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THE MIYAOKA-YAU INEQUALITY AND UNIFORMISATION OF CANONICAL MODELS

BY DANIEL GREB, STEFAN KEBEKUS, THOMAS PETERNELL AND BEHROUZ TAJI

ABSTRACT. – We establish the Miyaoka-Yau inequality in terms of orbifold Chern classes for the tangent sheaf of any complex projective variety of general type with klt singularities and nef canonical divisor. In case equality is attained for a variety with at worst terminal singularities, we prove that the associated canonical model is the quotient of the unit ball by a discrete group action.

RÉSUMÉ. – Nous établissons l'inégalité de Miyaoka-Yau en termes de classes de Chern orbifoldes pour le faisceau tangent d'une variété complexe projective de type général à singularités klt et diviseur canonique nef. Dans le cas d'égalité pour une variété à singularités terminales, nous établissons que le modèle canonique associé est un quotient de la boule unité par un groupe agissant discrètement.

1. Introduction

A classical result in complex geometry asserts that the Chern classes of any holomorphic, slope-semistable vector bundle \mathscr{E} of rank *r* on a compact Kähler manifold (X, ω) satisfy the *Bogomolov-Gieseker inequality*

$$\int_X \left(2r \cdot c_2(\mathscr{E}) - (r-1) \cdot c_1^2(\mathscr{E}) \right) \wedge \omega^{n-2} \ge 0.$$

Thanks to his solution of the Calabi conjecture, Yau established in [66] the following stronger, Miyaoka-Yau inequality for the holomorphic tangent bundle of any *n*-dimensional compact Kähler manifold X with ample canonical class K_X ,

(*)
$$\int_X \left(2(n+1) \cdot c_2(\mathscr{T}_X) - n \cdot c_1^2(\mathscr{T}_X) \right) \cdot [K_X]^{n-2} \ge 0.$$

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In case of equality, the natural symmetries imposed by the Kähler-Einstein condition lead to the uniformisation of X by the unit ball.

A fundamental result of Birkar, Cascini, Hacon and M^cKernan, [4], states that every projective manifold of general type admits a minimal model, which is a normal, \mathbb{Q} -factorial, projective variety with at most terminal singularities whose canonical divisor is big and nef. These varieties are however usually singular. It was expected that the Miyaoka-Yau inequality should also hold in this context, with applications to uniformisation in case of equality. This problem has attracted considerable interest; Section 1.5 gives a short account of the history.

1.1. Main results of this paper

The main result of this paper settles the problem in full generality, even in the broader context of varieties with Kawamata log-terminal (= klt) singularities and nef canonical divisor.

THEOREM 1.1 (Q-Miyaoka-Yau inequality). – Let X be an n-dimensional, projective, klt variety of general type whose canonical divisor K_X is nef. Then,

(1.1.1)
$$\left(2(n+1)\cdot\widehat{c}_2(\mathscr{T}_X)-n\cdot\widehat{c}_1(\mathscr{T}_X)^2\right)\cdot[K_X]^{n-2}\geq 0.$$

The formulation of Theorem 1.1 uses the fact that varieties with klt singularities have quotient singularities in codimension two. This allows to define Q-Chern classes (or "orbifold Chern classes") $\hat{c}_1(\mathscr{T}_X)$ and $\hat{c}_2(\mathscr{T}_X)$ for the tangent sheaf $\mathscr{T}_X = (\Omega_X^1)^*$ of X, which is reflexive and a Q-vector bundle on the open subset where X has quotient singularities. We refer to Section 3.7 for definitions and for a detailed discussion. If X is smooth in codimension two, which is the case when X has terminal singularities, these agree with the usual Chern classes $c_{\bullet}(\mathscr{T}_X)$. We call a projective variety of general type *minimal* if it has at worst terminal singularities and if its canonical divisor is nef, cf. [41, 2.13] and Definition 2.3 below.

THEOREM 1.2 (Characterisation of singular ball quotients, I). – Let X be an n-dimensional minimal variety of general type. If equality holds in (1.1.1), then the canonical model X_{can} is smooth in codimension two, there exists a ball quotient Y and a finite, Galois, quasi-étale morphism $f : Y \to X_{can}$. In particular, X_{can} has only quotient singularities.

We refer to Section 2.2 for a discussion of ball quotients and canonical models.

We expect that Theorem 1.2 holds without the additional assumption that X be terminal. In fact, we prove a result slightly stronger than Theorem 1.2, which applies to varieties with klt singularities that are smooth in codimension two, cf. Theorem 8.1 as well as Theorem and Definition 1.3 below. As already said above, Theorem 1.2 applies to all minimal models of smooth varieties of general type, which is the case most relevant for applications in the Minimal Model Program. In Section 10, we discuss further potential generalizations of the Miyaoka-Yau inequality and the uniformisation result.

Extending Theorem 1.2, we show that the canonical models of Theorem 1.2 admit a "singular uniformisation" by the unit ball \mathbb{B}^n . More precisely, they can be realized as

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quotients of \mathbb{B}^n by actions of discrete subgroups in PSU(1, n) that are not necessarily fixedpoint free. In particular, the geometry of these spaces can be studied using the theory of automorphic forms, cf. [40, Part II].

THEOREM AND DEFINITION 1.3 (Characterisation of singular ball quotients, II).

Let X be a normal, irreducible, compact, complex space of dimension n. Then the following statements are equivalent.

- (1.3.1) The space X is of the form $\mathbb{B}^n/\widehat{\Gamma}$ for a discrete, cocompact subgroup $\widehat{\Gamma} < \operatorname{Aut}_{\mathscr{O}}(\mathbb{B}^n)$ whose action on \mathbb{B}^n is fixed-point free in codimension two.
- (1.3.2) The space X is of the form Y/G, where Y is a ball quotient (cf. Definition 2.5), and G is a finite group of automorphisms of Y whose action is fixed-point free in codimension two.
- (1.3.3) The space X is projective, klt, and smooth in codimension two; the canonical divisor K_X is ample and we have equality in the Q-Miyaoka-Yau Inequality (1.1.1).

A compact complex space is called singular ball quotient if it satisfies these equivalent conditions.

COROLLARY 1.4 (Hyperbolicity of smooth loci of canonical models).

In the setting of Theorem 1.2, the canonical model X_{can} is a singular ball quotient. In particular, the smooth locus of X_{can} is Kobayashi-hyperbolic.

In fact, a more precise hyperbolicity statement holds, see Section 9.3. In addition, classical results concerning deformation rigidity [5], Mostow rigidity [66, Thm. 6], stability under Galois conjugation [56, Cor. 9.5], and the fact that ball quotients can be defined over number fields [54] have analogs for singular ball quotients. These aspects will be addressed in a future work.

1.2. Outline of the proof

Various earlier papers used differential-geometric techniques, such as orbifold Kähler-Einstein metrics, to obtain the Miyaoka-Yau inequality. Inspired by the work of Simpson [56] we take a different approach, partially generalizing Simpson's results on the Kobayashi-Hitchin correspondence for Higgs sheaves. For suitable manifolds X, Simpson equips $\mathscr{E} := \Omega_X^1 \oplus \mathscr{O}_X$ with a natural structure of a Higgs bundle, proves its stability and derives a Bogomolov-Gieseker inequality for \mathscr{E} . The Miyaoka-Yau inequality for \mathscr{T}_X is an immediate consequence. In case of equality, he constructs a variation of Hodge structures whose period map gives the desired uniformisation by the ball.

On a technical level, one main contribution of our paper is to establish a good definition of Higgs sheaves on singular spaces, and an associated notion of stability. These definitions may seem a little awkward at first, but for varieties with the singularities of the minimal model program they have just enough universal properties to make Simpson's approach work the list of properties includes restrictions theorems of Mehta-Ramanathan type, weakly functorial pull-back, and invariance of stability under resolution. As for a converse, earlier work on differential forms, [16, 32], suggests that spaces with klt singularities are the largest

class of varieties where functorial pull-back properties can possibly hold for any reasonable definition.

In our singular situation, the correct analog of the sheaf \mathscr{E} used by Simpson is $(\Omega_X^1)^{**} \oplus \mathscr{O}_X$. The starting point of our analysis is the fact that this Higgs sheaf is stable with respect to K_X in case X is klt and K_X is big and nef. This is a consequence of a recent result of Guenancia [24], which in turn generalizes a by now classical result of Enoki [9] to the klt setup. Using restriction theorems of Mehta-Ramanathan type, Theorem 1.1 follows as a consequence of a Bogomolov-Gieseker-type inequality for stable Higgs sheaves on surfaces with quotient singularities, Theorem 6.1.

To prove Theorem 1.2, let $Y \to X$ we consider a quasi-étale cover, where the étale fundamental groups $\hat{\pi}_1(Y)$ and $\hat{\pi}_1(Y_{reg})$ agree; the existence of such covers was established in [17, Thm. 1.5]. We aim to prove that Y is smooth. The proof is based on the second main technical contribution of this paper, a partial generalization of Simpson's Nonabelian Hodge Correspondence to the singular setting, see Section 5.8. Using the relation of special representations of fundamental groups to Higgs *bundles* and variations of Hodge structures, the choice of Y allows us to prove that $(\Omega_Y^1)^{**} \oplus \mathcal{O}_Y$ is in fact locally free. The confirmation of the Zariski-Lipman conjecture for spaces with klt singularities, [16, Thm. 6.1], then shows that Y is smooth. Using the original uniformisation theorem proven by Yau, we conclude that Y is a ball quotient.

1.3. Comparison with the torus-quotient case

A related uniformisation problem for klt varieties with vanishing first and second Chern class has been solved by the authors partly in joint work with Steven Lu, see [17] and [46]. We would like to emphasize that although there are some similarities between the strategies of the proof of Theorem 1.2 and those that were devised to solve the uniformisation problem in the case of vanishing Chern classes, the two approaches are significantly different. First of all, to prove Theorem 1.2, we need a suitable notion of Higgs sheaves over singular spaces verifying some important functoriality properties (see Subsections 5.3, 5.4 and 5.7). But, the difference between the two approaches in [17] and the current paper is not confined to the technicalities that arise from the setup of a theory of Higgs sheaves over singular spaces; a substantially refined strategy is required for a successful application of this new machinery to establish the uniformisation result, Theorem 1.2. We refer the reader to Remark 8.8 for a detailed comparison of the two strategies.

1.4. Structure of the paper

Section 2 establishes notation and reviews a few facts that will be used later. Building on the work of Mumford, Sections 3–3.7 establish basic properties pertaining to \mathbb{Q} -varieties and \mathbb{Q} -sheaves, and use these to construct \mathbb{Q} -Chern classes on klt spaces.

Sections 4–5 introduce the main objects of our study: sheaves with operators and (singular) Higgs sheaves on klt spaces. The extension theorem for reflexive differential forms and the existence of pull-back functors, [16, 32], allow to establish weak functoriality properties for Higgs sheaves, including variants of pulling-back for certain morphisms, as well as into and out of \mathbb{Q} -varieties. This allows to compare stability of Higgs sheaves

on different birational models. It also helps to establish a restriction theorem of Mehta-Ramanathan type, Theorem 5.22, which allows to reduce many of our problems to the surface case. In Section 5.8, we extend Simpson's correspondence between rigid representations of the fundamental group of a smooth projective variety and polarized complex variations of Hodge structures to our singular setup, thereby establishing the foundational steps of a Nonabelian Hodge Theory on klt spaces.

With these methods at hand, we establish a \mathbb{Q} -analog of the Bogomolov-Gieseker inequality in Section 6. Section 7 applies this, as well as a recent stability result of Guenancia [24], to establish the \mathbb{Q} -Miyaoka-Yau inequality, Theorem 1.1. The second main result, Theorem 1.2, is shown in Section 8.

The concluding Section 9 discusses quotients of the ball by cocompact subgroups of its automorphism group, in order to prove the characterisation of singular ball quotients given in Theorem 1.3, as well as the hyperbolicity result of Corollary 1.4. We conclude with an example of Keum, showing that many of our results are essentially sharp.

1.5. Earlier work

Generalizations of the Miyaoka-Yau inequality and uniformisation in case of equality have attracted considerable interest in the last few decades.

Inequality (*) and the uniformisation result were extended to the context of compact Kähler varieties with only quotient singularities by Cheng-Yau [7] using orbifold Kähler-Einstein metrics. Tsuji established Inequality (*) for *smooth minimal models* of general type in [62]. Enoki's result on the semistability of tangent sheaf of minimal models, [9], was used by Sugiyama [59] to establish the Bogomolov-Gieseker inequality for the tangent sheaf of any resolution of a given minimal model of general type with only canonical singularities, the polarization given by the pullback of the canonical bundle on the minimal model. By using a strategy very similar to ours, that is via results of Simpson [56], Langer in [42, Thm. 5.2] established the Miyaoka-Yau inequality in this context. He recently also gave the first purely algebraic proof of the Bogomolov Inequality for semistable Higgs sheaves (on smooth projective varieties over fields of arbitrary characteristic), see [44].

A strong uniformisation result, together with the Miyaoka-Yau inequality, was established by Kobayashi [35] in the case of open orbifold surfaces.

After the work of Tsuji, the past few years have witnessed significant developments in the theory of singular Kähler-Einstein metrics and Kähler-Ricci flow. These are evident, for example, in the works of Tian-Zhang [61], Eyssidieux-Guedj-Zeriahi [10], and Zhang [69]. In particular, Inequality (*) together with a uniformisation result for *smooth minimal models* of general type have been successfully established by Zhang [68].

Finally, we mention that the related uniformisation problem for klt varieties with vanishing first and second Chern class has been solved by the authors partly in a joint work with Steven Lu, see [17] and [46].

1.6. Acknowledgements

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2. Notation and standard facts

2.1. Global conventions

Throughout this paper, all schemes, varieties and morphisms will be defined over the complex number field. We follow the notation and conventions of Hartshorne's book [26]. In particular, varieties are always assumed to be irreducible. For all notation around Mori theory, such as klt spaces and klt pairs, we refer the reader to [41].

2.2. Varieties

In the course of the proofs, we need to switch between the Zariski– and the Euclidean topology at times. We will consistently use the following notation.

NOTATION 2.1 (Complex space associated with a variety). – Given a variety or projective scheme X, denote by X^{an} the associated complex space, equipped with the Euclidean topology. If $f : X \to Y$ is any morphism of varieties or schemes, denote the induced map of complex spaces by $f^{an} : X^{an} \to Y^{an}$. If \mathscr{F} is any coherent sheaf of \mathscr{O}_X -modules, denote the associated coherent analytic sheaf of $\mathscr{O}_{X^{an}}$ -modules by \mathscr{F}^{an} .

The notion of "Q-Chern class," which is used in the formulation of our main result, is usually defined for varieties with quotient singularities. However, the word "quotient singularity" is not consistently used in the literature and is often left undefined. We use the following terminology.

DEFINITION 2.2 (Quotient varieties and quotient singularities). – Let X be a normal, quasi-projective variety. We say that X is a quotient variety if there exists a finite group G, a smooth G-variety $\hat{X}^{(1)}$ such that $X \cong \hat{X}/G$ and such that the quotient map is étale over $X_{\text{reg.}}$. We say that X has quotient singularities, if there exists a covering of X^{an} by analytically-open sets $(U_{\alpha})_{\alpha \in A}$, and for each $\alpha \in A$ a quotient variety Y_{α} , and an analytically open set $V_{\alpha} \subseteq Y_{\alpha}^{\text{an}}$ that is biholomorphic to U_{α} .

Our main result pertains to canonical models of varieties of general type. We briefly recall the relevant definitions and facts.

⁽¹⁾ In other words, the group G acts on the variety \widehat{X} by automorphisms.

DEFINITION 2.3 (Minimal varieties). – A normal, projective variety X is called minimal if X has at worst terminal singularities and if K_X is nef.

REMINDER 2.4 (Basepoint-Free Theorem and Canonical models).

If X is a projective, klt variety of general type whose canonical divisor K_X is nef, the Basepoint-Free Theorem asserts that K_X is semiample, [41, Thm. 3.3]. A sufficiently high multiple of K_X thus defines a birational morphism $\phi : X \to Z$ to a normal projective variety with at worst klt singularities whose canonical divisor K_Z is ample, cf. [41, Lem. 2.30]. There exists a Q-linear equivalence $K_X \sim_Q \phi^* K_Z$. If X is a minimal variety of general type, Z has at worst canonical singularities, we set $Z = X_{can}$, and call it the *canonical model* of X.

DEFINITION 2.5 (Ball quotient). – A smooth projective variety X of dimension n is a ball quotient if the universal cover of X^{an} is biholomorphic to the unit ball $\mathbb{B}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 < 1\}$. Equivalently, there exists a discrete subgroup $\Gamma < \operatorname{Aut}_{\mathscr{O}}(\mathbb{B}^n)$ of the holomorphic automorphism group of \mathbb{B}^n such that the action of Γ on \mathbb{B}^n is cocompact and fixed-point free, and such that X is isomorphic to \mathbb{B}^n / Γ .

The following will be used for notational convenience.

NOTATION 2.6 (Big and small subsets). – Let X be a normal, quasi-projective variety. A closed subset $Z \subset X$ is called *small* if $\operatorname{codim}_X Z \ge 2$. An open subset $U \subseteq X$ is called *big* if $X \setminus U$ is small.

Fundamental groups are basic objects in our arguments. We will use the following notation.

DEFINITION 2.7 (Fundamental group and étale fundamental group).

If X is a complex, quasi-projective variety, we set $\pi_1(X) := \pi_1(X^{an})$, and call it the fundamental group of X. Moreover, the étale fundamental group of X will be denoted by $\hat{\pi}_1(X)$.

REMARK 2.8. – Recall that $\hat{\pi}_1(X)$ is isomorphic to the profinite completion of $\pi_1(X)$; e.g., see [47, §5 and references given there].

2.3. Morphisms

Galois morphisms appear prominently in the literature, but their precise definition is not consistent. We will use the following definition, which does not ask Galois morphisms to be étale.

DEFINITION 2.9 (Covers and covering maps, Galois morphisms).

A cover or covering map is a finite, surjective morphism $\gamma : X \to Y$ of normal, quasi-projective varieties. The covering map γ is called Galois if there exists a finite group $G \subset \operatorname{Aut}(X)$ such that γ is isomorphic to the quotient map.

NOTATION 2.10. – In the setting of Definition 2.9, we will frequently write

$$X \xrightarrow{\gamma} Y \qquad \text{or} \qquad X \xrightarrow{\gamma} Y$$

to indicate that γ is isomorphic to the quotient map. We will also write $G = \operatorname{Gal}(X/Y)$.

DEFINITION 2.11 (Quasi-étale morphisms). – A morphism $f : X \to Y$ between normal varieties is called quasi-étale if f is of relative dimension zero and étale in codimension one. In other words, f is quasi-étale if dim $X = \dim Y$ and if there exists a closed, subset $Z \subseteq X$ of codimension codim_X $Z \ge 2$ such that $f|_{X \setminus Z} : X \setminus Z \to Y$ is étale.

2.4. Sheaves

Reflexive sheaves are in many ways easier to handle than arbitrary coherent sheaves, and we will therefore frequently take reflexive hulls. The following notation will be used.

NOTATION 2.12 (Reflexive hull). – Given a quasi-projective variety X and a coherent sheaf \mathscr{E} on X, write

$$\Omega_X^{[p]} := (\Omega_X^p)^{**}, \quad \mathscr{E}^{[m]} := (\mathscr{E}^{\otimes m})^{**} \quad \text{and} \quad \det \mathscr{E} := (\Lambda^{\operatorname{rank} \mathscr{E}} \mathscr{E})^{**}.$$

Given any morphism $f: Y \to X$, write $f^{[*]} \mathscr{E} := (f^* \mathscr{E})^{**}$, etc.

One key notion in our argument is that of a *flat sheaf*.

DEFINITION 2.13 (Flat sheaf, [17, Def. 1.15]). – If X is any quasi-projective variety and \mathscr{F} is any locally free, analytic sheaf on the underlying complex space X^{an} , we call \mathscr{F} flat if it is defined by a representation of the topological fundamental group $\pi_1(X^{an})$. A locally free, algebraic sheaf \mathscr{E} on X is called flat if the associated analytic sheaf \mathscr{E}^{an} is flat.

We use [12, Chap. 3] as our main reference for Chern classes on singular spaces. The Bogomolov discriminant plays a central role.

NOTATION 2.14 (Bogomolov discriminant). – Let X be a projective variety and \mathscr{E} be a locally free sheaf on X, of rank r > 0. One defines the *Bogomolov discriminant* of \mathscr{E} as $\Delta(\mathscr{E}) := 2r \cdot c_2(\mathscr{E}) - (r-1) \cdot c_1(\mathscr{E})^2$.

2.5. G-sheaves

In the discussion of \mathbb{Q} -varieties one needs to consider varieties X that are equipped with a faithful action of a finite group G. Almost all sheaves that are relevant in our discussion come with a natural structure of a G-sheaf, also called G-linearized sheaf in the literature, [50]. A detailed discussion of G-sheaves, including full proofs of all relevant facts used here, is found in [50, § 1.3], [63, § 3.2], and [16, Appendix A].

NOTATION 2.15 (*G*-invariant push-forward). – Let X be a quasi-projective variety, equipped with a faithful action of a finite group G, and with associated quotient map $\pi : X \to X/G$. If \mathscr{E} is any G-sheaf on X, write $\pi_*(\mathscr{E})^G \subseteq \pi_*(\mathscr{E})$ to denote the G-invariant part of the push-forward.

If X has a G-action and \mathscr{E} is a G-sheaf, it is generally not true that any G-subsheaf $\mathscr{F} \subseteq \mathscr{E}$ comes from the quotient. The following standard proposition gives a criterion when this *is* true. The preprint version, available from the arXiv, contains a full proof.

