Rend. Sem. Mat. Univ. Padova, Vol. 142 (2019), 9–36
DOI 10.4171/RSMUP/27

On some variational measures related to the wide Denjoy integral and its counterparts

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Abstract – Weak and q-weak variational measures defined by Brian S. Thomson [*On* VBG *functions and the Denjoy–Khintchine integral*, Real Analysis Exchange, **41**(1) (2015/16), 173–226] are shown to coincide with variational measures resulting from Riemann definitions of some wide Denjoy type integrals. This fact is applied in characterizations of these integrals, via absolute continuity of weak and q-weak measures. In related results, it is discussed if these Riemann definitions can be essentially simplified. The paper is a follow-up to a paper by the author published some time ago in the Rendiconti [*On Riemann-type definition for the wide Denjoy integral*, Rendiconti del Seminario Matematico della Università di Padova, **126** (2011), 175–200].

Mathematics Subject Classification (2010). 26A39.

Keywords. Wide Denjoy integral, weak variational measure, composite gauge, local system, VBG, ACG, Lusin's condition (N).

1. Introduction and basics

Soon after discovering his renowned integral (equivalent to the *restricted Denjoy* integral, or *Denjoy–Perron integral*) [10], Ralph Henstock, in a remark in his textbook [11], suggested a similar Riemann-type approach that ought to lead to another Denjoy integral, now the *wide Denjoy integral* (or *Denjoy–Khintchine integral*). Since he provided there no detailed proof, this gave birth (much later) to a number of works on this problem [7, 16, 23]. The setting considered then was more general than Henstock's, with some generalized continuity (of primitives) used instead of ordinary continuity.

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In his previous paper on this topic (in the Rendiconti) [26] the author established how the definitions proposed and considered in [7, 16, 23] are related to various Lusin-type integrals, to the Denjoy–Khintchine integral in particular.

Being inspired by a recent exposition by Brian Thomson on measure-theoretic characterizations of ACG and VBG properties [30], in the present work we use Thomson's weak and q-weak measures in characterizations of various Lusin-type integrals defined with ACG and VBG, via absolute continuity of these measures (section 5.1). Moreover, we consider Riemann-type definitions directly related to weak and q-weak measures (section 5.2).

All reasonings are relatively standard as for measure and integration theory, with the main technique used being the Vitali covering lemma and Jordan decomposition theorem. The notation and terminology of the present work are consistent with those of [26].

1.1 – *Basic notation*

Let $E \subset \mathbb{R}$, then $|E|$ stands for the Lebesgue outer measure of E, while $\operatorname{int} E$ and $\operatorname{cl} E$ (respectively) for the interior and closure of E. Let $F\colon E \to \mathbb{R}$. We say F satisfies *Lusin's condition* (N) if $|F(D)| = 0$ as long as $D \subset E$ and $|D| = 0$. With $\operatorname{Var} F$ we denote the (total) variation of F.

1.2 – *Divisions, gauges, and forms*

The symbol $\langle x, y\rangle$, $x, y \in \mathbb{R}$, stands for the compact interval (segment) with endpoints at x and y; i.e., $\langle x, y\rangle = [x, y]$ if $x \le y$, $[y, x]$ otherwise. A pair $(\langle x, y\rangle, x)$ is called a *tagged interval*. This tagged interval is said to be *anchored* in a set $E \subset \mathbb{R}$ if $x \in E$, to be *in* E if $\langle x, y\rangle \subset E$, while it (and $\langle x, y\rangle$ alone) is called *E-fine* if both $x, y \in E$. If δ is a positive function defined at the *tag* x, then we say $(\langle x, y\rangle, x)$ is δ-fine if $|x - y| < \delta(x)$. A finite collection of tagged intervals $(\langle x_i, y_i\rangle, x_i)$, $i = 1, \dots, m$, with the property that $\langle x_i, y_i\rangle \cap \langle x_j, y_j\rangle$ has empty interior if $i \ne j$, is called a *division*. A division is anchored in E, E-fine or δ-fine if all its members are such. Consider an E-fine division $\mathcal{P} = \{(\langle x_i, y_i\rangle, x_i)\}_{i=1}^m$, a function F on E, and introduce the following notations:

$$\mathcal{I}(\mathcal{P}) = \bigcup_{i=1}^m \langle x_i, y_i\rangle, \quad |\Delta| F(\mathcal{P}) = \sum_{i=1}^m |F(y_i) - F(x_i)|,$$
$$\sigma(F, \mathcal{P}) = \sum_{i=1}^m F(x_i)|y_i - x_i|.$$

We will say a division $\mathcal{P}$ is a *partition* of a segment $[a, b]$ if $\mathcal{I}(\mathcal{P}) = [a, b]$.

A countable collection $\boldsymbol{\alpha} = \{E_n\}_{n=1}^{\infty}$ of sets is an E-*form* if $\bigcup_{n=1}^{\infty} E_n = E$. An E-form is called *closed* or *measurable* if all E_n are (respectively) closed and measurable. The symbol Is $\boldsymbol{\alpha}$ stands for the set of all x such that for some n, $x \in E_n$ and x is isolated from either side of E_n. For two E-forms $\boldsymbol{\alpha}, \boldsymbol{\beta}$ we say $\boldsymbol{\alpha}$ is *finer* than $\boldsymbol{\beta}$, and write $\boldsymbol{\alpha} \succ \boldsymbol{\beta}$ or $\boldsymbol{\beta} \prec \boldsymbol{\alpha}$, if every member of $\boldsymbol{\alpha}$ is contained in some member of $\boldsymbol{\beta}$.

A function $F\colon E \to \mathbb{R}$ is said to be VBG (resp. [VBG]) if there is a (closed) E-form $\{E_n\}_{n=1}^{\infty}$ such that the restriction $F \restriction E_n$ is of bounded variation for every n (that is, $\operatorname{Var}(F \restriction E_n) < \infty$). Analogously, the concepts of ACG and [ACG] are defined.

Consider an E-form $\boldsymbol{\alpha} = \{E_n\}_{n=1}^{\infty}$ and a related (i.e., $\delta_n\colon E_n \to (0,\infty)$) sequence of positive functions (called *gauges*) $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ (the term *gauge* will be used also for $\boldsymbol{\delta}$). Given n, we will say a tagged interval $(\langle x, y\rangle, x)$ is $\boldsymbol{\alpha}$-fine if it is E_n-fine for some n, (E_n, δ_n)-fine if it is both E_n-fine and δ_n-fine, and $\boldsymbol{\delta}$-fine if it is (E_n, δ_n)-fine for some n. Accordingly $\boldsymbol{\alpha}$-fine, (E_n, δ_n)-fine and $\boldsymbol{\delta}$-fine divisions are understood.

1.3 – *Differentiation*

Let $F\colon E \to \mathbb{R}$ and $x \in E$.

Upper and lower derivatives of F at x are denoted respectively by $\overline{\mathrm{D}}F(x)$ and $\underline{\mathrm{D}}F(x)$ (while the ordinary derivative of F at x by $F'(x)$); $\mathcal{D}_F$ is the set of all differentiability points of F, that is $x \in \mathcal{D}_F$ if it is an accumulation point of E and $F'(x)$ exists finite. Now, let an E-form $\boldsymbol{\alpha} = \{E_n\}_{n=1}^{\infty}$ be given. We say F is $\boldsymbol{\alpha}$-differentiable to $F'_{\boldsymbol{\alpha}}(x) \in \mathbb{R}$ at $x \in E$ if for every n such that $x \in E_n$ and x is an accumulation point of E_n (and for at least one n), $F'_{\boldsymbol{\alpha}}(x) = (F \restriction E_n)'(x)$. $\mathcal{D}_F^{\boldsymbol{\alpha}}$ is the set of all $x \in E$ such that $F'_{\boldsymbol{\alpha}}(x)$ exists finite. Lower ($\underline{D}_{\boldsymbol{\alpha}} F(x)$) and upper ($\overline{D}_{\boldsymbol{\alpha}} F(x)$) derivatives of F relative to $\boldsymbol{\alpha}$ are understood in a natural way; they exist, finite or infinite, at all except countably many $x \in E$.

F is said to be *approximately differentiable* at x, with $F'_{\mathrm{ap}}(x) \in \mathbb{R}$ the value of the derivative, if there is a measurable set $P \subset E$ with density one at x, i.e.,

$$d(P, x) = \lim_{h \searrow 0} \frac{|P \cap [x-h, x+h]|}{2h} = 1, \tag{1}$$

such that

$$F'_{\mathrm{ap}}(x) = \lim_{y \to x,\, y \in P} \frac{F(y) - F(x)}{y - x}.$$

With $\mathcal{D}_F^{\mathrm{ap}}$ we denote the set of all $x \in E$ at which F is approximately differentiable. By the Denjoy–Khintchine theorem [19, (4.3), p. 222], being a counterpart of the Lebesgue differentiability theorem, for every measurable VBG-function F, $|E \setminus \mathcal{D}_F^{\mathrm{ap}}| = 0$.

1.4 – *Variational measures*

Now, let $F\colon S \to \mathbb{R}$, S an ambivalent (i.e., at the same time an $\mathcal{F}_\sigma$ and $\mathcal{G}_\delta$) set. For a subset $E \subset S$, a closed S-form $\boldsymbol{\alpha}$ and a related gauge $\boldsymbol{\delta}$ denote

$$\bar{\mu}^F_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E) = \sup_{\mathcal{P}} |\Delta| F(\mathcal{P}),$$

where sup runs over all $\boldsymbol{\delta}$-fine divisions $\mathcal{P}$ anchored in E. Denote moreover

$$\bar{\mu}^F_{\boldsymbol{\alpha}}(E) = \inf_{\boldsymbol{\delta}} \bar{\mu}^F_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E) \quad \text{and} \quad \bar{\mu}^F(E) = \inf_{\boldsymbol{\alpha}} \bar{\mu}^F_{\boldsymbol{\alpha}}(E),$$

where inf's run over all $\boldsymbol{\delta}$ related to $\boldsymbol{\alpha}$ and over all closed S-forms $\boldsymbol{\alpha}$, respectively. The notation $\bar{\mu}^F_{\boldsymbol{\alpha},\infty}(E)$ for the sup of $|\Delta| F(\mathcal{P})$ over all $\boldsymbol{\alpha}$-fine $\mathcal{P}$ anchored in E, is consistent with the above. With all (instead of only closed) S-forms $\boldsymbol{\alpha}$ taken into account, we can define analogously $\mu^F_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E)$, $\mu^F_{\boldsymbol{\alpha},\infty}(E)$, $\mu^F_{\boldsymbol{\alpha}}(E)$, and $\mu^F(E)$. In this case values of F on E itself are essential only (so that one can consider E-forms instead of S-forms). This is also the case for "barred" μ's, but only as far as E is ambivalent.

$\bar{\mu}^F_{\boldsymbol{\alpha}}$, $\bar{\mu}^F$, $\mu^F_{\boldsymbol{\alpha}}$, and μ^F are easily shown to be metric outer measures, cf. [30, Lemma 10], so they give rise to Borel measures in $\mathbb{R}$. When endowed with the local system component, see section 5 of this paper, they arise in connection with Riemann-type definitions of $\mathcal{F}_i^{\blacktriangle}$-integrals [26].

2. Connection to VBG and Thomson's weak measures

Lemma 1. *Let $F\colon E \to \mathbb{R}$. Then,*

$$\mu^F(E) = \lim_{\boldsymbol{\alpha}} \mu^F_{\boldsymbol{\alpha},\infty}(E), \tag{2}$$

where the limit is taken over all E-forms $\boldsymbol{\alpha}$ ordered with $\prec$.

