A Phragmén–Lindelöf property of viscosity solutions to a class of doubly nonlinear parabolic equations. Bounded case

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Abstract – We study Phragmén–Lindelöf properties for viscosity solutions to a class of nonlinear parabolic equations of the type

$$H(Du, D^2u + Z(u)Du \otimes Du) + \chi(t)|Du|^{\sigma} - u_t = 0$$

under a certain boundedness condition on H. We also prove similar results for positive solutions to a class of doubly nonlinear equation $H(Du, D^2u) - f(u)u_t = 0$.

Contents

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1. Introduction

In this work we study Phragmén–Lindelöf properties of viscosity solutions $u(x, t)$ for a class of nonlinear parabolic, possibly degenerate, equations on the infinite strip $\mathbb{R}_T^n = \mathbb{R}^n \times (0, T)$, where $n \geq 2$ and $0 < T < \infty$. The current work may be viewed as a follow up of the works [5] and [7]. Our goal is to obtain maximum principles in infinite strips under certain growth conditions.

Phragmén–Lindelöf properties of solutions to elliptic and parabolic equations are of great interest. The classical reference see [21, Section 9 in Chapter 2 and Section 6 in Chapter 3] contains a detailed discussion of such properties for solutions of linear uniformly elliptic and parabolic equations. This also includes a discussion of the connection between removable singularities of such equations and the Phragmén–Lindelöf property. A similar discussion appears in see [20, Section 6 in Chapter 1]. See also [23] in this context.

Our work, however, studies equations that are nonlinear and, possibly, degenerate. In addition, the work includes some instances of doubly nonlinear equations which, as it turns out, provide the main motivation for the current work.

Let $g \colon \mathbb{R}^n \to (0, \infty)$ be continuous and $f \colon [0, \infty) \to [0, \infty)$ be an increasing continuous function. Consider

$$(1.1) \qquad H(Du, D^2u) - f(u)u_t = 0 \quad \text{in } \mathbb{R}_T^n$$

with

$$u(x, 0) = g(x) \quad \text{for all } x \in \mathbb{R}^n,$$

where $H = H(q, X)$, defined for any $q \in \mathbb{R}^n$ and X any symmetric $n \times n$ matrix, satisfies certain homogeneity conditions and $u \in C(\mathbb{R}^n \times [0, T))$ is a viscosity solution. See Section 2 for more details.

As noted in [7, Lemma 2.3], if f satisfies certain conditions then there is an increasing function ϕ and a non-increasing function $Z \geq 0$ such that the change of variable $u = \phi(v)$ transforms the differential equation in (1.1) into

$$(1.2) \qquad H(Dv, D^2v + Z(v)Dv \otimes Dv) - v_t = 0 \quad \text{in } \mathbb{R}_T^n,$$

with

$$v(x, 0) = \phi^{-1}(g(x)) \quad \text{for all } x \in \mathbb{R}^n.$$

Thus, the solutions of (1.2) and hence, the solutions of (1.1), satisfy a comparison principle, see [3, 4, 7]. Incidentally, Z may not be defined in all of $\mathbb{R}$, a matter that is taken up later. For the current discussion, we will overlook this issue.

We consider a somewhat more general setting and study Phragmén–Lindelöf type results for equations of the kind

$$(1.3) \qquad H(Dv, D^2v + Z(v)Dv \otimes Dv) + \chi(t)|Dv|^\sigma - v_t = 0 \quad \text{in } \mathbb{R}^n_T,$$

with

$$v(x, 0) = h(x) \quad \text{for all } x \in \mathbb{R}^n,$$

where $\sigma \geq 0$, and $\chi: (0, T) \to \mathbb{R}$ and $h: \mathbb{R}^n \to \mathbb{R}$ are both continuous and bounded. See also [8].

In this work, we assume that $\sup_\lambda[\max_{|e|=1} H(e, \lambda e \otimes e + I)] < \infty$, where e is a unit vector, I is the $n \times n$ identity matrix and λ is a parameter. Under this assumption, H is not uniformly elliptic. However, we will show that under certain growth conditions, the solutions of (1.3) satisfy the maximum principle.

An example of such an operator is $H(X) = \sum_{i=m}^n \mu_i(X)$, $m > 1$, where X is a symmetric $n \times n$ matrix and $\mu_1(X) \geq \mu_2(X) \geq \cdots \geq \mu_n(X)$ are the eigenvalues of X. See Remark 2.1 (ii). Observe that H is fully nonlinear and is of local partial curvature type. These and related operators are of great interest and we direct the reader to the works [2, 9, 10, 11, 15, 16, 17, 18, 24] for a more detailed discussion. See [24, Section 1], in particular.

The case $\sup_\lambda[\min_{|e|=1} H(e, \lambda e \otimes e - I)] = \infty$ is studied in [8]. This class includes instances of uniformly elliptic operators, nonlinear degenerate operators (e.g. the p-Laplacian and the infinity-Laplacian) and some fully nonlinear operators(e.g. the Pucci operator). See related works [1, 4, 6, 7, 12, 14, 19, 22, 23].

The maximum principles derived in this work result from imposing growth rates at infinity. As will be seen, these rates are influenced by the dueling terms $Z(v)(Dv \otimes Dv)$ and $\chi(t)|Dv|^\sigma$ and the power σ. We also show how the sign of χ influences the growth rates, see Section 6.

Another matter to be noted is that since $Z \geq 0$, by ellipticity,

$$H(Du, D^2u) \leq H(Du, D^2u + Z(u)Du \otimes Du).$$

Our work will show that, unlike [8], $Z(s)$ can be allowed to vanish as the value of Z does not influence the bound on $H(e, \lambda e \otimes e \pm I)$.

We have divided our work as follows. In Section 2, we introduce more notation and state the main results. Section 3 contains preliminary calculations and previously proven lemmas, useful for the current work. In sections 4 and 5, we construct super-solutions and sub-solutions respectively. Section 6 addresses some special situations. The proofs of the main results appear in Section 7.

As a final note, we do not address existence and uniqueness questions for such solutions. The question of optimality of the growth rates is also omitted from this work.

2. Hypotheses, notation and the main results

In this section, we introduce notation that are used throughout this work. We address problems (1.1) and (1.3) in infinite strips $\mathbb{R}_T^n$, for $n \geq 2$ and $T > 0$. The letter o denotes the origin in $\mathbb{R}^n$ and e represents a unit vector in $\mathbb{R}^n$. Let $S^{n \times n}$ be the set of all symmetric $n \times n$ real matrices. Let I be the $n \times n$ identity matrix and O the $n \times n$ zero matrix. The expressions usc and lsc stand for *upper semi-continuous* and *lower semi-continuous* respectively.

In this work, sub-solutions, super-solutions and solutions are meant in the sense of viscosity. See Section 3 for definitions. See also [7, 13].

We assume throughout that the operator H satisfies the following conditions.

CONDITION A (monotonicity). The operator $H \colon \mathbb{R}^n \times S^{n \times n} \to \mathbb{R}$ is continuous for any $(q, X) \in \mathbb{R}^n \times S^{n \times n}$. We require that

$$(2.1\mathrm{i}) \qquad H(q, X) \leq H(q, Y) \quad \text{for all } q \in \mathbb{R}^n \text{ and } X, Y \in S^{n \times n} \text{ with } X \leq Y,$$

and

$$(2.1\mathrm{ii}) \qquad H(q, O) = 0 \qquad \text{for all } q \in \mathbb{R}^n.$$

Clearly, for any $q \in \mathbb{R}^n$ and $X \in S^{n \times n}$, $H(q, X) \geq 0$ if $X \geq O$.

CONDITION B (homogeneity). There is a constant $k_1 \geq 0$ such that for any $(q, X) \in \mathbb{R}^n \times S^{n \times n}$,

$$(2.2\mathrm{i}) \qquad\qquad H(\theta q, X) = |\theta|^{k_1} H(q, X) \quad \text{for all } \theta \in \mathbb{R},$$

and

$$(2.2\mathrm{ii}) \qquad\qquad H(q, \theta X) = \theta H(q, X) \qquad \text{for all } \theta > 0.$$

Note that if $k_1 = 0$ then $H(q, X) = H(X)$.

We introduce two quantities before stating the next condition. For any unit vector $e \in \mathbb{R}^n$, note that $(e \otimes e)_{ij} = e_i e_j$, for any $i, j, = 1, 2, \ldots, n$. Moreover, $e \otimes e \geq O$. For $\lambda \in \mathbb{R}$, set

$$(2.3\mathrm{i}) \qquad\qquad \Lambda_{\min}(\lambda) = \min_{|e|=1} H(e, \lambda e \otimes e - I)$$

and

$$(2.3\mathrm{ii}) \qquad\qquad \Lambda_{\max}(\lambda) = \max_{|e|=1} H(e, \lambda e \otimes e + I).$$

By Condition A, both $\Lambda_{\min}(\lambda)$ and $\Lambda_{\max}(\lambda)$ are non decreasing functions of λ.

Define

(2.4i) $$\Lambda^{\mathrm{sup}} = \sup_\lambda \Lambda_{\mathrm{max}}(\lambda)$$

and

(2.4ii) $$\Lambda^{\mathrm{inf}} = \inf_\lambda \Lambda_{\mathrm{min}}(\lambda).$$

CONDITION C (growth at infinity). We impose that

(2.5i) $$\max_{|e|=1} H(e, -I) < 0 < \min_{|e|=1} H(e, I)$$

and

(2.5ii) $$\Lambda^{\mathrm{sup}} < \infty.$$

It follows easily from (2.5ii), Condition A and (2.2ii) of Condition B that

$$H(e, e \otimes e) = 0.$$

NOTE. In this work, requirement (2.5) applies throughout. For some of our results, we require additionally that

$$\Lambda^{\mathrm{inf}} > -\infty.$$

We now present examples of operators H that satisfy conditions A–C and include some observations. Set $k = k_1 + 1$, see (2.7) below.

REMARK 2.1. (i) An example of an H that satisfies conditions A–C is

$$H_p(q, X) = |q|^p \{|q|^2 \operatorname{Tr}(X) - q_i q_j X_{ij}\} \quad \text{for all } p \geq 0, (q, X) \in \mathbb{R}^n \times S^{n \times n},$$

where $\operatorname{Tr}(X)$ is the trace of X. Clearly, H is quasilinear and

$$H_p(Du, D^2 u) = |Du|^p (|Du|^2 \Delta u - \Delta_\infty u).$$

Thus, for any $c \in \mathbb{R}$,

$$\begin{aligned} H_p(q, X + cq \otimes q) &= |q|^p \{|q|^2 \operatorname{Tr}(X) + c|q|^4 - q_i q_j X_{ij} - c|q|^4\} \\ &= H_p(q, X). \end{aligned}$$

In particular,

$$H_p(e, \lambda e \otimes e \pm I) = H_p(e, \pm I) = \pm(n-1) \quad \text{for all } \lambda \in \mathbb{R}.$$

Here $k = k_1 + 1 \geq 1$.

(ii) A second example can be constructed as follows. Let $\mu_i = \mu_i(X)$, for $i = 1, 2, \ldots, n$, be the eigenvalues of a matrix $X \in S^{n \times n}$. We order these as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. Define

$$H_p^m(q, X) = |q|^p \Big(\sum_{i=m}^{n} \mu_i(X) \Big) \quad \text{for all } p \geq 0 \text{ and } 2 \leq m < n.$$

Clearly, H satisfies conditions A and B, $H_p^m(e, \pm I) = \pm(n - m + 1)$ and $k = p + 1$.

We compute $H_p^m(e, \lambda e \otimes e \pm I)$. First, observe that $\det(e \otimes e) = 0$ and $(e \otimes e)^2 = e \otimes e$. Next, $(e \otimes e - \mu I)x = 0$ if and only if $\mu = 0$ (x orthogonal to e) or $\mu = 1$ (x parallel to e) implying that the eigenvalues of $e \otimes e$ are 0, with multiplicity $n - 1$, and 1. Thus, the eigenvalues of $\lambda e \otimes e + I$ are 1 (multiplicity $n - 1$) and $\lambda + 1$. Similarly, the eigenvalues of $\lambda e \otimes e - I$ are -1 (multiplicity $n - 1$) and $\lambda - 1$. Thus,

$$\begin{cases} H_p^m(e, \lambda e \otimes e + I) = n + 1 - m & \text{if } \lambda \geq 0, \\ H_p^m(e, \lambda e \otimes e - I) = \lambda - (n - m + 1) & \text{if } \lambda \leq 0. \end{cases}$$

Theorems 2.2, 2.3, 2.4, 2.5(c), and 2.6(a) given below, hold for this operator. However, if $H(x) = \sum_{i=\ell}^{m} \lambda_i(X)$, $1 < \ell \leq m < n$ then $\max(\Lambda^{\sup}, |\Lambda^{\inf}|) < \infty$ since $H(e, \lambda e \otimes e \pm I) = \pm(m + 1 - \ell)$. In this case, all our results hold.

(iii) If H is odd in X i.e., $H(q, -X) = -H(q, X)$ then (2.5) shows that $\Lambda^{\sup} = -\Lambda^{\inf} < \infty$. All our results hold.

(iv) It is clear from conditions B (see (2.2)) and C (see (2.5ii) i.e., $\Lambda^{\sup} < \infty$) that $H(e, e \otimes e) = 0$. If H is odd then $H(e, \pm e \otimes e) = 0$. Note that, in general, the condition that $H(e, e \otimes e) = 0$ does not imply that $\Lambda^{\sup} < \infty$. As an example, take $H(X) = \det(X)$. Then $H(e \otimes e) = 0$ but $H(\lambda e \otimes e + I) = 1 + \lambda$. Our work does not include such operators.

We introduce some further notation. Let $\chi \colon (0, T) \to \mathbb{R}$ be a bounded continuous function and, for some $m \in \mathbb{R}$ (to be specified later) $Z \colon [m, \infty) \to [0, \infty)$ be a non-increasing continuous function. We point out that there will be instances where we extend Z to $(-\infty, m]$ by $Z(m)$, thus retaining continuity and monotonicity. Since m is in the domain of Z, $Z(m) = \sup_{[m,\infty)} Z < \infty$. However, if Z is defined on an open interval (m, ∞) then Z may be unbounded at m.

For $\sigma \geq 0$, set

(2.6) $\mathcal{P}_\sigma(t, u, u_t, Du, D^2u) = H(Du, D^2u + Z(u)Du \otimes Du) + \chi(t)|Du|^\sigma - u_t.$

Define

(2.7) $k = k_1 + 1$ and $\gamma = k + 1 = k_1 + 2.$

Clearly, $\gamma \geq 2$, and if $k = 1$ then $k_1 = 0$ and $\gamma = 2$. Next, define

(2.8i) $\sigma^* = \dfrac{\sigma}{\sigma - 1}$ for all $\sigma > 1,$

and

(2.8ii) $\gamma^* = \dfrac{\gamma}{k - 1} = \dfrac{\gamma}{\gamma - 2}$ for all $k > 1.$

For a fixed $z \in \mathbb{R}^n$ and for all $x \in \mathbb{R}^n$, set $r = |x - z|$. Define the cylinder $B_T^R = \{(x, t): |x - z| \leq R, \ 0 < t < T\}$. Set P_T^R as its parabolic boundary.

We now state the main results.

CASE I: $k > 1$. Assume that H satisfies conditions A–C, and $\mathcal{P}_\sigma$ be as defined in (2.6). We impose no restrictions on $\Lambda^{\inf}$ (see (2.4ii)) for theorems 2.2 and 2.3. We reiterate that

$$\sigma^* = \frac{\sigma}{\sigma - 1} \quad \text{for all } \sigma > 1$$

and

$$\gamma^* = \frac{\gamma}{\gamma - 2}.$$

Note that $\gamma > 2$.

