}) = \delta(\gamma \sigma_2 \gamma^{-1} \sigma_2^{-1}).$$

Thus, we identify the partition function of this S_n-TFT over the periodic cellular complex $F_2 \times S^1$ with our previous counting:

$$Z(F_2 \times S^1; S_n) = Z_3(n).$$

Now we have

$$Z_3(n) = Z(F_2 \times S^1; S_n) = Z(G_3 \times S^1; S_n). \tag{20}$$

Since the S_n-TFT $Z(X; S_n)$ simply counts homomorphisms $\pi_1(X) \to S_n$, the last equality is just the topological fact that

$$\pi_1(F_2 \times S^1) = \pi_1(G_3 \times S^1) \quad \text{and} \quad \pi_1(F_2) = \pi_1(G_3).$$

In more physical terms, these relations give an example of the statement that the S_n-TFT is a topological field theory, with partition function invariant under a coarsening of the lattice which leaves the fundamental group invariant. The transformation leading from $G_3 \times S^1$ to $F_2 \times S^1$ shrinks the middle 2-cell in Figure 6 thus identifying the two edges d and d'.

4.3. Conjugation equivalence, embedded bi-partite graphs, matrix models, and branched covers.
Let us return to the formulation of the counting in terms of conjugation equivalence of the pair (τ_1, τ_2) which is expressed, via the Burnside lemma in (18). We can manipulate this expression by introducing another permutation τ_0 constrained by $\tau_0 = (\tau_1 \tau_2)^{-1}$

$$
\begin{aligned}
Z_3(n) &= \frac{1}{n!} \sum_{\gamma \in S_n} \sum_{\tau_1, \tau_2 \in S_n} \delta(\gamma \tau_1 \gamma^{-1} \tau_1^{-1}) \delta(\gamma \tau_2 \gamma^{-1} \tau_2^{-1}) \\
&= \frac{1}{n!} \sum_{\gamma \in S_n} \sum_{\tau_0, \tau_1, \tau_2 \in S_n} \delta(\gamma \tau_1 \gamma^{-1} \tau_1^{-1}) \delta(\gamma \tau_2 \gamma^{-1} \tau_2^{-1}) \delta(\tau_0 \tau_1 \tau_2) \\
&= \frac{1}{n!} \sum_{\gamma \in S_n} \sum_{\tau_0, \tau_1, \tau_2 \in S_n} \delta(\gamma \tau_0 \gamma^{-1} \tau_0^{-1}) \delta(\gamma \tau_1 \gamma^{-1} \tau_1^{-1}) \delta(\gamma \tau_2 \gamma^{-1} \tau_2^{-1}) \delta(\tau_0 \tau_1 \tau_2).
\end{aligned}
\tag{21}
$$

In the last line we introduced an extra delta function, implied by the ones already there, to make the formula more symmetric. We can recognize that this is counting, according to the Burnside lemma, triples of permutations τ_0, τ_1, τ_2 obeying

$$
\tau_0 \tau_1 \tau_2 = 1.
\tag{22}
$$

More precisely, it is counting equivalence classes of these triples under the conjugation equivalence by $\gamma \in S_n$: $\tau_i \sim \gamma \tau_i \gamma^{-1}$. We recognize in (22) the group generated by three generators subject to one relation, which is the fundamental group of the two-sphere, with three punctures (equivalently 2-sphere with 3 discs removed). Our counting function $Z_3(n)$ thus counts the number of equivalence classes of branched covers of the 2-sphere, with 3-branch points, each equivalence class being counted once.[2] In two dimensions, branched covers are also holomorphic maps. These permutation triples thus have a very rich mathematics: maps with three branch points (which are often taken as $0, 1, \infty$) are called Belyi maps and are known to be definable over algebraic number fields; see [58]. Given such a map, the inverse image of the interval $[0, 1]$ gives an embedded bi-partite graph on the covering Riemann surface, where black vertices are inverse images of 1 and white vertices are inverse images of 0. These bi-partite graphs can be viewed as the large N graphs of matrix models; see [13] and [59]. Since branched covers in two dimensions are also holomorphic maps (defined by nice local equations which use the complex structure of the surfaces involved), this has led to investigations of links between these bi-partite graphs and topological string theory; see [13], [14], and [15]. Our present observations relating

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[2]This is to be contrasted with the statement that $Z_3(n)$ counts equivalence classes of covers of $G_3 \times S^1$, not with weight one, but with weight equal to inverse automorphism group of these covers. As observed in [24], counting with weight 1 and with inverse automorphism are related via Burnside's lemma to introduction of an extra circle associated with γ.

the counting of 3-index tensor model invariants to embedded bi-partite graphs suggests that there may be surprising connections between these tensor models and matrix models (and their associated gauge/string duals), with permutation-TFTs playing a key role. We will venture some more remarks in this direction in Section 8.

Eq. (21) was used as a starting point for refined counting of embedded bi-partite graphs in [24]. A very similar solving of delta functions, alongside Burnside's lemma, was used to uncover a surprising link between the counting of vacuum graphs in Quantum Electrodynamics and ribbon graphs; see [23].

4.4. General rank d. Most of the above discussion generalizes straightforwardly to higher rank. The counting of invariants built from n copies of a rank d colored tensor Φ and n copies of the conjugate $\bar{\Phi}$ is given by a function $Z_d(n)$ which coincides with the permutation-TFT partition function on $G_d \times S^1$. G_d is a graph with two vertices and d edges. This partition function can be simplified to that of $F_{d-1} \times S^1$, where F_{d-1} is the flower graph with $d-1$ edges and a single vertex, as

$$Z_d(n) = Z[F_{d-1} \times S^1; \ S_n] = Z[G_d \times S^1; \ S_n], \quad \pi_1(F_{d-1}) = \pi_1(S^2 \setminus d \text{ discs}). \tag{23}$$

By introducing an extra permutation equal to the inverse of the $d-1$ permutations, we recognize the counting of equivalence classes of branched covers of degree n of the sphere S^2 with d branch points (each counted with weight 1). The counting for the case of general d is not known to us to have a simple matrix model realization, of the kind discussed above for $d = 3$.

5. Color-symmetrized counting of tensor invariants

The simplest colored-tensor model, e.g. the Gaussian model, has a symmetry of permutations of the colors. It is natural to investigate the class of interaction terms invariant under this symmetry. Here we will investigate the enumeration of these color-symmetrized equivalence classes, express them in the language of permutations and obtain multi-variable generating functions for their counting.

5.1. Rank $d = 3$ case. We start by the rank $d = 3$ case which will serve as a guiding non trivial situation. The color symmetrization can be achieved after imposing another type of equivalence now acting on the permutation triple as

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\sigma_2, \sigma_1, \sigma_3) \sim (\sigma_1, \sigma_3, \sigma_2) \sim \cdots . \tag{24}$$

As it stands, this problem turns out to nicely addressed using the group algebra $\mathbb{C}(S_n)$ of S_n. Consider the element

$$[\sigma_1 \sigma_2 \sigma_3] \overset{\text{def}}{=} \sum_{\alpha \in S_3} \sigma_{\alpha(1)} \otimes \sigma_{\alpha(2)} \otimes \sigma_{\alpha(3)} \ \in \ \mathbb{C}(S_n)^{\otimes 3}. \tag{25}$$

Now we are investigating equivalence classes given by

$$[\sigma_1\sigma_2\sigma_3] \sim [\gamma_1^{\otimes 3}][\sigma_1\sigma_2\sigma_3][\gamma_2^{\otimes 3}] = \sum_{\alpha \in S_3} \gamma_1\sigma_{\alpha(1)}\gamma_2 \otimes \gamma_1\sigma_{\alpha(2)}\gamma_2 \otimes \gamma_1\sigma_{\alpha(3)}\gamma_2$$

and we intend to find $Z_{3;\,\text{sc}}(n)$ or the cardinal of

$$\text{Diag}(S_n)\backslash\text{Sym}(\mathbb{C}(S_n)^{\otimes 3})/\text{Diag}(S_n),$$

with $\text{Sym}(\mathbb{C}(S_n)^{\otimes 3})$ the group algebra generated by symmetric elements of the form (25).

Using Burnside's lemma on $\mathbb{C}(S_n)^{\otimes 3}$, we have

$$Z_{3;\,\text{sc}}(n)$$

$$= \frac{1}{(3!)^2(n!)^2} \sum_{\gamma_1,\gamma_2 \in S_n} \sum_{\sigma_i \in S_n} \delta([\gamma_1^{\otimes 3}][\sigma_1\sigma_2\sigma_3][\gamma_2^{\otimes 3}][\sigma_1\sigma_2\sigma_3]^{-1})$$

$$\stackrel{\text{def}}{=} \frac{1}{(3!)^2(n!)^2} \sum_{\gamma_1,\gamma_2 \in S_n} \sum_{\sigma_i \in S_n} \sum_{\alpha,\beta \in S_3} \delta(\gamma_1\,\sigma_{\alpha(1)}\gamma_2\sigma_{\beta(1)}^{-1})$$
$$\delta(\gamma_1\sigma_{\alpha(2)}\gamma_2\sigma_{\beta(2)}^{-1})\delta(\gamma_1\sigma_{\alpha(3)}\gamma_2\sigma_{\beta(3)}^{-1}).$$
$$(26)$$

We then use the same recipe introduced before and integrate one γ. Solving one delta function such that $\gamma_1 = \sigma_{\beta(1)}\gamma_2^{-1}\sigma_{\alpha(1)}^{-1}$, we rewrite (26) as

$$Z_{3;\,\text{sc}}(n)$$

$$= \frac{1}{(3!)^2(n!)^2} \sum_{\gamma_2 \in S_n} \sum_{\sigma_i \in S_n} \sum_{\alpha,\beta \in S_3} \delta(\sigma_{\beta(1)}\gamma_2^{-1}\sigma_{\alpha(1)}^{-1}\sigma_{\alpha(2)}\gamma_2\sigma_{\beta(2)}^{-1})$$
$$\delta(\sigma_{\beta(1)}\gamma_2^{-1}\sigma_{\alpha(1)}^{-1}\sigma_{\alpha(3)}\gamma_2\sigma_{\beta(3)}^{-1}).$$

We change dummy variables $i \leftrightarrow \alpha^{-1}(i)$ so that

$$Z_{3;\,\text{sc}}(n)$$

$$= \frac{1}{(3!)^2(n!)^2} \sum_{\gamma \in S_n} \sum_{\sigma_i \in S_n} \sum_{\alpha,\beta \in S_3} \delta(\sigma_{[\alpha^{-1}\beta](1)}\gamma^{-1}\sigma_1^{-1}\sigma_2\gamma\sigma_{[\alpha^{-1}\beta](2)}^{-1})$$
$$\delta(\sigma_{[\alpha^{-1}\beta](1)}\gamma^{-1}\sigma_1^{-1}\sigma_3\gamma\sigma_{[\alpha^{-1}\beta](3)}^{-1}).$$

Perform a last change in variable

$$\alpha^{-1}\beta \longrightarrow \beta$$

and generate

$$Z_{3;\,\text{sc}}(n)$$

$$= \frac{1}{3!(n!)^2} \sum_{\gamma \in S_n} \sum_{\sigma_i \in S_n} \sum_{\beta \in S_3} \delta(\sigma_{\beta(1)}\gamma^{-1}\sigma_1^{-1}\sigma_2\gamma\,\sigma_{\beta(2)}^{-1})$$
$$\delta(\sigma_{\beta(1)}\gamma^{-1}\sigma_1^{-1}\sigma_3\gamma\,\sigma_{\beta(3)}^{-1})$$

$$= \frac{1}{3!(n!)^2} \sum_{\gamma \in S_n} \sum_{\sigma_i \in S_n} \Big\{ \delta(\sigma_1\gamma^{-1}\sigma_1^{-1}\sigma_2\gamma\sigma_2^{-1})\delta(\sigma_1\gamma^{-1}\sigma_1^{-1}\sigma_3\gamma\sigma_3^{-1})$$
$$+ \delta(\sigma_2\gamma^{-1}\sigma_1^{-1}\sigma_2\gamma\sigma_1^{-1})\delta(\sigma_2\gamma^{-1}\sigma_1^{-1}\sigma_3\gamma\sigma_3^{-1})$$
$$+ \delta(\sigma_3\gamma^{-1}\sigma_1^{-1}\sigma_2\gamma\sigma_2^{-1})\delta(\sigma_3\gamma^{-1}\sigma_1^{-1}\sigma_3\gamma\sigma_1^{-1})$$
$$+ \delta(\sigma_1\gamma^{-1}\sigma_1^{-1}\sigma_2\gamma\sigma_3^{-1})\delta(\sigma_1\gamma^{-1}\sigma_1^{-1}\sigma_3\gamma\sigma_2^{-1})$$
$$+ \delta(\sigma_3\gamma^{-1}\sigma_1^{-1}\sigma_2\gamma\sigma_1^{-1})\delta(\sigma_3\gamma^{-1}\sigma_1^{-1}\sigma_3\gamma\sigma_2^{-1})$$
$$+ \delta(\sigma_2\gamma^{-1}\sigma_1^{-1}\sigma_2\gamma\sigma_3^{-1})\delta(\sigma_2\gamma^{-1}\sigma_1^{-1}\sigma_3\gamma\sigma_1^{-1}) \Big\}.$$