PROPOSITION 2.16 (*G*-sheaves coming from the quotient). – Let $\gamma : Y \to X$ be a Galois morphism with group *G*. Let \mathcal{B}_X be a reflexive sheaf on *X* and $\mathcal{B} := \gamma^{[*]}\mathcal{B}_X$. Observe that \mathcal{B} naturally carries the structure of a *G*-sheaf. If $\mathcal{A} \subseteq \mathcal{B}$ is any saturated *G*-subsheaf, then there exists a reflexive, saturated subsheaf $\mathcal{A}_X \subseteq \mathcal{B}_X$ such that $\mathcal{A} = \gamma^{[*]}\mathcal{A}_X$.

2.6. Intersection, slope and stability

Given a normal, quasi-projective variety X, we follow the notation of [12] and denote by $A_k(X)$ the groups of k-dimensional cycles modulo rational equivalence. The symbol $N^1(X)_{\mathbb{Q}}$ denotes the \mathbb{Q} -vector space of numerical Cartier divisor classes. Given any divisor D or any sheaf \mathscr{E} whose determinant is \mathbb{Q} -Cartier, we write the appropriate elements of $N^1(X)_{\mathbb{Q}}$ as [D] and $[\mathscr{E}] := [\det \mathscr{E}]$, respectively.

We recall the following standard construction of intersection numbers between Weil- and Cartier divisors.

CONSTRUCTION 2.17 (Intersection of Weil and Cartier divisors).

Let X be an *n*-dimensional, normal, projective variety and $0 \neq \mathscr{E}$ be any coherent sheaf. Its determinant is then a Weil divisorial sheaf on X, say det $\mathscr{E} = \mathscr{O}_X(D)$. The Weil divisor D defines an element $\Delta \in A_{n-1}(X)$. Given (n-1) line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_{n-1}$, we can then form the cap product and consider the number

$$\deg(\Delta \cap c_1(\mathscr{L}_1) \cap \cdots \cap c_1(\mathscr{L}_{n-1})) \in \mathbb{Z}.$$

Since its value depends only on the numerical classes of the line bundles \mathscr{L}_i , the sheaf \mathscr{E} induces a well-defined \mathbb{Q} -multilinear form $N^1(X)_{\mathbb{Q}}^{\times (n-1)} \to \mathbb{Q}$.

NOTATION 2.18. – Abusing notation somewhat, we denote the multilinear form of Construction 2.17 by $[\mathscr{E}]$, as if the sheaf \mathscr{E} had a numerical class. Given elements $\alpha_1, \ldots, \alpha_{n-1} \in N^1(X)_{\mathbb{O}}$, we denote the associated value by $[\mathscr{E}] \cdot \alpha_1 \cdots \alpha_{n-1} \in \mathbb{Q}$.

The abuse of notation is partially justified by the following remark.

REMARK 2.19. – In the setting of Construction 2.17, if $\pi : \tilde{X} \to X$ is any resolution of singularities, then $[\mathscr{E}] \cdot \alpha_1 \cdots \alpha_{n-1} = [\pi^* \mathscr{E}] \cdot \pi^* \alpha_1 \cdots \pi^* \alpha_{n-1}$. If det \mathscr{E} is Q-Cartier, then there is a numerical class $[\mathscr{E}] \in N^1(X)_{\mathbb{Q}}$, and Construction 2.17 gives the expected results.

DEFINITION 2.20 (Slope with respect to a nef divisor). – Let X be a normal, projective variety and H be a nef \mathbb{Q} -Cartier divisor on X. If $\mathscr{E} \neq 0$ is any torsion free, coherent sheaf on X, define the slope of \mathscr{E} with respect to H as

$$\mu_H(\mathscr{E}) := \frac{[\mathscr{E}] \cdot [H]^{\dim X - 1}}{\operatorname{rank} \mathscr{E}}.$$

Call \mathscr{E} semistable with respect to *H* if $\mu_H(\mathscr{F}) \leq \mu_H(\mathscr{E})$ for any subsheaf $\mathscr{F} \subseteq \mathscr{E}$ with $0 < \operatorname{rank} \mathscr{F} < \operatorname{rank} \mathscr{E}$. *Call* \mathscr{E} stable with respect to *H* if strict inequality always holds.

In the setup of Definition 2.20, the class $[H]^{\dim X-1}$ is a movable numerical curve class, cf. [18, Def. 2.2]. If X is Q-factorial, our definition of slope agrees with that of [18, Def. 2.10]. The standard proofs of the following elementary facts carry over from [18] essentially verbatim.

LEMMA 2.21 (Elementary properties of slope). – In the setup of Definition 2.20, if $\pi : \tilde{X} \to X$ is any generically finite morphism of normal, projective varieties, then the following holds.

- (2.21.1) We have $\mu_H(\mathscr{E}) = (\deg \pi)^{-1} \cdot \mu_{\pi^*H}(\pi^{[*]}\mathscr{E}).$
- (2.21.2) If $\tilde{\mathscr{E}}$ is any coherent sheaf on \tilde{X} that differs from $\pi^{[*]}\mathscr{E}$ at most over a small subset of X, then $\mu_H(\mathscr{E}) = (\deg \pi)^{-1} \cdot \mu_{\pi^*H}(\tilde{\mathscr{E}}).$

LEMMA 2.22 (Harder-Narasimhan filtration). – In the setup of Definition 2.20, there exists a unique filtration, $0 = \mathscr{E}_0 \subsetneq \mathscr{E}_1 \subsetneq \cdots \subsetneq \mathscr{E}_{\ell} = \mathscr{E}$, whose quotients $\mathscr{E}^i := \mathscr{E}_i / \mathscr{E}_{i-1}$ are torsion free, semistable with respect to H and where the sequence $\mu_H(\mathscr{E}^i)$ is strictly decreasing.

NOTATION 2.23 (Harder-Narasimhan filtration). – The filtration of Lemma 2.22 is called *Harder-Narasimhan filtration*. Call \mathscr{E}_1 the maximal *H*-destabilizing subsheaf of \mathscr{E} and write $\mu_H^{\max}(\mathscr{E}) := \mu_H(\mathscr{E}^1)$ and $\mu_H^{\min}(\mathscr{E}) := \mu_H(\mathscr{E}^{\ell-1})$.

PART I

FOUNDATIONAL MATERIAL

3. Q-varieties and Q-Chern classes

3.1. Q-varieties

The construction of the \mathbb{Q} -Chern classes that are used to formulate our main results relies on the notions of \mathbb{Q} -variety, also known as V-manifolds in the literature. While \mathbb{Q} -Chern classes on \mathbb{Q} -surfaces have been discussed in the literature at length, the (sometimes delicate) issues arising in higher dimensions are often not well covered. For the reader's convenience, this section gathers the main definitions, results and constructions concerning \mathbb{Q} -varieties that are used in our paper. For the sake of brevity, many of the more standard proofs are left to the reader. The preprint version of the paper, available from the arXiv, contains full proofs and explains all constructions in detail.

3.2. Q-varieties

We recall the definition of \mathbb{Q} -varieties, as given in the fundamental articles of Mumford and Gillet, [49, 14].

Note that there are other definitions in the literature ⁽²⁾. To avoid confusion, we restrict ourselves to Mumford's paper as a main reference and stick to the notation introduced there.

⁽²⁾ The definition found in Megyesi's well-known article [39, Sect. 10] is more restrictive than Definition 3.1 in that it requires the morphisms $X_{\alpha} \rightarrow U_{\alpha}$ to be quasi-étale.

DEFINITION 3.1 (Q-variety, cf. [49, Sect. 2] and [14, Def. 9.1]). -A Q-variety is a tuple consisting of a normal, quasi-projective variety X, a finite set A and for each $\alpha \in A$ a smooth, quasi-projective variety X_{α} and a diagram of morphisms between quasi-projective varieties

(3.1.1)
$$X_{\alpha} \xrightarrow{p_{\alpha}} U_{\alpha} \xrightarrow{p_{\alpha}', \text{ étale}} X$$

such that $X = \bigcup p_{\alpha}(X_{\alpha})$ and such that the following compatibility condition holds: for each $(\alpha, \beta) \in A \times A$, denoting by $X_{\alpha\beta}$ the normalization of $X_{\alpha} \times_X X_{\beta}$, then the natural morphisms

 $p_{\alpha\beta,\alpha}: X_{\alpha\beta} \to X_{\alpha} \quad and \quad p_{\alpha\beta,\beta}: X_{\alpha\beta} \to X_{\beta}$

are étale. In particular, $X_{\alpha\beta}$ is smooth. For brevity of notation, we refer to the Q-variety by $(X, \{p_{\alpha}\}_{\alpha \in A})$. We refer to the diagrams (3.1.1) as charts.

We refer to [49, Sect. 2.b] for the definition of a morphism of Q-varieties as well as for an explanation of the relation to general Deligne-Mumford stacks.

3.3. Quasi-étale Q-varieties

This paper is mainly concerned with \mathbb{Q} -varieties whose charts are quasi-étale. As we will see below, these have particularly good properties.

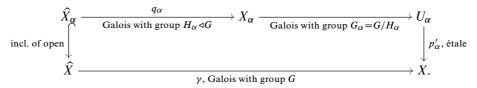
DEFINITION 3.2. – $A \mathbb{Q}$ -variety $(X, \{p_{\alpha}\}_{\alpha \in A})$ is called quasi-étale if all the morphisms p_{α} are quasi-étale.

REMARK 3.3 (Quasi-étale charts). – Let X be a normal, quasi-projective variety, let A be a finite set and for each $\alpha \in A$, assume we are given a diagram as in (3.1.1), such that $X = \bigcup p_{\alpha}(X_{\alpha})$. If all morphisms p_{α} are quasi-étale, then the morphisms $X_{\alpha\beta} \to X_{\alpha}$ are then likewise étale in codimension one and hence, by purity of the branch locus, étale, [67, Prop. 2] or [51, Thm. 1]. The condition on the $p_{\alpha\beta,\alpha}$ is therefore vacuous, and the p_{α} equip X with the structure of a Q-variety.

LEMMA 3.4 (Uniqueness of quasi-étale \mathbb{Q} -variety structures). – Let X be any normal, quasi-projective variety. Then, any two quasi-étale \mathbb{Q} -variety structures on X have a common refinement.

3.4. Global covers

Given an *n*-dimensional \mathbb{Q} -variety $X_{\mathbb{Q}} := (X, \{p_{\alpha}\}_{\alpha \in A})$ as in Definition 3.1, Mumford constructs in [49, Sect. 2] a *global cover of* $X_{\mathbb{Q}}$, that is, a normal variety \hat{X} (not necessarily smooth), a global Galois morphism $\gamma : \hat{X} \to X$, and for every $\alpha \in A$ a commutative diagram as follows,



We call \hat{X} a global cover of $X_{\mathbb{Q}}$.

OBSERVATION 3.5 (The importance of being Cohen-Macaulay, I).

If \hat{X} is Cohen-Macaulay, then the Galois morphisms q_{α} will automatically be flat, [8, Ex. 18.17]. In particular, pull-back of coherent sheaves is an exact functor. Recalling that a coherent sheaf \mathscr{F} is reflexive if and only if it is locally a 2^{nd} syzygy sheaf, [27, Prop. 1.1], it follows that for any $\alpha \in A$, the pull-back of any reflexive sheaf on X_{α} to \hat{X}_{α} is again reflexive.

3.5. Q-sheaves

The next relevant items are the definition of \mathbb{Q} -sheaves and the construction of \mathbb{Q} -sheaves by reflexive pull-back.

DEFINITION 3.6 (Q-sheaf and Q-bundle, cf. [49, § 2]). – Let $X_{\mathbb{Q}} := (X, \{q_{\alpha}\}_{\alpha \in A})$ be a Q-variety, as in Definition 3.1, a Q-sheaf \mathscr{F} on $X_{\mathbb{Q}}$ is a tuple

 $(\{\mathscr{F}_{\alpha}\}_{\alpha\in A},\{i_{\alpha\beta}\}_{(\alpha,\beta)\in A\times A})$

consisting of a family of coherent sheaves \mathscr{F}_{α} on X_{α} plus isomorphisms

$$i_{\alpha\beta}: p^*_{\alpha\beta,\alpha}(\mathscr{F}_{\alpha}) \to p^*_{\alpha\beta,\beta}(\mathscr{F}_{\beta})$$

that are compatible on the triple overlaps. The \mathbb{Q} -sheaf \mathscr{F} is called reflexive if all the \mathscr{F}_{α} are reflexive. It is called \mathbb{Q} -bundle if all the \mathscr{F}_{α} are locally free.

REMARK 3.7 (Induced sheaves on global covers). – In the setting of Definition 3.6, given a global cover as in Section 3.4, Mumford shows that the pull-back sheaves $q_{\alpha}^* \mathscr{F}_{\alpha}$ glue to give a coherent *G*-sheaf $\widehat{\mathscr{F}}$ on \widehat{X} , [49, Sect. 2]. If we assume in addition that \mathscr{F} is reflexive, then the \mathscr{F}_{α} are locally free in codimension two, [27, Cor. 1.4], and the same holds for $\widehat{\mathscr{F}}$. In particular, if \mathscr{F} is reflexive and dim X = 2, then $\widehat{\mathscr{F}}$ is locally free.

With this construction, Mumford proves that to give a \mathbb{Q} -sheaf on $X_{\mathbb{Q}}$, it is equivalent to give a *G*-sheaf on \hat{X} whose restrictions to \hat{X}_{α} are isomorphic (as H_{α} -sheaves) to pull-back sheaves from X_{α} .

CONSTRUCTION 3.8 (Reflexive pull-back). – Given a Q-variety $X_{\mathbb{Q}} := (X, \{p_{\alpha}\}_{\alpha \in A})$ and given any coherent sheaf \mathscr{F} on X, one defines a reflexive Q-sheaf $\mathscr{F}^{[\mathbb{Q}]}$ on $X_{\mathbb{Q}}$ by setting $\mathscr{F}_{\alpha} := p_{\alpha}^{[*]} \mathscr{F}$ —the existence of natural isomorphisms $i_{\alpha\beta}$ is guaranteed by étalité of $p_{\alpha\beta,\alpha}$ and $p_{\alpha\beta,\beta}$. The Q-sheaves $(\Omega_X^{[p]})^{[\mathbb{Q}]}$ and $(\mathscr{T}_X)^{[\mathbb{Q}]}$ are Q-bundles.

REMARK 3.9 (The importance of being Cohen-Macaulay, II). – In the setting of Construction 3.8, let \hat{X} be a global cover as in Section 3.4, and let $\hat{\mathscr{F}}$ be the sheaf induced by the \mathbb{Q} -sheaf $\mathscr{F}^{[\mathbb{Q}]}$, as in Remark 3.7. If \hat{X} is Cohen-Macaulay, it follows from Observation 3.5 that $\hat{\mathscr{F}} = \gamma^{[*]} \mathscr{F} = \gamma^{[*]} (\mathscr{F} / \text{tor})$. In particular, we obtain that these sheaves are locally free in codimension two. In a similar vein, observing that the two reflexive sheaves $\hat{\mathscr{F}}^*$ and $\gamma^{[*]} (\mathscr{F}^*) = \gamma^{[*]} ((\mathscr{F} / \text{tor})^*)$ agree over the big open set where the torsion free sheaf \mathscr{F} / tor is locally free, we see that $\hat{\mathscr{F}}^*$ and $\gamma^{[*]} (\mathscr{F}^*)$ do in fact agree.

3.6. Constructions

We recall three folklore constructions of Q-variety structures.

3.6.1. Varieties with quotient singularities. – Given any \mathbb{Q} -variety $(X, \{p_{\alpha}\}_{\alpha \in A})$ as in Definition 3.1, then X clearly has quotient singularities, in the sense of Definition 2.2. We briefly recall the fundamental fact that the converse is also true, cf. [49, p. 276].

PROPOSITION 3.10 (Varieties with quotient singularities admit Q-structures).

Let X by any quasi-projective variety with quotient singularities. Then, X admits the structure of a quasi-étale \mathbb{Q} -variety.

3.6.2. Cutting down. – If X is a quasi-projective variety that has been equipped with the structure of a \mathbb{Q} -variety, there is generally no natural \mathbb{Q} -variety structure on an arbitrary hypersurfaces or subvarieties of X, cf. [39, Warnings on p. 116]. We remark that this is different for general elements of basepoint free linear systems.

PROPOSITION 3.11 (Q-variety structures on general hyperplanes).

Let $X_{\mathbb{Q}} := (X, \{p_{\alpha}\}_{\alpha \in A}))$ be a \mathbb{Q} -variety, let \mathscr{L} be a line bundle on X and $V \subseteq |\mathscr{L}|$ a finitedimensional, basepoint free linear system whose generals element are irreducible. Then, there exists a dense, Zariski-open subset $V^{\circ} \subseteq V$ such any hypersurface H is irreducible and normal, and admits a structure $H_{\mathbb{Q}}$ of a \mathbb{Q} -variety, together with a morphism $\iota_{\mathbb{Q}} : H_{\mathbb{Q}} \to X_{\mathbb{Q}}$ whose induced morphism $\iota : H \to X$ is the inclusion. Under the following additional assumptions, more is true.

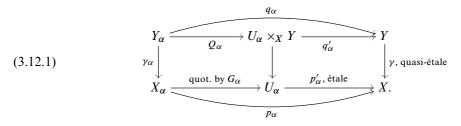
- (3.11.1) If \mathscr{E} is any reflexive sheaf on X and if $H \in V^{\circ}$ is general, then $\mathscr{E}_{|_{H}}$ is likewise reflexive, and $\iota_{\mathbb{Q}}^{*}(\mathscr{E}^{[\mathbb{Q}]}) \cong (\mathscr{E}_{|_{H}})^{[\mathbb{Q}]}$.
- (3.11.2) If $X_{\mathbb{Q}}$ is quasi-étale and $H \in V^{\circ}$ is general, then $H_{\mathbb{Q}}$ is quasi-étale.
- (3.11.3) If $X_{\mathbb{Q}}$ admits a global, Cohen-Macaulay cover and $H \in V^{\circ}$ is general, then $H_{\mathbb{Q}}$ admits a global, Cohen-Macaulay cover.

3.6.3. *Quasi-étale coverings.* – There is generally no notion of "pull-back" for \mathbb{Q} -variety structures, even for finite morphisms. If the morphism is quasi-étale, a pull-back structure does exist, however.

PROPOSITION 3.12 (Q-variety structures on quasi-étale coverings).

Let $X_{\mathbb{Q}}$ be a quasi-étale \mathbb{Q} -variety, and $\gamma : Y \to X$ a finite, quasi-étale cover. Then, Y admits a structure $Y_{\mathbb{Q}}$ of a quasi-étale \mathbb{Q} -variety, together with a morphism $\gamma_{\mathbb{Q}} : Y_{\mathbb{Q}} \to X_{\mathbb{Q}}$ whose induced morphism $Y \to X$ is γ . If \mathscr{E} is any reflexive sheaf on X, then $\gamma_{\mathbb{Q}}^{\infty}(\mathscr{E}^{[\mathbb{Q}]}) \cong (\gamma^{[*]}\mathscr{E})^{[\mathbb{Q}]}$.

Proof. – Write $X_{\mathbb{Q}} := (X, \{p_{\alpha}\}_{\alpha \in A})$ and, given any $\alpha \in A$, let Y_{α} be the normalization of $X_{\alpha} \times_X Y$. Base change gives a finite set of diagrams as follows,



It follows from stability of étalité under base change that q'_{α} is étale and that q_{α} is quasi-étale. In particular, $U_{\alpha} \times_X Y$ is normal. This in turn implies that Q_{α} is the quotient map for the natural G_{α} -action on Y_{α} . Lastly, note that the map γ_{α} is étale away from

 $\gamma_{\alpha}^{-1}(\text{Ramification } p_{\alpha}) \cup q_{\alpha}^{-1}(\text{Ramification } \gamma),$

which is a small subset of Y_{α} . Purity of the branch locus then implies that γ_{α} is étale and, in particular, that Y_{α} is smooth. Using Remark 3.3, we see that the top rows of the diagrams (3.12.1), restricted to the irreducible components of Y_{α} , equip Y with a structure $Y_{\mathbb{Q}}$ of a quasi-étale \mathbb{Q} -variety. The restrictions of the full Diagrams (3.12.1) to the irreducible components of Y_{α} define a morphism $\gamma_{\mathbb{Q}}: Y_{\mathbb{Q}} \to X_{\mathbb{Q}}$ whose induced morphism $Y \to X$ is γ .

It remains to consider the \mathbb{Q} -sheaves attached to a reflexive sheaf \mathscr{E} on X. To this end, observe that the \mathbb{Q} -sheaf $\gamma^*_{\mathbb{Q}}(\mathscr{E}^{[\mathbb{Q}]})$ is given at the level of the Y_{α} by the sheaves $\gamma^*_{\alpha}((\mathscr{E}^{[\mathbb{Q}]})_{\alpha})$, which are reflexive because the γ_{α} are étale. But we have canonical isomorphisms,

$$\begin{split} \gamma_{\alpha}^{*} \big((\mathscr{E}^{[\mathbb{Q}]})_{\alpha} \big) &\cong \gamma_{\alpha}^{*} p_{\alpha}^{[*]} \mathscr{E} & \text{definition of } \mathscr{E}^{[\mathbb{Q}]} \\ &\cong q_{\alpha}^{[*]} \gamma^{[*]} \mathscr{E} & \text{sheaves agree over big set where } \mathscr{E} \text{ is locally free,} \end{split}$$

which give the desired statement.