Proof. The argument is folklore. By definition, $\mu^F = \inf_{\boldsymbol{\alpha}} \mu^F_{\boldsymbol{\alpha}}$ and $\mu^F_{\boldsymbol{\alpha}} = \inf_{\boldsymbol{\delta}} \mu^F_{\boldsymbol{\alpha},\boldsymbol{\delta}}$. So, given $\varepsilon > 0$, there exist an E-form $\boldsymbol{\alpha} = \{E_n\}_{n=1}^\infty$ and a related gauge $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^\infty$ such that $\mu^F(E) \geq \mu^F_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E) - \varepsilon$. Define an E-form $\boldsymbol{\beta}$ finer than $\boldsymbol{\alpha}$ as follows:

$$E_{nkj} = \{x \in E_n\colon \delta_n(x) > 1/k\} \cap [j/k, (j+1)/k],$$

$n, k \in \mathbb{N}$, $j \in \mathbb{Z}$. Since $\bigcup_{k,j} E_{nkj} = E_n$, $\boldsymbol{\beta} = \{E_{nkj}\}_{n,k,j}$ is indeed an E-form and $\boldsymbol{\beta} \succ \boldsymbol{\alpha}$. Notice that every tagged interval that is $\boldsymbol{\beta}$-fine, is $\boldsymbol{\delta}$-fine. Hence $\mu^F_{\boldsymbol{\beta},\infty}(E) \le \mu^F_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E)$. So, for every E-form $\boldsymbol{\gamma} \succ \boldsymbol{\beta}$,

$$\mu^F_{\boldsymbol{\gamma},\infty}(E) \ge \mu^F(E) \ge \mu^F_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E) - \varepsilon \ge \mu^F_{\boldsymbol{\beta},\infty}(E) - \varepsilon \ge \mu^F_{\boldsymbol{\gamma},\infty}(E) - \varepsilon$$

and the claim follows. □

Remark 1. The right hand side of (2) is the (*quite weak* or *q-weak*) variational measure W^q_F from Thomson [30, Definition 22].

Lemma 2. *Let* $F: E \to \mathbb{R}$. *Then the following are equivalent*:

(i) *F is VBG,*

(ii) *μ^F is σ-finite,*

(iii) *$\mu^F_{\boldsymbol{\alpha}}$ is σ-finite for some E-form $\boldsymbol{\alpha}$.*

Proof. This follows from Lemma 1 and the fact that, given an E-form $\boldsymbol{\alpha}$, $\mu^F_{\boldsymbol{\alpha},\infty}(E_n) = \operatorname{Var}(F \restriction E_n)$ for every $E_n \in \boldsymbol{\alpha}$. □

(i) $\iff$ (ii) of the above lemma is 3. of Lemma 24 in [30].

Lemma 3. *Let $F: D \to \mathbb{R}$, D closed. Then, for a set $E \subset D$, we have* (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv), *where*

(i) *F is [VBG] on some ambivalent set S, $E \subset S \subset D$,*

(ii) *$\bar{\mu}^F_{\boldsymbol{\alpha}}$ is σ-finite on E for some closed D-form $\boldsymbol{\alpha}$,*

(iii) *$\bar{\mu}^F$ is σ-finite on E,*

(iv) *F is [VBG] on some $\mathcal{F}_\sigma$ set S, $E \subset S \subset D$.*

Proof. (ii) $\Longrightarrow$ (iii) is obvious by definition.

(i) $\Longrightarrow$ (ii). Let $\boldsymbol{\alpha} = \{S_n\}_{n=1}^\infty$ be a closed S-form such that $S \supset E$ is ambivalent and $\operatorname{Var}(F \restriction S_n) < \infty$ for every n. We have $\bar{\mu}^F_{\boldsymbol{\alpha}}(E \cap S_n) \le \bar{\mu}^F_{\boldsymbol{\alpha}}(S_n) \le \bar{\mu}^F_{\boldsymbol{\alpha},\infty}(S_n) = \operatorname{Var}(F \restriction S_n) < \infty$ (recall the remark after the definition of $\bar{\mu}^F$).

(iii) $\Longrightarrow$ (iv). Let $\{E_n\}_{n=1}^\infty$ be an E-form such that $\bar{\mu}^F(E_n) < \infty$ for every n. Fix n and take a closed D-form $\boldsymbol{\alpha} = \{D_k\}_{k=1}^\infty$ such that $\bar{\mu}^F_{\boldsymbol{\alpha}}(E_n) < \infty$. We can assume that there is $k_n \in \mathbb{N}$ such that $E_n \subset D_{k_n}$. Let $\boldsymbol{\delta} = \{\delta_k\}_{k=1}^\infty$ be a gauge related to $\boldsymbol{\alpha}$ such that $\bar{\mu}^F_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E_n) < \infty$. Denote

$$E_{nmj} = \{x \in E_n : \delta_{k_n}(x) > 2/m\} \cap [j/m, (j+1)/m],$$

$m \in \mathbb{N}$, $j \in \mathbb{Z}$, and $\operatorname{cl} E_{nmj} = C_{nmj}$. Of course, $C_{nmj} \subset D_{k_n}$. Fix m, j and take any collection $\{[x_i, y_i]\}_{i=1}^r$ of nonoverlapping C_{nmj}-fine segments, $y_{i-1} \le x_i$ for $i \ge 2$. Given i, there is a point $\xi_i \in E_{nmj}$ with $|\xi_i - x_i| < 1/m$ (and so $|y_i - \xi_i| < 2/m$). One can assume, moreover, that $\xi_i < y_i$ and, if $\xi_i < x_i$ and $i \ge 2$, then $\xi_i > x_{i-1}$. Let us write

$$\sum_{i=1}^{r} |F(y_i) - F(x_i)| \le \sum_{i=1}^{r} |F(y_i) - F(\xi_i)| + \sum_{i=1}^{r} |F(\xi_i) - F(x_i)|. \tag{3}$$

Notice that both tagged intervals $(\langle x_i, \xi_i\rangle, \xi_i)$ and $([\xi_i, y_i], \xi_i)$ are $\boldsymbol{\delta}$-fine (because $\delta_{k_n}(\xi_i) > 2/m$ and all x_i, y_i, ξ_i are in D_{k_n}) and anchored in E_n. Moreover, $(\langle x_i, \xi_i\rangle, \xi_i)$ and $([\xi_i, y_i], \xi_i)$ can mutually overlap and can be overlapping with other such intervals, but only with $(\langle x_{i-1}, \xi_{i-1}\rangle, \xi_{i-1})$ and $([\xi_{i-1}, y_{i-1}], \xi_{i-1})$. Therefore, using the estimate (3) separately for even and odd i, we can conclude that $\sum_{i=1}^r |F(y_i) - F(x_i)|$ does not exceed $4 \cdot \bar{\mu}^F_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E_n) < \infty$. We proved that F is VB on C_{nmj} and so [VBG] on $S = \bigcup_{n,m,j} C_{nmj} \supset E$. □

The implication (ii) $\Longrightarrow$ (i) (and so (iii) $\Longrightarrow$ (i) and (iv) $\Longrightarrow$ (i)) is not true. Take a continuous function F on a segment I whose variation is infinite on every subsegment of I. Then for $E = I \cap \mathbb{Q}$ and any closed I-form $\boldsymbol{\alpha}$, $\bar{\mu}^F_{\boldsymbol{\alpha}}(E) = 0$. On the other hand, if S is a $\mathcal{G}_\delta$ set dense in some segment in I, and $\{S_n\}_{n=1}^\infty$ a closed S-form, then from the Baire category theorem, for some n, S_n covers a portion of S. An easy argument shows that F cannot be VB on any portion of S. Thus $\operatorname{Var}(F \restriction S_n) = \infty$.

Problem 1. *Are the implications* (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) *reversible*?

3. Representations of variational measures

3.1 – *Representations of* μ^F

Let $F\colon E \to \mathbb{R}$.

Theorem 2. *Assume F is monotone. Then, for some E-form $\boldsymbol{\alpha}$,*

$$\mu^F_{\boldsymbol{\alpha}}(E) = \mu^F(E) = |F(E)|.$$

Moreover, then, $\mu^F_{\boldsymbol{\beta}}(E) = |F(E)|$ for every E-form $\boldsymbol{\beta} \succ \boldsymbol{\alpha}$.

Proof. Obviously, $\mu^F_{\boldsymbol{\alpha}}(E) \ge \mu^F(E)$ for every $\boldsymbol{\alpha}$. Hence, it is enough to prove that (a) $\mu^F(E) \ge |F(E)|$ and, for some $\boldsymbol{\alpha}$, (b) $\mu^F_{\boldsymbol{\alpha}}(E) \le |F(E)|$. For the sake of simplicity, in the proof we assume F is bounded.

(a) Take any E-form $\boldsymbol{\alpha} = \{E_n\}_{n=1}^{\infty}$ and, given $\varepsilon > 0$, a gauge $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ related to $\boldsymbol{\alpha}$ such that $\mu^F_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E) \le \mu^F_{\boldsymbol{\alpha}}(E) + \varepsilon$. Denote by E' the set of all $x \in E$ such that $F(x)$ is isolated in the image $F(E_n)$ on either side, for some $n \in \mathbb{N}$. $F(E')$ is, clearly, countable. Refine $\boldsymbol{\alpha}$ to $\tilde{\boldsymbol{\alpha}}$ by extracting E' from all E_n, that is, $\tilde{\boldsymbol{\alpha}} = \{E_n \setminus E'\}_{n=1}^{\infty} \cup \{F^{-1}(y) \cap E_n\}_{y \in F(E'), n \in \mathbb{N}}$. By definition, $\mu^F_{\boldsymbol{\alpha}}(E) \ge \mu^F_{\tilde{\boldsymbol{\alpha}}}(E)$. Let $\tilde{\boldsymbol{\delta}} = \{\tilde{\delta}_n\}_{n=1}^{\infty}$, where $\tilde{\delta}_n$ is δ_n restricted to $E_n \setminus E'$. (Then $\tilde{\boldsymbol{\delta}}$ can be regarded as related to $\tilde{\boldsymbol{\alpha}}$ as other members of $\tilde{\boldsymbol{\alpha}}$ are meaningless in connection with $\mu^F_{\tilde{\boldsymbol{\alpha}}}$). The collection of all intervals of the form $\langle F(x), F(y)\rangle$, where $(\langle x, y\rangle, x)$ is a $\tilde{\boldsymbol{\delta}}$-fine interval, is a Vitali cover of $F(E \setminus E')$, a co-countable subset of $F(E)$. By the Vitali covering lemma, given $\varepsilon > 0$, there exists a $\tilde{\boldsymbol{\delta}}$-fine division $\mathcal{P}$ anchored in $E \setminus E'$ such that

$$|\Delta| F(\mathcal{P}) \ge |F(E \setminus E')| - \varepsilon = |F(E)| - \varepsilon.$$

This proves that $\mu^F_{\boldsymbol{\alpha}}(E) + \varepsilon \ \ge \ \mu^F_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E) \ \ge \ \mu^F_{\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\delta}}}(E) \ \ge \ |F(E)| - \varepsilon$, i.e., $\mu^F_{\boldsymbol{\alpha}}(E) \ge |F(E)|$. As $\boldsymbol{\alpha}$ was arbitrary, $\mu^F(E) \ge |F(E)|$.

(b) Take any E-form $\boldsymbol{\alpha} = \{E_n\}_{n=1}^{\infty}$ and consider E' and the refinement $\tilde{\boldsymbol{\alpha}} \succ \boldsymbol{\alpha}$ as defined in the proof of (a). Let $\varepsilon > 0$ and pick an open set $O \supset F(E)$ such that $|O| \le |F(E)| + \varepsilon$. F is continuous at every point of $x \in E \setminus E'$, so, there exists $\delta(x)$ with the property that $\langle F(x), F(y)\rangle \subset O$ for every $y \in (x - \delta(x), x + \delta(x)) \cap E$. Let δ_n be δ on $E_n \setminus E'$ and set $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ (we neglect, as before, gauges on other members of $\tilde{\boldsymbol{\alpha}}$). For every $\boldsymbol{\delta}$-fine interval $(\langle x, y\rangle, x)$ with $x \in E \setminus E'$ we have $\langle F(x), F(y)\rangle \subset O$, so if $\mathcal{P}$ is a $\boldsymbol{\delta}$-fine division, then, in view of the monotonicity of F,

$$|\Delta| F(\mathcal{P}) \le |O| \le |F(E)| + \varepsilon.$$

So, $\mu^F_{\tilde{\boldsymbol{\alpha}}}(E) \le \mu^F_{\tilde{\boldsymbol{\alpha}},\boldsymbol{\delta}}(E) \le |F(E)| + \varepsilon$ and, as ε was arbitrary, $\mu^F_{\tilde{\boldsymbol{\alpha}}}(E) \le |F(E)|$. We proved that $\mu^F(E) = \mu^F_{\tilde{\boldsymbol{\alpha}}}(E) = |F(E)|$. □

Remark 2. The E-form $\tilde{\boldsymbol{\alpha}}$ we constructed in the proof of (a) above is good for all $D \subset E$, that is, $\mu^F_{\tilde{\boldsymbol{\alpha}}}(D) = \mu^F(D) = |F(D)|$.