These six terms come, respectively, from $\beta = \{\text{id}, (12), (13), (23), (132), (123)\}$. In each of the last three lines, σ_3 appears only once in at least one of the delta functions. So we can integrate these to be left with a single delta function. For the last line, we also do renaming of $\sigma_3 \to \sigma\gamma$ after the elimination of σ_2. The upshot is

$$Z_{3;\,\text{sc}}(n) = \frac{1}{6n!} \sum_{\gamma \in S_n} \sum_{\sigma_2,\sigma_3 \in S_n} \delta(\gamma^{-1}\sigma_2\gamma\sigma_2^{-1})\delta(\gamma^{-1}\sigma_3\gamma\sigma_3^{-1})$$

$$+ \frac{1}{2n!} \sum_{\gamma \in S_n} \sum_{\sigma \in S_n} \delta(\gamma^2\sigma\gamma^{-2}\sigma^{-1})$$

$$+ \frac{1}{3n!} \sum_{\gamma,\sigma \in S_n} \delta(\gamma^3\sigma^3).$$

We know how to calculate the first sum in terms of a sum over partitions. We should be able to derive something similar for the last two terms. As a first step, we write

$$Z_{3;\,\text{sc}}(n)$$

$$= \frac{1}{6n!} \sum_{p \vdash n} \text{Sym}(p) + \frac{1}{2n!} \sum_{\gamma \in S_n} \sum_{\sigma \in S_n} \delta(\gamma^2\sigma\gamma^{-2}\sigma^{-1}) + \frac{1}{3n!} \sum_{\gamma,\sigma \in S_n} \delta(\gamma^3\sigma^3).$$

$$(27)$$

Let us write this as

$$Z_{3;\text{sc}}(n) = \frac{1}{6}S^{(3)}_{[1^3]}(n) + \frac{1}{2}S^{(3)}_{[2,1]}(n) + \frac{1}{3}S^{(3)}_{[3]}(n) \tag{28}$$

where the superscript indicates that this is the $d = 3$ case, while the subscript is a partition of 3 corresponding to the conjugacy class of α which gives rise to the relevant term. We record here our most effective formulae for each term:

$$S_{[1^3]}^{(3)}(n) = \sum_{p \vdash n} \mathrm{Sym}(p) = \sum_{p \vdash n} \prod_{i=1}^{n} (i^{\mu_i})(\mu_i!), \qquad (29a)$$

$$S_{[2,1]}^{(3)}(n) = \sum_{p \vdash n} \mathrm{Coefficient}\, [Z^{(2)}(t, \vec{x}), t^n x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}] \times \Big[\prod_{i=1}^{n} i^{p_i} p_i! \Big], \quad (29b)$$

and

$$S_{[3]}^{(3)}(n), = \sum_{p \vdash n} \Big(\mathrm{Coefficient}\, [Z^{(3)}(t, \vec{x}), t^n \prod_{i=1}^{n} x_i^{p_i}] \Big)^2 \times \Big[\prod_{i=1}^{n} i^{p_i} p_i! \Big]. \qquad (29c)$$

The derivation of $S_{[1^3]}^{(3)}(n)$ was explained earlier; see (6). The formulae for $S_{[2,1]}^{(3)}(n)$ and $S_{[3]}^{(3)}(n)$ in terms of multi-variable generating functions $Z^{(2)}(t, \vec{x})$ and $Z^{(3)}(t, \vec{x})$, respectively, are explained and derived as (B.1) and (B.3) in Appendix B. These formulae can be evaluated to high orders using Mathematica (see Mathematica code 2 in Appendix D). The result for $S_{[2,1]}^{(3)}(n)$ is,[3] from $n = 1$ to $n = 10$,

$$1, 2, 5, 13, 31, 89, 259, 842, 2810, 10020, \ldots.$$

The sequence $S_{[3]}^{(3)}(n)$ evaluates in the same way as (see Appendix B and Mathematica code 2 in Appendix D), from $n = 1$ to $n = 10$,

$$1, 1, 2, 4, 5, 13, 29, 48, 114, 301.$$

Adding all these up with the right coefficients, we get, up to order $n = 10$,

$$1, 2, 5, 15, 44, 199, 1069, 7638, 64503, 628900. \qquad (30)$$

Note that the summands can be fractional, but the sum (28) is integral. This is the disconnected case. Having a closer look at Figure 4, we can associate the graphs to the first orders, $Z_{3;\mathrm{sc}}(1) = 1$ is simply the class given by A; for $Z_{3;\mathrm{sc}}(2) = 2$, there are two classes of graphs: the first is given by a disconnected graph formed by twice (A, A), and the second class is formed by the three remaining B1, B2, and B3 which are indeed form a closed set under the S_3 operations of permuting the three colors.

---

[3]This is recognized as the sequence A082733 by OEIS, and described there as the sum of all entries in the character table of S_n.

Now $Z_{3;\text{sc}}(3) = 5$ is generated by $\{(A,A,A)\}$ (disconnected), **AB**=$\{(A, Bi),$ i=1,2,3$\}$ (disconnected), **C**=$\{C1, C2, C3\}$, **D**=$\{D1, D2, D3\}$ and the last class given by **E**=$\{E\}$.

It turns out that the Plog does not give the correct relation between connected and disconnected for this color-symmetrized counting. For instance, at order $n = 4$ (graph with eight legs), the Plog gives 9. This means that it has subtracted six classes (from the initial fifteen classes) regarded as disconnected. Now, from the case $n = 3$, we can observe directly that these classes can be organized as follows: 3 disconnected graphs are formed by (A, **C**), (A, **D**) and (A, **E**); another case is given by twice a copy of A plus a connected piece with four legs, which gives (A, **AB**); then we must also include the graph made with four copies of A, which is (A, A, A, A). That yields five cases already out of six. So the remaining disconnected graph would be the one formed by twice a graph made with four legs (a double copy of Bi, i=1,2,3, see Figure 4). However, in the latter category of disconnected objects, the class obtained by the disjoint union of graphs Bi denoted by $\{(Bi, Bi)\}$ and the one $\{(Bi, Bj), i \neq j\}$ (see Figure 8) are not equivalent under (24). Thus, the ordinary Plog of the disconnected series does not give the correct answer. It would be interesting to work out an analog of the Plog formula for this case of color-symmetrized counting of invariants.
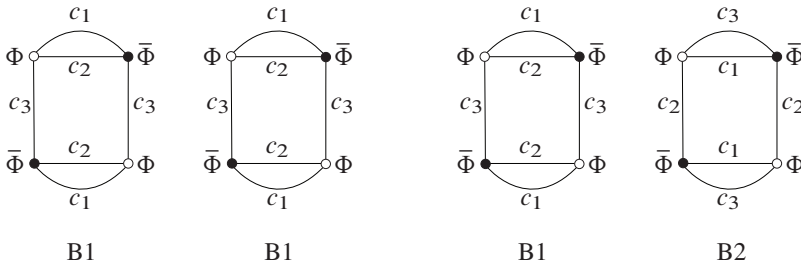
Figure 8. Non equivalent disconnected graphs.

A GAP program can however generate the sequence of connected graphs (see GAP code, in Appendix D). One finds

$$1, 1, 3, 8, 24, 72. \tag{31}$$

The case $n = 4$ giving $Z_{3;\text{sc}}^{\text{connected}}(n = 4) = 8$ has been illustrated in Figure 9.

5.2. Rank $d = 4$ case. The color symmetrization in general can be implemented by the equivalence of the d–tuples

$$(\sigma_1, \ldots, \sigma_d) \sim (\sigma_{\alpha(1)}, \ldots, \sigma_{\alpha(d)}), \quad \alpha \in S_d.$$
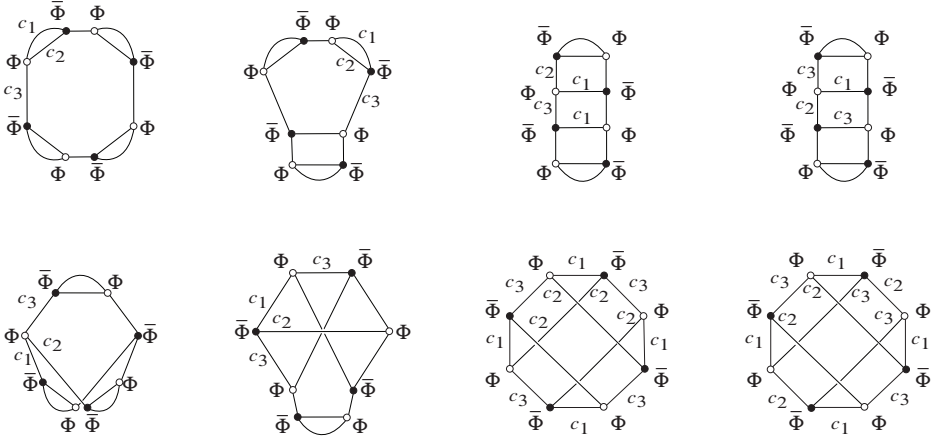
Figure 9. Rank 3 colored symmetric connected invariants at order $n = 4$.

Using now the group algebra $\mathbb{C}(S_n)$ of S_n, we consider the element

$$[\sigma_1 \ldots \sigma_d] := \sum_{\alpha \in S_d} \sigma_{\alpha(1)} \otimes \sigma_{\alpha(2)} \otimes \cdots \otimes \sigma_{\alpha(d)} \ \in \ \mathbb{C}(S_n)^{\otimes 3} \tag{32}$$

which leads us to the search of equivalent classes such that

$$[\sigma_1 \ldots \sigma_d] \sim [\gamma_1^{\otimes d}][\sigma_1 \ldots \sigma_d][\gamma_2^{\otimes d}] = \sum_{\alpha \in S_3} \gamma_1 \sigma_{\alpha(1)} \gamma_2 \otimes \cdots \otimes \gamma_1 \sigma_{\alpha(d)} \gamma_2.$$

This is counting the points of

$$\text{Diag}(S_n) \backslash \text{Sym}(\mathbb{C}(S_n)^{\otimes d}) / \text{Diag}(S_n),$$

with $\text{Sym}(\mathbb{C}(S_n)^{\otimes d})$ the group algebra generated by symmetric elements of the form (32).

Burnside's lemma on $\mathbb{C}(S_n)^{\otimes d}$ allows us to write

$$Z_{d;\text{sc}}(n)$$

$$= \frac{1}{(d!)^2 (n!)^2} \sum_{\gamma_1, \gamma_2 \in S_n} \sum_{\sigma_i \in S_n} \sum_{\alpha, \beta \in S_d} \delta(\gamma_1 \sigma_{\alpha(1)} \gamma_2 \sigma_{\beta(1)}^{-1}) \ldots \delta(\gamma_1 \sigma_{\alpha(d)} \gamma_2 \sigma_{\beta(d)}^{-1}).$$

$$\tag{33}$$

Integrating γ_1, $\gamma_1 = \sigma_{\beta(1)} \gamma_2^{-1} \sigma_{\alpha(1)}^{-1}$, eq. (33) re-expresses as

$$Z_{d;\text{sc}}(n) = \frac{1}{(s!)^2 (n!)^2} \sum_{\gamma_2 \in S_n} \sum_{\sigma_i \in S_n} \sum_{\alpha, \beta \in S_d} \delta(\sigma_{\beta(1)} \gamma_2^{-1} \sigma_{\alpha(1)}^{-1} \sigma_{\alpha(2)} \gamma_2 \sigma_{\beta(2)}^{-1}) \ldots$$

$$\ldots \delta(\sigma_{\beta(1)} \gamma_2^{-1} \sigma_{\alpha(1)}^{-1} \sigma_{\alpha(d)} \gamma_2 \sigma_{\beta(d)}^{-1}).$$

Changing variables as

$$i \longleftrightarrow \alpha^{-1}(i)$$

and performing

$$\alpha^{-1}\beta \longrightarrow \beta$$

generate

$$Z_{d;\,\mathrm{sc}}(n)$$

$$= \frac{1}{(d!)^2 (n!)^2} \sum_{\gamma \in S_n} \sum_{\sigma_i \in S_n} \sum_{\alpha,\beta \in S_d} \delta(\sigma_{[\alpha^{-1}\beta](1)} \gamma^{-1} \sigma_1^{-1} \sigma_2 \gamma \sigma_{[\alpha^{-1}\beta](2)}^{-1}) \cdots$$
$$\cdots \delta(\sigma_{[\alpha^{-1}\beta](1)} \gamma^{-1} \sigma_1^{-1} \sigma_d \gamma \sigma_{[\alpha^{-1}\beta](d)}^{-1})$$

$$= \frac{1}{d!(n!)^2} \sum_{\gamma \in S_n} \sum_{\sigma_i \in S_n} \sum_{\beta \in S_d} \delta(\sigma_{\beta(1)} \gamma^{-1} \sigma_1^{-1} \sigma_2 \gamma \sigma_{\beta(2)}^{-1}) \cdots$$
$$\delta(\sigma_{\beta(1)} \gamma^{-1} \sigma_1^{-1} \sigma_d \gamma \sigma_{\beta(d)}^{-1}).$$

We now specialize to the case $d = 4$. Expanding the last sum $\sum_{\beta \in S_d}$, one gets after some algebra:

$$Z_{4;\,\mathrm{sc}}(n) = \frac{1}{24n!} \sum_{\gamma \in S_n} \sum_{\sigma_i \in S_n} \delta(\gamma^{-1} \sigma_2 \gamma \sigma_2^{-1}) \delta(\gamma^{-1} \sigma_3 \gamma \sigma_3^{-1}) \delta(\gamma^{-1} \sigma_4 \gamma \sigma_4^{-1})$$

$$+ \frac{1}{4n!} \sum_{\gamma \in S_n} \sum_{\sigma_i \in S_n} \delta(\gamma \sigma_1 \gamma^{-1} \sigma_1^{-1}) \delta(\gamma^2 \sigma_2 \gamma^{-2} \sigma_2^{-1})$$

$$+ \frac{1}{3n!} \sum_{\gamma \in S_n} \sum_{\sigma \in S_n} \delta(\gamma^3 \sigma \gamma^{-3} \sigma^{-1}) \tag{34}$$

$$+ \frac{1}{8n!} \sum_{\gamma \in S_n} \sum_{\sigma_i \in S_n} \delta(\sigma_1^2 \gamma^2) \delta(\gamma^2 \sigma_2 \gamma^{-2} \sigma_2^{-1})$$

$$+ \frac{1}{4n!} \sum_{\gamma \in S_n} \sum_{\sigma \in S_n} \delta(\sigma^4 \gamma^4).$$

These five terms come, respectively, from the conjugacy classes represented by

$$\{\mathrm{id},\ (12),\ (123),\ (12)(34),\ (1234)\}.$$

As above, the first sum computes to a sum over partitions already known from (12).