3.7. Q-Chern classes on klt spaces

It is well understood that the base variety X of any klt surface pair (X, D) has quotient singularities. The geometry of X can then be studied using generalized Chern classes, known as Q-Chern classes or orbifold Chern classes—we refer to Kawamata's proof [31] of the abundance conjecture in dimension three for an example. In higher dimensions, the base variety of a klt pair does not necessarily have quotient singularities. However, once one removes a suitable subset $Z \subseteq X$ of codimension three, only quotient singularities remain and $X \setminus Z$ can be equipped with the structure of a Q-variety that admits a global, Cohen-Macaulay cover. In particular, following Mumford's fundamental paper [49], Chern classes can be defined. Since codim Z = 3, this allows to construct on any klt space useful intersection products with first and second Q-Chern classes of any reflexive sheaf on X. This applies in particular to the tangent sheaf \mathscr{T}_X . The (standard) proofs are omitted, but spelled out in the arXiv version of this paper.

THEOREM 3.13 (Q-Chern classes on klt spaces). – There exist a map that assigns to any projective, klt pair (X, D) of dimension $n \ge 2$ and any reflexive sheaf \mathscr{E} on X three symmetric, Q-multilinear forms, denoted as follows,

 $\begin{aligned} \hat{c}_1(\mathscr{E}) &: N^1(X)_{\mathbb{Q}}^{\times (n-1)} \to \mathbb{Q}, \\ \hat{c}_1(\mathscr{E})^2 &: N^1(X)_{\mathbb{Q}}^{\times (n-2)} \to \mathbb{Q}, \\ \hat{c}_2(\mathscr{E}) &: N^1(X)_{\mathbb{Q}}^{\times (n-2)} \to \mathbb{Q}, \end{aligned} \qquad (\alpha_1, \dots, \alpha_{n-2}) \mapsto \hat{c}_1(\mathscr{E})^2 \cdot \alpha_1 \cdots \alpha_{n-2} \\ (\alpha_1, \dots, \alpha_{n-2}) \mapsto \hat{c}_2(\mathscr{E}) \cdot \alpha_1 \cdots \alpha_{n-2} \end{aligned}$

such that the following properties hold for all X, all reflexive \mathscr{E} on X, and all $\alpha_1, \ldots, \alpha_{n-1} \in N^1(X)_{\mathbb{Q}}$.

(3.13.1) If n = 2, then X has quotient singularities, and $\hat{c}_1(\mathscr{E}) \in N^1(X)^*_{\mathbb{Q}}$ as well as $\hat{c}_1(\mathscr{E})^2$, $\hat{c}_2(\mathscr{E}) \in \mathbb{Q}$ are the classical \mathbb{Q} -Chern classes⁽³⁾ discussed in the literature, cf. [39,

⁽³⁾ The "multilinear forms" $\hat{c}_1(\mathscr{E})^2$ and $\hat{c}_2(\mathscr{E})$ take no arguments and are therefore identified with rational numbers.

Chapt. 10]. In particular, there exists a Galois cover $\gamma : \hat{X} \to X$ (not necessarily quasi-étale), where $\gamma^{[*]} \mathscr{E}$ is locally free for any reflexive sheaf \mathscr{E} on X, and where the following equalities hold, for all \mathscr{E} and for all numerical classes $\alpha_1 \in N^1(X)_{\mathbb{Q}}$,

$$\widehat{c}_1(\mathscr{E}) \cdot \alpha_1 = (\deg \gamma)^{-1} \cdot c_1(\gamma^{[*]} \mathscr{E}) \cdot \gamma^* \alpha_1.$$

Ditto for $\hat{c}_1(\mathscr{E})^2$ and $\hat{c}_2(\mathscr{E})$. The cover γ is a global, Cohen-Macaulay cover for a suitable, quasi-étale \mathbb{Q} -variety structure on X.

(3.13.2) If n > 2, if $\mathscr{L} \in \operatorname{Pic}(X)$ is a line bundle and $V \subseteq |\mathscr{L}|$ is a basepoint free linear system whose elements are all connected, then there exists a dense open subset $V^{\circ} \subseteq V$ such that for all $H \in V^{\circ}$, the hypersurface H is irreducible, not contained in supp D, the pair $(H, D|_{H})$ is klt, the restriction $\mathscr{E}|_{H}$ is reflexive, and

$$\hat{c}_1(\mathscr{E}) \cdot [\mathscr{L}] \cdot \alpha_2 \cdots \alpha_{n-1} = \hat{c}_1(\mathscr{E}|_{u}) \cdot \alpha_2 \cdots \alpha_{n-1}.$$

Ditto for $\hat{c}_1(\mathscr{E})^2$ and $\hat{c}_2(\mathscr{E})$.

The construction of \hat{c}_1 is compatible with classical definitions in case the determinant of \mathscr{E} is Q-Cartier. If \mathscr{E} is locally free, then the forms $\hat{c}_1(\mathscr{E})$, $\hat{c}_1(\mathscr{E})^2$ and $\hat{c}_2(\mathscr{E})$ equal the usual product with the Chern classes of \mathscr{E} .

REMARK 3.14. – Since every line bundle on X is the difference of two very ample ones, it follows from multilinearity that the forms are uniquely determined by Items (3.13.1), (3.13.2). As for the converse, it might seem tempting take these items as a definition of the forms, in order to avoid the Mumford's constructions. But then well-definedness needs to be shown, which will in essence lead to the same set of problems.

The following definition and notation will be used in most of the applications.

DEFINITION 3.15 (Q-Bogomolov discriminant). – Let (X, D) be a projective klt pair and \mathscr{E} be a reflexive sheaf on X of rank r > 0. One defines the Q-Bogomolov discriminant of \mathscr{E} as the multilinear form

$$\widehat{\Delta}(\mathscr{E}) := 2r \cdot \widehat{c}_2(\mathscr{E}) - (r-1) \cdot \widehat{c}_1(\mathscr{E})^2.$$

We end this section with a number of remarks and immediate corollaries that we will later use.

3.8. Calculus of Q-Chern classes

The following results help to compute \mathbb{Q} -Chern classes in practice. They follow fairly quickly from Mumford's construction and from basic properties of \mathbb{Q} -varieties.

LEMMA 3.16 (Behavior under quasi-étale covers). – If (X, D) is a projective klt pair of dimension $n \ge 2$ and if $\gamma : Y \to X$ is quasi-étale between projective varieties, then $(Y, \gamma^* D)$ is again klt, and the following equalities hold for all reflexive sheaves \mathscr{E} and all numerical classes $\alpha_1, \ldots, \alpha_{n-1} \in N^1(X)_{\mathbb{Q}}$,

$$\hat{c}_1(\gamma^{[*]}\mathscr{E}) \cdot (\gamma^* \alpha_1) \cdots (\gamma^* \alpha_{n-1}) = (\deg \gamma) \cdot \hat{c}_1(\mathscr{E}) \cdot \alpha_1 \cdots \alpha_{n-1}.$$

Ditto for $\hat{c}_1(\gamma^{[*]}\mathscr{E})^2$ and $\hat{c}_2(\gamma^{[*]}\mathscr{E})$.

LEMMA 3.17 (Q-Chern classes of flat sheaves). – If (X, D) is a projective klt pair of dimension $n \ge 2$ and \mathscr{E} a reflexive sheaf on X whose restriction to X_{reg} is locally free and flat, then the forms $\hat{c}_1(\mathscr{E})$, $\hat{c}_1(\mathscr{E})^2$ and $\hat{c}_2(\mathscr{E})$ are all zero.

3.9. Q-varieties as Deligne-Mumford stacks

In the language of stacks, Mumford's constructions of \mathbb{Q} -varieties recalled in this section reflect the existence of an algebraic Deligne-Mumford (DM) stack structure for algebraic varieties with only quotient singularities, cf. [14, Prop. 9.2]. More precisely, given a \mathbb{Q} -variety / quasi-étale \mathbb{Q} -variety ($X, \{p_{\alpha}\}_{\alpha \in A}$), there is an algebraic DM stack χ verifying the following properties.

(3.17.1) The stack χ is smooth and X is its coarse moduli space.

(3.17.2) The isotropy group at the generic point / every codimension-one point of χ is trivial.

In [14], Gillet developed an intersection theory for DM stacks and their coarse moduli spaces and showed that for \mathbb{Q} -varieties the theory is the same as Mumford's theory recalled above. Consequently, one could have used Gillet's results instead of Mumford's in our discussion of \mathbb{Q} -Chern classes for reflexive sheaves on klt varieties. By using Mumford's approach, we followed a tradition in higher-dimensional classification theory, e.g., see [39, 53].

4. Sheaves with operators

4.1. Definitions and elementary operations

In order to define and discuss Higgs sheaves on singular spaces in Section 5, this preliminary section discusses sheaves with operators. Our main emphasis lies on stability properties. Because of the singularities we cannot assume that any of the sheaves in question is locally free. We need to resort to the following, rather general definition. We also need to discuss the case of G-sheaves, but restrict ourselves to the minimal amount of material required to make our arguments work.

DEFINITION 4.1 (Sheaf with an operator). – Let X be a normal, quasi-projective variety and \mathcal{W} be a coherent sheaf of \mathcal{O}_X -modules. A sheaf with a \mathcal{W} -valued operator is a pair (\mathcal{E}, θ) where \mathcal{E} is a coherent sheaf and $\theta : \mathcal{E} \to \mathcal{E} \otimes \mathcal{W}$ is an \mathcal{O}_X -linear sheaf morphism.

DEFINITION 4.2 (*G*-sheaf with an invariant operator). – Let X be a normal, quasiprojective variety, equipped with the action of a finite group G, and let \mathcal{W} be a coherent G-sheaf of \mathcal{O}_X -modules. A G-sheaf with an invariant \mathcal{W} -valued operator is a sheaf with a \mathcal{W} -valued operator, (\mathcal{E}, θ) , where \mathcal{E} is a coherent G-sheaf and θ is a morphism of G-sheaves.

WARNING 4.3 (Incompatible definitions in the literature). – The literature contains no uniform definition of sheaves with operators. Our definition agrees with that of [43, p. 257] but differs from [42, Def. 1.1]. All definitions that we have seen agree if \mathscr{E} is torsion free and \mathscr{W} is locally free. We will be careful to quote the literature, in particular [42], only in settings where these conditions hold.

CONSTRUCTION 4.4 (Direct sum and tensor product). – Let X be a normal, quasiprojective variety and $(\mathscr{E}_1, \theta_1)$, $(\mathscr{E}_2, \theta_2)$ two sheaves with a \mathscr{W} -valued operator, as in Definition 4.1. Then, $(\mathscr{E}_1 \otimes \mathscr{E}_2, \theta_1 \otimes \operatorname{Id}_{\mathscr{E}_2} + \operatorname{Id}_{\mathscr{E}_1} \otimes \theta_2)$ and $(\mathscr{E}_1 \oplus \mathscr{E}_2, \theta_1 \oplus \theta_2)$ are again sheaves with a \mathscr{W} -valued operator

CONSTRUCTION 4.5 (Duals and endomorphisms). – Let X be a normal, quasi-projective variety and (\mathscr{E}, θ) a sheaf with a \mathscr{W} -valued operator, as in Definition 4.1. Assume that \mathscr{E} is locally free. The operator θ can then be seen as a section in the sheaf $(\mathcal{E}nd \mathscr{E}) \otimes \mathscr{W}$. Using the canonical identification $\mathcal{E}nd \mathscr{E} \cong \mathcal{E}nd(\mathscr{E}^*)$, we obtain an operator on the dual sheaf, $\theta^* : \mathscr{E}^* \to \mathscr{E}^* \otimes \mathscr{W}$.

NOTATION 4.6 (Elementary operations). – We denote the sheaves of Construction 4.4 briefly as $(\mathscr{E}_1, \theta_2) \oplus (\mathscr{E}_1, \theta_2)$ and $(\mathscr{E}_1, \theta_2) \otimes (\mathscr{E}_1, \theta_2)$. If \mathscr{L} is any coherent sheaf of \mathscr{O}_X -modules, taking zero-morphism gives sheaf $(\mathscr{L}, 0)$ with a \mathscr{W} -valued operator. We will briefly write $(\mathscr{E}_1, \theta_2) \otimes \mathscr{L}$ instead of $(\mathscr{E}_1, \theta_2) \otimes (\mathscr{L}, 0)$. In the setting of Construction 4.5, write $(\mathscr{E}, \theta)^* = (\mathscr{E}^*, \theta^*)$ and $\mathscr{E}nd(\mathscr{E}, \theta) = (\mathscr{E}, \theta)^* \otimes (\mathscr{E}, \theta)$.

CONSTRUCTION 4.7 (Pull-back and restriction). – Let X be a normal, quasi-projective variety and (\mathscr{E}, θ) a sheaf with a \mathscr{W} -valued operator. If $f : Y \to X$ is morphism of normal varieties, then $f^*\theta : f^*\mathscr{E} \to f^*\mathscr{E} \otimes f^*\mathscr{W}$ equips $f^*\mathscr{E}$ with the structure of a sheaf with an $f^*\mathscr{W}$ -valued operator, which we denote as $f^*(\mathscr{E}, \theta) = (f^*\mathscr{E}, f^*\theta)$. If f is a closed immersion, we will also write $(\mathscr{E}, \theta)|_{V} = (\mathscr{E}|_{V}, \theta|_{V})$.

4.2. Invariant subsheaves

Much of the classical literature discusses sheaves (\mathscr{E}, θ) with \mathscr{W} -valued operators only in settings where both \mathscr{E} and \mathscr{W} are locally free. Stability of (\mathscr{E}, θ) is then measured by looking at θ -invariant subsheaves of \mathscr{E} , that is, subsheaves $\mathscr{F} \subseteq \mathscr{E}$ where $\theta(\mathscr{F}) \subseteq \mathscr{F} \otimes \mathscr{W}$. If \mathscr{E} and \mathscr{W} are arbitrary, the tensor product $\mathscr{F} \otimes \mathscr{W}$ is not necessarily a subsheaf of $\mathscr{E} \otimes \mathscr{W}$ and the question whether $\theta(\mathscr{F})$ is contained in $\mathscr{F} \otimes \mathscr{W}$ no longer makes sense. In order to obtain a workable theory with good universal properties and a meaningful restriction theorem, the following more delicate definition needs to be used.

DEFINITION 4.8 (Invariant subsheaf). – Let X be a normal, quasi-projective variety and (\mathcal{E}, θ) a sheaf with a \mathcal{W} -valued operator, as in Definition 4.1. A coherent subsheaf $\mathcal{F} \subseteq \mathcal{E}$ is called θ -invariant if $\theta(\mathcal{F})$ is contained in the image of the natural map $\mathcal{F} \otimes \mathcal{W} \to \mathcal{E} \otimes \mathcal{W}$. Call \mathcal{F} generically invariant if the restriction $\mathcal{F}|_U$ is invariant with respect to $\theta|_U$, where $U \subseteq X$ is the maximal, dense, open subset where \mathcal{W} is locally free.

WARNING 4.9 (No operator on invariant subsheaves). – In the setting of Definition 4.8, if $\mathscr{F} \subseteq \mathscr{E}$ is θ -invariant and \mathscr{W} is locally free, then $\mathscr{F} \otimes \mathscr{W} \to \mathscr{E} \otimes \mathscr{W}$ is injective, the restricted map $\theta|_{\mathscr{F}}$ factors via $\mathscr{F} \otimes \mathscr{W}$ and therefore endows \mathscr{F} with the structure of a sheaf with a \mathscr{W} -valued operator. If \mathscr{W} is not locally free, then θ does in general not induce a natural \mathscr{W} -valued operator on \mathscr{F} . We refrain from discussing Harder-Narasimhan filtrations of sheaves with operators and do not attempt to define morphisms, or to construct an Abelian category. REMARK 4.10 (Invariance and tensor product). – In the setting of Construction 4.4, let $\mathscr{F} \subseteq \mathscr{E}$ be any given subsheaf. If \mathscr{L} is invertible, then $\mathscr{F} \subseteq \mathscr{E}$ is θ -invariant (resp. generically θ -invariant) if and only if $\mathscr{F} \otimes \mathscr{L} \subseteq \mathscr{E} \otimes \mathscr{L}$ is.

We end the present subsection with two lemmas, pointing out that invariance is wellbehaved with respect to saturation.

LEMMA 4.11 (Saturation of invariant subsheaf if *W* is locally free).

In the setting of Definition 4.8, assume that \mathscr{E} is torsion free and that \mathscr{W} is locally free. If $\mathscr{F} \subseteq \mathscr{E}$ is invariant, then so is its saturation $\mathscr{F}^{sat} \subseteq \mathscr{E}$.

Proof. – Since \mathcal{W} is locally free, $\mathcal{F}^{sat} \otimes \mathcal{W}$ is saturated in $\mathcal{E} \otimes \mathcal{W}$. The sheaf $\theta(\mathcal{F}^{sat})$, which is almost everywhere contained in $\mathcal{F}^{sat} \otimes \mathcal{W}$, is therefore entirely contained in $\mathcal{F}^{sat} \otimes \mathcal{W}$, and is hence θ -invariant.

LEMMA 4.12 (Saturations of sheaves that are invariant on an open subset).

In the setting of Definition 4.8, if there exists a dense open set $V \subseteq X$ such that $\mathscr{W}|_V$ is locally free and $\mathscr{F}|_V$ is θ -invariant, then is saturation \mathscr{F}^{sat} is generically θ -invariant.

Proof. – Aiming to prove that \mathscr{F}^{sat} is *generically* θ -invariant, we may assume without loss of generality that \mathscr{W} is locally free. The following composition of morphisms,

$$\mathscr{F}^{\mathrm{sat}} \xrightarrow{\theta|_{\mathscr{F}^{\mathrm{sat}}}} \mathscr{E} \otimes \mathscr{W} \xrightarrow{\mathrm{projection}} (\mathscr{E} \otimes \mathscr{W}) / (\mathscr{F}^{\mathrm{sat}} \otimes \mathscr{W}) = (\mathscr{E} / \mathscr{F}^{\mathrm{sat}}) \otimes \mathscr{W},$$

will then vanish identically over V. Since its target is torsion free as a tensor product of a torsion free and a locally free sheaf, it follows that the composition vanishes everywhere. This shows the claim.

4.3. Stability

The notion of stability of sheaves with operators will be crucial for all what follows. The definition may look rather technical and perhaps not intuitive, but has several advantages that will make our arguments work. For one, it agrees with the classical definition in cases where \mathscr{E} is torsion free and \mathscr{W} is locally free. Secondly, it has good universal properties. These will later enable us to prove a restriction theorem for Higgs sheaves on singular spaces, and compare stability of a Higgs sheaf on a singular space with that of its pull-back to a resolution of singularities.

DEFINITION 4.13 (Stability of sheaves with operator). – Let X be a normal, projective variety and H be any nef, Q-Cartier Q-divisor on X. Let (\mathcal{E}, θ) be a sheaf with an operator, as in Definition 4.1, were \mathcal{E} is torsion free. We say that (\mathcal{E}, θ) is semistable with respect to H if the inequality $\mu_H(\mathcal{F}) \leq \mu_H(\mathcal{E})$ holds for all generically θ -invariant subsheaves $\mathcal{F} \subseteq \mathcal{E}$ with $0 < \operatorname{rank} \mathcal{F} < \operatorname{rank} \mathcal{E}$. The pair (\mathcal{E}, θ) is called stable with respect to H if strict inequality holds. Direct sums of stable sheaves with operator are called polystable.

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DEFINITION 4.14 (G-Stability of G-sheaves with operator). – Let X be a normal, projective variety equipped with the action of a finite group G, and H be any nef, Q-Cartier Q-divisor on X. Let (\mathcal{E}, θ) be a G-sheaf with an invariant operator, as in Definition 4.2, were \mathcal{E} is torsion free. We say that (\mathcal{E}, θ) is G-semistable with respect to H if the inequality $\mu_H(\mathcal{F}) \leq \mu_H(\mathcal{E})$ holds for all generically θ -invariant G-subsheaves $\mathcal{F} \subseteq \mathcal{E}$ with $0 < \operatorname{rank} \mathcal{F} < \operatorname{rank} \mathcal{E}$. The pair (\mathcal{E}, θ) is called G-stable with respect to H if strict inequality holds.

REMARK 4.15. – The conditions spelled out in Definitions 4.13 and 4.14 are trivially satisfied if \mathscr{E} does not contain generically invariant subsheaves of the appropriate rank.

LEMMA 4.16 (Stability and tensor product). – In the setting of Definition 4.13, let \mathscr{L} be any invertible sheaf. Then, (\mathscr{E}, θ) is stable (resp. semistable) with respect to H if and only if $(\mathscr{E}, \theta) \otimes \mathscr{L}$ is.

Proof. – Lemma 4.16 follows from Remark 4.10 and the fact that slope is additive, $\mu_H(\mathscr{F} \otimes \mathscr{L}) = \mu_H(\mathscr{F}) + \mu_H(\mathscr{L})$ for all non-trivial subsheaves $\mathscr{F} \subseteq \mathscr{E}$.

We next address openness properties of stability, with the goal to generalize results for ample polarisations to the nef case. The following proposition is not the strongest possible, but suffices for our purposes.

PROPOSITION 4.17 (Openness of stability). – Let X be a normal, projective variety, equipped with an action of a finite group G. Let H be a nef \mathbb{Q} -Cartier \mathbb{Q} -divisor, $[H] \neq 0$, and (\mathscr{E}, θ) be a torsion free G-sheaf which an invariant operator, and assume that (\mathscr{E}, θ) is G-stable with respect to H. Given any nef \mathbb{Q} -Cartier \mathbb{Q} -divisor A, there exists a positive number ε_0 such that for all rational numbers $0 < \varepsilon < \varepsilon_0$, the G-sheaf with invariant operator (\mathscr{E}, θ) is G-stable with respect to $(H + \varepsilon \cdot A)$.