Assume now F is of bounded variation. Let $F = F_+ - F_-$ be the Jordan decomposition of F; $V = F_+ + F_-$ the function of total variation of F, that is, $V(y) - V(x) = \operatorname{Var}(F \restriction (E \cap [x, y]))$, $x \le y$, $x, y \in E$. V, F_+, F_- are unique up to an additive constant.

THEOREM 3. *There exists an E-form $\boldsymbol{\alpha}$ such that*

$$\mu^F(E) = \mu^F_{\boldsymbol{\alpha}}(E) = |F_+(E)| + |F_-(E)| = |V(E)|.$$

In the proof we will make use of the following well-known facts.

LEMMA 4. *Let $F\colon E \to \mathbb{R}$ be monotone, $Z = \{x \in E\colon \underline{D}F(x) = 0\}$. Then $|F(Z)| = 0$.*

PROOF. The Vitali covering lemma. □

LEMMA 5 (Hahn's decomposition). *Let F be of bounded variation and $x \in E$. If $\underline{D}F_\pm(x) > 0$, then $\underline{D}F_\mp(x) = 0$.*

PROOF OF THEOREM 3. Denote $E^\pm = \{x \in E\colon \underline{D}F_\pm(x) > 0\}$. By Lemma 5, $E^+ \cap E^- = \emptyset$. By Lemma 4 and Theorem 2, $\mu^{F_\pm}(E) = \mu^{F_\pm}_{\boldsymbol{\alpha}}(E) = |F_\pm(E)|$, $\mu^V(E) = \mu^V_{\boldsymbol{\alpha}}(E) = |V(E)|$, $\mu^{F_\mp}(E^\pm) = \mu^{F_\mp}_{\boldsymbol{\alpha}}(E^\pm) = |F_\mp(E^\pm)| = 0$ for some E-form $\boldsymbol{\alpha}$. For an $\varepsilon > 0$, take a gauge $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^\infty$ related to $\boldsymbol{\alpha}$ such that $\mu^{F_\pm}_{\boldsymbol{\alpha}}(E) + \varepsilon \geq \mu^{F_\pm}_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E)$, $\mu^V_{\boldsymbol{\alpha}}(E) + \varepsilon \geq \mu^V_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E)$, and $\mu^{F_\mp}_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E^\pm) < \varepsilon$. Consider any $\boldsymbol{\delta}$-fine division $\mathcal{R}$ anchored in E. We have

$$\begin{aligned}|\Delta|F(\mathcal{R}) &\leq |\Delta|F_+(\mathcal{R}) + |\Delta|F_-(\mathcal{R}) \leq \mu^{F_+}_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E) + \mu^{F_-}_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E) \\ &\leq \mu^{F_+}_{\boldsymbol{\alpha}}(E) + \mu^{F_-}_{\boldsymbol{\alpha}}(E) + 2\varepsilon = |F_+(E)| + |F_-(E)| + 2\varepsilon.\end{aligned}$$

This means that $\mu^F_{\boldsymbol{\alpha}}(E) \leq \mu^F_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E) \leq |F_+(E)| + |F_-(E)| + 2\varepsilon$, so, as ε was arbitrary, $\mu^F_{\boldsymbol{\alpha}}(E) \leq |F_+(E)| + |F_-(E)|$. Similarly,

$$\begin{aligned}|\Delta|F(\mathcal{R}) &\leq |\Delta|F_+(\mathcal{R}) + |\Delta|F_-(\mathcal{R}) = |\Delta|V(\mathcal{R}) \\ &\leq \mu^V_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E) \leq \mu^V_{\boldsymbol{\alpha}}(E) + \varepsilon = |V(E)| + \varepsilon\end{aligned}$$

and so, $\mu^F_{\boldsymbol{\alpha}}(E) \leq \mu^F_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E) \leq |V(E)| + \varepsilon$, whence $\mu^F(E) \leq |V(E)|$.

Now, given $\varepsilon > 0$, take sets $O^\pm \supset E^\pm$, open in E, having the property that $|F_\pm(O^\pm)| - |F_\pm(E^\pm)| \leq |F_\pm(O^\pm \setminus E^\pm)| < \varepsilon$, $|V(O^\pm)| - |V(E^\pm)| \leq |V(O^\pm \setminus E^\pm)| < \varepsilon$, and $|F_\mp(O^\pm)| = |F_\mp(O^\pm)| - |F_\mp(E^\pm)| < \varepsilon$. We use here the fact that $E^\pm$ are Borel relative to E. Prune all δ_n with $O^\pm$ at $x \in E^\pm$ and call the pruned gauge again by $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^\infty$.

All segments of the form $\langle F_\pm(x), F_\pm(y)\rangle$, $(\langle x, y\rangle, x)$ being $\boldsymbol{\delta}$-fine, $x \in E^\pm$, form a Vitali cover for a co-countable subset of $F_\pm(E^\pm)$. By the Vitali covering lemma, there exists a $\boldsymbol{\delta}$-fine division $\mathcal{R}^+$ anchored in E^+ with the property that

$$|F_+(E)| = |F_+(E^+)| \le |\Delta|F_+(\mathcal{R}^+) + \varepsilon. \tag{4}$$

Since $\mathcal{I}(\mathcal{R}^+) \cap E \subset O^+$ and $O^+ \cap E^- \subset O^+ \setminus E^+$,

$$\begin{aligned}\left|F_-(E^-) \setminus F_-(\mathcal{I}(\mathcal{R}^+) \cap E^-)\right| &\ge |F_-(E^-) \setminus F_-(O^+ \setminus E^+)| \\ &\ge |F_-(E^-)| - |F_-(O^+ \setminus E^+)| > |F_-(E^-)| - \varepsilon.\end{aligned}$$

Again by the Vitali covering lemma, there exists a $\boldsymbol{\delta}$-fine division $\mathcal{R}^-$ anchored in E^-, with all segments nonoverlapping with $\mathcal{I}(\mathcal{R}^+)$, such that

$$\begin{aligned}|\Delta|F_-(\mathcal{R}^-) + \varepsilon &\ge \left|F_-(E^-) \setminus F_-(\mathcal{I}(\mathcal{R}^+) \cap E^-)\right| \\ &\ge |F_-(E^-)| - \varepsilon = |F_-(E)| - \varepsilon.\end{aligned} \tag{5}$$

The division $\mathcal{R} = \mathcal{R}^+ \cup \mathcal{R}^-$ is $\boldsymbol{\delta}$-fine and so, by (4) and (5),

$$\begin{aligned}\mu^F_{\boldsymbol{\alpha}}(E) + \varepsilon \ge \mu^F_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E) &\ge |\Delta|F(\mathcal{R}) = |\Delta|F(\mathcal{R}^+) + |\Delta|F(\mathcal{R}^-) \\ &\ge |\Delta|F_+(\mathcal{R}^+) - |\Delta|F_-(\mathcal{R}^+) + |\Delta|F_-(\mathcal{R}^-) - |\Delta|F_+(\mathcal{R}^-) \\ &\ge |F_+(E)| - \varepsilon - \mu^{F_-}_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E^+) + |F_-(E)| - 2\varepsilon - \mu^{F_+}_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E^-) \\ &\ge |F_+(E)| + |F_-(E)| - 5\varepsilon,\end{aligned}$$

Hence, as $\boldsymbol{\alpha}$ did not depend on ε, $\mu^F_{\boldsymbol{\alpha}}(E) \ge |F_+(E)| + |F_-(E)|$. As the argument follows for any $\boldsymbol{\beta} \succ \boldsymbol{\alpha}$ instead of $\boldsymbol{\alpha}$ (cf. Theorem 2), $\mu^F(E) \ge |F_+(E)| + |F_-(E)|$.

Next we proceed similarly as above with V instead of $F_\pm$. Take a $\boldsymbol{\delta}$-fine division $\mathcal{P}^+$ anchored in E^+ so that $|V(E^+)| \le |\Delta|V(\mathcal{P}^+) + \varepsilon$. (As $\mathcal{I}(\mathcal{P}^+) \cap E \subset O^+$ and $O^+ \cap E^- \subset O^+ \setminus E^+$), one can then find a $\boldsymbol{\delta}$-fine division $\mathcal{P}^-$ anchored in E^-, with $\mathcal{I}(\mathcal{P}^+) \cap \operatorname{int} \mathcal{I}(\mathcal{P}^-) = \emptyset$, and such that

$$|\Delta|V(\mathcal{P}^-) + \varepsilon \ge \left|V(E^-) \setminus V(\mathcal{I}(\mathcal{P}^+) \cap E)\right| \ge |V(E^-)| - \varepsilon.$$

For the ($\boldsymbol{\delta}$-fine) division $\mathcal{P}^+ \cup \mathcal{P}^-$ one has then

$$\begin{aligned}|F_+(E)| + |F_-(E)| + 4\varepsilon &= \mu^{F_+}_{\boldsymbol{\alpha}}(E) + \varepsilon + \varepsilon + \varepsilon + \mu^{F_-}_{\boldsymbol{\alpha}}(E) + \varepsilon \\ &\ge \mu^{F_+}_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E) + \mu^{F_-}_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E^+) + \mu^{F_+}_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E^-) + \mu^{F_-}_{\boldsymbol{\alpha},\boldsymbol{\delta}}(E) \\ &\ge |\Delta|F_+(\mathcal{P}^+) + |\Delta|F_-(\mathcal{P}^+) + |\Delta|F_+(\mathcal{P}^-) + |\Delta|F_-(\mathcal{P}^-) \\ &= |\Delta|V(\mathcal{P}^+) + |\Delta|V(\mathcal{P}^-) \\ &\ge |V(E^+)| - \varepsilon + |V(E^-)| - 2\varepsilon = |V(E)| - 3\varepsilon.\end{aligned}$$

(The last equality follows from Lemma 4 and the fact that if $\underline{D}F_+(x) = \underline{D}F_-(x) = 0$, then $\underline{D}V(x) = 0$). Hence, $|F_+(E)| + |F_-(E)| \ge |V(E)|$. The proof is over. □

Corollary 1. *Suppose F is VBG on E. Then $\mu^F(D) = 0$ holds for every nullset $D \subset E$ if and only if F satisfies Lusin's condition (N).*

It follows from Theorem 3 and the fact that the class of VB-functions with (N) on (arbitrary) $E \subset \mathbb{R}$ is a linear space [6, Corollary 2], and that if F satisfies (N) then so does V [25, Lemma 3.3].

Notice that, for any E-form $\boldsymbol{\alpha} = \{E_n\}_{n=1}^{\infty}$, the interval function $I \mapsto \mu_{\boldsymbol{\alpha}}^F(E \cap I)$ is additive; i.e., $\mu_{\boldsymbol{\alpha}}^F(E\cap I) + \mu_{\boldsymbol{\alpha}}^F(E \cap J) = \mu_{\boldsymbol{\alpha}}^F(E \cap (I \cup J))$ provided the segments I and J are nonoverlapping. The same concerns, thus, μ^F.

