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Remark on the equations of axially symmetric gravitational fields

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Abstract – A local theorem of existence and uniqueness of solutions of the equations of stationary axially symmetric vacuum gravitational fields in the general theory of relativity close to the flat space solution is proved using the implicit function theorem in Banach spaces.

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1. Introduction

The stationary axially symmetric vacuum gravitational field equations of the general theory of relativity are highly nonlinear and pose a formidable mathematical challenge. Special solutions for cylindrical symmetric mass distributions were obtained in [18], [19], [1], [2], [8], [9], and [10]. Similar problems are also studied in different contexts, see [14], [6], [12], [17], and [3]. We quote, in particular, the papers of T. Lewis [11] and A. Papapetrou [13]. In Section 2 we study the boundary value problem for the field equations in a domain of $\mathbf{R}^3$ with cylindrical symmetry. A theorem of existence of solutions near the flat space solution is given using as tool the implicit function theorem in Banach spaces.

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2. A local theorem of existence and uniqueness

The equations of axially symmetric gravitational fields are derived from the quadratic form [13]

$$d s^2 = f\, d t^2 - e^{\mu}(d\rho^2 + d z^2) - l\, d\varphi^2 - 2m\, d\varphi\, d t,$$

where ρ, z and φ are cylindrical coordinates and the unknown functions f, μ, l and m depend only on ρ and z. Referring to [11] for the derivation and the physical background we have, using canonical coordinates, [1] the system of field equations

$$f_{\rho\rho} + f_{zz} - \frac{1}{\rho} f_\rho = -\frac{f}{\rho^2}(f_\rho l_\rho + f_z l_z + m_\rho^2 + m_z^2), \tag{2.1}$$

$$l_{\rho\rho} + l_{zz} - \frac{1}{\rho} l_\rho = -\frac{l}{\rho^2}(f_\rho l_\rho + f_z l_z + m_\rho^2 + m_z^2), \tag{2.2}$$

$$m_{\rho\rho} + m_{zz} - \frac{1}{\rho} m_\rho = -\frac{m}{\rho^2}(f_\rho l_\rho + f_z l_z + m_\rho^2 + m_z^2), \tag{2.3}$$

$$\mu_\rho = -\frac{1}{2\rho}(f_\rho l_\rho - f_z l_z + m_\rho^2 - m_z^2), \tag{2.4}$$

$$\mu_z = -\frac{1}{2\rho}(f_\rho l_z + f_z l_\rho + 2m_\rho m_z). \tag{2.5}$$

If f, l and m have been determined from (2.1)–(2.3), μ can be found from (2.5) and (2.4). For, if (2.1) and (2.2) are taken into account we find that the condition of integrability of the differential form

$$\frac{1}{2\rho}(f_\rho l_\rho - f_z l_z + m_\rho^2 - m_z^2)\, d\rho + \frac{1}{2\rho}(f_\rho l_z + f_z l_\rho + 2m_\rho m_z)\, d z$$

is satisfied and thus $\mu(\rho, z)$ is known except for an arbitrary constant. We note that if (f, l, m) is a solution of (2.1)–(2.3) also $(-f, -l, m)$ and $(f, l, -m)$ are solutions. The solution of (2.1)–(2.5), corresponding to the flat space, is $f = 1$, $l = \rho^2$, $m = 0$, $\mu = 0$. Moreover, the three equations (2.1)–(2.3) are not independent since, if (f, l, m) is a solution, we have, (see [11] and [18])

$$f l + m^2 = \rho^2. \tag{2.6}$$

This reflects the fact that the system of the Einstein equations is overdetermined. If we multiply (2.1) by l, (2.2) by $-f$ and add the resulting equations we obtain

$$\Big[\frac{1}{\rho}(l f_\rho - f l_\rho)\Big]_\rho + \Big[\frac{1}{\rho}(l f_z - f l_z)\Big]_z = 0. \tag{2.7}$$

[1] Also known as the Weyl-Lewis-Papapetrou coordinates [3].