Let us denote

$$
\begin{aligned}
&Z_{4;\text{sc}}(n) \\
&= \frac{1}{24} S^{(4)}_{[1^4]}(n) + \frac{1}{4} S^{(4)}_{[2,1^2]}(n) + \frac{1}{3} S^{(4)}_{[3,1]}(n) + \frac{1}{8} S^{(4)}_{[2^2]}(n) + \frac{1}{4} S^{(4)}_{[4]}(n),
\end{aligned} \tag{35}
$$

where, as in the rank 3 case, we can label each sum by a subscript giving by a particular partition of $d = 4$. In Appendix B, we manipulate these delta functions to arrive at expressions as sums over symmetry factors of partitions or in terms of multi-variable generating functions. We summarize the key formulae (see Appendix B, (B.4), (B.5), (B.6), and (B.7) for more details):

$$
S^{(4)}_{[1^4]}(n) = \sum_{p \vdash n} (\text{Sym } p)^2, \tag{36a}
$$

$$
\begin{aligned}
&S^{(4)}_{[2,1^2]}(n) \\
&= \sum_{p \vdash n} \left[\prod_{j=1}^{\lfloor \frac{n}{2} \rfloor} (2j)^{2p_{4j}} (2p_{4j})! \right] \left[\prod_{j=0}^{\lfloor \frac{n}{2} \rfloor} (2j+1)^{p_{2j+1}+2p_{4j+2}} (p_{2j+1} + 2p_{4j+2})! \right],
\end{aligned} \tag{36b}
$$

$$
S^{(4)}_{[3,1]}(n) = \sum_{p \vdash n} \text{Coefficient } [Z^{(3)}(t,\vec{x}), t^n x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}] \times \left[\prod_{i=1}^{n} i^{p_i} p_i! \right], \tag{36c}
$$

$$
S^{(4)}_{[2^2]}(n) = \sum_{p \vdash n} (\text{Coefficient}[Z^{(2)}(t,\vec{x}), t^n x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}] \text{Sym}(p))^2, \tag{36d}
$$

and

$$
S^{(4)}_{[4]}(n) = S^{(4)}_{[4]}(n) = \sum_{p \vdash n} \left(\text{Coefficient}[Z^{(4)}(t,\vec{x}), \prod_{i=1}^{n} x_i^{p_i}] \right)^2 \times \left[\prod_{i=1}^{n} i^{p_i} p_i! \right]. \tag{36e}
$$

With these formulae in hand, we can generate the sequences to high order with Mathematica. Direct evaluation of the delta functions with GAP at low orders agrees with these generating functions but at high orders calculation with the help of (36) is the only practical option. The sequences $S^{(4)}_{[\cdot]}(n)$ can be computed with Mathematica to give (see Appendix B and Appendix D, Mathematica code 2, 3, and 4 for further

details) up to order $n = 10$,

$$S_{[2,1^2]}^{(4)}: 1, 4, 15, 83, 385, 2989, 20559, 203922, 1827640, 21863590, \ldots,$$

$$S_{[3,1]}^{(4)}: 1, 2, 4, 12, 27, 103, 391, 1383, 6260, 32704, \ldots,$$

$$S_{[2^2]}^{(4)}: 1, 4, 17, 105, 685, 5825, 54013, 585018, 6873522, 90254150, \ldots,$$

$$S_{[4]}^{(4)}: 1, 2, 3, 11, 27, 93, 233, 978, 3156, 13280, \ldots.$$

Combining these sums yields

1, 3, 10, 69, 811, 23372, 1073376, 67963017, 5492554668, 548974274040, ...

This sequence corresponds to the disconnected case. The connected case sequence can be obtained with a GAP program extending the rank $d = 3$ case as given in Appendix D.

6. Counting tensor invariants without color

We address here countings of invariants for tensors without color, which are the tensor models of more traditional interest. In the first case, the tensor field $\Phi_{i_1 \ldots i_d}$ will have d indices and we will allow contraction of any i_a with any of the d indices of $\bar{\Phi}_{j_1 \ldots j_d}$. In the second case, there will again be no restriction on which j a given i can contract with, but the tensor field will be symmetric under S_d permutations of its indices. We will have a family of counting problems for each integer n corresponding to the number of Φ and $\bar{\Phi}$ fields.

6.1. Invariants without color: general tensors.
The counting above is equivalent to the counting of invariants in $\mathrm{Sym}_n(V^{\otimes d})^{\otimes n} \otimes \mathrm{Sym}_n(\bar{V}^{\otimes d})^{\otimes n}$ under a diagonal $U(N)$ action. Here, Sym_n indicates the symmetrization of the n copies, which arises from the fact that the n copies of Φ and the n copies of $\bar{\Phi}$ can be permuted without changing the invariant. The unitary group acts as

$$U^{\otimes nd} \otimes \bar{U}^{\otimes nd}$$

on $(V^{\otimes d})^{\otimes n} \otimes (\bar{V}^{\otimes d})^{\otimes n}$ which descends to an action on the symmetrized subspaces. The contractions are given by permutations $\sigma \in S_{dn}$ (mixing all dn indices) and the equivalences that we seek are encoded in

$$\sigma \sim \gamma_1 \sigma \gamma_2$$

where $\gamma_1, \gamma_2 \in S_n = \mathrm{Diag}(S_n^{\times d}) \subset S_{dn}$ (this is the embedding of S_n in S_{dn}). Equivalently we are counting points in the double coset

$$S_n \backslash S_{dn} / S_n.$$

We can again use the formula

$$\mathcal{Z}_{d;\,\text{noncolor}}(n) = \frac{1}{(n!)^2} \sum_C (Z_C^{S_n \to S_{dn}})^2 \text{Sym}(C),$$

where the sum is over conjugacy classes of S_{dn}. For a given conjugacy class C, $Z_C^{S_n \to S_{dn}}$ counts the number of elements $(\sigma, \ldots, \sigma)$ in $\text{Diag}(S_n^{\times d})$ that is in C. For $(\sigma, \ldots, \sigma)$ to be in C, C must have a cycle structure of $d \times$ the cycle structure of σ. The latter is entirely determined by partition of n so that $\text{Sym}(C) = \prod_i i^{dp_i}(dp_i)!$ (see Proposition 3, in Appendix A). We finally get

$$\mathcal{Z}_{d;\,\text{noncolor}}(n) = \sum_{p \vdash n} \frac{1}{(\text{Sym}(p))^2} \prod_i i^{dp_i}(dp_i)! = \sum_{p \vdash n} \prod_i \frac{i^{(d-2)p_i}(dp_i)!}{(p_i!)^2}.$$

A Mathematica program allows to compute this (see Appendix D, Mathematica code 5). Doing this for $d = 2$ (matrix models) we get the sequence, up to order $n = 10$,

$$2, 8, 26, 94, 326, 1196, 4358, 16248, 60854, 230184, \ldots . \tag{37}$$

This sequence is recognized by the OEIS as A067855 or the squared length of sum of s_p^2, where s_p is a Schur function and p ranges over all partitions of n.

For $d = 3$, we get, up to order $n = 10$,

$$6, 192, 10170, 834612, 90939630,$$
$$12360636540, 2012440468938, 381799921738584, \ldots . \tag{38}$$

This corresponds to the disconnected case. The Plog function can generate the connected situation along the lines (8), (9), and (10) (Mathematica code 1 see Appendix D).

6.2. Invariants without color: symmetric tensors. Consider a complex symmetric tensor Φ of rank d, such that

$$\Phi_{i_1 i_2 i_3 \ldots i_d} = \Phi_{i_{\rho(1)} i_{\rho(2)} \ldots i_{\rho(d)}}, \quad \rho \in S_d.$$

We want to know the number $\mathcal{Z}_{d;\,\text{sym}}(n)$ of bi-partite graphs that one can build by contracting n copies of Φ (seen as vertices of valence d) with n copies of $\bar{\Phi}$. In terms of a traditional invariant theory question, we are counting invariants of $U(N)$ acting on $\text{Sym}_n((\text{Sym}_d(V^{\otimes d}))^{\otimes n}) \otimes \text{Sym}_n((\text{Sym}_d(\bar{V}^{\otimes d}))^{\otimes n})$. The S_d symmetrization implicit in Sym_d comes from having symmetric tensors. The S_n symmetrization comes from having n copies of the same Φ and n of the same $\bar{\Phi}$.

The possible contractions between these fields can be drawn as the possible parings between two families of n vertices with d half-lines in the way given in Figure 10.
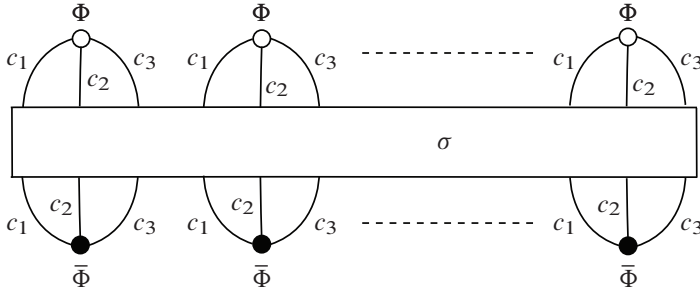
Figure 10. Symmetric tensor contraction.

In other words, the determination of possible graph amounts to count the number of permutations

$$\sigma \in S_{dn},$$

permutations subject to the equivalence

$$\sigma \sim \gamma_1 \cdot \sigma \cdot \gamma_2,$$

where

$$\gamma_i \in S_n \ltimes (S_d)^n \stackrel{\text{def}}{=} S_n[S_d]$$

(called the *wreath product*) act as follows. The $S_d^{\otimes n}$ permutes independently the d-tuples of indices for each of the n tensors (say Φ); the S_n acts by permuting the n tensors, equivalently it permutes the n d-tuples among each other, while not changing their internal structure. The permutation σ acts point-wise on the full set of these $\{1, \ldots, nd\}$ indices. If we write the nd indices on the Φ's as i_α^a where a runs from 1 to n and α runs from 1 to d, with all indices with fixed a attached to the same Φ, the action of $(\gamma; \gamma_1, \ldots, \gamma_n) \in S_n[S_d]$ with $\gamma \in S_n, \gamma_a \in S_d$ for $a \in \{1, \ldots, d\}$ acts as

$$i_\alpha^a \longrightarrow i_{\gamma_a(\alpha)}^{\gamma(a)}.$$

Hence, the counting we are interested in is given by the number of classes in the double coset space

$$S_n[S_d] \backslash S_{dn} / S_n[S_d].$$

Applying (4), the counting can be recast as

$$\mathcal{Z}_{d;\,\text{sym}}(n) = \frac{1}{(n!)^2 (d!)^{2n}} \sum_C (Z_C^{S_n[S_d] \to S_{dn}})^2 \text{Sym}(C).$$

In order to achieve this, we use similar generating function techniques as developed in [23]. We have the generating function of wreath products as

$$\mathcal{Z}_d^{S_\infty[S_d]}(t, \vec{x}) = \sum_n t^n Z^{S_n[S_d]}(\vec{x}) = e^{\sum_{i=1}^\infty \frac{t^i}{i} \left[\sum_{q \vdash d} \prod_{\ell=1}^d \left(\frac{x_{i\ell}}{\ell} \right)^{v_\ell} \frac{1}{v_\ell!} \right]},$$

where $\vec{x} = (x_1, x_2, \dots)$ and the partition $q = (\nu_\ell)_\ell$ of d generates $\sum_\ell \ell \nu_\ell = d$. Finally,

$$\mathcal{Z}_{d;\,\mathrm{sym}}(n) = \sum_{p \vdash dn} (\mathrm{Coefficient}\,[\mathcal{Z}_d^{S_\infty[S_d]}(t, \vec{x}), t^n x_1^{p_1} x_2^{p_2} \dots x_{dn}^{p_{dn}}])^2 \mathrm{Sym}(p).$$

A Mathematica program (see Mathematica code 6, in Appendix D) allows to obtain the sequences. For $d = 2$ (matrix model fully symmetric invariants),

$$1, 2, 3, 5, 7, 11, 15, 22, 30, 42, \tag{39}$$

which gives, according to the OEIS, simply the number of partition up to order of starting at $n = 1$ up to order 10. At order $n \geq 14$, the evaluation becomes challenging. This sequence might be coincide several known by OEIS (for instance A000041, A046054, etc.).

