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PINNING AND DISORDER RELEVANCE FOR THE LATTICE GAUSSIAN FREE FIELD II: THE TWO DIMENSIONAL CASE

BY HUBERT LACOIN

ABSTRACT. – This paper continues a study initiated in [35], on the localization transition of a lattice free field on $\mathbb{Z}^d$ interacting with a quenched disordered substrate that acts on the interface when its height is close to zero. The substrate has the tendency to localize or repel the interface at different sites. A transition takes place when the average pinning potential h goes past a threshold h_c: this critical value separates a delocalized phase $h < h_c$, where the field is macroscopically repelled by the substrate from a localized one $h > h_c$ where the field sticks to the substrate. Our goal is to investigate the effect of the presence of disorder on this phase transition. We focus on the two dimensional case ($d = 2$) for which we had obtained so far only limited results. We prove that the value of $h_c(\beta)$ is the same as for the annealed model, for all values of the disorder intensity β. Moreover we prove that, in contrast with the case $d \geq 3$ where the free energy has a quadratic behavior near the critical point, the phase transition is of infinite order

$$\lim_{u \to 0+} \frac{\log \mathrm{F}(\beta, h_c(\beta) + u)}{(\log u)} = \infty.$$

An analogous result is presented for the two dimensional co-membrane model.

RÉSUMÉ. – Cet article approfondit l'étude (commencée dans [35]) de la transition de localisation pour un champ libre gaussien défini sur le réseau $\mathbb{Z}^d$ en interaction avec un substrat désordonné qui affecte les points situés proches de la hauteur zéro. Le substrat peut avoir un effet attracteur ou répulsif selon le site considéré. Une transition a lieu lorsque le potentiel moyen d'interaction h dépasse un certain seuil h_c: cette valeur critique définit une phase délocalisée $h < h_c$, au sein de laquelle le champ est globalement repoussé par le substrat, et une phase localisée $h > h_c$ ou le champ adhère au substrat. Notre objectif est d'évaluer les effets de la présence de désordre pour cette transition de phase. Nous nous concentrons sur le cas bi-dimensionnel ($d = 2$), et démontrons que la valeur du point critique $h_c(\beta)$ coincide avec celle du modèle moyenné (ou *annealed*), et ce quelle que soit la valeur de l'intensité du désordre β. De plus, nous démontrons que, contrairement au cas $d \geq 3$ pour lequel l'énergie libre a un comportement quadratique au voisinage du point critique, la transition de phase est ici d'ordre infini

$$\lim_{u \to 0+} \frac{\log \mathrm{F}(\beta, h_c(\beta) + u)}{(\log u)} = \infty.$$

Un résultat analogue est exposé pour le modèle de co-membrane bi-dimensionnelle.

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1. Introduction

The aim of statistical mechanics is to obtain a qualitative understanding of natural phenomena of phase transitions by the study of simplified models, often built on a lattice. In general the Hamiltonian of a model of statistical mechanics is left invariant by the lattice symmetries: a prototypical example being the Ising model describing a ferromagnet.

However, one might argue that materials which are found in nature are usually not completely homogeneous and for this reason, physicists were led to considering systems in which the interaction terms, for example the potentials between nearest neighbor spins, are chosen by sampling a random field—which we call *disorder*—with good ergodic properties, often a field of independent identically distributed random variables. An important question which arises is thus whether the results concerning the phase transition obtained for a model with homogeneous interactions referred to as *the pure system* (e.g., Onsager's solution of the two dimensional Ising Model [47]) remain valid when a system where randomness of a very small amplitude is introduced.

In [39] A. B. Harris gave a strikingly simple heuristical argument, based on renormalization theory consideration, to predict the effect of the introduction of a small amount of the system: in substance Harris' criterion predicts that if the phase transition of the pure system is sufficiently smooth, it will not be affected by small perturbation (disorder is then said to be *irrelevant*), while in the other cases the behavior of the system is affected by an arbitrary small addition of randomness (disorder is *relevant*). To be complete, let us mention also the existence of a boundary case for which the criterion yields no prediction (the *marginal disorder* case). The criterion however does not give a precise prediction concerning the nature of the phase transition when the disorder is relevant.

The mathematical verification of the Harris criterion is a very challenging task in general. In the first place, it can only be considered for the few special models of statistical mechanics for which we have a rigorous understanding of the critical properties of the pure system. In the last twenty years this question has been addressed, first by theoretical physicists (see e.g., [27] and references therein) and then by mathematicians [4, 5, 3, 7, 26, 37, 36, 38, 42, 48] (see also [32, 33] for reviews) for a simple model of a 1-dimensional interface interacting with a substrate: for this model the interface is given by the graph of a random walk which takes random energy rewards when it touches a defect line. In this case, the pure system has the remarkable quality of being what physicists call *exactly solvable*, meaning that there exists an explicit expression for the free energy [29].

This model under consideration in the present paper can be seen as a high dimension generalization of the random walk pinning model. The random walk is replaced by a random field $\mathbb{Z}^d \to \mathbb{R}$, and the random energies are collected when the graph of the field is close to the hyper-plane $\mathbb{Z}^d \times \{0\}$. While the pure model is not exactly solvable in that case, it has been studied in details and the nature of the phase transition is well known [13, 15, 17, 19, 50].

On the other hand, the study of the disordered version of the model is much more recent [22, 23, 35, 34]. In [35], we gave a close to complete description of the free energy diagram of the disordered model when $d \geq 3$:

— We identified the value of the disordered critical point, which is shown to coincide with that of the associated annealed model, regardless of the amplitude of disorder.

— We proved that for Gaussian disorder, the behavior of the free energy close to h_c is quadratic, in contrast with the annealed model for which the transitition is of first order.

— In case of general disorder, we proved that the quadratic upper bound still holds, and found a polynomial lower bound with a different exponent.

Let us stress that the heuristic of our proof strongly suggests that the behavior of the free energy should be quadratic for a suitable large class of environments (those who satisfy a second moment assumption similar to (2.5)).

In the present paper, we choose to attack the case $d = 2$, for which only limited results were obtained so far. We have seen in the proof of the main result [35] that the critical behavior of the model is very much related to the extremal process of the field. The quadratic behavior of the free energy in [35, Theorem 2.2] comes from the fact that high level sets of the Gaussian free field for $d \geq 3$ look like a uniformly random set with a fixed density (see [21]). In dimension 2 however, the behavior of the extremal process is much more intricate, with a phenomenon of clustering in the level sets (see [11, 28, 24] or also [6] for a similar phenomenon for branching Brownian Motion). This yields results of a very different nature.

2. Model and results

Given Λ a finite subset of $\mathbb{Z}^d$, we let $\partial\Lambda$ denote the internal boundary of Λ, $\mathring{\Lambda}$ the set of interior points of Λ, and $\partial^-\Lambda$ the set of points which are adjacent to the boundary,

$$\begin{aligned} \partial\Lambda &:= \{x \in \Lambda : \exists y \notin \Lambda,\ x \sim y\}, \\ \mathring{\Lambda} &:= \Lambda \setminus \partial\Lambda, \\ \partial^-\Lambda &:= \{x \in \mathring{\Lambda} : \exists y \in \partial\Lambda,\ x \sim y\}. \end{aligned} \tag{2.1}$$

In general some of these sets could be empty, but throughout this work Λ is going to be a large square. Given $\widehat{\phi} : \mathbb{Z}^d \to \mathbb{R}$, we define $\mathbf{P}_\Lambda^{\widehat{\phi}}$ to be the law of the lattice Gaussian free field $\phi = (\phi_x)_{x\in\Lambda}$ with boundary condition $\widehat{\phi}$ on $\partial\Lambda$. The field ϕ is a random function from Λ to $\mathbb{R}$. It satisfies

$$\phi_x := \widehat{\phi}_x \quad \text{for every } x \in \partial\Lambda, \tag{2.2}$$

and the distribution of $(\phi_x)_{x\in\mathring{\Lambda}}$ is given by

$$\mathbf{P}_\Lambda^{\widehat{\phi}}(\mathrm{d}\phi) = \frac{1}{\mathcal{Z}_\Lambda^{\widehat{\phi}}} \exp\left(-\frac{1}{2} \sum_{\substack{(x,y)\in(\Lambda)^2\setminus(\partial\Lambda)^2 \\ x\sim y}} \frac{(\phi_x-\phi_y)^2}{2}\right) \prod_{x\in\mathring{\Lambda}} \mathrm{d}\phi_x, \tag{2.3}$$

where $\prod_{x\in\mathring{\Lambda}} \mathrm{d}\phi_x$ denotes the Lebesgue measure on $\mathbb{R}^{\mathring{\Lambda}}$ and

$$\mathcal{Z}_\Lambda^{\widehat{\phi}} := \int_{\mathbb{R}^{\mathring{\Lambda}}} \exp\left(-\frac{1}{2} \sum_{\substack{(x,y)\in(\Lambda)^2\setminus(\partial\Lambda)^2 \\ x\sim y}} \frac{(\phi_x-\phi_y)^2}{2}\right) \prod_{x\in\mathring{\Lambda}} \mathrm{d}\phi_x \tag{2.4}$$

(one of the two (1/2) factors is present to compensate the fact that the edges are counted twice in the sum, the other one being the one usually present for Gaussian densities). In what follows we consider the case

$$\Lambda = \Lambda_N := \{0, \dots, N\}^d,$$

for some $N \in \mathbb{N}$. Note that we have

$$\mathring{\Lambda}_N := \{1, \dots, N-1\}^d.$$

We also introduce the notation $\widetilde{\Lambda}_N := \{1, \dots, N\}^d$, and we simply write $\mathbf{P}_N^{\widehat{\phi}}$ for $\mathbf{P}_{\Lambda_N}^{\widehat{\phi}}$. We drop $\widehat{\phi}$ from our notation in the case where we consider zero boundary condition $\widehat{\phi} \equiv 0$.

We let $\omega = \{\omega_x\}_{x \in \mathbb{Z}^d}$ be the realization of a family of IID square integrable centered random variables (of law $\mathbb{P}$). We assume that they have finite exponential moments, or more precisely, that there exist constants $\beta_0, \overline{\beta} \in (0, \infty]$ such that

$$\lambda(\beta) := \log \mathbb{E}[e^{\beta \omega_x}] < \infty \quad \text{for every } \beta \in (-\beta_0, 2\overline{\beta}]. \tag{2.5}$$

For $x \in \Lambda_N$, we set $\delta_x := \mathbf{1}_{[-1,1]}(\phi_x)$. For $\beta > 0$ and $h \in \mathbb{R}$, we define a modified measure $\mathbf{P}_{N,h}^{\beta,\omega,\widehat{\phi}}$ via the density

$$\frac{\mathrm{d}\mathbf{P}_{N,h}^{\beta,\omega,\widehat{\phi}}}{\mathrm{d}\mathbf{P}_N^{\widehat{\phi}}}(\phi) := \frac{1}{Z_{N,h}^{\beta,\omega,\widehat{\phi}}} \exp\left(\sum_{x \in \widetilde{\Lambda}_N} (\beta\omega_x - \lambda(\beta) + h)\delta_x \right), \tag{2.6}$$

where

$$Z_{N,h}^{\beta,\omega,\widehat{\phi}} := \mathbf{E}_N^{\widehat{\phi}} \left[\exp\left(\sum_{x \in \widetilde{\Lambda}_N} (\beta\omega_x - \lambda(\beta) + h)\delta_x \right) \right]. \tag{2.7}$$

Note that in the definition of $\mathbf{P}_{N,h}^{\beta,\omega,\widehat{\phi}}$, the sum $\left(\sum_{x \in \widetilde{\Lambda}_N}\right)$ can be replaced by either $\left(\sum_{x \in \Lambda_N}\right)$ or $\left(\sum_{x \in \mathring{\Lambda}_N}\right)$ as these changes affect only the partition function. In the case where $\widehat{\phi} \equiv 0$, we drop the corresponding superscript from the notation. In the special case where $\beta = 0$, we simply write $\mathbf{P}_{N,h}^{\widehat{\phi}}$ and $Z_{N,h}^{\widehat{\phi}}$ for the pinning measure and partition function (as they do not depend on ω) respectively. This case is referred to as the *pure* (or homogeneous) model. When $\beta > 0$, (2.6) defines the pinning model with *quenched* disorder.

2.1. The free energy

The important properties of the system are given by the asymptotic behavior of the partition function, or more precisely by the free energy. The existence of quenched free energy for the disordered model has been proved in [22, Theorem 2.1]. We recall this result here together with some basic properties

PROPOSITION 2.1. – *The free energy*

$$\mathrm{F}(\beta, h) := \lim_{N \to \infty} \frac{1}{N^d} \mathbb{E}\left[\log Z_{N,h}^{\beta,\omega}\right] \overset{\mathbb{P}(\mathrm{d}\omega)-a.s.}{=} \lim_{N \to \infty} \frac{1}{N^d} \log Z_{N,h}^{\beta,\omega}, \tag{2.8}$$

exists (and is self-averaging). It is a convex, nonnegative, nondecreasing function of h. *Moreover there exists a* $h_c(\beta) \in (0, \infty)$ *which is such that*

$$\mathrm{F}(\beta, h) \begin{cases} = 0 \text{ for } h \le h_c(\beta), \\ > 0 \text{ for } h > h_c(\beta). \end{cases} \tag{2.9}$$

Let us briefly explain why $h_c(\beta)$ marks a transition on the large scale behavior of ϕ under $\mathbf{P}^{\beta,\omega}_{N,h}$. A simple computation gives

$$\partial_h \left(\frac{1}{N^d} \log Z^{\beta,\omega}_{N,h} \right) = \frac{1}{N^d} \sum_{x \in \widetilde{\Lambda}_N} \mathbf{E}^{\beta,\omega}_{N,h} [\delta_x] . \tag{2.10}$$

Hence by convexity, we have

$$\partial_h \mathrm{F}(\beta, h) = \lim_{N \to \infty} \frac{1}{N^d} \sum_{x \in \widetilde{\Lambda}_N} \mathbf{E}^{\beta,\omega}_{N,h} [\delta_x] , \tag{2.11}$$

for the values of h at which $\mathrm{F}(\beta, h)$ differentiable (for the hypothetical countable set where $\partial_h \mathrm{F}(\beta, h)$ may not exist, we can replace lim by lim inf resp. lim sup, $=$ by $\le$ resp. $\ge$ and consider the left- resp. right-derivative in the above equation).

For $h > h_c(\beta)$, we have $\partial_h \mathrm{F}(\beta, h) > 0$ by convexity and thus the expected number of points in contact with the substrate is asymptotically of order N^d. On the contrary when $h < h_c(\beta)$, the asymptotic expected contact fraction vanishes when N tends to infinity.

Note that the whole model is perfectly defined for all $d \ge 1$. However, the case $d = 1$ is a variant of the random walk pinning model, which, as mentioned in the introduction, was the object of numerous studies in the literature. However, the effect of disorder in dimension 1 being quite different, in the remainder of this introduction, we present the results we obtain in the present paper for the case $d = 2$ and discuss how they compare with those obtained in the more related case $d \ge 3$ [35].

2.2. The pure model

In the case $\beta = 0$, we simply write $\mathrm{F}(h)$ for $\mathrm{F}(0, h)$. In that case the behavior of the free energy is known in details (see [22, Fact 2.4] and also [35, Section 2.3 and Remark 7.10] for a full proof for $d \ge 3$). We summarize it below.

PROPOSITION 2.2. – *For all* $d \ge 1$, *we have* $h_c(0) = 0$ *and moreover*

(i) *For* $d = 2$

$$\mathrm{F}(h) \overset{h \to 0+}{\sim} \frac{2\sqrt{2}h}{\sqrt{|\log h|}}, \tag{2.12}$$

(ii) *For* $d \ge 3$

$$\mathrm{F}(h) \overset{h \to 0+}{\sim} c_d h, \tag{2.13}$$

where $c_d := \mathbf{P}[\sigma_d \mathcal{N} \in [-1, 1]]$ *and* σ_d *is the standard deviation for the infinite volume free field in* $\mathbb{Z}^d$.

To be more precise $\sigma_d := \sqrt{G^0(x,x)}$ where G^0 is the Green function defined in (3.16). The result in dimension 2 is well known folklore to people in the fields, but as to our knowledge, no proof of it is available in the literature. For this reason we present a short one in Appendix C.

2.3. The quenched/annealed free energy comparison

Using Jensen's inequality, we can for every $\beta \geq 0$, compare the free energy to that of the annealed system, which is the one associated to the averaged partition function $\mathbb{E}\left[Z_{N,h}^{\beta,\omega}\right]$,

$$\textsc{f}(\beta,h) = \lim_{N\to\infty} \frac{1}{N^d}\mathbb{E}\left[\log Z_{N,h}^{\beta,\omega}\right] \leq \lim_{N\to\infty} \frac{1}{N^d}\log \mathbb{E}\left[Z_{N,h}^{\beta,\omega}\right]. \tag{2.14}$$

Our choice of parametrization implies that

$$\mathbb{E}\left[Z_{N,h}^{\beta,\omega}\right] = \mathbf{E}_N\left[\mathbb{E}\left[e^{\sum_{x\in\widetilde{\Lambda}_N}(\omega_x-\lambda(\beta)+h)\delta_x}\right]\right] = \mathbf{E}_N\left[e^{\sum_{x\in\widetilde{\Lambda}_N} h\delta_x}\right] = Z_{N,h}, \tag{2.15}$$

and thus for this reason we have

$$\textsc{f}(\beta,h) \leq \textsc{f}(h) \quad \text{and} \quad h_c(\beta) \geq 0. \tag{2.16}$$

It is known that the inequality (2.16) is strict: for $h > 0$, we have $\textsc{f}(\beta,h) < \textsc{f}(h)$ in all dimensions (cf. [22]). However we can ask ourselves if the behavior of the model with quenched disorder is similar to that of the annealed one in several other ways:

(a) Is the critical point of the quenched model equal to that of the annealed model (i.e., is $h_c(\beta) = 0$)?

(b) Do we have a critical exponent for the free energy transition: Do we have

$$\textsc{f}(\beta, h_c(\beta)+u) \overset{u\to 0+}{\sim} u^{\nu+o(1)},$$

and is ν equal to one, like for the annealed model (cf. Proposition 2.2)?

This question has been almost fully solved in the case $d \geq 3$. Let us display the result here

Theorem A ([35, Theorem 2.2]). – *For $d \geq 3$, for every $\beta \in (0,\bar{\beta}]$ we have:*

(i) $h_c(\beta) = 0$.

(ii) *If ω is Gaussian, there exist positive constants $c_1(\beta) < c_2(\beta)$ such that for all $h \in (0,1)$*

$$c_1(\beta)h^2 \leq \textsc{f}(\beta,h) \leq c_2(\beta)h^2. \tag{2.17}$$

(iii) *In the case of general ω, there exist positive constants $c_1(\beta) < c_2(\beta)$ such that for all $h \in (0,1)$*

$$c_1(\beta)h^{66d} \leq \textsc{f}(\beta,h) \leq c_2(\beta)h^2. \tag{2.18}$$

Remark 2.3. – *We strongly believe that the quadratic behavior holds for every ω as soon as $\lambda(2\beta) < \infty$, and the Gaussian assumption is mostly technical. However, if $\lambda(2\beta) = \infty$, we believe that the model is in a different universality class and the critical exponent depends on the tail of the distribution of the variable $\xi := e^{\beta\omega_0}$.*

The aim of the paper is to provide answers in the case of dimension 2.

2.4. The main result

We present now the main achievement of this paper. We prove that similarly to the $d \geq 3$ case, the critical point $h_c(\beta)$ coincides with the annealed one for every value of β (which is in contrast with the case $d = 1$ where the critical points differ for every $\beta > 0$ [37]). However, we are able to prove also that the critical behavior of the free energy is not quadratic, $\textsc{f}(\beta, h)$ being smaller than any power of h in a (positive) neighborhood of $h = 0$. This indicates that the phase transition is of infinite order.

THEOREM 2.4. – *When $d = 2$, for every $\beta \in (0, \bar{\beta}]$ the following holds*

(i) *We have $h_c(\beta) = 0$.*

(ii) *We have*

$$\lim_{h \to 0+} \frac{\log \textsc{f}(\beta, h)}{\log h} = \infty. \tag{2.19}$$

More precisely, there exists $h_0(\beta)$ such that for all $h \in (0, h_0(\beta))$

$$\exp\left(-h^{-20}\right) \leq \textsc{f}(\beta, h) \leq \exp\left(-|\log h|^{3/2}\right). \tag{2.20}$$

REMARK 2.5. – *We do not believe that either bound in* (2.20) *is sharp. However it seems to us that the strategy used for the lower bound is closer to capture the behavior of the field. Our educated guess for the true behavior of the free energy would be*

$$\textsc{f}(\beta, h) \approx \exp(h^{-1+o(1)}).$$

While a lower bound of this type might be achieved by optimizing the proof presented in the present paper (but this would require some significant technical work), we do not know how to obtain a significant improvement on the upper bound (see additional comments in Section 4.9).

2.5. Some intuition for the infinite order phase transition

The infinite exponent in (2.19) is in strong contrast with the exponent 2 witnessed in higher dimension. Let us try to expose the reasons for such a difference.

If one fixes the density of contact (we look at the measure conditioned to $\sum_{x \in \Lambda_N} \delta_x = \lceil \rho N^2 \rceil$ for a fixed value of ρ) the strategy of localization to maximize the energy will be very different in the two cases: In dimension $d \geq 3$ we believe (and partially proved) that the optimal behavior for the spatial field is to stabilize at some height in order to match the prescribed density, as the obtained set of contact looks like identically distributed Bernouilli with the desired density ρ, this yields a quadratic behavior for the free energy. In dimension 2, while we have not fully identified the strategy, we can see that it has to be different from the one used in higher dimension: the level sets of the lattice free-field have a natural tendency to form clusters, and because of this imitating a Bernouilli measure would have an entropic cost that would overcome the benefit.

The clusters in the set of contact points $\{x : \phi_x \in [-1, 1]\}$, have the heuristic effect of reducing the contact fraction: if C is cluster of contact points, the associated reward $\exp\left(\sum_{x \in C} [\beta\omega_x - \lambda(\beta) + h\right)$ is a random variable which has a positive expectation but variance which grows exponentially with the size of $|C|$, and thus the system tends to select the clustered contact points among rare favorable regions, and this makes the phase transition much smoother: the contact fraction is very small when h gets small and thus so is the free-energy.

2.6. Co-membrane models in two dimension

Like in [35], it is worthwhile to notice that the proof of the results of the present paper can be adapted to a model with a different localization mechanism. It is the analog of the model of a copolymer in the proximity of the interface between selective solvents, see [12, 20] and references therein. For this model given a realization of ω and two fixed parameters $\varrho, h > 0$, the measure is defined via the following density

$$\frac{\mathrm{d}\check{\mathbf{P}}^{\omega,\varrho}_{N,h}}{\mathrm{d}\mathbf{P}_N} \propto \exp\left(\varrho \sum_{x\in\widetilde{\Lambda}_N} (\omega_x + h)\,\mathrm{sign}\,(\phi_x)\right), \tag{2.21}$$

where we assume sign(0) $= +1$. A natural interpretation of the model is that the graph of $(\phi_x)_{x\in\Lambda_N}$ models a membrane lying between two solvents A and B which fill the upper and lower half-space respectively: for each point of the graph, the quantity $\omega_x + h$ describes the energetic preference for one solvent of the corresponding portion of the membrane (A if $\omega_x + h > 0$ and B if $\omega_x + h < 0$). As h is positive and ω_x is centered, there is, on average, a preference for solvent A (by symmetry this causes no loss of generality).

If $\mathbb{P}[(\omega_x < -h) > 0]$, there is a non trivial competition between energy and entropy: the interaction with the solvent gives an incentive for the field ϕ to stay close to the interface so that its sign can match as much as possible that of $\omega + h$, but such a strategy might be valid only if the energetic rewards it brings is superior to the entropic cost of the localization.

A more evident analogy with the pinning measure (2.6) can be made by observing that we can write

$$\frac{\mathrm{d}\check{\mathbf{P}}^{\omega,\varrho}_{N,h}}{\mathrm{d}\mathbf{P}_N} = \frac{1}{\check{Z}^{\omega,\varrho}_{N,h}} \exp\left(-2\varrho \sum_{x\in\widetilde{\Lambda}_N} (\omega_x + h)\Delta_x\right), \tag{2.22}$$

where $\Delta_x := (1-\mathrm{sign}(\phi_x))/2$, that is Δ_x is the indicator function that ϕ_x is in the lower half plane, and

$$\check{Z}^{\omega,\varrho}_{N,h} := \mathbf{P}_N\left[\exp\left(-2\varrho \sum_{x\in\widetilde{\Lambda}_N} (\omega_x + h)\Delta_x\right)\right]. \tag{2.23}$$

It is probably worth stressing that from (2.21) to (2.22) there is a non trivial (but rather simple) change in energy. In particular, the strict analog of Proposition 2.1 holds, with

$$\check{\mathrm{F}}(\varrho, h) := \lim_{N\to\infty} \frac{1}{N^2} \log \check{Z}^{\omega,\varrho}_{N,h} \geq 0, \tag{2.24}$$

where the limit is considered in the almost sure sense. We then set $\check{h}_c(\varrho) := \inf\{h > 0 : \check{\mathrm{F}}(\varrho, h) = 0\}$. Adapting the proof of Theorem 2.4 we can identify the value of $\check{h}_c(\varrho)$, as well as the critical behavior of the system.

