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On the intersection of non-normal maximal subgroups of a finite group

Zahra Aghajari (∗) – Gholamreza Rezaeezadeh (∗∗)

Abstract – The subgroup $\Delta(G)$ of a group G is defined to be the intersection of all non-normal maximal subgroups of G (and $\Delta(G) = G$ if all maximal subgroups of G are normal). A group G is called a T_2-*group* if $G/\Delta(G)$ is a T-group. Ballester-Bolinches et al. [3] considered the class of T_2-groups and gave several results of such groups. In particular, they showed if G is a solvable group, the classes of T_2-groups and PST_2-groups (that is, a group in which $G/\Delta(G)$ is a PST-group) are equal. The present work, we introduce the class of SST_2-groups which are defined as the groups G for which $G/\Delta(G)$ is an SST-group and we show several results of the class SST_2-groups. Also, we discuss about equivalency the classes of solvable PST_2-groups and solvable SST_2-groups.

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1. Introduction

Throughout this paper, all groups are finite. $Z_\infty(G)$, $G^{\mathfrak{N}}$ and $\Phi(G)$ denote the hypercenter, nilpotent residual and the Frattini subgroup of G, respectively. G_p denotes the Sylow p-subgroup of G.

(∗) *Indirizzo dell'A.*: Department of Pure Mathematics, Shahrekord University, P.O. Box 115, Shahrekord, Iran
E-mail: Z.Aghajari@stu.sku.ac.ir

(∗∗) *Indirizzo dell'A.*: Department of Pure Mathematics, Shahrekord University, P.O. Box 115, Shahrekord, Iran
E-mail: rezaeezadeh@sci.sku.ac.ir

When authors investigate the structure of a finite group G, there exist different methods. They sometimes put assumptions on special kinds of the maximal subgroups of G. They sometimes study some information about the intersection of certain types of the maximal subgroups of G. With these methods, new interesting results were obtained (see [3], [13], [17], [18], ...). First, a few known definitions should be recalled.

A subgroup H of a group G is said *permutable* (*S-permutable*) in G if H permutes with all the subgroups (Sylow subgroups) of G. A group G is called a *T-group* (resp. *PT-group, PST-group*) if normality (resp. permutability, S-permutability) is a transitive relation, that is, if H and K are subgroups of G such that H is normal (resp. permutable, S-permutable) in K and K is normal (resp. permutable, S-permutable) in G, then H is normal (resp. permutable, S-permutable) in G. Kegel [14] showed that every S-permutable subgroup is subnormal. So we can say that T-groups (PT-groups, PST-groups) are exactly those groups in which every subnormal subgroup is normal (resp. permutable, S-permutable). Gaschütz [13], Zacher [19] and Agrawal [1] proved definitive results on solvable T-groups, PT-groups, and PST-groups.

Li et al. [15] introduced an important embedding property which is called *SS-quasinormal subgroup.* A subgroup H of a group G is said *SS-permutable* (*SS-quasinormal*) in G if H has a supplement K in G such that H permutes with every Sylow subgroup of K. In this case, K is called an *SS-permutable supplement* of H in G. A group G is called an *SST-group* if SS-permutability is a transitive relation.

It is clear that every S-permutable subgroup of a group G is SS-permutable in G. However, the converse does not hold in general. In addition, every solvable SST-group G is a solvable PST-group, but the converse is not true.

Recall a group G is said a *X_0-group* if $G/\Phi(G)$ is a X-group. Ragland [16] studied finite solvable X_0-groups for $X \in \{T, PT, PST\}$. In particular, he obtained characterizations for finite solvable T_0-groups, in the spirit of the theorems of Gaschütz, Zacher, and Agrawal. Also, T_0-groups have been studied in other papers (see [4], [6], ...).

A group is called a *T_1-group* if $G/Z_\infty(G)$ is a T-group. Beidleman [6] described some of the basic properties of solvable T_1-groups and some of the properties of these groups were also developed in [7]. For a group G, $\Delta(G)$ denotes the intersection of all non-normal maximal subgroups of G (and $\Delta(G) = G$ if all maximal subgroups of G are normal, that is, if G is nilpotent). Gaschütz [13] established many interesting properties of $\Delta(G)$. Also, he showed how these properties could be used to characterize of finite groups. Next, Ballester-Bolinches et.al [3] introduced a new concept of groups, that is, T_2-groups. We say that G is a T_2-group if $G/\Delta(G)$ is a T-group. They obtained some of the properties of these groups.