THEOREM 2.2 (maximum principle). *Let $0 < T < \infty$ and $m \in \mathbb{R}$. Suppose that $h: \mathbb{R}^n \to \mathbb{R}$ is continuous with $h > m$ and $\sup_{\mathbb{R}^n} h(x) < \infty$, and that $Z: [m, \infty) \to [0, \infty)$ is non-increasing and continuous. Let $u \in \text{usc}(\mathbb{R}_T^n)$ be such that $\inf u > m$. Suppose that*

$$\mathcal{P}_\sigma(t, u, u_t, Du, D^2 u) \geq 0 \quad \text{in } \mathbb{R}_T^n,$$

and

$$u(x) \leq h(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Let γ^* and σ^* be as in (2.8). Suppose that $\sup_{B_T^R} u(x, t) = o(R^\delta)$, as $R \to \infty$. Then the following hold:

(a) *if $0 \leq \sigma \leq \gamma/2$ and $\delta = \gamma^*$, then*

$$\sup_{\mathbb{R}_T^n} u(x, t) \leq \begin{cases} \sup_{\mathbb{R}^n} h(x) + (\sup_{[0,T]} |\chi(t)|)t & \text{if } \sigma = 0, \\ \sup_{\mathbb{R}^n} h(x) & \text{if } 0 < \sigma \leq \gamma/2; \end{cases}$$

(b) *if $\sigma > \gamma/2$ and $\delta = \sigma^*$, then*

$$\sup_{\mathbb{R}^n_T} u(x,t) \leq \sup_{\mathbb{R}^n} h(x).$$

Observe that if $m = -\infty$ then the restriction $\inf u > m$ may be dropped. Also, if $\sigma = \gamma/2$ we have that $\sigma^* = \gamma^*$. If $\chi < 0$ and $\sigma \geq k$ for all $k \geq 1$, then Theorem 2.2 holds without any growth rate requirement, see remarks 6.2 and 7.1.

THEOREM 2.3 (minimum principle). *Let $0 < T < \infty$, $h: \mathbb{R}^n \to \mathbb{R}$ be a continuous function, with $\inf_{\mathbb{R}^n} h(x) > -\infty$, and $Z: (-\infty, \infty) \to [0, \infty)$ be a non-increasing continuous function. Let $u \in \mathrm{lsc}(\mathbb{R}^n_T)$ solve*

$$\mathcal{P}_\sigma(t, u, u_t, Du, D^2u) \leq 0 \quad in \ \mathbb{R}^n_T$$

and

$$u(x) \geq h(x) \quad for \ all \ x \in \mathbb{R}^n.$$

Let γ^* and σ^* be as in (2.8). *Suppose that $\sup_{B^R_T}(-u(x,t)) = o(R^\delta)$ as $R \to \infty$. Then the following hold*:

(a) *if $0 \leq \sigma \leq \gamma/2$ and $\delta = \gamma^*$, then*

$$\inf_{\mathbb{R}^n_T} u(x,t) \geq \begin{cases} \inf_{\mathbb{R}^n} h(x) - (\sup_{[0,t]} |\chi(t)|)t & if \ \sigma = 0, \\ \inf_{\mathbb{R}^n} h(x) & if \ 0 < \sigma \leq \gamma/2; \end{cases}$$

(b) *if $\sigma > \gamma/2$ and $\delta = \sigma^*$, then*

$$\inf_{\mathbb{R}^n_T} u(x,t) \geq \inf_{\mathbb{R}^n} h(x).$$

If $\chi > 0$ then the above result holds for $\sigma \geq k$ for all $k \geq 1$, without any growth restrictions. See remarks 6.2 and 7.1.

CASE II: $k = 1$. We now state analogous results for $k = 1$. Assume that H satisfies conditions A–C. Note that $H(q, X) = H(X)$ and $\gamma = 2$.

We clarify that the statement that, for some $s > 0$, $v(r) = e^{o(r^s)}$ as $r \to \infty$, is the same as that $\log v^+ = o(r^s)$ as $r \to \infty$, where $v^+ = \max(v, 0)$.

THEOREM 2.4 (maximum principle). *Let $0 < T < \infty$ and $m \in \mathbb{R}$. Suppose that $h: \mathbb{R}^n \to \mathbb{R}$ is continuous with $h > m$ and $\sup_{\mathbb{R}^n} h(x) < \infty$, and that $Z: [m, \infty) \to [0, \infty)$ is non-increasing and continuous. Let $u \in \mathrm{usc}(\mathbb{R}^n_T)$, $\inf u > m$, solve*

$$H(D^2u + Z(u)Du \otimes Du) + \chi(t)|Du|^\sigma - u_t \geq 0 \quad in \ \mathbb{R}^n_T,$$

and

$$u(x) \leq h(x) \quad for \ all \ x \in \mathbb{R}^n.$$

Let σ^* be as in (2.8). *Then the following hold.*

(a) *Suppose that $\sigma = 0$. If $\sup_{B_T^R} u(x,t) = e^{o(R^2)}$ as $R \to \infty$, then*

$$u(x,t) \leq \sup_{\mathbb{R}^n} h(x) + (\sup_{(0,T)} |\chi(t)|)t \quad \text{for all } (x,t) \in \mathbb{R}_T^n.$$

(b) *Let $0 < \sigma \leq 1$. If $\sup_{B_T^R} u(x,t) = e^{o(R)}$ as $R \to \infty$, then*

$$u(x,t) \leq \sup_{\mathbb{R}^n} h(x) + K(1-\sigma)(\sup_{(0,T)} |\chi(t)|)t,$$

   where $K = K(\alpha, \Lambda^{\sup}, \sigma, T)$.

(c) *Let $1 < \sigma < \infty$ and assume that $\sup_{B_T^R} u(x,t) = o(R^{\sigma^*})$ as $R \to \infty$. Then*

$$u(x,t) \leq \sup_{\mathbb{R}^n} h(x).$$

In part (b), $K = 1$ for $\sigma = 0$, see (4.24) and (4.26).

We now present a minimum principle. Note that the condition $\Lambda^{\inf} > -\infty$ is needed only for parts (a) and (b) of the theorem and is not required for part (c).

THEOREM 2.5 (minimum principle). *Let $0 < T < \infty$, $h \colon \mathbb{R}^n \to \mathbb{R}$ be continuous, with $\inf_{\mathbb{R}^n} h(x) > -\infty$, and $Z \colon (-\infty, \infty) \to [0, \infty)$ be non-increasing and continuous.*

Let $u \in \mathrm{lsc}(\mathbb{R}_T^n)$ solve

$$H(D^2 u + Z(u)Du \otimes Du) + \chi(t)|Du|^\sigma - u_t \leq 0 \quad \text{in } \mathbb{R}_T^n,$$

and

$$u(x) \geq h(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Assume for parts (a) *and* (b) *that $\Lambda^{\inf} > -\infty$. Let σ^* be as in* (2.8). *Then the following hold.*

(a) *Suppose that $\sigma = 0$. If $\sup_{B_T^R}(-u(x,t)) = e^{o(R^2)}$ as $R \to \infty$, then*

$$u(x,t) \geq \inf_{\mathbb{R}^n} h(x) - (\sup_{(0,T)} |\chi(t)|)t \quad \text{for all } (x,t) \in \mathbb{R}_T^n.$$

(b) *Let $0 < \sigma \leq 1$. If $\sup_{B_T^R}(-u(x,t)) = e^{o(R)}$, as $R \to \infty$ then*

$$u(x,t) \geq \inf_{\mathbb{R}^n} h(x).$$

(c) *Let $1 < \sigma < \infty$ and assume that $\sup_{B_T^R}(-u(x,t)) = o(R^{\sigma^*})$ as $R \to \infty$. Then*

$$u(x,t) \geq \inf_{\mathbb{R}^n} h(x).$$

CASE III: DOUBLY NONLINEAR EQUATIONS. Finally, we present analogous results for a class of doubly nonlinear equations of the type

$$H(Du, D^2u) - f(u)u_t = 0 \quad \text{in } \mathbb{R}^n_T,$$

with

$$u(x, 0) = g(x) \quad \text{for all } x \in \mathbb{R}^n.$$

If $k = 1$, i.e., $k_1 = 0$, we assume that $f \equiv 1$ and the equation then reads

$$(2.9) \qquad\qquad H(D^2u) - u_t = 0 \quad \text{in } \mathbb{R}^n_T$$

with

$$u(x, 0) = g(x) \quad \text{for all } x \in \mathbb{R}^n.$$

The above is not doubly nonlinear but is contained in our work. Some of the results, stated here, follow from the theorems stated in cases I and II.

For $k > 1$, let $f : [0, \infty) \to [0, \infty)$ be an increasing C^1 function such that $f^{1/(k-1)}$ is concave. Consider equations of the type

$$(2.10) \qquad\qquad H(Du, D^2u) - f(u)u_t = 0 \quad \text{in } \mathbb{R}^n_T$$

with

$$(2.11) \qquad\qquad u(x, 0) = g(x) \quad \text{for all } x \in \mathbb{R}^n,$$

where $u > 0$.

For $k > 1$, let F be a primitive of $f^{-1/(k-1)}$. Since $f(s) > f(0) \geq 0$ for all $s > 0$, we consider the following two situations:

$$(2.12\text{i}) \qquad\qquad \lim_{\varepsilon \to 0^+} F(1) - F(\varepsilon) < \infty$$

and

$$(2.12\text{ii}) \qquad\qquad \lim_{\varepsilon \to 0^+} F(1) - F(\varepsilon) = \infty.$$

Clearly,

- (2.12i) implies that $\lim_{\varepsilon \to 0^+} F(\varepsilon) > -\infty$ and

- (2.12ii) leads to $\lim_{\varepsilon \to 0^+} F(\varepsilon) = -\infty$,

since F is increasing. A more detailed discussion appears in Remark 3.7 in Section 3.

We continue to assume that H satisfies conditions A–C. Set $\chi \equiv 0$.

THEOREM 2.6. *Let $f: [0, \infty) \to [0, \infty)$ be a C^1 increasing function and $g: \mathbb{R}^n \to (0, \infty)$, continuous, be such that $0 < \inf_x g(x) \leq \sup_x g(x) < \infty$. Recall from* (2.8) *that $\gamma^* = \gamma/(\gamma - 2)$, for $\gamma > 2$ or $k > 1$.*

(a) MAXIMUM PRINCIPLE. *Let $k > 1$ and $f^{1/(k-1)}$ be a concave function. Suppose that $\phi: \mathbb{R} \to [0, \infty)$ is a C^2 increasing function such that $\phi'(\tau) = f(\phi(\tau))^{1/(k-1)}$.*

   If $u \in \mathrm{usc}(\overline{\mathbb{R}^n_T})$, $u > 0$, solves

$$H(Du, D^2 u) - f(u) u_t \geq 0 \quad in \ \mathbb{R}^n_T,$$

   and

$$u(x, 0) \leq g(x) \quad for \ all \ x \in \mathbb{R}^n,$$

   and $\sup_{B^R_T} u(x, t) \leq \phi(o(R^{\gamma^*}))$ as $R \to \infty$, then

$$\sup_{\mathbb{R}^n_T} u(x, t) \leq \sup_{\mathbb{R}^n} g(x).$$

   Let $k = 1$ and $f \equiv 1$, i.e., $H(D^2 u) - u_t \geq 0$. If $\sup_{B^R_T} u(x, t) \leq e^{o(R^2)}$ as $R \to \infty$, then $\sup_{\mathbb{R}^n_T} u(x, t) \leq \sup_{\mathbb{R}^n} g(x)$.

(b) MINIMUM PRINCIPLE. *Let $k > 1$, f, and ϕ be as in part* (a).

   Suppose that $u \in \mathrm{lsc}(\overline{\mathbb{R}^n_T})$, $u > 0$, solves

$$H(Du, D^2 u) - f(u) u_t \leq 0 \quad in \ \mathbb{R}^n_T$$

   and

$$u(x, 0) \geq g(x) \quad for \ all \ x \in \mathbb{R}^n.$$

   If condition (2.12i) *holds, i.e., $\lim_{\varepsilon \to 0+} F(1) - F(\varepsilon) < \infty$ and $\Lambda^{\inf} > -\infty$, then*

$$u(x, t) \geq \inf_{\mathbb{R}^n} g(x) \quad for \ all \ (x, t) \in \mathbb{R}^n_T.$$

   If condition (2.12ii) *holds, i.e., $\lim_{\varepsilon \to 0+} F(1) - F(\varepsilon) = \infty$, and if one has $\inf_{B^R_T} u(x, t) \geq \phi(-o(R^{\gamma^*}))$ as $R \to \infty$ then*

$$u(x, t) \geq \inf_{\mathbb{R}^n} g(x) \quad for \ all \ (x, t) \in \mathbb{R}^n_T.$$

   Suppose that $k = 1$ and $f \equiv 1$, i.e., $H(D^2 u) - u_t \leq 0$. If one has $\inf_{B^R_T} u(x, t) \geq -e^{o(R^2)}$ as $R \to \infty$, then

$$u(x, t) \geq \inf_{\mathbb{R}^n} g(x) \quad for \ all \ (x, t) \in \mathbb{R}^n_T.$$

Note that in the above theorem, $F(\phi(\tau)) - F(\phi(\tau_0)) = \tau - \tau_0$, where F is a primitive of $f^{-1/(k-1)}$. For $k = 1$, g can have any sign in both parts (a) and (b). Also, part (b) holds if $g > 0$ and if $u \geq 0$ and satisfies a somewhat different lower bound, see Remark 7.2.

3. Further definitions and preliminaries

In this section, we present additional definitions, lemmas and remarks we use to prove the main results. Fix $z \in \mathbb{R}^n$ and set $r = |x - z|$ for all $x \in \mathbb{R}^n$. A unit vector in $\mathbb{R}^n$ is denoted by $e = (e_1, e_2, \ldots, e_n)$. Recall $\mathbb{R}^n_T = \{(x, t) : x \in \mathbb{R}^n, 0 < t < T\}$. We set $\overline{\mathbb{R}^n_T} = \{(x, t) : x \in \mathbb{R}^n, 0 \leq t < T\}$.

We begin by introducing the definitions of viscosity sub-solutions and super-solutions of (1.3). We say that $u \in \mathrm{usc}(\mathbb{R}^n_T)$ is a viscosity sub-solution of (1.3) in $\mathbb{R}^n_T$, if, for any ψ, C^2 in x and C^1 in t, and $(u - \psi)(x, t) \leq (u - \psi)(y, s)$ for some $(y, s) \in \mathbb{R}^n_T$, then

$$H(D\psi, D^2\psi + Z(u)D\psi \otimes D\psi)(y, s) + \chi(s)|D\psi|^\sigma(y, s) - (\psi_t)(y, s) \geq 0.$$

A function $v \in \mathrm{lsc}(\mathbb{R}^n_T)$ is a super-solution of (1.3) in $\mathbb{R}^n_T$, if, for any ψ, C^2 in x and C^1 in t, with $(v - \psi)(x, t) \geq (v - \psi)(y, s)$ for some $(y, s) \in \mathbb{R}^n_T$, we have that

$$H(D\psi, D^2\psi + Z(v)D\psi \otimes D\psi)(y, s) + \chi(s)|D\psi|^\sigma(y, s) - (\psi_t)(y, s) \leq 0.$$

Set $T_\sigma(t, r, q, X) = H(q, X + Z(r)q \otimes q) + \chi(t)|q|^\sigma$. We say that $u \in \mathrm{usc}(\overline{\mathbb{R}^n_T})$ is a sub-solution of the initial value problem in (1.3) if

$$T_\sigma(t, u, Du, D^2u) - u_t \geq 0 \quad \text{in } \mathbb{R}^n_T$$

and

$$u(x, 0) \leq h(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Similarly, $v \in \mathrm{lsc}(\overline{\mathbb{R}^n_T})$ is a super-solution of the initial value problem in (1.3) if

$$T_\sigma(t, v, Dv, D^2v) - v_t \leq 0 \quad \text{in } \mathbb{R}^n_T$$

and

$$v(x, 0) \geq h(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Lastly, u is a solution in $\mathbb{R}^n_T$ if it is both a sub-solution and a super-solution, and a solution of (1.3) if it is both a sub-solution and a super-solution of (1.3).