($\mu_{\boldsymbol{\alpha}}^F$–Saks–Henstock) Lemma 6. *Let $F \colon E \to \mathbb{R}$, $\mu^F(E) < \infty$. If an E-form $\boldsymbol{\beta}$ is such that $\mu_{\boldsymbol{\beta},\infty}^F(E) < \mu^F(E) + \varepsilon$, then for every $\boldsymbol{\beta}$-fine division $\mathcal{P}$, $|\Delta|F(\mathcal{P}) \leq \mu^F(E \cap \mathcal{I}(\mathcal{P})) + \varepsilon$.*

Proof. Suppose not; i.e., there is a $\boldsymbol{\beta}$-fine division $\mathcal{P}$ such that $|\Delta|F(\mathcal{P}) > \mu^F(E \cap \mathcal{I}(\mathcal{P})) + \varepsilon$. Split $\operatorname{cl}(\mathbb{R} \setminus \mathcal{I}(\mathcal{P}))$ into s nonoverlapping closed intervals J and pick a $\boldsymbol{\beta}$-fine division $\mathcal{P}_J$ in every such J, anchored in E, so that $|\Delta|F(\mathcal{P}_J) > \mu^F(J \cap E) - 1/s^2$. The division $\mathcal{R} = \mathcal{P} \cup \bigcup_J \mathcal{P}_J$ is $\boldsymbol{\beta}$-fine and

$$|\Delta|F(\mathcal{R}) = |\Delta|F(\mathcal{P}) + \sum_J |\Delta|F(\mathcal{P}_J) > \mu^F(E \cap \mathcal{I}(\mathcal{P})) + \varepsilon + \sum_J \mu^F(E \cap J) - \frac{1}{s}.$$

According to the remark preceding the proof, the right hand side equals $\mu^F(E) + \varepsilon - 1/s$. As s could have been arbitrarily large here, $\mu_{\boldsymbol{\beta},\infty}^F(E) \geq \mu^F(E) + \varepsilon$, a contradiction with the assumption on $\boldsymbol{\beta}$. □

Theorem 4. *For any $F \colon E \to \mathbb{R}$ there exists an E-form $\boldsymbol{\alpha}$ such that $\mu^F(E) = \mu_{\boldsymbol{\alpha}}^F(E)$.*

Proof. If F is not VBG, then given any E-form $\boldsymbol{\beta} = \{E_n\}_{n=1}^{\infty}$ there is an n such that $F \restriction E_n$ has unbounded variation. Then $\mu_{\boldsymbol{\beta},\infty}^F(E) \geq \mu_{\boldsymbol{\beta},\infty}^F(E_n) = \operatorname{Var}(F \restriction E_n) = \infty$. By Lemma 1, $\mu^F(E) = \infty$ and so also $\mu_{\boldsymbol{\alpha}}^F(E) = \infty$ for *every* E-form $\boldsymbol{\alpha}$. Hence we can assume that $F \in$ VBG. Take an E-form $\boldsymbol{\alpha} = \{E_n\}_{n=1}^{\infty}$ such that $F \restriction E_n$ is VB and continuous for every n. We can assume E_n's are pairwise disjoint. The proof of Theorem 3 (and 2), says that $\mu_{\boldsymbol{\alpha}}^F(E_n) = \mu^F(E_n)$, $n \in \mathbb{N}$. Given $\varepsilon > 0$, again by Lemma 1, take an E-form $\boldsymbol{\beta} \succ \boldsymbol{\alpha}$ such that, for every $n \in \mathbb{N}$,

$$\mu_{\boldsymbol{\beta},\infty}^F(E_n) < \mu^F(E_n) + \frac{\varepsilon}{2^n} = \mu_{\boldsymbol{\alpha}}^F(E_n) + \frac{\varepsilon}{2^n}.$$

Consider a $\boldsymbol{\beta}$-fine division $\mathcal{P}$ such that $|\Delta|F(\mathcal{P}) > \mu_{\boldsymbol{\beta},\infty}^F(E) - \varepsilon$. Denote $\mathcal{P}_n = \{(I, x) \in \mathcal{P} \colon (I, x)$ is E_n-fine$\}$, $n \in \mathbb{N}$ (since $\boldsymbol{\beta} \succ \boldsymbol{\alpha}$, $\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n$). From the definition of $\boldsymbol{\beta}$ and Lemma 6,

$$(6) \qquad |\Delta|F(\mathcal{P}_n) \leq \mu^F(E_n \cap \mathcal{I}(\mathcal{P}_n)) + \frac{\varepsilon}{2^n} = \mu^F_{\boldsymbol{\alpha}}(E_n \cap \mathcal{I}(\mathcal{P}_n)) + \frac{\varepsilon}{2^n}.$$

Take any E-form $\boldsymbol{\xi} \succ \boldsymbol{\beta}$. For every $(I,x) \in \mathcal{P}_n$ there is a $\boldsymbol{\xi}$-fine division $\mathcal{R}_I$ in I, anchored in E_n, such that

$$(7) \ |\Delta|F(\mathcal{R}_I) > \mu^F_{\boldsymbol{\xi}}(E_n \cap I) - \frac{\varepsilon}{|\mathcal{P}|} = \mu^F_{\boldsymbol{\alpha}}(E_n \cap I) - \frac{\varepsilon}{|\mathcal{P}|} = \mu^F(E_n \cap I) - \frac{\varepsilon}{|\mathcal{P}|},$$

where $|\mathcal{P}|$ is the cardinality of $\mathcal{P}$. Denote $\mathcal{S}_n = \bigcup_{(I,x)\in\mathcal{P}_n} \mathcal{R}_I$ and $\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$. We have, by the additivity of $I \mapsto \mu^F_{\boldsymbol{\alpha}}(E_n \cap I)$,

$$\begin{aligned} |\Delta|F(\mathcal{S}) &= \sum_{n=1}^{\infty} |\Delta|F(\mathcal{S}_n) \overset{(7)}{>} \sum_{n=1}^{\infty} \mu^F_{\boldsymbol{\alpha}}(E_n \cap \mathcal{I}(\mathcal{P}_n)) - \varepsilon \\ &\overset{(6)}{\geq} \sum_{n=1}^{\infty} |\Delta|F(\mathcal{P}_n) - \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} - \varepsilon = |\Delta|F(\mathcal{P}) - 2\varepsilon \\ &> \mu^F_{\boldsymbol{\beta},\infty}(E) - 3\varepsilon \geq \mu^F_{\boldsymbol{\beta}}(E) - 3\varepsilon = \mu^F_{\boldsymbol{\alpha}}(E) - 3\varepsilon. \end{aligned}$$

Hence, as $\mathcal{S}$ is $\boldsymbol{\xi}$-fine, $\mu^F_{\boldsymbol{\xi},\infty}(E) \geq \mu^F_{\boldsymbol{\alpha}}(E) - 3\varepsilon$. As $\boldsymbol{\xi} \succ \boldsymbol{\beta}$ was arbitrary, $\mu^F(E) \geq \mu^F_{\boldsymbol{\alpha}}(E) - 3\varepsilon$ (Lemma 1) and so $\mu^F(E) \geq \mu^F_{\boldsymbol{\alpha}}(E)$. □

3.2 – *Representations of* $\bar{\mu}^F$

Let $F\colon S \to \mathbb{R}$, S an ambivalent set, $E \subset S$. Denote with W_F the so-called *weak variational measure* [30, Definition 22]; i.e.,

$$W_F(E) = \lim_{\boldsymbol{\alpha}} \bar{\mu}^F_{\boldsymbol{\alpha},\infty}(E),$$

where the limit is taken with respect to the partial order $\prec$ among all *closed* S-forms $\boldsymbol{\alpha}$. Notice that, by definition, $\mu^F(E) \leq \bar{\mu}^F(E) \leq W_F(E)$.

Theorem 5. *Let F be monotone. Then,*

$$\bar{\mu}^F(E) = W_F(E) = |F(E)|.$$

Proof. By Theorem 2, $|F(E)| \leq \bar{\mu}^F(E)$, so it's enough to show that $W_F(E) \leq |F(E)|$. Given $\varepsilon > 0$, take an open set $O \supset F(E)$ such that $|O| < |F(E)| + \varepsilon$. The set $F^{-1}(O) \subset S$ can be represented in the form $S \cap \bigcup_i I_i$, where I_i form a collection of disjoint intervals (closed, open, or half-open). Denote with U the set of all endpoints of I_i that belong to $F^{-1}(O)$. Write down every intersection $S \cap \operatorname{int} I_i$ as a union of closed sets, $S \cap \operatorname{int} I_i = \bigcup_s E_s^{(i)}$, and set $\boldsymbol{\alpha} = \{E_s^{(i)}\}_{s,i}$. $\boldsymbol{\alpha}$ is a closed $(F^{-1}(O) \setminus U)$-form. Consider any $\boldsymbol{\alpha}$-fine division $(\langle x_j, y_j\rangle, x_j)$, $j = 1, \ldots, m$, anchored in $E \setminus U$. Notice that if $(\langle x_j, y_j\rangle, x_j)$ is $E_s^{(i)}$-fine, then $\langle x_j, y_j\rangle \subset I_i$ and so $\langle F(x_j), F(y_j)\rangle \subset O$. Since $\langle F(x_j), F(y_j)\rangle$ are pairwise nonoverlapping,

$$\sum_{j=1}^{m} |F(y_j) - F(x_j)| \le |O| < |F(E)| + \varepsilon.$$

This means that $\bar{\mu}^F_{\boldsymbol{\alpha},\infty}(E \setminus U) \le |F(E)| + \varepsilon$ (note that $F^{-1}(O) \setminus U$ is ambivalent), so $W_F(E \setminus U) \le |F(E)|$. Since $W_F(U) = 0$ and W_F is a Borel measure [30, Theorem 23], the result follows. □

Theorem 6. *Let F be VB. Then,*

$$\bar{\mu}^F(E) = W_F(E) = |F_+(E)| + |F_-(E)|.$$

Proof. Denote $E^\pm = \{x \in E : F'_\pm(x) > 0\}$, $C^\pm = E \setminus E^\pm$. By Lemma 5, $E^+ \cap E^- = \emptyset$. By Lemma 4 and Theorem 5 we have $\bar{\mu}^{F_\pm}(E^\pm) = W_{F_\pm}(E^\pm) = |F_\pm(E^\pm)|$ and $\bar{\mu}^{F_\pm}(C^\pm) = W_{F_\pm}(C^\pm) = |F_\pm(C^\pm)| = 0$. Given $\varepsilon > 0$, take a closed S-form $\boldsymbol{\alpha}$ such that $W_{F_\pm}(E^\pm) + \varepsilon > \bar{\mu}^{F_\pm}_{\boldsymbol{\alpha},\infty}(E^\pm)$; $\bar{\mu}^{F_\pm}_{\boldsymbol{\alpha},\infty}(C^\pm) < \varepsilon$, i.e., $|\Delta| F_\pm(\mathcal{P}_\pm) < \varepsilon$ holds for every $\boldsymbol{\alpha}$-fine division $\mathcal{P}_\pm$ anchored in $C^\pm$. Consider now any $\boldsymbol{\alpha}$-fine division $\mathcal{R}$ anchored in E. We have

$$\begin{aligned}
|\Delta| F(\mathcal{R}) &\le |\Delta| F_+(\mathcal{R}_+) + |\Delta| F_+(\mathcal{S}_+) + |\Delta| F_-(\mathcal{R}_-) + |\Delta| F_-(\mathcal{S}_-) \\
&\le \bar{\mu}^{F_+}_{\boldsymbol{\alpha},\infty}(E^+) + \varepsilon + \bar{\mu}^{F_-}_{\boldsymbol{\alpha},\infty}(E^-) + \varepsilon \\
&\le W_{F_+}(E^+) + W_{F_-}(E^-) + 4\varepsilon = |F_+(E^+)| + |F_-(E^-)| + 4\varepsilon \\
&= |F_+(E)| + |F_-(E)| + 4\varepsilon,
\end{aligned}$$

where $\mathcal{R}_\pm = \{(I, x) \in \mathcal{R} : x \in E^\pm\}$, $\mathcal{S}_\pm = \mathcal{R} \setminus \mathcal{R}_\pm$. It means that $\bar{\mu}^F_{\boldsymbol{\alpha},\infty}(E) \le |F_+(E)| + |F_-(E)| + 4\varepsilon$, so, as ε was arbitrary, $W_F(E) \le |F_+(E)| + |F_-(E)|$. From Theorem 3 it follows that $|F_+(E)| + |F_-(E)| = \mu^F(E) \le \bar{\mu}^F(E) \le W_F(E)$. The proof is over. □

For the next corollary one can argue like for Corollary 1.

Corollary 2. *Assume F is [VBG]. Then $\bar{\mu}^F(E) = 0$ holds for every nullset $E \subset S$ if and only if F satisfies Lusin's condition (N).*

Corollary 3. *Assume F is [VBG]. Then, for every subset $E \subset S$, $W_F(E) = \bar{\mu}^F(E)$.*

Proof. Recall, both W_F and $\bar{\mu}^F$ are Borel measures. Let $\{S_n\}_{n=1}^{\infty}$ be a closed S-form such that $F \upharpoonright S_n$ is VB for every n. Denote

$$S_1' = S_1, \quad S_{n+1}' = S_{n+1} \setminus (S_1 \cup \ldots \cup S_n),$$

$n \in \mathbb{N}$. By Theorem 6 we have $W_F(E \cap S_n') = \bar{\mu}^F(E \cap S_n')$, $n \in \mathbb{N}$, so (as S_n' are disjoint Borel sets),

$$W_F(E) = \sum_{n=1}^{\infty} W_F(E \cap S_n') = \sum_{n=1}^{\infty} \bar{\mu}^F(E \cap S_n') = \bar{\mu}^F(E). \qquad \square$$

Problem 7. *Does $W_F(E) = \bar{\mu}^F(E)$ hold without the [VBG] assumption for F?*

Now we aim to provide integral representations for $\bar{\mu}^F$ and μ^F. The results are counterparts of [30, Lemma 27], where the function F is assumed to be continuous on the closure of S.

