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Groups with few self-centralizing subgroups which are not self-normalizing

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Abstract – A self-normalizing subgroup is always self-centralizing, but the converse is not necessarily true. Given a finite group G, we denote by $w(G)$ the number of all self-centralizing subgroups of G which are not self-normalizing. We observe that $w(G) = 0$ if and only if G is abelian, and that if G is nonabelian nilpotent then $w(G) \geq 3$. We also prove that if $w(G) \leq 20$ then G is solvable. Finally, we provide structural information in the case when $w(G) \leq 3$.

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1. Introduction

Let G be a group, and H a subgroup of G. We say that H is a *self-normalizing subgroup* of G if $H = N_G(H)$, the normalizer of H in G. Moreover, H is a *self-centralizing subgroup* of G if $H \subseteq C_G(H)$, the centralizer of H in G. This is equivalent to require that $C_G(H) = Z(H)$, the center of H.

Self-normalizing or self-centralizing subgroups are widely used for recognition on groups. An interesting question is to study groups in which all subgroups not having a given property are self-centralizing. In [2] and [6] locally finite groups, in which all noncyclic subgroups are self-centralizing, are classified.

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A still unsolved problem posed by Berkovich [1, p.439, research problem 9] is to classify finite p-groups in which every nonabelian subgroup is self-centralizing. In [3], Delizia et al. studied such groups and provided some partial answer to the above question. Later, Pavel Zalesskii suggested another related problem: to classify finite groups in which every nonabelian subgroup is self-normalizing. This problem has been solved in [4]. Groups in which every nonnilpotent subgroup is self-normalizing have been considered in [5].

Clearly, every self-normalizing subgroup is self-centralizing. The converse is, in general, false. Our purpose is to study groups having only few self-centralizing subgroups which are not self-normalizing. All groups considered in this paper are finite. Given any group G, we denote by $w(G)$ the number of all self-centralizing subgroups of G which are not self-normalizing. In Section 2 we gather some preliminary results. In Section 3 we prove that $w(G) = 0$ if and only if G is abelian. Furthermore, if G is nilpotent and nonabelian then $w(G) \geq 3$. In Section 4 we show that if $w(G) \leq 20$ then G is solvable. In Section 5 we investigate the structure of finite groups G with $w(G) \leq 3$. In particular, if $w(G) \leq 2$ then G is an A-group, i.e. all Sylow subgroups of G are abelian.

2. Preliminaries

We say that a subgroup H of a group G is a *W-subgroup*, if it is a self-centralizing and non-self-normalizing subgroup, i.e. $C_G(H) \leq H \lneq N_G(H)$. We denote by $w(G)$ the number of all W-subgroups of G. It is obvious from the definition that the trivial subgroups are not W-subgroups, and that $w(G) = 0$ for all abelian groups G.

Proposition 2.1. *If G is a finite group, then for every subgroup H of G, either $\langle H, C_G(H)\rangle$ is a W-subgroup or $N_G(H) = \langle H, C_G(H)\rangle$.*

In particular, for every abelian subgroup $B \leq G$, either $C_G(B)$ is a W-subgroup or $N_G(B) = C_G(B)$.

Proof. Let $K = \langle H, C_G(H)\rangle$. Suppose K is not a W-subgroup, thus K is non-self-centralizing or self-normalizing. But K is always self-centralizing, because that $C_G(K) \leq C_G(H) \leq K$.

Thus K is self-normalizing, so $N_G(K) = K$ and we have $N_G(H) \leq N_G(K) = K \leq N_G(H)$, since $N_G(H) \leq N_G(C_G(H))$ and $K = \langle H, C_G(H)\rangle \leq N_G(H)$. Therefore $N_G(H) = K = \langle H, C_G(H)\rangle$.

In particular, if B is any abelian subgroup of G, we have $C_G(B) = \langle B, C_G(B)\rangle$. □

Lemma 2.2. *Let G be a finite group and H be a subgroup of G, then*

$$w(H) \leq w(G).$$

If K is a W-subgroup of H, then there exists $\widehat{K} \leq G$ such that $\widehat{K}$ is a W-subgroup in G and $K = H \cap \widehat{K}$.

Proof. Suppose $w(H) = s$ and $K_1, \ldots, K_s$ are pairwise distinct W-subgroups of H. Then by Proposition 2.1 for every K W-subgroup of H, we have $N_H(K) = \langle K, C_H(K)\rangle$ or $\langle K, C_H(K)\rangle = K_i$, for some $i \in \{1, \ldots, s\}$.