The aim of this paper is to study finite groups that $G/\Delta(G)$ is an SST-group. It seems reasonable investigate a connection between such groups and groups in which $G/\Delta(G)$ is a PST-group. We begin with the following definition.

Definition 1.1. A group G is called an SST_0-*group* (resp. SST_1-*group*, SST_2-*group*) if $G/\Phi(G)$ (resp. $G/Z_\infty(G)$, $G/\Delta(G)$) is an SST-group.

text slightly changed to avoid an overfull.

2. Preliminaries

In this section, we gather some results from the literature that will be used later.

Theorem 2.1 ([1] and [5], Theorem 3.3). *The soluble group G is a PST-group if and only if the following conditions hold*:

(1) $G^{\mathfrak{N}}$ *is a normal abelian Hall subgroup of G with odd order*;

(2) G *acts by conjugation as power automorphisms on* $G^{\mathfrak{N}}$.

Moreover, if G is a PST-group, then $F(G) = G^{\mathfrak{N}} \times Z_\infty(G)$.

Lemma 2.2 ([9], Lemma 3.1). *Let G be a solvable PST-group. If $G/Z_\infty(G)$ is a solvable SST-group, then G is a solvable SST-group* .

Lemma 2.3 ([9], Corollary 1.10). *The class of all solvable SST-groups is closed under taking subgroups and epimorphic images.*

Lemma 2.4. *A solvable SST_0-group is supersolvable.*

Proof. If G is a solvable SST_0-group, then $G/\Phi(G)$ is a solvable SST-group and so it is supersolvable. Hence G is supersolvable. □

Theorem 2.5 ([3]). *Let G be a group. Then*

(1) $\Delta(G)$ *is nilpotent,*

(2) $\Delta(G)/\Phi(G) = Z(G/\Phi(G))$.

Recall that a proper normal subgroup H of a group G is called a special generalized Frattini subgroup of G provided that $G = N_G(A)$ for each normal subgroup L of G and each Hall subgroup A of L such that $G = HN_G(A)$. We denote the collection of all special generalized Frattini subgroups of G by $s.g.f.(G)$.

Lemma 2.6 ([11], Corollary 3.12). *Let $H \in$ s. g. f.(G) and let K be a proper normal subgroup of G which contains H. Then $K \in$ s. g. f.(G) if and only if $K/H \in$ s. g. f.(G/H).*

3. Main Results

Lemma 3.1. *Let G be a solvable group, then $Z_\infty(G) \leq \Delta(G)$.*

Proof. First note that if M is a maximal subgroup of a solvable group G and H/K is a chief factor of G such that $K \leq M$ and $H \not\leq M$ then M is non-normal in G if and only if H/K is eccentric in G.

Assume that for some maximal non-normal subgroup M of G we have $Z_\infty(G) \not\leq M$. Since $Z_\infty(G/\Phi(G)) = Z_\infty(G)/\Phi(G)$, there is a chief factor H/K of G such that $\Phi(G) \leq K \leq M$, $H \not\leq M$ and $H/\Phi(G) \leq Z_\infty(G/\Phi(G))$. But then H/K is central in G which is contradicts non-normality of M. □

Theorem 3.2. *Let G be a solvable group and N be a normal subgroup of G. Then the following statements hold*:

(1) *if G is an SST_2-group, then G/N is an SST_2-group*;

(2) *if $N \leq \Delta(G)$ and G/N is an SST_2-group, then G is an SST_2-group*;

(3) *if G is an SST_2-group, then G is supersolvable*;

(4) *if $G/Z_\infty(G)$ is an SST_0-group, then G is an SST_2-group.*

Proof. Let G be a solvable group and N be a normal subgroup of G.

(1) Let G be an SST_2-group. Then the quotient group $G/\Delta(G)$ is a solvable SST-group. Therefore $G/\Delta(G)N$ is a solvable SST-group, too.

On the other hand, $\Delta(G)N/N \leq \Delta(G/N)$. Hence G/N is an SST_2-group.