THEOREM 2.6. – *For $d = 2$, for any ϱ such that $\lambda(-2\rho) < \infty$ we have*

$$\check{h}_c(\varrho) = \frac{1}{2\varrho}\lambda(-2\varrho). \tag{2.25}$$

Moreover we have

$$\exp\left(-u^{-20}\right) \le \check{\mathrm{F}}(\varrho, h_c(\varrho) - u) \le \exp\left(-|\log u|^{3/2}\right). \tag{2.26}$$

Note that while pure co-membrane model (i.e., with no disorder) displays a first order phase transition in h, the above result underlines that the transition becomes of infinite order in the presence of an arbitrary small quantity of disorder. This result differs both from the one obtained in dimension $d \ge 3$ (for which the transition is shown to be quadratic at least for Gaussian environment [35, Theorem 2.5]), and that in dimension 1: for the copolymer model based on renewals presented in [12], $\frac{1}{2\varrho}\lambda(-2\varrho)$ is in most cases a strict upper bound on $\check{h}_c(\varrho)$ (see e.g., the results in [49]).

The proof of Theorem 2.6 can be obtained by performing only a few modifications with the proof of Theorem 2.4: The reader can observe that although the proof of 2.4 is long most of the technical statements do not rely on the specific distribution of the set of contact. Thus δ_x can be replaced by Δ_x without any change in the proof for many intermediate lemmas. Some parts of the proof however, need more a substantial adaptation. All the details for the proof are provided in Appendix A.

2.7. Organization of the paper

The proof of the upper bound and of the lower bound on the free energy presented in Equation (2.20) are largely independent. However some general technical results concerning the covariance structure of the free field are useful in both proofs, and we present these results in Section 3 and prove most of them in Appendix B.

The proof of the upper bound is developed in Section 4. The proof of the lower bound spreads from Section 5 to Section 8. In Section 5 we present an estimate on the free energy in terms of a finite system with "stationary" boundary condition. In Section 6, we give a detailed sketch of the proof of the lower bound based on this finite volume criterion, divided into several steps. The details of these steps are covered in Section 7 and 8.

For the proof of both the upper and the lower bound, we need fine results on the structure of the free field. Although these results or their proof cannot directly be extracted from the existing literature, our proof (especially the techniques developed in Section 8) is largely based on tools that were developed in the numerous study on extrema and extremal processes of the two dimensional free field [14, 16, 24, 28] and other log-correlated Gaussian processes [1, 2, 6, 18, 46] (the list of references being far from being complete). In particular for the lower bound, we present an *ad-hoc* decomposition of the field in Section 6 and then exploit this decomposition to apply a conditioned second moment technique, similarly to what is done e.g., in [2].

For the upper bound, we also make use of a change of measure machinery inspired by a similar technique developed in the study of disordered pinning model [11, 26, 37, 36] and adapted successfully in other contexts [8, 9, 10, 43, 44, 51].

3. A toolbox

3.1. Notation and convention

Throughout the paper, to avoid a painful enumeration, we use C to denote an arbitrary constant which is not allowed to depend on the value of h or N nor on the realization of ω. Its value may change from one equation to another. For the sake of clarity, we try to write $C(\beta)$ when the constant may depend on β. When a constant has to be chosen small enough rather than large enough, we may use c instead of C.

For $x = (x_1, x_2) \in \mathbb{Z}^2$ we let $|x|$ denote the l_1 norm

$$|x| := |x_1| + |x_2|. \tag{3.1}$$

The notation $|\cdot|$ is also used to denote the cardinal of a finite set as this should yield no confusion.

If $A \subset \mathbb{Z}^2$ and $x \in \mathbb{Z}^2$ we set

$$d(x, A) := \min_{y \in A} |x - y|. \tag{3.2}$$

We use double brackets to denote an interval of integers, that for $i < j$ in $\mathbb{Z}$

$$[\![i, j]\!] := [i, j] \cap \mathbb{Z} = \{i, i+1, \dots, j\}. \tag{3.3}$$

If $(A_i)_{i=1}^k$ is a finite family of events, we refer to the following inequality as *the union bound*.

$$\mathbb{P}(\cup_{i=1}^k A_i) \leq \sum_{i=1}^k \mathbb{P}(A_i). \tag{3.4}$$

We let $(X_t)_{t \geq 0}$ denote continuous time simple random walk on $\mathbb{Z}^d$ whose generator Δ is the lattice Laplacian defined by

$$\Delta f(x) := \sum_{y \sim x} \big(f(y) - f(x) \big) \tag{3.5}$$

and we let P^x denote its law starting from $x \in \mathbb{Z}^d$. We use the notation P_t for the associated heat-kernel

$$P_t(x, y) = P^x(X_t = y). \tag{3.6}$$

If μ denotes a probability measure on a space Ω, and f a measurable function on Ω we denote the expectation of f by

$$\mu(f) = \int_\Omega f(\omega) \mu(\mathrm{d}\omega), \tag{3.7}$$

with an exception where the probability measure is denoted by the letter P: In that case E is used for expectations.

If $\mathcal{N}(\sigma)$ is a Gaussian of standard deviation σ, it is classical (and can directly be checked from the expression of the density) that

$$P\left[\mathcal{N}(\sigma) \geq u\right] \leq \frac{\sigma}{\sqrt{2\pi} u} e^{-\frac{u^2}{2\sigma^2}}. \tag{3.8}$$

We refer to the Gaussian tail bound when we use this inequality.

3.2. The massive free field

In this section we quickly recall the definition and some basic properties of the massive free field. Given $m > 0$, and a set $\Lambda \subset \mathbb{Z}^d$ and a function $\widehat{\phi}$, we define the law $\mathbf{P}_{\Lambda}^{m,\widehat{\phi}}$ of the massive free field on Λ with boundary condition $\widehat{\phi}$ and mass m as follows: it is absolutely continuous w.r.t $\mathbf{P}_{\Lambda}^{\widehat{\phi}}$ and

$$\frac{\mathrm{d}\mathbf{P}_{\Lambda}^{m,\widehat{\phi}}}{\mathrm{d}\mathbf{P}_{\Lambda}^{\widehat{\phi}}}(\phi) := \frac{1}{\mathbf{E}_{\Lambda}^{\widehat{\phi}}\left[\exp\left(-m^2 \sum_{x\in\mathring{\Lambda}} \phi_x^2\right)\right]} \exp\left(-m^2 \sum_{x\in\mathring{\Lambda}} \phi_x^2\right). \tag{3.9}$$

We let $\mathbf{P}_{N}^{m,\widehat{\phi}}$ denote the law of the massive field on Λ_N. (in the special case $\widehat{\phi}_x \equiv 0$, $\widehat{\phi}$ is omitted in the notation).

We let $\mathbf{P}^m$ denote the law of the centered infinite volume massive free field $\mathbb{Z}^d$, which is the limit of $\mathbf{P}_{\Lambda}^{m}$ when $\Lambda \to \mathbb{Z}^d$ (see Section 3.4 for a proper definition with the covariance function). We will in some cases have to choose the boundary condition $\widehat{\phi}$ itself to be random and distributed like an infinite volume centered massive free field (independent of ϕ), in which case we denote its law by $\widehat{\mathbf{P}}^m$ instead of $\mathbf{P}^m$.

Note that the free field and its massive version satisfy a Markov spatial property. In particular the law of $(\phi)_{x\in\Lambda_N}$ under $\widehat{\mathbf{P}}^m \times \mathbf{P}_{N}^{m,\widehat{\phi}}$ is the same as under the infinite volume measure $\mathbf{P}^m$.

3.3. Getting rid of the boundary condition

Even if the definition of the free energy given in Proposition 2.1 is made in terms of the partition function with $\widehat{\phi} \equiv 0$, it turns out that our methods to obtain upper and lower bounds involve considering non trivial boundary conditions (cf. Proposition 4.6 and Proposition 5.3).

However, it turns out to be more practical to work with a fixed law for the field and not one that depends on $\widehat{\phi}$. Fortunately, given a boundary condition $\widehat{\phi}$ the law of $\mathbf{P}_{N}^{m,\widehat{\phi}}$ can simply be obtained by translating the field with 0 boundary condition by a function that depends only on $\widehat{\phi}$. This is a classical property of the free field but let us state it in details. It only relies on rewriting the probability density with care, we refer to [31, Equation (2.9)-(2.10)] for a complete proof. We have

$$\mathbf{P}_{N}^{m,\widehat{\phi}}[\phi \in \cdot\,] = \mathbf{P}_{N}^{m}[\phi + H_{N}^{m,\widehat{\phi}} \in \cdot\,], \tag{3.10}$$

where

$$H_{N}^{m,\widehat{\phi}}(x) := \mathbf{E}_{N}^{m,\widehat{\phi}}[\phi(x)]. \tag{3.11}$$

In addition $H_{N}^{m,\widehat{\phi}}$ is a solution of the system (recall (3.5))

$$\begin{cases} H(x) = \widehat{\phi}(x), & x \in \partial\Lambda_N, \\ \Delta H(x) = m^2 H(x), & x \in \mathring{\Lambda}_N. \end{cases} \tag{3.12}$$

We simply write $H_N^{\widehat{\phi}}(x)$ when $m = 0$. The solution of (3.12) is unique and $H_N^{m,\widehat{\phi}}$ has the following representation: consider X_t the simple random walk on $\mathbb{Z}^d$ and for $A \subset \mathbb{Z}^d$ let τ_A denote the first hitting of A. We have

$$H_N^{m,\widehat{\phi}}(x) = E^x\left[e^{-m^2\tau_{\partial\Lambda_N}}\widehat{\phi}\left(X_{\tau_{\partial\Lambda_N}}\right)\right]. \tag{3.13}$$

Given $\widehat{\phi}$ and $x \in \widetilde{\Lambda}_N$, we introduce the notation

$$\delta_x^{\widehat{\phi}} := \mathbf{1}_{[-1,1]}(\phi_x + H_N^{\widehat{\phi}}(x)). \tag{3.14}$$

In view of (3.10) an alternative way of writing the partition function is

$$Z_{N,h}^{\beta,\omega,\widehat{\phi}} = \mathbf{E}_N\left[e^{\sum_{x\in\widetilde{\Lambda}_N}(\beta\omega_x-\lambda(\beta)+h)\delta_x^{\widehat{\phi}}}\right]. \tag{3.15}$$

In some situation the above expression turns out to be handier than the Definition (2.7).

3.4. Some estimates on Green functions and heat Kernels

In this section we present some estimates on the covariance function of the free field and massive free field in dimension 2, which will be useful in the course of the proof. These are not new results, but rather variants of existing estimates in the literature (see e.g [24, Lemma 2.1]).

The covariance kernel of the infinite volume free field with mass $m > 0$ in $\mathbb{Z}^2$ or $m \geq 0$ in Λ_N is given by the Green function G^m which is the inverse of $\Delta - m^2$ (this can in fact be taken as the definition of the infinite volume free field, requiring in addition that it is centered). The covariance function of the field under the measure $\mathbf{P}_N^m$ is $G^{m,*}$ which is the inverse of $\Delta - m^2$ with Dirichlet boundary condition on $\partial\Lambda_N$. Both of these functions can be represented as integral of the heat kernel (3.6), we have

$$\begin{aligned}\mathbf{E}^m[\phi(x)\phi(y)] &= \int_0^\infty e^{-m^2t}P_t(x,y)\mathrm{d}t =: G^m(x,y),\\ \mathbf{E}_N^m[\phi(x)\phi(y)] &= \int_0^\infty e^{-m^2t}P_t^*(x,y)\mathrm{d}t =: G^{m,*}(x,y),\end{aligned} \tag{3.16}$$

where P_t^* is the heat kernel on Λ_N with Dirichlet boundary condition on $\partial\Lambda_N$,

$$P_t^*(x,y) := P^x\left[X_t = y\ ;\ \tau_{\partial\Lambda_N} > t\right]. \tag{3.17}$$

Note that $P_t(x,\cdot)$ being a probability we have

$$\sum_{y\in\mathbb{Z}^2} G^m(x,y) = \int_0^\infty e^{-m^2t} = m^{-2}. \tag{3.18}$$

We simply write G^* in the case $m = 0$. Note that, because of the spatial Markov property (Section 3.2) and of (3.10), when $\widehat{\phi}$ has law $\widehat{\mathbf{P}}^m$ and ϕ has law $\mathbf{P}_N$, $(H_N^{m,\widehat{\phi}}(x) + \phi_x)_{x\in\Lambda_N}$ has the same law as the (marginal in Λ_N of the) infinite volume field. Hence as a consequence

$$\begin{aligned}\widehat{\mathbf{E}}^m\left[H_N^{m,\widehat{\phi}}(x)H_N^{m,\widehat{\phi}}(y)\right] &= G^m(x,y) - G^{m,*}(x,y)\\ &= \int_0^\infty e^{-m^2t}(P_t(x,y) - P_t^*(x,y))\mathrm{d}t.\end{aligned} \tag{3.19}$$

Before giving more involved estimates, let us mention first a quantitative version of the Local Central Limit Theorem (which can be deduced from [45, Theorem 2.1.1] stated for discrete time random walks) for the heat kernel which we use as an essential building brick to obtain them. There exists a constant C such that for all $t \geq 1$,

$$\left| P_t(x,x) - \frac{1}{4\pi t} \right| \leq \frac{C}{t^{3/2}}. \tag{3.20}$$

Let us recall the notation (3.2) for the distance between a set and a point. The following two lemmas are proved in Appendix B.

LEMMA 3.1. – *There exists a constant C such that:*

(i) *For all $m \leq 1$, for any $x \in \mathbb{Z}^2$*

$$\left| G^m(x,x) + \frac{1}{2\pi} \log m \right| \leq C. \tag{3.21}$$

(ii) *For all $m \leq 1$, for any $x \in \Lambda_N$*

$$\left| G_N^{m,*}(x,x) - \frac{1}{2\pi} \log \min(m^{-1}, d(x, \partial\Lambda_N)) \right| \leq C. \tag{3.22}$$

LEMMA 3.2. – *The following assertions hold*

(i) *There exists a constant C such that for all $t \geq 1$, $|x-y| \leq \sqrt{t}$, we have*

$$\begin{aligned} (P_t(x,x) - P_t(x,y)) &\leq \frac{C|x-y|^2}{t^2}, \\ \left(P_t^*(x,x) + P_t^*(y,y) - 2P_t^*(x,y)\right) &\leq \frac{C|x-y|^2}{t^2}. \end{aligned} \tag{3.23}$$

(ii) *There exists a constant C such that for all $t \geq 1$ and x, y satisfying $|x-y| \leq t$ we have*

$$P_t(x,y) \leq \begin{cases} \frac{C}{t} e^{-\frac{|x-y|^2}{Ct}}, & \text{for } |x-y| \leq t, \\ e^{-\frac{1}{C}|x-y| \log\left(\frac{|x-y|}{t}\right)} & \text{for } |x-y| \geq t, \end{cases} \tag{3.24}$$

and as a consequence

$$\sum_{y \in \mathbb{Z}^2} (G^m(x,y))^2 \leq C m^{-2}. \tag{3.25}$$

(iii) *We have for all x*

$$\frac{P_t^*(x,x)}{P_t(x,x)} \leq C \frac{[d(x, \partial\Lambda_N)]^2}{t}. \tag{3.26}$$

(iv) *We have for all x*

$$\begin{cases} P_t(x,x) - P_t^*(x,x) \leq \frac{C}{t} e^{-\frac{d(x,\partial\Lambda_N)^2}{Ct}}, & \text{for } t \geq d(x, \partial\Lambda_N), \\ P_t(x,x) - P_t^*(x,x) \leq e^{-\frac{1}{C} d(x,\partial\Lambda_N) \log\left(\frac{d(x,\partial\Lambda_N)}{t}\right)}, & \text{for } t \leq d(x, \partial\Lambda_N). \end{cases} \tag{3.27}$$

3.5. On the cost of positivity constraints for Gaussian random walks

Finally we conclude this preliminary section with an estimate for the probability to remain above a line for Gaussian random walks. The statement is not optimal and with some efforts the term $(\log k)$ could be replaced by 1 but as the rougher estimate is sufficient for our purpose we prefer to keep the proof (included in Appendix B) simpler.

LEMMA 3.3. – *Let $(X_i)_{i=1}^k$ be a random walk with independent centered Gaussian increments, each of which with variance bounded above by* 2 *and such that the total variance satisfies* $\mathrm{Var}(X_k) \geq k/2$. *Then we have for all $x \geq 0$*

$$1 - e^{-\frac{x^2}{k}} \leq \mathbf{P}\left[\max_i X_i \leq x \mid X_k = 0\right] \leq \frac{C(x + (\log k))^2}{k}. \tag{3.28}$$

4. The upper bound on the free energy

Let us briefly discuss the structure of the proof before going into more details. The main idea is presented in Section 4.2: we introduce a function which penalizes some environments ω which are too favorable, and use it to get a better annealed bound which penalizes the trajectories with clustered contact points in a small region (Proposition 4.5).

However, to perform the coarse-graining step of the proof, we need some kind of control on ϕ. For this reason, in Section 4.1 we start the proof by showing that restricting the partition function to a set of uniformly bounded trajectory affects the free energy only by a small amount.

4.1. Restricting the partition function

In this section, we show that restricting the partition function by limiting the maximal height of the field ϕ does not affect too much the free energy. This statement is to be used to control the boundary condition of each cell when performing a coarse-graining argument in Proposition 4.6. Let us set

$$\mathscr{A}_N^r := \{\forall x \in \Lambda_N,\ |\phi_x| \leq r\}, \tag{4.1}$$

and write

$$Z_{N,h}^{\beta,\omega}(\mathscr{A}_N^r) := \mathbf{E}_N\left[\exp\left(\sum_{x \in \widetilde{\Lambda}_N} (h + \beta\omega_x - \lambda(\beta))\delta_x\right)\mathbf{1}_{\mathscr{A}_N^r}\right]. \tag{4.2}$$

PROPOSITION 4.1. – *There exists a constant c such that for any $h > 0$ and $\beta > 0$ and $r \geq r_0$ sufficiently large, we have*

$$\liminf_{N\to\infty} \frac{1}{N^2}\mathbb{E}\log \mathbf{P}_{N,h}^{\beta,\omega}\left[\mathscr{A}_N^r\right] \geq -\exp(-cr). \tag{4.3}$$

As a consequence, we have

$$\textsc{f}(\beta, h) \leq \liminf_{N\to\infty} \frac{1}{N^2}\mathbb{E}\log Z_{N,h}^{\beta,\omega}(\mathscr{A}_N^r) + \exp(-cr). \tag{4.4}$$

Proof. – For practical purposes we introduce the two following events

$$
\begin{aligned}
\mathscr{A}_N^{r,1} &:= \left\{ \forall x \in \mathring{\Lambda}_N,\ \phi_x \le r \right\}, \\
\mathscr{A}_N^{r,2} &:= \left\{ \forall x \in \mathring{\Lambda}_N,\ \phi_x \ge -r \right\}, \\
\mathscr{B}_N &:= \left\{ \forall x \in \mathring{\Lambda}_N,\ \phi_x \ge 1 \right\}, \\
\mathscr{C}_N &:= \left\{ \forall x \in \mathring{\Lambda}_N,\ \phi_x \le -1 \right\}.
\end{aligned}
\tag{4.5}
$$

We have $\mathscr{A}_N^r = \mathscr{A}_N^{r,1} \cap \mathscr{A}_N^{r,2}$. In order to obtain a bound on the probability of $\mathscr{A}_N^r$ we need to use the FKG inequality for the Gaussian free field which we present briefly (we refer to [31, Section B.1] for more details). We denote by $\le$ the natural order on the set of functions $\{\phi,\ \Lambda_N \to \mathbb{Z}^d\}$ defined by

$$
\{\ \phi \le \phi'\ \} \quad \Leftrightarrow \quad \{\ \forall x \in \Lambda_N,\ \phi_x \le \phi'_x\ \}. \tag{4.6}
$$

An event A is said to be increasing if for $\phi \in A$ we have

$$
\phi' \ge \phi \Rightarrow \phi' \in A
$$

and decreasing if its complement is increasing. Let us remark that all the events described in (4.5) are either decreasing or increasing. A probability measure μ is said to satisfy the FKG inequality if for any pair of increasing events A, B we have $\mu(A \cap B) \ge \mu(A)\mu(B)$. Note that this yields automatically similar inequalities for any pairs of monotonic events which we also call FKG inequalities.

It is well known that $\mathbf{P}_N$ satisfies the FKG inequality: it is sufficient to check that Holley's criterion [30, 40] is satisfied by the Hamiltonian in (2.3). The same argument yields that $\mathbf{P}_{N,h}^{\beta,\omega}$ as well as the conditioned measures $\mathbf{P}_{N,h}^{\beta,\omega}\left(\cdot \mid \mathscr{A}_N^{r,1}\right)$ and $\mathbf{P}_{N,h}^{\beta,\omega}\left(\cdot \mid \mathscr{B}_N\right)$ also satisfy the FKG inequality. Hence using the FKG inequality for $\mathbf{P}_{N,h}^{\beta,\omega}$, we have

$$
\mathbf{P}_{N,h}^{\beta,\omega}(\mathscr{A}_N^{r,1}) \ge \mathbf{P}_{N,h}^{\beta,\omega}(\mathscr{A}_N^{r,1} \mid \mathscr{B}_N) = \mathbf{P}_N(\mathscr{A}_N^{r,1} \mid \mathscr{B}_N). \tag{4.7}
$$

Then, using the FKG inequality for $\mathbf{P}_{N,h}^{\beta,\omega}(\,\cdot \mid \mathscr{A}_N^{r,1})$ and we have

$$
\mathbf{P}_{N,h}^{\beta,\omega}\left(\mathscr{A}_N^{r,2} \mid \mathscr{A}_N^{r,1}\right) \ge \mathbf{P}_{N,h}^{\beta,\omega}\left(\mathscr{A}_N^{r,2} \mid \mathscr{C}_N\right) = \mathbf{P}_N\left(\mathscr{A}_N^{r,2} \mid \mathscr{C}_N\right) = \mathbf{P}_N(\mathscr{A}_N^{r,1} \mid \mathscr{B}_N), \tag{4.8}
$$

where we used symmetry to get the last equality. Then we can conclude that

$$
\mathbf{P}_{N,h}^{\beta,\omega}(\mathscr{A}_N^{r}) = \mathbf{P}_{N,h}^{\beta,\omega}(\mathscr{A}_N^{r,1} \cap \mathscr{A}_N^{r,2}) \ge \left[\mathbf{P}_N(\mathscr{A}_N^{r,1} \mid \mathscr{B}_N)\right]^2 \ge \left[\mathbf{P}_N(\mathscr{A}_N^{r,1} \cap \mathscr{B}_N)\right]^2. \tag{4.9}
$$

We are left with estimating the last term. Note that changing the boundary condition by a constant amount does not affect the leading order of the asymptotic thus, to conclude, it is sufficient to bound asymptotically the probability of the event

$$
\mathscr{A}_N^{r,3} := \left\{ \max_{x \in \Lambda_N} |\phi_x| \le \frac{r-1}{2} \right\}, \tag{4.10}
$$

which is a translated version of $\mathscr{A}_N^{r,1} \cap \mathscr{B}_N$. More precisely we have for an adequate constant K_r

$$
\mathbf{P}_{N,h}^{\beta,\omega}(\mathscr{A}_N^{r}) \ge \exp(-K_r N)\left[\mathbf{P}_N(\mathscr{A}_N^{r,3})\right]^2. \tag{4.11}
$$

To bound the probability of $\mathscr{H}_N^{r,3}$ we use the following result, whose proof is postponed to the end of the section.