Let $\widehat{K} = \langle K, C_G(K)\rangle$, then it is self-centralizing in G, since $C_G(\widehat{K}) \leq C_G(K) \leq \widehat{K}$.

Notice that $\widehat{K}$ is not self-normalizing in G. Indeed, there exists $h \in N_H(K)\setminus K$, so $h \notin C_H(K)$, hence $h \notin \widehat{K}$, but $h \in N_G(\widehat{K})$. Since $C_H(K) \leq K$, by the modular law we have

$$\widehat{K} \cap H = (K \cdot C_G(K)) \cap H = K \cdot (C_G(K) \cap H) = K \cdot C_H(K) = K.$$

Therefore $\widehat{K}$ is a W-subgroup in G and $K = H \cap \widehat{K}$.

The subgroups $\widehat{K}_1, \ldots, \widehat{K}_s$ are pairwise distinct, since if $\widehat{K}_i = \widehat{K}_j$, then $\widehat{K}_i \cap H = \widehat{K}_j \cap H$, thus $K_i = K_j$ and so $i = j$.

Therefore for W-subgroups $K_1, \ldots, K_s$ of H, there exist pairwise distinct W-subgroups $\widehat{K}_1, \ldots, \widehat{K}_s$ of G. Hence $s = w(H) \leq w(G)$. □

Lemma 2.3. *Let G be a finite group and N be a normal subgroup of G. Then*

$$w(G/N) \leq w(G).$$

If H/N is a W-subgroup of G/N, then H is a W-subgroup of G.

Proof. If H/N is a self-centralizing subgroup of G/N, then H is a self-centralizing subgroup of G, since

$$C_G(H)N/N \leq C_{G/N}(H/N).$$

Also H/N is a self-normalizing subgroup of G/N if and only if H is a self-normalizing subgroup of G, since

$$N_{G/N}(H/N) = N_G(H)N/N.$$

Therefore, if $w(G/N) = t$ and $H_1/N, \ldots, H_t/N$ are pairwise distinct W-subgroups of G/N, then $H_1, \ldots, H_t$ are pairwise distinct W-subgroups of G, thus $w(G/N) \leq w(G)$. □

Corollary 2.4. *If G is a finite group and N is a normal subgroup of G, then*

$$w(N) + w(G/N) \leq w(G).$$

Proof. Let $w(N) = s$ and $w(G/N) = t$. Let $K_1, \ldots, K_s$ be pairwise distinct W-subgroups of N and $H_1/N, \ldots, H_t/N$ be pairwise distinct W-subgroups of G/N. By Lemma 2.2 we know that, for all $i = 1, \ldots, s$, there exist W-subgroups $\widehat{K}_i$ of G such that $K_i = N \cap \widehat{K}_i$. Moreover, by Lemma 2.3, $H_1, \ldots, H_t$ are W-subgroups of G containing N.

If $\widehat{K}_i = H_j$, for some i, j, then $N < H_j = K_i$, so $K_i = N \cap \widehat{K}_i = N$, a contradiction. □

Proposition 2.5. *If G and H are finite groups, then*

$$w(G \times H) \geq w(G) \cdot w(H).$$

Proof. It is easy to see that

- $A \times B$ is a self-centralizing subgroup of $G \times H$ if and only if A and B are self-centralizing subgroups of G and H, respectively;
- $A \times B$ is a self-normalizing subgroup of $G \times H$ if and only if A and B are self-normalizing subgroups of G and H, respectively.

Therefore $A \times B$ is a W-subgroup of $G \times H$ if and only if A and B are self-centralizing subgroups of G and H, respectively, and one of them is W-subgroup. So that $G \times H$ has at least $w(G) \cdot w(H)$ W-subgroups. □

Corollary 2.6. *If G and A are finite groups and A is abelian, then*

$$w(G \times A) = w(G).$$

Proof. Let H be any W-subgroup of G. Then $H \times A$ is a W-subgroup of $G \times A$, as in the proof of Proposition 2.5. So $w(G) \leq w(G \times A)$. Now let K be any W-subgroup of $G \times A$. Then $C_{G\times A}(K) \leq K$. Since A is abelian, we get $A \leq Z(G\times A) \leq C_{G\times A}(K) \leq K$. Write $H = K \cap G$. Then $HA = (K \cap G)A = K$, so $K = H \times A$. Since $H \times A$ is a W-subgroup of $G \times A$, arguing as in the proof of Proposition 2.5 we have that H is a W-subgroup of G, so $w(G \times A) \leq w(G)$. Therefore $w(G \times A) = w(G)$. □

3. W-subgroups of nilpotent groups

We start by showing that if in a group G, all self-centralizing subgroups of G are self-normalizing i.e. G has no W-subgroups, then it is abelian.