We state an elementary remark that is used frequently in our work. Let $z \in \mathbb{R}^n$.

REMARK 3.1. Let $T > 0$ and $\kappa: [0, T) \to [0, \infty)$ be a C^1 function. Suppose that $w: \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a C^1 function of x and t. It is C^2 in x except, perhaps, at $x \neq z$. In what follows, we take $w(x, t) = w(r, t)$, where $r = |x - z|$. Suppose that $w_r(x, t) \neq 0$ for all $r > 0$. Note that $w_r(0, t) = 0$. Thus, for $r \neq 0$,

$$Dw = w_r e \quad \text{and} \quad D^2 w = \frac{w_r}{r}(I - e \otimes e) + w_{rr} e \otimes e,$$

where $e = (e_1, e_2, \ldots, e_n) = (x - z)/r$. Hence, for $r \neq 0$,

$$(3.1) \quad \begin{aligned} &H(Dw, D^2 w + Z(w) Dw \otimes Dw) \\ &= H\left(w_r e, \left(\frac{w_r}{r}\right) I + \left(w_{rr} - \frac{w_r}{r} + (w_r)^2 Z(w)\right) e \otimes e\right). \end{aligned}$$

CASE (a): $w_r > 0$. We apply Condition B, i.e. (2.2), to (3.1) and factor w_r from the first entry, w_r/r from the second and use $k = k_1 + 1$ to get

$$(3.2) \quad \begin{aligned} &H(Dw, D^2 w + Z(w) Dw \otimes Dw) \\ &= \left(\frac{w_r^k}{r}\right) H\left(e, I + \left(\frac{r w_{rr}}{w_r} - 1 + r w_r Z(w)\right) e \otimes e\right) \quad \text{for all } r > 0. \end{aligned}$$

If $w(x, t) = \kappa(t) v(r)$, with $v'(r) > 0$, then (3.2) implies that, in $r > 0$,

$$(3.3) \quad \begin{aligned} &H(Dw, D^2 w + Z(w) Dw \otimes Dw) \\ &= \frac{(\kappa(t) v'(r))^k}{r} H\left(e, I + \left(\frac{r v''(r)}{v'(r)} - 1 + r \kappa(t) v'(r) Z(w)\right) e \otimes e\right). \end{aligned}$$

CASE (b): $w_r < 0$. An application of Condition B to (3.1) leads to

$$(3.4) \quad \begin{aligned} &H(Dw, D^2 w + Z(w) Dw \otimes Dw) \\ &= \frac{|w_r|^k}{r} H\left(e, \left(1 - \frac{r w_{rr}}{w_r} + r |w_r| Z(w)\right) e \otimes e - I\right) \quad \text{for all } r > 0. \end{aligned}$$

If $w(x, t) = \kappa(t) v(r)$, with $v'(r) < 0$, then (3.4) leads to the following analogue of (3.3):

$$(3.5) \quad \begin{aligned} &H(Dw, D^2 w + Z(w) Dw \otimes Dw) \\ &= \frac{(\kappa(t) |v'(r)|)^k}{r} H\left(e, \left(r |v'(r)| \kappa(t) Z(w) + 1 - \frac{r v''(r)}{v'(r)}\right) e \otimes e - I\right). \end{aligned}$$

We now discuss an auxiliary function that is used in the work. See also [8].

LEMMA 3.2. *Let β and $\bar{\beta}$ be such that $1 < \bar{\beta} < \beta$ and $R > 0$. Fix $z \in \mathbb{R}^n$, set $r = |x - z|$ and*

$$v(r) = \int_0^{r^\beta} \frac{1}{1 + \tau^p} \, d\tau \quad \text{for all } r \geq 0,$$

where

$$p = \frac{\beta - \bar{\beta}}{\beta}.$$

Then

(i) $$0 < p < 1,$$

(ii) $$(1 - p)\beta = \bar{\beta},$$

and

(iii) $$\frac{r^\beta}{1 + r^{\beta p}} \leq v(r) \leq \min\left(r^\beta, \frac{\beta r^{\bar{\beta}}}{\bar{\beta}}\right) \quad \text{for all } r \geq 0.$$

If $R \geq 1$ then

(iv) $$\frac{\beta(r^{\bar{\beta}} - R^{\bar{\beta}})}{2\bar{\beta}} \leq v(r) - v(R) \leq \frac{\beta(r^{\bar{\beta}} - R^{\bar{\beta}})}{\bar{\beta}} \quad \text{for all } r \geq R.$$

Moreover,

(v) $$v'(r) = \frac{\beta r^{\beta - 1}}{1 + r^{p\beta}} \leq \beta \min\left(r^{\bar{\beta} - 1}, r^{\beta - 1}\right) \quad \text{for all } r \geq 0,$$

(vi) $$\frac{(v'(r))^k}{r} \leq \beta^k \min\left(r^{k\beta - \gamma}, r^{k\bar{\beta} - \gamma}\right) \quad \text{for all } r > 0,$$

and

(vii) $$v''(r) = \beta r^{\beta - 2}\left(\frac{(\beta - 1) + (\bar{\beta} - 1)r^{p\beta}}{(1 + r^{p\beta})^2}\right) \quad \text{for all } r > 0.$$

NOTE. Parts (iii) and (iv) of Lemma 3.2 show that $v(r)$ grows like r^β near $r = 0$ and like $r^{\bar{\beta}}$ for large values of r. Since $\beta \geq \bar{\beta}$, one can design the function to decay fast enough at $r = 0$ so that $(v'(r))^k/r$ exists at $r = 0$, while the growth rate for large values of r may be slower.

PROOF. Parts (i)–(iii) follow quite readily (use $(1 - p) = \bar{\beta}/\beta$). For part (iv), we take $R \geq 1$ and write

$$v(r) = \int_0^{r^\beta} (1 + \tau^p)^{-1} \, d\tau = v(R) + \int_{R^\beta}^{r^\beta} (1 + \tau^p)^{-1} d\tau$$

In the second integral, we use $(2\tau^p)^{-1} \leq (1 + \tau^p)^{-1} \leq \tau^{-p}$, for $\tau \geq 1$. This leads to part (iv). For part (v), use part (ii) and note that $1 + r^{p\beta} \geq \min(1,\ r^{p\beta})$. Part (vi) follows by recalling that $\gamma = k + 1 = k_1 + 2$.

Next,

$$v''(r) = \beta \left[\frac{(\beta - 1)r^{\beta-2}}{1 + r^{p\beta}} - \frac{p\beta r^{p\beta+\beta-2}}{(1 + r^{p\beta})^2} \right].$$

A simple calculation leads to part (vii). $\qquad\qquad\qquad\qquad\qquad\square$

In the following remark, we choose specific values for β and $\bar{\beta}$ and apply the above lemma. These values arise from the work in sections 4 and 5. Recall that $\gamma = k + 1$ and $k = k_1 + 1$.

REMARK 3.3. For Part I (iv) in Section 4, we take $k > 1$ (i.e., $\gamma > 2$) and $\sigma > \gamma/2$. We set

$$\beta = \gamma^* = \gamma/(\gamma - 2) \quad \text{and} \quad \bar{\beta} = \sigma^* = \sigma/(\sigma - 1),$$

and define

$$v(r) = \int\limits_{0}^{r^{\gamma^*}} \frac{1}{1 + \tau^p}\, d\tau, \quad \text{where } p = 1 - \frac{\bar{\beta}}{\beta} = 1 - \frac{\sigma^*}{\gamma^*} = \frac{2\sigma - \gamma}{\gamma(\sigma - 1)}.$$

From Lemma 3.2,

(i) $0 < p < 1$,

(ii) $(1 - p)\gamma^* = \sigma^*$,

(iii) for $r \geq 0$,

$$\frac{r^{\gamma^*}}{1 + r^{\gamma^*p}} \leq v(r) \leq \min\left(r^{\gamma^*}, \frac{\gamma^* r^{\sigma^*}}{\sigma^*} \right),$$

(iv) for any $R \geq 1$,

$$\frac{\gamma^*}{2\sigma^*} \leq \frac{v(r) - v(R)}{r^{\sigma^*} - R^{\sigma^*}} \leq \frac{\gamma^*}{\sigma^*} \quad \text{for all } r \geq R.$$

Moreover,

(v) $\qquad v'(r) = \dfrac{\gamma^* r^{\gamma^*-1}}{1 + r^{p\gamma^*}} \leq \gamma^* \min(r^{\sigma^*-1},\ r^{\gamma^*-1}) \quad \text{for all } r \geq 0,$

(vi) $\qquad \dfrac{(v'(r))^k}{r} \leq (\gamma^*)^k \min(r^{k\gamma^*-\gamma},\ r^{k\sigma^*-\gamma})$

$$= (\gamma^*)^k \min(r^{(\gamma-\sigma)/(\sigma-1)},\ r^{\gamma^*}), \quad \text{for all } r > 0,$$

and

(vii) $v''(r) = \gamma^* r^{\gamma^*-2}\left(\dfrac{(\gamma^*-1) + (\sigma^*-1)r^{p\gamma^*}}{(1 + r^{p\gamma^*})^2}\right)$ for all $r > 0$.

REMARK 3.4. The super-solutions and sub-solutions in this work involve a C^1 function of t and a radial function $v(r) \in C^{1,\alpha}$ (for some $\alpha > 0$) in x. Moreover, the radial functions $v(r)$ are C^∞ in x, except, perhaps, at $r = 0$. Also, $v'(0) = 0$ and $v'(r) \neq 0$ in $r \neq 0$. We show that the auxiliary functions satisfy the definitions at $r = 0$.

We consider only the case $k > 1$. The auxiliary functions for $k = 1$ are C^2 in x, for any x. See Part II in Section 4.

Let $\kappa(t) \geq 0$ be a C^1 function in $t \geq 0$. Set $r = |x - z|$ and $w(x,t) = \kappa(t)v(r)$. Recalling Remark 3.3 (refer also to Part I (i)–(iv) in Section 4), we see that

(3.6) $v(r) = \begin{cases} r^{\gamma^*} & \text{if } 0 \leq \sigma \leq \gamma/2, \\ \displaystyle\int_0^{r^{\gamma^*}} (1 + \tau^p)^{-1} d\tau & \text{if } \sigma > \gamma/2, \ p = 1 - \sigma^*/\gamma^*. \end{cases}$

These are used to obtain sub-solutions and super-solutions.

Note that in (3.6),

$$k > 1, \quad \gamma = k + 1 > 2, \quad \gamma^* = \frac{\gamma}{\gamma - 2} = \frac{k+1}{k-1}, \quad \sigma^* = \frac{\sigma}{\sigma - 1}.$$

We observe that, in $r > 0$ (see Remark 3.3 (v)),

$$\frac{(v'(r))^k}{r} = \begin{cases} (\gamma^*)^k r^{\gamma^*} & \text{if } 0 \leq \sigma \leq \gamma/2, \\ (\gamma^*)^k \dfrac{r^{\gamma^*}}{(1 + r^{p\gamma^*})^k} & \text{if } \sigma > \gamma/2, \ p = 1 - \sigma^*/\gamma^*, \end{cases}$$

since $k(\gamma^* - 1) - 1 = k\gamma^* - \gamma = \gamma^*$. Set $J(r) = (v'(r))^k/r$. Clearly, in both cases,

(3.7) $J(0) = \displaystyle\lim_{r \to 0} \frac{(v'(r))^k}{r} = 0$

Next, using Remark 3.3 (vii),

$$\frac{rv''(r)}{v'(r)} = \begin{cases} \gamma^* - 1 & \text{if } 0 \leq \sigma \leq \gamma/2, \\ \dfrac{\{(\gamma^*-1) + (\sigma^*-1)r^{p\gamma^*}\}}{1 + r^{p\gamma^*}} & \text{if } \sigma > \gamma/2, \ p = 1 - \sigma^*/\gamma^*. \end{cases}$$

The above is continuous at $r = 0$ by extension. Clearly,

$$(3.8) \qquad \lim_{r \to 0} \frac{r v''(r)}{v'(r)} - 1 = \gamma^* - 2.$$

Recall (3.1) and (3.3) in Remark 3.1. Taking $r > 0$, setting $e = x/r$ and $w = \kappa(t) v(r)$, where κ is C^1, we get after a slight rearrangement

$$(3.9)$$
$$H(Dw, D^2 w + Z(w) Dw \otimes Dw) + \chi(t) |Dw|^\sigma - w_t$$
$$= \chi(t) (\kappa(t))^\sigma (v'(r))^\sigma - \kappa'(t) v(r)$$
$$+ \kappa(t)^\kappa J(r) H \left(e, \kappa(t) (r v'(r)) Z(w) e \otimes e + \left\{ I + \left(\frac{r v''(r)}{v'(r)} - 1 \right) e \otimes e \right\} \right).$$

By employing the observations made above, it is clear that the right hand side of (3.16) may be extended continuously to $r = 0$. Observing that $v(0) = 0$, $v'(0) = 0$ and recalling (3.7) and (3.8), set the expression for the right hand side of (3.9), at $r = 0$, as

$$\chi(t) L(\sigma) + \widehat{H}(0),$$

where

$$L(\sigma) = \begin{cases} 1 & \text{if } \sigma = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\widehat{H}(0) = \kappa(0)^\kappa J(0) H(e, I + (\gamma^* - 2) e \otimes e).$$

However, $\widehat{H}(0) = 0$. This follows from (3.7).

Our goal is to show that at any $(0, s)$, $s > 0$,

$$(3.10) \qquad H(Dw, D^2 w + Z(w) Dw \otimes Dw) + \chi(t) |Dw|^\sigma - w_t = \chi(s) L(\sigma)$$

holds in the viscosity sense.

Suppose that ψ, C^1 in t and C^2 in x, is such that $(w - \psi)(x, t) \le (w - \psi)(o, s)$, for (x, t) near (o, s). Since $v(0) = 0$, we obtain that

$$\kappa(t) v(r) \le \langle D\psi(o, s), x \rangle + \psi_t(o, s)(t - s) + o(|x| + |t - s|) \quad \text{as } (x, t) \to (o, s).$$

Since $v(0) = 0$ and $v'(0) = 0$, we have that $\psi_t(o, s) = 0$ and $D\psi(o, s) = 0$. Using these values and Condition B ($k_1 > 0$), we get that

$$H(D\psi, D^2 \psi + Z(w) D\psi \otimes D\psi)(0, s) + \chi(s) |D\psi|^\sigma(0, s) - \psi_t(0, s) = \chi(s) L(\sigma),$$

since $H(0, X) = 0$ for all $X \in S^{n \times n}$. Hence w is a sub-solution of (3.10).

Now, let ψ, C^1 in t and C^2 in x, be such that $(w - \psi)(x, t) \geq (w - \psi)(o, s)$, for (x, t) near (o, s). Thus,

$$\kappa(t)v(r) \geq \langle D\psi(o, s), x \rangle + \psi_t(o, s)(t - s) + o(|x| + |t - s|),$$

as $(x, t) \to (o, s)$. As argued before, $D\psi(o, s) = 0$ and $\psi_t(o, s) = 0$. Hence,

$$\begin{aligned} H(D\psi, D^2\psi &+ Z(\psi)D\psi \otimes D\psi)(o, s) + \chi(s)|D\psi|^\sigma(o, s) - \psi_t(o, s) \\ &= H(0, D^2\psi)(o, s) + \chi(s)L(\sigma) = \chi(s)L(\sigma). \end{aligned}$$

Clearly, w is a super-solution of (3.10). By using (3.5) in Remark 3.1, a similar argument can be used in the case of the sub-solutions.