(2) We have $\Delta(G)/N \trianglelefteq G/N$, and so $(G/N)/(\Delta(G)/N) \simeq G/\Delta(G)$ is an SST_2-group by (1). It is clear that $\Delta(G/\Delta(G)) = 1$. Hence G is an SST_2-group.

(3) Suppose that G is an SST_2-group. We use induction on the order of G. By (1), $G/\Phi(G)$ is an SST_2-group. Thus $G/\Phi(G)$ is supersolvable and hence G is supersolvable.

I removed "then" to avoid an overfull.

(4) Suppose that $G/Z_\infty(G)$ be an SST_0-group. $(G/Z_\infty(G))/\Delta(G/Z_\infty(G))$ is an SST-group. Since $Z_\infty(G) \leq \Delta(G)$, it follows that $G/\Delta(G)$ is an SST-group and so G is an SST_2-group. □

Theorem 3.3. *G is an SST_2-group if and only if $G/\Phi(G)$ is an SST_1-group.*

PROOF. Assume that G is an SST_2-group. Since

$$(*) \qquad Z(G/\Phi(G)) \leq Z_\infty(G/\Phi(G)) \leq \Delta(G)/\Phi(G) = Z(G/\Phi(G)),$$

it follows that $G/\Phi(G)$ is an SST_1-group.

Conversely, let $G/\Phi(G)$ be an SST_1-group. So from ($*$) we get G is an SST_2-group. □

THEOREM 3.4. *Let G be a group. If every subgroup of G is an SST_2-group, then G is supersolvable.*

PROOF. Let G be a group which every its subgroup is an SST_2-group. Then, by induction, every proper subgroup of G is supersolvable. Hence G is solvable. Thus, by part (3) of Theorem (3.2), G is supersolvable. □

THEOREM 3.5. *If G is a solvable SST_0-group, then $G^{\mathfrak{N}}$ is a nilpotent Hall subgroup of G of odd order.*

PROOF. Let G be a solvable SST_0-group. By Lemma (2.4), G is supersolvable. Thus $G^{\mathfrak{N}}$ is nilpotent of odd order.

If G is nilpotent, then $G^{\mathfrak{N}} = 1$. Therefore $G^{\mathfrak{N}}$ is a Hall subgroup of G and the proof is complete. Therefore we can assume that G is not nilpotent. Let p be the largest prime divisor of $|G|$. By induction on $|G|$ the quotient group $G^{\mathfrak{N}}G_p/G_p$ is a Hall subgroup of G/G_p.

Now, we can distinguish two cases.

(1) THE PRIME p DOES NOT DIVIDE $|G^{\mathfrak{N}}|$. In this case, we can conclude $G^{\mathfrak{N}}$ is a Hall subgroup of G.

(2) THE PRIME p DIVIDES $|G^{\mathfrak{N}}|$. If $O_{p'}(G) \neq 1$, then $\overline{G^{\mathfrak{N}}} = G^{\mathfrak{N}}O_{p'}(G)/O_{p'}(G)$ is a Hall subgroup of $\overline{G} = G/O_{p'}(G)$ by induction on $|G|$. Since the prime p divides $|G^{\mathfrak{N}}|$ and $\overline{G^{\mathfrak{N}}}$ is a Hall subgroup of $\overline{G}$, it follows that $\overline{G_p} = G_pO_{p'}(G)/O_{p'}(G) \in \mathrm{Syl}_p(\overline{G^{\mathfrak{N}}})$. Hence $G_p \leq G^{\mathfrak{N}}$, so $G^{\mathfrak{N}}G_p/G_p = G^{\mathfrak{N}}/G_p$ is a Hall subgroup of G/G_p. Thus $G^{\mathfrak{N}}$ is a Hall subgroup of G.

Now, we suppose $O_{p'}(G) = 1$. Thus the Fitting subgroup of G, $F(G)$, is a p-subgroup of G and so $F(G) = G_p$. Since G is supersolvable, it follows that $G^{\mathfrak{N}} \leq G_p$.