This suggests to introduce two new independent unknown functions $\psi(\rho, z)$, $\tau(\rho, z)$ via the transformation

$$\psi = \frac{1}{2}\ln\Big(\frac{l}{f}\Big), \tag{2.8}$$

$$\tau = \frac{m}{\rho}, \tag{2.9}$$

where f, l, m, and ρ are connected by (2.6). This gives

$$f = \rho\sqrt{1-\tau^2}e^{-\psi}, \quad l = \rho\sqrt{1-\tau^2}e^{\psi}. \tag{2.10}$$

The equation (2.7) becomes, in terms of ψ and τ,

$$\frac{1}{\rho}(\rho\psi_\rho)_\rho + \psi_{zz} = \frac{2\tau}{1-\tau^2}(\tau_\rho\psi_\rho + \tau_z\psi_z). \tag{2.11}$$

Moreover, writing $\tau\rho$ for m in (2.3) we have the equation for τ

$$\frac{1}{\rho}(\rho\tau_r) + \tau_{zz} - \frac{1}{\rho^2}\tau = -\frac{\tau}{\rho^2}(f_\rho l_\rho + f_z l_z + m_\rho^2 + m_z^2), \tag{2.12}$$

where f and l are given by (2.10) and $m = \tau\rho$. The equations (2.9) and (2.10) should be more correctly written

$$\tau = \pm\frac{1}{\rho}\sqrt{\rho^2 - fl}, \quad f = \pm\rho\sqrt{1-\tau^2}e^{-\psi}, \quad l = \pm\rho\sqrt{1-\tau^2}e^{\psi}.$$

However, this ambiguity of sign is immaterial and disappears in the resulting equations (2.11), (2.12) as may be easily verified. The flat space solution becomes, in terms of ψ and τ,

$$\psi(\rho, z) = \log\rho, \quad \tau(\rho, z) = 0. \tag{2.13}$$

On the axis of symmetry, i.e for $\rho = 0$, the solutions of (2.12), (2.11) present a singularity which has been studied in details in [15], [16] for the case of the Curzon solution [5]. To avoid this difficulty we assume that all the matter producing the field is contained in the axially symmetric set

$$S = \{(\rho, z, \varphi),\ |z| < \infty,\ 0 < \varphi \leq 2\pi,\ 0 \leq \rho \leq h(z)\}$$

with

$$h(z) > 0,\ h(z) \in C^0(\mathbf{R}^1), \qquad \lim_{z\to\pm\infty} h(z) = h_0$$

and is supposed to determine the values of ψ and τ on the boundary of S via two given functions $\alpha(z)$ and $\beta(z)$. At infinity the metric approaches that of the special theory of relativity. Under these hypotheses we obtain the following exterior Dirichlet's problem for the determination of ψ and τ:

$$\frac{1}{\rho}(\rho\psi_\rho)_\rho + \psi_{zz} = \frac{2\tau}{1-\tau^2}(\tau_\rho\psi_\rho + \tau_z\psi_z) \quad \text{in } S^c, \tag{2.14}$$

$$\frac{1}{\rho}(\rho\tau_r) + \tau_{zz} - \frac{1}{\rho^2}\tau = -\frac{\tau}{\rho^2}(f_\rho l_\rho + f_z l_z + m_\rho^2 + m_z^2) \quad \text{in } S^c, \tag{2.15}$$

$$\psi(h(z), z) = \alpha(z), \tag{2.16}$$

$$\tau(h(z), z) = \beta(z), \tag{2.17}$$

$$\lim_{\rho\to\infty} \frac{\psi(\rho, z)}{\log\rho} = 1 \quad \text{uniformly with respect to } z, \tag{2.18}$$

$$\lim_{\rho\to\infty} \tau(\rho, z) = 0 \quad \text{uniformly with respect to } z. \tag{2.19}$$