LEMMA 4.2. – *There exists a constant C such that for any N, and for any set $\Gamma \subset \mathring{\Lambda}_N$ which is such that $\Gamma \cup \partial\Lambda_N$ is connected, we have*

$$\mathbf{P}_N\left[\max_{x\in\Gamma} |\phi_x| \le 1\right] \ge \exp(-C|\Gamma|). \tag{4.12}$$

We divide Λ_N in cells of side-length

$$N_0 := \exp(cr)$$

for some small constant c. We set

$$\Lambda(y, N_0) := yN_0 + \Lambda_{N_0}.$$

We apply Lemma 4.2 for the following set

$$\Gamma_N = \left(\bigcup_{y\in\mathbb{Z}^2} \partial\Lambda(y, N_0)\right) \cap \mathring{\Lambda}_N, \tag{4.13}$$

which is is a grid which splits Λ_N in cells of side-length N_0. We obtain that

$$\frac{1}{N^2}\log \mathbf{P}_N\left[\max_{x\in\Gamma_N} |\phi_x| \le 1\right] \ge -\frac{2C}{N_0} = -2C\exp(-cr), \tag{4.14}$$

where we used the inequality $|\Gamma_N| \le 2N^2/N_0$ valid for all N. To conclude we need to show that

$$\frac{1}{N^2}\log \mathbf{P}_N\left[\max_{x\in\Lambda_N} |\phi_x| \le \frac{r-1}{2} \;\Big|\; \max_{x\in\Gamma_N} |\phi_x| \le 1\right] \ge -(N_0)^{-2}. \tag{4.15}$$

To prove (4.15) it is sufficient to remark that conditioned to $(\phi_x)_{x\in\Gamma_N}$, the variance of the field $(\phi_x)_{x\in\Lambda(y,N_0)}$ is uniformly bounded by $\frac{1}{2\pi}\log N_0 + C$ (cf. (3.22) for $m = 0$). Thus, for any realization of ϕ satisfying $\max_{x\in\Gamma_N} |\phi_x| \le 1$, for any $z \in \Lambda_N \setminus \Gamma_N$, using the Gaussian tail bound (3.8) we have for h sufficiently small

$$\mathbf{P}_N\left[|\phi_z| \ge \frac{r-1}{2} \;\Big|\; (\phi_x)_{x\in\Gamma_N}\right] \le \exp\left(-\frac{\pi r^2}{4\log N_0}\right) \le \exp\left(-\frac{\pi}{4c}r\right). \tag{4.16}$$

Now with this in mind we can apply union bound in $\Lambda(y, N_0)$ and obtain

$$\mathbf{P}_N\left[\max_{z\in\Lambda(y,N_0)} |\phi_z| \le \frac{r-1}{2} \;\Big|\; (\phi_x)_{x\in\Gamma_N}\right] \ge 1 - (N_0-1)^2\exp\left(-\frac{\pi}{4c}r\right) \ge e^{-1/2}, \tag{4.17}$$

where the last inequality is valid provided the constant c is chosen sufficiently small. As conditioned to the realization of $(\phi_x)_{x\in\Gamma_N}$, the fields $(\phi_x)_{x\in\Lambda(y,N_0)}$ are independent for different values of y, we prove that the inequality (4.15) holds by multiplying (4.17) for all distinct $\Lambda(y, N_0)$ which fit (at least partially) in Λ_N (there are at most $(N/N_0)^2$ full boxes, to which one must add at most $2N/N_0 + 1$ uncompleted boxes), and taking the expectation with respect to $(\phi_x)_{x\in\Gamma_N}$ conditioned on the event $\max_{x\in\Gamma_N} |\phi_x| \le 1$. This ends the proof of Proposition 4.1. □

Proof of Lemma 4.2. – We can prove it by induction on the cardinality of Γ. Assume that the result is valid for Γ and let us prove it for $\Gamma \cup \{z\}$.

$$\mathbf{P}_N\left[\phi_z \in [-1,1] \mid \max_{x\in\Gamma}|\phi_x| \le 1\right] \ge \exp(-C). \tag{4.18}$$

Note that conditioned to $(\phi_x)_{x\in\Gamma}$, ϕ_z is a Gaussian variable. Its variance is given by

$$E^z\left[\int_0^{\tau_{\partial\Lambda_N\cup\Gamma}} \mathbf{1}_{\{X_t=z\}}\mathrm{d}t\right] \le 1. \tag{4.19}$$

The reason being that as by assumption $\partial\Lambda_N \cup \Gamma \cup \{z\}$ is connected, the walk X is killed with rate one while it lies on z. In addition, if $\max_{x\in\Gamma}|\phi_x| \le 1$, then necessarily

$$\mathbf{E}_N\left[\phi_z \mid (\phi_x)_{x\in\Gamma}\right] \in [-1,1]. \tag{4.20}$$

For this reason, the above inequality is valid if one chooses

$$C := -\min_{u\in[-1,1]} \log P\left(\mathcal{N} \in [-1+u, 1+u]\right) = -\log P\left(\mathcal{N} \in [0,2]\right), \tag{4.21}$$

where $\mathcal{N}$ is a standard normal. □

4.2. Change of measure

For the remainder of the proof we fix the value of r to be

$$r_h := (\log h)^2. \tag{4.22}$$

This value is chosen large enough in particular so that the error term in (4.4) is negligible and sufficiently small for Lemma 4.9 to work.

To bound the expectation of $\mathbb{E}[\log Z^{\beta,\omega}_{N,h}(\mathcal{H}^r_N)]$ we use a “change of measure” argument. The underlying idea is that the annealed bound obtained by Jensen's inequality (2.14) is not sharp because some very atypical ω's (a set of ω of small probability) give the most important contribution to the annealed partition function. Hence our idea is to identify these bad environments and to introduce a function $f(\omega)$ that penalizes them. This idea originates from [37] where it was used to prove the non-coincidence of critical point for a hierarchical variant of the pinning model and was then improved many times in the context of pinning [11, 26, 36] and found application for other models like random-walk pinning, directed polymers, random walk in a random environment or self-avoiding walk in a random environment [8, 9, 10, 43, 44, 51].

In [11, 26, 36], we used the detailed knowledge that we have on the structure of the set of contact points, (which is simply a renewal process) in order to find the right penalization function $f(\omega)$.

Here we have a much less precise knowledge on the structure $(\delta_x)_{x\in\Lambda_N}$ under $\mathbf{P}_N$ (especially because we have to consider possibly very wild boundary condition), but we know that one typical feature of the two-dimensional free field is that the level sets tend to have a clustered structure. We want to perform a change of measure that has the consequence of penalizing these clusters of contact points: we do so by looking at the empirical mean of ω in some small regions and by giving a penalty when it takes an atypically high value.

Let us be more precise about what we mean by penalizing with a function $f(\omega)$. Using Jensen inequality, we remark that

$$\mathbb{E}\left[\log Z_{N,h}^{\beta,\omega}(\mathscr{A}_N^r)\right] = 2\mathbb{E}\left[\log\sqrt{Z_{N,h}^{\beta,\omega}(\mathscr{A}_N^r)}\right] \le 2\log\mathbb{E}\left[\sqrt{Z_{N,h}^{\beta,\omega}(\mathscr{A}_N^r)}\right]. \tag{4.23}$$

If we let $f(\omega)$ be an arbitrary positive function of $(\omega_x)_{x\in\widetilde{\Lambda}_N}$, we have by Cauchy-Schwartz inequality

$$\mathbb{E}\left[\sqrt{Z_{N,h}^{\beta,\omega}(\mathscr{A}_N^r)}\right]^2 \le \mathbb{E}[f(\omega)^{-1}]\mathbb{E}\left[f(\omega)Z_{N,h}^{\beta,\omega}(\mathscr{A}_N^r)\right], \tag{4.24}$$

and hence

$$\frac{1}{N^2}\mathbb{E}\left[\log Z_{N,h}^{\beta,\omega}(\mathscr{A}_N^r)\right] \le \frac{1}{N^2}\log\mathbb{E}[f(\omega)^{-1}] + \frac{1}{N^2}\log\mathbb{E}\left[f(\omega)Z_{N,h}^{\beta,\omega}(\mathscr{A}_N^r)\right]. \tag{4.25}$$

Let us now present our choice of $f(\omega)$. Our idea is to perform some kind of coarse-graining argument: we divide Λ_N into cells of fixed side-length N_1

$$N_1(h) := h^{-1/4}, \tag{4.26}$$

and perform a change of measure inside of each cell. We assume that N_1 is an even integer (the free energy being monotone this causes no loss of generality), and that $N = kN_1$ is a sufficiently large multiple of N_1. Given $y \in \mathbb{Z}^2$, we let $\widetilde{\Lambda}_{N_1}(y)$ denote the translation of the box $\widetilde{\Lambda}_{N_1}$ which is (approximately) centered at yN_1 (see Figure 1)

$$\widetilde{\Lambda}_{N_1}(y) := N_1\left[y - \left(\frac{1}{2},\frac{1}{2}\right)\right] + \widetilde{\Lambda}_{N_1}.$$

In the case $y = (1,1)$ we simply write $\widetilde{\Lambda}'_{N_1}$ (note that it is not identical to $\widetilde{\Lambda}_{N_1}$). We define the event

$$\mathscr{E}_{N_1}(y) := \left\{\exists x \in \widetilde{\Lambda}_{N_1}(y),\ \sum_{\{z\in\widetilde{\Lambda}_{N_1}(y)\,:\,|z-x|\le(\log N_1)^2\}} \omega_z \ \ge \frac{\lambda'(\beta)(\log N_1)^3}{2}\right\}, \tag{4.27}$$

which is simply denoted by $\mathscr{E}_{N_1}$ in the case when $y = (1,1)$. Here $\lambda'(\beta)$ denotes the derivative of λ defined in (2.5). Finally we set (recall (3.3))

$$f(\omega) := \exp\left(-2\sum_{y\in[\![1,k-1]\!]^2}\mathbf{1}_{\mathscr{E}_{N_1}(y)}\right). \tag{4.28}$$

The effect of $f(\omega)$ is to give a penalty (multiplication by e^{-2}) for each cell in which one can find a region of ω with diameter $(\log N_1)^2$ and atypically high empirical mean.

Combining Proposition 4.1 and (4.25), we have (provided that the limit exists)

$$\textsc{f}(\beta,h) \le e^{-c(\log h)^2} + \lim_{k\to\infty}\frac{1}{N^2}\log\mathbb{E}[(f(\omega))^{-1}] + \liminf_{k\to\infty}\frac{1}{N^2}\log\mathbb{E}\left[f(\omega)Z_{N,h}^{\beta,\omega}(\mathscr{A}_N^r)\right]. \tag{4.29}$$

We can conclude the proof with the two following results, which evaluate respectively the cost and the benefit of our change of measure procedure.

PROPOSITION 4.3. – *There exists positive constants $c(\beta)$ and $h_0(\beta)$ and such that for all $h \in (0, h_0(\beta))$ sufficiently small, for all k*

$$\log \mathbb{E}\left[(f(\omega))^{-1}\right] \leq (k-1)^2 e^{-c(\beta)(\log h)^2}. \tag{4.30}$$

As a consequence we have for all N of the form kN_1

$$\frac{1}{N^2} \log \mathbb{E}[(f(\omega))^{-1}] \leq e^{-c(\beta)(\log h)^2}. \tag{4.31}$$

PROPOSITION 4.4. – *There exists $h_0(\beta) > 0$ such that for all $h \in (0, h_0(\beta))$ and r given by* (4.22)

$$\limsup_{k\to\infty} \frac{1}{N^2} \log \mathbb{E}\left[f(\omega) Z^{\beta,\omega}_{N,h}(\mathscr{A}^r_N)\right] \leq e^{-2|\log h|^{3/2}}. \tag{4.32}$$

As a consequence of (4.29) and of the two propositions above, we obtain that for $h \in (0, h_0(\beta))$, we have

$$\textsc{f}(\beta, h) \leq e^{-|\log h|^{3/2}}. \tag{4.33}$$

The proof of Proposition 4.3 is simple and short and is presented below. The proof Proposition 4.4 requires a significant amount of work. We decompose it in important steps in the next subsection.

Proof of Proposition 4.3. – Because of the product structure, we have

$$\mathbb{E}\left[(f(\omega))^{-1}\right] = \left(\mathbb{E}\left[\exp\left(2\mathscr{E}_{N_1}\right)\right]\right)^{(k-1)^2}. \tag{4.34}$$

Hence it is sufficient to obtain a bound on

$$\log \mathbb{E}\left[\exp\left(2\mathbf{1}_{\mathscr{E}_{N_1}}\right)\right] \leq (e^2-1)\mathbb{P}[\mathscr{E}_{N_1}(0)]. \tag{4.35}$$

As an easy consequence of the proof of Cramér's Theorem (see e.g., [25, Chapter 2]), there exists a constant $c(\beta)$ that for any $x \in \widetilde{\Lambda}_{N_1}$

$$\mathbb{P}\left[\sum_{\{z\in\widetilde{\Lambda}'_{N_1}\,:\,|z-x|\leq(\log N_1)^2\}} \omega_z \geq \frac{\lambda'(\beta)(\log N_1)^3}{2}\right] \leq e^{-c(\beta)(\log N_1)^2}, \tag{4.36}$$

and by union bound we obtain that $\mathbb{P}[\mathscr{E}_{N_1}] \leq N_1^2 \exp(-c(\log N_1)^2)$, which in view of (4.35) and (4.34) is sufficient to conclude □

4.3. Decomposing the proof of Proposition 4.4

The proof is split in three steps, whose details are performed in Section 4.4, 4.5 and 4.6 respectively. In the first one we show that our averaged partition function $\mathbb{E}\left[f(\omega) Z^{\beta,\omega}_{N,h}(\mathscr{A}^r_N)\right]$, can be bounded from above by the partition of an homogeneous system where an extra term is added in the Hamiltonian to penalize the presence of clustered contact in a small region

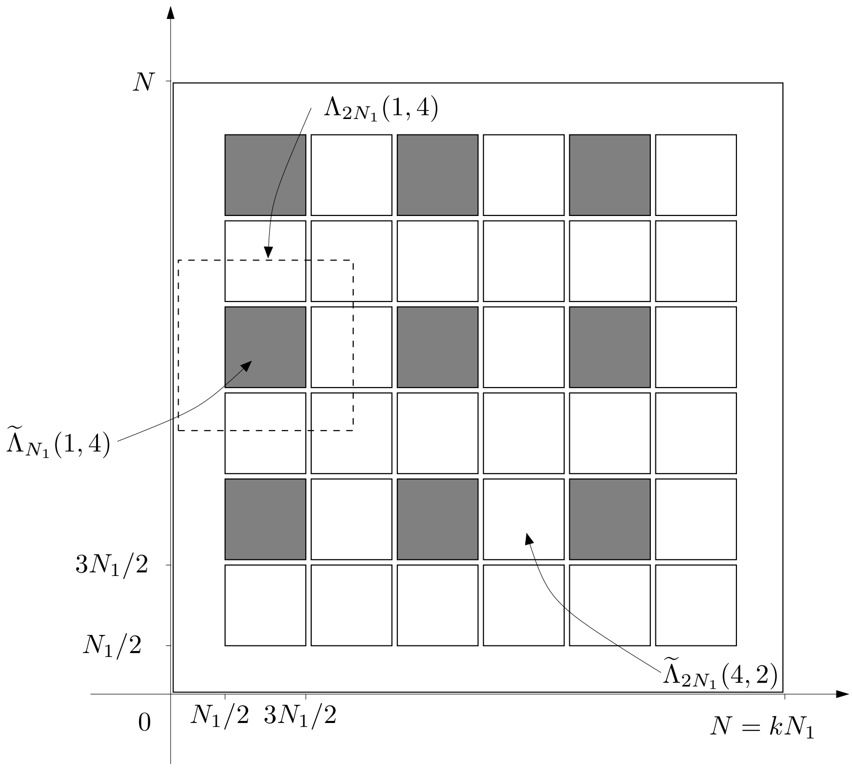

FIGURE 1. A schematic representation of our coarse graining procedure. We have chosen $k = 7$. The small squares of side length N_1 represent the disjoint boxes $\widetilde{\Lambda}_{N_1}(y)$, $y \in [\![1, k-1]\!]^2$. The dark squares represent the boxes $\widetilde{\Lambda}_{N_1}(y)$ for which $y \in \Xi(2)$. The dotted square corresponds to the box $\Lambda_{2N_1}(1,4)$.

(here a region of diameter $(\log N_1)^2$). We introduce the event $\mathcal{C}_{N_1}(y)$ which indicates the presence of such a cluster in $\widetilde{\Lambda}_{N_1}(y)$,

$$\mathcal{C}_{N_1}(y) := \left\{ \exists x \in \widetilde{\Lambda}_{N_1}(y), \sum_{\{z \in \widetilde{\Lambda}_{N_1}(y)\,:\,|z-x| \le (\log N_1)^2\}} \delta_z \ge (\log N_1)^3 \right\}. \tag{4.37}$$

We simply write $\mathcal{C}_{N_1}$ for the case $y = (1,1)$.

PROPOSITION 4.5. – *We have*

$$\mathbb{E}\left[f(\omega) Z^{\beta,\omega}_{N,h}(\mathscr{A}^r_N)\right] \le \mathbf{E}_N\left[\exp\left(h \sum_{x \in \widetilde{\Lambda}_N} \delta_x - \sum_{y \in [\![1,k-1]\!]^2} \mathbf{1}_{\mathcal{C}_{N_1}(y)}\right) \mathbf{1}_{\mathscr{A}^r_N}\right] =: \widehat{Z}(N, N_1, h). \tag{4.38}$$

In the second step, we perform a factorization in order to reduce the estimate of $\widehat{Z}(N, N_1, h)$ to that of similar system with only one cell. Let us set (see Figure 1)

$$\Lambda_{2N_1}(y) := N_1\big(y-(1,1)\big) + \Lambda_{2N_1}. \tag{4.39}$$

Note that for every for $y \in [\![1, k-1]\!]^2$ we have $\Lambda_{2N_1}(y) \subset \Lambda_N$ and that $\Lambda_{2N_1}((1,1)) = \Lambda_{2N_1}$.

PROPOSITION 4.6. – *We have*

$$\widehat{Z}(N, N_1, h) \le e^{2N_1 N h}\left(\max_{\{\widehat{\phi}\,:\,\|\widehat{\phi}\|_\infty \le r\}} \mathbf{E}^{\widehat{\phi}}_{2N_1}\left[e^{4h\sum_{x\in\widetilde{\Lambda}'_{N_1}}\delta_x - 4\mathbf{1}_{\mathcal{C}_{N_1}}}\right]\right)^{\frac{(k-1)^2}{4}}. \tag{4.40}$$

Let us notice two important features in our factorization which are present to reduce possible nasty boundary effects:

— There is a restriction on the boundary condition $\|\widehat{\phi}\|_\infty \le r$, which forbids wild behavior of the field. This restriction is directly inherited from the restriction to $\mathscr{H}^r_N$ in the partition function and brings some light on the role of Proposition 4.1 in our proof.

— The Hamiltonian

$$4h\sum_{x\in\widetilde{\Lambda}'_{N_1}}\delta_x - 4\mathbf{1}_{\mathcal{C}_{N_1}}$$

is a functional of $(\phi_x)_{x\in\widetilde{\Lambda}'_{N_1}}$ i.e., of the field restricted to a region which is distant from the boundary of the box $\partial\Lambda_{2N_1}$.

The final step of the proof consists in evaluating the contribution of one single cell to the partition function.

PROPOSITION 4.7. – *For all h sufficiently small for all $\widehat{\phi}$ satisfying $\|\widehat{\phi}\|_\infty \le r_h$ we have*

$$\log \mathbf{E}^{\widehat{\phi}}_{2N_1}\left[e^{4h\sum_{x\in\widetilde{\Lambda}_{N_1}}\delta_x - 4\mathbf{1}_{\mathcal{C}_{N_1}}}\right] \le e^{-2|\log h|^{3/2}}. \tag{4.41}$$

Combining the three results presented above, we have

$$\log \mathbb{E}\left[f(\omega) Z^{\beta,\omega}_{N,h}(\mathscr{H}^r_N)\right] \le 2N_1 N h + \frac{(k-1)^2}{4} e^{-2|\log h|^{3/2}}, \tag{4.42}$$

and this is sufficient to conclude the proof of Proposition 4.4.

4.4. Proof of Proposition 4.5

Given a realization ϕ, we let $\mathbb{P}^\phi$ be a probability law which is absolutely continuous with respect to $\mathbb{P}$ and whose the density is given by

$$\frac{\mathrm{d}\mathbb{P}^\phi}{\mathrm{d}\mathbb{P}}(\omega) := \exp\left(\sum_{x\in\widetilde{\Lambda}_N}(\beta\omega_x - \lambda(\beta))\,\delta_x\right). \tag{4.43}$$

Under $\mathbb{P}^\phi$, the variables $(\omega_x)_{x\in\mathbb{Z}^d}$ are still independent but they are not IID, as the law of the ω_xs for which $\delta_x = 1$ have been tilted. In particular it satisfies

$$\mathbb{E}^\phi[\omega_x] = \lambda'(\beta)\delta_x \quad \text{ and } \operatorname{Var}_{\mathbb{P}^\phi}[\omega_x] = 1 + (\lambda''(\beta) - 1)\delta_x, \tag{4.44}$$

where $\lambda'(\beta)$ and $\lambda''(\beta)$ denote the two first derivatives of λ the function defined in (2.5). This notation gives us another way of writing the quantity that we must estimate

$$\mathbb{E}\left[f(\omega)Z^{\beta,\omega}_{N,h}(\mathscr{A}^r_N)\right]=\mathbf{E}_N\left[\mathbb{E}^{\phi}[f(\omega)]e^{h\sum_{x\in\widetilde{\Lambda}_N}\delta_x}\mathbf{1}_{\mathscr{A}^r_N}\right]. \tag{4.45}$$

To conclude it is sufficient to prove that

$$\mathbb{E}^{\phi}[f(\omega)]\le\exp\left(-\sum_{y\in[\![0,k-1]\!]}\mathbf{1}_{\mathcal{C}_{N_1}(y)}\right). \tag{4.46}$$

Note that because both $\mathbb{E}^{\phi}$ and $f(\omega)$ have a product structure, it is in fact sufficient to prove that for any $y\in[\![0,k-1]\!]^2$ we have

$$\mathbb{E}^{\phi}\left[e^{-2\mathbf{1}_{\mathcal{E}_{N_1}(y)}}\right]\le e^{-\mathbf{1}_{\mathcal{C}_{N_1}(y)}}. \tag{4.47}$$

With no loss of generality we assume that $y=(1,1)$. The result is obvious when $\phi\notin\mathcal{C}_{N_1}$ hence we can also assume $\phi\in\mathcal{C}_{N_1}$. Let $x_0\in\widetilde{\Lambda}'_{N_1}$ be a vertex satisfying

$$\sum_{\{z\in\Lambda'_{N_1}\,:\,|z-x_0|\le(\log N_1)^2\}}\delta_z\ge(\log N_1)^3$$

(e.g., the smallest one for the lexicographical order). We have

$$\begin{aligned}&\mathbb{E}^{\phi}\left[\sum_{\{z\in\widetilde{\Lambda}'_{N_1}\,:\,|z-x_0|\le(\log N_1)^2\}}\omega_z\right]=\lambda'(\beta)\sum_{\{z\in\widetilde{\Lambda}'_{N_1}\,:\,|z-x_0|\le(\log N_1)^2\}}\delta_z\ \ge\ \lambda'(\beta)(\log N_1)^3,\\&\mathrm{Var}_{\mathbb{P}^{\phi}}\left[\sum_{\{z\in\widetilde{\Lambda}'_{N_1}\,:\,|z-x_0|\le(\log N_1)^2\}}\omega_z\right]\le\left[2(\log N_1)^2+1\right]^2\max(\lambda''(\beta),1).\end{aligned} \tag{4.48}$$

Hence in particular if N_1 is sufficiently large, Chebychev's inequality gives

$$\mathbb{P}^{\phi}\left[\mathcal{E}_{N_1}\right]\ge\frac{1-e^{-1}}{1-e^{-2}}, \tag{4.49}$$

which is equivalent to (4.47). □

4.5. Proof of Proposition 4.6

We start by taking care of the contribution of the contact points located near the boundary $\partial\Lambda_N$, as they are not included in any $\widetilde{\Lambda}_{N_1}(y)$. Assuming that all these points are contact points we obtain the following crude bound

$$\sum_{x\in\widetilde{\Lambda}_N}\delta_x\le\left[N^2-(k-1)^2N_1^2\right]+\sum_{y\in[\![1,k-1]\!]^2}\sum_{x\in\widetilde{\Lambda}_{N_1}(y)}\delta_x, \tag{4.50}$$

and the first term is smaller than $2NN_1$. Hence we have

$$\widehat{Z}(N,N_1,h)\le e^{2N_1Nh}\mathbf{E}_N\left[e^{\sum_{y\in[\![1,k-1]\!]^2}\left(h\sum_{x\in\widetilde{\Lambda}_{N_1}(y)}\delta_x-\mathbf{1}_{\mathcal{C}_{N_1}(y)}\right)}\mathbf{1}_{\mathscr{A}^r_N}\right]. \tag{4.51}$$

We partition the set of indices $[\![1,k-1]\!]^2$ into 4 subsets, according to the parity of the of the coordinates. If we let $\alpha_1(i)$ and $\alpha_2(i)$ denote the first and second diadic digits of $i-1$. We set

$$\Xi(i) := \left\{ y=(y_1,y_2) \in [\![1,k-1]\!]^2 \ : \ \forall j\in\{1,2\},\ y_j \overset{(\mathrm{mod}\ 2)}{=} \alpha_j(i) \right\}. \tag{4.52}$$

Using Hölder's inequality we have

$$\mathbf{E}_N\left[e^{\sum_{y\in[\![1,k-1]\!]^2}\left(h\sum_{x\in\widetilde{\Lambda}_{N_1}(y)}\delta_x - \mathbf{1}_{\mathcal{C}_{N_1}(y)}\right)} \mathbf{1}_{\mathcal{A}^r_N} \right]^4 \le \prod_{i=1}^4 \mathbf{E}_N\left[e^{4\sum_{y\in\Xi(i)}\left(h\sum_{x\in\widetilde{\Lambda}_{N_1}(y)}\delta_x - \mathbf{1}_{\mathcal{C}_{N_1}(y)}\right)} \mathbf{1}_{\mathcal{A}^r_N} \right]. \tag{4.53}$$