Theorem 3.1 (Zassenhaus). *If, in a finite group G, the normalizer of every abelian subgroup coincides with the centralizer of that subgroup, then the group G is abelian.*

Proof. See [15, Theorem 7]. □

Theorem 3.2. *Let G be a finite group. Then $w(G) = 0$ if and only if G is abelian.*

Proof. It is a consequence of Proposition 2.1 and Theorem 3.1. We also give a direct proof.

Let G be a minimal counterexample. By Lemma 2.2, all proper subgroups of G are abelian, hence G is a minimal nonabelian group.

We have two cases: $G' < G$ or $G' = G$.

If $G' < G$, then there exists a maximal subgroup M of G, such that $G' \leq M$, hence M is normal. By minimality of G, M is a maximal abelian subgroup of G, so it is self-centralizing. Thus M is W-subgroup, so $w(G) > 0$, a contradiction.

Assume now that $G' = G$. We show that G is simple. If N is a nontrivial normal subgroup of G, then by Lemma 2.3, $w(G/N) = 0$, so by minimality of G, we have G/N is abelian, hence $G' \leq N$, a contradiction.

Therefore G is simple and minimal nonabelian. By [9], minimal nonabelian groups are non-simple, a contradiction. □

Proposition 3.3. *Let G be a finite nilpotent group, and assume that G is not abelian. Then $w(G) \geq 3$.*

Proof. Since G is nilpotent, every proper subgroup of G is properly contained in its normalizer in G. Hence every self-centralizing subgroup of G is a W-subgroup of G. Each element of G is contained in a maximal abelian subgroup of G, and G is the union of them. As G is nonabelian, G has at least three maximal abelian subgroups. Clearly, the latter are self-centralizing subgroups of G. Therefore $w(G) \geq 3$. □

Corollary 3.4. *Let G be a finite group with $w(G) < 3$. Then all nilpotent subgroups of G are abelian.*

Proof. Let H be any nilpotent subgroup of G. By Lemma 2.2 we have $w(H) \leq w(G)$. Since H is nilpotent, by Proposition 3.3 we obtain $w(H) = 0$. So H is abelian by Theorem 3.2. □

4. Groups with at most 20 W-subgroups

In this section we show that finite groups G with $w(G) \leq 20$, are solvable. One can see, for example with GAP [13] that $w(A_5) = 21$, so our bound is sharp.

If G is a nonsolvable group of minimum order with respect to $w(G) \leq 20$, then it is a minimal simple group. A group is called a *minimal simple* group if it is a nonabelian simple group and every proper subgroup of it is solvable. Thompson in 1968 classified these groups:

THEOREM 4.1. [14, Cor.1] *All the finite minimal simple groups (up to isomorphism) are*

- *the projective special linear group* $\mathrm{PSL}(2, 2^p)$, *where* p *is a prime number*;
- *the projective special linear group* $\mathrm{PSL}(2, 3^p)$, *where* p *is an odd prime*;
- *the projective special linear group* $\mathrm{PSL}(2, p)$, *where* $p > 3$ *is a prime such that* $5 | p^2 + 1$;
- *the Suzuki group* $Sz(2^p) = {}^2B_2(2^p)$, *where* p *is an odd prime*;
- *the projective special linear group* $\mathrm{PSL}(3, 3)$.

Thus we only have to deal with projective special linear groups and Suzuki groups.

PROPOSITION 4.2. *Let* $G = \mathrm{PSL}(2, q)$ *with* $q \geq 41$. *Then* $w(G) \geq 21$.

PROOF. Let $H = \left\{ \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix} \right] \mid x \in F_q \right\}$. By an easy calculation, one can see that H is an abelian subgroup of G and $C_G(H) = H$, so H is self-centralizing.

Similarly we have $N_G(H) = \left\{ \left[\begin{smallmatrix} a & x \\ 0 & a^{-1} \end{smallmatrix} \right] \mid a, x \in F_q, a \neq 0 \right\}$. Thus $N_G(H) \neq H$, hence H is not self-normalizing, therefore H is a W-subgroup.