Theorem 8. *Let F be arbitrary. For every measurable set $T \subset \mathcal{D}_F$ and every closed S-form $\boldsymbol{\alpha}$, $\bar{\mu}^F(T) = \bar{\mu}^F_{\boldsymbol{\alpha}}(T) = \int_T |F'|$.*

The proof of Theorem 8 boils down, essentially, to the following lemma.

Lemma 7. *Given $0 \le r \le s$, $D_{rs} = \{x \in \mathcal{D}_F : r \le |F'(x)| \le s\}$, for any closed S-form $\boldsymbol{\alpha}$ and any measurable $T \subset D_{rs}$,*

$$r|T| \le \bar{\mu}^F(T) \le \bar{\mu}^F_{\boldsymbol{\alpha}}(T) \le s|T|. \tag{8}$$

We will prove Lemma 7 in a slightly wider setting, in terms of derivatives relative to a form, cf. § 1.3.

Lemma 8. *Let $\boldsymbol{\beta} = \{E_n\}_{n=1}^{\infty}$ be a closed S-form. Given $0 \le r \le s$, $D_{rs} = \{x \in S : r \le \underline{D}_{\boldsymbol{\beta}} F(x) \le \overline{D}_{\boldsymbol{\beta}} F(x) \le s$ or $-s \le \underline{D}_{\boldsymbol{\beta}} F(x) \le \overline{D}_{\boldsymbol{\beta}} F(x) \le -r\}$, for any closed S-form $\boldsymbol{\alpha} \succ \boldsymbol{\beta}$ and any measurable $T \subset D_{rs}$, (8) holds.*

Proof. Let $\boldsymbol{\alpha} = \{S_n\}_{n=1}^{\infty} \succ \boldsymbol{\beta}$ be any closed S-form and $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ a gauge related to $\boldsymbol{\alpha}$.

Take an $\varepsilon > 0$ and for an $x \in S_n$ fetch a positive number $\gamma_n(x) \le \delta_n(x)$ so that $(r - \varepsilon)|y - x| \le |F(y) - F(x)|$ provided $x \in D_{rs}$, $y \in S_n$, $|y - x| < \gamma_n(x)$. For any division $\mathcal{P}$ that is $\boldsymbol{\gamma}$-fine, $\boldsymbol{\gamma} = \{\gamma_n\}_{n=1}^{\infty}$, and anchored in D_{rs} we have $(r - \varepsilon)|\mathcal{I}(\mathcal{P})| \le |\Delta| F(\mathcal{P})$. The collection of all segments $\langle x, y \rangle$, where $(\langle x, y \rangle, x)$

is $\boldsymbol{\gamma}$-fine and $x \in D_{rs}$ forms a Vitali cover of a co-countable subset of D_{rs}. Thus there is a $\boldsymbol{\gamma}$-fine division $\mathcal{P}$ anchored in T with $|\mathcal{I}(\mathcal{P})| > |T| - \varepsilon$. Hence, by definition,

$$\bar{\mu}^F_{\boldsymbol{\alpha},\boldsymbol{\delta}}(T) \geq \bar{\mu}^F_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(T) \geq |\Delta| F(\mathcal{P}) \geq (r - \varepsilon)|\mathcal{I}(\mathcal{P})| > (r - \varepsilon)(|T| - \varepsilon).$$

Passing with $\varepsilon \to 0$ we obtain the inequality $\bar{\mu}^F_{\boldsymbol{\alpha},\boldsymbol{\delta}}(T) \geq r|T|$. Hence $\bar{\mu}^F_{\boldsymbol{\alpha}}(T) \geq r|T|$ and so $\bar{\mu}^F_{\boldsymbol{\alpha}}(T) \geq \bar{\mu}^F(T) \geq r|T|$.

For the second inequality, take $\varepsilon > 0$ and an open set $O \supset T$ with $|O| < |T| + \varepsilon$. There is a gauge $\boldsymbol{\gamma} = \{\gamma_n\}_{n=1}^{\infty}$ related to $\boldsymbol{\alpha}$ such that $|x - y| < \gamma_n(x)$, $y \in S_n$, $x \in T \cap S_n$ imply $|F(y) - F(x)| \leq (s + \varepsilon)|y - x|$ and $\langle x, y\rangle \subset O$. Take an arbitrary $\boldsymbol{\gamma}$-fine division $\mathcal{P}$ anchored in T. By the definition of $\boldsymbol{\gamma}$, $|\mathcal{I}(\mathcal{P})| < |T| + \varepsilon$. Hence

$$|\Delta| F(\mathcal{P}) \leq (s + \varepsilon)|\mathcal{I}(\mathcal{P})| \leq (s + \varepsilon)(|T| + \varepsilon).$$

Therefore $\bar{\mu}^F_{\boldsymbol{\alpha},\boldsymbol{\gamma}}(T) \leq (s + \varepsilon)(|T| + \varepsilon)$ and so $\bar{\mu}^F_{\boldsymbol{\alpha}}(T) \leq s|T|$. □

A generalization of Theorem 8 that corresponds to Lemma 8 would be

Theorem 9. *Let F be arbitrary. For every measurable set $T \subset \mathcal{D}^{\boldsymbol{\beta}}_F$ and every closed S-form $\boldsymbol{\alpha} \succ \boldsymbol{\beta}$, $\bar{\mu}^F(T) = \bar{\mu}^F_{\boldsymbol{\alpha}}(T) = \int_T |F'_{\boldsymbol{\beta}}|$.*

Of course, $F'_{\boldsymbol{\beta}} = F'_{\boldsymbol{\alpha}}$ over a co-countable subset of T.

Example 1. The equality $\bar{\mu}^F(T) = \bar{\mu}^F_{\boldsymbol{\alpha}}(T)$ need not hold if $T \not\subset \mathcal{D}_F$. Take an increasing function $F\colon [0, 1] \to [0, 1]$ defined with $F(x) = \sum_{m: q_m < x} 2^{-m}$, where $\{q_m\}_{m=1}^{\infty} = \mathbb{Q} \cap [0, 1]$. Given a closed $[0, 1]$-form $\boldsymbol{\alpha} = \{E_n\}_{n=1}^{\infty}$, there are n and a nondegenerate segment $J \subset [0, 1]$ such that $\operatorname{cl} E_n \supset J$. With any gauge $\boldsymbol{\delta}$ related to $\boldsymbol{\alpha}$ one has $\bar{\mu}^F_{\boldsymbol{\alpha},\boldsymbol{\delta}}(\mathbb{Q} \cap [0, 1]) \geq \Delta F(J) > 0$, so $\bar{\mu}^F_{\boldsymbol{\alpha}}(\mathbb{Q} \cap [0, 1]) \geq \Delta F(J) > 0$, while $\bar{\mu}^F(\mathbb{Q} \cap [0, 1]) = 0$ and $W_F(\mathbb{Q} \cap [0, 1]) = 0$.

Problem 10. *Let F be [VBG]. Does Theorem 8 hold for $T \subset \mathcal{D}^{\mathrm{ap}}_F$ (and F'_{ap} instead of F')?*

Corollary 4. *Let $F\colon S \to \mathbb{R}$ be a [VBG]-function satisfying (N). Then for every measurable set $T \subset S$, $\bar{\mu}^F(T) = \int_T |F'_{\mathrm{ap}}|$.*

Proof. It follows from Corollary 2 and Theorem 9, the fact that F is a.e. approximately differentiable, and that if $\boldsymbol{\alpha}$ is a closed S-form related to the property [VBG] of F, then $F'_{\boldsymbol{\alpha}}$ agrees with F'_{ap} almost everywhere in S. □

Along the same lines one can prove analogous theorems for variational measures μ^F (and μ^F_{α}).

THEOREM 11. *Let $E \subset \mathbb{R}$ and $F: E \to \mathbb{R}$ be arbitrary. For every measurable set $T \subset \mathcal{D}_F$, $\mu^F(T) = \int_T |F'|$.*

This is a special case of Thomson's Theorem 36 of [30].

COROLLARY 5. *Let $F: E \to \mathbb{R}$ be a measurable VBG-function satisfying (N). Then, for every measurable set $T \subset E$, $\mu^F(T) = \int_T |F'_{\text{ap}}|$.*

A consequence of Corollaries 4 and 5 is that for every [VBG]-function F on an ambivalent S, that satisfies (N), $\bar{\mu}^F$ and μ^F agree over the class of measurable subsets of $\mathcal{D}^{\text{ap}}_F$.

EXAMPLE 2. There is an approximately continuous ACG-function F defined on $[0, 1]$ such that μ^F and $\bar{\mu}^F$ differ for some subsets of $\mathcal{D}^{\text{ap}}_F$. We take the function from Example 3.1 of [24]. Notice that the (countable) set $\{x_i^{(n)}\}_{i,n} = \mathbb{C} \setminus F^{-1}(0) \subset \mathcal{D}^{\text{ap}}_F$. It is an easy exercise to check that for every portion $P \neq \emptyset$ of $\mathbb{C} \setminus F^{-1}(0)$, $\bar{\mu}^F(P) = \infty$ while $\mu^F(P) = 0$. It follows from the fact that $\sum_{x \in P} |F(x)| = \infty$ for every such P.

4. Absolute continuity of variational measures μ^F and $\bar{\mu}^F$

Let $F: D \to \mathbb{R}$, D closed.

THEOREM 12. *$F \in [\text{VBG}] \cap (N)$ if and only if $\bar{\mu}^F(E) = 0$ for every nullset $E \subset D$.*

PROOF. ($\Longrightarrow$) Corollary 4.

($\Longleftarrow$) Assume $\bar{\mu}^F(E) = 0$ for every nullset $E \subset D$. From Lemma 3, for every nullset $E \subset D$ there exists an $\mathcal{F}_\sigma$ set $S \supset E$ such that F is [VBG] on S. Note the following condition being equivalent to $F \in$ [VBG]: for every compact $P \subset S$ there is a portion $P \cap I \neq \emptyset$ of P on which F is VB. We will show this is fulfilled by F. Suppose, to the contrary, the condition does not hold for some perfect $P \subset S$. It means F is VB on no portion of P. As F is not VB on P one can find a collection of nonoverlapping segments $I_1^{(1)}, \dots, I_{k_1}^{(1)}$ all with both endpoints in P such that

$$\sum_{i=1}^{k_1} |\Delta F(I_i^{(1)})| > 1. \tag{9}$$

Claim. We can assume that both endpoints of every $I_i^{(1)}$ are accumulation points of $P \cap I_i^{(1)}$.