This difficult problem is further complicated by the fact that in (2.14) and (2.15) we do not have the "full" laplacian in cylindrical coordinates. Thus in this paper we limit ourselves to study a simplified form of problem (2.14)–(2.19) in a bounded domain which excludes the z-axis. Let $\rho = h(z) \in C^{0,\lambda}([-L, L])$, $0 < \lambda < 1$ satisfy $h(z) > 0$ in $[-L, L]$. Define (see Figure 1), with $R > \max_{[-L,L]} h(z)$,

$$\Omega = \{(\rho, z, \varphi); |z| < L,\ 0 < \varphi \leq 2\pi,\ h(z) < \rho < R\},$$

$$\Gamma_1 = \{(\rho, z, \varphi);\ h(L) \leq \rho \leq R,\ z = L,\ 0 < \varphi \leq 2\pi\},$$

$$\Gamma_2 = \{(\rho, z, \varphi);\ \rho = R,\ |z| < L,\ 0 < \varphi \leq 2\pi\},$$

$$\Gamma_3 = \{(\rho, z, \varphi);\ h(-L) \leq \rho \leq R,\ z = -L,\ 0 < \varphi \leq 2\pi\},$$

$$\Gamma_4 = \{(\rho, z, \varphi);\ \rho = h(z),\ |z| < L,\ 0 < \varphi \leq 2\pi\}.$$

All the matter producing the fields is contained in the set

$$S = \{(\rho, z, \varphi); |z| \leq L,\ 0 < \varphi \leq 2\pi,\ 0 \leq \rho \leq h(z)\}$$

and is supposed to determine the values of ψ and τ on Γ_4 via two functions $\alpha(z)$ and $\beta(z)$. Our "horizon"will be the set $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Correspondingly we assume on $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ the boundary condition of the flat space solution i.e. $\psi = \log\rho$ on $\Gamma_1 \cup \Gamma_3$, $\psi = \log R$ on Γ_2, $\tau = 0$ on $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. On Γ_4 we have $\psi = \alpha$ and

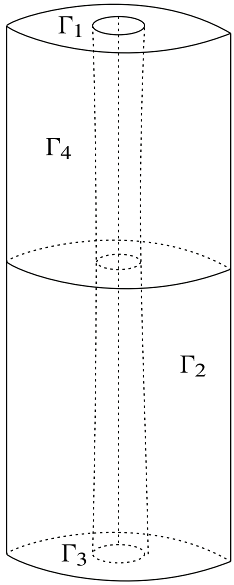

Figure 1

$\tau = \beta$ where $\alpha(z)$ and $\beta(z)$ are assigned $C^{0,\lambda}([-L, L])$ functions which satisfy the compatibility conditions

$$\alpha(L) = \log h(L), \quad \alpha(-L) = \log h(-L), \quad \beta(L) = 0, \quad \beta(-L) = 0.$$

For the determination of ψ and τ we have therefore the problem

$$\psi_{\rho\rho} + \psi_{zz} + \frac{1}{\rho}\psi_\rho = \frac{2\tau}{1-\tau^2}(\tau_\rho\psi_\rho + \tau_z\psi_z) \quad \text{in } \Omega, \tag{2.20}$$

$$\tau_{\rho\rho} + \tau_{zz} + \frac{1}{\rho}\tau_\rho - \frac{1}{\rho^2}\tau = -\frac{\tau}{\rho^2}(f_\rho l_\rho + f_z l_z + m_\rho^2 + m_z^2) \quad \text{in } \Omega, \tag{2.21}$$