Proof. – For simplicity of notation, write $n := \dim X$ and $r := \operatorname{rank} \mathscr{E}$. We may assume that *H* and *A* are integral and Cartier. In particular, recalling that the intersection numbers of Weil and Cartier divisors of Construction 2.17 take values in the integers, the *H*-stability of (\mathscr{E}, θ) implies that for any *G*-subsheaf $0 \neq \mathscr{F} \subset \mathscr{E}$ with rank $\mathscr{F} < r$ we have

(4.17.1)
$$\mu_H(\mathscr{E}) - \mu_H(\mathscr{F}) \ge r^{-1}.$$

Generalizing Definition 2.20 slightly, given any number $0 \le k < n$, write

$$\mu_{A^k H^{n-1-k}}(\mathscr{E}) := \frac{[\mathscr{E}] \cdot [A]^k \cdot [H]^{n-1-k}}{\operatorname{rank} \mathscr{E}}.$$

Fix a resolution of singularities, $\pi : \widetilde{X} \to X$ and observe that the curve class $\alpha_k := [\pi^* A]^k \cdot [\pi^* H]^{n-1-k} \in N_1(\widetilde{X})_{\mathbb{Q}}$ is movable. In particular, if $\mathscr{F} \subseteq \mathscr{E}$ is any coherent subsheaf, then $\pi^{[*]}\mathscr{F} \subseteq \pi^{[*]}\mathscr{E}$, we have an equality of slopes, $\mu_{A^k H^{n-1-k}}(\mathscr{F}) = \mu_{\alpha_k}(\pi^{[*]}\mathscr{F})$, and it follows from [18, Prop. 2.21] that

$$\mu_{A^{\bullet}H^{\bullet}}^{\max}(\mathscr{E}) := \sup \{ \mu_{A^{k}H^{n-1-k}}(\mathscr{F}) \mid \mathscr{F} \subseteq \mathscr{E} \text{ a coherent subsheaf and } 0 \le k < n \}$$

is finite. Now, given any rational $0 \le \varepsilon \ll 1$ and any *G*-subsheaf $0 \ne \mathscr{F} \subset \mathscr{E}$ with rank $\mathscr{F} < r$, owing to (4.17.1) we have

$$\mu_{H+\varepsilon \cdot A}(\mathscr{F}) = \mu_{H}(\mathscr{F}) + \sum_{k=1}^{n-1} \binom{n-1}{k} \varepsilon^{k} \cdot \mu_{A^{k}H^{n-1-k}}(\mathscr{F})$$
$$\leq \mu_{H}(\mathscr{E}) - \frac{1}{r} + \mu_{A^{\bullet}H^{\bullet}}^{\max}(\mathscr{E}) \cdot \sum_{k=1}^{n-1} \binom{n-1}{k} \varepsilon^{k}$$
$$< \mu_{H+\varepsilon \cdot A}(\mathscr{E}),$$

for ε sufficiently small, which proves the claim.

5. Higgs sheaves

This section introduces Higgs sheaves on singular varieties and establishes their basic properties. We include a discussion of Higgs \mathbb{Q} -sheaves on \mathbb{Q} -varieties, investigate functoriality of Higgs sheaves, define stability and prove a restriction theorem of Mehta-Ramanathan type. We conclude with a section on Higgs bundles and variations of Hodge structures that summarizes some work of Simpson and fits it into the framework of minimal model theory.

5.1. Fundamentals

On a singular variety, some attention has to be paid concerning the definition of "Higgs sheaf" at singular points. We will see in Section 5.3–5.7 that Higgs sheaves in the sense of the following definition have just enough universal properties to make our strategy of proof work. In the converse direction, it seems that Definition 5.1 and our notion of stability are in essence uniquely dictated if we ask all these universal properties to hold.

DEFINITION 5.1 (Higgs sheaf and Higgs *G*-sheaf). – Let *X* be a normal variety. A Higgs sheaf is a pair (\mathscr{E}, θ) of a coherent sheaf \mathscr{E} of \mathscr{O}_X -modules, together with an $\Omega_X^{[1]}$ -valued operator $\theta : \mathscr{E} \to \mathscr{E} \otimes \Omega_X^{[1]}$, called Higgs field, such that the composed morphism

$$\mathscr{E} \xrightarrow{\quad \theta \quad } \mathscr{E} \otimes \Omega_X^{[1]} \xrightarrow{\quad \theta \otimes \operatorname{Id} \quad } \mathscr{E} \otimes \Omega_X^{[1]} \otimes \Omega_X^{[1]} \xrightarrow{\quad \operatorname{Id} \otimes [\Lambda] \quad } \mathscr{E} \otimes \Omega_X^{[2]}$$

vanishes. Following tradition, the composed morphism will be denoted by $\theta \Lambda \theta$. If X is equipped with the action of a finite group G, a Higgs G-sheaf on X is a Higgs sheaf (\mathcal{E}, θ), where \mathcal{E} is a G-sheaf, and where the Higgs field θ is a morphism of G-sheaves.

DEFINITION 5.2 (Morphism of Higgs sheaves). – In the setting of Definition 5.1, a morphism of Higgs sheaves (resp. morphism of Higgs G-sheaves), written $f : (\mathscr{E}_1, \theta_1) \rightarrow (\mathscr{E}_2, \theta_2)$, is a morphism $f : \mathscr{E}_1 \rightarrow \mathscr{E}_2$ of sheaves (resp. G-sheaves) that commutes with the Higgs fields, $(f \otimes \mathrm{Id}_{\Omega_1^{[1]}}) \circ \theta_1 = \theta_2 \circ f$.

The above definitions extend to \mathbb{Q} -Higgs sheaves on \mathbb{Q} -varieties. These will be introduced in Section 5.5 once the existence of the necessary pull-back functors has been established.

EXAMPLE 5.3 (A natural Higgs sheaf attached to a normal variety). Let X be a normal variety. Set $\mathscr{E} := \Omega_X^{[1]} \oplus \mathscr{O}_X$ and define an operator θ as follows,

$$\begin{aligned} \theta : \ \Omega_X^{[1]} \ \oplus \ \mathscr{O}_X \ \to \left(\Omega_X^{[1]} \oplus \ \mathscr{O}_X\right) \otimes \Omega_X^{[1]}, \\ a \ + \ b \ \mapsto \ (0 \ + \ 1) \ \otimes \ a. \end{aligned}$$

An elementary computation shows that $\theta \Lambda \theta = 0$, so that (\mathscr{E}, θ) forms a Higgs sheaf. If X is a G-variety, then \mathscr{E} has a natural structure of a G-sheaf, and (\mathscr{E}, θ) is in fact a G-Higgs sheaf. Observe that the direct summand $\mathscr{O}_X \subseteq \mathscr{E}$ is generically θ -invariant. Non-zero subsheaves of the direct summand $\Omega_X^{[1]}$ are not generically θ -invariant.

EXAMPLE 5.4 (Tensor, dual and endomorphisms). – The direct sum and tensor operation of Construction 4.4 transforms Higgs sheaves into Higgs sheaves. Ditto for the dual sheaf and the endomorphism sheaf that are constructed in 4.5 if the Higgs sheaf is locally free.

5.2. Explanation

The reader might wonder why Definition 5.1 requires the Higgs field to take its values in $\mathscr{E} \otimes \Omega_X^{[1]}$. At least two other potential choices for the target come to mind. At first sight, it might seem most natural and functorial to take $\mathscr{E} \otimes \Omega_X^1$ for a target. However, in the main application to Miyaoka-Yau inequalities and to uniformisation for varieties of general type, the naturally induced sheaf of geometric origin is $\mathscr{E} := \Omega_X^{[1]} \oplus \mathscr{O}_X$, as discussed in Example 5.3 above. For this particular \mathscr{E} to be a Higgs sheaf, we have to allow the target of the Higgs field to be $\mathscr{E} \otimes \Omega_X^{[1]}$. Also, note that looking at $\Omega_X^1 \oplus \mathscr{O}_X$ instead would render a discussion of semistability moot, as semistability requires torsion freeness and even the most simple klt singularities lead to torsion in Ω_X^1 , see [20] for examples.

On the other hand, the reader might wonder why θ takes its values in $\mathscr{E} \otimes \Omega_X^{[1]}$ and not in its reflexive hull. The advantages of our choice will become apparent in the following Section 5.3, where pull-back functors are defined: in general, none of the constructions there will work for reflexive hulls.

5.3. Pull-back

To pull back Higgs sheaves is at least as difficult as to pull-back reflexive differentials. Functorial pull-back for reflexive differentials does, however, not exist in general unless the target space supports a divisor that makes it klt.

CONSTRUCTION 5.5 (Pull-back of Higgs sheaves). – Let (X, D) be a klt pair and let (\mathscr{E}, θ) be a Higgs sheaf on X. Given any normal variety Y and any morphism $f : Y \to X$, recall from [32, Thms. 1.3 and 5.2] that there exists a natural pull-back functor for reflexive differentials on klt pairs that is compatible with the usual pull-back of Kähler differentials and gives rise to a sheaf morphism

$$d_{\text{refl}}f: f^*\Omega_X^{[1]} \to \Omega_Y^{[1]}.$$

We claim that θ' , defined as the composition of the following morphisms,

(5.5.1)
$$f^*\mathscr{E} \xrightarrow{f^*\theta} f^*\left(\mathscr{E} \otimes \Omega_X^{[1]}\right) = f^*\mathscr{E} \otimes f^*\Omega_X^{[1]} \xrightarrow{\operatorname{Id}_{f^*\mathscr{E}} \otimes d_{\operatorname{refl}}f} f^*\mathscr{E} \otimes \Omega_Y^{[1]},$$

equips $f^* \mathscr{E}$ with the structure of a Higgs sheaf. To check that $\theta' \Lambda \theta' = 0$, one uses the compatibility of reflexive pull-back with wedge products, [32, Prop. 5.13], to verify that the following diagram is commutative,

$$\begin{array}{c} f^{*}(\theta\Lambda\theta) \\ f^{*}\mathscr{E} & & f^{*}\mathscr{E} \otimes f^{*}\Omega_{X}^{[1]} \otimes f^{*}\Omega_{X}^{[1]} \longrightarrow f^{*}\mathscr{E} \otimes f^{*}\Omega_{X}^{[2]} \\ & & & \downarrow^{\mathrm{Id} \otimes d_{\mathrm{refl}}} & \downarrow^{\mathrm{Id} \otimes d_{\mathrm{refl}}} \\ f^{*}\mathscr{E} & \xrightarrow{(\theta' \otimes \mathrm{Id}) \circ \theta'} & f^{*}\mathscr{E} \otimes \Omega_{Y}^{[1]} \otimes \Omega_{Y}^{[1]} \longrightarrow f^{*}\mathscr{E} \otimes \Omega_{Y}^{[2]}. \\ & & \theta' \Lambda \theta' \end{array}$$

By minor abuse of notation, this Higgs sheaf will be denoted as $f^*(\mathcal{E}, \theta)$ or $(f^*\mathcal{E}, f^*\theta)$.

NOTATION 5.6 (Restriction of Higgs sheaves). – In the setting of Construction 5.5, if f is a closed or open immersion, we will also write $(\mathscr{E}, \theta)|_Y$ or $(\mathscr{E}|_Y, \theta|_Y)$. To keep notation reasonably short, we will in the remainder of the paper tacitly equip restrictions of Higgs sheaves with their natural Higgs fields.

We mentioned above that the pull-back functor $d_{\text{refl}}f : f^*\Omega_X^{[1]} \to \Omega_Y^{[1]}$ is compatible with the usual pull-back of Kähler differentials. If X and Y are smooth and f is a closed immersion, the pull-back $f^*(\mathcal{E}, \theta)$ of Construction 5.5 will therefore agree with the standard pull-back (resp. restriction) of Higgs sheaves discussed in the literature.

LEMMA 5.7 (Pull-back of invariant subsheaves). – In the setting of Construction 5.5, if $\mathscr{F} \subseteq \mathscr{E}$ is θ -invariant in the sense of Definition 4.8, then $\mathscr{F}' := \operatorname{img}(f^*\mathscr{F} \to f^*\mathscr{E}) \subseteq f^*\mathscr{E}$ is θ' -invariant.

Proof. – Denote the natural inclusion map as $i : \mathscr{F} \to \mathscr{E}$. If \mathscr{F} is θ -invariant, then $\theta|_{\mathscr{F}}$ will factor via $\operatorname{img}(\mathscr{F} \otimes \Omega_X^{[1]} \to \mathscr{E} \otimes \Omega_X^{[1]})$. Pulling back, we obtain a commutative diagram

$$\begin{array}{c} f^* \mathscr{F} \xrightarrow{a} & f^* \operatorname{img} \left(\mathscr{F} \otimes \Omega_X^{[1]} \to \mathscr{E} \otimes \Omega_X^{[1]} \right) \xrightarrow{b} & f^* \mathscr{E} \otimes f^* \Omega_X^{[1]} \\ f^* i \\ & & & & \\ f^* \mathscr{E} \xrightarrow{f^* \theta} & & f^* \mathscr{E} \otimes f^* \Omega_X^{[1]} \end{array}$$

and, by an elementary computation, an inclusion

$$(f^*\theta)(\mathscr{F}') \subseteq \operatorname{img} b = \operatorname{img} \left(f^*\mathscr{F} \otimes f^*\Omega_X^{[1]} \to f^*\mathscr{E} \otimes f^*\Omega_X^{[1]} \right).$$

The following commutative diagram,

$$\begin{array}{cccc} f^{*}\mathscr{F} \otimes f^{*}\Omega_{X}^{[1]} & \xrightarrow{f^{*}i \otimes \mathrm{Id}} & f^{*}\mathscr{E} \otimes f^{*}\Omega_{X}^{[1]} & \xrightarrow{f^{*}\theta} & f^{*}\mathscr{E} \\ \mathrm{Id} \otimes d_{\mathrm{refl}}f & & & & & & \\ \mathrm{Id} \otimes d_{\mathrm{refl}}f & & & & & & \\ f^{*}\mathscr{F} \otimes \Omega_{Y}^{[1]} & \xrightarrow{f^{*}i \otimes \mathrm{Id}} & f^{*}\mathscr{E} \otimes \Omega_{Y}^{[1]}, & \xrightarrow{\theta':=(\mathrm{Id} \otimes d_{\mathrm{refl}}f) \circ f^{*}\theta} \end{array}$$

then yields the claim.

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The following two lemmas are almost immediate.

LEMMA 5.8 (Pull-back as criterion for invariance). – In the setting of Construction 5.5, assume that f is étale. If $\mathscr{F} \subseteq \mathscr{E}$ is any subsheaf such that $f^*\mathscr{F} \subseteq f^*\mathscr{E}$ is θ' -invariant, then \mathscr{F} is θ -invariant.

LEMMA 5.9 (Functoriality with respect to morphisms between spaces).

Given klt pairs (X, D_X) and (Y, D_Y) , a normal space Z, a Higgs sheaf (\mathcal{E}, θ) on X and morphisms $g : Z \to Y$ and $f : Y \to X$, then $g^* f^*(\mathcal{E}, \theta) = (f \circ g)^*(\mathcal{E}, \theta)$.

5.4. Reflexive pull-back

In the setting of Construction 5.5, assume that (\mathscr{E}, θ) is a reflexive Higgs sheaf on X and $f: Y \to X$ is a resolution of singularities. The pull-back $f^*(\mathscr{E}, \theta)$ is then a Higgs sheaf on Y, but $f^*\mathscr{E}$ is generally not torsion free. In particular, we cannot ask if $f^*(\mathscr{E}, \theta)$ is stable as a sheaf with an Ω^1_Y -valued operator. Using smoothness of Y, the following construction avoids this problem by equipping the *reflexive* pull-back $f^{[*]}\mathscr{E}$ with the structure of a Higgs sheaf.

CONSTRUCTION 5.10 (Reflexive pull-back of Higgs sheaves). – If the variety Y of Construction 5.5 is smooth, then $\Omega_Y^{[1]} = \Omega_Y^1$ is locally free. Taking reflexive hulls on either end of (5.5.1), we obtain an operator

$$f^{[*]}\theta: f^{[*]}\mathscr{E} \to \left(f^*\mathscr{E} \otimes \Omega_Y^{[1]}\right)^{**} = f^{[*]}\mathscr{E} \otimes \Omega_Y^1.$$

The associated map $f^{[*]}\theta \wedge f^{[*]}\theta$: $f^{[*]}\mathscr{E} \to f^{[*]}\mathscr{E} \otimes \Omega_Y^2$ clearly agrees with $0 = \theta' \wedge \theta'$ wherever $f^*\mathscr{E}$ is locally free. It follows that $f^{[*]}\theta \wedge f^{[*]}\theta$ vanishes generically and hence, since $f^{[*]}\mathscr{E} \otimes \Omega_Y^2$ is torsion free, identically. In summary, we see that $f^{[*]}\theta$ equips the reflexive pull-back $f^{[*]}\mathscr{E}$ with the structure of a Higgs sheaf. We will use the symbols $f^{[*]}(\mathscr{E}, \theta)$ or $(f^{[*]}\mathscr{E}, f^{[*]}\theta)$.

LEMMA 5.11 (Reflexive pull-back of invariant subsheaves). – In the setting of Construction 5.10, if $\mathscr{F} \subseteq \mathscr{E}$ is θ -invariant, write

$$\mathscr{F}' := \operatorname{img}(f^*\mathscr{F} \to f^*\mathscr{E}) \subseteq f^*\mathscr{E} \quad and \quad \mathscr{F}'' := (\mathscr{F}')^{**} \subseteq f^{[*]}\mathscr{E}.$$

Then, \mathscr{F}'' *is* $f^{[*]}\theta$ *-invariant.*

Proof. – Since Ω_Y^1 is locally free, $\mathscr{F}' \otimes \Omega_Y^1$ is a subsheaf of $f^{[*]} \mathscr{E} \otimes \Omega_Y^1$, and Lemma 5.7 gives a commutative diagram

(5.11.1)
$$\begin{array}{c} \mathscr{F}' & \longrightarrow & \mathscr{F}' \otimes \Omega^1_Y \\ & & & & & & \\ & & & & & \\ f^* \mathscr{E} & \xrightarrow{\theta'} & f^* \mathscr{E} \otimes \Omega^1_Y. \end{array}$$

Taking reflexive hulls is a left-exact functor. Applied to (5.11.1), it will thus give the desired inclusion $f^{[*]}\theta(\mathscr{F}'') \subseteq \mathscr{F}'' \otimes \Omega^1_Y \subseteq f^{[*]}\mathscr{E} \otimes \Omega^1_Y$.

OBSERVATION 5.12 (Weak functoriality with respect to morphisms between spaces).

Assume we are given klt pairs (X, D_X) and (Y, D_Y) , a smooth space Z, a sheaf \mathscr{E} on X and morphisms $g : Z \to Y$ and $f : Y \to X$. Then, there exists a canonical morphism $c : (f \circ g)^{[*]} \mathscr{E} \to g^{[*]} f^{[*]} \mathscr{E}$. If we assume additionally that $f^* \mathscr{E}$ is reflexive, then c is isomorphic and given any Higgs-field θ , one verifies immediately that $g^{[*]} f^*(\mathscr{E}, \theta) = (f \circ g)^{[*]} (\mathscr{E}, \theta)$.

WARNING 5.13 (No full functoriality with respect to morphisms between spaces).

We have seen in Lemma 5.9 that pull-back of Higgs sheaves is fully functorial with respect to morphisms between spaces. There is no full analog of this for reflexive pull-back. In fact, taking reflexive hulls does in general not commute with pull-back, the morphism c of Observation 5.12 will in general not be isomorphic, and functoriality fails already at the level of sheaves, without any additional Higgs structure. The arXiv version of this paper discusses an example in detail.

5.5. Higgs sheaves on Q-varieties

The definition of \mathbb{Q} -sheaves given in Section 3.5 has an obvious analog for Higgs sheaves.

DEFINITION 5.14 (Higgs Q-sheaf and Q-bundle). – Setup and notation as in Definition 3.1. A Higgs Q-sheaf (\mathcal{E}, θ) on $X_{\mathbb{Q}}$ is a tuple

$$(\{(\mathscr{E}_{\alpha},\theta_{\alpha})\}_{\alpha\in A},\{i_{\alpha\beta}\}_{(\alpha,\beta)\in A\times A})$$

consisting of a family of Higgs sheaves ($\mathscr{E}_{\alpha}, \theta_{\alpha}$) on X_{α} plus isomorphisms

$$i_{\alpha\beta}: p^*_{\alpha\beta,\alpha}(\mathscr{E}_{\alpha},\theta_{\alpha}) \to p^*_{\alpha\beta,\beta}(\mathscr{E}_{\beta},\theta_{\beta})$$

that are compatible on the triple overlaps. The Higgs \mathbb{Q} -sheaf (\mathcal{E}, θ) is called reflexive if all the \mathcal{E}_{α} are reflexive. It is called Higgs \mathbb{Q} -bundle if all the \mathcal{E}_{α} are locally free.

In complete analogy to Construction 3.8, any Higgs sheaf on X pulls back to a reflexive Higgs \mathbb{Q} -sheaf on $X_{\mathbb{Q}}$.

CONSTRUCTION 5.15 (Construction of Higgs Q-sheaf by reflexive pull-back).

Given a quasi-étale \mathbb{Q} -variety $X_{\mathbb{Q}} := (X, \{p_{\alpha}\}_{\alpha \in A})$, recall from [41, Prop. 5.20] that X is necessarily klt. In particular, there exists reflexive pull-back from Higgs sheaves on X to reflexive Higgs sheaves on the manifolds X_{α} . We can thus define a reflexive Higgs \mathbb{Q} -sheaf $(\mathscr{E}, \theta)^{[\mathbb{Q}]}$ on $X_{\mathbb{Q}}$, setting $(\mathscr{E}_{\alpha}, \theta_{\alpha}) := p_{\alpha}^{[*]}(\mathscr{G}, \theta)$ —the existence of natural isomorphisms $i_{\alpha\beta}$ is guaranteed by étalité of $p_{\alpha\beta,\alpha}$ and $p_{\alpha\beta,\beta}$.