For $d = 3$, one gets, for $n = 1, \dots, 8$,

$$1, 2, 5, 12, 31, 103, 383, 1731. \tag{40}$$

Last for $d = 4$, we obtain for $n = 1, \dots, 8$,

$$1, 3, 9, 43, 264, 2804, 44524, 1012456. \tag{41}$$

Both of (40) and (41) are new sequences according to the OEIS website.

7. Correlators of tensor observables

We have already motivated the enumeration of the tensor model invariants in terms of classifying the possible interaction terms that can be added to the Gaussian term. Other perspectives suggest that there should be additional algebraic structures on these tensor invariants. In the context of matrix models, the string duals lead one to consider a state space with basis corresponding to the traces of matrices; see [60]. On this state space, there can be an interesting non-degenerate pairing or inner product. The pairings are related to correlators involving insertions of two of these general observables in the path integral. A ring structure is also a fruitful object of study containing information about the dual geometry; see [61]. With this in mind, we can define a vector space with basis labeled by the tensor model invariants and study correlators involving insertions of two or more of the general invariants. We write some formulae for correlators with two insertions of the observables we have classified in a Gaussian integral for colored tensors. We obtain some formulae in terms of permutation groups, with structure similar to the delta function sums that appeared in the previous counting. We will restrict attention to $d = 3$.

Consider the Gaussian model

$$\mathcal{Z} = \int d\Phi d\bar{\Phi} \, e^{-\frac{1}{2}\Phi^{i_1 i_2 i_3} \bar{\Phi}^{i_1 i_2 i_3}}.$$

The index i_a runs over $\{1 \dots N_a\}$, for $a \in \{1, 2, 3\}$. The 2-point function is

$$\langle \Phi^{i_1 i_2 i_3} \bar{\Phi}^{j_1 j_2 j_3} \rangle = \delta^{i_1 j_1} \delta^{i_2 j_2} \delta^{i_3 j_3}.$$

The observables, invariant under $U(N) \times U(N) \times U(N)$, are labeled by permutations $(\sigma_1, \sigma_2, \sigma_3)$ subject to equivalence $(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_1 \sigma_1 \gamma_2, \gamma_1 \sigma_2 \gamma_2, \gamma_1 \sigma_3 \gamma_2)$. We will write these observables as $\mathcal{O}_{\sigma_1, \sigma_2, \sigma_3}$ with the understanding that

$$\mathcal{O}_{\sigma_1, \sigma_2, \sigma_3} = \mathcal{O}_{\gamma_1 \sigma_1 \gamma_2, \gamma_1 \sigma_2 \gamma_2, \gamma_1 \sigma_3 \gamma_2}.$$

The two-point function obtained by inserting in the above tensor model integral the product of two such operators

$$\langle \mathcal{O}_{\sigma_1, \sigma_2, \sigma_3} \bar{\mathcal{O}}_{\tau_1, \tau_2, \tau_3} \rangle = \frac{1}{\mathcal{Z}} \int d\Phi d\bar{\Phi} \, e^{-\frac{1}{2} \Phi^{ijk} \bar{\Phi}^{ijk}} \mathcal{O}_{\sigma_1, \sigma_2, \sigma_3} \mathcal{O}_{\tau_1^{-1}, \tau_2^{-1}, \tau_3^{-1}}.$$

We consider the operators to be normal-ordered, i.e when we sum over Wick contractions, we do not include contractions between the Φ's within an operator $\mathcal{O}_{\sigma_1, \sigma_2, \sigma_3}$ and $\bar{\Phi}$'s within the same operator. The two-point function will be a function of σ_i, τ_i which is invariant when either the σ_i or the τ_i are multiplied by γ_1 on the left and γ_2 on the right. The answer is (see Appendix C for the derivation)

$$\langle \mathcal{O}_{\sigma_1, \sigma_2, \sigma_3} \bar{\mathcal{O}}_{\tau_1, \tau_2, \tau_3} \rangle$$
$$= \sum_{\mu_1, \mu_2 \in S_n} N_1^n N_2^n N_3^n \delta(\mu_1 \sigma_1 \mu_2 \tau_1^{-1} \Omega_1) \delta(\mu_1 \sigma_2 \mu_2 \tau_2^{-1} \Omega_2) \delta(\mu_1 \sigma_3 \mu_2 \tau_3^{-1} \Omega_3).$$

$$(42)$$

Here $\Omega_a = \sum_{\sigma \in S_n} N_a^{C_{\sigma a} - n} \sigma$ and is in the group algebra of S_n. A transformation

$$\sigma_i \longrightarrow \gamma_1 \sigma_i \gamma_2, \quad \tau_i \longrightarrow \gamma_1' \tau_i \gamma_2'$$

can be absorbed by changing variables in the sums

$$\mu_1 \longrightarrow \gamma_2' \mu_1 \gamma_1, \quad \mu_2 \longrightarrow \gamma_1' \mu_2 \gamma_2.$$

This shows that the correlator gives a pairing of the equivalence classes of permutation triples which we counted in Section 3. Note that Ω_a commute with all permutations in S_n. In the large N_a limit, $\Omega_a \to 1$. Then the 2-point correlator becomes an inner product which is diagonal on the equivalence classes, with positive diagonal values. This is an analog of the familiar *large N factorization* of matrix model, where different trace structures do not mix in the leading large N limit. Here the two equivalence classes of invariants inserted (which are the analogs of trace structure for one-matrix invariants) have to be identical for a non-vanishing 2-point correlator. At subleading orders $\frac{1}{N_a}$, different equivalence classes mixing under the inner product, with the mixing being controlled by the group multiplication in S_n.

Eq. (42) can be further simplified by defining

$$\alpha_2 = \sigma_1^{-1}\sigma_2, \quad \alpha_3 = \sigma_1^{-1}\sigma_3$$

and

$$\beta_2 = \tau_1^{-1}\tau_2, \quad \beta_3 = \tau_1^{-1}\tau_3.$$

We have

$$\langle \mathcal{O}_{\sigma_1,\sigma_2,\sigma_3}\overline{\mathcal{O}}_{\tau_1,\tau_2,\tau_3}\rangle = n! \sum_{\mu \in S_n} \delta(\beta_2^{-1}\mu^{-1}\alpha_2\mu\Omega_1\Omega_2)\delta(\beta_3^{-1}\mu^{-1}\alpha_3\mu\Omega_1\Omega_3).$$

This simplification is analogous to the one that happened in the counting delta functions (18) and (20). The close parallels between counting and correlators exhibited by the permutation-TFT approach is a recurrent theme that has been encountered for example in [62], [11], and [12] in the context of AdS/CFT. The algebraic structures present in correlators, such as non-degenerate pairing (inner product) and product structure (related to insertion of three observables), are also of interest in the context of CFTs. We will comment on a 4D CFT context for studying the combinatorics and correlators of 3-index fields in Section 8.

8. Summary and discussion

In this section, we will summarize the main results of this paper and outline extensions thereof. We then discuss some conceptual questions raised by the results of this paper, and describe associated technical investigations that can be carried out.

8.1. Summary of main results.

- There is a counting of invariants made from n copies of a colored $d-$tensor, along with n copies of the conjugate tensor given in terms of a sum over partition of n; see eq. (11). This counting includes disconnected invariants (analogous to multi-traces in matrix models). With this disconnected counting as input, the plethystic log function is used to generate the connected invariants. Using these formulae, we generated the counting sequences to high order: (7) and (12) give the disconnected counting for the rank 3 and rank 4 case, respectively, whereas (10) and (13) give the connected counting for the rank 3 and 4, respectively.

- We have shown that the counting of invariants of the $d-$tensors, with n copies of Φ and n of $\bar{\Phi}$, is equivalent to the counting of degree n branched covers of the sphere with d branch points (summed over the possible genera of the covering space). Other geometrical interpretations in terms of covering spaces are also discussed in Section 4. Permutation-TFTs, in conjunction with the Burnside lemma from combinatorics and the links between fundamental groups, permutations and covering spaces given by algebraic topology, form a unifying framework for exhibiting the different geometrical interpretations.

- For the case $d = 3$, the counting of tensor invariants is equivalent to the counting of embedded bi-partite graphs with n edges and is also related to the computation of correlators of complex matrix models.

- We studied a color-symmetrized counting, obtaining explicit formulae in terms of multi-variable generating functions. Key results are (27), (29), (34), and (36).

- The permutation techniques were used to give counting formulae for the tensor invariants in the cases of the more traditional non-colored tensor models.

- As a start towards investigating algebraic structures on the space of tensor observables provided by the Gaussian tensor model, we gave permutation group formulae for the 2-point correlator of the general invariants. We noted that the normal-ordered 2-point correlator gives an inner product, which is diagonalized by the equivalence classes of tensor invariants (or, expressed another way, by the equivalence classes of branched covers of the 2-sphere) in the large N limit. This diagonality is a tensor model analog of large N factorization of matrix models.

8.2. Discussion. In this section, we discuss some conceptual questions raised by our results and list some related problems for investigation.

8.2.1. Braid orbits. Given a permutation triple, (τ_1, τ_2, τ_3) obeying

$$\tau_1 \tau_2 \tau_3 = 1$$

color-symmetrization proceeds by group actions generated by (C_1, C_2),

$$C_1(\tau_1, \tau_2, \tau_3) = (\tau_2, \tau_1, \tau_1^{-1}\tau_2^{-1})$$

and

$$C_2(\tau_1, \tau_2, \tau_3) = (\tau_1^{-1}, \tau_1^{-1}\tau_2, \tau_2^{-1}\tau_1^2).$$

One checks that

$$C_1^2 = 1, \quad C_2^2 = 1, \quad C_1 C_2 C_1 = C_2 C_1 C_2.$$

This means that the group generated by $\{C_1, C_2\}$ contains

$$\{1, C_1, C_2, C_1 C_2, C_2 C_1, C_1 C_2 C_1\}$$

and is S_3, the symmetric group of permutations of 3 elements.

Recall that this came from gauge-fixing $(\sigma_1, \sigma_2, \sigma_3)$, using the gauge equivalence in (3)

$$(\sigma_1, \sigma_2, \sigma_3) \longrightarrow (1, \sigma_1^{-1}\sigma_2, \sigma_1^{-1}\sigma_3) \equiv (1, \tau_1, \tau_2).$$

There is another S_3 action on triples τ_1, τ_2, τ_3 which multiply to 1, which is generated by two braiding generators B_1, B_2 which act as

$$B_1(\tau_1, \tau_2, \tau_3) = (\tau_2, \tau_2^{-1}\tau_1\tau_2, \tau_3) \quad \text{and} \quad B_2(\tau_1, \tau_2, \tau_3) = (\tau_1, \tau_3, \tau_3^{-1}\tau_2\tau_3).$$

Again we have $B_1^2 = B_2^2 = 1$ and $B_1 B_2 B_1 = B_2 B_1 B_2$, so that the group generated is S_3.

From the above description, there appear to be two similar but distinct S_3 actions - one coming from color-symmetrization and one from braiding. Yet when we compute the number of braid orbits using Burnside's lemma, applying delta functions and simplifying, we get the same answer as with color-symmetrized equivalence classes. Also computation with GAP gives the same counting. This means that formulae (27) and (29) give the counting of braid orbits. Braid orbits are of interest from the point of view of the topological classification of polynomials; see [63].

It is natural to ask if the connection between color-symmetrized equivalence classes and braid orbits goes beyond the counting and holds for the actual orbits themselves. This would hold if a more direct connection between the two actions of S_3 on (τ_1, τ_2, τ_3) could be found, e.g. by some appropriate change of variables. Even at the level of counting, there is the question of whether the equality holds for d higher than 3. The cases $d = 4, 5$ should be a somewhat tedious but very doable problem.

8.2.2. Higher dimensional topology and low-dimensional covers.
The primary motivations for the study of tensor models by physicists has been its connections to higher dimensional topology and geometry. With an improved understanding of counting problems associated with tensor models and with the aid of modern computational tools for group theoretic computations, one may ask if tensor models can provide a new perspective on counting problems in topology studied in the mathematical literature, e.g. [64] and [65]. For example, can we use tensor models to count triangulations of 3-sphere with specified numbers of vertices? Such a question is strongly related to Gromov's conjecture on sphere triangulations; see [66]. Another goal would be to try and extract information about continuum geometry from discrete computations, through mathematical connections such as that provided by the Riemann existence theorem – we would need some form of higher dimensional generalizations of it.

What is intriguing in the connection between tensor models and branched covers of the two-sphere we have developed here, is that it suggests that two dimensional holomorphic maps know about higher dimensional combinatoric topology. The study of dimer models - and the associated bi-partite graphs and Belyi maps – in connection with toric Calabi-Yau geometries is another example of physical links between low-dimensional holomorphic maps and higher dimensional geometry; see [67], [68], [69], [70], [71], and [72].