Next, we recall a comparison principle needed for our work, see [13, Theorem 8.2]. See also [7] and [8].

For $(t, r, q, X) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^n \times S^{n \times n}$, set

$$(3.11) \qquad F_\sigma(t, r, q, X) = H(q, X + Z(r)q \otimes q) + \chi(t)|q|^\sigma,$$

where H satisfies conditions A–C, Z is a non-increasing continuous function and $\sigma \geq 0$. Then F_σ is continuous and satisfies, for all $X, Y \in S^{n \times n}$ with $X \leq Y$,

$$(3.12) \quad F_\sigma(t, r_1, q, X) \leq F_\sigma(t, r_2, q, Y), \quad \text{for all } (t, q) \in \mathbb{R}^+ \times \mathbb{R}^n \text{ and } r_1 \geq r_2.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\Omega_T = \Omega \times (0, T)$ and P_T be the parabolic boundary of Ω_T.

LEMMA 3.5 (comparison principle). *Let F_σ be as in (3.11) and $\hat{f} : \mathbb{R}^+ \to \mathbb{R}^+$ be a bounded continuous function. For some m, let $Z : [m, \infty) \to \mathbb{R}$ be a non-increasing continuous function.*

Let $u \in \mathrm{usc}(\Omega_T \cup P_T)$ and $v \in \mathrm{lsc}(\Omega_T \cup P_T)$ be such that $\min(\inf u, \inf v) > m$. Suppose that u and v solve

$$(3.13\mathrm{i}) \qquad\qquad F_\sigma(t, u, Du, D^2u) - \hat{f}(t)u_t \geq 0$$

and

$$(3.13\mathrm{ii}) \qquad\qquad F_\sigma(t, v, Dv, D^2v) - \hat{f}(t)v_t \leq 0$$

in Ω_T. If $u \leq v$ on P_T, then $u \leq v$ in Ω_T.

PROOF. We adapt the proof of Theorem 8.2 in [13] and provide an outline of the ideas involved.

For $\varepsilon > 0$, set $v_\varepsilon = v + \varepsilon/(T - t)$. Thus, $\inf(u, v_\varepsilon) > m$ and $Z(v_\varepsilon) \leq Z(v)$. Using (3.12), (3.13), and referring to the proof of Theorem 8.2 in [13],

(3.14)
$$F_\sigma(t, v_\varepsilon, Dv_\varepsilon, D^2 v_\varepsilon) - \hat{f}(t)(v_\varepsilon)_t \leq F_\sigma(t, v, Dv, D^2 v) - \hat{f}(t)\Big(v_t + \frac{\varepsilon}{(T-t)^2}\Big)$$
$$\leq -\frac{\varepsilon \hat{f}(t)}{(T-t)^2}$$
$$< 0.$$

Note that $u \leq v_\varepsilon$ on P_T.

Rest of the proof is the same as in [13] (see (8.10), in particular). Suppose that for $\varepsilon > 0$, small enough, $u - v_\varepsilon > 0$ some where in Ω_T. It can then be shown that there are points $(\hat{t}, \hat{x})$ and $(\hat{t}, \hat{y})$ in Ω_T(depending on α and ε), scalars a and b with $a = b$, an $\alpha > 0$ large enough, and $\hat{X}, \hat{Y} \in S^{n \times n}$(depending on α and ε) with $\hat{X} \leq \hat{Y}$ such that $u(\hat{t}, \hat{x}) > v_\varepsilon(\hat{t}, \hat{y})$,

(3.15i) $\qquad F_\sigma(\hat{t}, u(\hat{x}, \hat{t}), \alpha(\hat{x} - \hat{y}), \hat{X}) - a\hat{f}(\hat{t}) \geq 0$

and

(3.15ii) $\qquad F_\sigma(\hat{t}, v_\varepsilon(\hat{y}, \hat{t}), \alpha(\hat{x} - \hat{y}), \hat{Y}) - b\hat{f}(\hat{t}) \leq -\frac{\varepsilon \hat{f}(\hat{t})}{(T - \hat{t})^2},$

where we have used (3.13) and (3.14). Noting that $u(\hat{t}, \hat{x}) > v_\varepsilon(\hat{t}, \hat{y})$, $\hat{X} \leq \hat{Y}$, we use (3.12) to conclude that

$$F_\sigma(\hat{t}, u(\hat{x}, \hat{t}), \alpha(\hat{x} - \hat{y}), X) - a\hat{f}(\hat{t}) \leq F_\sigma(\hat{t}, v_\varepsilon(\hat{y}, \hat{t}), \alpha(\hat{x} - \hat{y}), Y) - b\hat{f}(\hat{t}).$$

This contradicts (3.15). Thus, $u \leq v_\varepsilon$ in Ω_T and the lemma follows by letting $\varepsilon \to 0$. $\qquad\square$

REMARK 3.6. The proof in [13] uses $u_\varepsilon = u - \varepsilon/(T - t)$. In our case, since the domain of Z is $[m, \infty)$, we extend Z to all of $(-\infty, \infty)$ by defining $Z(\theta) = Z(m)$ in $-\infty < \theta \leq m$. In case, the domain of Z is an open set (m, ∞) then we use v_ε as Z may be unbounded at m. Hence, Lemma 3.5 holds in both cases.

Next, we discuss a change of variables that is used in the proof of Theorem 2.6 for doubly nonlinear equations of the kind

(3.16) $\qquad\qquad H(Du, D^2 u) - f(u)u_t = 0 \quad \text{in } \mathbb{R}^n_T,$

with $u > 0$ such that $u(x, 0) = g(x)$ for all $x \in \mathbb{R}^n$.

REMARK 3.7. Let $f : [0, \infty) \to [0, \infty)$ be an increasing C^1 function. Suppose that $k > 1$ and $f^{1/(k-1)}$ is concave. See [7, Lemma 2.3].

The interval I denotes either the interval $[0, \infty)$ or the interval $(-\infty, \infty)$, see (b) and (c) below. We select $\phi : I \to [0, \infty)$, an increasing C^2 function, such that

$$\phi'(\tau) = f(\phi(\tau))^{1/(k-1)} \quad \text{for all } \tau \in I$$

or

$$\int_{\phi(\tau_0)}^{\phi(\tau)} f^{-1/(k-1)}(\theta) \, d\theta = \tau - \tau_0.$$

We define the change of variable $u = \phi(v)(u \geq 0)$ by

$$(3.17) \qquad \phi^{-1}(u) = v(u) = v(u_0) + \int_{u_0}^{u} f^{-1/(k-1)}(\theta) \, d\theta, \quad u \geq u_0,$$

for some $u_0 \geq 0$.

We discuss some examples. Let $\alpha > 0$, $\delta \geq 0$ and $f(s) = (s + \delta)^\alpha$ for all $s \geq 0$. Then $f(s)^{1/(k-1)}$ is concave if $0 < \alpha \leq k - 1$. Set $c_k = 1 - \alpha/(k-1)$. We take $u_0 = 0$ in (3.17) and set $v(0) = 0$ to get that

$$u = \phi(v) = \begin{cases} [c_k v + \delta^{c_k}]^{1/c_k} - \delta & \text{for } 0 < \alpha < k - 1, \delta \geq 0, \\ \delta e^v - \delta & \text{for } \alpha = k - 1, \delta > 0. \end{cases}$$

See also part (b) below. If $\delta = 0$ then $f(s) = s^{k-1}$ and $u = \hat{k} e^v$ for any $\hat{k} > 0$. But, $u_0 \neq 0$, see part (c).

We make some observations about (3.17).

(a) It is clear that v is an increasing concave function of u. The concavity follows since f is non-decreasing. Since v is increasing, u is a convex function of v.

(b) If the integral in (3.17) is convergent for $u_0 = 0$ we then define

$$v = \phi^{-1}(u) = \int_0^u f^{-1/(k-1)}(\theta) \, d\theta.$$

Thus, $v(0) = 0$ and $v \geq 0$. We choose $I = [0, \infty)$ and $\phi : [0, \infty) \to [0, \infty)$. This applies to examples like

$$f(s) = \begin{cases} s^\alpha & \text{for } 0 \leq \alpha < k - 1, \\ (s + \delta)^\alpha & \text{for } 0 \leq \alpha \leq k - 1, \end{cases}$$

where $\delta > 0$.

(c) If the integral in (3.17) is divergent for $u_0 = 0$ then $v(u_0) \to -\infty$ as $u_0 \to 0^+$. In this case, we select a primitive

$$v = \phi^{-1}(u) = \int^u f^{-1/(k-1)}(\theta) \, d\theta, \quad u > 0.$$

We choose $I = (-\infty, \infty)$ and $\phi: (-\infty, \infty) \to (0, \infty)$. This includes examples such as $f(s) = s^{k-1}$, $(s + \log(s+1))^{k-1}$ etc.

(d) We show that, in parts (b) and (c), $v \to \infty$ if $u \to \infty$. Set $v(s) = f^{1/(k-1)}(s)$. Since $v(s)$ is concave in $(0, \infty)$, it is clear that, for a fixed $\varepsilon > 0$,

$$v(s) \le v(\varepsilon) + (s - \varepsilon)v'(\varepsilon), \quad s \ge \varepsilon.$$

Using (3.17), we get that

$$v(u) = v(\varepsilon) + \int_\varepsilon^u \frac{1}{v(s)} ds \ge v(\varepsilon) + \int_\varepsilon^u \frac{1}{v(\varepsilon) + (s - \varepsilon)v'(\varepsilon)} ds.$$

The claim holds.

(e) It is clear from (3.17) that

$$\frac{\phi''(v)}{\phi'(v)} = \left. \left(\frac{d}{ds} f^{1/(k-1)}(s) \right) \right|_{\phi(v)},$$

and $\phi''(v)/\phi'(v)$ is non-increasing in v since $f^{1/(k-1)}$ is concave and $\phi(v)$ is increasing in v. Suppose that there are constants $0 < \omega_1 \le \omega_2 < \infty$ such that

(3.18) $$\omega_1 \le \frac{\phi''(v)}{\phi'(v)} \le \omega_2.$$

Integrating from $s = 0$ to any $s > 0$, we get that,

$$\omega_1 s \le f^{1/(k-1)}(s) - f^{1/(k-1)}(0) \le \omega_2 s \quad \text{for all } s \ge 0.$$

Since $f(0) \ge 0$, we get that, for some $\omega \ge 0$,

$$(\omega_1 s + \omega)^{k-1} \le f(s) \le (\omega_2 s + \omega)^{k-1} \quad \text{for all } s \ge 0.$$

If $\omega > 0$ then we use v as in part (b). If $\omega = 0$ then we use part (c).

(f) The change of variable $u = \phi(v)$, as given by (3.17), transforms (3.16) into

$$H(Dv, D^2v + Z(v)Dv \otimes Dv) - v_t = 0 \quad \text{in } \mathbb{R}_T^n$$

with

$$v(x, 0) = \phi^{-1}(g(x)) \quad \text{for all } x \in \mathbb{R}^n,$$

where $Z(v) = \phi''(v)/\phi'(v)$, see [7, Lemma 2.3]. By part (e), $Z(v)$ is non-increasing in v. In part (b), the domain of Z is $(0, \infty)$ or $[0, \infty)$. It is $(-\infty, \infty)$ in part (c).

We now state a comparison principle for doubly nonlinear equations. The statement includes the case $k = 1$, i.e., the equation $H(D^2u) - u_t = 0$. We assume that H satisfies conditions A–C.

LEMMA 3.8. *Let $T > 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\Omega_T = \Omega \times (0, T)$ and P_T be its parabolic boundary. Suppose that $u \in \text{usc}(\Omega_T \cup P_T)$ and that $v \in \text{lsc}(\Omega_T \cup P_T)$.*

(a) *Suppose that $k > 1$ and $f : [0, \infty) \to [0, \infty)$ is a non-decreasing C^1 function such that $f^{1/(k-1)}$ is concave. Let $u > 0$ and $v > 0$ solve*

$$H(Du, D^2u) - f(u)u_t \geq 0 \quad \text{and} \quad H(Dv, D^2v) - f(v)v_t \leq 0$$

in Ω_T. If $u \leq v$ on P_T, then $u \leq v$ in Ω_T.

(b) *Suppose that $k = 1$. If u and v solve $H(D^2u) - u_t \geq 0$ and $H(D^2v) - v_t \leq 0$ in Ω_T, respectively, and $u \leq v$ on P_T, then $u \leq v$ in Ω_T.*

PROOF. We employ Lemma 3.5 and Remark 3.7. Let u and v be as in the statement of the theorem. We begin with the case $k > 1$ and prove part (a). Set

$$G(u, s) = \int_s^u f^{-1/(k-1)}(\theta)d\theta \quad \text{for all } u \geq s \geq 0.$$

We define $G(u, 0) = \lim_{s \to 0+} G(u, s)$, if it exists.

(i) Suppose that $G(1, 0) < \infty$ then we define

$$\bar{u} = \phi^{-1}(u) = G(u, 0) \quad \text{and} \quad \bar{v} = \phi^{-1}(v) = G(v, 0).$$

By parts (a) and (b) of Remark 3.7, $\bar{u} > 0$ and $\bar{v} > 0$. Also, by part (f) of Remark 3.7,

$$H(D\bar{u}, D^2\bar{u} + Z(\bar{u})D\bar{u} \otimes D\bar{u}) - \bar{u}_t \geq 0$$

and

$$H(D\bar{v}, D^2\bar{v} + Z(\bar{v})D\bar{v} \otimes D\bar{v}) - \bar{v}_t \leq 0,$$

in Ω_T, where $Z(s) = \phi''(s)/\phi'(s)$ is non-increasing in s and the domain of Z contains $(0, \infty)$. Since, $\bar{u} \leq \bar{v}$ on P_T, applying Lemma 3.5, $\bar{u} \leq \bar{v}$ in Ω_T. This implies that $u \leq v$ in Ω_T.

(ii) Suppose now that $G(1, 0)$ is divergent, see part (c) of Remark 3.7. Fix a primitive

$$\widehat{G}(s) = \int^s f^{-1/(k-1)}(\theta)d\theta, \quad s > 0.$$

Define

$$\bar{u} = \phi^{-1}(u) = \widehat{G}(u) \quad \text{and} \quad \bar{v} = \phi^{-1}(v) = \widehat{G}(v).$$

Then $-\infty < \bar{u}, \bar{v} < \infty$, and by parts (e) and (f) of Remark 3.7, we get in Ω_T,

$$H(D\bar{u}, D^2\bar{u} + Z(\bar{u})D\bar{u} \otimes D\bar{u}) - \bar{u}_t \geq 0$$

and

$$H(D\bar{v}, D^2\bar{v} + Z(\bar{v})D\bar{v} \otimes D\bar{v}) - \bar{v}_t \leq 0,$$

where the domain of Z is $(-\infty, \infty)$. Using Lemma 3.5, $\bar{u} \leq \bar{v}$ in Ω_T thus implying that $u \leq v$ in Ω_T.

For the case $k = 1$, a direct application of Lemma 3.5 (take $Z \equiv 0$) leads to the conclusion. $\qquad\square$

4. Super-solutions

In this section, we construct super-solutions of (1.3). These will be employed to prove theorems 2.2, 2.4, and 2.6.