As with \mathbb{Q} -sheaves, any Higgs \mathbb{Q} -sheaf on a \mathbb{Q} -variety pulls back to an honest Higgs sheaf on any global cover. The following are direct analogs of the appropriate statements for \mathbb{Q} -sheaves that are found in Section 3.5.

FACT 5.16 (Induced Higgs G-sheaf on global cover). – In the setting of Definition 5.14, assume we are given a global cover $\gamma : \hat{X} \to X$ as in Section 3.4, which is Galois with group G. Then, the pull-back Higgs sheaves $q_{\alpha}^*(\mathscr{E}_{\alpha}, \theta_{\alpha})$ glue to give a Higgs G-sheaf $(\hat{\mathscr{E}}, \hat{\theta})$ on \hat{X} . If the Higgs Q-sheaf (\mathscr{E}, θ) is reflexive, then $\hat{\mathscr{E}}$ is locally free in codimension two. If (\mathscr{E}, θ) is reflexive and \hat{X} is Cohen-Macaulay, then $(\hat{\mathscr{E}}, \hat{\theta})$ is likewise reflexive. A Higgs Q-sheaf does not only induce an honest Higgs-sheaf on any global cover, but also on any resolution of singularities of global covers that are Cohen-Macaulay. This can again be seen as a form of reflexive pull-back, this time from the global cover (which need not be klt) to the resolution of singularities.

LEMMA 5.17 (Induced Higgs G-sheaf on resolution of global cover).

Given a Q-variety X, a reflexive Higgs Q-sheaf (\mathscr{E}, θ) , a global cover \hat{X} with Galois group G and induced Higgs sheaf $(\hat{\mathscr{E}}, \hat{\theta})$, let $\pi : \tilde{X} \to \hat{X}$ be a G-equivariant resolution of singularities. Set $\tilde{\mathscr{E}} := \pi^{[*]}\hat{\mathscr{E}}$. If \hat{X} is Cohen-Macaulay, then there exists a G-invariant Higgs field $\tilde{\theta}$ on $\tilde{\mathscr{E}}$, such that the Higgs G-sheaf $(\tilde{\mathscr{E}}, \tilde{\theta})$ agrees with the reflexive π -pull back of $(\hat{\mathscr{E}}, \hat{\theta})$ over the maximal open set where \hat{X} is klt (and where reflexive π -pull pull-back is therefore defined).

Proof. – To define a *G*-invariant Higgs field on $\widetilde{\mathscr{E}}$, we denote the charts of the \mathbb{Q} -variety $X_{\mathbb{Q}}$ by $(X, \{p_{\alpha}\}_{\alpha \in A})$, and use the notation for global covers introduced in Section 3.4. Setting $\widetilde{X}_{\alpha} := \pi^{-1}(\widehat{X}_{\alpha})$, the following diagrams summarize our situation

$$\widetilde{X}_{\alpha} \xrightarrow{\pi_{\alpha} := \pi \mid_{\widetilde{X}_{\alpha}}} \widehat{X}_{\alpha} \xrightarrow{q_{\alpha}} X_{\alpha} \xrightarrow{p_{\alpha}} U_{\alpha} \xrightarrow{p_{\alpha}', \text{ étale}} X.$$

Set $(\tilde{\mathscr{E}}_{\alpha}, \tilde{\theta}_{\alpha}) := (q_{\alpha} \circ \pi_{\alpha})^{[*]}(\mathscr{E}_{\alpha}, \theta_{\alpha})$. Using the assumption that \hat{X} is Cohen-Macaulay, recall from Observation 3.5 that $q_{\alpha}^* \mathscr{E}_{\alpha}$ is reflexive. In particular, it follows directly that $\tilde{\mathscr{E}}_{\alpha} = \tilde{\mathscr{E}}_{|\tilde{X}_{\alpha}}$. More is true. Over the open set where \hat{X} is smooth and pull-back of Higgs sheaves is therefore defined, it follows from weak functoriality, Observation 5.12, that

$$\left(\widetilde{\mathscr{E}}_{\alpha},\widetilde{\theta}_{\alpha}\right) = \pi_{\alpha}^{[*]}q_{\alpha}^{*}(\mathscr{E}_{\alpha},\theta_{\alpha}) = \pi_{\alpha}^{[*]}\left(\widehat{\mathscr{E}}_{|\widehat{X}_{\alpha}},\widehat{\theta}_{\alpha}|_{\widehat{X}_{\alpha}}\right).$$

In particular, we see that the *G*-invariant Higgs fields $\tilde{\theta}_{\alpha}$ agree over this dense open set. Since \tilde{X} is smooth, two Higgs fields on the torsion free sheaf $\tilde{\mathscr{E}}$ agree if they agree on an open set. It follows that the $\tilde{\theta}_{\alpha}$ glue to give a globally defined Higgs *G*-sheaf $(\tilde{\mathscr{E}}, \tilde{\theta})$ that agrees with the reflexive π -pull back of $(\hat{\mathscr{E}}, \hat{\theta})$ wherever that pull-back is defined.

NOTATION 5.18 (Reflexive pull-back from global cover). – In the setting of Lemma 5.17, we write $\pi^{[*]}(\hat{\mathscr{E}}, \hat{\theta}) := (\tilde{\mathscr{E}}, \tilde{\theta})$, and refer to this sheaf as the reflexive pull-back.

Since this new piece of terminology agrees with the old one as soon as \hat{X} is klt, we do not expect this to lead to any confusion.

5.6. Stability

A Higgs sheaf is stable if it is stable as a sheaf with an $\Omega_X^{[1]}$ -valued operator, cf. Definition defn:swostab1. For later use, the following propositions, describing the behavior of stability under pull-backs, will be useful.

PROPOSITION 5.19 (*G*-stability under birational pull-back). – Let (X, D) be a projective klt pair, where X is equipped with an action of a finite group G. Let H be any nef, \mathbb{Q} -Cartier \mathbb{Q} -divisor on X and (\mathcal{E}, θ) be any Higgs G-sheaf, where \mathcal{E} is torsion free. Given a birational morphism $\pi : \tilde{X} \to X$ of projective G-varieties with \tilde{X} smooth, then (\mathcal{E}, θ) is G-stable (resp.

semistable) with respect to H if and only if $\pi^{[*]}(\mathcal{E}, \theta)$ is G-stable (resp. semistable) with respect to π^*H .

Proof. – Given any number $s \in \mathbb{Q}$, we need to show that the following two statements are equivalent.

- (5.19.1) There exists a G-subsheaf $0 \neq \mathscr{F} \subseteq \mathscr{E}$ with slope $\mu_H(\mathscr{F}) \geq s$ that is generically θ -invariant.
- (5.19.2) There exists a *G*-subsheaf $0 \neq \widetilde{\mathscr{F}} \subseteq \pi^{[*]} \mathscr{E}$ with slope $\mu_{\pi^*H}(\widetilde{\mathscr{F}}) \geq s$ that is generically $\pi^{[*]}\theta$ -invariant.

To this end, let $X^{\circ} \subseteq X_{\text{reg}}$ be the maximal open set where π is isomorphic, and observe that X° is a big, *G*-invariant subset of *X*. We set $\tilde{X}^{\circ} := \pi^{-1}(X^{\circ})$.

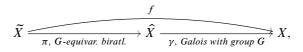
 $(5.19.1) \Rightarrow (5.19.2)$. – Given a sheaf \mathscr{F} as in (5.19.1), set $\widetilde{\mathscr{F}}' := f^{[*]}\mathscr{F}$. This is a *G*-invariant subsheaf of $f^{[*]}\mathscr{E}$ whose restriction to \widetilde{X}° is $f^{[*]}\theta$ -invariant. Its saturation $\widetilde{\mathscr{F}}$ is *G*-invariant, and, by Lemma 4.12, generically $\pi^{[*]}\theta$ -invariant. The ranks of $\widetilde{\mathscr{F}}$ and \mathscr{F} agree, the slope only increases in the process.

 $(5.19.2) \Rightarrow (5.19.1).$ – Given a sheaf $\widetilde{\mathscr{F}}$ as in (5.19.2), use the identification $\widetilde{X}^{\circ} \cong X^{\circ}$ to view $\widetilde{\mathscr{F}}_{|\widetilde{X}^{\circ}}$ as a $\theta_{|_{X^{\circ}}}$ -invariant sheaf \mathscr{F}° on X° . Recall from [21, I.Thm. 9.4.7 and 0.Sect. 5.3.2] that there exists a coherent subsheaf extension of \mathscr{F}° to X, that is, a coherent subsheaf $\mathscr{F}' \subseteq \mathscr{E}$ whose restriction to X° equals \mathscr{F}° . As before, Lemma 4.12 guarantees that its saturation $\mathscr{F} := (\mathscr{F}')^{\text{sat}}$ is generically θ -invariant. The ranks of $\widetilde{\mathscr{F}}$ and \mathscr{F} agree, the slope only increases in the process.

The following is an analog for morphisms that are generically Galois, say with group G. It differs from Proposition 5.19 in that it compares G-stability on the domain to normal stability on the target of the morphism. Its proof uses Proposition 2.16 to descent sheaves from \tilde{X} to X, but is otherwise completely similar to that of Proposition 5.19. The arXiv version of this paper contains the full argument.

PROPOSITION 5.20 (Stability under generically Galois pull-back).

Let (X, D) be a projective, klt pair, let H be any nef, \mathbb{Q} -Cartier \mathbb{Q} -divisor on X and (\mathcal{E}, θ) be any Higgs sheaf, where \mathcal{E} is torsion free. Given a sequence of morphisms between normal, projective varieties,



with \tilde{X} smooth, then (\mathcal{E}, θ) is stable (resp. semistable) with respect to H if and only if $f^{[*]}(\mathcal{E}, \theta)$ is G-stable (resp. semistable) with respect to π^*H .

Consider the setting of Proposition 5.20 in the special case where γ is quasi-étale. The pair $(\hat{X}, \gamma^* D)$ is then klt, and reflexive pull-back $\pi^{[*]}$ from \hat{X} to \tilde{X} exists. The Higgs sheaves $f^{[*]}(\mathscr{E}, \theta)$ and $\pi^{[*]}\gamma^{[*]}(\mathscr{E}, \theta)$, however, need not agree, cf. Warning 5.13. More generally, given a commutative diagram of morphisms between supporting spaces of klt pairs, failure

of functoriality will frequently lead to a large number of potentially different reflexive pullback Higgs sheaves, each corresponding to one particular path through the diagram. The following proposition will often be used to compare their stability properties.

PROPOSITION 5.21 (Comparison of *G*-stability). – Let *X* be a normal, projective variety, let *G* be a finite group that acts on *X*, let *H* be any nef, \mathbb{Q} -Cartier \mathbb{Q} -divisor on *X* and $\pi : \widetilde{X} \to X$ be a projective, birational, *G*-equivariant morphism, where \widetilde{X} is smooth. Let $E \subseteq \widetilde{X}$ be the π -exceptional set and assume that we are given two Higgs *G*-sheaves on \widetilde{X} , say (\mathscr{E}^1, θ^1) and (\mathscr{E}^2, θ^2), that agree as Higgs *G*-sheaves away from *E*. Then, the following two statements are equivalent for any pair of numbers $r \in \mathbb{N}$, $s \in \mathbb{Q}$.

- (5.21.1) There exists a G-invariant subsheaf $\mathscr{F}^1 \subseteq \mathscr{E}^1$ with rank $\mathscr{F} = r$ and slope $\mu_{\pi^*H}(\mathscr{F}^1) \geq s$ that is θ^1 -invariant.
- (5.21.2) There exists a G-invariant subsheaf $\mathscr{F}^2 \subseteq \mathscr{E}^2$ with rank $\mathscr{F} = r$ and slope $\mu_{\pi^*H}(\mathscr{F}^2) \geq s$ that is θ^2 -invariant.

In particular, $(\mathscr{E}^1, \theta^1)$ is G-stable (resp. G-semistable) with respect to π^*H if and only if $(\mathscr{E}^2, \theta^2)$ is.

Proof. – By symmetry, it suffices to show (5.21.1) \Rightarrow (5.21.2). Given a subsheaf \mathscr{F}^1 as in (5.21.1), consider the open set $\widetilde{X}^\circ := \widetilde{X} \setminus E$ and recall from [21, I.Thm. 9.4.7 and 0.Sect. 5.3.2] that there exists a subsheaf $\mathscr{G} \subseteq \mathscr{E}^2_{\widetilde{X}}$ whose restriction to \widetilde{X}° equals \mathscr{F}^1 . Replacing \mathscr{G} by $\sum_{g \in G} g^* \mathscr{G} \subseteq \mathscr{E}^2_{\widetilde{X}}$ if needed, we may assume without loss of generality that \mathscr{G} is *G*-invariant. Next, recall from Item (2.21.2) of Lemma 2.21 that $\mu_{\pi^*H}(\mathscr{G}) =$ $\mu_{\pi^*H}(\mathscr{F}^1) \geq s$. Let $\mathscr{F}^2 \subseteq \mathscr{E}^2$ be the saturation of \mathscr{G} , observe that $\mathscr{F}^2 \subseteq \mathscr{E}^2$ is again *G*-invariant, and recall from Lemma 4.12 that \mathscr{F}^2 is invariant with respect to θ^2 .

5.7. The restriction theorem for Higgs sheaves

This subsection establishes the restriction theorem for stable Higgs sheaves, which will be crucial for the proof of our main results. For Higgs bundles on manifolds with ample polarization, the theorem appears in Simpson's work, [58, Lem. 3.7], referring to "arguments of Mehta and Ramanathan" for a restriction theorem for sheaves with operators. Our proof instead cites a restriction theorem for sheaves with operators from the work of Langer, [44, Thm. 9]. He works in positive characteristic but says that, *mutatis mutandis*, his arguments will also work in characteristic zero, cf. [44, Page 906]. For clarity's sake, the arXiv version of this paper contains a statement of the precise result needed and a short, self-contained proof.

THEOREM 5.22 (Restriction theorem for stable Higgs sheaves). – Let (X, Δ) be a projective klt pair of dimension $n \ge 2$, let $H \in \text{Div}(X)$ be an ample, \mathbb{Q} -Cartier \mathbb{Q} -divisor and let (\mathscr{E}, θ) be a torsion free Higgs sheaf on X of positive rank. Assume that (\mathscr{E}, θ) is stable with respect to H. If $m \gg 0$ is sufficiently large and divisible, then there exists a dense open set $U \subseteq |m \cdot H|$ such that the following holds for any hyperplane $D \in U$ with associated inclusion map $\iota : D \to X$.

- (5.22.1) The hyperplane D is normal, connected and not contained in supp Δ . The pair $(D, \Delta_{\mid D})$ is klt.
- (5.22.2) The sheaf $\mathscr{E}_{|_{D}}$ is torsion free. The Higgs sheaf $\iota^*(\mathscr{E}, \theta)$ is stable with respect to $H_{|_{D}}$.

Proof. Step 1: Setup. – Choose a strong, log resolution of singularities, say $\pi : \tilde{X} \to X$. We have seen in Proposition 5.19 that $\pi^{[*]}(\mathscr{E}, \theta)$ is stable with respect to $\tilde{H} := \pi^* H$. Set $\widetilde{\mathscr{E}} := \pi^{[*]}\mathscr{E}$ and $r := \operatorname{rank} \mathscr{E}$.

NOTATION 5.23. – Given sheaves \mathscr{A} on X, \mathscr{B} on \widetilde{X} and \mathscr{C} on a subvariety $\widetilde{D} \subseteq \widetilde{X}$, we write deg $\mathscr{A} := \deg_{H} \mathscr{A}$, deg $\mathscr{B} := \deg_{\widetilde{H}} \mathscr{B}$, deg $\mathscr{C} := \deg_{\widetilde{H}|_{\widetilde{D}}} \mathscr{C}$ and similarly with μ and μ^{\max} .

Twisting \mathscr{E} with a sufficiently ample, invertible sheaf, Lemma 4.16 allows to assume that the following condition holds in addition.

ASSUMPTION W.L.O.G. 5.24. – The numbers $\mu(\mathscr{E})$ and $\mu^{\max}(\widetilde{\mathscr{E}})$ are positive.

Step 2: Choice of m. – Choosing $m \gg 0$ sufficiently large and divisible, the following will hold.

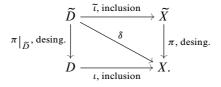
- (5.25.1) The divisor $m \cdot H$ is integral, Cartier and very ample.
- (5.25.2) Flenner's restriction theorem holds for \mathscr{E} , cf. [11, Thm. 1.2]. In particular, if $D \in |m \cdot H|$ is general, then $\mu^{\max}(\mathscr{E}|_D) = \mu^{\max}(\mathscr{E})$.
- (5.25.3) The number *m* satisfies the condition spelled out in the restriction theorem for sheaves with an operator, when applying the theorem to the Higgs sheaf $\pi^{[*]}(\mathscr{E}, \theta)$ as a sheaf with an $\Omega^1_{\widetilde{X}}$ -valued operator, [44, Thm. 9] but see also the appendix in the arXiv version of this paper.
- (5.25.4) We have a strict inequality $2r \cdot \mu^{\max}(\tilde{\mathscr{E}}) < [m \cdot H]^n$.

Step 3: Choice of U. – Next, observe that there exists an open subset $U \subseteq |m \cdot H|$ such that the following holds for all hyperplanes $D \in U$ and their preimages $\tilde{D} := \pi^{-1}(D)$.

- (5.26.1) The hyperplane D is reduced, irreducible, normal and not contained in supp Δ , Item (5.25.1) and Seidenberg's theorem [3, Thm. 1.7.1]. Its preimage \tilde{D} is smooth, Item (5.25.1) and Bertini. The pair $(D, \Delta_{|_D})$ is klt, Item (5.25.1) and [41, Lem. 5.17]. In particular, there exists a reflexive pull-back functor from Higgs sheaves on Dto Higgs sheaves on \tilde{D} .
- (5.26.2) The restrictions $\mathscr{E}_{|_{\widetilde{D}}}$ and $\widetilde{\mathscr{E}}_{|_{\widetilde{D}}}$ are reflexive, [22, Thm. 12.2.1].

(5.26.3) If $\mathscr{A} \subseteq \widetilde{\mathscr{E}}_{|\widetilde{D}}$ is any subsheaf, then $\mu(\mathscr{A}) \leq \mu^{\max}(\widetilde{\mathscr{E}})$, Item (5.25.2) and Lemma 2.21.

Choose one hyperplane $D \in U$ and fix that choice for the remainder of the proof. As before, write $\tilde{D} := \pi^{-1}(D)$ and consider the diagram



Pulling back, we can equip all spaces considered so far with naturally defined Higgs sheaves, which we list here for the reader's convenience.

 (\mathscr{E}, θ) ... Higgs sheaf on X that is initially given

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$$\begin{aligned} & (\mathscr{E}_{D}, \theta_{D}) &:= \iota^{*}(\mathscr{E}, \theta) & \dots \text{ Higgs sheaf on } D, \text{ equals } \iota^{[*]}(\mathscr{E}, \theta) \text{ by } (5.26.2) \\ & (\widetilde{\mathscr{E}}, \widetilde{\theta}) &:= \pi^{[*]}(\mathscr{E}, \theta) & \dots \text{ Higgs sheaf on } \widetilde{X} \\ & (\widetilde{\mathscr{E}}_{D}, \widetilde{\theta}_{D}) &:= \widetilde{\iota}^{*}(\widetilde{\mathscr{E}}, \widetilde{\theta}) & \dots \text{ Refl. sheaf on } \widetilde{D} \text{ with } \Omega^{1}_{\widetilde{X}|\widetilde{D}} \text{-valued operator} \\ & (\widetilde{\mathscr{E}}_{D}, \widetilde{\theta}_{D}) &:= \widetilde{\iota}^{*}(\widetilde{\mathscr{E}}, \widetilde{\theta}) & \dots \text{ Higgs sheaf on } \widetilde{D}, \text{ equals } \widetilde{\iota}^{[*]}(\widetilde{\mathscr{E}}, \widetilde{\theta}) \text{ by } (5.26.2) \\ & (\widetilde{\mathscr{E}}_{D}, \widetilde{\theta}_{D}) &:= (\pi_{|\widetilde{D}})^{[*]}(\mathscr{E}_{D}, \theta_{D}) & \dots \text{ Higgs sheaf on } \widetilde{D}, \text{ equals } \delta^{[*]}(\mathscr{E}, \theta) \text{ by } (5.26.2) \\ & \text{ and Observation } 5.12. \end{aligned}$$

We do not claim that the two Higgs sheaves on \tilde{D} , namely $(\tilde{\mathscr{E}}_{\tilde{D}}, \tilde{\theta}_{\tilde{D}})$ and $(\tilde{\mathscr{E}}_{D}, \tilde{\theta}_{D})$ necessarily agree, although they certainly agree outside of the $\pi_{|\tilde{D}}$ -exceptional set. We will compare these sheaves in the last step of this proof.

Step 4: Numerical computations. – We aim to show that $(\mathscr{E}_D, \theta_D)$ is stable, or equivalently, that $(\widetilde{\mathscr{E}}_D, \widetilde{\theta}_D)$ is stable. For this, we will first establish stability of $(\widetilde{\mathscr{E}}_D, \widetilde{\theta}_D)$ in Step 5 of this proof. The following numerical computation is instrumental.

CLAIM 5.27. – If $\mathscr{F} \subseteq \widetilde{\mathscr{E}}_{|\widetilde{D}}$ is any saturated subsheaf with $\mu(\mathscr{F}) \geq \mu(\mathscr{E}) = \mu(\widetilde{\mathscr{E}})$ and if \mathscr{A} is any subsheaf of the quotient $\mathscr{Q} := (\widetilde{\mathscr{E}}_{|\widetilde{D}})/\mathscr{F}$, then deg $\mathscr{A} \leq r \cdot \mu^{\max}(\widetilde{\mathscr{E}})$.