$$\psi = \log\rho, \quad \tau = 0 \quad \text{on } \Gamma_1, \tag{2.22}$$

$$\psi = \log R, \quad \tau = 0 \quad \text{on } \Gamma_2, \tag{2.23}$$

$$\psi = \log\rho, \quad \tau = 0 \quad \text{on } \Gamma_3, \tag{2.24}$$

$$\psi = \alpha, \quad \tau = \beta \quad \text{on } \Gamma_4. \tag{2.25}$$

If R and L are very large in comparison with $\max h(z)$ the present model would give a good approximation to the model with conditions of flat space at infinity, i.e. (2.14)–(2.19). With the choice of boundary conditions

$$\alpha = \log \rho, \quad \beta = 0 \tag{2.26}$$

the solution of problem (2.20)–(2.25) is simply $(\psi, \tau) = (\log \rho, 0)$. We wish to prove that perturbing the boundary values (2.26) the problem (2.20)–(2.25) still has one and only one solution. To this end we use the implicit theorem in Banach space which we quote below.

THEOREM 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $\mathcal{N}$ an open subset of $\mathcal{X}$, $u^* \in \mathcal{N}$ and $F \in C^1(\mathcal{N}, \mathcal{Y})$. Assume the Frechet's differential $F'(u^*)$ to be invertible. Then there exists a neighbourhood U of u^* in $\mathcal{X}$ and a neighbourhood V of $v^* = F(u^*)$ in $\mathcal{Y}$ such that F is a diffeomorphism from U to V.*

To apply this theorem we restate the problem (2.20)–(2.25) with null boundary conditions on $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Let $\phi(\rho, z) = \psi(\rho, z) - \log \rho$. In terms of ϕ and τ we arrive at the problem

$$\frac{1}{\rho}(\rho\phi_\rho)_\rho + \phi_{zz} = \frac{2\tau}{1-\tau^2}\left(\tau_\rho\phi_\rho + \phi_z\tau_z + \frac{1}{\rho}\tau_\rho\right) \quad \text{in } \Omega, \tag{2.27}$$

$$\frac{1}{\rho}(\rho\tau_\rho)_\rho + \tau_{zz} - \frac{1}{\rho^2}\tau = -\frac{\tau}{\rho}\left[(\tau^2-1)(2\rho\phi_\rho + \rho^2|\nabla\phi|^2) + \frac{\rho^2|\nabla\tau|^2}{\tau^2-1}\right] \quad \text{in } \Omega, \tag{2.28}$$

$$\phi = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \qquad \phi = \alpha - \log \rho \quad \text{on } \Gamma_4, \tag{2.29}$$

$$\tau = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \qquad \tau = \beta \qquad \text{on } \Gamma_4. \tag{2.30}$$

At a first sight it may appear reasonable, in view of the fact that all the data in problem (2.27)–(2.30) do not depend on φ, to study this problem in the plane domain $D = \{(\rho, z);\ |z| < L,\ h(z) < \rho < R\}$. This would simplify the discussion; however in this way it is impossible to exclude the, a priori possible, dependence of the solution from φ because the problem is in itself three-dimensional. Let $\gamma_1 = \{(\rho, z, \varphi);\ \rho = h(L),\ z = L,\ 0 < \varphi \leq 2\pi\}$, $\gamma_3 = \{(\rho, z, \varphi);\ \rho = h(-L),\ z = -L,\ 0 < \varphi \leq 2\pi\}$. To apply Theorem 2.1 we define the following spaces of functions

$$\mathcal{A} = \{\eta \in C^{2,\lambda}(\overline{\Omega}),\ 0 < \lambda < 1,\ \eta = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,\ \eta \text{ not depending on } \varphi\},$$

$$\mathcal{B} = \{\eta \in C^{0,\lambda}(\overline{\Omega}),\ 0 < \lambda < 1,\ \eta \text{ not depending}^{2} \text{ on } \varphi\},$$

$$\mathcal{C} = \{\eta \in C^{0,\lambda}(\Gamma_4),\ 0 < \lambda < 1,\ \eta = 0 \text{ on } \gamma_1 \cup \gamma_3,\ \eta \text{ not depending on } \varphi\},$$

$$\mathcal{X} = \mathcal{A} \times \mathcal{A}, \quad \mathcal{Y} = (\mathcal{B} \times \mathcal{C}) \times (\mathcal{B} \times \mathcal{C}),$$

$$\mathcal{N} = \{(\phi, \tau) \in \mathcal{X},\ \tau^2 < 1/2 \text{ in } D\}, \quad u^* = (0,0) \in \mathcal{N}.$$