We have divided our work into two parts. Part I addresses the case $k > 1$ (or $\gamma > 2$) and Part II discusses the case $k = 1$ or $\gamma = 2$. Each part is further sub-divided to address various situations based on the values of σ.

Part I has four sub-parts:

(i) $\sigma = 0$,

(ii) $0 < \sigma < \gamma/2$,

(iii) $\sigma = \gamma/2$ and

(iv) $\sigma > \gamma/2$,

and Part II has three sub-parts:

(i) $0 \leq \sigma \leq 1$,

(ii) $1 < \sigma \leq 2$, and

(iii) $\sigma > 2$.

We recall from (2.6) that

(4.1)
$$\begin{aligned}
&\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) \\
&\quad := H(Dw, D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t,
\end{aligned}$$

where $\sigma \geq 0$, and $Z(s) \geq 0$ is a non-increasing continuous function of s.

Let $m = m(u) < \min(\inf_{\mathbb{R}^n} h, \inf_{\mathbb{R}^n_T} u)$, where h is the initial data in (1.3) and u is the sub-solution to (1.3), see Theorems 2.2 and 2.4. We assume that the domain of Z is at least $[m, \infty)$.

Recall from (2.3) and (2.5) that $\Lambda^{\mathrm{sup}} = \sup_\lambda(\max_{|e|=1} H(e, I + \lambda e \otimes e))$. We set

(4.2)
$$\alpha = \sup_{[0,T]} |\chi(t)| \quad \text{and} \quad M = \max(\Lambda^{\mathrm{sup}}, 1).$$

We assume that α and M are bounded. We also recall from (2.7) and (2.8) that

$$k = k_1 + 1, \quad \gamma = k_1 + 2 = k + 1, \quad \gamma \geq 2,$$

and

$$\gamma^* = \frac{\gamma}{\gamma - 2} \quad \text{if } \gamma > 2.$$

Moreover, $\gamma = 2$ if and only if $k = 1$ ($k_1 = 0$).

Super-solutions

NOTE. The auxiliary functions w, constructed in Parts I and II, are viscosity super-solutions in $\mathbb{R}^n_T$. A justification may be found in Remark 3.4. The radial functions $v(r)$ satisfy $v'(0) = 0$ and $v'(r) \neq 0$, for all $r > 0$.

Let $z \in \mathbb{R}^n$ be fixed and set $r = |x - z|$ for all $x \in \mathbb{R}^n$. Recall that $B_T^R = B_R(z) \times (0, T)$.

PART I: $k > 1$. Since $\gamma > 2$, we see that

(4.3i)
$$\gamma^* - 1 = \frac{2}{\gamma - 2}, \quad \gamma^* - 2 = \frac{4 - \gamma}{\gamma - 2} > -1,$$

and

(4.3ii)
$$k\gamma^* - \gamma = \frac{\gamma}{\gamma - 2} = \gamma^*.$$

We start with the case $0 \leq \sigma \leq \gamma/2$ and present some calculations that hold for the entire interval $[0, \gamma/2]$. We then discuss the cases $\sigma = 0$, $0 < \sigma < \gamma/2$ and $\sigma = \gamma/2$ separately.

Define for $a > 0$, $0 < b \leq 1$, to be determined in what follows, and any fixed $d > m$,

(4.4) $\qquad w(x, t) = d + at + b(1 + t)v(r) \quad$ for all $(r, t) \in \mathbb{R}_T^n$,

where $v(r) \geq 0$ and $v'(r) \geq 0$. Our choice of $v(r)$ will imply that $v'(r) > 0$ in $r > 0$. Since $w \geq d > m$, $Z(w)$ and $Z(d)$ are defined and $Z(w) \leq Z(d)$. Also, we calculate $\lim_{b \to 0} a$, wherever applicable.

We use (3.3), (4.1), and (4.2) to get

(4.5)
$$
\begin{aligned}
\mathcal{P}_\sigma & (t, w, w_t, Dw, D^2w) \\
&= \frac{[b(1+t)v'(r)]^k}{r} H\Big(e, I + \Big(\frac{rv''(r)}{v'(r)} - 1 + b(1+t)rv'(r)Z(w)\Big)e \otimes e\Big) \\
&\quad + \chi(t)[b(1+t)v'(r)]^\sigma - a - bv(r) \\
&\leq \frac{M[b(1+T)]^k v'(r)^k}{r} + \alpha[b(1+T)]^\sigma (v'(r))^\sigma - a - bv(r)
\end{aligned}
$$

for all $r > 0$. We use the above inequality in both Parts I and II.

For Part I (i–iii), i.e., for $0 \leq \sigma \leq \gamma/2$, we take $v(r) = r^{\gamma^*}$. Using (4.3) and $k = \gamma - 1$ in (4.5), we find that

(4.6)
$$
\begin{aligned}
\mathcal{P}_\sigma & (t, w, w_t, Dw, D^2w) \\
&\leq M[b\gamma^*(1+T)]^k \frac{(r^{\gamma^*-1})^k}{r} + \alpha[b\gamma^*(1+T)]^\sigma (r^{\gamma^*-1})^\sigma - a - br^{\gamma^*} \\
&\leq M[\gamma^*(1+T)]^k (b^k r^{\gamma^*}) + \alpha[\gamma^*(1+T)]^\sigma (b^\sigma r^{2\sigma/(\gamma-2)}) - a - (br^{\gamma^*}).
\end{aligned}
$$

In order to write more compactly, we set

$$
E = M[\gamma^*(1+T)]^k \quad \text{and} \quad F = [\gamma^*(1+T)]^\sigma.
$$

Thus, (4.6) reads

(4.7) $\quad \mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) \leq E(b^k r^{\gamma^*}) + \alpha F(b^\sigma r^{2\sigma/(\gamma-2)}) - a - (br^{\gamma^*}).$

PART I (i): $\sigma = 0$. In (4.7), we take $\sigma = 0$ and $F = 1$ to obtain that

$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2 w) \le b(Eb^{k-1} - 1)r^{\gamma^*} + \alpha - a.$$

Select $a = \alpha$ and $0 < b < \min(1, E^{1/(1-k)})$. Clearly, $w(x, t)$ is a super-solution in $\mathbb{R}_T^n$ and

$$(4.8) \qquad w(x, t) = d + \alpha t + b(1 + t)r^{\gamma^*} \quad \text{for all } d > m.$$

PART I (ii): $0 < \sigma < \gamma/2$. Since $\gamma^* = \gamma/(\gamma - 2)$, inequality (4.7) yields that

$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2 w) \le Eb^k r^{\gamma^*} - br^{\gamma^*} + \alpha Fb^\sigma r^{2\sigma/(\gamma-2)} - a$$
$$(4.9)$$
$$= br^{\gamma^*}\left(Eb^{k-1} - 1 + \frac{\alpha Fb^{\sigma-1}}{r^{(\gamma-2\sigma)/(\gamma-2)}}\right) - a.$$

We choose

$$(4.10) \qquad \bullet \ 0 < b^{k-1} < \min(1, \ (4E)^{-1}),$$
$$\bullet \ R = (4\alpha Fb^{\sigma-1})^{(\gamma-2)/(\gamma-2\sigma)}, \text{ and}$$
$$\bullet \ a = Eb^k R^{\gamma^*} + \alpha Fb^\sigma R^{2\sigma/(\gamma-2)}.$$

The choice for a shows that w is a super-solution in B_T^R. In $r \ge R$, using $0 < \sigma < \gamma/2$ and the selections for b and R, stated in (4.10), in (4.9), we get

$$Eb^{k-1} - 1 + \frac{\alpha Fb^{\sigma-1}}{r^{(\gamma-2\sigma)/(\gamma-2)}} \le -\frac{3}{4} + \frac{\alpha Fb^{\sigma-1}}{R^{(\gamma-2\sigma)/(\gamma-2)}} = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}.$$

Thus, w is a super-solution in $\mathbb{R}_T^n$ for any $a > 0$ and $b > 0$ satisfying (4.10), and any fixed $d > m$.

We now evaluate $\lim_{b \to 0} a$. If $\sigma \ge 1$, it is clear from (4.10) that $\lim_{b \to 0} a = 0$. Note that $R \to 0$ as $b \to 0$, if $\sigma > 1$. For $\sigma = 1$, $R = 4\alpha F$.

Let $0 < \sigma < 1$. Using (4.10), $\gamma^* = \gamma/(\gamma - 2)$ and $k = \gamma - 1$, we obtain that, for some K_1 and K_2, independent of b,

$$b^k R^{\gamma^*} = K_1 b^{\gamma-1}(b^{(\sigma-1)(\gamma-2)/(\gamma-2\sigma)})^{\gamma/(\gamma-2)} = K_1 b^{(\gamma-\sigma)(\gamma-2)/(\gamma-2\sigma)},$$

and

$$b^\sigma R^{2\sigma/(\gamma-2)} = K_2 b^\sigma (b^{(\sigma-1)(\gamma-2)/(\gamma-2\sigma)})^{2\sigma/(\gamma-2)} = K_1 b^{\sigma(\gamma-2)/(\gamma-2\sigma)}.$$

It is clear that

$$(4.11) \qquad \lim_{b \to 0} a = 0 \quad \text{for all } d > m.$$

PART I (iii): $\sigma = \gamma/2$. We modify w as follows:

$$(4.12) \qquad\qquad w(x,t) = d + b(t+1)r^{\gamma^*},$$

where $b > 0$ is to be determined. Note that

$$\gamma^* = \frac{\gamma}{\gamma - 2} = \frac{2\sigma}{\gamma - 2}.$$

Taking $a = 0$ in (4.7) and observing that $k > 1$ and $\gamma > 2$, we get

$$\begin{aligned}
\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) &\le Eb^k r^{\gamma^*} + \alpha F b^\sigma r^{2\sigma/(\gamma-2)} - b r^{\gamma^*} \\
&= Eb^k r^{\gamma^*} + \alpha F b^{\gamma/2} r^{\gamma^*} - b r^{\gamma^*} \\
&= b r^{\gamma^*}(Eb^{k-1} + \alpha F b^{(\gamma-2)/2} - 1) \\
&\le 0,
\end{aligned}$$

if $0 < b \le b_0$, for some $b_0 = b_0(\alpha, k, \gamma, E, F)$ chosen small enough. Thus,

$$(4.13) \qquad w(x,t) = d + b(1+t)r^{\gamma^*} \quad \text{for all } 0 < b \le b_0 \text{ and } d > m,$$

is a super-solution in $\mathbb{R}^n_T$.

PART I (iv): $\sigma > \gamma/2$. We use Remark 3.3 and take

$$(4.14) \qquad\qquad w(x,t) = d + at + b(1+t)v(r),$$

where

$$v(r) = \int\limits_0^{r^{\gamma^*}} \frac{1}{1 + \tau^p}\, d\tau, \quad p = \frac{\gamma^* - \sigma^*}{\gamma^*}, \quad \gamma^* = \frac{\gamma}{\gamma-2}, \quad \sigma^* = \frac{\sigma}{\sigma - 1}.$$

Note that $v(r) \ge 0$, grows like r^{γ^*} near $r = 0$ and like r^{σ^*} for large r.

Recalling (4.5) and the definitions of E and F(see the line following (4.6)), we see that

$$\begin{aligned}
\mathcal{P}_\sigma&(t, w, w_t, Dw, D^2w) \\
(4.15) \quad &\le \frac{M[b(1+T)]^k v'(r)^k}{r} + \alpha[b(1+T)]^\sigma v'(r)^\sigma - a - bv(r) \\
&= \frac{E(bv'(r))^k}{(\gamma^*)^k\, r} + \frac{\alpha F(bv'(r))^\sigma}{(\gamma^*)^\sigma} - a - bv(r).
\end{aligned}$$

We use parts (ii)–(vii) of Remark 3.3, $k = \gamma - 1$ and $(\sigma^* - 1)\sigma = \sigma^*$. Note that

(iv) for all $r \geq 0$, $v(r) \leq r^{\gamma^*}$ and

$$\frac{\gamma^*}{2\sigma^*} \leq \frac{v(r) - v(R)}{r^{\sigma^*} - R^{\sigma^*}} \leq \frac{\gamma^*}{\sigma^*} \quad \text{for all } r \geq R \geq 1;$$

(v) $(v'(r))^\sigma \leq (\gamma^*)^\sigma [\min(r^{\sigma^*-1}, r^{\gamma^*-1})]^\sigma = (\gamma^*)^\sigma \min(r^{\sigma^*}, r^{2\sigma/(\gamma-2)})$; and

(vi) $(v'(r))^k / r \leq (\gamma^*)^k \min(r^{k\sigma^*-\gamma}, r^{k\gamma^*-\gamma}) = (\gamma^*)^k \min(r^{(\gamma-\sigma)/(\sigma-1)}, r^{\gamma^*})$.

Remark 3.3 (iii) shows that $v(r) \geq r^{\gamma^*}/2$ in $0 \leq r \leq 1$. Thus, using (v) and (vi) in (4.15), we get that

$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w)$$

(4.16)
$$= \begin{cases} Eb^k r^{\gamma^*} + \alpha Fb^\sigma r^{\sigma^*} - a - (br^{\gamma^*})/2 & \text{if } 0 \leq r \leq 1, \\ Eb^k r^{(\gamma-\sigma)/(\sigma-1)} + \alpha Fb^\sigma r^{\sigma^*} - a - bv(r) & \text{if } r \geq 1. \end{cases}$$

In $0 \leq r \leq 1$, w is a super-solution if we select

$$(4.17) \qquad\qquad a = Eb^k + \alpha Fb^\sigma + \frac{b\gamma^*}{\sigma^*}.$$

In $r \geq 1$, a lower bound for $v(r)$ is obtained by setting $R = 1$ in (iv) above and ignoring $v(1)$. Taking $r \geq 1$, (4.16) yields that

(4.18)

$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) \leq Eb^k r^{(\gamma-\sigma)/(\sigma-1)} + \alpha Fb^\sigma r^{\sigma^*} - a - \frac{b\gamma^*(r^{\sigma^*} - 1)}{2\sigma^*}$$

$$= Eb^k r^{(\gamma-\sigma)/(\sigma-1)} + \alpha Fb^\sigma r^{\sigma^*} + \frac{b\gamma^*}{2\sigma^*} - a - \frac{\gamma^* br^{\sigma^*}}{2\sigma^*}.$$

Since $r^{(\gamma-\sigma)/(\sigma-1)} \leq r^{\sigma^*}$ in $r \geq 1$, using (4.17) in (4.18) implies that

$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) \leq Eb^k r^{\sigma^*} + \alpha Fb^\sigma r^{\sigma^*} + \frac{b\gamma^*}{2\sigma^*} - a - \frac{\gamma^* br^{\sigma^*}}{2\sigma^*}$$

$$\leq br^{\sigma^*} \left(Eb^{k-1} + \alpha Fb^{\sigma-1} - \frac{\gamma^*}{2\sigma^*} \right)$$

$$\leq 0,$$

if we select $0 < b \leq b_0$, for some $b_0 = b_0(\alpha, \gamma, \sigma, E, F)$, small enough. Thus, w is super-solution in $\mathbb{R}^n_T$ and, for all $d > m$,

$$(4.19) \qquad\qquad \lim_{b \to 0} a = 0.$$

PART II: $k = 1$. Clearly, $\gamma = 2$ and $k_1 = 0$. By Condition B, we have that $H(q, X) = H(X)$ for all $(q, X) \in \mathbb{R}^n \times S^{n \times n}$. Thus, the operator in (4.1) reads

$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) = H(D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t \geq 0,$$

We treat separately the three possibilities:

(i) $0 \leq \sigma \leq 1$,

(ii) $1 < \sigma \leq 2$, and

(iii) $2 < \sigma < \infty$.