8.2.3. Fourier transforms and finite N effects.
In all the counting problems we have treated in this paper, we have treated N – the range of values taken by the tensor index – to be large sufficiently large compared to n – the number of tensors in the observable. There are qualitative changes in the counting when N is finite. For the case of matrices, this is a consequence of Caley–Hamilton theorem which allows us

to write $\text{tr}(X^{N+1})$ for an $N \times N$ matrix in terms of products of lower traces. This has important implications in string theory in the form the stringy exclusion principle; see [73]. These finite N effects have been studied in a variety of multi-matrix systems; see [8], [10], and [6]. The key lesson is that they are neatly characterized by using permutations to describe invariants (as we have done here) and then performing the Fourier transform on permutation groups to go from "permutations subject to constraints" to appropriate representation theoretic data given by representations of permutation groups. The finite N cutoffs are simple in terms of Young diagrams. The reason why representation theory of the permutation groups knows about the finite N of $U(N)$ is Schur–Weyl duality. For an overview of how Schur–Weyl duality enters gauge-string duality see [19], [74], [75], and [12].

8.2.4. A gauge theory perspective on counting and correlators of tensor invariants. Consider a gauge theory, say in 4 dimensions, with gauge group $U(N)^{\times 3}$. Choose the matter to be a Lorentz scalar which is complex and transforms in the (N, N, N) of the gauge group. It is then a four dimensional field $\Phi_{ijk}(x)$. We may ask how to enumerate all the gauge invariant observables made from $\Phi_{ijk}(x)$ in the large N limit. The zero coupling limit is a conformal field theory (CFT), so we have an operator-state correspondence. The enumeration we gave in Section 3 is then counting physical (gauge-invariant) states that can be built from the scalar. The two-point correlators we computed give the CFT-inner-product on these states; for uses of operator-states corresponding in the context of AdS/CFT see for example [76]. 3-index fields have recently been of interest in the context of supersymmetric gauge theories; see [77], and [78].

8.2.5. Complex matrix models and 3-index tensor models: An intriguing relation. Consider a complex matrix model with Gaussian measure, with

$$\int dZ e^{-\frac{1}{2}\text{tr}ZZ^\dagger},$$

where we have

$$\langle Z^i_j (Z^\dagger)^k_l \rangle = \delta^i_l \delta^k_j.$$

The holomorphic traces of Z can be parametrized by permutations τ,

$$\mathcal{O}_\tau(Z) = \sum_{i_1 \dots i_n = 1}^{N} Z^{i_1}_{i_{\tau(1)}} \dots Z^{i_n}_{i_{\tau(n)}},$$

subject to constraints

$$\mathcal{O}_\tau = \mathcal{O}_{\gamma \tau \gamma^{-1}},$$

for $\gamma \in S_n$. This parametrization includes both single traces such as $\text{tr}(Z^3)$ and multi-traces such as $[\text{tr}(Z^2)](\text{tr}Z)$. The cycle structure of τ determines the numbers

of single traces, double traces etc. Of particular interest in AdS/CFT are the correlators with one holomorphic and one anti-holomorphic observable. We have

$$\frac{|T_1|}{n!}\frac{|T_2|}{n!}\langle \mathcal{O}_{\tau_1}(Z)\mathcal{O}_{\tau_2}(Z^\dagger)\rangle$$

The natural normalization factors involve the sizes of the conjugacy classes corresponding τ_1, τ_2 which have been denoted T_1, T_2. It can be shown that the correlator is a sum over triples of permutations, see [3], [13], and [59],

$$\frac{|T_1|}{n!}\frac{|T_2|}{n!}\langle \mathcal{O}_{\tau_1}(Z)\mathcal{O}_{\tau_2}(Z^\dagger)\rangle = \frac{1}{n!}\sum_{\tau_1\in T_1}\sum_{\tau_2\in T_2}\sum_{\tau_0\in S_n}\delta(\tau_1\tau_2\tau_0)N^{C_{\tau_0}}.$$

This shows that the correlator is a sum over branched covers of the 2-sphere, branched over three points. The covers are summed with weight given by the inverse order automorphism group of the covers. This is a geometrical description of the Feynman graphs (more bi-partite embedded graphs) of the matrix model.

In this paper, we have found that observables of the 3-index tensor model are parametrized by permutations τ_1, τ_2 subject to conjugation equivalence; see equations (18) and (21). These equivalence classes are precisely the Feynman graphs for the correlators of the complex matrix model described above. Feynman graphs of the matrix model become physical states (observables) of the tensor model. As we saw the (normal-ordered) two-point correlators of the tensor model provide an inner product on these observables. So in this case, in a more than superficial sense, *Feynman graphs of a matrix model have become states of a tensor model*. It would be interesting to unravel the proper interpretation and implications of this connection. How general is it? It has a flavor of being a dimensional uplift, which is often related to categorification (see further discussion of the connection between (refined) graph counting and three-dimensional permutation-TFTs in [24]). This should be better understood both from a physical and a mathematical point of view. Note that the usual physical argument for tensor models being a higher dimensional generalization of matrix models relies on interpreting the indices as being dual to simplexes. Here we are seeing an extra dimension from the tensor model by considering counting and correlators of invariants, which are objects built after contracting away all the indices.

Appendices

A. Orbit-stabilizer theorem, Burnside's lemma, size of conjugacy classes

We gather, in this appendix, basic facts about conjugacy classes in the symmetric group S_n, the group of all $n!$ permutations of n objects, and about finite groups acting on finite sets. Further discussion of these topics can be found, for example, in [79].

Definition 1 (Cycle-type). Two permutations are of the same *cycle type* or have the same *cycle structure* if the unordered list of sizes of their cycles coincide.

Example. Consider σ_1 a permutation defined by its cycles $(123)(4)(5)(67)$ the list of the sizes of the cycles of σ_1 is $(3, 1, 1, 2)$. Note that the order in which appear 3,1,1,2 is not relevant. Consider another permutation σ_2 such that $(12)(3)(456)(7)$, then σ_1 and σ_2 have the same cycle type.

The cycle type of a permutation in S_n determines a list $p = (p_1, \ldots, p_n)$ of numbers p_i of cycles of length i. The list p is a partition of n:

$$n = \sum_i i\, p_i.$$

Proposition 1 (Conjugacy class). Two permutations α and α' have the same cycle type if and only if they are related to each other by conjugation, i.e $\alpha' = \sigma\alpha\sigma^{-1}$ for some σ.

Proof. ($\Longrightarrow$) The action under conjugation preserves the cycle structure. Indeed, consider α and σ two permutations. From $\sigma\alpha\sigma^{-1}(\sigma(x)) = \sigma(\alpha(x))$, one has for any cycle of a given permutation

$$\sigma(a_1, \ldots, a_2)\sigma^{-1} = (\sigma(a_1), \ldots, \sigma(a_2)). \tag{A.1}$$

($\Longleftarrow$) Consider σ_1 and σ_2 with same cycle type. Construct first a bijection φ between the cycles of these permutations mapping cycles with the same size one onto another (φ may be not unique). For a pair of cycles $s_1 = (a_1, \ldots, a_q)$ of σ_1 and $s_2 = (b_1, \ldots, b_q)$ of σ_2 linked by φ, namely $\varphi(s_1) = s_2$, construct a bijection σ such that $\sigma(a_i) = b_i$ (σ may be not unique as well). Then one checks that $\sigma s_1 \sigma^{-1} = s_2$ and that $\sigma\sigma_1\sigma^{-1} = \sigma_2$. $\qquad\square$

Burnside's lemma. Consider a finite set X and a finite group G acting on X. Consider $x \in X$ and the application

$$F_x \colon G \longrightarrow X, \ g \longmapsto gx.$$

Note that the image of F_x, $\Im(F_x) = Gx$, is the orbit of x in X whereas the kernel $\ker(F_x) = G_x$ of F_x is the stabilizer of x in G. The *orbit-stabilizer theorem* states that the size of the orbit generated by the group action on an element x is the ratio of the group size divided by the size of the subgroup which leaves the element x fixed. In equations

$$|Gx| = [G : G_x] = \frac{|G|}{|G_x|}. \tag{A.2}$$

The following statement holds.

Proposition 2 (Burnside's lemma). The number of orbits of the G-action on X, denoted $|X/G|$ is given by average number of fixed points of the group action. More explicitly,

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} X^g,$$

where $X^g = \{x \in X, \, gx = x\}$ is the set of fixed point of g.

Proof. Let us observe that

$$\sum_{x \in X} |G_x| = \sum_{x \in X} \left[\sum_{g \in G / gx = x} 1 \right] = \sum_{g \in G} \left[\sum_{x \in X / gx = x} 1 \right] = \sum_{g \in G} X^g.$$

Then inverting the relation (A.2) and summing over x yields

$$\sum_{g \in G} X^g = \sum_{x \in X} |G_x| = |G| \sum_{x \in X} \frac{1}{|Gx|} = |G| \sum_{x \in X} \sum_{A \in X/G / x \in A} \frac{1}{|A|}$$

$$= |G| \sum_{A \in X/G} \sum_{x \in A} \frac{1}{|A|} = |G| \, |X/G|,$$

where we used the fact that the classes in X/G determine a partition of X. $\square$

If one is interested in the number of elements in a conjugacy class of symmetric group, then, by Proposition 1, it is enough to look at their unique cycle type. Precisely, the following statement holds.

Proposition 3 (Size of conjugacy classes). Consider the conjugacy class T_p in the symmetric group $G = S_n$ with cycle type entirely determined by the list $p = (p_1, p_2, \ldots, p_n)$, where p_i gives the number of cycles of size i. This list forms a partition of n since $n = \sum_i i p_i$. Then the size of the conjugacy class $|T_p|$ is given by

$$|T_p| = \frac{n!}{\mathrm{Sym}(p)}, \quad \mathrm{Sym}\, p = \prod_{i=1}^{n} (i^{p_i})(p_i!),$$

where $\mathrm{Sym}(p)$ is the number of elements of S_n commuting with any permutation in the conjugacy class T_p.

This is an application of the orbit stabilizer theorem, for the case where the group S_n acts on itself by conjugation.

Sym(p) can be computed as follows. For $\alpha \in T_p$, we are looking for $\sigma \in S_n$ such that $\sigma \alpha \sigma^{-1} = \alpha$. As we saw in (A.1), conjugating α with σ amounts to replacing the integers j in the cycles of α by $\sigma(j)$. This σ transformation of the cycles of α can leave the cycles fixed, or exchange cycles of the same length. Focusing on cycles of length i, of which there are p_i, σ can cycle the numbers within a cycle. For a cycle of length i there are i of these cyclic permutations. So there are i^{p_i} cyclic permutations σ which just cycle the integers within cycles of length i in α, thus leaving α unchanged. Then, there are permutations of exchanging the p_i different cycles. In all, we get $\prod_{i=1}^{n}(i^{p_i})(p_i!)$ as stated above.

B. Symmetric group delta functions to generating functions for countings

In this appendix, we address the evaluation of formal sums appearing as $S_{[2,1]}^{(3)}$ and $S_{[3]}^{(3)}(n)$ in (27) and $S_{[2,1^2]}^{(4)}$, $S_{[3,1]}^{(4)}$, $S_{[2^2]}^{(4)}$, and $S_{[4]}^{(4)}$ appearing in (35).