Moreover $|N_G(H)| = q(q-1)$. But it is well-known that

$$|G| = |\mathrm{PSL}(2, q)| = \frac{q(q^2-1)}{\gcd(2, q-1)}.$$

Hence

$$|G : N_G(H)| = \frac{q+1}{\gcd(2, q-1)} \geq \frac{q+1}{2} \geq \frac{41+1}{2} = 21.$$

Thus H has at least 21 conjugate subgroups in G, and they are W-subgroups. Therefore $w(G) \geq 21$. □

PROPOSITION 4.3. *Let* $G = Sz(2^{2m+1})$ *with* $m \geq 1$. *Then* $w(G) \geq 21$.

PROOF. By Suzuki's work in 1960 [12, p.3], the Suzuki group $Sz(q)$ has a cyclic subgroup A of order $q - 2r + 1$, where $q = 2^{2m+1}$ and $r = 2^m$, such that A is the centralizer of its non-identity elements. So that A is self-centralizing. Moreover, A is of index 4 in its normalizer, hence A is not self-normalizing. Therefore A is a W-subgroup.

But the order of G is $q^2(q-1)(q^2+1)$ and $q = 2^{2m+1} \geq 8$, so that

$$\begin{aligned}
[G : N_G(A)] &= \frac{|G|}{|N_G(A)|} \\
&= \frac{|G|}{4|A|} \\
&= \frac{1}{4}q^2(q-1)\frac{q^2+1}{q-2r+1} \\
&= \frac{1}{4}q^2(q-1)(q+2r+1) \\
&\geq \frac{1}{4}q^2(q-1) \\
&\geq \frac{1}{4}8^2(8-1) \geq 21.
\end{aligned}$$

Therefore $[G : N_G(A)] \geq 21$, thus H has at least 21 conjugate subgroups in G, and they are W-subgroups. Therefore G has at least 21 W-subgroups, i.e. $w(G) \geq 21$. □

THEOREM 4.4. *Every finite group* G *with* $w(G) \leq 20$, *is solvable.*

PROOF. Let G be a minimal counterexample. Thus G is nonsolvable and $w(G) \leq 20$. By Lemma 2.2, for any proper subgroup H of G, $w(H) \leq w(G) \leq 20$. Thus, minimality of G implies that H is solvable.

Similarly by Lemma 2.3, we obtain that G/N is solvable, for any proper normal subgroup N of G. Now if G is not simple, then there exists a normal subgroup $1 \neq N < G$, hence N and G/N are solvable, therefore G is solvable, a contradiction.

So that G is a (nonabelian) simple group such that all proper subgroups are solvable, i.e. G is a minimal simple group. Thus, by the Thompson's result [14, Theorem 4.1], we can assume that G is either a projective special linear group or a Suzuki group.

By Proposition 4.3, G cannot be a Suzuki group. So assume that G is a projective special linear group. If $G = \mathrm{PSL}(2, q)$, then by Proposition 4.2 we have $q \leq 40$. On the other side, for $4 \leq q \leq 40$ we can easily check, using GAP [13], that $w(G) \geq 21$, a contradiction. Similarly if $G = \mathrm{PSL}(3, 3)$ then $w(G) \geq 21$. Therefore G is solvable. □

5. Groups with few W-subgroups

In this section we describe groups with at most three W-subgroups. From now on, the Fitting subgroup of a group G will be denoted by $F(G)$. First we consider groups G with $w(G) = 1$.

Proposition 5.1. *Let G be a finite group, and assume that $w(G) = 1$. Then $F(G)$ has prime index in G. Moreover $F(G)$ is the unique proper normal self-centralizing subgroup of G, and the unique W-subgroup of G.*

Proof. Let H_0 be the unique W-subgroup of G. Then all conjugate subgroups of H_0 in G are W-subgroups of G. Since $w(G) = 1$, H_0 is normal. By Lemma 2.3, $w(G/H_0) \leq w(G) = 1$. Thus either G/H_0 is abelian or $w(G/H_0) = 1$.

If $w(G/H_0) = 1$, with W-subgroup H/H_0, then by Lemma 2.3, H is a W-subgroup of G, so $H = H_0$. But this is a contradiction, because the trivial subgroup $H/H_0 = H_0/H_0$ is not self-centralizing.

Hence G/H_0 is abelian and $G' \leq H_0$. Thus every proper subgroup K of G containing H_0 is normal and self-centralizing (and W-subgroup), since $G' \leq H_0 \leq K$ and upward-closedness property of self-centralizing subgroups.