Proof of the claim. Of course, we can assume that $\sum_{i=1}^{k_1} |\Delta F(I_i^{(1)})| > 5$. By adding more segments to $\mathcal{I} = \{I_i^{(1)}\}_{i=1}^{k_1}$ if necessary, one can have $\bigcup_{i=1}^{k_1} I_i^{(1)} = [\min P, \max P]$. Splitting some segments $I_i^{(1)}$ into two, one can claim that only at most every second segment can fail to fulfil the requirement (we refer to such segments as "bad"). Splitting (again) some good segments into two, we can assume that there are at least two good segments between any two bad ones. Assume the enumeration in $\mathcal{I}$ agrees with the order of real line and consider any three adjacent segments $I_{k-1}^{(1)} = [x, y]$, $I_k^{(1)} = [y, w]$, and $I_{k+1}^{(1)} = [w, z]$ from $\mathcal{I}$, with only the middle one, $[y, w]$, being bad. If $|F(w)-F(y)| \geq 2|F(z)-F(w)|+2|F(y)-F(x)|$, then we replace the triple $[x, y], [y, w], [w, z]$ in $\mathcal{I}$ with their union $[x, z]$ (which is a good segment). The loss in the sum in (9) is at most $2/3$ of the sum[1] $|F(w) - F(y)| + |F(z) - F(w)| + |F(y) - F(x)|$. Otherwise we remove $[y, w]$ from the triple; the loss is less than $4/5$ of the above sum. Having modified all triples with a bad segment in the middle this way, the collection $\mathcal{I}$ consists of good segments only; moreover, $\sum_{i=1}^{k_1} |\Delta F(I_i^{(1)})| > 1$, as required. △

We proceed inductively. Assume we have defined a collection of nonoverlapping segments $I_i^{(m)}$, $i = 1, \ldots, k_m$, $m \geq 1$, all with both endpoints being accumulation points of $P \cap I_i^{(m)}$. Using the fact that F is VB on no portion of P, and arguing as in the claim above, one can find a collection of nonoverlapping segments $I_j^{(m+1)}$, $j = 1, \ldots, k_{m+1}$, with both endpoints of $I_j^{(m+1)}$ being accumulation points of $P \cap I_j^{(m+1)}$, $j = 1, \ldots, k_{m+1}$, that fulfils the following three conditions:

- i_m every segment $I_i^{(m)}$ contains at least two segments of the $(m+1)$ st rank—one sharing the left hand and one sharing the right hand endpoints with $I_i^{(m)}$;
- ii_m for every $i = 1, \ldots, k_m$,

$$\sum_{j \in \mathcal{J}_i^{(m)}} |\Delta F(I_j^{(m+1)})| > m + 1, \quad \text{where } \mathcal{J}_i^{(m)} = \{j : I_j^{(m+1)} \subset I_i^{(m)}\};$$

- iii_m $\displaystyle\sum_{j=1}^{k_{m+1}} |I_j^{(m+1)}| < \frac{1}{m}.$

[1] Indeed, $|F(x) - F(z)| \geq |F(w) - F(y)| - |F(z) - F(w)| - |F(x) - F(y)| \geq |F(w) - F(y)|/2 = (1/3+1/6)|F(w)-F(y)| \geq (|F(w)-F(y)|+|F(z)-F(w)|+|F(y)-F(x)|)/3$.

We define

$$N = \bigcap_{m=1}^{\infty} \bigcup_{i=1}^{k_m} I_i^{(m)}.$$

As P is closed, $N \subset P$. Due to i_m, N is perfect, while from iii_m, $|N| = 0$. So, by assumption, F is [VBG] on N. Thus there exists a closed N-form $\{N_n\}_{n=1}^{\infty}$ such that $\mathrm{Var}\,(F \restriction N_n) < \infty$ for every n. By the Baire category theorem, some N_n covers a portion of N, say $N \cap I_j^{(m)}$. Consider the segments $I_i^{(m+1)}$, $i \in \mathcal{J}_j^{(m)}$. In virtue of ii_m,

$$\sum_{i \in \mathcal{J}_j^{(m)}} |\Delta F(I_i^{(m+1)})| > m + 1.$$

So, $\mathrm{Var}\,(F \restriction N_n) > m + 1$. As m could have been taken arbitrarily large here, $\mathrm{Var}\,(F \restriction N_n) = \infty$, a contradiction. We proved F is [VBG]. The fact that F satisfies (N) follows now from Corollary 2. □

Theorem 13. *$F \in$ [ACG] if and only if there is a closed D-form $\boldsymbol{\alpha}$ such that $\bar{\mu}_{\boldsymbol{\alpha}}^F(E) = 0$ for every nullset $E \subset D$.*

Proof. ($\Longrightarrow$) Let $\boldsymbol{\alpha} = \{D_n\}_{n=1}^{\infty}$ be a closed D-form such that $F \restriction D_n \in \mathrm{AC}$ for all n. Consider any nullset $E \subset D$. Fix $\varepsilon > 0$ and n and let a number $\eta_n > 0$ correspond to $\varepsilon/2^n$ in connection with absolute continuity of F on D_n. Set $E_n = E \cap D_n$. Let $O_n \supset E_n$ be open and such that $|O_n| < \eta_n$. Define a gauge δ_n on D_n so that $\langle x, y\rangle \subset O_n$ if $(\langle x, y\rangle, x)$, $x \in E_n$, is δ_n-fine. Consider $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ as related to $\boldsymbol{\alpha}$. If $\{(\langle x_i, y_i\rangle, x_i)\}_{i=1}^{m}$ is a $\boldsymbol{\delta}$-fine division anchored in E, then

$$\sum_{i=1}^{m} |F(y_i) - F(x_i)| = \sum_{n=1}^{\infty} \sum_{i \in \mathcal{I}_n} |F(y_i) - F(x_i)| < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon,$$

where $\mathcal{I}_n = \{i : (\langle x_i, y_i\rangle, x_i) \text{ is } (D_n, \delta_n)\text{-fine}\}$. Therefore $\bar{\mu}_{\boldsymbol{\alpha},\boldsymbol{\delta}}^F(E) \le \varepsilon$ and so $\bar{\mu}_{\boldsymbol{\alpha}}^F(E) = 0$.

($\Longleftarrow$) From the previous theorem, F is $[\mathrm{VBG}] \cap (N)$. Moreover, it is easily seen that the condition implies continuity of $F \restriction D_n$ for every $D_n \in \boldsymbol{\alpha}$. Therefore, from the Banach–Zarecki theorem [19, Chapter VII, (6.8)], F is [ACG] on D. This ends the proof. □

Let now $D \subset \mathbb{R}$ and $F : D \to \mathbb{R}$ be measurable.

Theorem 14. *$F \in \mathrm{VBG} \cap (N)$ if and only if $\mu^F(E) = 0$ for every nullset $E \subset D$.*

The ($\Longleftarrow$) part of the above equivalence doesn't hold without the measurability assumption on F. The function $F\colon [0,1] \to \mathbb{R}$ constructed in [22, Example, § 3] has the property that $F(B)$ is countable as long as $B \subset [0,1]$ is a nullset (so, $\mu^F_{\boldsymbol{\alpha}}(B) = 0$ where F is constant on every member of $\boldsymbol{\alpha}$); F is not VBG, though.

Proof. ($\Longleftarrow$) By Lemma 2, the condition implies F is VBG on every nullset. Since F is measurable, by [7, Theorem 3], F is VBG. Condition (N) follows from Corollary 1. ($\Longrightarrow$) Corollary 1 again. □

Corollary 6. *$F \in \mathrm{VBG} \cap (N)$ if and only if there is a D-form $\boldsymbol{\alpha}$ such that $\mu^F_{\boldsymbol{\alpha}}(E) = 0$ for every nullset $E \subset D$.*

Proof. See Remark 2. □

5. Application to some Lusin-type integrals

Among continuous functions (on a segment or, more generally, an $\mathcal{F}_\sigma$ set) the conditions ACG, [ACG], VBG or [VBG] (the latter two together with the condition (N)) are equivalent (Banach–Zarecki theorem). Without the continuity constraint, however, all the four conditions give rise to different classes of functions. Pruning these classes with some extra regularity conditions (weaker than continuity), can result in subclasses having monotonicity property, that is, in classes of primitives for some generalized integrals in the real line. This approach in generalizing the wide Denjoy integral goes back to Ridder [17, 18], who first considered the class of approximately continuous [ACG]-functions as the class of primitives encompassing both the wide Denjoy integral and Burkill's approximately continuous Perron integral. The other classes arrived in a similar connection in [9, 12, 13, 14, 20, 21].

Since our presentation concerns a Riemann-type approach to the aforementioned generalized integrals, the continuity-like constraints we put on primitives should be of local flavor. The setting we use in this connection are, like in [26], *local systems* [29]. By a local system (or a *simple system of sets* [27, 28]) we mean a family $\blacktriangle = \{\blacktriangle(x)\}_{x\in\mathbb{R}}$ such that every $\blacktriangle(x)$ is a nonvoid collection of subsets of $\mathbb{R}$ with the following properties:

(i) $\{x\} \notin \blacktriangle(x)$;
(ii) if $S \in \blacktriangle(x)$, then $x \in S$;
(iii) if $S \in \blacktriangle(x)$ and $R \supset S$, then $R \in \blacktriangle(x)$;
(iv) if $S \in \blacktriangle(x)$ and $\delta > 0$, then $(x-\delta, x+\delta) \cap S \in \blacktriangle(x)$.

(In order to avoid confusion with an increment of a function, we resort here, unlike in [26], to the character "$\blacktriangle$" instead of "Δ" whenever it stands for a local system).

Every S belonging to $\blacktriangle(x)$ we call a *path* leading to x. A function $\mathcal{C}$ on $A \subset \mathbb{R}$ such that $\mathcal{C}(x) \in \blacktriangle(x)$ for every $x \in A$, we call *a* $\blacktriangle$*-choice on* A. Given $\mathcal{C}$, we say a tagged interval $(\langle x, y\rangle, x)$ is $\mathcal{C}$-fine if $y \in \mathcal{C}(x)$.

We say that a local system $\blacktriangle$ is *filtering down*, if for each $x \in \mathbb{R}$ and each two paths $S_1, S_2 \in \blacktriangle(x)$ one has $S_1 \cap S_2 \in \blacktriangle(x)$. We say that $\blacktriangle$ is *bilateral* if $(x-\delta, x) \cap S \neq \emptyset$ and $(x, x+\delta) \cap S \neq \emptyset$ for each $x \in \mathbb{R}$, $S \in \blacktriangle(x)$, $\delta > 0$. We say that $\blacktriangle$ satisfies the *intersection condition* (abbr. IC), if for every choice $\mathcal{C}$, there exists a gauge δ such that

$$0 < y - x < \min\{\delta(x), \delta(y)\} \implies \mathcal{C}(x) \cap \mathcal{C}(y) \cap [x, y] \neq \emptyset.$$

As the most significant examples of local systems let us mention the local system $\blacktriangle_{\mathrm{e}}$ that consists of neighborhoods in the Euclidean topology and the (Lebesgue) density local system $\blacktriangle_{\mathrm{ap}}$ defined as follows:

$$E \in \blacktriangle_{\mathrm{ap}}(x) \iff x \in E \text{ and some measurable } P \subset E \text{ has density 1 at } x,$$

cf. (1). Given $\blacktriangle$, we say that a function $f : \mathbb{R} \to \mathbb{R}$ is $\blacktriangle$*-continuous at* $x \in \mathbb{R}$, if for each $\varepsilon > 0$ there exists a path $S \in \blacktriangle(x)$ such that

$$f(x) - \varepsilon < f(t) < f(x) + \varepsilon$$

for each $t \in S$. We say $f : \mathbb{R} \to \mathbb{R}$ is $\blacktriangle$*-continuous* if it is $\blacktriangle$-continuous at each $x \in \mathbb{R}$.

Lemma 9 ([29]). *If a bilateral local system* $\blacktriangle$ *satisfies* IC*, then every* $\blacktriangle$*-continuous function* $f : \mathbb{R} \to \mathbb{R}$ *is Darboux Baire one.*

From now on, we assume the local system $\blacktriangle$ considered is filtering down and such that every $\blacktriangle$-continuous function on $[a, b]$ is Darboux Baire one. From the above lemma, the latter can follow from bilaterality and IC. Let us consider the following four classes of $\blacktriangle$-continuous functions defined on a segment $[a, b]$:

- $\mathcal{F}_1^{\blacktriangle}$: [ACG]-functions,
- $\mathcal{F}_2^{\blacktriangle}$: [VBG]-functions satisfying (N),
- $\mathcal{F}_3^{\blacktriangle}$: measurable ACG-functions,
- $\mathcal{F}_4^{\blacktriangle}$: measurable VBG-functions satisfying (N).

For each i, the class $\mathcal{F}_i^{\blacktriangle}$ forms a linear space. This is a consequence of the filtering down property of $\blacktriangle$ and the fact that measurable VBG functions with the (N) condition form a linear space as well [6, Corollary 2]. It is well-known [19, Chapter VII, (4.3)] that each member of $\mathcal{F}_i^{\blacktriangle}$, $i = 1, 2, 3, 4$, is approximately differentiable almost everywhere.

Definition 15. We call a function $f:[a,b]\to\mathbb{R}$, $\mathcal{F}_i^{\blacktriangle}$-integrable, $i=1,2,3,4$, if there exists a function $F\in\mathcal{F}_i^{\blacktriangle}$, on $[a,b]$, such that $F'_{\mathrm{ap}}(x)=f(x)$ for almost all $x\in[a,b]$. The $\mathcal{F}_i^{\blacktriangle}$-integral of f is defined as $F(b)-F(a)$.

The $\mathcal{F}_i^{\blacktriangle}$-integral is uniquely defined since $\mathcal{F}_i^{\blacktriangle}$ is a linear space and, e.g., up to the following, quite general, monotonicity result.