As in Part I, we take $a > 0$ and $0 < b \leq 1$, to be determined, and any fixed $d > m$.

PART II (i): $0 \leq \sigma \leq 1$. Take

(4.20) $\qquad w(x, t) = d + at + b(1 + t)v(r) \quad$ for all $(x, t) \in \mathbb{R}_T^n$.

(a) CASE $\sigma = 0$. We choose

$$v(r) = e^{cr^2}.$$

where $c > 0$ is to be determined. We note the following elementary facts.

$$v'(r) = 2cre^{cr^2}, \quad \frac{v'(r)}{r} = 2ce^{cr^2}, \quad \frac{rv''(r)}{v'(r)} = 1 + 2cr^2.$$

Using these in (4.5) and using $\sigma = 0$, we get

$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) \leq b(1 + T)M\left(\frac{v'(r)}{r}\right) + \alpha - a - bv(r)$$

$$= 2bc(1 + T)Me^{cr^2} + \alpha - a - be^{cr^2}.$$

Set $a = \alpha$, $\bar{E} = 2(1 + T)M$ and $c = 1/\bar{E}$ to obtain

$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) \leq 0 \quad \text{in } \mathbb{R}_T^n.$$

Thus, for any $d > m$,

$$w(x, t) = d + \alpha t + b(1 + t)e^{r^2/\bar{E}} \quad \text{for all } (x, t) \in \mathbb{R}_T^n,$$

is a super-solution in $\mathbb{R}_T^n$ for any $b > 0$. Moreover,

(4.21) $\qquad\qquad\qquad \lim_{b \to 0} w(x, t) = d + \alpha t.$

(b) CASE $0 < \sigma \le 1$. For $c > 0$, to be determined, we define

(4.22) $$w(x, t) = d + at + b(1 + t)v(r) \quad \text{in } \mathbb{R}_T^n,$$

where

$$v(r) = e^{cr} - (1 + cr).$$

Thus,

$$v'(r) = c(e^{cr} - 1),$$

$$c^2 \le \frac{v'(r)}{r} \le c^2 e^{cr},$$

and

$$1 \le \frac{rv''(r)}{v'(r)} \le \frac{e\{\max(1, cr)\}}{e - 1}.$$

For the last estimate, use that

  (i) $\theta e^\theta / (e^\theta - 1)$ is increasing in $0 < \theta < 1$, and

  (ii) $e^\theta / (e^\theta - 1)$ is decreasing in $\theta > 1$.

Applying the above to (4.5) with $k = 1$, we obtain

(4.23)
$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2 w)$$

$$\le b(1 + T)M\left(\frac{v'(r)}{r}\right) + \alpha[b(1 + T)v'(r)]^\sigma - a - bv(r)$$

$$\le bc^2(1 + T)Me^{cr} + \alpha[c(1 + T)]^\sigma[b(e^{cr} - 1)]^\sigma - a - b(e^{cr} - 1 - cr).$$

Set $\bar{E} = (1 + T)M$ and $\bar{F} = \alpha(1 + T)^\sigma$. A rearrangement of the above leads to

(4.24)
$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2 w)$$
$$\le b(1 + cr) + (c^2 \bar{E})be^{cr} + (c^\sigma \bar{F})(be^{cr})^\sigma - be^{cr} - a.$$

Since $0 < \sigma \le 1$, applying Young's inequality, $(be^{cr})^\sigma \le (1 - \sigma) + \sigma be^{cr}$, inequality (4.24) implies that

$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2 w)$$
$$\le b(1 + cr) + be^{cr}(c^2 \bar{E} + \sigma c^\sigma \bar{F} - 1) + (1 - \sigma)c^\sigma \bar{F} - a$$
$$\le b[(1 + cr) + e^{cr}(c^2 \bar{E} + \sigma c^\sigma \bar{F} - 1)] + [(1 - \sigma)c^\sigma \bar{F} - a].$$

Select $c > 0$ such that $c^2 \bar{E} + \sigma c^\sigma \bar{F} = 1 - \varepsilon$, for a small fixed $0 < \varepsilon < 1$. Hence,

(4.25) $\mathcal{P}_\sigma(t, w, w_t, Dw, D^2 w) \leq [(1 - \sigma)c^\sigma \bar{F} - a] + b[(1 + cr) - \varepsilon e^{cr}].$

The maximum of the function $1 + cr - \varepsilon e^{cr}$ occurs at $r_0 = c^{-1} \log(1/\varepsilon)$ and the maximum value is $\log(1/\varepsilon)$. Select

$$a = b \log(1/\varepsilon) + (1 - \sigma)c^\sigma \bar{F}.$$

Using the choice for a in (4.25), we get that $\mathcal{P}_\sigma(t, w, w_t, Dw, D^2 w) \leq 0$ in $\mathbb{R}_T^n$. Thus, w is a super-solution in $\mathbb{R}_T^n$ and

(4.26) $$\lim_{b \to 0} a = (1 - \sigma)c^\sigma \bar{F}, \quad 0 < \sigma \leq 1.$$

Observe that if $\sigma = 0$ then $\lim_{b \to 0} a = \alpha$. This agrees with part (a) but the growth rate in part (a) is greater. Also, if we take $\sigma = 1$, $\lim_{b \to 0} a = 0$. It is also clear that $c \leq \sqrt{(1 - \varepsilon)/\bar{\bar{E}}}$.

PART II (ii): $1 < \sigma \leq 2$. We select, for any fixed $d > m$,

(4.27) $w(x, t) = d + at + b(1 + t)r^{\sigma^*} \quad$ for all $(x, t) \in \mathbb{R}_T^n$,

where

$$\sigma^* = \frac{\sigma}{\sigma - 1}.$$

Note that $\sigma^* \geq 2$. Setting $v(r) = r^{\sigma^*}$, we find that

$$\frac{v'(r)}{r} = \sigma^* r^{\sigma^* - 2} = \sigma^* r^{(2-\sigma)/(\sigma-1)},$$

$$v'(r)^\sigma = (\sigma^*)^\sigma r^{\sigma^*},$$

and

$$\frac{r v''(r)}{v'(r)} = \sigma^* - 1.$$

Using the above in (4.23) or (4.5) and recalling that we have $\bar{E} = (1 + T)M$ and $\bar{F} = \alpha(1 + T)^\sigma$, we obtain that

(4.28)

$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2 w) \leq \bar{E}\left(\frac{bv'(r)}{r}\right) + \bar{F}(bv'(r))^\sigma - a - bv(r)$$

$$= \sigma^* \bar{E}(br^{\sigma^* - 2}) + (\sigma^*)^\sigma \bar{F}(b^\sigma r^{\sigma^*}) - a - (br^{\sigma^*}).$$

Choose

- $R = \sqrt{4\sigma^* \overline{E}}$,
- $0 < b^{\sigma-1} < \dfrac{1}{4(\sigma^*)^\sigma \overline{F}}$, and
- $a = \sigma^* \overline{E}(bR^{\sigma^*-2}) + (\sigma^*)^\sigma \overline{F}(b^\sigma R^{\sigma^*})$.

Employing the above values in (4.28) and noting that $\sigma^* \geq 2$, we see that w is a super-solution in B_T^R. In $r \geq R$,

$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) \leq \sigma^* \overline{E}(br^{\sigma^*-2}) + (\sigma^*)^\sigma \overline{F}(b^\sigma r^{\sigma^*}) - a - (br^{\sigma^*})$$

$$= br^{\sigma^*} \left(\frac{\sigma^* \overline{E}}{r^2} + (\sigma^*)^\sigma \overline{F} b^{\sigma-1} - 1 \right) - a.$$

Using the values of R and b, it is clear that w is a super-solution in $\mathbb{R}_T^n$. Moreover,

$$(4.29) \qquad\qquad\qquad \lim_{b \to 0} a = 0.$$

PART II (iii): $2 < \sigma < \infty$. Note that $1 < \sigma^* < 2$. We choose

$$(4.30) \qquad w(x, t) = d + at + b(1 + t)v(r) \quad \text{for all } (x, t) \in \mathbb{R}_T^n,$$

where

$$v(r) = \int_0^{r^2} \frac{1}{1 + \tau^p} \, d\tau \quad \text{with } p = 1 - \frac{\sigma^*}{2} = \frac{\sigma - 2}{2(\sigma - 1)}.$$

Thus, $v(r)$ is like r^2 near $r = 0$ and like r^{σ^*} for large r.

In Lemma 3.2, we set $\beta = 2$ and $\bar{\beta} = \sigma^*$. Thus, parts (iv), (v), and (vi) yield

$$(iv) \qquad\qquad \frac{1}{\sigma^*} \leq \frac{v(r) - v(1)}{r^{\sigma^*} - 1} \leq \frac{2}{\sigma^*} \quad \text{for all } r \geq 1,$$

$$(v) \qquad\qquad\qquad v'(r) \leq 2\min(r^{1/(\sigma-1)}, r),$$

and

$$(vi) \qquad\qquad\qquad \frac{v'(r)}{r} \leq 2.$$

Using the above in (4.23) or (4.5) and recalling that $\overline{E} = (1 + T)M$ and that $\overline{F} = \alpha(1 + T)^\sigma$, we get

$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w)$$

$$(4.31) \qquad \leq b(1 + T)M \left(\frac{v'(r)}{r} \right) + [\alpha(b(1 + T))^\sigma](v'(r))^\sigma - a - bv(r)$$

$$\leq 2b\overline{E} + (2b)^\sigma \overline{F} \min(r^{\sigma^*}, r^\sigma) - a - bv(r).$$

We choose

$$a = 2b\bar{E} + (2b)^\sigma \bar{F} + \frac{b}{\sigma^*} \quad \text{and} \quad 0 < b^{\sigma-1} < \frac{1}{2^\sigma \sigma^* \bar{F}}.$$

Using the above, w is a super-solution in $0 \le r \le 1$ and $0 \le t \le T$.

In $r \ge 1$, we employ the values of a, b and the bound $v(r) \ge (r^{\sigma^*} - 1)/\sigma^*$ (see (iv) above) in (4.31) to find that

$$
\begin{aligned}
\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) &\le 2b\bar{E} + (2b)^\sigma \bar{F} \min(r^{\sigma^*}, r^\sigma) - a - \frac{br^{\sigma^*}}{\sigma^*} + \frac{b}{\sigma^*} \\
&\le (2b)^\sigma \bar{F} r^{\sigma^*} - \frac{br^{\sigma^*}}{\sigma^*} \\
&\le br^{\sigma^*}\left(2^\sigma b^{\sigma-1} \bar{F} - \frac{1}{\sigma^*}\right) \\
&\le 0.
\end{aligned}
$$

Thus, w is super-solution in $\mathbb{R}^n_T$. Moreover,

$$(4.32) \qquad\qquad\qquad \lim_{b \to 0} a = 0.$$

WE SUMMARIZE. Select $w(x, t) = d + at + b(1 + t)v(r)$, where $d > m$ and $v(r)$ is as follows

(4.33I) PART I ($k > 1$):

$$
v(r) = \begin{cases}
r^{\gamma^*} & \text{if } 0 \le \sigma \le \gamma/2, \\
\displaystyle\int_0^{r^{\gamma^*}} (1 + \tau^p)^{-1} d\tau & \text{if } \sigma > \gamma/2,
\end{cases}
$$

$$
\lim_{b \to 0} a = \begin{cases}
\alpha & \text{if } \sigma = 0, \\
0 & \text{if } \sigma > 0,
\end{cases}
$$

where $p = 1 - (\sigma^*/\gamma^*)$;

(4.33II) PART II ($k = 1$):

$$
v(r) = \begin{cases}
e^{cr^2} & \text{if } \sigma = 0, \\
e^{cr} - 1 - cr & \text{if } 0 < \sigma \le 1, \\
r^{\sigma^*} & \text{if } 1 < \sigma \le 2, \\
\displaystyle\int_0^{r^2} (1 + \tau^p)^{-1} d\tau & \text{if } \sigma > 2,
\end{cases}
$$

$$\lim_{b \to 0} a = \begin{cases} \alpha & \text{if } \sigma = 0, \\ (1 - \sigma)c^\sigma \bar{F} & \text{if } 0 < \sigma \leq 1, \\ 0 & \text{if } \sigma > 1, \end{cases}$$

where $p = 1 - (\sigma^*/2)$ and $\bar{F} = \alpha(1 + T)^\sigma$.

See (4.8), (4.11), (4.13) and (4.19), (4.21), (4.26), (4.29), and (4.32). Note that $v(r)$ grows like r^{σ^*} in Part I for $\sigma > \gamma/2$, and in Part II for $\sigma > 2$. The functions w are super-solutions for any $0 < b \leq b_0$, where $b_0 = b_0(\sigma, \gamma, k, T, \Lambda^{\text{sup}})$ is small enough.

Also, any super-solution $w \geq d > m$. Since the domain of Z is $[m, \infty)$, we have that $0 \leq Z(w) \leq Z(d) \leq Z(m) < \infty$.

5. Sub-solutions

The work in this section is similar to that in Section 4. Even though H is not assumed to be odd in X, the auxiliary functions used in Section 4 continue to apply here. Thus, we provide an outline here. The domain for Z is taken to be $(-\infty, \infty)$. Remark 3.4 shows that these auxiliary functions are viscosity sub-solutions. The case where the domain Z is $(0, \infty)$ or $[0, \infty)$ is treated in Case (i) in Section 6.

We use functions of the type $w(x, t) = -[d + at + b(1 + t)v(r)]$, where $a > 0$ and $0 < b \leq 1$, small, $d \in \mathbb{R}$, $v(r) > 0$ and $v'(r) \geq 0$. Recalling (3.5), we see that

(5.1)
$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w)$$
$$= \frac{[b(1 + t)v'(r)]^k}{r} H\left(e, \left(1 - \frac{rv''(r)}{v'(r)} + b(1 + t)rv'(r)Z(w)\right)e \otimes e - I\right)$$
$$+ \chi(t)[b(1 + t)v'(r)]^\sigma + a + bv(r).$$

We recall the definition of Λ^{inf} in (2.4) and the note right after Condition C, see (2.5). We set

(5.2) $\alpha = \sup_{[0,T]} |\chi(t)|$ and $N = \Lambda^{\text{inf}} = \inf_\lambda(\min_{|e|=1} H(e, \lambda e \otimes e - I))$.

By Condition C, $N < 0$ since $H(e, -I) < 0$.

As done in Section 4, we take $v(r)$ to be a power of r(power greater than 1) or e^{cr^2} or e^{cr}, for some $c > 0$. For exponential functions, $1 - (rv''(r))/v'(r)$ could become unbounded. Thus, a lower bound on H is needed and we require that $N > -\infty$. However, if $v(r)$ is a power of r then $1 - (rv''(r))/v'(r)$ is

bounded from below and the bound depends on the power. We get a natural lower bound depending on the power of r. We use N to denote the lower bound in both situations.

With the above discussion in mind, (5.1) then implies

$$
\begin{aligned}
\mathcal{P}_\sigma&(t, w, w_t, Dw, D^2w)\\
(5.3)\quad &\geq \frac{[b(1+T)v'(r)]^k N}{r} - \alpha[b(1+T)v'(r)]^\sigma + a + bv(r)\\
&= -\Big(\frac{[b(1+T)v'(r)]^k |N|}{r} + \alpha[b(1+T)v'(r)]^\sigma - a - bv(r)\Big).
\end{aligned}
$$

We now use auxiliary functions $v(r)$ that are similar to those in Section 4. The goal is to choose $a \geq 0$ and $0 < b \leq 1$ such that (5.3) is non-negative i.e.,

$$
(5.4)\quad \frac{[b(1+T)v'(r)]^k}{r}|N| + \alpha[b(1+T)v'(r)]^\sigma - a - bv(r) \leq 0.
$$

This is similar to (4.5) and the analysis in Section 4 applies. We list the choice for $w(x,t)$ for the various values of σ.