Let us start by $S_{[2,1]}^{(3)}$ and find a way to perform this sum. We have

$$S_{[2,1]}^{(3)} = \sum_{\gamma,\sigma \in S_n} \delta(\gamma^2 \sigma \gamma^{-2} \sigma^{-1}).$$

For every partition p of n, $n = p_1 + 2p_2 + \dots$, there is a permutation σ of cycle of type p, i.e. σ has p_1 cycles of length 1, p_2 cycles of length 2, etc. Let us denote this by $\sigma \in p$. Let T_p be the sum of permutations in the cycle-type p in the group algebra $\mathbb{C}(S_n)$:

$$T_p = \sum_{\sigma \in p} \sigma.$$

Consider the sum, still with value in $\mathbb{C}(S_n)$,

$$Z^{(2)}(n) = \sum_{\gamma \in S_n} \gamma^2 = \sum_{p \vdash n} Z_p^{(2)} \frac{T_p}{|T_p|}.$$

The sum of γ^2 commutes with each element of S_n (for all $\sigma \in S_n$, $\sum_\gamma \sigma\gamma^2\sigma^{-1} = \sum_\gamma (\sigma\gamma\sigma^{-1})^2 = \sum_\gamma \gamma^2$), so it is a sum over complete conjugacy classes T_p, each which some weight. We have defined $\frac{Z_p^{(2)}}{|T_p|}$ to be the coefficient of T_p in the sum of γ^2, where $|T_p|$ is the number of permutations in the conjugacy class corresponding to cycle-type given by p (see Proposition 1). Similarly, we can define

$$\sum_{\gamma \in S_n} \gamma = \sum_{p \vdash n} Z_p^{(1)} \frac{T_p}{|T_p|}.$$

In this case we have

$$Z_p^{(1)} = |T_p| = \frac{n!}{\prod_i i^{p_i} p_i!} = \frac{n!}{\mathrm{Sym}(p)}.$$

Now there is a generating function for $Z_p^{(1)}$ given by

$$Z^{(1)}(t, \vec{x}) = Z^{(1)}(t, x_1, x_2, \dots) = e^{\sum_{i=0}^{\infty} \frac{t^i x_i}{i}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{p \vdash n} Z_p^{(1)} \prod_i x_i^{p_i},$$

where $\vec{x} = (x_1, x_2, \dots)$. When we square a permutation, all odd cycles become odd cycles again, whereas all even cycles split in two of half the length of the former ones. As a result the generating function for $Z_p^{(2)}$ is

$$Z^{(2)}(t, \vec{x}) = Z^{(1)}(t, x_1, x_2 = x_1^2, x_3, x_4 = x_2^2, \dots) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{p \vdash n} Z_p^{(2)} \prod_i x_i^{p_i}.$$

We can finally write

$$\begin{aligned}
S_{[2,1]}^{(3)} &= \frac{1}{n!} \sum_{p \vdash n} Z_p^{(2)} \, \mathrm{Sym}(p) \\
&= \sum_{p \vdash n} \mathrm{Coefficient}\,[Z^{(2)}(t, \vec{x}), t^n x_1^{p_1} x_2^{p_2} \dots x_n^{p_n}] \times \Big[\prod_{i=1}^{n} i^{p_i} p_i! \Big],
\end{aligned}$$
(B.1)

where, given a partition p of n, Coefficient $[Z^{(2)}(t, \vec{x}), t^n x_1^{p_1} x_2^{p_2} \dots x_n^{p_n}]$ is the coefficient of the monomial $t^n x_1^{p_1} x_2^{p_2} \dots x_n^{p_n}$ in the series $Z^{(2)}$. This is easily programmable in Mathematica (see Mathematica code 2, in Appendix D) and one gets (from $n = 1$ to $n = 13$)

$$1,\ 2,\ 5,\ 17,\ 59,\ 265,\ 1095,\ 6342,\ 33966,$$

$$219968,\ 1333654,\ 9930505,\ 70419371, \dots.$$

Another quantity which appears in the computation of the invariants constructed from 3-index invariants is

$$S_{[3]}^{(3)} = \frac{1}{n!} \sum_{\gamma, \sigma \in S_n} \delta(\gamma^3 \sigma^3).$$

Consider the following element of the group algebra of S_n,

$$Z^{(3)}(n) = \sum_{\gamma \in S_n} \gamma^3 = \sum_{p \vdash n} Z_p^{(3)} \frac{T_p}{|T_p|}.$$

In terms of these, we have

$$S_{[3]}^{(3)} = \frac{1}{n!} \sum_{p \vdash n} \left(\frac{Z_p^{(3)}}{|T_p|} \right)^2 |T_p| = \frac{1}{(n!)^2} \sum_{p \vdash n} (Z_p^{(3)})^2 \mathrm{Sym}(p).$$

In an analogous way than before, there is a generating function

$$Z^{(3)}(t, \vec{x}) = \sum_{n=0}^{\infty} t^n \sum_{p \vdash n} \frac{Z_p^{(3)}}{n!} \prod_{i=1}^{n} x_i^{p_i}.$$

When we take the cube of a permutation, any cycle of length divisible by 3 becomes a triple of 1-cycles. Any other cycle stays a cycle of the same length. Hence, one has

$$Z^{(3)}(t, x_1, x_2, \dots) = Z^{(1)}(t, x_i|_{x_i \to (x_i/3)^3 \text{ if and only if } i \text{ is divisible by } 3}). \quad (B.2)$$

So, we have

$$S_{[3]}^{(3)} = \sum_{p \vdash n} \left(\mathrm{Coefficient} \left[Z^{(3)}(t, \vec{x}), t^n \prod_i x_i^{p_i} \right] \right)^2 \times \left[\prod_i i^{p_i} p_i! \right]. \quad (B.3)$$

This is easily calculable in Mathematica (see Mathematica code 2 in Appendix D) and we list the numbers starting at $n = 1$ up to $n = 13$ as

$$1, 1, 2, 4, 5, 13, 29, 48, 114, 301, 579, 1462, 4198, \dots .$$

The first few terms can be easily checked in GAP by directly summing pairs of permutations subject to $\gamma^3 \sigma^3 = \mathrm{id}$.

Counting the case of tensors with 4-indices, say $Z_{4;\,\mathrm{sc}}(n)$, we encounter the sum $S_{[3,1]}^{(4)}(n)$ which is similar to that $S_{[2,1]}^{(3)}(n)$, but $Z^{(2)}$ is replaced with $Z^{(3)}$:

$$S_{[3,1]}^{(4)}(n) = \frac{1}{n!} \sum_{p \vdash n} Z_p^{(3)} \mathrm{Sym}(p)$$

$$= \sum_{p \vdash n} \mathrm{Coefficient} \, [Z^{(3)}(t, \vec{x}), t^n x_1^{p_1} x_2^{p_2} \dots x_n^{p_n}] \times \left[\prod_{i=1}^{n} i^{p_i} p_i! \right], \quad (B.4)$$

where $Z^{(3)}$ is as given above in (B.2). Using still Mathematica, this can be programmed and we get (see Appendix D, Mathematica code 2), for $n = 1$ to $n = 13$,

$$1, 2, 4, 12, 27, 103, 391, 1383, 6260, 32704, 149045, 812696, 5034682 \dots .$$

Still in the rank 4 case, one finds the sum

$$S_{[4]}^{(4)}(n) = \frac{1}{n!} \sum_{\sigma, \gamma \in S_n} \delta(\gamma^4 \sigma^4).$$

Following the above arguments, we will define $Z^{(4)}(t, \vec{x})$ by substituting in $Z^{(1)}(t, \vec{x})$

$$x_i \longrightarrow (x_{i/4})^4 \quad \text{for } i = 4q \text{ with integer } q,$$
$$\longrightarrow (x_{i/2})^2 \quad \text{for } i = 4q + 2,$$
$$\longrightarrow x_i \qquad \text{for } i = 4q + 1 \text{ or } i = 4q + 3.$$

Then we have

$$S_{[4]}^{(4)}(n) = \sum_{p \vdash n} \left(\text{Coefficient}\left[Z^{(4)}(t, \vec{x}), \prod_i x_i^{p_i} \right] \right)^2 \times \left[\prod_i i^{p_i} p_i! \right]. \tag{B.5}$$

Some terms of this sequence, starting from $n = 1$ up to $n = 13$ (see Mathematica code 2 in Appendix D)

1, 2, 3, 11, 27, 93, 233, 978, 3156, 13280, 44476, 205611, 796091, ...

The first few terms are quickly checked by directly summing the delta function over the symmetric group with GAP, but this soon becomes prohibitive, and the generating function method is much more efficient.

Counting rank 4 tensor invariants up to color permutation leads to another sum given by

$$S_{[2,1^2]}^{(4)}(n) = \frac{1}{n!} \sum_{\gamma, \sigma_1, \sigma_2 \in S_n} \delta(\gamma \sigma_1 \gamma^{-1} \sigma_1^{-1}) \delta(\gamma^2 \sigma_2 \gamma^{-2} \sigma_2^{-1}).$$

For γ in a conjugacy class given by p, let us define $Sq(p)$ to be the cycle structure of γ^2:

$$(Sq(p))_{2j} = 2p_{4j} \quad \text{and} \quad (Sq(p))_{2j+1} = p_{2j+1} + 2p_{4j+2}.$$

As γ runs over all the partitions p, we have

$$S_{[2,1^2]}^{(4)}$$

$$= \frac{1}{n!} \sum_{p \vdash n} |T_p| \, \text{Sym}(p) \, \text{Sym}(Sq(p))$$

$$= \sum_{p \vdash n} \text{Sym}(Sq(p))$$

$$= \sum_{p \vdash n} \prod_{j=1}^{\lfloor \frac{n}{2} \rfloor} (2j)^{2p_{4j}} (2p_{4j})! \prod_{j=0}^{\lfloor \frac{n}{2} \rfloor} (2j+1)^{p_{2j+1}+2p_{4j+2}} (p_{2j+1} + 2p_{4j+2})!.$$

$$\tag{B.6}$$

This is calculated with Mathematica for n in the range 1 to 10 (see Mathematica code 3 in Appendix D) as

$$1, 4, 15, 83, 385, 2989, 20559, 203992, 1827640, 21864590, \ldots.$$

The first few terms are checked against GAP which calculates the delta functions directly.

Finally, one can transform $S^{(4)}_{[2^2]}(n)$ in the following way:

$$
\begin{aligned}
S^{(4)}_{[2^2]}(n) &= \frac{1}{n!} \sum_{\gamma \in S_n} \sum_{\sigma_i \in S_n} \delta(\sigma_1^2 \gamma^2) \delta(\gamma^2 \sigma_2 \gamma^{-2} \sigma_2^{-1}) \\
&= \frac{1}{n!} \sum_{\gamma, \alpha \in S_n} \sum_{\sigma_i \in S_n} \delta(\sigma_1^2 \alpha) \delta(\alpha^{-1} \gamma^2) \delta(\alpha \sigma_2 \alpha^{-1} \sigma_2^{-1}) \\
&= \frac{1}{n!} \sum_{\alpha \in S_n} \delta(\alpha Z^{(2)}(n)) \delta(\alpha^{-1} Z^{(2)}(n)) \mathrm{Sym}(\alpha) \\
&= \frac{1}{n!} \sum_{p \vdash n} \sum_{\alpha \in [p]} \delta(\alpha Z^{(2)}(n)) \delta(\alpha^{-1} Z^{(2)}(n)) \mathrm{Sym}(\alpha) \\
&= \frac{1}{n!} \sum_{p \vdash n} \sum_{\alpha \in p} \left(\frac{Z_p^{(2)}}{|T_p|} \right)^2 \mathrm{Sym}(p) \\
&= \frac{1}{n!} \sum_{p \vdash n} \frac{n!}{\mathrm{Sym}(p)} (Z_p^{(2)})^2 \frac{1}{|T_p|^2} \mathrm{Sym}(p) \\
&= \frac{1}{(n!)^2} \sum_{p \vdash n} (Z_p^{(2)})^2 (\mathrm{Sym}(p))^2 \\
&= \sum_{p \vdash n} (\mathrm{Coefficient}[Z^{(2)}(t, \vec{x}), t^n x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}] \mathrm{Sym}(p))^2.
\end{aligned}
\tag{B.7}
$$

Doing this with Mathematica (see Appendix D code), we get for n from 1 to 12.

$$1, 4, 17, 105, 685, 5825, 585018, 54013,$$
$$6873522, 90254150, 1275023778, 19651966895.$$

The first few agree with GAP.

C. Derivations for correlator computations

In this section, we explain the derivations of the formulae for correlators in terms of delta functions over symmetric groups, which are best expressed in diagrammatic form. For some CFT applications of such techniques for correlators see [4], [6], [8], [10], and [11]. When we use the basic 2-point correlator and apply Wick's theorem to calculate the correlator of n copies of Φ with n copies of $\bar{\Phi}$, we get a sum over Wick contractions. This is a sum over permutations which expresses as

$$
\langle \Phi^{i_1,j_1,k_1} \ldots \Phi^{i_n,j_n,k_n} \bar{\Phi}_{i_1,j_1,k_1} \ldots \bar{\Phi}_{i_n,j_n,k_n} \rangle
$$

$$
= \sum_{\mu_1,\mu_2,\mu_3 \in S_n} \delta^{i_1,i_{\mu_1(1)}} \ldots \delta^{i_n,i_{\mu_1(1)}} \ \delta^{j_1,j_{\mu_2(1)}} \ldots \delta^{j_n,j_{\mu_2(n)}}
$$
$$
\delta^{k_1,k_{\mu_3(1)}} \ldots \delta^{k_n,k_{\mu_3(n)}} .
$$

It is convenient to describe this diagrammatically as in Figure 11.
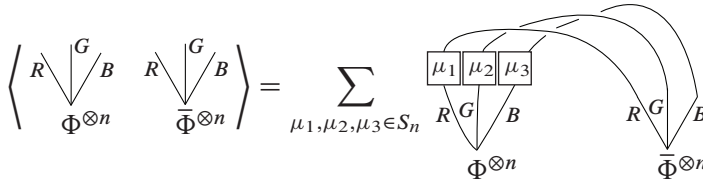

Figure 11. Basic correlator in a diagrammatic form.

Let us also draw the observable parametrized by $\sigma_1, \sigma_2, \sigma_3$ using a similar diagram of Figure 12. This is simplified version of the diagram in Figure 3.
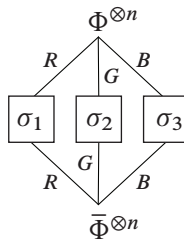

Figure 12. Observable as a diagram.