Lemma 10 ([15, Theorem 1]). *Assume that an $F:\mathbb{R}\to\mathbb{R}$ satisfies (N) and is Baire one Darboux. If $F'(x)\geq 0$ at almost every point $x\in\mathbb{R}$ at which F is differentiable (in the usual sense), then F is nondecreasing.*

5.1 – *Characterizations via absolute continuity of variational measures*

All Riemann- and variational-type characterizations of $\mathcal{F}_i^{\blacktriangle}$-integrals, $i=1,2,4$, presented in this section we borrow from [26]. For more background in this connection, see that work of the author.

Given a $\blacktriangle$-choice $\mathcal{C}$ on $A\subset\mathbb{R}$, an $[a,b]$-form $\boldsymbol{\alpha}$ together with a related gauge $\boldsymbol{\delta}$, a tagged interval $(\langle x,y\rangle,x)$ is called $(\mathcal{C},\boldsymbol{\delta})$-fine if it is either $\mathcal{C}$-fine or $\boldsymbol{\delta}$-fine. Accordingly $(\mathcal{C},\boldsymbol{\delta})$-fine divisions/partitions are defined. Pairs $(\mathcal{C},\boldsymbol{\delta})$ (so-called *composite gauges*) can be used for Riemann-type integration due to the following partitioning lemma.

Lemma 11 ([8, Lemma 4.2]). *Let an $[a,b]$-form $\boldsymbol{\alpha}$ be closed and a local system $\blacktriangle$ be bilateral. For every $\blacktriangle$-choice $\mathcal{C}$ on $\operatorname{Is}\boldsymbol{\alpha}$, and every gauge $\boldsymbol{\delta}$ related to $\boldsymbol{\alpha}$ there exists a $(\mathcal{C},\boldsymbol{\delta})$-fine partition of $[a,b]$.*

In what follows, we assume $\blacktriangle$ is a local system being bilateral and filtering down, and having `IC`. With the partitioning property from Lemma 11 in hand, the following Definitions 16 and 18 are meaningful.

Definition 16 (AH-integral of [16] if $\blacktriangle=\blacktriangle_{\mathrm{ap}}$). A function $f:[a,b]\to\mathbb{R}$ is said to be $\mathrm{LL}_{\blacktriangle}$-integrable if there exist a number $\mathbf{I}\in\mathbb{R}$ (the value of $\int_a^b f$) and a closed $[a,b]$-form $\boldsymbol{\alpha}$ with the property that: for every $\varepsilon>0$ there exists a gauge $\boldsymbol{\delta}$ related to $\boldsymbol{\alpha}$ and, given a countable set $A\supset\operatorname{Is}\boldsymbol{\alpha}$, a $\blacktriangle$-choice $\mathcal{C}$ on A, such that for every partition π of $[a,b]$ that is $(\mathcal{C},\boldsymbol{\delta})$-fine,

$$|\sigma(\pi,f)-\mathbf{I}|<\varepsilon. \tag{10}$$

Theorem 17 ([26, Theorem 28]). *The $\mathcal{F}_1^{\blacktriangle}$-integral is equivalent to the $\mathrm{LL}_{\blacktriangle}$-integral.*

Definition 18 ([$\mathcal{S}_1\mathcal{S}_2\mathcal{R}$]- or [$\mathcal{S}\mathcal{R}$]-integral of [8]). A function $f\colon[a,b]\to\mathbb{R}$ is said to be $E_\blacktriangle$-integrable if there exists a number $\mathbf{I}\in\mathbb{R}$ such that for every $\varepsilon>0$ there exists a closed $[a,b]$-form $\boldsymbol{\alpha}$, together with a related gauge $\boldsymbol{\delta}$, and, given a countable set $A\supset \operatorname{Is}\boldsymbol{\alpha}$, a $\blacktriangle$-choice $\mathcal{C}$ on A, such that for every partition π of $[a,b]$ that is $(\mathcal{C},\boldsymbol{\delta})$-fine, the inequality (10) holds.

Theorem 19 ([26, Corollary 26]). *The $\mathcal{F}_2^\blacktriangle$-integral is equivalent to the $E_\blacktriangle$-integral.*

For non-closed $[a,b]$-forms the claim of Lemma 11 does not hold. Thus, in order to characterize $\mathcal{F}_4^\blacktriangle$-integral via Riemann sums, one has to resort to variational approach.

Definition 20 ([26, Definition 21]). A function $f\colon[a,b]\to\mathbb{R}$ is said to be $\mathrm{vwE}_\blacktriangle$-integrable if there exists a function $F\colon[a,b]\to\mathbb{R}$ (an indefinite integral of f) such that for every $\varepsilon>0$ there exists an $[a,b]$-form $\boldsymbol{\alpha}$, together with a related gauge $\boldsymbol{\delta}$, and, given a countable set $A\supset\operatorname{Is}\boldsymbol{\alpha}$, a $\blacktriangle$-choice $\mathcal{C}$ on A, such that for every division $\mathcal{P}$ in $[a,b]$ that is $(\mathcal{C},\boldsymbol{\delta})$-fine,

$$\sum_{(I,x)\in\mathcal{P}} |f(x)|I| - \Delta F(I)| < \varepsilon. \tag{11}$$

Theorem 21 ([26, Corollary 34]). *The $\mathcal{F}_4^\blacktriangle$-integral is equivalent to the $\mathrm{vwE}_\blacktriangle$-integral.*

Remark 3. Equivalently, as has been proven in [26], in Definition 20 the $[a,b]$-form $\boldsymbol{\alpha}$ can be assumed independent of ε. Moreover, in [26] we assume $\boldsymbol{\alpha}$ there to be measurable, a condition which can be dropped. Indeed, if f, F satisfy Definition 20, then F is a $\blacktriangle$-continuous, thus Baire one (assumption on $\blacktriangle$) and so measurable VBG-function. An $[a,b]$-form $\boldsymbol{\alpha}=\{E_n\}_{n=1}^\infty$ such that $F\restriction E_n$ is VB and continuous for all n can be then found measurable. The proof of [26, Theorem 30] shows f is $\mathrm{vwE}_\blacktriangle$-integrable using this $\boldsymbol{\alpha}$.

Remark 4. In Definition 20, and in Theorem 25, the condition that $A\supset\operatorname{Is}\boldsymbol{\alpha}$ is kept for the sake of uniformity only.

One of basic topics in contemporary generalized Riemann integration theory is related to the so-called *strong Lusin condition*. The condition concerns variational measure $|.|_F$ determined by an *integration basis* and a function F; it says the variational measure is absolutely continuous with respect to the Lebesgue measure.[2] A general result from the theory of integration w.r.t. a basis says that if F is an indefinite integral then it necessarily satisfies the corresponding strong Lusin condition. The opposite implication is problematic in many particular cases and not true in general (see, e.g., § 5 of [5]). If it holds, the corresponding integral allows a descriptive characterization with its variational measure: indefinite integrals are exactly the functions generating absolutely continuous variational measures (a result in the spirit of Lebesgue integration theory). For several characterizations of this kind see e.g. [1, 2, 3, 4]. In case of $LL_\blacktriangle$-, $E_\blacktriangle$-, and $vwE_\blacktriangle$-integrals the question of complete descriptive characterization is not, in fact, an issue, since original definitions of these integrals ($\mathcal{F}_i^\blacktriangle$-integrals, $i = 1, 2, 4$) are themselves descriptive. Nevertheless, one can ask whether corresponding strong Lusin conditions do characterize indefinite $\mathcal{F}_i^\blacktriangle$-integrals, $i = 1, 2, 4$, that is, $\blacktriangle$-continuous: [ACG]-functions, [VBG]-functions satisfying (N), and VBG-functions satisfying (N). The results from § 4 allow us to formulate respective characterizations with no extra proof. First of all, however, we need to enhance variational measures $\bar{\mu}_{\boldsymbol{\alpha}}^F$, $\mu_{\boldsymbol{\alpha}}^F$, $\bar{\mu}^F$, and μ^F in order to make them fully correspond to Definitions 16, 18, and 20.

Let $F\colon I \to \mathbb{R}$, I an interval. For a subset $E \subset I$, a closed I-form $\boldsymbol{\alpha}$, a related gauge $\boldsymbol{\delta}$, and a $\blacktriangle$-choice $\mathcal{C}$ on some countable $A \supset \operatorname{Is}\boldsymbol{\alpha}$, denote

$$\bar{m}_{\boldsymbol{\alpha},\boldsymbol{\delta},\mathcal{C}}^F(E) = \sup_{\mathcal{P}} |\Delta| F(\mathcal{P}),$$

where the sup runs over all $(\mathcal{C}, \boldsymbol{\delta})$-fine divisions $\mathcal{P}$ in I, anchored in E. Denote moreover

$$\bar{m}_{\boldsymbol{\alpha}}^F(E) = \inf_{\boldsymbol{\delta}} \sup_{A} \inf_{\mathcal{C}} \bar{m}_{\boldsymbol{\alpha},\boldsymbol{\delta},\mathcal{C}}^F(E) \quad \text{and} \quad \bar{m}^F(E) = \inf_{\boldsymbol{\alpha}} \bar{m}_{\boldsymbol{\alpha}}^F(E),$$

where the first inf runs over all gauges $\boldsymbol{\delta}$ related to $\boldsymbol{\alpha}$, the sup concerns all countable $A \supset \operatorname{Is}\boldsymbol{\alpha}$, while $\mathcal{C}$ is here an arbitrary $\blacktriangle$-choice on a given A. Analogously one defines $m_{\boldsymbol{\alpha}}^F(E)$ and $m^F(E)$.

The sole role of parameter $\mathcal{C}$ in the above definitions is to guarantee that a function F with the variational measure $\bar{m}_{\boldsymbol{\alpha}}^F$, $\bar{m}^F$, $m_{\boldsymbol{\alpha}}^F$, or m^F being absolutely continuous, is $\blacktriangle$-continuous (indeed, any of these measures is absolutely continuous if and only if the corresponding μ is absolutely continuous and F is $\blacktriangle$-continuous).

[2] The name *strong Lusin condition* comes from the fact that, in many cases, this condition implies (Lusin's) condition (N).

From Theorems 12, 13, and 14 (and also Lemma 1, Theorem 4, and Corollary 3) we have the following (alternative) descriptive characterizations for $\mathcal{F}_i^{\blacktriangle}$-integrals, $i = 1, 2, 4$.

THEOREM 22. *Let* $F\colon [a,b] \to \mathbb{R}$. *The following are equivalent*:

- *F is an indefinite $\mathcal{F}_1^{\blacktriangle}$-integral*;
- *for some closed $[a,b]$-form $\boldsymbol{\alpha}$, $\overline{m}_{\boldsymbol{\alpha}}^F$ is absolutely continuous*;
- *F is $\blacktriangle$-continuous and for some closed $[a,b]$-form $\boldsymbol{\alpha}$, $\bar{\mu}_{\boldsymbol{\alpha}}^F$ is absolutely continuous.*

THEOREM 23. *Let* $F\colon [a,b] \to \mathbb{R}$. *The following are equivalent*:

- *F is an indefinite $\mathcal{F}_2^{\blacktriangle}$-integral*;
- *$\overline{m}^F$ is absolutely continuous*;
- *F is $\blacktriangle$-continuous and $\bar{\mu}^F$ is absolutely continuous*;
- *F is $\blacktriangle$-continuous and W_F is absolutely continuous.*

THEOREM 24. *Let* $F\colon [a,b] \to \mathbb{R}$. *The following are equivalent*:

- *F is an indefinite $\mathcal{F}_4^{\blacktriangle}$-integral*;
- *m^F is absolutely continuous*;
- *for some $[a,b]$-form $\boldsymbol{\alpha}$, $m_{\boldsymbol{\alpha}}^F$ is absolutely continuous*;
- *F is $\blacktriangle$-continuous and μ^F is absolutely continuous*;
- *F is $\blacktriangle$-continuous and for some $[a,b]$-form $\boldsymbol{\alpha}$, $\mu_{\boldsymbol{\alpha}}^F$ is absolutely continuous*;
- *F is $\blacktriangle$-continuous and W_F^q is absolutely continuous.*

5.2 – *Improved Riemann and variational definitions*

Thomson [30, Theorems 46 and 49] provided a Riemann (and variational) definition for the wide Denjoy integral in terms of the relation $\succ$. That Moore–Smith approach can be seen as a simplified version of Theorem 17 (in the same way as Thomson's definitions of weak and quite weak variational measures differ from our definitions of $\bar{\mu}^F$ and μ^F; cf. Lemma 1 and Theorem 6) and, when applied under relaxed continuity restrictions, leads to (as well simplified) variational description of the $\mathcal{F}_4^{\blacktriangle}$-integral (Theorem 25). An analogous result for the integral defined with closed forms is however far from being true.

