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Corrigendum and addendum to "Centralizers of finite subgroups in Hall's universal group"

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Abstract – In Hall's universal group every non-trivial conjugacy class satisfies $CC = U$. Hence generalized version of J. G. Thompson's conjecture is true for every non-trivial conjugacy class C in U. Moreover Ore's conjecture (every element is a commutator) is true for U is added to [4]. In [4, Theorem 2.4] $C_U(F)/Z(F) \cong U$ is true if $Z(F) = 1$.

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Let A be a periodic abelian group. In [1, p. 10] and [2], a group G is called a *universal locally finite central extension* of A if the following conditions are satisfied.

(i) $A \leq Z(G)$.

(ii) G is locally finite.

(iii) A-injectivity. Suppose that $A \leq B \leq D$ with $A \leq Z(D)$, that D/A is finite, and that $\psi\colon B \to G$ is an A-monomorphism (that is $\psi(a) = a$ for all $a \in A$). Then there exists an extension $\overline{\psi}\colon D \to G$ of ψ to a monomorphism of D into G. The class of all groups satisfying the above three conditions is denoted by ULF(A). Hall's universal group $U \in$ ULF(1).

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Let F be a finite subgroup of U. The group U is an existentially closed group in the class of locally finite groups. Using this property, Hickin and Macintyre [3, Theorem 5] proved that $C_U(F)/Z(F)$ is a simple group. Our proof also shows that $C_U(F)$ is an extension of $Z(F)$. Hence if F is finite abelian, then by [2, p. 53] $C_U(F) \in \mathrm{ULF}(F)$.

Remark 1. If F is finite and $Z(F) = 1$, then our proof shows that $C_U(F)$ is isomorphic to U. In the general case if F is a finite subgroup of U with non-trivial center, then $C_U(F)/Z(F)$ is not necessarily isomorphic to U. But quotient $C_U(F)/Z(F)$ is simple and it is a subgroup of $C_U(Z(F))/Z(F)$ where $C_U(Z(F)) \in \mathrm{ULF}(Z(F))$. In particular in [4, Corollary 2.5], $C_U(F)$ has an epimorphic image isomorphic to U, should be replaced by $C_U(F)$ has a subgroup isomorphic to U.

In [5, Theorem 4.2] we use the same technique as in [4]. Therefore we notice that $C_G(F)/Z(F)$ is not necessarily isomorphic to G for a subgroup F contained in G_i for some $i \in I$, unless $Z(F) = 1$. But by [5, Lemma 3.8] $C_G(F)/Z(F)$ is simple.

Addendum

Since in Hall's universal group U every finite subgroup F is contained in a finite subgroup B with $Z(B) = 1$, we have $U \cong C_U(B) \leq C_U(F)$. Then $U \cong C_U(B)Z(F)/Z(F) \leq C_U(F)/Z(F)$.

Corollary 2. *The centralizer $C_U(F)$ of every finite subgroup F of U contains an isomorphic copy of U. Moreover $C_U(F)/Z(F)$ has a subgroup isomorphic to U.*

Corollary 3. *U can be written as a direct limit of finite simple groups $G_1 \leq G_2 \leq G_i \leq \cdots$ where $U = \bigcup_{i \in \mathbb{N}} G_i$. Then U has a descending chain of centralizers $C_U(G_i)$ where $C_U(G_1) \geq C_U(G_2) \geq C_U(G_3) \geq \cdots \geq C_U(G_i) \geq \cdots$ and for each $i \in \mathbb{N}$, $C_U(G_i) \cong U$ and $\bigcap_{i \in \mathbb{N}} C_U(G_i) = 1$*

The property that U is existentially closed in the class LF of locally finite groups implies that every group E, existentially closed in any class $\mathcal{C}$ of groups satisfying $\mathcal{C} \supseteq LF$ will contain isomorphic copies of U.

One of the properties of U is that, for every non-trivial conjugacy class C in U we have $C^2 = U$. It follows, clearly from this property that Generalized version of Thompson's conjecture [6, p.@ 1069-2] for U is true for any non-trivial conjugacy class C of U. The classification of finite simple groups is not used in the proof. Then the Ore conjecture: every element of U is a commutator, follows immediately from Thompson’s conjecture.

By using free product, every infinite group A generated by fewer than κ-elements can be embedded into a group B generated by fewer than κ-elements with $Z(B) = 1$. Then we may repeat the above arguments for U, to κ-existentially closed groups and state the following.

Corollary 4. *Let G be the unique κ-existentially closed group of inaccessible cardinality κ and F be any proper subgroup of G. Then $C_G(F)$ contains a subgroup isomorphic to G. In particular if $Z(F) = 1$, then $C_G(F)$ is isomorphic to G. Moreover $C_G(F)/Z(F)$ has a subgroup isomorphic to G.*

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