We have $G/\Phi(G)$ is a solvable SST-group and hence $G^{\mathfrak{N}}\Phi(G)/\Phi(G)$ is a Hall subgroup of $G/\Phi(G)$. Since $G^{\mathfrak{N}}\Phi(G)/\Phi(G)$ is a p-group, we conclude that $G^{\mathfrak{N}}\Phi(G) = G_p$. Suppose that H is a subgroup of G such that G_p is complemented by H in G. Then $G = HG_p = HG^{\mathfrak{N}}$ and hence $G^{\mathfrak{N}}$ is a Sylow p-subgroup of G, that is, $G^{\mathfrak{N}}$ is a Hall subgroup of G. □

Theorem 3.6. *If G is a solvable SST_1-group such that* $(|G^{\mathfrak{N}}|, |Z_\infty(G)|) = 1$, *then G is a solvable PST-group.*

Proof. Suppose that $G/Z_\infty(G)$ is an SST-group. Then $G^{\mathfrak{N}} Z_\infty(G)/Z_\infty(G)$ is an abelian Hall subgroup of $G/Z_\infty(G)$ of odd order on which $G/Z_\infty(G)$ acts by conjugation as power automorphisms on $G^{\mathfrak{N}} Z_\infty(G)/Z_\infty(G)$. Since $G^{\mathfrak{N}}$ is G-isomorphic to $G^{\mathfrak{N}} Z_\infty(G)/Z_\infty(G)$, it follows that G acts as a group of power automorphisms on $G^{\mathfrak{N}}$, and $G^{\mathfrak{N}}$ is abelian. On the other hand, $G^{\mathfrak{N}} Z_\infty(G)/Z_\infty(G)$ is a Hall subgroup of $G/Z_\infty(G)$ and $(|G^{\mathfrak{N}}|, |Z_\infty(G)|) = 1$. Thus $G^{\mathfrak{N}}$ is a Hall subgroup of G and G is a PST-group by Theorem (2.1). □

Theorem 3.7. *Let G be a solvable SST_1-group. Then the following statements hold*:

(1) *$G^{\mathfrak{N}}$ is nilpotent of class at most* 2;

(2) *G acts by conjugation on $G^{\mathfrak{N}}/(G^{\mathfrak{N}})'$ as a group of power automorphisms.*

Proof. (1) By hypothesis $G/Z_\infty(G)$ is a solvable SST-group. Thus the nilpotent residual of $G/Z_\infty(G)$, $G^{\mathfrak{N}} Z_\infty(G)/Z_\infty(G)$, is abelian. Therefore

$$(**) \qquad (G^{\mathfrak{N}})' \subseteq G^{\mathfrak{N}} \cap Z_\infty(G)$$

Since $[G^{\mathfrak{N}}, Z_\infty(G)] = 1$, it follows from $(**)$ that $(G^{\mathfrak{N}})' \subseteq Z(G^{\mathfrak{N}})$. Hence $G^{\mathfrak{N}}$ is nilpotent of class at most 2.

(2) Since $[G^{\mathfrak{N}}, G] = G^{\mathfrak{N}}$, it follows that $G^{\mathfrak{N}} \cap Z_\infty(G) = (G^{\mathfrak{N}})'$ and G operates on $G^{\mathfrak{N}} Z_\infty(G)/Z_\infty(G) \simeq G^{\mathfrak{N}}/(G^{\mathfrak{N}})'$ as a group of power automorphisms. □

Theorem 3.8. *Let G be a solvable SST_1-group. Then G is a solvable SST_0-group if and only if $G^{\mathfrak{N}}$ is a Hall subgroup of G.*

Proof. Let G be a solvable SST_1-group. The necessity of the condition has already been proved by Theorem (3.5). Thus, we need only prove that if $G^{\mathfrak{N}}$ is a Hall subgroup of G, then G is an SST_0-group.

By Theorem (3.7), G acts by conjugation as a group of power automorphisms on $G^{\mathfrak{N}}/(G^{\mathfrak{N}})'$ and so it acts in the same way on $G^{\mathfrak{N}}/\Phi(G^{\mathfrak{N}})$ since $(G^{\mathfrak{N}})' \subseteq \Phi(G^{\mathfrak{N}})$. We have $\Phi(G^{\mathfrak{N}}) = \Phi(G) \cap G^{\mathfrak{N}}$ and hence G acts as a group of power automorphisms on $G^{\mathfrak{N}}\Phi(G)/\Phi(G)$ since it is G-isomorphic to $G^{\mathfrak{N}}/\Phi(G^{\mathfrak{N}})$. Hence, by Theorem (2.1), $G/\Phi(G)$ is a PST-group. Hence G is an SST_0-group by Theorem (2.2). □

Theorem 3.9. *Let G be a solvable SST_2-group. Then G is a solvable SST_0-group if and only if $G^{\mathfrak{N}}$ is a Hall subgroup of G.*

Proof. Let G be a solvable SST_2-group. The necessity of the condition has already been proved Theorem (3.5). So, we need only prove that if $G^{\mathfrak{N}}$ is a Hall subgroup of G, then G is an SST_0-group.