Theorem 25. *A function* $f\colon[a,b]\to\mathbb{R}$ *is* $\mathcal{F}_4^{\blacktriangle}$*-integrable, with indefinite integral* F*, if and only if the following condition holds.*

(12) *For every* $\varepsilon>0$ *there exists an* $[a,b]$*-form* $\boldsymbol{\alpha}$ *and, given a countable set* $A\supset \operatorname{Is}\boldsymbol{\alpha}$*, a* $\blacktriangle$*-choice* $\mathcal{C}$ *on* A*, such that every division* $\mathcal{P}$ *in* $[a,b]$ *that is* $(\mathcal{C},\boldsymbol{\alpha})$*-fine fulfils* (11).

Proof. With Theorem 21, it's enough to prove the necessity part. Let $F\colon[a,b]\to\mathbb{R}$ be a $\blacktriangle$-continuous VBG-function satisfying (N) and an $[a,b]$-form $\{E_n\}_{n=1}^{\infty}$ be such that $F_n=F\restriction E_n$ is VB for every n. Denote with D_n the set of points $x\in E_n$ such that $F_n'(x)$ exists and equals $f(x)$; $|E_n\setminus D_n|=0$. Let $B_{nm}=\{x\in E_n\setminus D_n\colon|f(x)|\le m\}$, $m\in\mathbb{N}$, and, given $\varepsilon>0$, pick an open set $O_{nm}\supset B_{nm}$ so that $m2^{m+n}|O_{nm}|<\varepsilon$. Let $\boldsymbol{\alpha}_{nm}$ be a B_{nm}-form with $2^{n+m}\mu^F_{\boldsymbol{\alpha}_{mn},\infty}(B_{nm})<\varepsilon$ (Theorem 3 and Lemma 1). Passing to a B_{nm}-form finer than $\boldsymbol{\alpha}_{nm}$, if necessary, one can assume that every member of $\boldsymbol{\alpha}_{nm}$ is contained in a single component of O_{nm}. Take a gauge δ_n on D_n defined so that $|F(x)-F(z)-f(x)(x-z)|\le\varepsilon|x-z|$ if $|z-x|<\delta_n(x)$, $z\in E_n$. Define $D_{nkj}=\{x\in D_n\colon\delta_n(x)>1/k\}\cap[j/k,(j+1)/k]$, $n,k\in\mathbb{N}$, $j\in\mathbb{Z}$. The collection $\boldsymbol{\alpha}=\bigcup_{n,m}\boldsymbol{\alpha}_{nm}\cup\{D_{nkj}\}_{n,k,j}$ is an $[a,b]$-form.

As F is $\blacktriangle$-continuous, for (12) it's enough to assume the division $\mathcal{P}$ is $\boldsymbol{\alpha}$-fine. Denote $\mathcal{P}_e=\{(\langle x,y\rangle,x)\in\mathcal{P}\colon\langle x,y\rangle$ is $\boldsymbol{\alpha}_{nm}$-fine for some $n,m\}$, $\mathcal{R}_{nkj}=\{(\langle x,y\rangle,x)\in\mathcal{P}\colon\langle x,y\rangle$ is D_{nkj}-fine$\}$, $n,k\in\mathbb{N}$, $j\in\mathbb{Z}$. Clearly,

$$|\sigma(\mathcal{P}_e,f)|\le\sum_{n,m}m|O_{nm}|<\sum_{n,m}\frac{\varepsilon}{2^{n+m}}=\varepsilon,$$

$$|\Delta|F(\mathcal{P}_e)\le\sum_{n,m}\mu^F_{\boldsymbol{\alpha},\infty}(B_{nm})<\sum_{n,m}\frac{\varepsilon}{2^{n+m}}=\varepsilon,$$

and

$$\sum_{(\langle x,y\rangle,x)\in\mathcal{R}_{nkj}}|F(x)-F(y)-f(x)(x-y)|\le\varepsilon\cdot|\mathcal{I}(\mathcal{R}_{nkj})|,$$

so

$$\sum_{n,k,j}\ \sum_{(\langle x,y\rangle,x)\in\mathcal{R}_{nkj}}|F(x)-F(y)-f(x)(x-y)|\le\varepsilon(b-a),$$

as long as $\mathcal{R}_{nkj}$'s are considered disjoint. This shows the condition (12) holds for f and F. □

REMARK 5. Consider the following condition, expressed in terms of closed $[a,b]$-forms, analogous to (12).

(13) For every $\varepsilon > 0$ there exists a closed $[a,b]$-form $\boldsymbol{\alpha}$ and, given a countable set $A \supset \operatorname{Is}\boldsymbol{\alpha}$ a $\blacktriangle$-choice $\mathcal{C}$ on A, such that for every partition $\mathcal{P}$ of $[a,b]$ that is $(\mathcal{C},\boldsymbol{\alpha})$-fine, the inequality (10) holds.

Unlike for measurable forms, this condition is essentially stronger than even the one for $\mathcal{F}_1^{\blacktriangle}$-integrability. Indeed, it's easy to provide an example of summable function g such that (13) is fulfilled (under any local system $\blacktriangle$) for no f equal almost everywhere to g! Just take a summable function $g \geq 0$ on $[a,b]$ that is essentially (i.e., a.e.) bounded on no subinterval[3] of $[a,b]$. Given any closed $[a,b]$-form $\boldsymbol{\alpha} = \{E_n\}_{n=1}^{\infty}$, there is n such that $E_n \supset [c,d]$, $a < c < d < b$. Given any M, there is $x \in E_n$ with $2x > c + d$ and $f(x) > M$. The interval $([c,x],x)$ is $\boldsymbol{\alpha}$-fine, however the term

$$|F(x) - F(c) - f(x)(x-c)| \geq |f(x)(x-c)| - |F(x) - F(c)| \geq M \cdot \frac{c+d}{2} - 2B$$

cannot be bounded over various choices of $x \in [c,d]$ (here $F = \int f = \int g$ and $B = \max_{t\in[a,b]} |F(t)|$).

On the other hand, in the same connection, we have the following positive results. The first one is obvious. Let $f\colon [a,b] \to \mathbb{R}$.

THEOREM 26. *Assume f is Riemann integrable. Then, f and $\mathbf{I} = \int_a^b f$ fulfil* (13) (*for any* $\blacktriangle$).

THEOREM 27. *Assume f is $\mathcal{F}_2^{\blacktriangle}$-integrable and Baire one. Then, f and $\mathbf{I} = \int_a^b f$ fulfil* (13) (*for any* $\blacktriangle$).

PROOF. Let $F = \int f$ and fix an arbitrary $\varepsilon > 0$. Take a closed $[a,b]$-form $\boldsymbol{\alpha} = \{E_n\}_{n=1}^{\infty}$ together with a related gauge $\boldsymbol{\delta} = \{\delta_n\}_{n=1}^{\infty}$ such that the $\mathcal{F}_2^{\blacktriangle}$-integrability condition holds for $f, \boldsymbol{\alpha}, \boldsymbol{\delta}$ (and a $\blacktriangle$-choice $\mathcal{C}_A$ corresponding to A). As f is Baire one, we can assume that the oscillation of f on E_n is $< \varepsilon$ for all n. Moreover, as F is [VBG], $\boldsymbol{\alpha}$ can be taken so that

$$\sum_{n=1}^{\infty} \sum_{x\in E_n} \omega_n(x) < \varepsilon; \tag{14}$$

[3] E.g., let G_n be a descending sequence of dense open subsets of $[a,b]$ with $|G_n| \leq \frac{1}{n2^n}$. Set $g(x) = n$ when $x \in G_n \setminus G_{n+1}$, $n \in \mathbb{N}$, $g(x) = 0$ otherwise.

here $\omega_n(x)$ stands for the oscillation of $F \restriction E_n$ *at* $x \in E_n$, that is,

$$\omega_n(x) = \inf_{\eta>0} \sup\{|F(x) - F(x')| : |x - x'| < \eta,\ x' \in E_n\}.$$

Denote

$$E_{nkj} = \{x \in E_n : \delta_n(x) > 2/k\} \cap [j/k, (j+1)/k] \quad \text{and} \quad D_{nkj} = \operatorname{cl} E_{nkj},$$

for $k, n \in \mathbb{N}$, $j \in \mathbb{Z}$. The collection $\boldsymbol{\beta} = \{D_{nkj}\}_{n,k,j}$ is a closed $[a,b]$-form finer than $\boldsymbol{\alpha}$. Consider any $\boldsymbol{\beta}$-fine division $(\langle x_i, y_i\rangle, x_i)$, $i = 1, \ldots, m$, and agree that the enumeration in it is orderwise. For an i consider the interval $(\langle x_i, y_i\rangle, x_i)$ and assume it is D_{nkj}-fine. Note that $|x_i - y_i| \le 1/k$ and pick a point $x_i' \in E_{nkj}$ such that

$$|x_i' - y_i| < \frac{2}{k} < \delta_n(x_i'), \quad |F(x_i) - F(x_i')| < \frac{\varepsilon}{m} + \omega_n(x_i), \quad |f(x_i)| \cdot |x_i' - x_i| < \frac{\varepsilon}{m},$$

and $x_i' < \min\{x_{i+2}, y_{i+2}\}$ if $i \le m-2$, $x_i' > \max\{x_{i-2}, y_{i-2}\}$ if $i \ge 3$. Estimate

$$\begin{aligned} &|F(y_i) - F(x_i) - f(x_i)(y_i - x_i)| \\ &\quad \le |F(y_i) - F(x_i') - f(x_i')(y_i - x_i')| \\ &\qquad + |f(x_i') - f(x_i)| \cdot |y_i - x_i'| + |F(x_i) - F(x_i')| + |f(x_i)| \cdot |x_i' - x_i| \end{aligned}$$

and summing up over all i,

$$\begin{aligned} &\sum_{i=1}^{m} |F(y_i) - F(x_i) - f(x_i)(y_i - x_i)| \\ &\quad = \Big(\sum_{\text{odd } i} + \sum_{\text{even } i}\Big) |F(y_i) - F(x_i) - f(x_i)(y_i - x_i)| \\ &\quad \le \Big(\sum_{\text{odd } i} + \sum_{\text{even } i}\Big) |F(y_i) - F(x_i') - f(x_i')(y_i - x_i')| \\ &\qquad + \Big(\sum_{\text{odd } i} + \sum_{\text{even } i}\Big) |f(x_i') - f(x_i)| \cdot |y_i - x_i'| \\ &\qquad + \sum_{i=1}^{m} (|F(x_i) - F(x_i')| + |f(x_i)| \cdot |x_i' - x_i|). \end{aligned}$$

The last sum is $\le 3\varepsilon$ by inequality (14). The first couple of sums is $< 2\varepsilon$ since $\{(\langle x_i', y_i\rangle, x_i')\}_{\text{odd } i}$, $\{(\langle x_i', y_i\rangle, x_i')\}_{\text{even } i}$ form $\boldsymbol{\delta}$-fine divisions. The second couple of sums over odd and even i is less than $2\varepsilon(b-a)$, because both x_i and x_i' belong to one E_n and the oscillation of f over E_n is $< \varepsilon$. Thus,

$$\sum_{i=1}^{N} |F(y_i) - F(x_i) - f(x_i)(y_i - x_i)| < 5\varepsilon + 2\varepsilon(b-a).$$

Since F is a $\blacktriangle$-continuous function, the proof is finished. □

Remark 6. It follows, if $F\colon [a,b] \to \mathbb{R}$ is an everywhere differentiable function, then F and $f = F'$ fulfil (13) (for any $\blacktriangle$). The same is true for approximately differentiable functions: if $f(x) = F'_{\mathrm{ap}}(x)$ for all $x \in [a,b]$, then F and f fulfil (13) for $\blacktriangle = \blacktriangle_{\mathrm{ap}}$. Indeed, f is then Baire one, while F an approximately continuous [ACG]-function.

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Manoscritto pervenuto in redazione il 19 marzo 2018.