Proof of Claim 5.27. – Let $q: \tilde{\mathscr{E}}_{|\tilde{D}} \to \mathscr{Q}$ be the natural projection and consider the exact sequence

 $0 \to \mathscr{F} \to q^{-1}\mathscr{A} \to \mathscr{A} \to 0.$

We obtain that $\operatorname{rank}(q^{-1}\mathscr{A}) = \operatorname{rank}(\mathscr{A}) + \operatorname{rank}(\mathscr{F})$ and

$$deg \mathscr{A} = deg q^{-1} \mathscr{A} - deg \mathscr{F}$$

$$= \operatorname{rank}(q^{-1} \mathscr{A}) \cdot \mu(q^{-1} \mathscr{A}) - \operatorname{rank}(\mathscr{F}) \cdot \mu(\mathscr{F}) \qquad \text{Definition of } \mu$$

$$\leq \operatorname{rank}(q^{-1} \mathscr{A}) \cdot \mu^{\max}(\widetilde{\mathscr{E}}) - \operatorname{rank}(\mathscr{F}) \cdot \mu(\mathscr{F}) \qquad \text{Item (5.26.3)}$$

$$\leq \operatorname{rank}(q^{-1} \mathscr{A}) \cdot \mu^{\max}(\widetilde{\mathscr{E}}) - \operatorname{rank}(\mathscr{F}) \cdot \mu(\widetilde{\mathscr{E}}) \qquad \text{Assumption on } \mathscr{F}$$

$$\leq r \cdot \mu^{\max}(\widetilde{\mathscr{E}}) \qquad \text{Assumption 5.24,}$$

and Claim 5.27 follows.

CONSEQUENCE 5.28. – In the setting of Claim 5.27, if $\mathscr{B} \subseteq \mathscr{Q} \otimes \mathscr{O}_{\widetilde{D}}(-\widetilde{D})$ is of positive rank, then deg $\mathscr{B} \leq r \cdot \mu^{\max}(\widetilde{\mathscr{E}}) - \operatorname{rank}(\mathscr{B}) \cdot [\widetilde{D}]^{\dim X} \leq r \cdot \mu^{\max}(\widetilde{\mathscr{E}}) - [\widetilde{D}]^{\dim X}$.

Step 5: Stability of $(\tilde{\mathscr{E}}_{\tilde{D}}, \tilde{\theta}_{\tilde{D}})$. – With Consequence 5.28 at hand, stability of $(\tilde{\mathscr{E}}_{\tilde{D}}, \tilde{\theta}_{\tilde{D}})$ can now be established following the line of argument outlined by Simpson, [58, p. 38].

CLAIM 5.29. – The Higgs sheaf $(\tilde{\mathscr{E}}_{\widetilde{D}}, \tilde{\theta}_{\widetilde{D}})$ is stable with respect to $\tilde{H}_{|\widetilde{D}}$.

Proof of Claim 5.29. – Argue by contradiction and assume that there exists a generically $\tilde{\theta}_{\tilde{D}}$ -invariant subsheaf $\mathscr{F} \subseteq \tilde{\mathscr{E}}_{\tilde{D}}$ with $\mu(\mathscr{F}) \geq \mu(\mathscr{E})$. Lemma 4.11 allows to assume that \mathscr{F} is a saturated subsheaf of $\tilde{\mathscr{E}}_{|\tilde{D}}$. For this, note that the slope of a sheaf increases when passing

to the saturation. Consider the standard conormal bundle sequence for the submanifold $\widetilde{D} \subset \widetilde{X}$, twisted by $\mathcal{Q} := (\widetilde{\mathscr{E}}_{\widetilde{D}})/\mathscr{F}$,

(5.29.1)
$$0 \longrightarrow \mathscr{Q} \otimes \mathscr{O}_{\widetilde{D}}(-\widetilde{D}) \xrightarrow{\alpha} \mathscr{Q} \otimes \Omega^{1}_{\widetilde{X}}|_{\widetilde{D}} \xrightarrow{\beta} \mathscr{Q} \otimes \Omega^{1}_{\widetilde{D}} \longrightarrow 0$$

and the composition γ of the following two morphisms,

$$\mathscr{F} \xrightarrow{\widetilde{\theta}}_{|\widetilde{D}} \widetilde{\mathscr{E}}_{|\widetilde{D}} \otimes \Omega^1_{\widetilde{X}}|_{\widetilde{D}} \longrightarrow \mathscr{Q} \otimes \Omega^1_{\widetilde{X}}|_{\widetilde{D}}.$$

Recalling from Condition (5.25.3) that $\widetilde{\mathscr{E}}|_{\widetilde{D}}$ is stable as a sheaf with the $\Omega^1_{\widetilde{X}}|_{\widetilde{D}}$ -valued operator $\widetilde{\theta}|_{\widetilde{D}}$, it follows that \mathscr{F} is *not* generically invariant under that operator. In other words, the composed map γ is not generically zero. In contrast, the assumption that the sheaf \mathscr{F} is generically a Higgs subsheaf implies that the map $\beta \circ \gamma$ is necessarily zero. Exactness of (5.29.1) then gives a non-zero map $\tau : \mathscr{F} \to \mathscr{Q} \otimes \mathscr{O}_{\widetilde{D}}(-\widetilde{D})$. We will now show by way of numerical computation that such a map cannot exist. To this end, observe on the one hand that

$$deg(img \tau) = deg(\mathscr{F}) - deg(\ker \tau)$$

$$\geq \operatorname{rank} \mathscr{F} \cdot \mu(\widetilde{\mathscr{E}}) - deg(\ker \tau) \qquad \text{Choice of } \mathscr{F}$$

$$\geq \operatorname{rank} \mathscr{F} \cdot \mu(\widetilde{\mathscr{E}}) - \operatorname{rank}(\ker \tau) \cdot \mu^{\max}(\widetilde{\mathscr{E}}) \qquad \text{Item (5.26.3)}$$

$$\geq -r \cdot \mu^{\max}(\widetilde{\mathscr{E}}) \qquad \text{Assumption 5.24.}$$

On the other hand,

$$\deg(\operatorname{img} \tau) \le r \cdot \mu^{\max}(\widetilde{\mathscr{E}}) - [\widetilde{D}]^{\dim X} \qquad \text{Consequence 5.28.}$$

We obtain a contradiction to the choice of m in Assumption (5.25.4). This finishes the proof of Claim 5.29.

Step 6: End of proof. – We aim to show that the Higgs sheaf $(\mathscr{E}_D, \theta_D) = \iota^{[*]}(\mathscr{E}, \theta)$ is stable with respect to $H_{|_D}$. Applying Proposition 5.19 to the resolution morphism $\pi_{|_{\widetilde{D}}}$, this is equivalent to showing that $(\widetilde{\mathscr{E}}_D, \widetilde{\theta}_D)$ is stable with respect to $\widetilde{H}_{|_{\widetilde{D}}}$. But since $(\widetilde{\mathscr{E}}_D, \widetilde{\theta}_D)$ and $(\widetilde{\mathscr{E}}_{\widetilde{D}}, \widetilde{\theta}_{\widetilde{D}})$, agree outside of the $\pi_{|_{\widetilde{D}}}$ -exceptional set, Proposition 5.21 says that one is stable if and only if the other is. Stability of $(\widetilde{\mathscr{E}}_{\widetilde{D}}, \widetilde{\theta}_{\widetilde{D}})$ was, however, established in Claim 5.29.

5.8. Higgs bundles and variations of Hodge structures

In a series of fundamental works, including [56, 58], Simpson relates locally free Higgs sheaves on projective manifolds to representations of the fundamental group, and to variations of Hodge structures. We will use these results later to prove our uniformisation result, Theorem 1.2. For the reader's convenience, we briefly recall the most relevant definitions and explain how they fit into the framework of minimal model theory.

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DEFINITION 5.30 (Polarized, complex variation of Hodge structures).

Let X be a complex manifold, and $w \in \mathbb{N}$ a natural number. A polarized, complex variation of Hodge structures of weight w, or pCVHS in short, is a \mathcal{C}^{∞} -vector bundle \mathcal{D} with a direct sum decomposition $\mathcal{D} = \bigoplus_{r+s=w} \mathcal{D}^{r,s}$, a flat connection D that decomposes as follows (5.30.1)

 $D_{|_{\mathcal{V}^{r,s}}} : \mathcal{V}^{r,s} \to \mathcal{A}^{0,1}(\mathcal{V}^{r+1,s-1}) \oplus \mathcal{A}^{1,0}(\mathcal{V}^{r,s}) \oplus \mathcal{A}^{0,1}(\mathcal{V}^{r,s}) \oplus \mathcal{A}^{1,0}(\mathcal{V}^{r-1,s+1}),$

and a *D*-parallel Hermitian metric on \mathcal{V} that makes the direct sum decomposition orthogonal and that on $\mathcal{V}^{r,s}$ is positive definite if r is even and negative definite if r is odd.

Given a $p\mathbb{C}VHS$, one constructs an associated Higgs bundle. In fact, there are two equivalent constructions that produce isomorphic results.

CONSTRUCTION 5.31 (Higgs sheaves induced by a pCVHS). – Given a pCVHS as in Definition 5.30, use (5.30.1) to decompose D as $D = \overline{\theta} \oplus \partial \oplus \overline{\partial} \oplus \theta$.

- **FIRST CONSTRUCTION:** The operators $\overline{\partial}$ equip the \mathcal{C}^{∞} -bundles $\mathcal{V}^{r,s}$ with complex structures. We write $\mathscr{E}^{r,s}$ for the associated locally free sheaves of \mathscr{O}_X -modules, and set $\mathscr{E} := \bigoplus \mathscr{E}^{r,s}$. The operators θ then define an \mathscr{O}_X -linear morphism $\mathscr{E} \to \mathscr{E} \otimes \Omega^1_X$. As D is flat, this is a Higgs field.
- **SECOND CONSTRUCTION:** The operators $\overline{\partial} + \overline{\theta}$ equip the \mathcal{C}^{∞} -bundle \mathcal{V} with a complex structure where the (1,0)-part of the connection D becomes holomorphic. We call this holomorphically flat bundle \mathscr{H} . The complex subbundles $\mathscr{F}^p := \bigoplus_{r \geq p} \mathcal{V}^{r,s}$ are holomorphic and hence give a decreasing filtration of \mathscr{H} by holomorphic subbundles, cf. [65, Thm. 10.3]. Condition (5.30.1) then translates into $D(\mathscr{F}^p) \subset \mathscr{F}^{p-1} \otimes \Omega^1_X$. Hence, D induces an \mathscr{O}_X -linear morphism $\mathscr{E} \to \mathscr{E} \otimes \Omega^1_X$ on the associated graded sheaf $\mathscr{E} := \bigoplus \mathscr{F}^p / \mathscr{F}^{p+1}$. As D is flat, this is a Higgs bundle.

While part of Simpson's work refers to the first construction, we will use the second construction throughout. The formulation using filtrations is closer to standard textbooks on Hodge theory and allows to quote [65] or [6] without conflict of notation.

DEFINITION 5.32 (Higgs bundles induced by a p $\mathbb{C}VHS$). – Let X be a complex manifold and (\mathscr{E}, θ) a Higgs bundle on X. We say that (\mathscr{E}, θ) is induced by a p $\mathbb{C}VHS$ if there exists a p $\mathbb{C}VHS$ on X such that (\mathscr{E}, θ) is isomorphic to the Higgs bundle obtained from it via the second construction in 5.31.

REMARK 5.33. – In the setting of Definition 5.32, the pCVHS \mathcal{D} is in general not uniquely determined by (\mathscr{E}, θ) .

5.8.1. Criteria for a Higgs bundle to be induced by a p $\mathbb{C}VHS$. – Scaling the Higgs field induces an action of \mathbb{C}^* on the set of isomorphism classes of Higgs bundles. Under suitable assumptions, Simpson shows that Higgs bundles induced by a p $\mathbb{C}VHS$ correspond exactly to \mathbb{C}^* -fixed points. The following theorem summarizes his results.

THEOREM 5.34 (Higgs bundles induced by a pCVHS, I, [58, Cor. 4.2]).

Let X be a complex, projective manifold of dimension n and $H \in Div(X)$ be an ample divisor. Let (\mathcal{E}, θ) be a Higgs bundle on X. Then, (\mathcal{E}, θ) comes from a variation of Hodge structures in the sense of Definition 5.32 if and only if the following three conditions hold.

(5.34.1) The Higgs bundle (\mathcal{E}, θ) is H-polystable.

(5.34.2) The intersection numbers $ch_1(\mathscr{E}) \cdot [H]^{n-1}$ and $ch_2(\mathscr{E}) \cdot [H]^{n-2}$ both vanish.

(5.34.3) For any $t \in \mathbb{C}^*$, the Higgs bundles (\mathcal{E}, θ) and $(\mathcal{E}, t \cdot \theta)$ are isomorphic.

REMARK 5.35. – With X and H as in Theorem 5.34, any Higgs bundle (\mathscr{E}, θ) that satisfies (5.34.1) and (5.34.2) carries a flat \mathscr{C}^{∞} -connection, [58, Thm. 1(2) and Cor. 1.3]. In particular, all its Chern classes vanish.

As one immediate consequence of Theorem 5.34, we obtain the following strengthening of [58, Cor. 4.3].

COROLLARY 5.36 (Higgs bundles induced by a p $\mathbb{C}VHS$, II). – Let X be a projective manifold, and $H \in Div(X)$ an ample divisor. Let $\iota : S \hookrightarrow X$ be a submanifold. The push-forward map $\iota_* : \pi_1(S) \to \pi_1(X)$ induces a restriction map

$$r: \left\{ \begin{array}{l} \text{Isomorphism classes of } H\text{-semi-}\\ \text{stable Higgs bundles } (\mathscr{E}, \theta) \text{ on } X\\ \text{with vanishing Chern classes.} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{Isomorphism classes of } H\text{-semi-}\\ \text{stable Higgs bundles } (\mathscr{E}, \theta) \text{ on } S\\ \text{with vanishing Chern classes.} \end{array} \right\}$$

In particular, if (\mathcal{E}, θ) is any *H*-semistable Higgs bundle (\mathcal{E}, θ) on *X* with vanishing Chern classes, then $(\mathcal{E}, \theta)|_{S}$ is again *H*-semistable. The map *r* has the following properties.

- (5.36.1) If ι_* is surjective, then r is injective. In particular, if (\mathscr{E}, θ) is a Higgs bundle on X such that $(\mathscr{E}, \theta)|_{S}$ comes from a pCVHS, then (\mathscr{E}, θ) comes from a pCVHS.
- (5.36.2) If in addition the induced push-forward map $\hat{\iota}_* : \hat{\pi}_1(S) \to \hat{\pi}_1(X)$ of algebraic fundamental groups is isomorphic, then r is surjective.

Proof. – Simpson's Nonabelian Hodge Correspondence, [58, Cor. 3.10] or [57, Thm. 1], gives an equivalence between the categories of representations of the fundamental group $\pi_1(X)$ (resp. $\pi_1(S)$) and *H*-semistable Higgs bundles on *X* (resp. *S*) with vanishing Chern classes. The correspondence is functorial in morphisms between manifolds, and pull-back of Higgs bundles corresponds to the push-forward of fundamental groups, [58, Rem. 1 on p. 36]. In particular, we see that the restriction of an *H*-semistable Higgs bundle with vanishing Chern classes is again *H*-semistable.

In the setting of (5.36.1) where the push-forward map $\pi_1(S) \to \pi_1(X)$ is surjective, this immediately implies that the restriction r is injective. The restriction map r is clearly equivariant with respect to the actions of \mathbb{C}^* obtained by scaling the Higgs fields. Injectivity therefore implies that the isomorphism class of a Higgs bundle (\mathscr{E}, θ) is \mathbb{C}^* -fixed if and only if the same is true for $(\mathscr{E}, \theta)|_S$. Theorem 5.34 thus proves the second clause of (5.36.1).

Now assume that we are in the setting of (5.36.2), where in addition the push-forward map $\hat{\pi}_1(S) \rightarrow \hat{\pi}_1(X)$ is assumed to be isomorphic. Since fundamental groups of algebraic varieties are finitely generated, this implies via Malcev's theorem that every representation of $\pi_1(S)$ comes from a representation of $\pi_1(X)$, [23, Thm. 1.2b] or see [17, Sect. 8.1] for a detailed pedestrian proof. The claim thus again follows from Simpson's Nonabelian Hodge Correspondence.

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5.8.2. The period map. – A p $\mathbb{C}VHS$ on a simply connected complex manifold X induces a map to the period domain. Here, we will show that Higgs bundles that are induced by a p $\mathbb{C}VHS$ come from the period domain. If X is the desingularisation of a klt variety, this implies that the relevant bundle comes from the singular space.

CONSTRUCTION 5.37 (Period map, cf. [65, Sect. 10.1.2-3] or [6, Sect. 4.3]).

Given a pCVHS on a simply connected complex manifold X, we obtain a period map $\rho : X \to \mathcal{D}$ into the classifying space \mathcal{D} for Hodge structures of the given type, the so-called *period domain*. Let us quickly recall the construction. Let F be the flag manifold parametrising complex flags of the type given by the filtration \mathscr{F}^{\bullet} . The projective manifold F embeds into the product P of Grassmannians that parametrise subspaces of those dimensions that occur in the filtration \mathscr{F}^{\bullet} . As X is simply connected, the holomorphically flat bundle \mathscr{H} trivializes, and so the filtration \mathscr{F}^{\bullet} yields a family of flags in a fixed complex vector space parametrised by X. Assigning to each point in X the corresponding point in P yields the period map $\rho : X \to F \hookrightarrow P$, which is actually holomorphic, cf. [65, Thm. 10.9]. The image of ρ can be seen to lie in a special domain \mathscr{D} inside the closed complex submanifold $\widetilde{\mathscr{D}}$ of F that is defined by the orthogonality condition required in Definition 5.30, the *period domain*.

PROPOSITION 5.38. – Let X be a simply connected manifold and (\mathcal{E}, θ) be a Higgs bundle on X that comes from a pCVHS. Let $\rho : X \to \mathcal{D}$ be the associated period map. Then, there exists a holomorphic vector bundle $\mathcal{E}_{\mathcal{D}}$ on \mathcal{D} such that $\mathcal{E} \cong \rho^* \mathcal{E}_{\mathcal{D}}$.

Proof. – It follows from Construction 5.37 that \mathcal{D} is an open subset in a flag manifold (whose type is determined by the filtration \mathscr{F}^{\bullet}), which in turn can be embedded into a product of Grassmannians. Each of the Grassmannians carries a tautological vector bundle, which can be restricted to \mathcal{D} , yielding a holomorphic vector bundle \mathscr{T}^p on \mathcal{D} . By definition and holomorphy of the period map, we have $\rho^*(\mathscr{T}^p) \cong \mathscr{F}^p$, cf. [65, p. 250]. It follows that $\mathscr{F}^p/\mathscr{F}^{p+1}$ is a pullback from the period domain, and hence so is $\mathscr{E} = \bigoplus \mathscr{F}^p/\mathscr{F}^{p+1}$. \Box

COROLLARY 5.39. – Let (X, D) be a klt pair and $\pi : \tilde{X} \to X$ a resolution of singularities. Let (\mathscr{E}, θ) be a Higgs bundle on \tilde{X} that is induced by a pCVHS. Then, \mathscr{E} comes from X. More precisely, there exists a locally free sheaf \mathscr{E}_X on X such that $\mathscr{E} = \pi^* \mathscr{E}_X$. Necessarily, we then have $\mathscr{E}_X \cong \pi_*(\mathscr{E})^{**}$.

Proof. – It suffices to construct \mathscr{E}_X locally in the analytic topology, near any given point of X. Now, given any $x \in X$, recall from [60, p. 827] that there exists a contractible, open neighborhood $U = U(x) \subseteq X^{an}$ whose preimage $\tilde{U} := \pi^{-1}(U)$ is simply connected. By assumption, (\mathscr{E}, θ) is induced from a pCVHS \mathcal{V} . Let $\rho : \tilde{U} \to \mathcal{D}$ be the corresponding period map.

We claim that ρ factors through the resolution $\pi : \tilde{U} \to U$. Indeed, since the fibers of π are rationally chain-connected by, it suffices to show that given any morphism $\eta : \mathbb{P}^1 \to \tilde{U}$, the composed map $\rho \circ \eta : \mathbb{P}^1 \to \mathcal{D}$ is constant. Pulling back \mathcal{D} via η yields a pCVHS on \mathbb{P}^1 whose associated period map equals $\rho \circ \eta$. However, due to hyperbolicity properties of the period domain \mathcal{D} , this map has to be constant, [6, Application 13.4.3].

By Proposition 5.38, we know that $\mathscr{E} \cong \rho^*(\mathscr{E}_{\mathcal{D}})$ for some vector bundle $\mathscr{E}_{\mathcal{D}}$ on the period domain \mathcal{D} . If $\rho_U : U \to \mathcal{D}$ is the holomorphic map whose existence was shown in the previous paragraph, the vector bundle $\mathscr{E}_U := \rho_U^*(\mathscr{E}_{\mathcal{D}})$ hence fulfills $\pi^*(\mathscr{E}_U) \cong \mathscr{E}$, as desired.

REMARK 5.40. – Corollary 5.39 is actually true in a much more general setting. In fact, the bundle \mathscr{E} is trivial on the fibers of π . Then, regardless whether \mathscr{E} carries a Higgs structure or not, \mathscr{E} is the pull-back of a bundle on X, as X has only klt singularities. As the proof is much more involved than the one presented in the previous paragraphs, with our main application in mind we have decided to restrict to the case of Higgs bundles coming from pCVHSs here. Details for the general case will appear in a forthcoming paper.