Since G is an SST_2-group, we may assume $\Delta(G) \neq \Phi(G)$. First, we suppose that $\Phi(G) = 1$. Then $\Delta(G) = Z(G) = Z_\infty(G)$ and so G is an SST_1-group. Hence, by Theorem (3.8), G is an SST_0-group.

Now, we may assume that $\Phi(G) \neq 1$. By part (1) of Theorem (3.2), $G/\Phi(G)$ is a solvable SST_2-group. Also, we have $G^{\mathfrak{N}}\Phi(G)/\Phi(G)$ is the nilpotent residual of $G/\Phi(G)$ and a Hall subgroup of $G/\Phi(G)$. By induction on $|G|$, it follows that $G/\Phi(G)$ is an SST_0-group. Therefore G is a solvable SST_0-group. □

Theorem 3.10. *Let G be a solvable group. If $G/\Delta(G)$ is a PST-group, then G is an SST_2-group.*

Proof. Let $G/\Delta(G)$ be a solvable PST-group. We use induction on the order of G. Since $(G/Z_\infty(G))/\Delta(G/Z_\infty(G))$ is a solvable PST-group, we can assume that $G/Z_\infty(G)$ is an SST_2-group. By part (2) of Theorem (3.2), G is a solvable SST_2-group. □

Theorem 3.11. *Let G be a solvable group. Then G is an SST_2-group if and only if it satisfies*:

(1) *$G^{\mathfrak{N}}\Delta(G)/\Delta(G)$ is an abelian Hall subgroup of $G/\Delta(G)$*;

(2) *G acts by conjugation on $G^{\mathfrak{N}}/\Delta(G) \cap G^{\mathfrak{N}}$ as a group of power automorphisms.*

I removed "then" to avoid an overfull.

Proof. Assume that G is a solvable SST_2-group. $G/\Delta(G)$ is an SST-group. By Theorem (2.1), (1), and (2) hold.

Conversely, assume that $G/\Delta(G)$ satisfies (1) and (2). By Theorem (2.1), $G/\Delta(G)$ is a solvable PST-group and by Theorem (3.10), G is a solvable SST_2-group. □

Theorem 3.12. *The classes of solvable T_2-groups, solvable PT_2-groups, solvable PST_2-groups and solvable SST_2-groups are equal.*

I removed "then" to avoid an overfull.

Proof. Let G be a solvable PST_2-group. $G/\Delta(G)$ is a solvable PST-group and, by Theorem (3.10), G is a solvable SST_2-group. The equality of the classes follows by [3, Theorem E] and $\mathfrak{S} \cap SST_2 \subseteq \mathfrak{S} \cap PST_2$, where $\mathfrak{S}$ is the class solvable groups. □

Recall that a subgroup H of a group G is said *semipermutable* [9] in G if H permutes with every subgroup X of G such that $(|H|, |X|) = 1$. A group G is called a *BT-group* [9] if semipermutability is a transitive relation.

THEOREM 3.13. *Let G be a solvable group. If $G/\Delta(G)$ is a BT-group, then G is an SST_2-group.*

PROOF. Let $G/\Delta(G)$ be a solvable BT-group. Then $G/\Delta(G)$ is a solvable PST-group. Hence $G/\Delta(G)$ is a SST-group, by theorem (3.12). □

COROLLARY 3.14. *The classes of solvable T_2-groups, solvable PT_2-groups, solvable PST_2-groups, solvable SST_2-groups and solvable BT_2-groups are equal.*

PROOF. Equality of the classes follows by Theorem (3.12), Theorem (3.13), and $\mathfrak{G} \cap SST \subseteq \mathfrak{G} \cap BT$, where $\mathfrak{G}$ is the class of all solvable groups. □

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