Similarly, draw the two-point function in a diagrammatic form given in Figure 13 and use the diagrammatic expression of the Wick contractions (Figure 11) in this correlator. As stated before, we are taking the observables to be "normal ordered" so we only allow contractions to take place between the Φ's from the first observable to the $\bar{\Phi}$'s from the second (parametrized by μ_a) and between the $\bar{\Phi}$'s from the first observable to the Φ's from the second (parametrized by ν_a).

$$\langle \mathcal{O}_{\sigma_1,\sigma_2,\sigma_3} \overline{\mathcal{O}}_{\tau_1,\tau_2,\tau_3} \rangle \sigma =$$
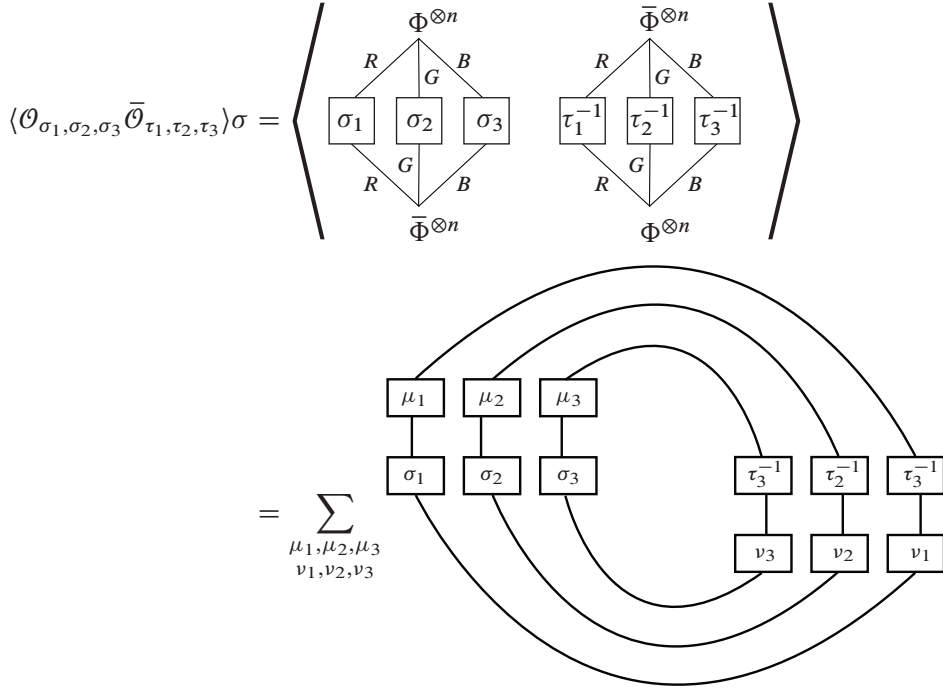

Figure 13. Two-point function as a diagram.

The final step is a simple diagrammatic straightening, to recognize that the correlator is a product of three traces of sequences of permutations

$$\langle \mathcal{O}_{\sigma_1,\sigma_2,\sigma_3} \overline{\mathcal{O}}_{\tau_1,\tau_2,\tau_3} \rangle$$
$$= \sum_{\mu_i \in S_n} \sum_{v_i \in S_n} \mathrm{tr}_{V_1^{\otimes n}}(\sigma_1 \mu_1 \tau_1^{-1} v_1) \mathrm{tr}_{V_2^{\otimes n}}(\sigma_2 \mu_2 \tau_2^{-1} v_2) \mathrm{tr}_{V_3^{\otimes n}}(\sigma_3 \mu_3 \tau_3^{-1} v_3).$$

Now if V is an N-dimensional space with basis e_i for $i = 1 \ldots N$ we have

$$\mathrm{tr}_{V^{\otimes n}}(\sigma) = \langle e^{i_1} \otimes \cdots \otimes e^{i_n} \mid \sigma \mid e_{i_1} \otimes e_{i_n} \rangle$$
$$= \langle e^{i_1} \otimes \cdots \otimes e^{i_n} \mid e_{i_{\sigma(1)}} \otimes e_{i_{\sigma(n)}} \rangle$$
$$= \delta^{i_1}_{i_{\sigma(1)}} \ldots \delta^{i_n}_{i_{\sigma(n)}}$$
$$= N^{C_\sigma}.$$

The repeated i indices are summed since we are taking a trace. C_σ is the number of cycles in the permutation σ. It is instructive to see how the last step works in a simple example, where $n = 2$. If $\sigma = (1)(2)$ is the identity permutation, then

$$\delta^{i_1}_{i_{\sigma(1)}} \delta^{i_2}_{i_{\sigma(2)}} = \delta^{i_1}_{i_1} \delta^{i_2}_{i_2} = N^2.$$

If $\sigma = (12)$ is the swop, we have instead

$$\delta^{i_1}_{i_{\sigma(1)}} \delta^{i_2}_{i_{\sigma(2)}} = \delta^{i_1}_{i_2} \delta^{i_2}_{i_1} = \delta^{i_1}_{i_1} = N.$$

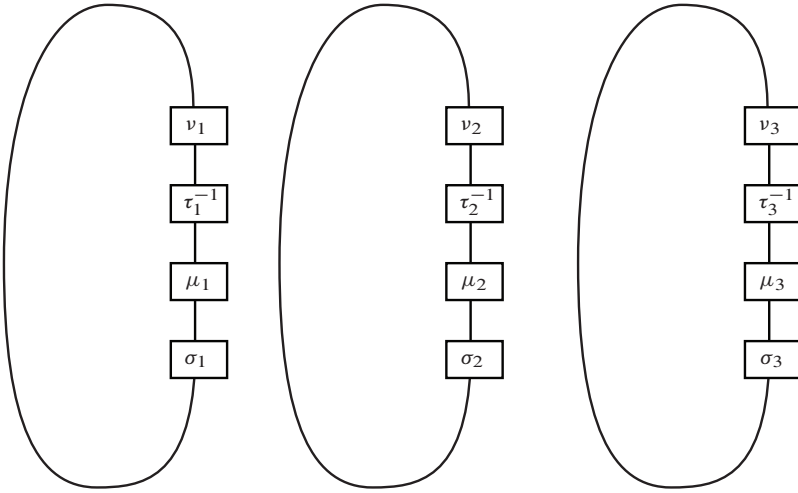Thus, we see that the power of N is the number of cycles in the permutation.

Figure 14. Straightening the traces.

Since we have allowed the 3-tensor indices to have different ranks, we can write

$$\langle \mathcal{O}_{\sigma_1,\sigma_2,\sigma_3} \overline{\mathcal{O}}_{\tau_1,\tau_2,\tau_3} \rangle$$

$$= \sum_{\mu_i \in S_n} \sum_{\nu_i \in S_n} N_1^{C_{\sigma_1 \mu_1 \nu_1 \tau_1^{-1}}} N_2^{C_{\sigma_2 \mu_1 \nu_1 \tau_2^{-1}}} N_3^{C_{\sigma_3 \mu_1 \nu_1 \tau_3^{-1}}}$$

$$= \sum_{\mu_i \in S_n} \sum_{\nu_i \in S_n} \sum_{\alpha_i \in S_n} N_1^{C_{\alpha_1}} N_2^{C_{\alpha_2}} N_3^{C_{\alpha_3}} \delta(\sigma_1 \mu_1 \tau_1^{-1} \nu_1 \alpha_1)$$
$$\delta(\sigma_2 \mu_2 \tau_2^{-1} \nu_2 \alpha_2) \delta(\sigma_3 \mu_3 \tau_3^{-1} \nu_3 \alpha_3)$$

$$= \sum_{\mu_i \in S_n} \sum_{\nu_i \in S_n} N_1^n N_2^n N_3^n \delta(\sigma_1 \mu_1 \tau_1^{-1} \nu_1 \Omega_1) \delta(\sigma_2 \mu_2 \nu_2 \tau_2^{-1} \Omega_2) \delta(\sigma_3 \mu_3 \nu_3 \tau_3^{-1} \Omega_3).$$

In the second line, we introduced three extra permutations constrained by delta functions to re-write the previous line. Note that $C_\alpha = C_{\alpha^{-1}}$. In the third line, we have extracted the leading power of N which comes from the permutation with the largest number of cycles, namely the identity permutations. The $\Omega(N)$ factor is an element of the group algebra of S_n of the form

$$N^n \Omega = N^n \left(1 + \sum_{\alpha \in S_n \setminus \{1\}} \frac{N^{C_\alpha}}{N^n} \alpha \right). \tag{C.1}$$

This element plays a key role in large N expansions of two dimensional YM; see [17] and [18]. The device of introducing delta functions makes the connection to the counting of branched covers transparent. Thus, we have derived (42) stated in Section 7.

D. GAP and Mathematica codes

We provide here some programming codes of GAP and Mathematica. These have allowed the determination of several sequences in the text. The number $Z_d(n)$ of rank d tensor invariants, made with n covariant tensors T and n contravariant tensor $\bar{T}$, is the one we primarily focused. Then other numbers are derived from its group data or its structure. After entering a given line, the line starting by (out) should be obtained.

GAP code for $Z_d(n)$ and $Z_{d;\,\mathrm{sc}}(n)$.   We provide here a code for evaluating $Z_3(n)$, number of rank 3 tensor invariants, and $Z_{3;\,\mathrm{sc}}(n)$ number of rank 3 color-symmetrized tensor invariants. In the following program, we use the particular value $n = 4$. Changing that parameter n or introducing a procedure for any finite range of value of n will allow one to recover the full sequences (7), (10), (30), and (31) in the text. Meanwhile changing the rank d of the tensor will require little extra work and allow to find (11) giving, in particular for $d = 4$, eq. (12).

The sequence of lines starting by the prompt gap> denotes the lines entered. The following lines with (out) are the outputs of that entry. The procedure starts by the computation of $Z_3(n = 4)$ using the formula (18). This allows us to reduce the number of steps because we simply avoid another sum over S_4. Then, from this, we can evaluate the number of connected invariants $Z_3^{\mathrm{connect}}(n = 4)$ (10), the number of color-symmetrized invariants $Z_{3;\,\mathrm{sc}}(n = 4)$ (30) and then the number $Z_{3;\,\mathrm{sc}}^{\mathrm{connect}}(n = 4)$ of color-symmetrized connected invariants (31). Interestingly, in order to obtain connected graphs, we use the command IsTransitive (G, [1..4]) checking if the action of the group G on $\{1, 2, 3, 4\}$ is transitive.

```
gap> TT := [ ];

(out) [ ]

gap> n := 4;;
     for tau1 in SymmetricGroup(n)
           do for tau2 in SymmetricGroup(n)
               do Add (TT, [tau1, tau2]);
               od;
           od;

gap> TT[3];

(out) [ (), (1,2,4) ]

gap> OTT := OrbitsDomain (SymmetricGroup(n), TT, OnPairs);;
     Ln := Length (OTT);;
               Print("Z_3(n=4) = ", Ln);

(out) Z_3(n=4) = 43

gap> OS := [];;  for k in [1..Ln]
                    do Add (OS , Set(OTT[k]));
                     od;

gap> OU := [];; for p in [1..Length(Unique(OS))]
                    do Add (OU, Unique(OS)[p][1]);
                     od;

gap> cnx := [];; for j in [1..Length(Unique(OS))]
                    do if IsTransitive (Group (OU[j][1], OU[j][2]) , [1..m])
                         then Add (cnx, OU[j]);
                       fi;
                    od;

gap> Print("Z^{connect}_3(n=4) = " , Length (cnx) );

(out) Z^{connect}_3(n=4) = 26


gap> P23 := function (List2P)
               local LL ;
               LL := [];
               Add (LL , List2P[2]);
               Add (LL , List2P[1]);
               return LL;
        end;

(out) function( List2P ) ... end
```

```
gap> P12 := function (List2P)
              local LL ;
              LL := [];
              Add (LL , Inverse ( List2P[1] ) );
              Add (LL, Inverse ( List2P[1] ) * List2P[2] );
            return LL;
        end;

(out) function( List2P ) ... end

gap> QROTT := [];;
        for i in [1 .. Ln]
        do Add (QROTT, [ ]);
         od;

      for i in [1.. Ln]
        do for j in [1 .. Length (OTT[i]) ]
          do Add (QROTT[i] , OTT[i][j] );
                              Add (QROTT[i] , P12 (OTT[i][j]));
             Add (QROTT[i] , P23 (OTT[i][j]));
             Add (QROTT[i] , P12( P23 (OTT[i][j]) ) );
             Add (QROTT[i] , P23( P12 (OTT[i][j]) ) );
             Add (QROTT[i] , P12( P23 (P12 (OTT[i][j]) )));
          od;
        od;


gap> Length (QROTT);

(out) 43

gap> SetQROTT := [];;  for i in [1..Ln]
                       do Add (SetQROTT, Set (QROTT[i]));
                         od;
         LUsq := Length ( Unique (SetQROTT) );;

gap> Print ("Z_{3;color}(n=4)=", LUsq);

Z_{3;color}(n=4) = 15

gap> UQROTT := [];;  for i in [1..LUsq]
                      do Add (UQROTT, Unique (SetQROTT)[i][1]);
                        od;

gap> CnX := [];;  for i in [1..LUsq]
                  do if IsTransitive (Group (UQROTT[i][1],
                                             UQROTT[i][2]), [1..n])
                     then Add (CnX, UQROTT[i]);
                    fi;
                  od;
```

```
gap> Print ("Z^{connect}_{3;color}(n=4) = ", Length(CnX));

Z^{connect}_{3;color}(n=4) = 8
```