For a fixed $i\in[\![1,4]\!]$, the interiors of the boxes $\Lambda_{2N_1}(y)$, $y\in\Xi(i)$ are disjoint (neighboring boxes overlap only on their boundary, we refer to Figure 1). This gives us a way to factorize the exponential: let us condition the expectation to the realization of $(\phi_x)_{x\in\Gamma(i)}$ where

$$\Gamma(i) := \bigcup_{y\in\Xi(i)} \partial\Lambda_{2N_1}(y). \tag{4.54}$$

The spatial Markov property implies that conditionally on $(\phi_x)_{x\in\Gamma(i)}$, the restrictions $\left[(\phi_x)_{x\in\Lambda_{2N_1}(y)}\right]_{y\in\Xi(i)}$ are independent. Hence we can factorize the expectation and get

$$\mathbf{E}_N\left[e^{4\sum_{y\in\Xi(i)}\left(h\sum_{x\in\widetilde{\Lambda}_{N_1}(y)}\delta_x - \mathbf{1}_{\mathcal{C}_{N_1}(y)}\right)} \mid (\phi_x)_{x\in\Gamma(i)} \right] \le \prod_{y\in\Xi(i)} \mathbf{E}_N\left[e^{4\left(h\sum_{x\in\widetilde{\Lambda}_{N_1}(y)}\delta_x - \mathbf{1}_{\mathcal{C}_{N_1}(y)}\right)} \mid (\phi_x)_{x\in\Gamma(i)} \right]. \tag{4.55}$$

On the event

$$\mathcal{A}^r(i) := \left\{ \max_{x\in\Gamma(i)} |\phi_x| \le r \right\},$$

we have for any $y\in\Xi(i)$, by translation invariance,

$$\mathbf{E}_N\left[e^{4\left(h\sum_{x\in\widetilde{\Lambda}_{N_1}(y)}\delta_x - \mathbf{1}_{\mathcal{C}_{N_1}(y)}\right)} \mid (\phi_x)_{x\in\Gamma(i)} \right] \le \max_{\{\widehat{\phi}\,:\,\|\widehat{\phi}\|_\infty\le r\}} \mathbf{E}^{\widehat{\phi}}_{2N}\left[e^{4h\left(\sum_{x\in\widetilde{\Lambda}'_{N_1}}\delta_x\right) - 4\mathbf{1}_{\mathcal{C}_{N_1}}} \right], \tag{4.56}$$

and hence we can conclude by taking the expectation of (4.55) restricted to the event $\mathcal{A}^r(i)$ (which includes $\mathcal{A}^r_N$). □

4.6. Proof of Proposition 4.7

Note that because of our choice of $N_1 = h^{-1/4}$ we always have

$$h\sum_{x\in\widetilde{\Lambda}'_{N_1}} \delta_x \le hN_1^2 \le h^{1/2}, \tag{4.57}$$

which is small. Hence for that reason, if h is sufficiently small, the Taylor expansion of the exponential gives

$$\begin{aligned}(4.58)\quad \log \mathbf{E}^{\widehat{\phi}}_{2N_1}\left[e^{4h\sum_{x\in\widetilde{\Lambda}'_{N_1}}\delta_x - 4\mathbf{1}_{\mathcal{C}_{N_1}}}\right] &\le \log \mathbf{E}^{\widehat{\phi}}_{2N_1}\Big[1+5h\sum_{x\in\widetilde{\Lambda}'_{N_1}}\delta_x - \frac{1}{2}\mathbf{1}_{\mathcal{C}_{N_1}}\Big]\\ &\le 5h\mathbf{E}^{\widehat{\phi}}_{2N_1}\Big[\sum_{x\in\widetilde{\Lambda}'_{N_1}}\delta_x\Big] - \frac{1}{2}\mathbf{P}^{\widehat{\phi}}_{2N_1}[\mathcal{C}_{N_1}]\\ &\le 5N_1^{-2}\max_{x\in\widetilde{\Lambda}'_{N_1}}\mathbf{P}^{\widehat{\phi}}_{2N_1}[\phi_x\in[-1,1]] - \frac{1}{2}\mathbf{P}^{\widehat{\phi}}_{2N_1}[\mathcal{C}_{N_1}].\end{aligned}$$

We have to prove that the r.h.s. is small. Before going into technical details let us quickly expose the main idea of the proof. For the r.h.s. of (4.58) to be positive, we need

$$(4.59)\qquad \frac{\max_{x\in\widetilde{\Lambda}'_{N_1}}\mathbf{P}^{\widehat{\phi}}_{2N_1}(\phi_x\in[-1,1])}{\mathbf{P}^{\widehat{\phi}}_{2N_1}[\mathcal{C}_{N_1}]} \ge \frac{N_1^2}{10}.$$

What we are going to show is that for this ratio to be large we need the boundary condition $\widehat{\phi}$ to be very high above the substrate (or below by symmetry), but that in that case the quantity $\left(\max_{x\in\widetilde{\Lambda}'_{N_1}}\mathbf{P}^{\widehat{\phi}}_{2N_1}[\phi_x\in[-1,1]]\right)$ itself has to be very small and this should allow ourselves to conclude.

To understand the phenomenon better we need to introduce quantitative estimates. Let G^* denote the Green function (3.16) in the box Λ_{2N_1} with 0 boundary condition, and set

$$(4.60)\qquad V_{N_1} := \max_{x\in\widetilde{\Lambda}'_{N_1}} G^*(x,x).$$

We have from Lemma 3.1

$$(4.61)\qquad \left|V_{N_1} - \frac{1}{2\pi}\log N_1\right| \le C.$$

Recall that from (3.10) we have

$$(4.62)\qquad \mathbf{P}^{\widehat{\phi}}_{2N_1}(\,\phi_x\in[-1,1]\,) \le \mathbf{P}_{2N_1}\left(\phi_x\in\left[-1-H^{\widehat{\phi}}_{2N_1}(x), 1-H^{\widehat{\phi}}_{2N_1}(x)\right]\right).$$

With this in mind we fix

$$(4.63)\qquad u = u(\widehat{\phi},N_1) := \min_{x\in\widetilde{\Lambda}_N}|H^{\widehat{\phi}}_{2N_1}(x)|.$$

Hence using basic properties of the Gaussian distribution, we obtain (provided that h is sufficiently small)

$$(4.64)\qquad \max_{x\in\widetilde{\Lambda}'_{N_1}}\mathbf{P}^{\widehat{\phi}}_{2N_1}(\phi_x\in[-1,1]) \le e^{-\frac{(u-1)^2}{2V_{N_1}}}.$$

It requires a bit more work to obtain a good lower bound for $\mathbf{P}^{\widehat{\phi}}_{2N_1}[\mathcal{C}_{N_1}]$ which is valid for all values of u. Fortunately we only need a rough estimate as the factor N_1^2 in (4.59) gives us a significant margin in the computation.

Recall that P_t^* denotes the two-dimensional heat-kernel with zero boundary condition on $\partial\Lambda_{2N_1}$. Let us set

$$V'_{N_1} := \min_{x\in\widetilde{\Lambda}'_{N_1}} \int_{(\log N_1)^8}^{\infty} P_t^*(x,x)\mathrm{d}t. \tag{4.65}$$

From the estimates in Lemma 3.2, we can deduce that

$$\left| V'_{N_1} - \frac{1}{2\pi}\left(\log N_1 - 4\log\log N_1\right)\right| \leq C. \tag{4.66}$$

For instance we have

$$\left| \int_0^{(\log N_1)^8} P_t^*(x,x)\mathrm{d}t - \frac{2}{\pi}\log\log N_1 \right| \leq \frac{C}{2}, \tag{4.67}$$

for some appropriate C (the estimate is obtained using (3.27) and (3.20)) so that the result can be deduced from the estimate in the Green-function (3.22).

Proposition 4.8. – *For all h sufficiently small, for all $\widehat{\phi}$ satisfying $\|\widehat{\phi}\|_\infty < r_h$, we have*

$$\mathbf{P}_{2N_1}^{\widehat{\phi}}[\mathcal{C}_{N_1}] \geq c(\log N_1)^{-1} e^{-\frac{u^2}{2V'_{N_1}}}. \tag{4.68}$$

Combining the above result with (4.64) and (4.58) We have

$$\begin{aligned} \max_{\{\widehat{\phi}\,:\,|\widehat{\phi}|\leq r\}} \log \mathbf{E}_{2N_1}^{\widehat{\phi}} &\left[\exp\left(4h\sum_{x\in\widetilde{\Lambda}_{N_1}} \delta_x - 4\mathbf{1}_{\mathcal{C}_{N_1}}\right)\right] \\ &\leq \sup_{u\in[0,(\log N_1)^2]} \left(5N_1^{-2}e^{-\frac{(u-1)^2}{2V_{N_1}}} - c(2\log N_1)^{-1}e^{-\frac{u^2}{2V'_{N_1}}}\right) \\ &= \sup_{u\in[0,(\log N_1)^2]} \frac{5e^{-\frac{(u-1)^2}{2V_{N_1}}}}{N_1^2}\left[1 - \frac{cN_1^2}{10(\log N_1)}e^{-\frac{u^2(V_{N_1}-V'_{N_1})}{2V'_{N_1}V_{N_1}} - \frac{2u-1}{2V_{N_1}}}\right]. \end{aligned} \tag{4.69}$$

Now note that for the second factor to be positive, we need one of the terms in the exponential to be at least of order $\log N_1$ in absolute value. Using the estimates we have for V'_{N_1} and V_{N_1}, we realize that the exponential term is larger than

$$c\exp\left(-\frac{cu^2(\log\log N_1)}{(\log N_1)^2}\right),$$

and hence the expression is negative if $u^2 \leq c(\log N_1)^3(\log\log N_1)^{-1}$, for some small c. For the other values of u we can just consider the first factor which already gives a satisfying bound, and we can conclude that the l.h.s. of (4.69) is smaller than

$$e^{-\frac{c(\log N_1)^2}{\log\log N_1}} \leq e^{-c|\log h|^{3/2}}. \tag{4.70}$$

4.7. Decomposing the proof of Proposition 4.8

We show here how to split the proof the proposition into three lemmas which we prove in the next subsection. Set

$$x_{\min} := \operatorname*{argmin}_{x\in\widetilde{\Lambda}'_{N_1}} |H^{\widehat{\phi}}_{2N_1}(x)|, \tag{4.71}$$

(it is not necessarily unique but in the case it is not we choose one minimizer in a deterministic manner) and

$$\widehat{\Lambda} := \{z \in \widetilde{\Lambda}'_{N_1} \ : \ |x_{\min} - z| \le (\log N_1)^2\}. \tag{4.72}$$

We bound from below the probability of $\mathcal{C}_{N_1}$ by only examining the possibility of having a cluster of contact around $x_{\min}$. Using (3.16) we have

$$\begin{aligned} \mathbf{P}^{\widehat{\phi}}_{2N_1}[\mathcal{C}_{N_1}] &\ge \mathbf{P}^{\widehat{\phi}}_{2N_1}\left[\sum_{z\in\widehat{\Lambda}} \delta_z \ge (\log N_1)^3\right] \\ &= \mathbf{P}_{2N_1}\left[\sum_{z\in\widehat{\Lambda}} \mathbf{1}_{[-1,1]}\left(\phi_z + H^{\widehat{\phi}}_{2N_1}(x)\right) \ge (\log N_1)^3\right]. \end{aligned} \tag{4.73}$$

To estimate the last probability, we first remark that for $x \in \widehat{\Lambda}$, $H^{\widehat{\phi}}_{2N_1}(x)$ is very close to $H^{\widehat{\phi}}_{2N_1}(x_{\min})$ which we assume to be equal to $-u$ for the rest of the proof (the case $H^{\widehat{\phi}}_{2N_1}(x_{\min}) = +u$ is exactly similar). The factor $\log N_1$ in the estimate is not necessary, but it yields a much simpler proof.

LEMMA 4.9. – *We have for all $x, y \in \widetilde{\Lambda}'_{N_1}$*

$$\left|H^{\widehat{\phi}}_{2N_1}(x) - H^{\widehat{\phi}}_{2N_1}(y)\right| \le \frac{C\|\widehat{\phi}\|_\infty(\log N_1)|x-y|}{N_1}. \tag{4.74}$$

In particular if h is sufficiently small, $|x - y| \le (\log N_1)^2$ and $\|\widehat{\phi}\|_\infty \le r_h$, we have

$$|H^{\widehat{\phi}}_{2N_1}(x) - H^{\widehat{\phi}}_{2N_1}(y)| \le 1/4. \tag{4.75}$$

Then to estimate the probability for ϕ to form a cluster of point close to height u, we decompose the field $(\phi_x)_{x\in\widehat{\Lambda}}$ into a rough field ϕ_1 which is almost constant on the scale $(\log N_1)^2$ and an independent field ϕ_2 which accounts for the local variations of ϕ. We set

$$\begin{aligned} Q^1(x,y) &:= \int_{(\log N)^8}^{\infty} P^*_t(x,y)\mathrm{d}t, \\ Q^2(x,y) &:= \int_0^{(\log N)^8} P^*_t(x,y)\mathrm{d}t. \end{aligned} \tag{4.76}$$

We let $(\phi_1(x))_{x\in\Lambda_{2N_1}}$ and $(\phi_2(x))_{x\in\Lambda_{2N_1}}$ denote two independent centered Gaussian fields with respective covariance function Q^1 and Q^2. By construction the law of $\phi_1+\phi_2$ has a law given by $\mathbf{P}_{2N_1}$, and thus we set for the remainder of the proof

$$\phi := \phi_1 + \phi_2, \tag{4.77}$$

and use $\mathbf{P}_{2N_1}$ to denote the law of (ϕ_1, ϕ_2). We have by standard properties of Gaussian variables that for every $u > 0$, and for h sufficiently small

$$\mathbf{P}_{2N_1}\left[\phi_1(x_{\min}) \in [u-1/4, u+1/4]\right] \geq \frac{1}{4\sqrt{2\pi V_{N_1}}} e^{-\frac{u^2}{2V'_{N_1}}} \geq \frac{1}{5\sqrt{\log N_1}} e^{-\frac{u^2}{2V'_{N_1}}}. \tag{4.78}$$

Now we have to check that the field ϕ_1 remains around level u on the whole box $\widehat{\Lambda}$.

Lemma 4.10. – *There exists a constant c such that for all h sufficiently small we have*

$$\mathbf{P}_{2N_1}\left[\exists y \in \widehat{\Lambda},\ |\phi_1(y) - \phi_1(x_{\min})| > 1/4\right] \leq e^{-c(\log N_1)^4}. \tag{4.79}$$

Finally we show that it is rather likely for ϕ_2 to have a lot of points around level zero.

Lemma 4.11. – *There exists a constant c such that for all h sufficiently small we have*

$$\mathbf{P}_{2N_1}\left[\sum_{z\in\widehat{\Lambda}} \mathbf{1}_{\{|\phi_2(z)|\leq 1/4\}} \geq (\log N_1)^3\right] \geq c(\log\log N_1)^{-1/2}. \tag{4.80}$$

We can now combine all these ingredient into a proof

Proof of Proposition 4.8. – According to Lemma 4.9, if $|x_{\min} - z| \leq (\log N_1)^2$ we have

$$\begin{aligned} \left\{(\phi_z + H^{\widehat{\phi}}_{2N_1}(z)) \in [-1,1]\right\} &\supset \{\phi_z \in [-3/4+u, 3/4+u]\} \\ &\supset \{\,|\phi_1(x_{\min}) - u| \leq 1/4\} \cap \{|\phi_1(x_{\min}) - \phi_1(z)| \leq 1/4\} \cap \{|\phi_2(z)| \leq 1/4\}. \end{aligned} \tag{4.81}$$

Thus we obtain as a consequence

$$\begin{aligned} &\left\{\sum_{z\in\widehat{\Lambda}} \mathbf{1}_{[-1,1]}(\phi_z + H^{\widehat{\phi}}_{2N_1}(x)) \geq (\log N_1)^3\right\} \\ &\quad\supset \{\,|\phi_1(x_{\min}) - u| \leq 1/4\} \cap \left\{\forall z \in \widehat{\Lambda},\ |\phi_1(x_{\min}) - \phi_1(z)| \leq 1/4\right\} \\ &\qquad\cap \left\{\sum_{z\in\widehat{\Lambda}} \mathbf{1}_{\{|\phi_2(z)|\leq(1/4)\}} \geq (\log N_1)^3\right\}. \end{aligned} \tag{4.82}$$

Using (4.78) combined with Lemmas 4.10 and 4.11 and the independence of ϕ_1 and ϕ_2 we conclude that

$$\begin{aligned} &\mathbf{P}_{2N_1}\left[\sum_{z\in\widehat{\Lambda}} \mathbf{1}_{[-1,1]}\left(\phi_z + H^{\widehat{\phi}}_{2N_1}(x)\right) \geq (\log N_1)^3\right] \\ &\quad\geq \left[\frac{c}{\sqrt{\log N_1}} e^{-\frac{u^2}{2V'_{N_1}}} - e^{-c(\log N_1)^4}\right] c(\log\log N_1)^{-1/2} \geq \frac{c'}{(\log N_1)} e^{-\frac{u^2}{2V'_{N_1}}}, \end{aligned} \tag{4.83}$$

where the last inequality is holds if h is sufficiently small. We used that from the Definition (4.63) and the assumption on $\widehat{\phi}$

$$u \leq \|\widehat{\phi}\|_\infty \leq r_h = (4\log N_1)^2.$$

We can thus conclude using (4.73). □

4.8. Proof of the technical lemmas

Proof of Lemma 4.9. – Given $x, y \in \widetilde{\Lambda}'_{N_1}$, let X^x and X^y be two simple random walk starting from $x = (x_1, x_2)$ and $y = (y_1, y_2)$, and coupled as follows: the coupling is made as the product of two one-dimensional couplings, along each coordinate the walk are independent until the coordinate match, then they move together. Let $\tau_{x,y}$ be the time where the two walks meet and $\tau^x_{\partial\Lambda_{2N_1}}$ be the time when X^x hits the boundary. Recalling (3.13) we have

$$\left| H^{\widehat{\phi}}_{2N_1}(y) - H^{\widehat{\phi}}_{2N_1}(x) \right| \le \|\widehat{\phi}\|_\infty 2P\left[\tau_{x,y} > \tau^x_{\partial\Lambda_{2N_1}}\right]. \tag{4.84}$$

We conclude by showing that

$$P\left[\tau_{x,y} > \tau^x_{\partial\Lambda_{2N_1}}\right] < \frac{C|x-y|(\log N_1)}{N_1}. \tag{4.85}$$

By union bound, we can reduce to the one dimensional case. Let Y^x and Y^y denote the first coordinates of X^x and X^y. Until the collision time, they are two independent one dimensional random walk in $[\![0, 2N_1]\!]$ with initial condition x_1 and y_1 in $[\![N_1/2, 3N_1/2]\!]$. Let $T_{x,y}$ and T' denote respectively their collision time and the first hitting time of $\{0, 2N_1\}$ for Y^x. We are going to show that

$$P\left[T' < T_{x,y}\right] < C\frac{|x_1 - y_1|(\log N_1)}{N_1}. \tag{4.86}$$

As before collision, $Y^x - Y^y$ is a nearest neighbor random-walk with jump rate equal to 2. Estimates for the tail distribution of the hitting time of zero for such a random walk are classical. From [45, Proposition 5.15] (with a little extra work to adapt the result to continuous time) we have for any $t > 0$

$$P[T_{x,y} > t] \le C|x_1 - y_1| t^{-1/2}. \tag{4.87}$$

On the other hand, T' is dominated by the hitting time of $N_1/2$ by the absolute value of a continuous time simple random walk $(S_t)_{t\ge 0}$ starting from zero. A standard large deviation estimates for the associated simple random walk in discrete time $(\mathbf{S}_n)_{n\ge 0}$ (see e.g., standard proof of Cramér's theorem [25, Theorem 2.2.3]) and on the number κ_t of jumps performed by S until time t (which is simply a Poisson variable of parameter P), yields for $t \ge N_1$

$$P[T' \le t] \le P[\kappa_t \ge \lceil 2t \rceil] + \sum_{n=1}^{\lceil 2t \rceil} P[|\mathbf{S}_n| \ge N_1/2] \le 2\left(e^{-ct} + e^{-\frac{cN_1}{t}}\right). \tag{4.88}$$

We can conclude choosing $t = N_1^2(\log N_1)^{-2}$. □

Proof of Lemma 4.10. – We obtain the result simply by performing a union bound on $y \in \widehat{\Lambda}$. Hence we only need to prove a bound on the variance

$$\begin{aligned} \mathbf{E}_{2N_1}\left[(\phi_1(y) - \phi_1(x_{\min}))^2\right] \\ \le \int_{(\log N_1)^8}^\infty \left[P^*_t(x_{\min}, x_{\min}) - 2P^*_t(x_{\min}, y) + P^*(y,y)\right] \mathrm{d}t. \end{aligned} \tag{4.89}$$

Using (3.23), we obtain that for any $y \in \widehat{\Lambda}$

$$\mathbf{E}_{2N_1}\left[(\phi_1(y)-\phi_1(x_{\min}))^2\right] \le C(\log N_1)^{-4}, \tag{4.90}$$

and thus that

$$\mathbf{P}_{2N_1}\left[\,|\phi_1(y)-\phi_1(x_{\min})| \ge 1/4\right] \le |\widehat{\Lambda}|e^{-c(\log N_1)^4}, \tag{4.91}$$

which allows to conclude. □

Proof of Lemma 4.11. – We set

$$J := \sum_{\{z\,:\,|x_{\min}-z|\le(\log N_1)^2\}} \mathbf{1}_{\{\phi_2(z)\in[-1/4,1/4]\}}.$$

Using the fact that the sum is deterministically bounded by $C(\log N_1)^4$, we have

$$\mathbf{P}_{2N_1}\left[J \ge \frac{\mathbf{E}_N[J]}{2}\right] \ge \frac{\mathbf{E}_N[J]}{2C(\log N_1)^4}. \tag{4.92}$$

From (4.67), we have for small h,

$$\mathrm{Var}(\phi_2(x)) = Q^2(x,x) \le \log\log N_1. \tag{4.93}$$

Then as $\phi_2(x)$ are centered Gaussians, we have

$$\mathbf{E}_{2N_1}\left[\sum_{\{z\,:\,|z-x_{\min}|\le(\log N_1)^2\}} \mathbf{1}_{\{\phi_2(z)\in[-1/4,1/4]\}}\right] \ge c(\log N_1)^4(\log\log N_1)^{-1/2}, \tag{4.94}$$

which combined with (4.92) allows to conclude. □

4.9. Some comments on the lower bound's proof

We try to expose in a concise manner why our method fails to give a better upper bound than $\exp(|\log h|^{\alpha})$ (as can be seen from the proof the value $3/2$ is not important here and could be taken arbitrarily close to 2).

A first observation is that there is some freedom in the choice of r (4.22). The two places where the value of r is used are (4.75) and (4.83). In (4.75) clearly it would be sufficient to take h to be a negative power of (N_1) while while (4.83) could probably be refined. Thus the error term e^{-cr} from Proposition 4.1 is not a limiting factor in our proof.

The important point turns out to be the choice of N_1: in order to be able to linearize the exponential as we do in (4.58) using (4.57), we need to have $hN_1^2 \ll 1$. All the estimates after (4.58) are basically sharp and not much could be gained there.

We believe that this limitation is purely technical and that refinement of the technique might allow to consider larger values of N_1 which would improve both Proposition 4.3 and 4.4.

5. Finite volume criteria: adding mass and changing the boundary condition

Let us remark that it seems technically easier to get a lower bound for $N^{-2}\mathbb{E}\left[\log Z^{\beta,\omega}_{N,h}\right]$ for a given N than to prove one directly for the limit. However there is no obvious sub-additivity property which allows to compare the two.

In [35], for $d \geq 3$ we introduced the idea of replacing the boundary condition by an infinite volume free field in order to recover sub-additivity . In dimension 2, the infinite volume free field does not exist as the variance diverges with the distance to the boundary of the domain. A way to bypass the problem it to artificially introduce mass and then to find a comparison between the free energy of the system with massive free field and the original one. This is the method that we adopted in our previous paper (see [35, Proposition 7.1 and Lemma 7.2]). However our previous results turn out out to be a bit too rough for our proof. We present here an improvement of it (Proposition 5.3) on which we build the proof of Theorem 2.4.

5.1. A first finite volume criterion

Let us recall the comparison used in [35]. Even if it is not sufficient for our purpose in this paper, it will help us to explain the improvement presented in Section 5.2. Given $u > 0$ and $m > 0$, we introduce the notation

$$\delta^u_x := \mathbf{1}_{[u-1,u+1]}(\phi_x) \tag{5.1}$$

and set

$$Z^{\beta,\omega,m}_{N,h,u} := \mathbf{E}^m_N\left[\exp\left(\sum_{x\in\widetilde{\Lambda}_N}(\beta\omega_x-\lambda(\beta)+h)\delta^u_x\right)\right] \tag{5.2}$$

and

$$\textsc{f}(\beta,h,m,u) := \lim_{N\to\infty}\frac{1}{N^d}\log Z^{\beta,\omega,m}_{N,h,u}. \tag{5.3}$$

The existence of the above limit is proved in [35]. We can compare this free energy to the original one using the following result.