PART II

MIYAOKA-YAU INEQUALITY AND UNIFORMISATION

6. The Q-Bogomolov-Gieseker inequality

We establish the \mathbb{Q} -Bogomolov-Gieseker inequality for Higgs sheaves on klt spaces. Section 7 applies this result to the natural Higgs sheaf of Example 5.3, in order to establish the \mathbb{Q} -Miyaoka-Yau inequality for the tangent sheaf of a klt variety of general type whose canonical divisor is nef.

THEOREM 6.1 (Q-Bogomolov-Gieseker inequality). – Let (X, D) be a projective, klt pair of dimension $n \ge 2$, and let P be a nef Q-Cartier Q-divisor on X. If (\mathcal{E}, θ) is any reflexive Higgs sheaf of rank $\mathcal{E} \ge 2$ on X that is stable with respect to P, then \mathcal{E} verifies

(6.1.1)
$$\widehat{\Delta}(\mathscr{E}) \cdot [P]^{n-2} \ge 0.$$

We refer to (6.1.1) as the Q-Bogomolov-Gieseker inequality.

We expect that Theorem 6.1 will also hold for semistable sheaves. Again, with our main application in mind, we restrict ourselves to the stable case.

6.1. Preparations for the proof of Theorem 6.1

Cutting by hyperplanes, the proof of the Q-Bogomolov-Gieseker inequality will quickly reduce to the surface case, which is handled first.

PROPOSITION 6.2 (Q-Bogomolov-Gieseker Inequality on klt surfaces).

Let (X, D) be a projective, klt pair of dimension two, and let H be a nef \mathbb{Q} -Cartier \mathbb{Q} -divisor on X. If (\mathcal{E}, θ) is any reflexive Higgs sheaf of rank $\mathcal{E} \geq 2$ on X that is stable with respect to H, then the sheaf \mathcal{E} satisfies the \mathbb{Q} -Bogomolov-Gieseker inequality $\hat{\Delta}(\mathcal{E}) \geq 0$.

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Proof. – Openness of stability, Proposition 4.17, allows to assume without loss of generality that *H* is integral, Cartier, and ample. Theorem 3.13 gives both a Q-variety structure $X_{\mathbb{Q}}$ on *X*, and a global, Cohen-Macaulay and Galois cover $\gamma : \hat{X} \to X$ that allows to compute Q-Chern classes on *X* in terms of honest Chern classes of pull-back sheaves. Let $G := \operatorname{Gal}(\hat{X}/X)$ be the corresponding Galois group and set $\hat{H} := \gamma^* H$.

Applying Construction 5.15 to (\mathscr{E}, θ) , we obtain a reflexive Higgs \mathbb{Q} -sheaf $(\mathscr{E}, \theta)^{[\mathbb{Q}]}$ on $X_{\mathbb{Q}}$. Let $(\widehat{\mathscr{E}}, \widehat{\theta})$ be the induced Higgs *G*-sheaf on \widehat{X} , as discussed in Fact 5.16. Since \widehat{X} is of dimension two, Fact 5.16 asserts that $(\widehat{\mathscr{E}}, \widehat{\theta})$ is actually a Higgs *G*-bundle. Finally, let $\pi : \widetilde{X} \to \widehat{X}$ be a strong, *G*-invariant resolution of \widehat{X} . The following diagram summarizes the situation:

(6.2.1)
$$\widetilde{X} \xrightarrow{\pi} \widehat{X} \xrightarrow{\gamma} X$$

resolution of sings. $\widehat{X} \xrightarrow{\gamma} X$

We obtain two locally free Higgs G-sheaves on \tilde{X} , namely $\pi^{[*]}(\hat{\mathscr{E}}, \hat{\theta})$ and $\psi^{[*]}(\mathscr{E}, \theta)$ —we refer to Section 5.4 for the construction of the reflexive pull-back $\psi^{[*]}$ and to Lemma 5.17 and Notation 5.18 for all matters concerning $\pi^{[*]}$. These Higgs sheaves are not necessarily equal, but they do agree over the big open set of X_{reg} where \mathscr{E} is locally free. By reflexivity, the two Higgs sheaves will then coincide outside the exceptional set of π .

It follows from Proposition 5.20 that $\psi^{[*]}(\mathscr{E},\theta)$ is *G*-stable with respect to $\pi^*(\hat{H})$. Since both sheaves agree outside the π -exceptional set, Proposition 5.21 implies that the *G*-Higgs bundle $\pi^*(\hat{\mathscr{E}},\hat{\theta})$ is *G*-stable with respect to the nef polarization $\pi^*(\hat{H})$ as well. Openness of *G*-stability, Proposition 4.17, allows to modify $\pi^*(\hat{H})$, and find a *G*-stable, ample divisor \tilde{A} on \tilde{X} , such that $\pi^*(\hat{\mathscr{E}},\hat{\theta})$ is *G*-stable with respect to \tilde{H} .

Since $\hat{\mathscr{E}}$ is locally free, we can discuss the standard Bogomolov discriminant $\Delta(\hat{\mathscr{E}})$, as introduced in Notation 2.14. The functorial properties of Chern classes, [12, Thm. 3.2(d)], and the choice of γ imply

(6.2.2)
$$\Delta(\pi^*\widehat{\mathscr{E}}) = \Delta(\widehat{\mathscr{E}}) = (\deg \gamma) \cdot \widehat{\Delta}(\mathscr{E})$$

Simpson's Bogomolov-Gieseker Inequality for *G*-Higgs bundles that are stable with respect to an ample polarization, [56, Thm. 1 and Prop. 3.4], applies to $\pi^*(\hat{\mathscr{E}}, \hat{\theta})$ and \tilde{A} , showing that $\Delta(\pi^*\hat{\mathscr{E}}) \ge 0$. Together with (6.2.2), this finishes the proof of Proposition 6.2.

6.2. Proof of Theorem 6.1

By multilinearity of the form $\hat{\Delta}$, it suffices to prove the claim in the case where *P* is an integral Cartier divisor. Using that the function

$$N^1(X)_{\mathbb{R}} \to \mathbb{R}, \qquad \alpha \mapsto \widehat{\Delta}(\mathscr{E}) \cdot \alpha^{n-2}$$

is continuous, Proposition 4.17 allows to assume without loss of generality that P is integral, Cartier, and ample. Choosing $m \gg 0$ sufficiently large, the Restriction theorem for stable Higgs sheaves, Theorem 5.22, allows to find a tuple of hyperplanes $(H_1, \ldots, H_{n-2}) \in |m \cdot P|^{\times (n-2)}$ with associated complete intersection surface $S := H_1 \cap \cdots \cap H_{n-2}$ such that the following holds.

(6.3.1) The scheme S is a normal and irreducible surface, and not contained in the support of D. The pair $(S, D|_{S})$ is klt, [41, Lem. 5.17].

- (6.3.2) The restriction $\mathscr{E}_{|_{S}}$ is reflexive, [22, Thm. 12.2.1].
- (6.3.3) Denoting the inclusion by $\iota: S \to X$, the Higgs sheaf $\iota^*(\mathscr{E}, \theta)$ is stable with respect to $P_{|_S}$, Theorem 5.22.

(6.3.4) We have an equality $\hat{\Delta}(\mathscr{E}) \cdot [P]^{n-2} = m^{n-2} \cdot \hat{\Delta}(\mathscr{E}_{|_S})$, Item (3.13.2) of Theorem 3.13. The result hence follows from Proposition 6.2 above.

7. The Q-Miyaoka-Yau inequality

7.1. Proof of Theorem 1.1

Theorem 1.1 will follow from the results of Section 6, once we can apply them to the natural Higgs sheaf $(\mathscr{E}_X, \theta_X)$ of Example 5.3, where $\mathscr{E}_X = \Omega_X^{[1]} \oplus \mathscr{O}_X$. We hence establish stability of $(\mathscr{E}_X, \theta_X)$ first. This is a consequence of the following minor generalization of a recent semistability result of Guenancia, [24, Thm. A], which in turn generalizes a classical result of Enoki, [9, Cor. 1.2].

THEOREM 7.1 (Semistability of tangent sheaves). – Let X be a projective, klt variety of general type whose canonical divisor K_X is nef. Then, \mathcal{T}_X and $\Omega_X^{[1]}$ are semistable with respect to K_X .

Proof. – It suffices to show semistability for \mathscr{T}_X . Recall from Reminder 2.4 that K_X is semiample and induces a birational morphism $\phi : X \to Z$, where Z is klt, and K_Z is ample. By [24, Thm. A], the tangent sheaf \mathscr{T}_Z is semistable with respect to K_Z . Since \mathscr{T}_X coincides with $\phi^{[*]}(\mathscr{T}_Z)$ outside of the ϕ -exceptional set, \mathscr{T}_X is hence semistable with respect to $K_X = \phi^*(K_Z)$, cf. Lemma 2.21. This concludes the proof.

COROLLARY 7.2 (Higgs-stability for varieties of general type). – Let X be a projective, klt variety of general type whose canonical divisor K_X is nef. Then, the natural Higgs sheaf $(\mathcal{E}_X, \theta_X)$ of Example 5.3 is stable with respect to K_X .

Proof. – Write $n := \dim X$ and $d := [K_X]^n \in \mathbb{Q}^+$, which is positive since K_X is nef and big. Aiming for a contradiction, assume that $(\mathscr{E}_X, \theta_X)$ is not stable with respect to K_X . Hence, there exists a subsheaf $0 \neq \mathscr{F} \subsetneq \mathscr{E}_X$ that is generically θ -invariant and satisfies

(7.2.1)
$$\mu_{K_X}(\mathscr{F}) \ge \mu_{K_X}(\mathscr{E}_X) = d/(n+1).$$

Lemma 4.12 allows to assume that \mathscr{F} is saturated in \mathscr{E}_X . In particular, \mathscr{F} is reflexive. Write $r := \operatorname{rank} \mathscr{F}$ and note that r < n + 1.

Let $\alpha : \mathscr{F} \to \mathscr{O}_X$ be the morphism induced by the projection to the \mathscr{O}_X -summand of \mathscr{E} . Recalling from Example 5.3 that no subsheaf of the direct summand $\Omega_X^{[1]}$ is ever generically θ_X -invariant, it follows that α is not the zero map. We also notice that α is *not* an injection, for otherwise \mathscr{F} is the Weil-divisorial sheaf of an anti-effective Weil divisor, and $[\mathscr{F}] \cdot [K_X]^{n-1} \leq 0$, contradicting Inequality (7.2.1). It follows that r > 1 and rank(ker α) = r-1 > 0. More can be said. Since det(img α) is Weil divisorial for an anti-effective divisor, we have

$$[\ker \alpha] \cdot [K_X]^{n-1} = [\mathscr{F}] \cdot [K_X]^{n-1} - [\operatorname{img} \alpha] \cdot [K_X]^{n-1} \ge [\mathscr{F}] \cdot [K_X]^{n-1}$$

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and, dividing by r - 1,

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$$\begin{aligned}
\mu_{K_X}(\ker \alpha) &\geq \frac{[\mathscr{F}] \cdot [K_X]^{n-1}}{r-1} = \mu_{K_X}(\mathscr{F}) \cdot \frac{r}{r-1} \\
&\geq \frac{d}{n+1} \cdot \frac{r}{r-1} \qquad \qquad \text{by (7.2.1)} \\
&= \frac{d}{n} \cdot \frac{nr}{(n+1)(r-1)} > \frac{d}{n} \qquad \qquad \text{since } n+1 > r \text{ and } d > 0.
\end{aligned}$$

It follows that $\mu_{K_X}(\ker \alpha) > \mu_{K_X}(\Omega_X^{[1]})$, the latter one being equal to d/n. Since ker α injects into $\Omega_X^{[1]}$ by definition of α , we hence obtain a contradiction to the semistability of $\Omega_X^{[1]}$ proven in Theorem 7.1.

7.2. Proof of Theorem 1.1

By elementary calculus of Chern classes, Inequality (1.1.1) is equivalent to

$$\widehat{\Delta}\big(\mathscr{T}_X \oplus \mathscr{O}_X\big) \cdot [K_X]^{n-2} = \widehat{\Delta}\big(\Omega_X^{[1]} \oplus \mathscr{O}_X\big) \cdot [K_X]^{n-2} \ge 0,$$

which follows from Theorem 6.1 and Corollary 7.2.

8. Uniformisation

8.1. Proof of Theorem 1.2

Theorem 1.2 follows directly from the subsequent, more general result.

THEOREM 8.1. – Let X be an n-dimensional, projective, klt variety of general type whose canonical divisor K_X is nef. Assume that X is smooth in codimension two. Recall from Reminder 2.4 that K_X is semiample, and induces a morphism $\varphi : X \to Z$, where Z is klt, and K_Z is ample. If equality holds in the Q-Miyaoka-Yau inequality (1.1.1), then Z is smooth in codimension two, there exists a ball quotient Y and a finite, Galois, quasi-étale morphism $f : Y \to Z$. In particular, Z has only quotient singularities.

Proof of Theorem 1.2. – As varieties with terminal singularities are smooth in codimension two, the result follows by applying Theorem 8.1. \Box

8.2. Preparation for the proof of Theorem 8.1

The proof of Theorem 8.1 is based on the following two propositions.

PROPOSITION 8.2. – Let X be a projective, klt variety of general type whose canonical divisor is nef. Suppose that X is smooth in codimension two and that equality holds in the \mathbb{Q} -Miyaoka-Yau inequality (1.1.1). Recall from Reminder 2.4 that K_X is semiample and induces a morphism $\varphi : X \to Z$, where Z is klt, and K_Z is ample. Then, Z is smooth in codimension two, and equality holds in the \mathbb{Q} -Miyaoka-Yau inequality for Z.

Proof. – Choose a strong resolution of singularities, say $\pi : \tilde{X} \to X$, and observe that the composed map $\varphi \circ \pi : \tilde{X} \to Z$ is a resolution of Z that is minimal in codimension two. Let S_Z be a surface cut out by general sections of $|m \cdot K_Z|$, for $m \gg 0$, and let S_X , $S_{\tilde{X}}$ denote the strict transforms in X and \tilde{X} , respectively. Since X is smooth in codimension two, S_X is entirely contained in the smooth locus X_{reg} , and π is therefore isomorphic near S_X . We obtain:

(8.2.1)
$$\hat{c}_2(\mathscr{T}_Z) \cdot [K_Z]^{n-2} \le c_2(\mathscr{T}_{\widetilde{X}}) \cdot [(\varphi \circ \pi)^* K_Z]^{n-2}$$
 by [53, Prop. 1.1]
$$= c_2(\mathscr{T}_{\widetilde{X}}|_{S_{\widetilde{X}}}) = c_2(\mathscr{T}_X|_{S_X})$$
$$= c_2(\mathscr{T}_X) \cdot [K_X]^{n-2},$$

with strict inequality if and only if Z does have singularities in codimension two, [53, Prop. 1.1]. In a similar vein,

(8.2.2)
$$\widehat{c}_1(\mathscr{T}_Z) \cdot [K_Z]^{n-1} = c_1(\mathscr{T}_X) \cdot [K_X]^{n-1}.$$

The \mathbb{Q} -Miyaoka-Yau inequality for Z thus forces equality in (8.2.1). This shows both that Z is smooth in codimension two, and that equality holds in the \mathbb{Q} -Miyaoka-Yau inequality for Z.

PROPOSITION 8.3. – Let X be a projective, klt variety of dimension n that is smooth in codimension two and such that the étale fundamental group of X and of its smooth locus agree, $\hat{\pi}_1(X_{reg}) \cong \hat{\pi}_1(X)$. If K_X is ample and if equality holds in the Q-Miyaoka-Yau Inequality (1.1.1), then X is smooth.

REMARK 8.4. – The main reason for the assumption on the codimension of the singular set is to guarantee smoothness of complete intersection surfaces and hence their isomorphic lifting to a strong resolution of singularities, where we are then able to use functoriality properties of Simpson's Nonabelian Hodge Correspondence; for details, see the subsequent proof.

Proof of Proposition 8.3. – For the reader's convenience, the proof is subdivided into a number of relatively independent steps.

Step 1. Setup. – The main object of study in our proof is the canonical Higgs sheaf $(\mathscr{E}_X, \theta_X)$ on X, introduced in Example 5.3. Recall that $\mathscr{E}_X = \Omega_X^{[1]} \oplus \mathscr{O}_X$ and that $(\mathscr{E}_X, \theta_X)$ is K_X -stable due to Corollary 7.2. Choose a strong log resolution of singularities, $\pi : \tilde{X} \to X$, such that there exists a π -ample Cartier divisor supported on the exceptional locus of π .

CLAIM 8.5. – Write $r := (n + 1)^2$. Let B_r denote the set of locally free sheaves \mathscr{F} on X that have rank r, satisfy $\mu_{K_X}^{\max}(\mathscr{F}) = \mu_{K_X}^{\max}(\mathscr{E}nd \mathscr{E}_X)$, and have Chern classes $c_i(\pi^*\mathscr{F}) = 0$ for all $0 < i \le r$. Then, B_r is bounded.

Proof of Claim 8.5. – Since X has rational singularities, the Euler characteristics $\chi_X(\mathscr{G})$ and $\chi_{\widetilde{X}}(\pi^*\mathscr{G})$ agree for all locally free sheaves \mathscr{G} on X. The assumption on Chern classes thus guarantees that the Hilbert polynomials of the members $\mathscr{F} \in B_r$ are constant, cf. [12, Cor. 15.2.1]. Boundedness thus follows from [29, Thm. 3.3.7]. This ends the proof of Claim 8.5.

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Next, take general divisors in the linear system $|m \cdot K_X|$, for *m* sufficiently large, and cut down to a surface. To be precise, observe the following.

Choosing a sufficiently increasing and divisible sequence of numbers $0 \ll m_1 \ll \cdots \ll m_{n-2}$ and a general tuple of elements $(H_1, \ldots, H_{n-2}) \in \prod_i |m_i \cdot K_X|$ the following will hold when we set $S := H_1 \cap \cdots \cap H_{n-2}$.

- (8.5.1) The intersection S is a smooth surface, and entirely contained in X_{reg} ; this is because X is smooth in codimension two by assumption.
- (8.5.2) The restriction $(\mathscr{E}_X, \theta_X)|_S$ is stable with respect to $K_X|_S$, cf. the Restriction Theorem 5.22.
- (8.5.3) The natural morphism $\iota_* : \pi_1(S) \to \pi_1(X_{\text{reg}})$, induced by the inclusion $\iota : S \hookrightarrow X_{\text{reg}}$, is isomorphic, cf. Goresky-MacPherson's Lefschetz-theorem [15, Thm. in Sect. II.1.2].
- (8.5.4) Let $\mathscr{F} \in B_r$. Then, \mathscr{F} is isomorphic to $\mathscr{E}_{Nd} \mathscr{E}_X$ if and only if the restrictions $\mathscr{F}|_S$ and $(\mathscr{E}_{Nd} \mathscr{E}_X)|_S$ are isomorphic, cf. the Bertini-type theorem for isomorphism classes in bounded families [17, Cor. 5.3].

REMARK 8.6. – The natural morphism $\pi_1(X_{reg}) \rightarrow \pi_1(X)$ is surjective, [13, 0.7.B on p. 33], and induces an isomorphism of profinite completions by assumption. Composed with the inclusion $S \hookrightarrow X_{reg}$, it follows from (8.5.3) that the morphism $\pi_1(S) \rightarrow \pi_1(X)$ is surjective and induces an isomorphism of profinite completions.

Step 2. The endomorphism bundle. – Since S is entirely contained in the smooth locus of X, the restricted Higgs sheaf $(\mathscr{E}_X, \theta_X)|_S$ is actually a Higgs bundle, and Construction 4.5 allows to equip the corresponding endomorphism bundle with a Higgs field. For brevity of notation, write $(\mathscr{F}_S, \Theta_S) := \mathcal{E}nd((\mathscr{E}_X, \theta_X)|_S)$. The rank of \mathscr{F}_S equals $r = (n + 1)^2$.

CLAIM 8.7. – The Higgs bundle (\mathscr{F}_S, Θ_S) is induced by a p $\mathbb{C}VHS$, in the sense of Definition 5.32.

Proof of Claim 8.7. – We need to check the properties listed in Theorem 5.34.

Item (5.34.1): polystability with respect to $K_{X|_S}$. By Theorem 5.22, we know that both $(\mathscr{E}_X, \theta_X)|_S$ and its dual are $K_{X|_S}$ -stable Higgs bundles on the smooth surface S. In particular, it follows from [58, Thm. 1(2)] that both bundles carry a Hermitian-Yang-Mills metric with respect to $K_{X|_S}$, and thus so does $(\mathscr{F}_S, \Theta_S)$. Hence it follows from [58, Thm. 1] that $(\mathscr{F}_S, \Theta_S)$ is polystable with respect to $K_{X|_S}$.

Item (5.34.2): vanishing of Chern classes. As the endomorphism bundle of the locally free sheaf $\mathscr{E}_{X|_S}$, the first Chern class of \mathscr{F}_S clearly vanishes. Vanishing of $c_2(\mathscr{F}_S)$ is then an immediate consequence of the assumed equality in (1.1.1). Together with polystability, this implies that \mathscr{F}_S is flat, [58, Thm. 1], and hence all its Chern classes vanish.

Item (5.34.3): the Higgs bundle $(\mathscr{E}_X, \theta_X)|_S$ has the structure of a system of Hodge bundles, [58, Sect. 4]. Its isomorphism class is therefore fixed under the action of \mathbb{C}^* , [58, p. 45]. Observing that the same holds for its dual and its endomorphism bundle, this ends the proof of Claim 8.7.