PART I: $k > 1$. Recall that $\gamma > 2$ and $\gamma^* = \gamma/(\gamma - 2)$. Set $r = |x - z|$, for some fixed $z \in \mathbb{R}^n$, and take, for $d \in \mathbb{R}$,

$$
w(x,t) = \begin{cases}
-d - at - b(1+t)r^{\gamma^*} & \text{when } 0 \leq \sigma < \gamma/2,\\
-d - b(1+t)r^{\gamma^*} & \text{when } \sigma = \gamma/2\\
-d - at - b(1+t)v(r) & \text{when } \sigma > \gamma/2,
\end{cases}
$$

where

$$
v(r) = \int_0^{r^{\gamma^*}} \frac{1}{1+\tau^p}, \quad \text{with } \sigma^* = \frac{\sigma}{\sigma - 1}, p = 1 - \frac{\sigma^*}{\gamma^*}.
$$

It is easily checked that (see Remark 3.3 (v,vii) and use $\sigma^* < \gamma^*$)

$$
1 - \frac{rv''(r)}{v'(r)} = \frac{2 - \gamma^* + (2 - \sigma^*)r^{p\gamma^*}}{1 + r^{p\gamma^*}} \geq 2 - \gamma^* > -\infty.
$$

We choose N to be an appropriate lower bound for H, see the right hand side of (5.1) and (5.2). Thus, (5.3) holds without any restrictions on $\Lambda^{\inf}$. Moreover, from (4.33),

$$
(5.5)\quad \lim_{b\to 0} a = \begin{cases} \alpha & \text{if } \sigma = 0,\\ 0 & \text{if } \sigma > 0. \end{cases}
$$

PART II: $k = 1$. In this case, $\gamma = 2$ and $k_1 = 0$. Set $\sigma^* = \sigma/(\sigma - 1)$ and $d \in \mathbb{R}$ is any constant. We choose $a \geq 0$, $0 < b < 1$ and $c > 0$ such that (5.4) holds. We select

$$
w(x,t) = \begin{cases}
-d - at - b(1+t)e^{cr^2} & \text{when } \sigma = 0, \\
-d - at - b(1+t)\left(e^{cr} - 1 - cr\right) & \text{when } 0 < \sigma \leq 1, \\
-d - at - b(1+t)r^{\sigma^*} & \text{when } 1 < \sigma \leq 2, \\
-d - at - b(1+t)v(r) & \text{when } 2 < \sigma < \infty,
\end{cases}
$$

where

$$
v(r) = \int\limits_0^{r^2} \frac{1}{1 + \tau^p} \, d\tau \quad \text{with } p = 1 - \frac{\sigma^*}{2}.
$$

For $0 \leq \sigma \leq 1$, $1 - rv''(r)/v'(r) \leq 0$ and is unbounded as $r \to \infty$. Thus, we impose that $|\Lambda^{\inf}| < \infty$. For $\sigma > 1$, however, no such requirement is made.

Moreover, from (4.33),

$$
(5.6) \qquad \lim_{b \to 0} a = \begin{cases}
\alpha & \text{when } \sigma = 0, \\
\alpha(1 - \sigma)c^\sigma (1 + T)^\sigma & \text{when } 0 < \sigma \leq 1, \\
0 & \text{when } \sigma > 1.
\end{cases}
$$

6. Some special cases

In this section we consider some special cases. Recall that

$$
(6.1) \qquad \begin{aligned}
&\mathcal{P}_\sigma(t, w, w_t, Dw, D^2 w) \\
&= H(Dw, D^2 w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t.
\end{aligned}
$$

As in (5.2), set

$$
N = \Lambda^{\inf} = \inf_\lambda [\min_{|e|=1} H(e, \lambda e \otimes e - I)].
$$

We address the following three cases.

CASE (i): $k \geq 1$ AND $\chi \equiv 0$. Then the equation (6.1) reads

$$
H(Dv, D^2 v + Z(v)Dv \otimes Dv) - v_t = 0 \quad \text{in } \mathbb{R}^n_T, \, v > 0,
$$

with $v(x, 0) = h(x)$ for all $x \in \mathbb{R}^n$.

As observed in (1.2) and part (f) of Remark 3.7, the above equation arises when a change of variables is applied to a doubly nonlinear equation. Moreover, as noted in Remark 3.7 and Lemma 3.8, the convergence or the divergence of the integral

$$(6.2) \qquad I = \int_0^1 f^{-1/(k-1)}(\theta)\, d\theta, \quad k > 1,$$

determines the domain of Z. In particular, if $I < \infty$ then the domain of Z is $(0, \infty)$ or $[0, \infty)$, and if $I = \infty$ then the domain is $(-\infty, \infty)$.

In Section 4, the super-solutions w are constructed so that $w > m$, since the domain of Z is $[m, \infty)$, see (4.33). If $m = 0$ then we use $d > 0$ and if $m = -\infty$ we use any d. Thus, we obtain super-solutions of (6.1). However, the domain of Z needs to be stated more precisely for sub-solutions. If I diverges then Part I in Section 5 applies since the domain of Z is $(-\infty, \infty)$. If I converges then the domain is $(0, \infty)$ or $[0, \infty)$ and a different sub-solution is needed. We discuss this in this section.

We also include a sub-solution for $k = 1$ when Z is defined on $(0, \infty)$ or $[0, \infty)$.

CASE (ii). We discuss super-solutions for $\chi \leq 0$ and we derive a maximum principle.

CASE (iii). We study sub-solutions for $\chi \geq 0$ and this leads to a minimum principle.

The cases (ii) and (iii) are related and bring out the influence of the sign of χ. We begin with the following cases.

CASE (i). Let $z \in \mathbb{R}^n$ and set $r = |x - z|$, for all $x \in \mathbb{R}^n$. We assume that the domain of Z is at least $(0, \infty)$. This applies to the case $I < \infty$, see (6.2).

CASE (i–1). We take $k > 1$, $\chi \equiv 0$ and $\sigma = 0$. Thus, the equation reads

$$\mathcal{P}_0(t, w, w_t, Dw, D^2w) = H(Dw, D^2w + Z(w)Dw \otimes Dw) - w_t.$$

Since our goal is to construct positive sub-solutions w, it suffices to find a w such that $H(Dw, D^2w) - w_t \geq 0$ since ellipticity ($Z \geq 0$) implies the desired conclusion.

Let $R > 0$ and set $B_T^R = B_R(z) \times (0, T)$. We construct a sub-solution w for any large R. More precisely, $w \geq 0$ solves

(6.3i) $H(Dw, D^2w) - w_t \geq 0$ in B_T^R

and

(6.3ii) $w(x, 0) \leq g(x)$ for all $x \in B_R(z)$.

We define for $D > 0$ and $E > 0$, to be determined, and for all $(x, t) \in B_T^R$,

(6.4)   • $\psi(t) = \dfrac{1}{(E + t)^{1/(k-1)}}$,

          • $v(r) = [R^{(k+1)/k} - r^{(k+1)/k}]^{k/(k-1)}$,

          • $w(x, t) = D\psi(t)v(r)$.

Noting that $v'(r) \neq 0$ for $r \neq 0$, we recall from (3.4) that if $w = D\psi(t)v(r)$ then, in $r > 0$,

$$
H(Dw, D^2w) - w_t
$$

(6.5)
$$
= \frac{D^k(|\psi(t)v'(r)|)^k}{r} H\left(e, \left(1 - \frac{rv''(r)}{v'(r)}\right)e \otimes e - I\right) - Dv(r)\psi'(t)
$$

$$
\geq -\frac{D^k|N|(|\psi(t)v'(r)|)^k}{r} - Dv(r)\psi'(t).
$$

Using (6.4), setting $c_k = [(k+1)/(k-1)]^k$ and $|v'(r)|^k/r = c_k v(r)$, (6.5) yields

$$
-Dv(r)\psi'(t) - \frac{D^k|N|(|\psi(t)v'(r)|)^k}{r}
$$

$$
= \frac{Dv(r)}{(k-1)(E + t)^{k/(k-1)}} - \frac{c_k D^k|N|v(r)}{(E + t)^{k/(k-1)}}
$$

$$
= \frac{Dv(r)}{(k-1)(E + t)^{k/(k-1)}}[1 - (k-1)c_k D^{k-1}|N|].
$$

Choosing
$$
D = \left(\frac{1}{c_k(k-1)|N|}\right)^{1/(k-1)},
$$

and using the value in (6.5), we get a viscosity sub-solution $w \geq 0$ in B_T^R such that $w(R, t) = 0$. A proof follows in Remark 6.1. Next, we calculate E by requiring that
$$
w(z, 0) = w(0, 0) = \frac{DR^{(k+1)/(k-1)}}{E^{1/(k-1)}} = \inf_x h(x) = \mu.
$$

Using (6.4), we get that

$$w = \frac{DR^{(k+1)/(k-1)}}{E^{1/(k-1)}} \frac{[1 - (r/R)^{(k+1)/k}]^{k/(k-1)}}{(1 + (t/E))^{1/(k-1)}} = \frac{\mu[1 - (r/R)^{(k+1)/k}]^{k/(k-1)}}{(1 + (t/E))^{1/(k-1)}}.$$

Note that $E = O(R^{k+1})$ and

$$w(z,t) = w(0,t) = \frac{\mu}{(1 + (t/E))^{1/(k-1)}} \longrightarrow \mu \quad \text{as } R \to \infty.$$

We record that in $0 \le r \le R$,

$$(6.6) \qquad w(x,t) = \frac{\mu[1 - (r/R)^{(k+1)/k}]^{k/(k-1)}}{(1 + (t/E))^{1/(k-1)}},$$

where

$$E = \frac{R^{k+1}}{c_k \mu^{k-1}(k-1)|N|}.$$

REMARK 6.1. We check that the function w, as defined in (6.6), is a viscosity sub-solution at points $(0, s)$, i.e., at $r = 0$ and for any $0 < s < T$. Write $w(x,t) = w(r,t) = \hat{\kappa}(t)\hat{v}(r)$ in B_T^R, where E is as in (6.6),

$$\hat{\kappa}(t) = (1 + (t/E))^{-1/(k-1)} \quad \text{and} \quad \hat{v}(r) = \mu[1 - (r/R)^{(k+1)/k}]^{k/(k-1)}.$$

Clearly, $\hat{v}'(0) = 0$ but $\hat{v}''(r)$ is not defined at $r = 0$. This requires us to verify that w is a viscosity sub-solution at $r = 0$.

Let $\psi(x,t)$ be C^2 in x and C^1 in t. Suppose that $(w - \psi)(x,t) \le (w - \psi)(0,s)$ for some $0 < s < T$. Then $w(x,t) - w(0,s) \le \psi(x,t) - \psi(0,s)$ and

$$\hat{\kappa}(t)\hat{v}(r) - \hat{\kappa}(s)\hat{v}(0) \le \langle D\psi(0,s), x \rangle + \psi_t(0,s)(t-s) + o(|x| + |t-s|),$$

as $(x,t) \to (0,s)$. Hence, $D\psi(0,s) = 0$ as $\hat{v}'(0) = 0$, and $\psi_t(0,s) = \hat{\kappa}'(s)\hat{v}(0)$. Then

$$H(D\psi, D^2\psi)(0,s) - \psi_t = H(0, D^2\psi)(0,s) - \hat{\kappa}'(s)\hat{v}(0).$$

Next, $k_1 > 0$ as $k > 1$. Using Condition B, we see that $H(0, D^2\psi)(0,s) = 0$. Thus,

$$H(D\psi, D^2\psi)(0,s) - \psi_t = -\hat{\kappa}'(s)\hat{v}(0) = \frac{(1 + (s/E))^{-k/(k-1)}\hat{v}(0)}{(k-1)E} > 0.$$

This shows that w is a sub-solution of (6.3) at $(0,s)$.

CASE (i–2). Take $k = 1$. We take $w(x,t) = De^{-Er^2}e^{-Ft}$, where $D > 0$, $E > 0$ and $F > 0$ are to be determined. Set $v(r) = e^{-Er^2}$ and $\psi(t) = e^{-Ft}$. Using (6.5), we get

$$-|N|\frac{D\psi(t)|v'(r)|}{r} - Dv(r)\psi'(t) = DFe^{-Er^2}e^{-Ft} - 2|N|DEe^{-Er^2}e^{-Ft}$$

$$= De^{-Er^2}e^{-Ft}(F - 2|N|E).$$

We take $F = 2|N|E$ and $D = \mu$ and obtain a sub-solution

$$(6.7) \qquad\qquad w(x,t) = \mu e^{-Er^2}e^{-2|N|Et} \quad \text{for all } E > 0.$$

It is clear that $w \to \mu$ as $E \to 0$.

We now consider the influence of the sign of χ on (1.3). In Case (ii), we construct a super-solution that implies theorems 2.2 and 2.4 for $\chi < 0$.

CASE (ii): $\chi < 0$. We consider

$$(6.8) \qquad \begin{aligned} &\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) \\ &= H(Dw, D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t, \end{aligned}$$

where $\chi < 0$. Set $\hat{\alpha} = \sup_{(0,T)} \chi(t)$ and assume that

$$(6.9) \qquad\qquad\qquad \hat{\alpha} < 0, \quad \sigma \geq k, \quad k \geq 1.$$

Our goal here is to construct super-solutions $w \geq \sup_{\mathbb{R}^n} h$ in cylinders B_T^R. The domain of Z is $[m, \infty)$ or (m, ∞), where $m = 0$ or $m = -\infty$.

Set $v = \sup_{\mathbb{R}^n} h$. We select $w(x,t) = v + at + (1+t)v(r)$, $v' > 0$, in $r > 0$. Set $M = \max(1, \Lambda^{\sup})$ (see(4.2)); recalling (3.3) and (6.8), we find that in $r \neq 0$,

$$(6.10)$$
$$\begin{aligned} &\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) \\ &\quad \leq \frac{[(1+t)v'(r)]^k}{r} H\left(e, I + \left(\frac{rv''(r)}{v'(r)} - 1 + (1+t)rv'(r)Z(w)\right)e \otimes e\right) \\ &\qquad - |\hat{\alpha}|[(1+t)v'(r)]^\sigma - a - v(r) \\ &\quad \leq \frac{[(1+t)v'(r)]^k}{r}\{M - r|\hat{\alpha}|[(1+t)v'(r)]^{\sigma-k}\} - a - v(r). \end{aligned}$$

For $R > 0$, set
$$v(r) = (R^2 - r^2)^{-1}, \quad 0 \le r < R.$$

Clearly, $v \in C^2$ in x, in $0 \le r < R$. Using

$$v'(r) = \frac{2r}{(R^2 - r^2)^2} \quad \text{and} \quad \frac{rv''(r)}{v'(r)} = \frac{R^2 + 3r^2}{R^2 - r^2},$$

in (6.10) yields that, in $0 \le r < R$,

(6.11)
$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w)$$
$$\le \left(\frac{2(1+t)}{(R^2 - r^2)^2} \right)^k r^{k-1} \left(M - |\hat{\alpha}| \left(\frac{2(1+t)}{(R^2 - r^2)^2} \right)^{\sigma-k} r^{\sigma-k+1} \right) - a - \left(\frac{1}{R^2 - r^2} \right).$$

It is easily shown that (6.11) holds in the viscosity sense in $0 \le r < R$.