The operator $F : \mathcal{N} \to \mathcal{Y}$ of Theorem 2.1 will be

$$F(\phi, \tau) = ((M(\phi,\tau), \phi|_{\Gamma_4}), (N(\phi,\tau), \tau|_{\Gamma_4})),$$

where

$$M(\phi,\tau) = \frac{1}{\rho}(\rho\phi_\rho)_\rho + \phi_{zz} - \frac{2\tau}{1-\tau^2}\Big(\tau_\rho\phi_\rho + \phi_z\tau_z + \frac{1}{\rho}\tau_\rho\Big),$$

$$N(\phi,\tau) = \frac{1}{\rho}(\rho\tau_\rho)_\rho + \tau_{zz} - \frac{1}{\rho^2}\tau + \frac{\tau}{\rho}\Big[(\tau^2-1)(2\rho\phi_\rho + \rho^2|\nabla\phi|^2) + \frac{\rho^2|\nabla\tau|^2}{\tau^2-1}\Big].$$

We claim that F is well-defined; this requires in particular $M(\phi,\tau)$ and $N(\phi,\tau) \in \mathcal{B}$. The linear parts of M and N certainly belong to $C^{0,\lambda}(\overline{\Omega})$. On the other hand, the nonlinear part of M belongs to $C^{0,\lambda}(\overline{\Omega})$. In fact, since $\tau^2 < 1/2$ we have $(1-\tau^2)^{-1} \in C^{0,\lambda}(\overline{\Omega})$, but the product of four functions of class $C^{0,\lambda}(\overline{\Omega})$ is still a function of class $C^{0,\lambda}(\overline{\Omega})$. Moreover, the last term of $M(\phi,\tau)$ creates no trouble since $\rho > \min h(z) > 0$. Similarly we see that $N(\phi,\tau) \in \mathcal{B}$, recalling that for $(\phi,\tau) \in \mathcal{N}$ we have $\sqrt{1-\tau^2} \in C^{0,\lambda}(\overline{\Omega})$. The Frechet's differential of $F(\phi,\tau)$, computed for $(\phi,\tau) = (0,0)$, is given by

$$F'(0,0)[\Phi, T] = \Big(\Big(\frac{1}{\rho}(\rho\Phi_\rho)_\rho + \Phi_{zz},\ \Phi|_{\Gamma_4}\Big),\ \Big(\frac{1}{\rho}(\rho T_\rho)_\rho + T_{zz} - \frac{1}{\rho^2}T,\ T|_{\Gamma_4}\Big)\Big).$$

It remains to prove that $F'(0,0)[\Phi, T]$, as an operator from $\mathcal{X}$ to $\mathcal{Y}$, is invertible. This is equivalent to say that, if

$$a \in \mathcal{B}, \quad c \in \mathcal{B}, \quad b \in \mathcal{C}, \quad d \in \mathcal{C},$$

the two linear problems

$$\text{(2.31)} \qquad \Phi \in \mathcal{A}, \qquad \frac{1}{\rho}(\rho\Phi_\rho)_\rho + \Phi_{zz} = a \quad \text{in } \overline{\Omega}, \qquad \Phi = b \quad \text{on } \Gamma_4,$$

[2] We say that a function $\eta\colon \Omega \to \mathbf{R}^1$ is not depending on φ if η is constant along the flow lines ∂_φ.