Proposition 5.1. – *We have for every u and m*

$$\textsc{f}(\beta,h,m,u) \leq \textsc{f}(\beta,h) + f(m). \tag{5.4}$$

where

$$f(m) := \frac{1}{2}\int_{[0,1]^2}\log\left(1+\frac{m^2}{4\left[\sin^2(\pi x/2)+\sin^2(\pi y/2)\right]}\right)\mathrm{d}x\mathrm{d}y. \tag{5.5}$$

There exists $C > 0$ such that for every $m \leq 1$ we have

$$\left|f(m)-\frac{1}{4\pi}m^2|\log m|\right| \leq Cm^2. \tag{5.6}$$

Moreover for all N we have

$$\textsc{f}(\beta,h,m,u) \geq \frac{1}{N^2}\widehat{\mathbf{E}}^m\mathbb{E}\left[\log\mathbf{E}^{m,\widehat{\phi}}_N\left[\exp\left(\sum_{x\in\widetilde{\Lambda}_N}(\beta\omega_x-\lambda(\beta)+h)\delta^u_x\right)\right]\right]. \tag{5.7}$$

Sketch of proof. – The result is proved in [35] (as Proposition 7.1 and Lemma 7.2) but let us recall briefly how it is done. For the first point, we have to remark that changing the height of the substrate (i.e., replacing δ_x by δ_x^u in (2.7)) for the original model does not change the value of the free energy, that is ,

$$\mathrm{F}(\beta,h,0,u)=\mathrm{F}(\beta,h,0,0),\qquad \text{for all values of } u.$$

Heuristically this is because the free field Hamiltonian is translation invariant but a proof is necessary to show that the boundary effect are indeed negligible (see [35, Proposition 4.1]). Note that for the massive free field, the limit (5.3) really depends on u because adding an harmonic confinement breaks the translation invariance.

Then we can compare the partition function associated with two free fields by noticing that the density of the massive field with respect to the original one (recall (3.9)) satisfies

$$\text{(5.8)}\qquad \frac{\mathrm{d}\mathbf{P}_N^m}{\mathrm{d}\mathbf{P}_N}(\phi):=\frac{\exp\left(-\frac{m^2}{2}\sum_{x\in\mathring{\Lambda}_N}\phi_x^2\right)}{\mathbf{E}_N\left[\exp\left(-\frac{m^2}{2}\sum_{x\in\mathring{\Lambda}_N}\phi_x^2\right)\right]}\le\frac{1}{\mathbf{E}_N\left[\exp\left(-\frac{m^2}{2}\sum_{x\in\mathring{\Lambda}_N}\phi_x^2\right)\right]},$$

and that

$$\text{(5.9)}\qquad \lim_{N\to\infty}\frac{1}{N^2}\log\mathbf{E}_N\left[\exp\left(-\frac{m^2}{2}\sum_{x\in\mathring{\Lambda}_N}\phi_x^2\right)\right]=:\lim_{N\to\infty}\frac{1}{N^2}\log W_N^m=-f(m).$$

Equation (5.7) then follows from of a sub-additive argument (see the proof of Proposition 4.2. in [35] or that of (5.20) below). □

REMARK 5.2. – *Note that Proposition 5.1 gives a bound on* $\mathrm{F}(\beta,h)$ *which depends only on the partition function of a finite system .*

$$\text{(5.10)}\qquad \mathrm{F}(\beta,h)\ge\frac{1}{N^2}\widehat{\mathbf{E}}^m\mathbb{E}\left[\log\mathbf{E}_N^{m,\widehat{\phi}}\left[\exp\left(\sum_{x\in\widetilde{\Lambda}_N}(\beta\omega_x-\lambda(\beta)+h)\delta_x^u\right)\right]\right]-f(m).$$

In particular we can prove Theorem 2.4, if for any $h>0$, $\beta\in(0,\overline{\beta})$ *we can find values for* u *and* m *and* N *such that the r.h.s. is positive. However it turns out that with our techniques, we cannot prove that the r.h.s. is positive for very small* h. *This is mostly because of the presence of a* $|\log m|$ *factor in the asymptotic behavior of* $f(m)$ *around* 0. *Therefore we need a better criterion in which the subtracted term is proportional to* m^2.

5.2. A finer comparison

To obtain a more efficient criterion, we want to restrict the partition function to a set of ϕ where $(\mathrm{d}\mathbf{P}_N^m/\mathrm{d}\mathbf{P}_N)(\phi)$ is much smaller than $\exp(N^2 f(m))$. We define $\mathscr{D}_N^0$ as a set where the density $(\mathrm{d}\mathbf{P}_N^m/\mathrm{d}\mathbf{P}_N)(\phi)$ takes "typical" values (see Proposition 6.1). For some constant $K>0$, we set

$$\text{(5.11)}\qquad \mathscr{D}_N^0:=\left\{\sum_{x\in\Lambda_N}\phi(x)^2\ge N^2\left(\frac{2f(m)}{m^2}-K\right)\right\}.$$

Recall that $\widehat{\mathbf{P}}^m$ denotes the law of the infinite volume massive free field (see Section 3.2) for the boundary condition $\widehat{\phi}$.

PROPOSITION 5.3. – *For any value of N, K and m we have*

$$\textsc{f}(\beta,h) \geq \frac{1}{N^2}\widehat{\mathbf{E}}^m \mathbb{E}\left[\log \mathbf{E}_N^{m,\widehat{\phi}}\left[\exp\left(\sum_{x\in\widetilde{\Lambda}_N}(\beta\omega_x - \lambda(\beta)+h)\,\delta_x^u\right)\mathbf{1}_{\mathcal{D}_N^0}\right]\right] - Km^2. \tag{5.12}$$

With the idea of working with a measure that does not depend on the boundary condition, we set similarly to (3.14)

$$\delta_x^{\widehat{\phi},u} := \mathbf{1}_{[u-1,u+1]}(\phi(x) + H_N^{m,\widehat{\phi}}(x)), \tag{5.13}$$

and

$$\mathcal{D}_N := \left\{\phi \ : \ \sum_{x\in\widetilde{\Lambda}_N}(\phi_x + H_N^{m,\widehat{\phi}}(x))^2 \geq N^2\left(\frac{2f(m)}{m^2} - K\right)\right\}. \tag{5.14}$$

With this notation and in view of the considerations of Section 3.3 the expected value in the r.h.s. in (5.12) is equal to

$$\widehat{\mathbf{E}}^m \mathbb{E}\left[\log \mathbf{E}_N^{m}\left[\exp\left(\sum_{x\in\widetilde{\Lambda}_N}(\beta\omega_x - \lambda(\beta)+h)\,\delta_x^{u,\widehat{\phi}}\right)\mathbf{1}_{\mathcal{D}_N}\right]\right]. \tag{5.15}$$

5.3. Using the criterion

Before giving a proof of Proposition 5.3 let us show how we are going to use it to prove our lower bound on the free energy (2.20). for the remainder of the proof we set

$$\begin{aligned} N_h &:= \exp(h^{-20}),\\ m_h &:= N_h^{-1}(\log N_h)^{1/4},\\ u_h &:= \sqrt{\frac{2}{\pi}}\log N_h - \frac{2+\alpha}{2\sqrt{2\pi}}\log\log N_h, \end{aligned} \tag{5.16}$$

where $\alpha = 3/4$ (we find that the computations are easier to follows with the letter α instead of a specific number, in fact any value in the interval $(11/20, 1)$ would also work). With Proposition 5.3, the proof of the lower bound in (2.20) is reduced to the following statement, whose proof will be detailed in the next three sections.

PROPOSITION 5.4. – *For any $\beta \leq \overline{\beta}$, there exists $h_0(\beta)$ such that for any $h \in (0, h_0(\beta))$*

$$\widehat{\mathbf{E}}^{m_h} \mathbb{E}\left[\log \mathbf{E}_{N_h}^{m_h,\widehat{\phi}}\left[\exp\left(\sum_{x\in\widetilde{\Lambda}_{N_h}}(\beta\omega_x - \lambda(\beta)+h)\,\delta_x^{u_h}\right)\mathbf{1}_{\mathcal{D}_{N_h}^0}\right]\right] - K(m_hN_h)^2 \geq 1. \tag{5.17}$$

Indeed the result directly implies that

$$\textsc{f}(\beta,h) \geq (N_h)^{-2}. \tag{5.18}$$

5.4. Proof of Proposition 5.3

Let us start by setting

$$Z'_N(\widehat{\phi}) = Z'_N := \mathbf{E}_N^m \left[\exp\left(\sum_{x\in\widetilde{\Lambda}_N} (\beta\omega_x - \lambda(\beta) + h)\, \delta_x^{\widehat{\phi},u} \right) \mathbf{1}_{\mathscr{D}_N} \right] \tag{5.19}$$
$$= \mathbf{E}_N^{m,\widehat{\phi}} \left[\exp\left(\sum_{x\in\widetilde{\Lambda}_N} (\beta\omega_x - \lambda(\beta) + h)\, \delta_x^{u} \right) \mathbf{1}_{\mathscr{D}^0_N} \right].$$

A simple computation (see below) is sufficient to show that for any $k \geq 0$ we have

$$\widehat{\mathbf{E}}^m \mathbb{E}\left[\log Z'_{2^k N}\right] \geq 4^k \widehat{\mathbf{E}}^m \mathbb{E}\left[\log Z'_N\right]. \tag{5.20}$$

Hence that it is sufficient to prove (5.12) with N replaced by $2^k N$ for an arbitrary integer k, or in the limit when k tends to infinity.

Let us prove (5.20). We divide the box Λ_{2N} into 4 boxes, Λ_N^i, $i = 1, \ldots, 4$. Set

$$\begin{aligned} \Lambda_N^i &:= \Lambda_N + (\alpha_1(i), \alpha_2(i))N \\ \widetilde{\Lambda}_N^i &:= \widetilde{\Lambda}_N + (\alpha_1(i), \alpha_2(i))N, \end{aligned} \tag{5.21}$$

where $\alpha_j(i) \in \{0,1\}$ is the j-th digit of the dyadic development of $i-1$. Set

$$\mathscr{D}_N^{0,i} := \left\{ \sum_{x\in\widetilde{\Lambda}_N^i} \phi(x)^2 \geq \left(\frac{2f(m)}{m^2} - K \right) N^2 \right\}. \tag{5.22}$$

We notice that

$$\bigcap_{i=1}^4 \mathscr{D}_N^{0,i} \subset \mathscr{D}_{2N}^0. \tag{5.23}$$

We define

$$\Gamma_N := \left(\bigcup_{i=1}^4 \partial\Lambda_N^i \right) \setminus \partial\Lambda_{2N}. \tag{5.24}$$

If we condition on the realization on ϕ in Γ_N, the partition functions of the system of size $2N$ factorizes into 4 partition functions of systems of size N, whose boundary conditions are determined by $\widehat{\phi}$ and $\phi|_{\Gamma_N}$, and we obtain

$$\mathbf{E}_{2N}^{m,\widehat{\phi}} \left[\exp\left(\sum_{x\in\widetilde{\Lambda}_{2N}} (\beta\omega_x - \lambda(\beta) + h)\delta_x^u \right) \mathbf{1}_{\bigcap_{i=1}^4 \mathscr{D}_N^{0,i}} \,\middle|\, \phi|_{\Gamma_N} \right] \tag{5.25}$$
$$= \prod_{i=1}^4 \mathbf{E}_{2N}^{m,\widehat{\phi}} \left[\exp\left(\sum_{x\in\widetilde{\Lambda}_N^i} (\beta\omega_x - \lambda(\beta) + h)\delta_x^u \right) \mathbf{1}_{\mathscr{D}_N^{0,i}} \,\middle|\, \phi|_{\Gamma_N} \right] =: \prod_{i=1}^4 \widetilde{Z}^i(\widehat{\phi}, \phi|_{\Gamma_N}, \omega).$$

By the spatial Markov property for the infinite volume field, each $\widetilde{Z}^i(\widehat{\phi}, \phi|_{\Gamma_N}, \omega)$ has the same distribution as Z'_N (if $\widehat{\phi}$ and $\phi|_{\Gamma_N}$ have distribution $\widehat{\mathbf{E}}^m$ and $\mathbf{E}^{m,\widehat{\phi}}_{2N}$ respectively and the ω_xs are IID). Using (5.23) and Jensen's inequality for $\mathbf{E}^{m,\widehat{\phi}}_{2N}\left[\,\cdot \mid \phi|_{\Gamma_N}\right]$ we have (recall

$$\mathbb{E}\widehat{\mathbf{E}}^m\left[\log Z'_{2N}\right] \geq \sum_{i=1}^{4} \mathbb{E}\widehat{\mathbf{E}}^m \mathbf{E}^{m,\widehat{\phi}}_{2N}\left[\log \widetilde{Z}^i(\widehat{\phi}, \phi|_{\Gamma_N})\right] = 4\mathbb{E}\widehat{\mathbf{E}}^m\left[\log Z'_N\right], \tag{5.26}$$

which ends the proof of (5.20).

Now we set $M := 2^k N$ with k large. In the computation, we write sometimes H for $H^{m,\widehat{\phi}}_M$ for simplicity. We remark that for $\phi \in \mathcal{D}_M$ we have

$$\begin{aligned}
&\log\left(\frac{\mathrm{d}\mathbf{P}^m_M}{\mathrm{d}\mathbf{P}_M}(\phi)\right) \\
&= \frac{m^2}{2}\left(\sum_{x\in\Lambda_M} H^2(x) - \sum_{x\in\Lambda_M}(\phi_x + H(x))^2 + 2\sum_{x\in\Lambda_M}\phi_x H(x)\right) - \log W_M \\
&\leq \left[M^2\left(\frac{m^2 K}{2} - f(m)\right) - \log W_M\right] + m^2\left[\sum_{x\in\Lambda_M}\phi_x H(x) + \frac{1}{2}\sum_{x\in\Lambda_M} H^2(x)\right] \\
&\leq m^2 M^2 K + m^2\left[\sum_{x\in\Lambda_M}\phi_x H(x) + \frac{1}{2}\sum_{x\in\Lambda_M} H^2(x)\right],
\end{aligned} \tag{5.27}$$

where the first inequality follows from the definition of $\mathcal{D}_M$ (5.14) and the last one from (5.9) and is valid provided k is sufficiently large. From this inequality we deduce that

$$\begin{aligned}
Z'_M &\leq e^{m^2 K M^2}\mathbf{E}_M\left[e^{\sum_{x\in\widetilde{\Lambda}_M}(\beta\omega_x - \lambda(\beta)+h)\delta^{\widehat{\phi},u}_x} e^{m^2\sum_{x\in\Lambda_M}\left[H(x)\phi(x) + \frac{H(x)^2}{2}\right]}\right] \\
&:= e^{m^2 K M^2} Z''_M.
\end{aligned} \tag{5.28}$$

To conclude the proof, we must show that the r.h.s. is not affected, in the limit, by the presence of H (which produces the two last terms and enters in the definition of $\delta^{\widehat{\phi},u}_x$) i.e., that

$$\lim_{M\to\infty}\frac{1}{M^2}\mathbb{E}\widehat{\mathbf{E}}^m\left[\log Z''_M\right] = \mathrm{F}(\beta, h). \tag{5.29}$$

We can replace $\delta^{\widehat{\phi},u}_x$ by δ^u_x at the cost of a Girsanov-type term in the density. For computations, it is practical to define

$$H^0(x) := H(x)\mathbf{1}_{\{x\in\mathring{\Lambda}_N\}}. \tag{5.30}$$

The distribution $\phi + H^0$ under $\mathbf{P}_M$ is absolutely continuous with respect to that of ϕ. The density of its distribution $\widetilde{\mathbf{P}}_M$ with respect to $\mathbf{P}_M$ is given by (recall the definition of the internal boundary $\partial^-\Lambda$ (2.1))

$$
\begin{aligned}
\frac{\mathrm{d}\widetilde{\mathbf{P}}_M}{\mathrm{d}\mathbf{P}_M^0}(\phi) &= \exp\Big(\frac{1}{2}\sum_{\Lambda_M}(\nabla\phi)^2 - (\nabla\phi - \nabla H^0)^2\Big)\\
&= \exp\Big(-\frac{1}{2}\sum_{\Lambda_M}(\nabla H_0)^2 + \sum_{\Lambda_M}\nabla\phi\nabla H^0\Big)\\
&= \exp\Big(-m^2\sum_{x\in\Lambda_M}\Big(H(x)\phi(x) - \frac{H^0(x)^2}{2}\Big)\\
&\quad + \sum_{x\in\partial\Lambda_M}\sum_{\substack{y\in\partial^-\Lambda_M\\ y\sim x}}\Big(H(x)\phi(y) - \frac{H(x)H(y)}{2}\Big)\Big),
\end{aligned}
\tag{5.31}
$$

where we used the notation

$$
\sum_{\Lambda_M}\nabla R\nabla T := \frac{1}{2}\sum_{\substack{x,y\in\Lambda_M\\ x\sim y}}(R(x)-R(y))(T(x)-T(y)).
\tag{5.32}
$$

To obtain the second line in (5.31) we have used the summation by part formula (which is valid without adding boundary terms since the functions we are integrating have zero boundary condition) and (3.12) to obtain

$$
\begin{aligned}
\sum_{\Lambda_M}\nabla H\nabla\phi &= -\sum_{x\in\mathring{\Lambda}_M}\Delta H(x)\phi(x) = -m^2\sum_{x\in\Lambda_M}H(x)\phi(x),\\
\sum_{\Lambda_M}\nabla H\nabla H^0 &= -\sum_{x\in\mathring{\Lambda}_M}\Delta H(x)H^0(x) = -m^2\sum_{x\in\Lambda_M}H^0(x)^2.
\end{aligned}
\tag{5.33}
$$

The substitution of H by H^0 produces the second term (boundary effects). Hence the expectation in (5.28) is equal to (assume $u > 1$)

$$
\begin{aligned}
&\exp\Big(\sum_{x\in\partial\Lambda_M\cap\widetilde{\Lambda}_M}(\beta\omega_x - \lambda(\beta) + h)\,\mathbf{1}_{[u-1,u+1]}(\widehat{\phi}(x))\\
&\quad + m^2\sum_{x\in\widetilde{\Lambda}_M}\frac{H(x)^2 + H^0(x)^2}{2} - \sum_{x\in\partial\Lambda_M}\sum_{\substack{y\in\partial^-\Lambda_M\\ y\sim x}}\frac{H(x)H(y)}{2}\Big)\\
&\quad\times\mathbf{E}_M^0\Big[\exp\Big(\sum_{x\in\widetilde{\Lambda}_M}(\beta\omega_x - \lambda(\beta) + h)\,\delta_x^u + \sum_{\substack{x\in\partial\Lambda_M, y\in\partial^-\Lambda_M\\ y\sim x}}H(x)\phi(y)\Big)\Big].
\end{aligned}
\tag{5.34}
$$

Let us show first that the exponential term in front of the expectation in (5.34) does not affect the limit of $M^{-2}\log Z''_M$. We have

$$\lim_{M\to\infty} \mathbb{E}\widehat{\mathbf{E}}^m \left| \frac{1}{M^2} \sum_{x\in\partial\Lambda_M\cap\widetilde{\Lambda}_M} (\beta\omega_x-\lambda(\beta)+h)\,\mathbf{1}_{[u-1,u+1]}(\widehat{\phi}(x)) \right| \le \lim_{M\to\infty}\frac{1}{M^2}\mathbb{E}\sum_{x\in\partial\Lambda_M\cap\widetilde{\Lambda}_M}|\beta\omega_x-\lambda(\beta)+h|=0. \tag{5.35}$$

For the other terms, set

$$\mathcal{M}_M := \max_{x\in\partial\Lambda_M}|\widehat{\phi}(x)|.$$

Being a maximum over $4M$ Gaussian variables of finite variance, it is not difficult to check that for all M sufficiently large,

$$\widehat{\mathbf{E}}[\mathcal{M}_M^2]\le(\log M)^2. \tag{5.36}$$

Moreover, recalling (3.13), we have

$$|H_M^{m,\widehat{\phi}}(x)| = \frac{1}{2d+m^2}\left|\sum_{y\sim x} H_M^{m,\widehat{\phi}}(y)\right| \le \frac{2d}{2d+m^2}\max_{y\sim x}|H_M^{m,\widehat{\phi}}(y)|. \tag{5.37}$$

This implies that the maximum of H is attained on the boundary and that

$$|H_M^{m,\widehat{\phi}}(x)| \le \mathcal{M}_M\left(\frac{2d}{2d+m^2}\right)^{d(x,\partial\Lambda_M)}. \tag{5.38}$$

This implies that

$$\left| m^2\sum_{x\in\widetilde{\Lambda}_M}\frac{H(x)^2+H^0(x)^2}{2} - \sum_{x\in\partial\Lambda_M}\sum_{\substack{y\in\partial^-\Lambda_M\\ y\sim x}}\frac{H(x)H(y)}{2}\right| \le C_m M\,\mathcal{M}_M^2. \tag{5.39}$$

In particular we have

$$\lim_{M\to\infty}\frac{1}{M^2}\widehat{\mathbf{E}}^m\left| m^2\sum_{x\in\widetilde{\Lambda}_M}\frac{H(x)^2+H^0(x)^2}{2} - \sum_{x\in\partial\Lambda_M}\sum_{\substack{y\in\partial^-\Lambda_M\\ y\sim x}}\frac{H(x)H(y)}{2}\right| =0. \tag{5.40}$$

Hence from (5.34), (5.35) and (5.40), Equation (5.29) holds provided we can show that

$$\lim_{M\to\infty}\frac{1}{M^2}\mathbb{E}\widehat{\mathbf{E}}^m\log\mathbf{E}_M\left[\exp\left(\sum_{x\in\widetilde{\Lambda}_M}(\beta\omega_x-\lambda(\beta)+h)\,\delta_x^u+T(\widehat{\phi},\phi)\right)\right]=\mathrm{F}(\beta,h), \tag{5.41}$$

where we have used the notation

$$T(\widehat{\phi},\phi):=\sum_{x\in\partial\Lambda_M}\sum_{\substack{y\in\partial^-\Lambda_M\\ y\sim x}}\widehat{\phi}(x)\phi(y) \tag{5.42}$$

(recall that $H(x)=\widehat{\phi}(x)$ for $x\in\partial\Lambda_M$). Note that conditioned to $\widehat{\phi}$, $T(\widehat{\phi},\phi)$ is a centered Gaussian random variable.

The proof of (5.41) is extremely similar to that of [35, Proposition 4.2] but we include the main line of the computation for the sake of completeness. We show in fact $\widehat{\mathbf{P}}^m \otimes \mathbb{P}$ almost sure convergence

$$\lim_{M\to\infty} \frac{1}{M^2} \log \mathbf{E}_M \left[\exp\left(\sum_{x\in\widetilde{\Lambda}_M} (\beta\omega_x - \lambda(\beta) + h)\,\delta_x^u + T(\widehat{\phi},\phi) \right)\right] = \mathrm{F}(\beta,h), \tag{5.43}$$

rather than convergence of the expectation of (5.41). However, since

$$\begin{aligned} &\left| M^{-2} \log \mathbf{E}_M \left[\exp\left(\sum_{x\in\widetilde{\Lambda}_M} (\beta\omega_x - \lambda(\beta) + h)\,\delta_x^u + T(\widehat{\phi},\phi) \right)\right]\right| \\ &\quad \le M^{-2} \sum_{x\in\widetilde{\Lambda}_M} |\beta\omega_x - \lambda(\beta) + h| + M^{-2} \log \mathbf{E}_M \left[e^{T(\widehat{\phi},\phi)} \right] \\ &\quad = M^{-2} \sum_{x\in\widetilde{\Lambda}_M} |\beta\omega_x - \lambda(\beta) + h| + \frac{1}{2} M^{-2} \mathrm{Var}_{\mathbf{P}_M} \left(T(\widehat{\phi},\phi) \right), \end{aligned} \tag{5.44}$$

and the sequence is uniformly integrable (cf. (5.45)), almost sure convergence implies convergence in L_1.

Now to prove (5.43), we remark that, as the covariance function of ϕ is positive, we have

$$\mathbf{E}_M\left[T(\widehat{\phi},\phi)^2 \right] \le \mathcal{M}_M^2 \mathbf{E}_M \left[\left(\sum_{x\in\partial\Lambda_M} \sum_{\substack{y\in\partial^-\Lambda_M \\ y\sim x}} \phi(y) \right)^2 \right] = 4(M-1)\,\mathcal{M}_M^2, \tag{5.45}$$

where the last equality is obtained similarly to [35, Equation (4.5)]. We define

$$A_M := \left\{ |T(\widehat{\phi},\phi)| \le M^{7/4}\,\mathcal{M}_M \right\}.$$

Combining our bound on the variance and standard Gaussian estimates, we obtain

$$\begin{aligned} \mathbf{P}_M\left[A_M^{\complement} \right] &\le e^{-cM^{5/2}}, \\ \mathbf{E}_M\left[e^{T(\widehat{\phi},\phi)} \mathbf{1}_{A_M^{\complement}} \right] &\le e^{-cM^{5/2}}. \end{aligned} \tag{5.46}$$

As we have

$$\mathbb{E}\mathbf{E}_M \left[e^{\sum_{x\in\widetilde{\Lambda}_M} (\beta\omega_x - \lambda(\beta)+h)\delta_x^u + T(\widehat{\phi},\phi)} \mathbf{1}_{A_M^{\complement}} \right] \le e^{hM^2} \mathbf{E}_M \left[e^{T(\widehat{\phi},\phi)} \mathbf{1}_{A_M^{\complement}} \right], \tag{5.47}$$

the second line of (5.46) implies in particular that

$$\lim_{M\to\infty} \frac{1}{M^2} \log \mathbb{E}\mathbf{E}_M \left[e^{\sum_{x\in\widetilde{\Lambda}_M} (\beta\omega_x - \lambda(\beta)+h)\delta_x^u + T(\widehat{\phi},\phi)} \mathbf{1}_{A_M^{\complement}} \right] = -\infty. \tag{5.48}$$

Applying Borel-Cantelli's Lemma, we can deduce that almost surely

$$\lim_{M\to\infty} \frac{1}{M^2} \log \mathbf{E}_M \left[e^{\sum_{x\in\widetilde{\Lambda}_M} (\beta\omega_x - \lambda(\beta)+h)\delta_x^u + T(\widehat{\phi},\phi)} \mathbf{1}_{A_M^{\complement}} \right] = -\infty, \tag{5.49}$$

and hence (5.43) is equivalent to

$$\lim_{M\to\infty} \frac{1}{M^2} \log \mathbf{E}_M \left[e^{\sum_{x\in\widetilde{\Lambda}_M} (\beta\omega_x - \lambda(\beta)+h)\delta_x^u + T(\widehat{\phi},\phi)} \mathbf{1}_{A_M} \right] = \mathrm{F}(\beta,h). \tag{5.50}$$

To prove (5.50), we first note using the first line of (5.46) that (2.8) implies that

$$\lim_{M\to\infty}\frac{1}{M^2}\log \mathbf{E}_M\left[\exp\left(\sum_{x\in\widetilde{\Lambda}_M}(\beta\omega_x-\lambda(\beta)+h)\,\delta^u_x\right)\mathbf{1}_{A_M}\right]=\mathrm{F}(\beta,h). \tag{5.51}$$

By definition of A_M we have

$$\frac{1}{M^2}\left|\log\frac{\mathbf{E}_M\left[e^{\sum_{x\in\widetilde{\Lambda}_M}(\beta\omega_x-\lambda(\beta)+h)\delta^u_x+T(\widehat{\phi},\phi)}\mathbf{1}_{A_M}\right]}{\mathbf{E}_M\left[e^{\sum_{x\in\widetilde{\Lambda}_M}(\beta\omega_x-\lambda(\beta)+h)\delta^u_x}\mathbf{1}_{A_M}\right]}\right|\le M^{-1/4}\,\mathcal{M}_M. \tag{5.52}$$

Hence to conclude we just need to show that

$$\lim_{M\to\infty} M^{-1/4}\,\mathcal{M}_M=0. \tag{5.53}$$

This follows from the definition of $\mathcal{M}_M$ and Borel-Cantelli's Lemma. □

6. Decomposition of the proof of Proposition 5.4

The overall idea for the proof is to restrict the partition function to a set of typical trajectories ϕ and to control the first two moments of the restricted partition function to get a good estimate for the expected log. However the implementation of this simple idea requires a lot of care. We decompose the proof in three steps.