Mathematica code 1 for $Z_d(n)$.   In this paragraph, we provide a Mathematica code
for evaluating the number $Z_d(n)$ (denoted Z[n,d]) of rank d tensor invariants made
with $2n$ tensors. Specifically, we evaluate $Z_3(n)$ and $Z_4(n)$ for the rank 3 and 4,
respectively. We use the built-in function Count[list, pattern] which count the
number of elements in a list matching a pattern. We also give the code for the
generating functions $Z_d(x)$ (denoted Zseries[x,d]) from which the Plog function
$\mathrm{Plog} Z_d(x)$ (denoted PLogZ[F,d,x]) is derived. Then we can obtain the number of
connected invariants from the later function using the built-in MoebiusMu or Möbius
function.

```
IntegerPartitions [ 4 ]
IntegerPartitions [ 4 ][[1]]

(out) {{4}, {3, 1}, {2, 2}, {2, 1, 1}, {1, 1, 1, 1}}
(out) {4}

Count [{1,1}, 2]
Count [{1,1,2}, 1]

(out) 0
(out) 2

Sym [p_ , n_] := Product [ i^( Count [p , i] ) ( Count [p , i] )! , {i, 1, n} ]

Sym [{1, 1} , 2]

(out) 2

Z [n_ , d_] := Sum [ ( Sym [ IntegerPartitions[n][[i]] , n ] )^(d - 2) ,
                    {i, 1, Length [ IntegerPartitions[n] ] } ]

Zseries [x_, d_] := Sum [ Z [n , d] x^n , {n, 0, 10} ]

Zseries [x , 3]

(out) 1 + x + 4 x^2 + 11 x^3 + 43 x^4 + 161 x^5 + 901 x^6 + 5579 x^7 +
 43206 x^8 + 378360 x^9 + 3742738 x^10

Zseries [x , 4]

1 + x + 8 x^2 + 49 x^3 + 681 x^4 + 14721 x^5 + 524137 x^6 +
 25471105 x^7 + 1628116890 x^8 + 131789656610 x^9 + 13174980291658 x^10


PLog [F_, d_, t_] := Sum [ MoebiusMu [ k ] / k Log [F [t^k , d] ] , {k, 1, 10} ]
```

```
Do[ Print[ "Plog[Z(", x, ",", d, ")] = ",
    Series [ PLog [ Zseries, d, x] , {x , 0, 10} ] ] , {d, 3, 4} ]
```

(out) Plog[Z(x,3)] = x + 3 x^2 + 7 x^3 + 26 x^4 +97 x^5 + 624 x^6 + 4163 x^7
+ 34470 x^8 + 314493 x^9 + 3202839 x^10+ O [x^11]

(out) Plog[Z(x,4)] = x + 7 x^2 + 41 x^3 + 604 x^4 + 13753 x^5 + 504243 x^6
+ 24824785 x^7 + 1598346352 x^8 + 129958211233 x^9 + 13030565312011 x^10
+ O [x^11]

Mathematica code 2 for $S^{(3)}_{[2,1]}$, $S^{(3)}_{[3]}(n)$, $S^{(4)}_{[3,1]}(n)$, and $S^{(4)}_{[4]}(n)$. In this paragraph, we provide Mathematica codes useful for the evaluation of $S^{(3)}_{[2,1]}$ and $S^{(3)}_{[3]}(n)$ appearing in $Z_{3;\,\mathrm{sc}}(n)$ (28) and $S^{(4)}_{[3,1]}(n)$ and $S^{(4)}_{[4]}(n)$ appearing in $Z_{4;\,\mathrm{sc}}(n)$ (35). The sums can be programmed in a very similar way. We start by $S^{(3)}_{[2,1]}$ denoted as S2ans (evaluated up to order 10) and then, using small modifications, we will give the program for the rest of the above sums.

```
X = Array [x , 15]
```

(out) {x[1], x[2], x[3], x[4], x[5], x[6], x[7], x[8], x[9], x[10],
x[11], x[12], x[13], x[14], x[15]}

```
Z [X , t] := Product [ Exp [ t^i  x[i]/i ] , {i, 1, 15} ]

RR =  Table [ x[2 i] -> x[i]^2 ,  {i, 1, 5} ]
```

(out) {x[2] -> x[1]^2, x[4] -> x[2]^2, x[6] -> x[3]^2, x[8] -> x[4]^2,
x[10] -> x[5]^2, x[12] -> x[6]^2, x[14] -> x[7]^2}

```
Z2 [X , t] = Z [X , t] /. RR
```

(out) e^(t x[1] + 1/2 t^2 x[1]^2 + 1/4 t^4 x[2]^2 + 1/3 t^3 x[3]
+ 1/6 t^6 x[3]^2 + 1/8 t^8 x[4]^2 + 1/5 t^5 x[5] + 1/10 t^10 x[5]^2
+ 1/12 t^12 x[6]^2 + 1/7 t^7 x[7] + 1/14 t^14 x[7]^2 + 1/9 t^9 x[9]
+ 1/11 t^11 x[11] + 1/13 t^13 x[13] + 1/15 t^15 x[15])

```
PP [ n_ ] := IntegerPartitions [n]

Z2ans [ n_ ] := Coefficient [ Series [ Z2[X , t] , {t, 0, n} ], t^n ]

Z2ans [3]
```

(out) 1/3 (2x[1]^3 + x[3])

```
Symm [q_ , n_] := Product [ i^( Count [q , i] ) ( Count [q , i] )! , {i, 1, n} ]

Symm [ {2, 2, 1} , 5]
```

```
(out) {8}

CC [n_ , q_] := Coefficient [ Z2ans[n] ,
                             Product [x[i]^(Count[q , i]) , {i, 1, n} ]  ]

CC [3 , {1, 1, 1}]
CC [3 , {3}]
Z2ans [ 3 ]

(out) 2/3
(out) 1/3
(out) 1/3 (2 x[1]^3 + x[3])

S2ans [ n_ ] := Sum [ CC [ n , PP[n][[i]] ] * Symm [ PP[n][[i]], n ] ,
                      {i, 1, Length [ PP[n] ] } ]

Do [Print ["S_2(", i, ") = ", S2ans[i][[1]]], {i, 10}]

Table [ S2ans [i] , {i, 1, 10} ]

(out) {1, 2, 5, 13, 31, 89, 259, 842, 2810, 10020}
```

The Mathematica code for $S^{(3)}_{[3]}(n)$ can be obtained from the above code by simply replacing RR and S2ans (by S3ans) entries as follows

```
RR  =  Table [ x[3 i] -> x[i]^3 , {i, 1, 5} ]

S3ans [n_] := Sum [ ( CC [ n , PP[n][[i]] ] )^2 * Symm [ PP[n][[i]], n ] ,
                    {i, 1, Length [ PP [n] ] } ]

Table [ S3ans [i] , {i, 1, 10} ]

(out) {1, 1, 2, 4, 5, 13, 29, 48, 114, 301}
```

The sum $S^{(4)}_{[3,1]}(n)$ can be programmed using the above and by substituting the definition of S3ans as follows.

```
S3ans [ n_ ] := Sum [ CC [ n , PP[n][[i]] ] * Symm [ PP[n][[i]], n ] ,
                      {i, 1, Length [ PP [n] ] } ]

Table [ S3ans [i] , {i, 1, 10} ]

(out) {1, 2, 4, 12, 27, 103, 391, 1383, 6260, 32704}
```

The Mathematica code for $S^{(4)}_{[4]}(n)$ can be also obtained by replacing (as well where necessary afterwards) RR, Z2 (by Z4) and S2ans (by S5ans) entries as follows.

```
RR  =  Table [ { x[4 i] -> x[i]^4 , x[4 i - 2] -> x[2 i - 1]^2 }, {i, 1, 4} ]

FRR := Flatten [ RR ]

Z5 [X , t] = Z [X , t] /. FRR
```

```
S5ans [ n_ ] := Sum [ ( CC [ n , PP[n][[i]] ] )^2 * Symm [ PP[n][[i]], n ] ,
                       {i, 1, Length [ PP [n] ] } ]

Table [ S5ans [i] , {i, 1, 10} ]

(out) {1, 2, 3, 11, 27, 93, 233, 978, 3156, 13280}
```

Mathematica code 3 for $S^{(4)}_{[2,1^2]}(n)$.   In this paragraph, we provide a Mathematica program for evaluating $S^{(4)}_{[2,1^2]}(n)$ occurring in $Z_{4;\,\text{sc}}(n)$; see (35).

```
SymH [n_ , p_] := Product [ (2 j)^( 2 Count [p , 4 j])
                           Factorial [ 2 Count [p , 4 j] ] ,
                           {j, 1, Floor [ n/2 ] } ]

Product [ (2 j + 1)^( Count [p , 2 j + 1] + 2 Count [p , 4 j + 2] )
          Factorial [Count [ p , 2 j + 1] + 2 Count [ p , 4 j + 2 ] ] ,
          {j, 0, Floor [  n/2  ] } ]

SymH [ 3 , {3} ]
SymH [ 3 , {1,1,1} ]
SymH [ 4 , {4} ]

(out) 3
(out) 6
(out) 8

PP [ n_ ] := IntegerPartitions [ n ]

Sp2  [ n_ ] := Sum [ SymH [ n , PP[n][[i]] ] , {i , 1, Length [ PP [n] ] } ]

Table [ Sp2 [i] , {i, 1, 10} ]

(out) {1, 4, 15, 83, 385, 2989, 20559, 203922, 1827640, 21863590}
```

Mathematica code 4 for $S^{(4)}_{[2^2]}(n)$.   The code for $S^{(4)}_{[2^2]}(n)$ (important for the evaluation of $Z_{4;\,\text{sc}}(n)$ (35)) is again very similar to the above 2 for $S^{(3)}_{[2,1]}$. We simply remove some lines and adjust the final S2ans in order to evaluate $S^{(4)}_{[2^2]}(n)$.

```
X = Array [x , 15]

Z [X , t] := Product [ Exp [ t^i  x[i]/i ] , {i, 1, 15} ]

RR = Table [ x[2 i] -> x[i]^2, {i, 1, 7} ]

Z2 [X , t] = Z [X , t] /. RR

PP [ n_ ] := IntegerPartitions [n]
```

```
Z2ans [n_] := Coefficient [ Series [ Z2 [X,  t] , {t, 0, n} ], t^n ]

CC [n_ , q_] := Coefficient [ Z2ans [n] ,
                                Product [ x[i]^(Count[q , i]) , {i, 1, n} ] ]

S4prime [ n_ ] := Sum [ (CC [ n , PP[n][[i]] ] Symm [ PP [n][[i]] , n ] )^2 ,
                        {i, 1, Length [ PP [n] ] } ]

Table [ S4prime [i] , {i, 1, 10} ]

(out) {1, 4, 17, 105, 685, 5825, 54013, 585018, 6873522, 90254150}
```

Mathematica code 5 for $\mathcal{Z}_{d\,;\text{noncolor}}(n)$.   Here, we provide a program which yields eq. (37) and eq. (38).

```
PP [n_] := IntegerPartitions  [ n ]

CC [d_ , n_] := Sum [ Product [ i^( (d - 2) Count [PP[n][[j]] , i] )
                                * (d Count [PP[n][[j]], i ])!
                                / (Count [PP[n][[j]] , i ]!)^2 ,
                                {i, 1, n}] , {j, 1, Length [ PP[n] ] } ]

Table [ CC [2, j] , {j, 1, 10} ]

(out) {2, 8, 26, 94, 326, 1196, 4358, 16248, 60854, 230184}

Table [ CC [3, j] , {j, 1, 10} ]

(out) {6, 192, 10170, 834612, 90939630, 12360636540, 2012440468938,
381799921738584}
```

Mathematica code 6 for $\mathcal{Z}_{d\,;\text{sym}}(n)$.   The following codes allow us to obtains the sequences (39), (40), and (41) for $\mathcal{Z}_{d\,;\text{sym}}(n)$, for any rank $d \geq 2$ and order $n \geq 1$.

```
X = Array [x , 15]

PP [n_] := IntegerPartitions [n]

Sym [q_  , n_] := Product [ i^(Count [q , i]) Count [q , i] ! , {i , 1, n}]

Symd [X, k_, q_, d_] := Product [ ( X [[ k *l ]] / l )^(Count [q , l])
                                  /( Count [q , l] !) ,
                                  {l, 1, d} ]

Z  [X, t, d_] := Product [ Exp [ ( t^i /i )
                                 * Sum[ Symd [X, i, PP [d][[j]], d],
                                        {j, 1, Length [ PP [d] ]}] ] ,
                           {i, 1, 15} ]
```

```
Zprim [n_, d_] := Coefficient [ Series [ Z [X,  t, d] , {t, 0, n} ] , t^n ]

CC[ n_ , q_ , d_] := Coefficient [ Zprim [n , d] ,
                                   Product [ X[[i]]^( Count [q , i] ) ,
                                   {i, 1, dn} ] ]

Zdsym [n_, d_]  := Sum [ ( CC [n , PP [d n][[i]], d])^2
                          * Sym [ PP [n d][[i]] , dn ] ,
                       {i, 1, Length [ PP [nd] ] } ]

Table [ Zdsym [i, 2] , {i, 1, 13} ]

(out) {1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101}

Table [ Zdsym [i, 3] , {i, 1, 7} ]

(out) {1, 2, 5, 12, 31, 103, 383, 1731}
```

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