SUBCASE (ii−1): $\sigma = k$. Set $r^* = M/|\hat{\alpha}|$ and take $R > r^*$. Then (6.11) yields that

(6.12) $$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) \le \left(\frac{2(1+t)}{(R^2 - r^2)^2} \right)^k (M - |\hat{\alpha}|r)r^{k-1} - a.$$

Thus, (6.12) yields that w is a super-solution if we select

$$a = M \left(\frac{2(1 + T)}{(R^2 - (r^*)^2)^2} \right)^k (r^*)^{k-1}.$$

Since $w(x, t) = v + at + (1 + t)/(R^2 - r^2)$, we get that

(6.13) $$w(x, t) \ge v, \quad \lim_{R \to \infty} a = 0 \quad \text{and} \quad \lim_{R \to \infty} w(0, t) = v.$$

SUBCASE (ii−2): $\sigma > k$. From (6.11) we have that

$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w)$$
$$\le \left(\frac{2(1+t)}{(R^2 - r^2)^2} \right)^k r^{k-1} \left(M - |\hat{\alpha}| \left(\frac{2}{(R^2 - r^2)^2} \right)^{\sigma-k} r^{\sigma-k+1} \right) - a.$$

Since the function $f(r) = r^{\sigma-k+1}/(R^2 - r^2)^{2(\sigma-k)}$ is continuous, increasing in $0 \le r < R$, $f(0) = 0$ and $f(r) \to \infty$, as $r \to R$, there is a unique $r^* = r^*(R) < R$ such that $2^{\sigma-k}|\hat{\alpha}|f(r^*) = M$. Choose

$$a = M \left(\frac{2(1 + T)}{(R^2 - (r^*)^2)^2} \right)^k (r^*)^{k-1}.$$

Clearly, w is super-solution in $0 \le r < R$. Recall that

$$f(r^*) = \frac{(r^*)^{\sigma-k+1}}{[R^2 - (r^*)^2]^{2(\sigma-k)}} = \frac{M}{2^{\sigma-k}|\hat{\alpha}|}.$$

Clearly, r^* is increasing in R and $r^* \to \infty$, as $R \to \infty$. For calculating $\lim_{R\to\infty} a$, we use the formula for $f(r^*)$ and observe that for an appropriate constant D, we have

$$\frac{(r^*)^{k-1}}{[R^2 - (r^*)^2]^{2k}} = \frac{D(r^*)^{k-1}}{(r^*)^{k(\sigma-k+1)/(\sigma-k)}} = \frac{D}{(r^*)^{\sigma/(\sigma-k)}}.$$

Thus,

(6.14) $$\lim_{R\to\infty} a = 0 \quad \text{and} \quad \lim_{R\to\infty} w(0,t) = v.$$

Case (iii): $\chi > 0$. This is similar to Case (ii). We require that $\sigma \ge k$ and find a sub-solution $w(x,t)$ i.e.,

$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w) = H(Dw, D^2w + Z(w)Dw \otimes Dw) + \chi(t)|Dw|^\sigma - w_t$$
$$\ge 0 \quad \text{in } \mathbb{R}^n_T,$$

where $\chi > 0$. Set

$$N = \Lambda^{\inf} = \inf_\lambda [\min_{|e|=1} H(e, \lambda e \otimes e - I)] \quad \text{and} \quad \hat{\alpha} = \inf \chi(t),$$

where $\hat{\alpha} > 0$. We take the domain of Z to be $(-\infty, \infty)$.

Select $w(x,t) = \mu - at - (1+t)v(r)$, $v' \ge 0$ and recall (3.5):

(6.15)
$$\mathcal{P}_\sigma(t, w, w_t, Dw, D^2w)$$

$$\ge \frac{[(1+t)v'(r)]^k}{r} H\left(e, \left((1+t)rv'(r)Z(w) + 1 - \frac{rv''(r)}{v'(r)}\right)e \otimes e - I\right)$$

$$+ \hat{\alpha}[(1+t)v'(r)]^\sigma + a + v(r)$$

$$\ge -\left[\frac{[(1+t)v'(r)]^k}{r}|N| - \hat{\alpha}[(1+t)v'(r)]^\sigma - a - v(r)\right].$$

Defining

$$v(r) = \frac{1}{R^2 - r^2} \quad \text{for all } 0 \le r < R,$$

and proceeding as in Case (ii), one can construct a sub-solution w with similar properties.

REMARK 6.2. We summarize. We employ the auxiliary function in Case (i-l) to prove the minimum principle in Theorem 2.6 if $k > 1$ and

$$\int_0^1 f^{-1/(k-1)}(s)ds < \infty.$$

This requires no lower bound except $u > 0$. In this case, the domain of Z is either $(0, \infty)$ or $[0, \infty)$. However, for $k = 1$ we require a lower bound for u.

Case (ii) implies the maximum principle in theorems 2.2 and 2.4 (for any $k \geq 1$) without imposing an upper bound. However, we require that $\sigma \geq k$.

Case (iii) leads to the minimum principle in theorems 2.3 and 2.5 (for any $k \geq 1$ and $\sigma \geq k$) without requiring any lower bound.

7. Proofs of the main results

Assume that $-\infty < \inf_{\mathbb{R}^n} h \leq \sup_{\mathbb{R}^n} h < \infty$ and set

$$\mu = \inf_{\mathbb{R}^n} h \quad \text{and} \quad \nu = \sup_{\mathbb{R}^n} h.$$

For a fixed $z \in \mathbb{R}^n$, set $r = |x - z|$, $x \in \mathbb{R}^n$, $B_T^R = B_R(z) \times (0, T)$ and P_T^R its parabolic boundary.

PROOFS OF THEOREMS 2.2 AND 2.3: $k > 1$. We first present the proof of Theorem 2.2. Select $\varepsilon > 0$ small and $R_0 > 0$ such that

(7.1) $$\sup_{[0,R] \times [0,T]} u(x,t) \leq \varepsilon R^\delta \quad \text{for all } R \geq R_0.$$

where $\delta(\gamma^* \text{ or } \sigma^*)$ is as described in the statement of the theorem.

Recall from (4.4) and (4.33) that a super-solution $w(x, t)$ is given by

(7.2) $$w(x, t) = d + at + b(1 + t)v(r) \quad \text{where } d > m,$$

$v(0) = 0$, and $v(r) > 0$. Also, w is a super-solution for any small $b > 0$, and v grows like r^δ, see (4.33), Lemma 3.2 and Remark 3.3. Take $d = \nu$ in (7.2) and define

$$w(x, t) = \nu + at + b(1 + t)v(r).$$

Let $\hat{k} > 2$ be a constant so that $\hat{k}v \geq r^\delta$, for all $r \geq R_1$, where R_1 is large enough. Take $b = \hat{k}\varepsilon$ in $w(x, t)$ and consider the cylinder B_T^R, where $R \geq \max(R_0, R_1)$. At $t = 0$, $w(x, 0) = \nu + \hat{k}\varepsilon v(r) \geq \nu \geq u(x, 0)$. On $|x - z| = R$,

$$w(x, t) \geq \hat{k}\varepsilon v(R) \geq \varepsilon R^\delta.$$

254 Tilak Bhattacharya – Leonardo Marazzi

Thus, $w \geq u$ on P_T^R. Next, Lemma 3.5 implies that $w \geq u$ in B_T^R for any large R, i.e.,

$$u(x,t) \leq v + at + \hat{k}\varepsilon(1+t)v(r) \quad \text{for all } |x-z| \leq R.$$

Taking $x = z$, we get that $u(z,t) \leq at + v$. We let $R \to \infty$ and then $\varepsilon \to 0$ (i.e., $b \to 0$). Recalling the value of $\lim_{b\to 0} a$ in (4.33), we obtain the theorem.

The proof of Theorem 2.3 follows by using Part I of Section 5 and arguing analogously. □

PROOFS OF THEOREMS 2.4 AND 2.5: $k = 1$. We first prove Theorem 2.4.

We recall (4.33II) and Part II (i) in Section 4. Take $\sigma = 0$. Let $0 < \varepsilon < c/10$ be small and fixed (for a definition of c, see Part II of Section 4). Set

$$w(x,t) = v + \alpha t + \varepsilon(1+t)e^{cr^2} \quad \text{for all } (x,t) \in \mathbb{R}_T^n.$$

Then w is super-solution for any $\varepsilon > 0$, small enough.

Choose $R_0 > 0$ such that

$$\sup_{B_R(z)\times[0,T]} u(x,t) \leq e^{\varepsilon R^2} \quad \text{and} \quad \varepsilon e^{cR^2} > e^{\varepsilon R^2} \quad \text{for all } R > R_0.$$

We apply the comparison principle Lemma 3.5 to prove the claim in the theorem. Observe that $w(x,0) \geq v \geq u(x,0)$ for all $x \in \mathbb{R}^n$. On $|x-z| = R > R_0$, $w(x,t) \geq \varepsilon e^{cR^2} \geq e^{\varepsilon R^2}$. By Lemma 3.5, $u(x,t) \leq w(x,t)$ for all $(x,t) \in B_T^R$ for any $R > R_0$. Hence, taking $x = z$ or $r = 0$,

$$u(z,t) \leq w(z,t) = v + \alpha t + \varepsilon(1+t).$$

Since the above holds for any large R, we let $\varepsilon \to 0$ to obtain the claim in part (a).

Part (b) may now be shown by arguing as above. Part (c) is shown by following the ideas in the Proof of Theorem 2.2. Theorem 2.5 follows analogously, see Part II in Section 5. □

REMARK 7.1. We assume that $\sigma \geq k$ and χ has one sign.

The case $\chi < 0$ follows by using the auxiliary function in Case (ii) in Section 6. By using comparison, $w(x,t) \geq u(x,t)$ in B_T^R. We obtain theorems 2.2 and 2.4 by letting $R \to \infty$ and using (6.14).

The case $\chi > 0$ uses Case (iii) in Section 6. We argue as done above and theorems 2.3 and 2.5 follow.

We now present the proof of Theorem 2.6. We start with the maximum principle.

PROOF OF THEOREM 2.6 (a): MAXIMUM PRINCIPLE. Let $k > 1$. We refer to Remark 3.7 and the comparison principle in Lemma 3.8. We set $\alpha = 0$ in part (a) of Theorem 2.2. Recalling Remark 3.7 (f), we use the change of variables $u = \phi(v)$. If

$$H(Du, D^2u) - f(u)u_t \geq 0 \quad \text{in } \mathbb{R}^n_T,$$

with $u > 0$ such that

$$u(x, 0) \leq g(x) \quad \text{for all } x \in \mathbb{R}^n,$$

then

$$H(Dv, D^2v + Z(v)Dv \otimes Dv) - v_t \geq 0, \quad \text{in } \mathbb{R}^n_T,$$

with

$$v(x, 0) \leq \phi^{-1}(g(x)) \quad \text{for all } x \in \mathbb{R}^n,$$

where $Z(s) = \phi''(s)/\phi'(s)$ and the domain of Z contains $(0, \infty)$ (see Remark 3.7 (f)). Note that both ϕ and ϕ^{-1} are increasing.

The super-solution w used in the proof of Theorem 2.2 is positive when $d = \phi^{-1}(\sup g) > 0$, see (7.2). Clearly, $Z(w)$ is well defined. Using Lemma 3.5 (or Lemma 3.8) and arguing as in the proof of Theorem 2.2 we get that $v \leq \phi^{-1}(\sup g)$, if

$$\sup_{B_R(z) \times (0,T)} v(x, t) = o(R^{\gamma^*}) \quad \text{as } R \to \infty.$$

Thus, the claim holds for u.

For $k = 1$ and $f \equiv 1$, we take $Z = 0$ and argue as in Theorem 2.4. $\qquad \square$

PROOF OF THEOREM 2.6 (b): MINIMUM PRINCIPLE. Let $k > 1$.

(i) Suppose that

$$\lim_{\delta \to 0^+} F(1) - F(\delta) < \infty.$$

We choose

$$v = \phi^{-1}(u) = \int_0^u f^{-1/(k-1)}(\theta) \, d\theta, \quad u > 0.$$

Then $v > 0$ and the domain of Z is either $(0, \infty)$ or $[0, \infty)$ (see Remark 3.7 (f)) and

$$H(Dv, D^2v + Z(v)Dv \otimes Dv) - v_t \leq 0 \quad \text{in } \mathbb{R}^n_T,$$

for $v > 0$ such that

$$v(x, 0) \geq \phi^{-1}(g(x)) \quad \text{for all } x \in \mathbb{R}^n.$$

We set $\mu = \phi^{-1}(\inf g)$ and recall Case (i−1) from Section 6 and (6.6) i.e.,

$$w(x, t) = \frac{\mu[1 - (r/R)^{(k+1)/k}]^{k/(k-1)}}{(1 + (t/E))^{1/(k-1)}},$$

where

$$E = \frac{R^{k+1}}{c_k \mu^{k-1}(k-1)|N|},$$

with $0 \leq r \leq R$, for any large $R > 0$.

We use comparison in B_T^R. It is clear that $v(x, 0) \geq \phi^{-1}(g(x)) \geq w(x, 0)$, for all $|x - z| < R$. Since $v > 0$ in $\mathbb{R}_T^n$, working with $R' < R$, close to R, we see that $v(x, t) \geq w(x, t)$. Considering the cylinder $B_T^{R'}$ and its parabolic boundary $P_T^{R'}$, we apply Lemma 3.5 to get that $v(x, t) \geq w(x, t)$ in $B_T^{R'}$, for any $R' < R$. Thus,

$$v(z, t) \geq w(z, t) = \frac{\mu}{(1 + (t/E))^{1/(k-1)}},$$

Letting $R \to \infty$ (i.e. $E \to \infty$), we get that $v(z, t) \geq \mu$ and the claim follows for u.

(ii) Suppose that

$$\lim_{\delta \to 0^+} F(1) - F(\delta) = \int_0^1 f^{-1/(k-1)}(\theta) d\theta = \infty.$$

We choose a primitive

$$v = \phi^{-1}(u) = \int^u f^{-1/(k-1)}(\theta) d\theta, \quad u > 0.$$

Then the domain of Z is $(-\infty, \infty)$ (see Remark 3.7 (f)) and

$$H(Dv, D^2v + Z(v)Dv \otimes Dv) - v_t \leq 0 \quad \text{in } \mathbb{R}_T^n$$

with

$$v(x, 0) \geq \phi^{-1}(g(x)) \quad \text{for all } x \in \mathbb{R}^n.$$

The rest is similar to the proof of Theorem 2.3 (a).

For $k = 1$ and $f \equiv 1$, take $\chi \equiv 0$ and $Z \equiv 0$ and argue as in Theorem 2.5 (a). $\qquad\square$

REMARK 7.2. Let $k = 1$ and $f \equiv 1$. Suppose that $\hat{m} = \inf g > 0$ and, for every $k > 0$, there is an $R_k > 0$ such that $\inf_{B_T^R} u(x, t) \geq e^{-kR^2}$ for all $R \geq R_k$. We use Case (i−2) in Section 6 and (6.7). For a given $\varepsilon > 0$, $\inf_{B_T^R} u(x, t) \geq \hat{m} e^{-\varepsilon R^2}$, for large enough $R > 0$. Recall from (6.7) that

$$w(x, t) = \hat{m} e^{-Er^2} e^{-2|N|Et} \quad \text{for all } E > 0,$$

is a sub-solution in $\mathbb{R}_T^n$. Take $E = 2\varepsilon$. Working in cylinders B_T^R, for large R, we find that $u(x, 0) \geq \hat{m} \geq w(x, 0)$. On $|x - z| = R$, $w(x, t) \leq u(x, t)$ implying that $w(x, t) \leq u(x, t)$ in B_T^R, for any large R, and, hence, in $\mathbb{R}_T^n$. Thus,

$$w(z, t) = \hat{m} e^{-4\varepsilon|N|t} \leq u(z, t).$$

The above holds for any R and for any $\varepsilon > 0$. Thus, $u \geq \hat{m}$.

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