In Section 6.1, we briefly present these steps and combine them to obtain the proof and in Section 6.2 we perform the first step of the proof, which is the simpler one. The two other steps need some detailed preparatory work which is only introduced in Section 7.

6.1. Sketch of proof

The first step is to show that $\mathcal{D}_N$, defined in (5.14), is a typical event in order to ensure that our restriction to $\mathcal{D}_N$ in the partition function does not cost much.

PROPOSITION 6.1. – *We can choose K in a way that for all $m\le 1$ sufficiently small, for all $N\ge m^{-1}|\log m|^{1/4}$, and for all realization of $\widehat{\phi}$*

$$\mathbf{P}^m_N[\mathcal{D}^{\complement}_N]\le C(\log N)^{-1/2}. \tag{6.1}$$

The result is not used directly in the proof of Proposition 5.4 but is a crucial input for the proof of Proposition 6.2 below.

The aim of the second step is to show that at a moderate cost one can restrict the zone of the interaction to a sub-box Λ'_N defined by

$$\Lambda'_N:=\mathbb{Z}^2\cap[N(\log N)^{-1/8},N(1-(\log N)^{-1/8})]^2. \tag{6.2}$$

The reason for which we want to perform this restriction is that it is difficult to control the effect of the boundary condition (i.e., of $H^{m,\widehat{\phi}}_N$) in $\partial\Lambda_N\setminus\Lambda'_N$. Inside Λ'_N however, due to the choice of the relative values of m and N in (5.16), $H^{m,\widehat{\phi}}_N$ is very small and has almost no effect.

PROPOSITION 6.2. – *There exists an event $\mathcal{C}_N \subset \mathcal{D}_N$ satisfying*

$$\mathbf{P}_N^m[\mathcal{C}_N^{\complement}] \le C(\log N)^{-1/16} \tag{6.3}$$

and a constant $C(\beta)$ such that

$$\begin{aligned}\mathbf{E}_N^m & \left[\exp\left(\sum_{x\in\widetilde{\Lambda}_N} (\beta\omega_x - \lambda(\beta) + h)\,\delta_x^{\widehat{\phi},u}\right)\mathbf{1}_{\mathcal{D}_N}\right] \\ &\ge \mathbf{E}_N^m\left[\exp\left(\sum_{x\in\Lambda'_N} (\beta\omega_x - \lambda(\beta) + h)\,\delta_x^{\widehat{\phi},u}\right)\mathbf{1}_{\mathcal{C}_N}\right] - C(\beta)(\log\log N)^4(\log N)^{\alpha-1/16}.\end{aligned} \tag{6.4}$$

Finally we have to show that the expected log of the restricted partition function in the r.h.s. of (6.4) is indeed sufficiently large to compensate for the second term. We actually only prove that this is the case for the set of good boundary conditions $\widehat{\phi}$ which have no significant influence in the bulk of the box

$$\widehat{\mathcal{A}}_N := \{\forall x \in \Lambda'_N, |H_N^{m,\widehat{\phi}}(x)| \le 1\}, \tag{6.5}$$

and show that the contribution of bad boundary condition is irrelevant.

We have chosen u_h in a way such that the expected density of contact is very small (the total expected number of contact in the box is a power of $\log N$, see (7.28) below), but the unlikely event that ϕ has a lot of contact is sufficient to make the second moment of the partition very large. Hence for our analysis to work, it is necessary to restrict the partition function to trajectories which have few contacts. We set

$$\begin{aligned} L_N &:= \sum_{x\in\Lambda'_N} \delta_x^{\widehat{\phi},u}, \\ \mathcal{B}_N &:= \mathcal{C}_N \cap \left\{L_N \le (\log N)^{\frac{\alpha+1}{2}}\right\}.\end{aligned} \tag{6.6}$$

We need to prove the following estimates concerning the restricted partition function in order to conclude.

PROPOSITION 6.3. – *We have*

(i) *For N sufficiently large*

$$\widehat{\mathbf{P}}^m[\widehat{\mathcal{A}}_N^{\complement}] \le N^{-4}. \tag{6.7}$$

(ii) *For any $\widehat{\phi} \notin \widehat{\mathcal{A}}_N$*

$$\mathbb{E}\log\mathbf{E}_N^m\left[\exp\left(\sum_{x\in\Lambda'_N} (\beta\omega_x - \lambda(\beta) + h)\,\delta_x^{\widehat{\phi},u}\right)\mathbf{1}_{\mathcal{C}_N}\right] \ge -N^2\lambda(\beta) - \log 2. \tag{6.8}$$

(iii) *There exists a constant $c > 0$ such that for any $\widehat{\phi} \in \widehat{\mathcal{A}}_N$*

$$\mathbb{E}\log\mathbf{E}_N^m\left[\exp\left(\sum_{x\in\Lambda'_N} (\beta\omega_x - \lambda(\beta) + h)\,\delta_x^{\widehat{\phi},u}\right)\mathbf{1}_{\mathcal{B}_N}\right] \ge ch(\log N)^{\alpha} - 2. \tag{6.9}$$

Proof of Proposition 5.4.. – Using Proposition 6.3, we have

$$\begin{aligned}
&\mathbb{E}\widehat{\mathbf{E}}^m \log \mathbf{E}_N^m \left[\exp\left(\sum_{x \in \Lambda'_N} (\beta\omega_x - \lambda(\beta) + h)\, \delta_x^{\widehat{\phi},u} \right) \mathbf{1}_{\mathcal{C}_N} \right] \\
&\geq -\widehat{\mathbf{P}}^m[\widehat{\mathscr{A}}_N^{\complement}] \left(N^2\lambda(\beta) + \log 2\right) \\
&\quad + \mathbb{E}\widehat{\mathbf{E}}^m \left[\log \mathbf{E}_N^m \left[\exp\left(\sum_{x \in \Lambda'_N} (\beta\omega_x - \lambda(\beta) + h)\, \delta_x^{\widehat{\phi},u} \right) \mathbf{1}_{\mathscr{B}_N} \right] \mathbf{1}_{\widehat{\mathscr{A}}_N} \right] \\
&\geq ch(\log N)^\alpha - 1.
\end{aligned} \tag{6.10}$$

Using Proposition 6.2 and recalling our choice of parameters (5.16), we have, for h sufficiently small

$$\begin{aligned}
&\mathbb{E}\widehat{\mathbf{E}}^m \log \mathbf{E}_N^m \left[\exp\left(\sum_{x \in \widetilde{\Lambda}_N} (\beta\omega_x - \lambda(\beta) + h)\, \delta_x^{\widehat{\phi},u} \right) \mathbf{1}_{\mathscr{D}_N} \right] - K(mN)^2 \\
&\geq ch(\log N)^\alpha - C(\beta)(\log\log N)^4 (\log N)^{\alpha - \frac{1}{16}} - K(\log N)^{1/2} - 1 \\
&\geq (c/2)(\log N)^{\alpha - \frac{1}{20}},
\end{aligned} \tag{6.11}$$

where in the last line we used that $\alpha - \frac{1}{20} > 1/2$. This is sufficient to conclude. □

6.2. Proof of Proposition 6.1

Again in this proof simply write H for $H_N^{m,\widehat{\phi}}$ The proof simply relies on computing the expectation and variance of $\sum_{x \in \widetilde{\Lambda}_N} (\phi(x) + H(x))^2$. We have

$$\mathbf{E}_N^m \left[\sum_{x \in \widetilde{\Lambda}_N} [\phi(x) + H(x)]^2 \right] = \mathbf{E}_N^m \left[\sum_{x \in \Lambda_N} \phi(x)^2 \right] + \sum_{x \in \widetilde{\Lambda}_N} H(x)^2. \tag{6.12}$$

From (3.22), for an appropriate choice of C, the following holds

$$\frac{1}{N^2} \mathbf{E}_N^m \left[\sum_{x \in \Lambda_N} \phi(x)^2 \right] \geq \frac{1}{2\pi} |\log m| - C \geq \frac{2f(m)}{m^2} - C. \tag{6.13}$$

Now let us estimate the variance. With the cancelation of odd moments of Gaussians, the expansion of the products gives

$$\begin{aligned}
&\mathbf{E}_N^m \left[\left(\sum_{x \in \widetilde{\Lambda}_N} (\phi(x) + H(x))^2 \right)^2 \right] - \left(\mathbf{E}_N^m \left[\sum_{x \in \widetilde{\Lambda}_N} (\phi(x) + H(x))^2 \right] \right)^2 \\
&= \mathbf{E}_N^m \left[\left(\sum_{x \in \Lambda_N} \phi(x)^2 \right)^2 \right] - \left(\mathbf{E}_N^m \left[\sum_{x \in \Lambda_N} \phi(x)^2 \right] \right)^2 \\
&\quad + 4\mathbf{E}_N^m \left[\sum_{x,y \in \widetilde{\Lambda}_N} \phi(x)\phi(y) H(x) H(y) \right].
\end{aligned} \tag{6.14}$$

We treat the last term separately and first concentrate on the two firsts which correspond to the zero boundary condition case. We have

$$\mathbf{E}_N^m\left[\phi(x)^2\phi(y)^2\right] - \mathbf{E}_N^m\left[\phi(x)^2\right]\mathbf{E}_N^m\left[\phi(y)^2\right] = 2\left[G^{m,*}(x,y)\right]^2, \tag{6.15}$$

and hence from (3.25) we can deduce that

$$\mathbf{E}_N^m\left[\left(\sum_{x\in\Lambda_N}\phi(x)^2\right)^2\right] - \left(\mathbf{E}_N^m\left[\sum_{x\in\Lambda_N}\phi(x)^2\right]\right)^2 = 2\sum_{x,y\in\Lambda_N}(G^{m,*}(x,y))^2 \le CN^2m^{-2}. \tag{6.16}$$

Concerning the last term in (6.14), we bound it as follows

$$\begin{aligned}\mathbf{E}_N^m\left[\sum_{x,y\in\widetilde{\Lambda}_N}\phi(x)\phi(y)H(x)H(y)\right] &= \sum_{x,y\in\widetilde{\Lambda}_N} G^{m,*}(x,y)H(x)H(y)\\ &\le \sum_{x\in\widetilde{\Lambda}_N}H(x)^2\sum_{y\in\Lambda_N}G^{m,*}(x,y) \le Cm^{-2}\sum_{x\in\widetilde{\Lambda}_N}H(x)^2,\end{aligned} \tag{6.17}$$

where in the last inequality we used (3.18). This gives

$$\mathrm{Var}_{\mathbf{P}_N^m}\left(\sum_{x\in\widetilde{\Lambda}_N}(\phi(x)+H(x))^2\right) \le Cm^{-2}\left(N^2 + \sum_{x\in\widetilde{\Lambda}_N}H(x)^2\right). \tag{6.18}$$

Hence, as long as K is chosen sufficiently large, using (6.18) and (6.12)-(6.13) we obtain

$$\begin{aligned}&\mathbf{P}_N^m\left[\frac{1}{N^2}\sum_{x\in\widetilde{\Lambda}_N}(\phi(x)+H(x))^2 \le \frac{2f(m)}{m^2} - K\right]\\ &\le \frac{\mathrm{Var}_{\mathbf{P}_N^m}\left(\sum_{x\in\widetilde{\Lambda}_N}(\phi(x)+H(x))^2\right)}{\left(\mathbf{E}_N^m\left[\sum_{x\in\widetilde{\Lambda}_N}(\phi(x)+H(x))^2\right] - N^2\left[\frac{2f(m)}{m^2}-K\right]\right)^2}\\ &\le \frac{Cm^{-2}\left(\sum_{x\in\widetilde{\Lambda}_N}H(x)^2+N^2\right)}{\left((K-C)N^2+\sum_{x\in\widetilde{\Lambda}_N}H(x)^2\right)^2} \le Cm^{-2}N^{-2}.\end{aligned} \tag{6.19}$$

The result thus follows for our choice for the range of N. □

7. Preliminary work for the proofs of Propositions 6.2 and 6.3

Both proofs require a detailed knowledge on the distribution of the number of contact in $\Lambda_N \setminus \Lambda'_N$ and in Λ'_N. The highly correlated structure of the field makes this kind of information difficult to obtain.

We have chosen u quite high in order to obtain a very low empirical density of contact. For this reason our problem is quite related to that of the study of the maximum and of the extremal process of the 2-dimensional free field, which has been the object of numerous studies in the past [14, 16, 24, 46] together with the related subject of Branching Random Walk [1, 2, 41] or Brownian Motion [6]. We borrow two key ideas from this literature:

(1) The Gaussian Free Field can be written as a sum of independent fields whose correlation spread on different scales. This makes the process very similar to the branching random walk.

(2) The number of points present at a height close to the expected maximum of the field is typically much smaller than its expectation (that is: by a factor $\log N$) but this log factor disapears if one conditions to a typical event.

These two points are respectively developed in Section 7.1 and 7.2.

7.1. Decomposing the free field in a martingale fashion

Let us decompose the massive free field into independent fields in order to separate the different scales in the correlation structure. The idea of decomposing the GFF is not new was used a lot to study the extremum and there are several possible choices (see [14] where a coarser decomposition is introduced or more recently [16]). Our choice of decomposition is made in order to have a structure similar to that present in [46].

There are several possible choices for the decomposition. The advantage of the one we present below is that the kernel of all the fields are expressed in terms of the heat-kernel, for which we have good estimates (cf. Section 3.4). Set (recall 3.16)

$$k := \lfloor G^m(x,x) \rfloor \tag{7.1}$$

(it does not depend on x as G is translation invariant). We perform the decomposition of ϕ into a sum k subfield, each of which having (roughly) unit variance. With this construction, $\phi(x)$ is the final step of a centered Gaussian random walk with k steps. With this in mind we define a decreasing sequence of times t_i, $i \in [\![0,k]\!]$ as follows

$$\begin{cases} t_0 := \infty, \\ \int_{t_1}^{\infty} e^{-m^2 t} P_t(x,x)\mathrm{d}t := 1, \\ \int_{t_{i+1}}^{t_i} e^{-m^2 t} P_t(x,x)\mathrm{d}t := 1, \quad i \in [\![1,k-2]\!], \\ t_k := 0. \end{cases} \tag{7.2}$$

This definition implies that

$$\int_0^{t_{k-1}} e^{-m^2 t} P_t(x,x)\mathrm{d}t \in [1,2). \tag{7.3}$$

From the Local Central Limit Theorem (3.20) we can deduce that there exists a constant $C > 0$ such that

$$\begin{aligned} \sup_{i \in [\![1,k-1]\!]} |\log t_i - 4\pi(k-i)| \le C, \\ \left| k + \frac{1}{2\pi} \log m \right| \le C. \end{aligned} \tag{7.4}$$

As a consequence of our choice of parameters (5.16) and of (7.4), we have the following asymptotic estimates

$$\begin{aligned} \frac{u}{k} &= 2\sqrt{2\pi} - \sqrt{\frac{\pi}{2}} \frac{\log\log N}{\log N} + O(\log N^{-1}), \\ \frac{u^2}{2k} &= 2\log N - \left(\frac{3}{2} + \alpha\right) \log\log N + O(1). \end{aligned} \tag{7.5}$$

We define $(\xi_i)_{i\in[\![1,k]\!]}$ to be a sequence of independent centered Gaussian fields (we use $\mathbf{P}$ to denote their joint law) indexed by Λ_N, each with covariance functions given by

$$Q_i^*(x,y) := \int_{t_i}^{t_{i-1}} e^{-m^2 t} P_t^*(x,y)\mathrm{d}t, \tag{7.6}$$

and set

$$\phi_i := \sum_{j=1}^{i} \xi_i. \tag{7.7}$$

Note that the covariance of ϕ_k is given by $G_N^{m,*}$ and for this reason we simply set $\phi := \phi_k$ and work from now with this extended probability space. For this reason we use simply $\mathbf{P}$ instead of $\mathbf{P}_N^m$ (this should bring no confusion as m and N a now fixed by (5.16)).

Note that the distribution of the field ξ_i in the bulk of Λ_N is "almost" translation invariant and its variance is very close to one. When x is close to the boundary $Q_i^*(x,x)$ becomes smaller, and this effect starts at distance $\exp(2\pi(k-i))$ from the boundary. The distance $\exp(2\pi(k-1))$ is also the scale on which covariance function $Q_i^*(x,y)$ varies in the bulk. For this reason it is useful to set

$$j(x) := \left(k - \left\lceil \frac{1}{2\pi}\log d(x,\partial\Lambda_N)\right\rceil\right)_+. \tag{7.8}$$

As a consequence of (3.22), (7.4) and of the definition of $j(x)$, we have

$$|\mathbf{E}[\phi^2(x)] - (k - j(x))| \le C. \tag{7.9}$$

We can deduce from this an estimate of the variance of $\phi_i(x)$, up to a correction of constant order: There exists a constant C such that

$$\forall x \in \Lambda_N,\ \forall i \in [\![0,k]\!], \quad \left|\mathbf{E}[\phi_i^2(x)] - (i - j(x))_+\right| \le C. \tag{7.10}$$

Indeed from Lemma 3.2 (iii), we have

$$\int_{t_{j(x)}}^{\infty} e^{-m^2 t} P_t^*(x,x)\mathrm{d}t \le C. \tag{7.11}$$

As the variance of $\xi_i(x)$ is bounded by 1 (or 2 when $i = k$) this implies

$$\mathbf{E}[\phi_i^2(x)] \le C + (i - j(x))_+. \tag{7.12}$$

Finally we obtain the other bound using the fact that, as the increments have variance smaller than one (ore two for the last one) we have

$$\mathbf{E}[\phi^2(x)] - \mathbf{E}[\phi_i^2(x)] \le k - i + 1, \tag{7.13}$$

and we conclude using (7.9).

7.2. The conditional expectation for the number of contact

Now we are going to use the decomposition in order to obtain finer results on the structure of the field ϕ. The idea is to show that with high probability the trajectory of $(\phi_i(x))_{i\in[\![0,k]\!]}$ tends to stay below a given line, for all $x \in \Lambda_N$, and thus if $\phi(x)$ reaches a value close to the maximum of the field, then conditioned to its final point, $(\phi_i(x))_{i=0}^k$ look more like a Brownian excursion than like a Brownian bridge, as it "feels" a constraint from above. If one restricts to the typical event described above, this constraint yields a loss of a factor k (hence $\log N$) in the probability of contact.

Note that for technical reasons, points near the boundary are a bit delicate to handle and thus we choose to prove a property in a sub-box Λ''_N which excludes only a few points of Λ_N. We set

$$\Lambda''_N := \mathbb{Z}^2 \cap \left[N(\log N)^{-2}, N(1-(\log N)^{-2})\right], \tag{7.14}$$

and

$$\gamma := 2\sqrt{2\pi}.$$

With our normalization, the constant γ is chosen so that γk gives the leading order for the asymptotic behavior for the maximum of our Gaussian Field in Λ_N with 0 boundary condition (see [14, 16] for the massless case, in our case m is chosen sufficiently small so that it does not alter this fact). Note (7.5) that u has been chosen in a way such that $u/k \overset{h\to 0}{\sim} \gamma$. From the Definition (7.8), $j(x)$ is of a smaller order as k when $x \in \Lambda''_N$, more precisely

$$\forall x \in \Lambda''_N, \quad j(x) \le \frac{1}{\pi} \log\log N. \tag{7.15}$$

We define

$$\mathscr{A}_N := \left\{\forall x \in \Lambda''_N,\ \forall i \ge j(x),\ \phi_i(x) \le \gamma(i - j(x)) + 100 \log\log N\right\}. \tag{7.16}$$

We show that this event is very typical. This is a crucial step to define the event $\mathscr{C}_N$ and to estimate the probability of $\mathscr{B}_N$.

PROPOSITION 7.1. – *We have*

$$\mathbf{P}[\mathscr{A}_N] \ge 1 - (\log N)^{-99}, \tag{7.17}$$

Proof. – We define for $i \in [\![0,k]\!]$

$$M_i := \frac{1}{|\Lambda''_N|} \sum_{x\in\Lambda''_N} \exp\left(\gamma\phi_i(x) - \frac{\gamma^2}{2}\mathbf{E}\left[\phi_i^2(x)\right]\right). \tag{7.18}$$

It is trivial to check that it is a martingale for the filtration

$$\mathscr{F}_i := \sigma(\phi_j(x), j \le i, x \in \Lambda''_N). \tag{7.19}$$

Integrating the second inequality in (3.23) on the interval $[t_i, \infty)$, we have for all $x, y \in \Lambda_N$ which satisfies $|x - y| \le e^{2\pi(k-i)}$

$$\mathbf{E}\left[(\phi_i(x) - \phi_i(y))^2\right] \le C|x-y|^2 e^{-4\pi(k-i)}. \tag{7.20}$$

Using a union bound and the Gaussian tail estimate (3.8), this implies that for N sufficiently large

$$\mathbf{P}\left[\max_{i\in[\![0,k-1]\!]} \max_{\left\{(x,y)\in(\Lambda_N)^2 \,:\, |x-y|\le e^{2\pi(k-i)}(\log N)^{-1}\right\}} |\phi_i(x) - \phi_i(y)| > 1\right] \le \frac{1}{N}. \tag{7.21}$$

On the complement of this event, if for a fixed $x \in \Lambda''_N$ we have

$$\phi_i(x) \ge \gamma(i - j(x)) + 100 \log\log N,$$

then

$$M_i \geq \frac{1}{|\Lambda''_N|} \sum_{\{y\,:\,|y-x|\leq e^{2\pi(k-i)}(\log N)^{-1}\}} e^{\gamma^2(i-j(x))+100\gamma\log\log N-\frac{\gamma^2}{2}\mathbf{E}[\phi_i^2(y)]-\gamma}. \tag{7.22}$$

Now as $i \geq j(x)$, we realize that in the range of y which is considered $j(y) \geq j(x) - 1$ and hence from (7.10) we have

$$\mathbf{E}[\phi_i^2(y)] \leq i - j(x) + C + 1.$$

For this reason, if N is sufficiently large, (7.22) implies that

$$\begin{aligned} M_i &\geq \frac{c}{|\Lambda''_N|(\log N)^2} \exp\left(4\pi(k-i) + \frac{\gamma^2}{2}(i - j(x)) + 100\gamma(\log\log N)\right) \\ &\geq c' e^{-4\pi j(x)}(\log N)^{100\gamma-5/2} \geq (\log N)^{100}, \end{aligned} \tag{7.23}$$

where for the first inequality we used (7.4) and the definition of m (5.16) to obtain

$$e^{4\pi k} \geq \frac{c}{m^2} \geq \frac{c' N^2}{(\log N)^{1/2}}, \tag{7.24}$$

and for the second one we used (7.15). Using (7.21) and the fact that M is a positive martingale with mean one, we conclude that

$$\mathbf{P}[\mathscr{A}_N] \leq \frac{1}{N} + \mathbf{P}\left[\exists i,\ M_i \geq (\log N)^{100}\right] \leq \frac{1}{N} + (\log N)^{-100}. \tag{7.25}$$ □

To conclude this section, we note that conditioning on the event $\mathscr{A}_N$ the probability of having a contact drops almost by a factor $(\log N)$, in the bulk of the box.