$$(2.32)\qquad T \in \mathcal{A}, \qquad \frac{1}{\rho}(\rho T_\rho)_\rho + T_{zz} - \frac{1}{\rho^2}T = c \quad \text{in } \overline{\Omega}, \qquad T = d \quad \text{on } \Gamma_4,$$

have one and only one solution. We note that the operator entering in (2.31) is only a "piece" of the three-dimensional laplacian which in cylindrical coordinates reads $\frac{1}{\rho}(\rho\Phi_\rho)_\rho + \Phi_{zz} + \frac{1}{\rho^2}\Phi_{\varphi\varphi}$ and the same is true for the problem (2.32). Thus in a domain less special of Ω the problems (2.31) or (2.32) would not have, in general, solution. Hence an *ad hoc* proof is needed.

THEOREM 2.2. *The problems* (2.31) *and* (2.32) *have one and only one solution.*

PROOF. We consider the auxiliary problem

$$(2.33)\qquad \overline{\Phi} \in C^{2,\lambda}(\overline{\Omega}), \qquad \frac{1}{\rho}(\rho\overline{\Phi}_\rho)_\rho + \overline{\Phi}_{zz} + \frac{1}{\rho^2}\overline{\Phi}_{\varphi\varphi} = a(\rho, z) \quad \text{in } \overline{\Omega}$$

$$(2.34)\qquad \overline{\Phi} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \qquad \overline{\Phi} = b \quad \text{on } \Gamma_4$$

$$(2.35)\qquad \overline{T} \in C^{2,\lambda}(\overline{\Omega}), \qquad \frac{1}{\rho}(\rho\overline{T}_\rho)_\rho + \overline{T}_{zz} + \frac{1}{\rho^2}\overline{T}_{\varphi\varphi} - \frac{1}{\rho^2}\overline{T} = c \quad \text{in } \overline{\Omega}$$

$$(2.36)\qquad \overline{T} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \qquad \overline{T} = d \quad \text{on } \Gamma_4.$$

Both problems (2.33)–(2.34) and (2.35)–(2.36) are uniquely solvable (see [7], p. 106). Moreover, by the Schauder estimates on uniformly elliptic equations of the second order, we have

$$(2.37)\qquad |\overline{\Phi}|_{2,\lambda,\bar{\Omega}} \leq C(|a|_{0,\lambda,\bar{\Omega}} + |b|_{2,\lambda,\bar{\Omega}}$$

$$(2.38)\qquad |\overline{T}|_{2,\lambda,\bar{\Omega}} \leq C(|c|_{0,\lambda,\bar{\Omega}} + |d|_{2,\lambda,\bar{\Omega}}.$$

These inequalities, together with the open mapping theorem, imply the continuity of the operator F' and of its inverse $(F')^{-1}$ since these operators act on linear subspaces of the spaces on which $\overline{\Phi}$ and $\overline{T}$ naturally live. "A priori" the solutions of problems (2.33)–(2.34) and (2.35)–(2.36) depend also on φ. On the other hand, it is easily seen that $\overline{\Phi}(\rho, z, \varphi + K_1)$ and $\overline{T}(\rho, z, \varphi + K_2)$, with K_1 and K_2 arbitrary constants, are solution of problems (2.33)–(2.34) and (2.35)–(2.36) in view of the special form of the domain $\overline{\Omega}$ and of the fact that a, b, c and d do not depend on φ. Thus $\overline{\Phi}$ and $\overline{T}$ are also solutions of problems (2.31) and (2.32), respectively. Since these problems have an unique solution we conclude that they have one and only one solution as required. □

We are now in a position to apply Theorem 2.1 and to conclude with the following

THEOREM 2.3. *There exists a constant $C > 0$ such that, if*

$$|\alpha|_{2,\lambda,[-L,L]} \leq C, \quad |\beta|_{2,\lambda,[-L,L]} \leq C,$$

the problem (2.27)–(2.30) *has one and only one solution.*

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