Therefore H_0 has prime index in G. Moreover H_0 is the unique proper normal self-centralizing subgroup of G. By Theorem 4.4 and Proposition 3.3, G is solvable and not nilpotent. Then $F(G)$ is a proper normal self-centralizing subgroup of G (see, for instance, [11, 7.4.7]). Therefore $F(G) = H_0$, as required. □

Theorem 5.2. *Let G be a finite group with $w(G) = 1$, and let $p = |G : F(G)|$. Then $F(G)$ is abelian, and $G = P \ltimes A$, where P is any Sylow p-subgroup and A is an abelian normal p'-subgroup of G. Moreover all Sylow subgroups of G are abelian.*

Proof. By Proposition 5.1, $|G : F(G)| = p$, a prime number. Write $|G| = p^t p_1^{t_1} \ldots p_k^{t_k}$, where $t \geq 1$, $k \geq 1$, p_i is a prime and $t_i \geq 0$ for all $i = 1, \ldots, k$.

Then $|F(G)| = p^{t-1}p_1^{t_1}\dots p_k^{t_k}$. Let P be a Sylow p-subgroup of G. For all $i = 1,\dots,k$, let P_i be a Sylow p_i-subgroup of $F(G)$. Since $F(G)$ is nilpotent, each P_i is characteristic in $F(G)$ and so it is normal in G. Hence $A = P_1\times\dots\times P_k$ is a normal p'-subgroup of G. Therefore $G = P \ltimes A$.

By Corollary 3.4, $F(G)$ and all Sylow subgroups of G are abelian. In particular, A is abelian, as required. □

Now we consider groups G with $w(G) = 2$.

Lemma 5.3. *If G is a finite group with $w(G) = 2$, then $F(G)$ is W-subgroup and maximal abelian in G.*

Proof. By Theorem 4.4, G is solvable. So $F(G)$ is a W-subgroup of G and, by Corollary 3.4, $F(G)$ is abelian. Assume there exists an abelian subgroup A of G with $F(G) \leq A$. Then from $F(G) < A$ would follow that $F(G)$ is a W-subgroup of A, a contradiction since $w(A) = 0$. Thus $F(G) = A$, as required. □

Theorem 5.4. *Let G be a finite group with $w(G) = 2$, and let $H_1 = F(G)$ and H_2 be the two W-subgroups of G. Then H_2 is a normal subgroup of G having prime index q in G, and one of the following holds*:

(1) *H_1 has prime index p in G, and $G = P \ltimes A$, where P is a Sylow p-subgroup of G and A is an abelian normal p'-subgroup of G;*

(2) *$H_1 < H_2$, H_1 has prime index $p \neq q$ in H_2, and $G = (P \times Q) \ltimes A$, where P is a Sylow p-subgroup of G, Q is a Sylow q-subgroup of G, and A is an abelian normal $\{p,q\}'$-subgroup of G.*

Proof. By Theorem 4.4, G is solvable. Any conjugate subgroup of H_2 in G is a W-subgroup of G. Since H_1 is normal in G and $w(G) = 2$, it follows that H_2 is also normal in G. Moreover H_1 is abelian by Corollary 3.4.

Note that the case $H_2 < H_1 = F(G)$ can not occur, since $F(G)$ is a maximal abelian subgroup of G by Lemma 5.3, so it is minimal self-centralizing [8, Proposition 2]. Hence, by Lemma 2.3, $w(G/H) = 0$, and thus G/H_2 is abelian by Theorem 3.2. Moreover, since H_2 is self-centralizing, every proper subgroup of G containing H_2 is a W-subgroup of G. It follows that G/H_2 is simple, so it has prime order, say q.

Since H_1 is also self-centralizing, every proper normal subgroup of G containing H_1 is a W-subgroup of G. Hence either G/H_1 is simple or $H_1 < H_2$.

In the former case $|G : H_1| = p$, a prime number. Thus, arguing as in the proof of Theorem 5.2, we obtain $G = P \ltimes A$, where P is a Sylow p-subgroup of G and A is an abelian normal p'-subgroup of G. Therefore (1) holds.