Lemma 7.2. – *There exists a constant C such that:*

(i) *For all $x \in \Lambda_N$ we have*

$$\frac{1}{C} N^{-2}(\log N)^{1+\alpha} \leq \widehat{\mathbf{E}}^m \mathbf{E}\left[\delta_x^{\widehat{\phi},u}\right] \leq C N^{-2}(\log N)^{1+\alpha}. \tag{7.26}$$

(ii) *For all $x \in \Lambda''_N$, we have*

$$\mathbf{E}\left[\delta_x^{\widehat{\phi},u}\mathbf{1}_{\mathscr{A}_N}\right] \leq C N^{-2}(\log N)^{\alpha}\left[H(x)^2 + (\log\log N)^2\right]\exp\left(\gamma H(x) - \frac{\gamma^2}{2} j(x)\right). \tag{7.27}$$

In particular

$$\widehat{\mathbf{E}}^m \mathbf{E}\left[\delta_x^{\widehat{\phi},u}\mathbf{1}_{\mathscr{A}_N}\right] \leq C N^{-2}(\log N)^{\alpha}(\log\log N)^2. \tag{7.28}$$

Proof. – Note that H being defined as linear function of $\widehat{\phi}$ (3.12), it is a Gaussian process which is independent of ϕ. To estimate the variance of $H(x)$ we recall that due to the spatial Markov property under $\widehat{\mathbf{P}}^m \otimes \mathbf{P}$, $\phi_x + H(x)$ is distributed an infinite volume free field and hence has variance $G^m(x,x) \in [k, k+1)$ meaning that $\widehat{\mathbf{E}}^m[H(x)^2] = G^m(x,x) - G^{m,*}(x,x)$. Hence from (7.9), we have

$$|\widehat{\mathbf{E}}^m[H(x)^2] - j(x)| \leq C. \tag{7.29}$$

To compute the l.h.s. of (7.27) we simply use the expression of the Gaussian density.

$$\widehat{\mathbf{E}}^m \mathbf{E}\left[\delta_x^{\widehat{\phi},u}\right] = \int_{u-1}^{u+1} \frac{e^{-\frac{-t^2}{2G^m(x,x)}}\mathrm{d}t}{\sqrt{2\pi G^m(x,x)}} \leq \frac{2}{\sqrt{2\pi G^m(x,x)}} e^{-\frac{-(u-1)^2}{2G^m(x,x)}}. \tag{7.30}$$

The result (the upper bound, but the lower bound is proved similarly, replacing $(u-1)$ by $(u+1)$ above) is obtained by using (7.5) after checking that

$$\left|\frac{(u-1)^2}{2G^m(x,x)}-\frac{u^2}{2k}\right|\le C. \tag{7.31}$$

Let us now focus on the second point. First we note that the result is completely obvious when $H(x)\ge 4u/5$: In that case, using (7.15) and the definition of u we see that the r.h.s. of (7.27) is larger than one. Hence we assume $H(x)\le 4u/5$ and notice that

$$\mathbf{E}\left[\delta_x^{\widehat{\phi},u}\mathbf{1}_{\mathscr{A}_N}\right]\le \mathbf{P}\big[\forall i\in [\![j(x),k]\!],\ \phi_i(x)\le\gamma(i-j(x))+100(\log\log N)\ ;\ \phi(x)+H(x)\in[u-1,u+1]\big]. \tag{7.32}$$

A first step is to show that

$$\mathbf{P}\big[\phi_x+H(x)\in[u-1,u+1]\big]\le CN^{-2}(\log N)^{\alpha+1}\exp\left(\gamma H(x)-\frac{\gamma^2}{2}j(x)\right). \tag{7.33}$$

Using the Gaussian tail estimate (3.8) and (7.9) we have

$$\mathbf{P}\big[\phi_k(x)+H(x)\in[u-1,u+1]\big]\le\frac{C\sqrt{k}}{u-H(x)}\exp\left(-\frac{(u-1-H(x))^2}{2(k-j(x)+C)}\right). \tag{7.34}$$

Note that the factor in front of the exponential is smaller than $C(\log N)^{-1/2}$ when $H(x)\le 4u/5$. Concerning the exponential term, using (7.5) and (7.15) we notice that

$$\begin{aligned}\frac{(u-1-H(x))^2}{2(k-j(x)+C)}&=\frac{u^2}{2k}+\frac{u^2(j(x)-C)}{2k(k-j(x)+C)}-\frac{(1+H(x))u}{k-j(x)+C}+\frac{(1+H(x))^2}{2(k-j(x)+C)}\\&\ge 2\log N-(\alpha+3/2)(\log\log N)+\frac{\gamma^2}{2}j(x)-\gamma H(x)-C'.\end{aligned} \tag{7.35}$$

This yields (7.33). To conclude the proof we need to show that for all $t\in[u-H(x)-1,u-H(x)+1]$

$$\mathbf{P}\big[\forall i\in[\![0,k]\!],\phi_i(x)\le\gamma(i-j(x))_+ +100(\log\log N)\mid\phi(x)=t\big]\le C(\log N)^{-1}\left(H(x)^2+(\log\log N)^2\right). \tag{7.36}$$

We use Lemma 3.3, for the re-centered walk

$$(X_i)_{i=1}^k:=(\phi_i(x)-\mathbf{E}[\phi_i(x)\mid\phi(x)=t])_{i=1}^k.$$

Let $V_i=V_i(x)$ denote the variance of $\phi_i(x)$ and $V=V(x)$ that of $\phi(x)$. We have by standard properties of Gaussian variables

$$\mathbf{E}[\phi_i(x)\mid\phi(x)=t]=(V_i/V)t.$$

From (7.5), and the fact that $V\le k+1$ we have for all the considered values of t and N sufficiently large

$$\frac{t}{V}\ge\frac{u-1}{k+1}-\frac{H(x)}{V}\ge\gamma-\frac{|H(x)|}{V}-C(\log N)^{-1}. \tag{7.37}$$

Hence using (7.10) and $(V_i/V)\le 1$ we obtain

$$\gamma(i-j(x))_+ +100(\log\log N)-(V_i/V)t\le 200(\log\log N)+|H(x)|. \tag{7.38}$$

Hence we have

$$\mathbf{P}[\forall i \in [\![0,k]\!], \phi_i(x) \le \gamma(i-j(x))_+ + 100(\log\log N) \mid \phi(x)=t] \\ \le \mathbf{P}\big[\ \forall i \in [\![0,k]\!], X_i \le 200(\log\log N) + |H(x)| \mid X_k = 0\ \big], \tag{7.39}$$

and we conclude using Lemma 3.3.

Finally (7.28) is deduced from (7.27) simply by using that $H(x)$ is Gaussian and that its variance satisfies (7.29) so that (recall (7.15))

$$\begin{aligned} \widehat{\mathbf{E}}^m\left[\exp\left(\gamma H(x) - \frac{\gamma^2}{2} j(x)\right)\right] &\le C, \\ \widehat{\mathbf{E}}^m\left[H(x)^2\exp\left(\gamma H(x) - \frac{\gamma^2}{2} j(x)\right)\right] &\le C(j(x))^2 \le C'(\log\log N). \end{aligned} \tag{7.40}$$

□

7.3. Proof of Proposition 6.2

We are now ready to define the event $\mathcal{C}_N$. We set

$$\mathcal{C}_N := \mathcal{D}_N \cap \mathcal{C}'_N, \tag{7.41}$$

where

$$\mathcal{C}'_N := \left\{\left(\sum_{x\in\widetilde{\Lambda}_N\setminus\Lambda'_N} \delta_x^{\widehat{\phi},u}\right) \le (\log N)^{1/16}\mathbf{E}\left[\sum_{x\in\widetilde{\Lambda}_N\setminus\Lambda'_N} \delta_x^{\widehat{\phi},u} \mid \mathcal{A}_N\right]\right\}. \tag{7.42}$$

From Markov's inequality, it is obvious that

$$\mathbf{P}[(\mathcal{C}')_N^{\complement} \mid \mathcal{A}_N] \le (\log N)^{-1/16}, \tag{7.43}$$

and we can conclude (provided that N is large enough) by using Propositions 6.1 and 7.1, that

$$\mathbf{P}[\mathcal{C}_N^{\complement}] \le \mathbf{P}[(\mathcal{C}')_N^{\complement} \mid \mathcal{A}_N] + \mathbf{P}[\mathcal{A}_N^{\complement}] + \mathbf{P}[\mathcal{D}_N^{\complement}] \le C(\log N)^{-1/16}. \tag{7.44}$$

Let us turn to the proof of (6.4). We want to get rid of the environment outside Λ'_N. The reader can check (by computing the second derivative that can be expressed as a variance)

$$\beta_2 \mapsto \mathbb{E}\left[\log\mathbf{E}\left[\exp\left(\sum_{x\in\Lambda'_N}(\beta\omega_x + h - \lambda(\beta))\delta_x^{\widehat{\phi},u} \right.\right.\right. \\ \left.\left.\left. + \sum_{x\in\widetilde{\Lambda}_N\setminus\Lambda'_N}(\beta_2\omega_x + h - \lambda(\beta))\delta_x^{\widehat{\phi},u}\right)\mathbf{1}_{\mathcal{D}_N}\right]\right] \tag{7.45}$$

is convex in β_2 and has zero derivative at 0. Hence reaches its minimum when β_2 equals zero, and

$$
\begin{aligned}
&\mathbb{E}\left[\log \mathbf{E}\left[\exp\left(\sum_{x\in\widetilde{\Lambda}_N}(\beta\omega_x-\lambda(\beta)+h)\,\delta_x^{\widehat{\phi},u}\right)\mathbf{1}_{\mathcal{D}_N}\right]\right]\\
&\geq \mathbb{E}\left[\log \mathbf{E}\left[\exp\left(\sum_{x\in\Lambda'_N}(\beta\omega_x+h-\lambda(\beta))\delta_x^{\widehat{\phi},u}-\lambda(\beta)\sum_{x\in\widetilde{\Lambda}_N\setminus\Lambda'_N}\delta_x^{\widehat{\phi},u}\right)\mathbf{1}_{\mathcal{D}_N}\right]\right]\\
&\geq \mathbb{E}\left[\log \mathbf{E}\left[\exp\left(\sum_{x\in\Lambda'_N}(\beta\omega_x+h-\lambda(\beta))\delta_x^{\widehat{\phi},u}\right)\mathbf{1}_{\mathcal{C}_N}\right]\right]\\
&\quad-(\log N)^{1/16}\lambda(\beta)\mathbf{E}\left[\sum_{x\in\widetilde{\Lambda}_N\setminus\Lambda'_N}\delta_x^{\widehat{\phi},u}\mid\mathcal{A}_N\right],
\end{aligned}
\tag{7.46}
$$

where the last line is obtained by restricting the expectation to $\mathcal{C}_N$ in order to bound $(\sum_{x\in\widetilde{\Lambda}_N\setminus\Lambda'_N}\delta_x^{\widehat{\phi},u})$ from below. Finally, using Lemma 7.2, more precisely (7.26) for $x\in\widetilde{\Lambda}_N\setminus\Lambda'_N$ and (7.28) for $x\in\Lambda'_N\setminus\Lambda''_N$, and the definition of Λ'_N (6.2) we obtain that

$$
\widehat{\mathbf{E}}^m\mathbf{E}\left[\sum_{x\in\widetilde{\Lambda}_N\setminus\Lambda'_N}\delta_x^{\widehat{\phi},u}\mid\mathcal{A}_N\right]\leq C(\log\log N)^2(\log N)^{\alpha-1/8},
\tag{7.47}
$$

which is sufficient to conclude. □

8. Proof of Proposition 6.3

8.1. Control of bad boundary conditions: Proof of (6.7) and (6.8)

We start with the easy part of the proposition by showing that the probability of a bad boundary condition is scarce (6.7), and that for this reason, a quite rough bound (6.8) is sufficient to bound their contribution to the total expectation.

To prove (6.7), we use Lemma 3.2. For a fixed $x\in\Lambda'_N$, we set in the next equation $d:=d(x,\partial\Lambda_N)$. We have

$$
\begin{aligned}
\widehat{\mathbf{E}}^m[(H_N^{m,\widehat{\phi}}(x))^2]&=\int_0^\infty e^{-m^2t}[P_t(x,x)-P_t^*(x,x)]\mathrm{d}t\\
&\leq\int_0^\infty\frac{C}{t}e^{-m^2t}\exp\left(-C^{-1}\min\left(\frac{d^2}{t},d\log[(d/t)+1]\right)\right)\mathrm{d}t\\
&\leq e^{-c'dm}\leq\exp\left(-c'(\log N)^{1/8}\right).
\end{aligned}
\tag{8.1}
$$

To check the penultimate inequality, the integration domain can be split at points $0<d<d/m<\infty$. We have used in the last inequality that $d(x,\Lambda_N)\geq N(\log N)^{-1/8}$ for $x\in\Lambda'_N$. Hence we have for any $x\in\Lambda'_N$

$$
\widehat{\mathbf{P}}^m\left[|H_N^{m,\widehat{\phi}}(x)|\geq 1\right]\leq\exp\left(-e^{c(\log N)^{1/8}}\right),
\tag{8.2}
$$

and we can conclude using a union bound.

To prove (6.8), we use Jensen's inequality and obtain

$$\mathbb{E}\log\mathbf{E}\left[\exp\left(\sum_{x\in\Lambda'_N}(\beta\omega_x-\lambda(\beta)+h)\,\delta_x^{\widehat{\phi},u}\right)\;\Big|\;\mathcal{C}_N\right]\\ \geq \mathbb{E}\mathbf{E}\left[\sum_{x\in\Lambda'_N}(\beta\omega_x-\lambda(\beta)+h)\,\delta_x^{\widehat{\phi},u}\;\Big|\;\mathcal{C}_N\right]\geq -\lambda(\beta)N^2. \tag{8.3}$$

Hence the conclusion follows from $\mathbf{P}[\mathcal{C}_N]\geq 1/2$.

8.2. Decomposing the proof of (6.9)

Proving that good boundary conditions give a good contribution to the expected log partition function (6.9), is the most delicate point. We divide the proof in several steps. First we want to show that conditioned on the event $\mathcal{B}_N$, the expected log partition function is close to the corresponding annealed bound (obtained by moving the expectation w.r.t. ω inside the log). This result is obtained by controlling the second moment of the restricted partition function.

LEMMA 8.1. – *For any* $\widehat{\phi}\in\widehat{\mathcal{A}_N}$ *we have*

$$\mathbb{E}\log\mathbf{E}\left[\exp\left(\sum_{x\in\Lambda'_N}(\beta\omega_x-\lambda(\beta)+h)\,\delta_x^{\widehat{\phi},u}\right)\;\Big|\;\mathcal{B}_N\right]\geq h\mathbf{E}\left[L_N\mid\mathcal{B}_N\right]-1. \tag{8.4}$$

The second point is to show that $\mathbf{E}\left[L_N\mid\mathcal{B}_N\right]$ is large. What makes this difficult is that L_N typically does not behave like its expectation $\mathbf{E}_N[L_N]$ (cf. Lemma 7.2). We are going to prove that conditioned to $\mathcal{A}_N$, L_N almost behaves like its expectation. To prove such a statement, we impose a restriction on the trajectories which is slightly stronger than $\mathcal{A}_N$, as this makes computation easier.

LEMMA 8.2. – *We have for any* $\widehat{\phi}\in\widehat{\mathcal{A}_N}$

$$\mathbf{E}\left[L_N\mid\mathcal{B}_N\right]\geq c(\log N)^{\alpha}. \tag{8.5}$$

Proof of (6.9). – First let us get a rough estimate on the probability of $\mathcal{B}_N$, valid for N sufficiently large

$$\mathbf{P}[\mathcal{B}_N^{\complement}]\leq C(\log N)^{-\frac{1}{16}}. \tag{8.6}$$

As $\mathbf{P}[\mathcal{A}_N]$ tends to one very fast (Proposition 7.1), it is sufficient to check the inequality for $\mathbf{P}[\mathcal{B}_N^{\complement}\mid\mathcal{A}_N]$. According to (7.27), for all $\widehat{\phi}\in\widehat{\mathcal{A}_N}$

$$\mathbf{E}\left[L_N\mathbf{1}_{\mathcal{A}_N}\right]\leq C(\log N)^{\alpha}(\log\log N)^2. \tag{8.7}$$

Hence using the definition of $\mathcal{B}_N$ (6.6), Markov's inequality and (6.3) we have

$$\begin{aligned}\mathbf{P}[\mathcal{B}_N^{\complement}\mid\mathcal{A}_N]&\leq \mathbf{P}[L_N\geq(\log N)^{\frac{1+\alpha}{2}}\mid\mathcal{A}_N]+\mathbf{P}[\mathcal{C}_N^{\complement}\mid\mathcal{A}_N]\\ &\leq C\left[(\log N)^{-\frac{1-\alpha}{2}}(\log\log N)^2+(\log N)^{-\frac{1}{16}}\right]\leq C'(\log N)^{-\frac{1}{16}},\end{aligned} \tag{8.8}$$

where in the last line we used that $\alpha = 3/4$. Combining (8.4) and (8.5), we have for $\widehat{\phi} \in \widehat{\mathcal{A}}_N$,

$$\begin{aligned}
&\mathbb{E}\log\mathbf{E}\left[\exp\left(\sum_{x\in\Lambda'_N}(\beta\omega_x-\lambda(\beta)+h)\,\delta_x^{\widehat{\phi},u}\right)\mathbf{1}_{\mathcal{C}_N}\right]\\
&\quad\geq \mathbb{E}\log\mathbf{E}\left[\exp\left(\sum_{x\in\Lambda'_N}(\beta\omega_x-\lambda(\beta)+h)\,\delta_x^{\widehat{\phi},u}\right)\mid\mathcal{B}_N\right]+\log\mathbf{P}\left[\mathcal{B}_N\right]\\
&\quad\geq h\mathbf{E}\left[L_N\mid\mathcal{B}_N\right]-2\geq ch(\log N)^{\alpha}-2.
\end{aligned}\tag{8.9}$$

□

8.3. Proof of Lemma 8.1

Proof. – Let us set

$$Y_N := \mathbb{E}\left[\exp\left(\sum_{x\in\Lambda'_N}(\beta\omega_x-\lambda(\beta)+h)\,\delta_x^{\widehat{\phi},u}\right)\mathbf{1}_{\mathcal{B}_N}\right],\tag{8.10}$$

and $\zeta := Y_N/\mathbf{E}[Y_N]$. We have

$$\mathbb{E}\left[\log Y_N\right]=\log\mathbb{E}\left[Y_N\right]+\mathbb{E}\log[\zeta].\tag{8.11}$$

We can bound the first term from below using Jensen's inequality as follows

$$\log\mathbb{E}[Y_N]=\log\mathbf{E}\left[\exp\left(hL_N\right)\mathbf{1}_{\mathcal{B}_N}\right]\geq h\mathbf{E}\left[L_N\mid\mathcal{B}_N\right]+\log\mathbf{P}[\mathcal{B}_N].\tag{8.12}$$

By (8.6), the second term is larger than $-\log 2$. To estimate $\mathbb{E}\log[\zeta]$ we simply compute the second moment of ζ. We have

$$\mathbb{E}[\zeta^2]=\widetilde{\mathbf{E}}_h^{\otimes 2}\left[\exp\left(\sum_{x\in\Lambda'_N}\chi(\beta)\delta_x^{(1)}\delta_x^{(2)}\right)\right],\tag{8.13}$$

where $\chi(\beta) := \lambda(2\beta)-2\lambda(\beta)$ and

$$\frac{\mathrm{d}\widetilde{\mathbf{P}}_h}{\mathrm{d}\mathbf{P}}(\phi) := \frac{1}{\mathbb{E}[Y_N]}\exp\left(hL_N\right)\mathbf{1}_{\mathcal{B}_N}.\tag{8.14}$$

Note that as a consequence of the definition of $\mathcal{B}_N$ for N sufficiently large, the density is bounded from above as follows

$$\frac{\mathrm{d}\widetilde{\mathbf{P}}_h}{\mathrm{d}\mathbf{P}}(\phi)\leq\frac{1}{\mathbf{P}[\mathcal{B}_N]}\exp\left(h(\log N)^{\frac{1+\alpha}{2}}\right)\leq N^{1/4}.$$

Using the inequality

$$\exp\left(\chi X\right)\leq 1+\frac{[e^{\chi K}-1]}{K}X\tag{8.15}$$

valid for $X\in[0,K]$, we obtain

$$\begin{aligned}
\mathbb{E}[\zeta^2]&\leq 1+e^{\chi(\beta)(\log N)^{\frac{1+\alpha}{2}}}\sum_{x\in\Lambda'_N}\widetilde{\mathbf{E}}_h^{\otimes 2}[\delta_x^{(1)}\delta_x^{(2)}]\\
&\leq 1+N^{1/2}e^{\chi(\beta)(\log N)^{\frac{1+\alpha}{2}}}\sum_{x\in\Lambda'_N}(\mathbf{E}[\delta_x^{\widehat{\phi}}])^2\leq 1+N^{3/4}\sum_{x\in\Lambda'_N}(\mathbf{E}[\delta_x^{\widehat{\phi}}])^2.
\end{aligned}\tag{8.16}$$

Note that from (3.22) and our choice for m (5.16), the variance of ϕ satisfies

$$\forall x \in \Lambda'_N, \quad \left| G^{*,m}(x,x) + \frac{1}{2\pi}\log m \right| \leq C. \tag{8.17}$$

Thus using our assumption on $|H(x)| \leq 1$, and (7.4) we have

$$\left| \frac{(u-1-H(x))^2}{2G^{*,m}(x,x)} - \frac{u^2}{2k} \right| \leq C \tag{8.18}$$

and thus we obtain that for all $x \in \Lambda'_N$

$$\mathbf{E}[\delta_x^{\widehat{\phi},u}]^2 \leq \left[\frac{2}{\sqrt{2\pi G^{*,m}(x,x)}} \exp\left(-\frac{(u-1-H(x))^2}{2G^{*,m}(x,x)} \right) \right]^2 \leq CN^{-4}(\log N)^{2(1+\alpha)}. \tag{8.19}$$

Thus we deduce from (8.16) that

$$\mathbb{E}[\zeta^2] - 1 \leq N^{-1}. \tag{8.20}$$

This ensures that ζ is close to one with a large probability. However to estimate $\mathbb{E}[\log \zeta]$, we also need some estimate on the right-tail distribution of $\log \zeta$. We use a rather rough one

$$|\log \zeta| \leq \max_{x \in \Lambda'_N} |\beta\omega_x - \lambda(\beta)|(\log N)^{\frac{1+\alpha}{2}}. \tag{8.21}$$

To conclude we note that for $\zeta \geq 1/2$ we have

$$\log(\zeta) + 1 - \zeta \geq -(\zeta-1)^2, \tag{8.22}$$

and hence that

$$\mathbb{E}[\log \zeta] = \mathbb{E}[\log(\zeta) + 1 - \zeta] \geq -\mathbb{E}[(\zeta-1)^2] + \mathbb{E}\left[(\log(\zeta) + 1 - \zeta)\mathbf{1}_{\{\zeta \leq 1/2\}} \right]. \tag{8.23}$$

The first term in the r.h.s. can be controlled using (8.20). By Cauchy-Schwartz, the second term is smaller in absolute value than

$$(\mathbb{P}[\zeta \leq 1/2])^{1/2} \left(\mathbb{E}\left[(\log \zeta + 1 - \zeta)^2 \mathbf{1}_{\{\zeta \leq 1/2\}} \right] \right)^{1/2} \leq (\mathbb{P}[\zeta \leq 1/2])^{1/2} \left(\mathbb{E}\left[(\log \zeta)^2 \right] \right)^{1/2}. \tag{8.24}$$

Using Chebychev inequality together with (8.20), we get that

$$\mathbb{P}[\zeta \leq 1/2] \leq 4N^{-1}.$$

Using (8.21) and the fact that ω have exponential tails (cf. assumption (2.5)), we have

$$\mathbb{E}\left[(\log \zeta)^2 \right] \leq C(\log N)^4. \tag{8.25}$$

Altogether we obtain that

$$\log \mathbb{E}[Y_N] \geq h\mathbf{E}\left[L_N \mid \mathcal{B}_N \right] + \log \mathbf{E}[\mathcal{B}_N] - CN^{-1/2}(\log N)^2, \tag{8.26}$$

and we can conclude using (8.6). □

8.4. Proof of Lemma 8.2

Instead of counting all the contacts, we decide to consider only a subset of them: those for which the trajectory $(\phi_i(x))_{i\in[\![0,k]\!]}$ stays below a given line. We choose the restriction to be a bit stronger than the one used in the definition of the event $\mathscr{A}_N$ (7.16). We set

$$\begin{aligned}\delta'_x &:= \mathbf{1}_{\left\{(\phi(x)-u+H(x))\in[-1,1],\ \forall i\in[\![1,k]\!],\ \phi_i(x)\le \frac{ui}{k}+10\right\}},\\ L'_N &:= \sum_{x\in\Lambda'_N}\delta'_x.\end{aligned} \tag{8.27}$$

Let us first show how to reduce the proof of Lemma 8.2 to a control on the two first moment of L'_N. We have

$$\begin{aligned}\mathbf{E}\left[L_N\mathbf{1}_{\mathscr{B}_N}\right] \ge \mathbf{E}\left[L'_N\mathbf{1}_{\mathscr{B}_N}\right] &= \mathbf{E}\left[L'_N\right]-\mathbf{E}\left[L'_N\mathbf{1}_{\mathscr{B}_N^{\complement}}\right]\\ &\ge \mathbf{E}\left[L'_N\right]-\sqrt{\mathbf{E}\left[(L'_N)^2\right]}\sqrt{\mathbf{P}\left[\mathscr{B}_N^{\complement}\right]}.\end{aligned} \tag{8.28}$$

Thus we can conclude provided that one can prove the two following bounds on the expectation and variance of L'_N

$$\begin{aligned}\mathbf{E}[L'_N] &\ge c(\log N)^{\alpha},\\ \mathbf{E}[(L'_N)^2] &\le C(\log N)^{2\alpha}(\log\log N)^6.\end{aligned} \tag{8.29}$$

It is then sufficient to combine these results with (8.28) and (8.6). Hence we need to prove the two following results.