Step 3. End of proof. – Since S is entirely contained in the smooth locus of X, it is canonically isomorphic to its preimage $\tilde{S} := \pi^{-1}(S)$ in the resolution \tilde{X} . Let $(\mathscr{F}_{\tilde{S}}, \Theta_{\tilde{S}})$ be the Higgs bundle on \tilde{S} that corresponds to $(\mathscr{F}_{S}, \Theta_{S})$ under this isomorphism.

There exists a \mathbb{Q} -divisor $E \in \mathbb{Q}$ Div (\tilde{X}) , supported entirely on the π -exceptional locus, such that $\tilde{H} := \pi^*(K_X) + E$ is ample. Since \tilde{S} and supp E are disjoint, the Higgs bundle $(\mathscr{F}_{\tilde{S}}, \Theta_{\tilde{S}})$ is clearly semistable with respect to \tilde{H} .

Recall from [60, Thm. 1.1] that the natural map of fundamental groups, $\pi_1(\tilde{X}) \to \pi_1(X)$ is isomorphic. Together with Remark 8.6, this implies that $\pi_1(\tilde{S}) \to \pi_1(\tilde{X})$ is surjective, and induces an isomorphism of profinite completions. Item (5.36.2) of Corollary 5.36 therefore allows to find a Higgs bundle $(\mathscr{F}_{\tilde{X}}, \Theta_{\tilde{X}})$ on \tilde{X} that restricts to $(\mathscr{F}_{\tilde{S}}, \Theta_{\tilde{S}})$, and is hence induced by a pCVHS due to Corollary 5.36, Item (5.36.1). We have seen in Remark 5.35 that all Chern classes of $\mathscr{F}_{\tilde{X}}$ vanish.

Corollary 5.39 implies that $\mathscr{F}_{\widetilde{X}}$ comes from X. More precisely, there exists a locally free sheaf \mathscr{F}_X on X such that $\mathscr{F}_{\widetilde{X}} = \pi^* \mathscr{F}_X$. The restriction $\mathscr{F}_X|_S$ agrees with $\mathscr{F}_S = \mathscr{E}_{nd} \mathscr{E}_X|_S$, which together with the observation on the Chern classes of $\mathscr{F}_{\widetilde{X}}$ made above implies that \mathscr{F}_X is a member of the family B_r that was introduced in Claim 8.5. Item (8.5.4) thus gives an isomorphism $\mathscr{E}_{nd} \mathscr{E}_X \cong \mathscr{F}_X$, showing that $\mathscr{E}_{nd} \mathscr{E}_X$ is locally free. But $\mathscr{E}_{nd} \mathscr{E}_X$ contains \mathscr{T}_X as a direct summand. It follows that \mathscr{T}_X is locally free and thus X is smooth by the solution of the Zariski-Lipman problem for klt spaces, [16, Thm. 6.1].

8.3. Proof of Theorem 8.1

By Proposition 8.2 we know that the variety Z is smooth in codimension two. Now, let $\gamma: Y \to Z$ be a quasi-étale, Galois cover such that $\hat{\pi}_1(Y_{\text{reg}}) \cong \hat{\pi}_1(Y)$. By [17, Thm. 1.14], such a cover exists. Since γ branches only over the singular set of Z, it follows from [41, Prop. 5.20] that Y is still klt and smooth in codimension two. Since γ is finite, the Q-Cartier divisor $K_Y = \gamma^* K_Z$ is still ample. Moreover, as both $f^{[*]}\mathcal{T}_X$ and \mathcal{T}_Y are reflexive and agree on the big open set of Y where γ is étale, we conclude that $f^{[*]}\mathcal{T}_X = \mathcal{T}_Y$. Consequently, Lemma 3.16 guarantees that equality holds in the Q-Miyaoka-Yau Inequality for Y. Proposition 8.3 hence applies and Y is smooth. We may thus use the classical uniformisation theorem of Yau [66, Rem. (iii) on p. 1799] to conclude that Y is a ball quotient, as claimed.

REMARK 8.8 (Comparison with the torus-quotient case). – Let us now briefly explain the difference between the above strategy and those that appear in the proof of the uniformisation theorem in the case of vanishing Chern classes, [17] and [46]. For simplicity, we assume that X is smooth in codimension two with only klt singularities and that $\hat{\pi}_1(X_{reg}) \cong \hat{\pi}_1(X)$.

In the setting where $c_1(X) \cdot H^{n-1} = 0$ and $c_2(X) \cdot H^{n-2} = 0$ for some ample divisor *H*, one uses the (slope) semistability of $\mathcal{T}_{X|_S}$, where *S* is a sufficiently general, complete intersection, smooth, projective surface determined by *H*, to construct a (holomorphic) flat connection

$$\nabla: \mathscr{T}_{X|_{S}} \to \Omega^{1}_{S} \otimes \mathscr{T}_{X|_{S}}.$$

Here ∇ is compatible with the holomorphic structure of $\mathscr{T}_{X|_S}$, that is $\nabla^{0,1}$ is defined by the holomorphic structure of $\mathscr{T}_{X|_S}$. After extending the linear representation $\pi_1(S)$ corresponding to ∇ to a representation of $\pi_1(X)$, one can construct a flat locally free analytic sheaf \mathscr{F} on X verifying the isomorphism $\mathscr{F}|_S \cong \mathscr{T}_{X|_S}$, as analytic sheaves.

On the other hand, when K_X is ample and the equality in the Miyaoka-Yau inequality is attained, the holomorphic structure of the harmonic bundle $\mathcal{E}nd(\mathcal{E}_{X|_S}, \theta_{X|_S})$ defined above is different from the one given by the representation of $\pi_1(S)$ associated to the underlying flat connection. This is simply because the (0, 1) part of the HYM connection on $\mathcal{E}nd(\mathcal{E}_{X|_S}, \theta_{X|_S})$ is of the form

$$\bar{\partial} + (\theta_{X|s})^h,$$

where $\bar{\partial}$ is the holomorphic structure of $\mathcal{E}nd(\mathcal{E}_{X|S})$, *h* is the harmonic metric and $(\theta_{X|S})^h$ is the conjugate of the Higgs field $\theta_{X|S}$ with respect to *h*. As a result, the argument in the torus-quotient case breaks down: If one naively extends the representation of $\pi_1(S)$ defined by $\mathcal{E}nd(\mathcal{E}_{X|S})$ to a representation ρ of $\pi_1(X)$, the flat analytic sheaf \mathscr{F} constructed from ρ does not satisfy the isomorphism $\mathcal{E}nd(\mathcal{E})|_S \cong \mathscr{F}|_S$, as analytic sheaves; the holomorphic structures are simply not compatible.

9. Characterisation of singular ball quotients

In this section, we prove Theorem 1.3 and Corollary 1.4, and concerning optimality of our results discuss an example of a singular ball quotient in Section 9.4. First, we recall a few standard definitions and elementary properties. Throughout the present section, all complex spaces will be reduced and are assumed to have a countable basis of topology.

DEFINITION 9.1 (Properly discontinuous action). – Let X be a complex space, and Γ a group of holomorphic automorphisms of X. We say that Γ acts properly discontinuously on X, if for any points $x, y \in X$, there exist neighborhoods U = U(x) and V = V(y) such that the set $\{g \in \Gamma \mid g \cdot U \cap V \neq \emptyset\} \subset \Gamma$ is finite.

REMARK 9.2. – Note that there exist several, not necessarily equivalent definitions of "properly discontinuous" in the literature, especially in a purely topological context. We follow [64, Sect. 2.1], where the terminology "discrete group of transformations" is used for the same concept. A further general reference is [45, Chap. 12].

LEMMA 9.3 (Criteria for actions to be properly discontinous). – Let Γ be a subgroup of Aut_{\mathcal{O}}(\mathbb{B}^n) = PSU(1, n). Then, the following statements are equivalent.

- (9.3.1) The group Γ acts properly discontinuously on \mathbb{B}^n .
- (9.3.2) The group Γ is discrete in PSU(1, n).
- (9.3.3) Every Γ -orbit in \mathbb{B}^n is a discrete subset of \mathbb{B}^n , and for every $z \in \mathbb{B}^n$ the isotropy group $\Gamma_z = \{\gamma \in \Gamma \mid \gamma \cdot z = z\}$ is finite.

Proof. – This is classical, see for example [64, Sect. 2.1], or [30, Sect. 2.2] for the proto-typical case n = 1.

9.1. Proof of Theorem 1.3

We will prove the implications $(1.3.2) \Rightarrow (1.3.3) \Rightarrow (1.3.1) \Rightarrow (1.3.2)$ separately.

 $(1.3.2) \Rightarrow (1.3.3)$. – As G is a finite group, and as Y is projective and smooth, X is projective. Moreover, it follows from the assumptions on the G-action that $f : Y \to X$ is quasi-étale. This implies that K_X is Q-Cartier, that X is klt, and that $K_Y = f^*K_X$. Moreover, by the same argument as in the first paragraph of Section 8.3 we have $\mathscr{T}_Y = f^{[*]}\mathscr{T}_X$.

Now, recall that K_Y is ample, and that the Chern classes of \mathscr{T}_Y satisfy the Miyaoka-Yau equality, see e.g., [40, (8.8.3)]. It follows that K_X is ample. The Q-Miyaoka-Yau equality for \mathscr{T}_X then follows from Lemma 3.16.

 $(1.3.3) \Rightarrow (1.3.1)$. – Let $f: Y \to X$ be the finite, Galois, quasi-étale morphism from a ball quotient Y to X guaranteed by Theorem 1.2. Let G be the Galois group of $f: Y \to X$ and define $\tilde{\pi} : \mathbb{B}^n \to X$ as $\tilde{\pi} = f \circ \pi$, where $\pi : \mathbb{B}^n \to Y$ is the universal cover of Y^{an} . Let $\Gamma := \pi_1(Y^{an})$ be the deck transformation group of π . Then, the restriction of $\tilde{\pi}$ to $\tilde{U} := \tilde{\pi}^{-1}(X_{\text{reg}}^{an})$ is a topological covering map, which we call $\tilde{\pi}_{\text{reg}}$. Additionally, as the codimension of \tilde{U} in the manifold \mathbb{B}^n is more than two, \tilde{U} is simply connected. Consequently, $\pi_{\text{reg}} := \pi|_{\tilde{U}} : \tilde{U} \to f^{-1}(X_{\text{reg}}^{an})$ and $\tilde{\pi}_{\text{reg}}$ are universal covering maps. It follows that $\hat{\Gamma} = \pi_1(X_{\text{reg}}^{an})$ acts on \tilde{U} by holomorphic automorphisms, and the action is properly discontinuous and fixed-point free. As $\Gamma = \pi_1(Y^{an}) = \pi_1(f^{-1}(X_{\text{reg}})^{an})$, and since $f^{-1}(X_{\text{reg}})/G = X_{\text{reg}}$, we have an exact sequence of groups

$$(9.4.1) 1 \to \Gamma \to \widehat{\Gamma} \to G \to 1,$$

and the action of $\hat{\Gamma}$ on \tilde{U} extends the action of Γ on \tilde{U} . Our situation can hence be summarized in the following commutative diagram,

(9.4.2)
$$\widetilde{U} \xrightarrow{\pi_{\text{reg}}, \text{ quot. by } \widehat{\Gamma}} f^{-1}(X_{\text{reg}})^{\text{an}} \xrightarrow{\text{quot. by } G} X_{\text{reg}}^{\text{an}} \xrightarrow{f^{\text{an}}, \text{ quot. by } G} \widetilde{U} \xrightarrow{f^{\text{an}}, \text{ quot. by } G} X_{\text{reg}}^{\text{an}}$$

As the inclusion $\tilde{U} \hookrightarrow \mathbb{B}^n$ realizes \mathbb{B}^n as the envelope of holomorphy of \tilde{U} , the action of $\hat{\Gamma}$ on \tilde{U} uniquely extends to a holomorphic action of $\hat{\Gamma}$ on \mathbb{B}^n , see [52, Lem. 4.1]. This extended action is fixed-point free in codimension two by construction. It now follows from the exact Sequence (9.4.1) and from Diagram (9.4.2) that the topological quotient $\mathbb{B}^n/\hat{\Gamma} \simeq (\mathbb{B}^n/\Gamma)/G$ is homeomorphic to X^{an} , and therefore Hausdorff. As \mathbb{B}^n and X^{an} are both normal complex spaces, and as we already know that X^{an}_{reg} is biholomorphic to $\tilde{U}/\hat{\Gamma}$, [28, Satz on p. 328] hence implies that $\mathbb{B}^n/\hat{\Gamma}$ is in a natural way a normal complex space, which is in fact biholomorphic to X^{an} . In particular, $\tilde{\pi} : \mathbb{B}^n \to X^{an}$ is the quotient map for the $\hat{\Gamma}$ -action. To conclude the proof, we will show that this action is properly discontinuous.

As $\tilde{\pi}$ is holomorphic, for every $z \in \mathbb{B}^n$ the fiber $\tilde{\pi}^{-1}(\tilde{\pi}(z)) = \hat{\Gamma} \cdot z$ is a zero-dimensional analytic, and hence discrete, subset of \mathbb{B}^n . Moreover, we claim that all isotropy groups $\hat{\Gamma}_z$ of points $z \in \mathbb{B}^n$ are finite. From this, it will follow that the $\hat{\Gamma}$ -action is properly discontinuous, see Lemma 9.3. So, suppose that there is a point $z_0 \in \mathbb{B}^n$ such that Γ_{z_0} is infinite. As the isotropy of z_0 in the full automorphism group PSU(1, n) is compact, $\hat{\Gamma}_{z_0}$ is not a discrete subgroup of PSU(1, n), i.e., there exists a sequence of elements $\gamma_n \in \hat{\Gamma}_{z_0}$ converging to the

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identity element, cf. [64, p. 7]. Now, if z_1 is any point in \tilde{U} , where the $\hat{\Gamma}$ -action is free, it follows that $\hat{\Gamma} \cdot z_1 = \tilde{\pi}_{reg}^{-1}(\tilde{\pi}_{reg}(z_1))$ is not discrete, a contradiction.

 $(1.3.1) \Rightarrow (1.3.2).$ – Recall that compact quotients of \mathbb{B}^n by discrete subgroups of PSU(1, n) are projective algebraic, see e.g., [2]. Let $\hat{\pi} : \mathbb{B}^n \to X = \mathbb{B}^n/\hat{\Gamma}$ be the quotient map. As the action of $\hat{\Gamma}$ is fixed-point free in codimension two, the restriction $\hat{\pi}|_{\hat{\pi}^{-1}(X_{\text{sing}})}$ is unramified and hence a topological covering map. Moreover, the preimage $\hat{\pi}^{-1}(X_{\text{sing}})$ has complement of complex codimension at least three in the smooth manifold \mathbb{B}^n , and is therefore simply connected. As X_{reg} is (the complex space associated with) a quasi-projective algebraic variety, its fundamental group, which is isomorphic to $\hat{\Gamma}$, is finitely generated. It therefore follows from Selberg's Lemma, e.g., see [1], that $\hat{\Gamma}$ has a normal subgroup Γ of finite index that acts without fixed points on \mathbb{B}^n . From this, we obtain the following factorisation of the $\hat{\Gamma}$ -quotient map:

$$\mathbb{B}^n \longrightarrow \mathbb{B}^n / \Gamma \xrightarrow{f} \mathbb{B}^n / \widehat{\Gamma} = X.$$

Here, f is the quotient for the action of the finite group $G := \hat{\Gamma} / \Gamma$ on the projective manifold $Y := \mathbb{B}^n / \Gamma$, which by the assumption on the $\hat{\Gamma}$ -action is fixed-point free in codimension two. It follows that f is quasi-étale.

9.2. Proof of Corollary 1.4

If X is a singular ball quotient, let $\pi : \mathbb{B}^n \to X$ be the quotient map for the corresponding discrete group action. Then,

$$\pi|_{\pi^{-1}(X_{\mathrm{reg}})} : \pi^{-1}(X_{\mathrm{reg}}) \to X_{\mathrm{reg}}$$

is an unramified covering map. By [37, Prop. 3.2.2(1)] the manifold $\pi^{-1}(X_{\text{reg}}) \subset \mathbb{B}^n$ is Kobayashi-hyperbolic, as it is contained in the *n*-dimensional polydisk $\mathbb{D} \times \cdots \times \mathbb{D}$, which is Kobayashi-hyperbolic by [37, Prop. 3.2.3]. Hence, the statement follows from [37, Thm. 3.2.8(2)].

9.3. Further comments on Corollary 1.4

Let π : $\mathbb{B}^n \to X$ be a singular ball quotient. Then, X has slightly more general hyperbolicity properties than those stated in Corollary 1.4, as we will explain now. Let $d_{\mathbb{B}^n} : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{R}^{\geq 0}$ be the Kobayashi distance on the ball. Then, we can define a natural distance on X as follows: if $p, q \in X$, and if $\tilde{p} \in \mathbb{B}^n$ satisfies $\pi(\tilde{p}) = p$, we set

$$d'_X := \inf_{\widetilde{q}} d_{\mathbb{B}^n}(\widetilde{p}, \widetilde{q}),$$

where the infimum runs over all points $\tilde{q} \in \mathbb{B}^n$ such that $\pi(\tilde{q}) = q$. In fact, analogous to the Kobayashi pseudodistance, d'_X can be defined using chains of locally liftable holomorphic maps from the unit disk $\mathbb{D} \subset \mathbb{C}$ to X, see [38, p. 101]. Here, a holomorphic map f from a complex space Z into X is called *locally liftable* if every point $z \in Z$ has an analytically open neighborhood U such that $f|_U$ factors via π . As \mathbb{B}^n is Kobayashi-hyperbolic, d'_X is indeed a distance, see [38, Chap. VII, Prop. 6.3]. It follows that every locally liftable holomorphic map from \mathbb{C} to X is constant. This property does not imply that X is Kobayashi-hyperbolic, see the subsequent subsection for an example. However, many of the properties known for

holomorphic maps into Kobayashi-hyperbolic manifolds hold for locally liftable holomorphic maps into X. At this time, we are not aware of any singular ball quotient with *canonical* singularities that fails to be Kobayashi-hyperbolic.

9.4. Keum's singular ball quotient

The following example illustrates three points:

- (9.5.1) The fundamental group of singular ball quotients might be trivial.
- (9.5.2) Kobayashi-hyperbolicity in general will not extend over klt singularities.
- (9.5.3) The resolution of a klt singular ball quotient X might have a geometry that is very different from X.

Keum found a two-dimensional ball quotient Y together with an order 7 automorphism g that acts with isolated fixed points on Y such that the minimal resolution $\pi : \tilde{X} \to X$ of the quotient $X = Y/\langle g \rangle$ is simply connected, of Kodaira dimension one, and admits an elliptic fibration $\eta : \tilde{X} \to C$, see [34, Thm. 1.1(2) and Prop. 2.4] and [33]. The general fiber F of η is an elliptic curve. Composing the universal covering map $f : \mathbb{C} \to F$ with π yields a non-constant (not locally liftable) holomorphic map from \mathbb{C} to X^{an} , which is therefore not Kobayashi-hyperbolic. On the other hand, note that the smooth locus X^{an}_{reg} is hyperbolic by the proof of Corollary 1.4. As \tilde{X} is simply connected and as the singularities of X are klt, X itself is also simply connected, while its smooth locus has infinite fundamental group. Note that the singularities of X are worse than canonical, as the resolution \tilde{X} is not of general type. Hence, X is certainly not the minimal model of any smooth projective variety of general type.

10. Further directions

10.1. The general klt case

While we established the Miyaoka-Yau inequality, Theorem 1.1, for a general klt variety with big and nef canonical divisor, the uniformisation theorem, Theorem 1.2, assumes the variety to be smooth in codimension two.⁽⁴⁾ This is used crucially in its proof when we construct a complex variation of Hodge structures on the restriction of the natural Higgs sheaf to the *smooth* complete intersection surface and subsequently extend this variation of Hodge structure to a strong resolution of singularities, into which the complete intersection surface embeds.

The authors are currently working towards removing this additional assumption by comparing the Nonabelian Hodge correspondence on a resolution of a given klt variety X to Mochizuki's Nonabelian Hodge correspondence, [48], on the smooth part of X. An alternative approach consists in proving that a generalization of the Nonabelian Hodge correspondence to the case of smooth projective DM-stacks, as discussed in [55], applies to the restriction of the canonical Higgs sheaf to a complete intersection surface, and then again in comparing with a resolution.

⁽⁴⁾ Note added in proof: In the meantime, the corresponding paper has appeared, see [19].

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10.2. The case of klt pairs

In [25, Thm. B], partially building upon the foundational work done in the present paper, the following generalization of Theorem 1.1 has been proved.

THEOREM 10.1. – Let (X, D) be a projective, klt pair, where D is of the form $D := \sum (1 - \frac{1}{a_i})D_i$ for prime divisors D_{\bullet} and positive integers a_{\bullet} . Assume that $K_X + D$ is ample. Writing \check{c}_1, \check{c}_2 for the relevant orbifold Chern classes for pairs, [25, Sect. 2], the following "Miyaoka-Yau inequality" is satisfied,

$$(2(n+1) \cdot \check{c}_2(X, D) - n \cdot \check{c}_1^2(X, D)) \cdot [K_X + D]^{n-2} \ge 0.$$

Based on this result, one expects the following generalization of our result on uniformisation, Theorem 1.2.

EXPECTATION. – In the setting of Theorem 10.1, if the equality is achieved, then X has only quotient singularities and (X, D) admits a (global) smooth Deligne-Mumford stack structure \mathcal{X} whose universal cover (in the sense of Deligne-Mumford stacks) is the ball.

We note that this in particular implies that the local isotropy groups of \mathcal{X} act trivially in codimension one with the exception of the divisors D_i along which the isotropy groups are isomorphic to $\mathbb{Z}/(a_i\mathbb{Z})$. We conclude by pointing out that the Expectation was confirmed for surface pairs (X, D) by Kobayashi, Nakamura, and Sakai in [36, Thm. 12].

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