In the latter case, by Lemma 2.2, $w(H_2) = 1$ and H_1 is the unique W-subgroup of H_2. It follows, by Theorem 5.2, that $H_1 = F(H_2)$ and $|H_2 : H_1|$ is a prime number, say p. Hence $|G : H_1| = pq$. Since H_2/H_1 is a W-subgroup of G/H_1, by Theorem 3.2 G/H_1 is not abelian. It follows that $p \neq q$. Write $|G| = p^\alpha q^\beta r_1^{\gamma_1} \dots r_k^{\gamma_k}$, where α and β are positive integers, r_i is a prime number different from p and q, and γ_i is a non-negative integer, for all $i = 1, \dots, k$. Thus $|H_2| = p^\alpha q^{\beta-1} r_1^{\gamma_1} \dots r_k^{\gamma_k}$ and $|H_1| = p^{\alpha-1} q^{\beta-1} r_1^{\gamma_1} \dots r_k^{\gamma_k}$. For all $i = 1, \dots, k$, let R_i denote a Sylow r_i-subgroup of H_1. Since H_1 is abelian, each R_i is normal in G. Hence $A = R_1 \times \dots \times R_k$ is an abelian normal $\{p, q\}'$-subgroup of G. As $|G : A| = p^\alpha q^\beta$, by Schur–Zassenhaus Theorem (see, for instance, [11, 9.3.6]) G has a subgroup D of order $p^\alpha q^\beta$. Thus $G = D \ltimes A$. Let P and Q be a Sylow p-subgroup and a Sylow q-subgroup of D, respectively. Since

$$|PQ| = \frac{|P| \cdot |Q|}{|P \cap Q|}$$

we have $PQ = D$. Hence P and Q are permutable. Therefore $D = P \times Q$, and (2) holds. □

Finally, we consider groups G having exactly three W-subgroups. Again, G is solvable by Theorem 4.4. Our first result shows that if G is a p-group with $w(G) = 3$, then $p = 2$.

Proposition 5.5. *Let G be a finite p-group. The following conditions are equivalent*:

(1) $w(G) = 3$;

(2) $G/Z(G) \cong C_2 \times C_2$;

(3) $|G : Z(G)| = 4$.

In particular, $p = 2$.

Proof. Arguing as in the proof of Proposition 3.3 we obtain that $w(G) = 3$ if and only if G has exactly three maximal abelian subgroups, and of course it is the union of them. Thus our statement follows from well-known results due to Scorza [10] (see also [7]). □

Proposition 5.6. *If G is a finite group with $w(G) = 3$, then all Sylow subgroups of odd order of G are abelian and G has a Sylow 2-subgroup P such that P is abelian or $|P : Z(P)| = 4$.*

Proof. If P is a Sylow p-subgroup of G, then by Lemma 2.2, $w(P) \leq w(G) = 3$. But P is nilpotent and by Proposition 3.3, $w(P) = 0$ or 3. Therefore by Theorem 3.2 and Proposition 5.5, P is abelian or $|P : Z(P)| = 4$. □

Proposition 5.7. *Let G be a finite nilpotent group. Then $w(G) = 3$ if and only if $G = P \ltimes A$, where P is a 2-group with $w(P) = 3$ and A is an abelian group of odd order.*

Proof. Let $w(G) = 3$. Then G is not abelian, so by Proposition 5.6 the Sylow 2-subgroup P of G is not abelian. Hence G has the required structure. Conversely, let $G = P \ltimes A$, where P is a 2-group with $w(P) = 3$ and A is an abelian group of odd order. Then $w(G) = 3$ by Corollary 2.6. □

Similar to Theorems 5.2 and 5.4 we have:

Theorem 5.8. *Let G be a finite group with exactly three W-subgroups. Then they are normal in G.*

Moreover, let G be nonnilpotent. Then the Fitting subgroup $F(G)$ is W-subgroup and maximal abelian in G.

Proof. If G is a p-group then by Proposition 5.5 the W-subgroups of G have index $p = 2$ and thus they are normal.

If G is a nilpotent group, similar to the p-group case and by Proposition 5.7 the W-subgroups of G have index 2.

Let G be a nonnilpotent group. Then $F(G)$ is a normal and self-centralizing subgroup of G, thus it is a W-subgroup of G. Assume that H_1, H_2 and $F(G)$ are distinct W-subgroups of G.

If H_1 is not normal in G, then the conjugate subgroups of H_1 are W-subgroups of G. Since $w(G) = 3$ and by normality of $F(G)$, the number of conjugate subgroups of H_1 equals 2. Hence $|G : N_G(H_1)| = 2$ and so $N_G(H_1)$ is normal in G. Therefore, $N_G(H_1)$ is a W-subgroup of G, since it is a proper, normal and self-centralizing subgroup in G.

But $w(G) = 3$ so $N_G(H_1) = F(G)$, a contradiction, because that by the proof of Theorem 5.4, $F(G)$ is a maximal abelian subgroup of G, so it is minimal self-centralizing.

Therefore H_1 is normal in G. Similarly H_2 is normal in G. □

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