Lemma 8.3. – *For all $x\in\Lambda'_N$ and $\widehat{\phi}\in\widehat{\mathscr{A}}_N$, we have*

$$cN^{-2}(\log N)^{\alpha} \le \mathbf{E}[\delta'_x] \le CN^{-2}(\log N)^{\alpha}(\log\log N)^2. \tag{8.30}$$

Lemma 8.4. – *We have for all $x,y\in\Lambda'_N$ (including $x=y$) and $\widehat{\phi}\in\widehat{\mathscr{A}}_N$,*

$$\mathbf{E}[\delta'_x\delta'_y] \le \frac{CN^{-4}(\log N)^{2\alpha+3}(\log\log N)^6}{(j(x,y)+1)^{3/2}(k-j(x,y)+1)^3}e^{\frac{j(x,y)u^2}{2k^2}}. \tag{8.31}$$

where

$$j(x,y) := \left\lceil \left(k-\frac{1}{2\pi}\log(|x-y|+1)\right)_+\right\rceil. \tag{8.32}$$

The quantity $j(x,y)$ can be interpreted as the step around which the increments of $(\phi_i(x))_{i=1}^k$ and $(\phi_j(x))_{i=1}^k$ decorrelate.

Remark 8.5. – *Note that has one has* $\mathbf{E}[\delta'_x\delta'_y]\le\mathbf{E}[\delta'_x]$, *the inequality* (8.31) *holds when* $(k-j)$ *is small as a consequence of the upper bound in* (8.30) *(use* (7.5) *to compute the exponential term* $e^{\frac{u^2}{2k^2}}$ *which appears in that case). Hence in our proof of Lemma 8.4 we will assume throughout the proof that $(k-j)$ is sufficiently large, meaning larger than an adequately chosen constant.*

Before giving the details of these lemmas, let us prove (8.29). The bound on the expectation follows immediately from (8.30). Concerning the bound on the variance, as for a fixed $l \in [\![1,k]\!]$, we have

(8.33) $$\#\{\,(x,y) \in (\Lambda'_N)^2 \;:\; j(x,y) = l\} \le CN^2 e^{4\pi(k-l)} = CN^4 (\log N)^{-1/2} e^{-4\pi l},$$

and a trivial bound of N^4 for the case $l = 0$. Hence we have

(8.34)
$$\begin{aligned}\mathbf{E}[(L'_N)^2] &= \sum_{x,y\in\Lambda'_N} \mathbf{E}[\delta'_x\delta'_y] \\ &\le C(\log N)^{2\alpha+3}(\log\log N)^6 \left[(\log N)^{-3} + \sum_{l=1}^{k} \frac{(\log N)^{-1/2} e^{-l\left(4\pi - \frac{u^2}{2k^2}\right)}}{(l+1)^{3/2}(k-l+1)^3}\right].\end{aligned}$$

We must then control the above sum. From (7.5) we have

(8.35) $$\frac{u^2}{2k^2} = 4\pi - 2\pi\,(1+\alpha)\,\frac{\log\log N}{\log N} + O((\log N)^{-1}),$$

and splitting the sum at the points $0 < k/2 < k - (\log N)^{5/6} < k$, we can check that

(8.36) $$\sum_{l=1}^{k} \frac{e^{-l\left(4\pi - \frac{u^2}{2k^2}\right)}}{(j+1)^{3/2}(k-j+1)^3} \le C(\log N)^{-\min(3,\frac{5}{2}+\alpha)}.$$

This implies (8.29).

8.5. Proof of Lemma 8.3

If $(u - H(x) - 1) \ge 0$ (which is satisfied if h is small enough because as $\widehat{\phi} \in \mathcal{A}_N$ we have $|H(x)| \le 1$), we obtain from the expression of the Gaussian density

(8.37) $$\begin{aligned}\frac{2}{\sqrt{2\pi G^{*,m}(x,x)}} e^{-\frac{(u-2)^2}{2G^{*,m}(x,x)}} &\le \mathbf{P}\big[\phi(x) \in [-1,1] + u - H(x)\big] \\ &\le \frac{2}{\sqrt{2\pi G^{*,m}(x,x)}} e^{-\frac{(u+2)^2}{2G^{*,m}(x,x)}}.\end{aligned}$$

We have (recall (8.18))

(8.38) $$\left|\frac{(u\pm 2)^2}{2G^{*,m}(x,x)} - \frac{u^2}{2k}\right| \le C$$

and thus using (7.5) and (8.17) we obtain

(8.39) $$cN^{-2}(\log N)^{1+\alpha} \le \mathbf{P}\,[\phi(x) \in [-1,1] + u - H(x)] \le CN^{-2}(\log N)^{1+\alpha}.$$

Now we can conclude provided we show that for all t in the interval $[u - 1 - H(x), u + 1 - H(x)]$, we have

(8.40) $$\frac{c}{\log N} \le \mathbf{P}\left[\forall i \in [\![1,k]\!],\ \phi_i(x) \le \frac{ui}{k} + 10 \mid \phi(x) = t\right] \le \frac{C(\log\log N)^2}{\log N}.$$

Let us recall the notation of Section 7.2: $V_i = V_i(x)$ denotes the variance of $\phi_i(x)$. For $i \le k-1$, we have

$$V_i(x) = \int_{t_i}^{\infty} e^{-m^2 t} P_t^*(x,x)\mathrm{d}t = i - \int_{t_i}^{\infty} e^{-m^2 t}\left[P_t(x,x) - P_t^*(x,x)\right]\mathrm{d}t. \tag{8.41}$$

Hence from (8.1) we have

$$\forall x \in \Lambda_N', \forall i \in [\![1,k]\!], \quad V_i(x) \in [i-1, i+1]. \tag{8.42}$$

We can check that (8.42) and $t \in [u-2, u+2]$ implies

$$1 \le \frac{ui}{k} + 10 - (V_i/V)t \le C. \tag{8.43}$$

To prove (8.40), we use simply Lemma 3.3 for the re-centered process

$$(X_i)_{i=1}^k := (\phi_i(x) - (V_i/V)t)_{i=1}^k .$$

We have

$$\mathbf{P}\left[\forall i \in [\![1,k]\!],\ \phi_i(x) \le \frac{ui}{k} + 10 \mid \phi(x) = t\right] \ge \mathbf{P}\left[\forall i \in [\![1,k]\!],\ X_i(x) \le 1 \mid X_k = 0\right] \ge \frac{c}{k}. \tag{8.44}$$

and

$$\mathbf{P}\left[\forall i \in [\![1,k]\!],\ \phi_i(x) \le \frac{ui}{k} + 10 \mid \phi(x) = t\right] \le \mathbf{P}\left[\forall i \in [\![1,k]\!],\ X_i(x) \le C \mid X_k = 0\right] \le \frac{C'(\log k)^2}{k}. \tag{8.45}$$

□

8.6. A simplified version of Lemma 8.4

We replace $(\phi_i(x))_{i=1}^k$ and $(\phi_i(y))_{i=1}^k$ and their intricate correlation structure by a simplified picture. Let $(X_i^{(1)})_{i=1}^k$, $(X_i^{(2)})_{i=1}^k$ be two walks, with IID standard Gaussian increments which are totally correlated until step $j \in [\![0,k]\!]$ and independent afterwards. More formally the covariance structure is given by

$$\begin{aligned} &\mathbf{E}[X_{i_1}^{(1)} X_{i_2}^{(2)}] := \min(i_1, i_2, j),\\ &\mathbf{E}[X_{i_1}^{(1)} X_{i_2}^{(1)}] = \mathbf{E}[X_{i_1}^{(2)} X_{i_2}^{(2)}] := \min(i_1, i_2). \end{aligned} \tag{8.46}$$

For $i \le j$ we set $X_i = X_i^{(1)} = X_i^{(2)}$. The simplified version of (8.31) we are going to prove is the following

$$\mathbf{P}\left[\forall l \in \{1,2\}, \forall i \in [\![1,k]\!],\ X_i^{(l)} \le \left(\frac{iu}{k} + 10\right),\ X_k^{(l)} \in [u-2, u+2]\right] \le \frac{C(\log\log N)^6}{(j+1)^{3/2}(k-j+1)^3} \exp\left(-\frac{(2k-j)u^2}{2k^2}\right). \tag{8.47}$$

The inequality above gets very similar to (8.31) if the term u/k appearing in the exponential is replaced by the asymptotic equivalent in the first line of (7.5).

Note that we replaced the interval $[u - H(x) - 1, u - H(x) + 1]$ and $[u - H(y) - 1, u - H(y) + 1]$ by $[u-2, u+2]$, and we also do so in the true proof of Lemma 8.4. This is ok since we are looking for an upper bound, as $\widehat{\phi} \in \widehat{\mathcal{A}}_N$, the latter interval includes the other two.

The strategy is to first evaluate the probability

$$\mathbf{P}\Big[X_j \in \mathrm{d}t\ ;\ \forall l \in \{1,2\},\ X_k^{(l)} \in [u-2, u+2]\Big],$$

and then compute the cost of the constraint $X_i^{(l)} \leq \frac{iu}{k} + 10$ using Lemma 3.3 and the fact that conditioned to X_j, $X_k^{(1)}$ and $X_k^{(2)}$, the processes $(X_i)_{i=1}^{j}$, $(X_i^{(1)})_{i=j}^{k}$ and $(X_i^{(2)})_{i=j}^{k}$ are three independent Brownian bridges. For the first step, notice that we have

$$\begin{aligned}(8.48)\quad &\mathbf{P}\Big[X_j \in \mathrm{d}t,\ X_k^{(1)} \in \mathrm{d}s_1,\ X_k^{(2)} \in \mathrm{d}s_2\Big]\\ &= \frac{1}{(2\pi)^{3/2}(k-j)\sqrt{j}} \exp\left(-\frac{t^2}{2j} - \frac{(s_1-t)^2 + (s_2-t)^2}{2(k-j)}\right) \mathrm{d}t\,\mathrm{d}s_1\mathrm{d}s_2.\end{aligned}$$

With the constraint $s_1, s_2 \in [u-2, u+2]$ and $t \leq \left(\frac{ju}{k} + 10\right)$, at the cost of loosing a constant factor we can replace s_1 and s_2 by $u-2$. We obtain, after integrating over s_1 and s_2,

$$\begin{aligned}(8.49)\quad &\mathbf{P}\Big[X_j \in \mathrm{d}t,\ X_k^{(1)}, X_k^{(2)} \in [u-2, u+2]\Big] \leq \frac{C}{(k-j)\sqrt{j}} \exp\left(-\frac{t^2}{2j} - \frac{(u-2-t)^2}{k-j}\right) \mathrm{d}t\\ &\leq \frac{C}{(k-j)\sqrt{j}} \exp\left(-\frac{(2k-j)u^2}{2k^2} - \left(\frac{u}{k} - \frac{4}{k-j}\right)\left(\frac{uj}{k} - t\right)\right) \mathrm{d}t.\end{aligned}$$

One way to obtain the second inequality is to define δ by setting $t = uj/k + \delta$, to expand the two squares $(uj/k + \delta)^2$ and $(u(k-j)/k - 2 + \delta)^2$ in the exponential and to neglect the δ^2 terms.

Note that due to our choice for u (5.16) and value of k we have $\left(\frac{u}{k} - \frac{4}{k-j}\right) \in [\gamma/2, \gamma]$ provided that h is sufficiently small and $k - j$ is sufficiently large (and hence the term can be replaced by $\gamma/2$ at the cost of changing the value of C).

Now using Lemma 3.3 (after re-centering the process), we obtain that

$$\begin{aligned}(8.50)\quad &\mathbf{P}\left[X_i \leq \left(\frac{ui}{k} + 10\right),\ \forall i \in [\![0, j]\!] \mid X_j = t\right]\\ &= \mathbf{P}\left[X_i \leq \left(\frac{ui}{k} + 10\right) - \frac{it}{j},\ \forall i \in [\![0, j]\!] \mid X_j = 0\right]\\ &\leq C j^{-1}\left(\left(\frac{uj}{k} - t\right)^2 + (\log j)^2\right),\end{aligned}$$

where we have used that for $t \leq \left(\frac{uj}{k} + 10\right)$ and $i \leq j$

$$\left(\frac{ui}{k} + 10\right) - \frac{it}{j} = \frac{i}{j}\left(\frac{uj}{k} - t\right) + 10 \leq \left(\frac{uj}{k} - t\right) + 20.$$

In the same manner we obtain that for $l \in \{1,2\}$

$$\mathbf{P}\left[X_i^{(l)} \le \frac{ui}{k} + 10,\ \forall i \in [\![j,k]\!] \mid X_j = t, X_k^{(l)} \in [u-2,u+2]\right] \le C(k-j)^{-1}\left(\left(\frac{uj}{k}-t\right)^2 + (\log(k-j))^2\right). \tag{8.51}$$

Hence using (8.49)-(8.50)-(8.51) and conditional independence we obtain that

$$\mathbf{P}\left[\forall l \in \{1,2\},\ \forall i \in [\![1,k]\!],\ X_i^{(l)} \le \frac{iu}{k} + 10\,;\ X_k^{(l)} \in [u-2,u+2]\,;\ X_j \in \mathrm{d}t\right] \le C(k-j)^{-3} j^{-3/2} (\log k)^6 \exp\left(-\frac{(2k-j)u^2}{2k^2} - (\gamma/4)\left(\frac{uj}{k}-t\right)\right)\mathrm{d}t, \tag{8.52}$$

(where $\gamma/4$ appears instead of $\gamma/2$ in order to absorb the powers of $\left(\frac{uj}{k}-t\right)$ appearing in front of the exponential) which after integration over $t \le (\frac{uj}{k}+10)$ gives

$$\mathbf{P}\left[\forall l \in \{1,2\}\forall i \in [\![1,k]\!],\ X_i^{(l)} \le \frac{iu}{k} + 10, X_k^{(l)} \in [u-2,u+2]\right] \le C(k-j)^{-3} j^{-3/2} (\log k)^6 \exp\left(-\frac{(2k-j)u^2}{2k^2}\right). \tag{8.53}$$

8.7. Proof of Lemma 8.4

Now, we are ready to handle the case where $X_i^{(1)}$ and $X_i^{(2)}$ are replaced by $\phi_i(x)$ and $\phi_i(y)$. Some adaptations are needed since the increments of $\phi_i(x)$ and $\phi_i(y)$ have a less simple correlation structure but the method presented above is hopefully robust enough to endure such mild modifications. Given x and y set

$$Z_i(x,y) = Z_i := \frac{\phi_i(x)+\phi_i(y)}{2} \quad \text{and} \quad U_i := \mathbb{E}\left[Z_i\right]^2. \tag{8.54}$$

Let us prove that, for $j = j(x,y)$ defined in (8.32), there exists a constant C such that

$$\begin{cases} |U_i - i| \le C, & \forall i \in [\![0,j]\!], \\ \left|U_i - \frac{i+j}{2}\right| \le C, & \forall i \in [\![j,k]\!]. \end{cases} \tag{8.55}$$

To see this, it is sufficient to remark that

$$\begin{aligned} U_i &:= \frac{1}{4}\int_{t_i}^{\infty} e^{-m^2 t}\left[P_t^*(x,x) + P_t^*(y,y) + 2P_t^*(x,y)\right]\mathrm{d}t \\ &= \frac{1}{2}\int_{t_i}^{\infty} e^{-m^2 t}\left[P_t(x,x) + P_t(x,y)\right]\mathrm{d}t - r_i(x,y), \end{aligned} \tag{8.56}$$

where

$$r_i(x,y) := \frac{1}{4}\int_{t_i}^{\infty} e^{-m^2 t}\left[(P_t - P_t^*)(x,x) + (P_t - P_t^*)(y,y) + 2(P_t - P_t^*)(x,y)\right]\mathrm{d}t. \tag{8.57}$$

Using (8.1), we see that P_t^* can be replaced by P_t at the cost of a small correction i.e., that r_i is small. More precisely, recalling (3.19) and (8.1) we observe that

$$\begin{aligned}(8.58)\quad \int_0^\infty e^{-m^2 t}\left[(P_t-P_t^*)(x,x)+(P_t-P_t^*)(y,y)+2(P_t-P_t^*)(x,y)\right]\\ = \widehat{\mathbf{E}}^m[(H_N^{m,\widehat{\phi}}(x)+H_N^{m,\widehat{\phi}}(y))^2] \le 2\widehat{\mathbf{E}}^m[(H_N^{m,\widehat{\phi}}(x))^2+(H_N^{m,\widehat{\phi}}(y))^2].\end{aligned}$$

Using the definition of t_i (7.2), we have for $i \in [\![0,j]\!]$

$$(8.59)\quad \frac{1}{2}\int_{t_i}^\infty e^{-m^2 t}\left[P_t(x,x)+P_t(x,y)\right]\mathrm{d}t = i - \frac{1}{2}\int_{t_i}^\infty e^{-m^2 t}\left[P_t(x,x)-P_t(x,y)\right]\mathrm{d}t,$$

while for $i \in [\![j,k]\!]$ we have

$$\begin{aligned}(8.60)\quad &\frac{1}{2}\int_{t_i}^\infty e^{-m^2 t}\left[P_t(x,x)+P_t(x,y)\right]\mathrm{d}t\\ &= \frac{i+j}{2} - \frac{1}{2}\int_{t_j}^\infty e^{-m^2 t}\left[P_t(x,x)-P_t(x,y)\right]\mathrm{d}t + \frac{1}{2}\int_{t_i}^{t_j} e^{-m^2 t}P_t(x,y)\mathrm{d}t.\end{aligned}$$

The kernel estimates (3.24) and (3.23) then allow to conclude that the integrals in the r.h.s of (8.59) and (8.60) are bounded by a constant and thus that (8.55) holds. Similarly to (8.49), we are first going to show that we have, for all $t \le (\frac{uj}{k}+10)$,

$$\begin{aligned}(8.61)\quad &\mathbf{P}\left[Z_j \in \mathrm{d}t \;\; \phi(x),\phi(y) \in [u-2,u+2]\right]\\ &\le \frac{C}{(k-j)\sqrt{j}}\exp\left(-\frac{(2k-j)u^2}{2k^2}-(\gamma/2)\left(\frac{uj}{k}-t\right)\right)\mathrm{d}t.\end{aligned}$$

Using the independence of Z_j and $Z_k - Z_j$ and the fact that, up to correction of a constant order their respective variances are respectively equal to j and $(k-j)/2$ (cf (8.59)-(8.60)), we can obtain (provided that $(k-j)$ is large enough), similary to (8.49) that

$$\begin{aligned}(8.62)\quad &\mathbf{P}\left[Z_j \in \mathrm{d}t\;;\; Z_k \in [u-2,u+2]\right]\\ &\le \frac{C}{\sqrt{j(k-j)}}\exp\left(-\frac{(2k-j)u^2}{2k^2}-(\gamma/2)\left(\frac{uj}{k}-t\right)\right)\mathrm{d}t.\end{aligned}$$

Now, on top of that, we want to show that

$$(8.63)\quad \mathbf{P}\left[\,(\phi(x)-\phi(y)) \in [-4,4] \mid Z_j = t, Z_k \in [u-2,u+2]\,\right] \le C(k-j)^{-1/2}.$$

As $(\phi(x)-\phi(y))$ is a Gaussian we can prove (8.63) by showing that

$$(8.64)\quad \mathrm{Var}_{\mathbf{E}[\cdot\,\mid\, Z_j,Z_k]}\left[\phi(x)-\phi(y)\right] \ge c(k-j),$$

at least when $(k-j)$ is large: it implies that conditional density is bounded by $(2\pi c(k-j))^{-1/2}$ and thus that (8.63) holds. In fact we prove this bound for the variance conditioned to $\phi_j(x),\phi_j(y)$ and Z_k (which is smaller as the conditioning is stronger) as it is easier to compute.

Setting

$$Z'_i = \phi_i(x) - \phi_i(y),$$

we notice, first using the fact that the increments of (Z, Z') are independent and then the usual formula for the conditional variance of Gaussian variable, that

$$\mathrm{Var}_{\mathbf{E}[\cdot\mid\phi_j(x),\phi_j(y),Z_k]} = \mathbf{E}[(Z'_k - Z'_j)^2] - \frac{\left(\mathbf{E}\left[(Z'_k - Z'_j)(Z_k - Z_j)\right]\right)^2}{\mathbf{E}[(Z_k - Z_j)^2]}. \tag{8.65}$$

Using (8.1) (to replace P^*_t by P_t) and (3.24) (to control the term $P^*_t(x, y)$) we have

$$\begin{aligned}\mathbf{E}[(Z'_k - Z'_j)^2] &= \int_0^{t_j} e^{-m^2 t}\left[P^*_t(x,x) + P^*_t(y,y) - 2P^*_t(x,y)\right]\mathrm{d}t\\ &\geq \int_0^{t_j} e^{-m^2 t}\left[P_t(x,x) + P_t(y,y)\right]\mathrm{d}t - C \geq 2(k-j) - C.\end{aligned} \tag{8.66}$$

Obviously $\mathbf{E}\left[(Z_k - Z_j)^2\right]$ is of the same order, and from (8.1) again.

$$\left|\mathbf{E}\left[(Z'_k - Z'_j)(Z_k - Z_j)\right]\right| = \left|\frac{1}{2}\int_0^{t_j} e^{-m^2 t}(P^*_t(x,x) - P^*_t(y,y))\mathrm{d}t\right| \leq 1. \tag{8.67}$$

Hence combining these inequalities in (8.65) we obtain that (8.64) holds. To conclude the proof we need to show that

$$\mathbf{P}\left[\forall i \in [\![0, j]\!], Z_i \leq \frac{ui}{k} + 10 \mid Z_j = t\right] \leq C\left[\left(\frac{uj}{k} - t\right)^2 + (\log j)^2\right] j^{-1} \tag{8.68}$$

and

$$\begin{aligned}\mathbf{P}\left[\ \forall i \in [\![j, k]\!], \phi_i(x), \phi_i(y) \leq \frac{ui}{k} + 10 \mid Z_j = t,\ \phi(x), \phi(y) \in [u-2, u+2]\ \right]&\\ \leq \left[\left(\frac{uj}{k} - t\right)^2 + (\log j)^2\right]^2 (k-j)^{-2}.&\end{aligned} \tag{8.69}$$

Indeed using conditional independence we can multiply the inequalities (8.68) and (8.69) with (8.61) to obtain

$$\begin{aligned}&\mathbf{P}[\delta'_x\delta'_y,\ Z_j \in \mathrm{d}t]\\ &\quad\leq \frac{C\left[\left(\frac{uj}{k} - t\right)^2 + (\log k)^2\right]^3}{(k-j)^3 j^{3/2}} \exp\left(-\frac{(2k-j)u^2}{2k^2} - (\gamma/2)\left(\frac{uj}{k} - t\right)\right)\mathrm{d}t,\end{aligned} \tag{8.70}$$

and conclude by integrating over t. The proof of (8.68) is quite similar to that of (8.50).

$$\begin{aligned}&\mathbf{P}\left[\forall i \leq j,\ Z_i \leq \frac{ui}{k} + 10 \mid Z_j = t\right]\\ &\qquad = \mathbf{P}\left[\forall i \leq j,\ Z_i \leq \frac{ui}{k} + 10 - (U_i/U_j)t \mid Z_j = 0\right].\end{aligned} \tag{8.71}$$

We use (8.59) to obtain for all $i \in [\![0, j]\!]$,

$$\frac{ui}{k} - \frac{U_i t}{U_j} \leq \frac{U_i}{U_j}\left(\frac{uj}{k} - t\right) + C \leq \left(\frac{uj}{k} - t\right) + C' \tag{8.72}$$

and applying Lemma 3.3, we obtain

$$\mathbf{P}\left[\forall i \le j,\ Z_i \le \frac{ui}{k} + 10 \mid Z_j = t\right] \le Cj^{-1}\left(\left(\frac{uj}{k} - t\right)^2 + (\log j)^2\right). \tag{8.73}$$

To prove (8.69) we have to be more careful as the increments of $\phi(x)$ and $\phi(y)$ are correlated. It is more practical in the computation to condition to the constraint $(\phi_j(x), \phi_j(y)) = (t_1, t_2)$ than to $Z_j = t$. To obtain a bound we then take the maximum over the constraint $(t_1 + t_2) = 2t$. We consider only the case $\phi(x) = \phi(y) = u - 2$ in the conditioning as the others can be deduced by monotonicity (which follows from positive correlations in the Gaussian processes that are considered). We can consider without loss of generality that

$$\frac{ju}{k} - C(k-j) \le t_1, t_2 \le \frac{ju}{k} + 10, \tag{8.74}$$

the upper bound is due to the conditioning, and if the lower bound is violated, the r.h.s. of (8.69) is larger than one. Similarly to (8.73), using (8.42) to control the value of V_i we can prove

$$\begin{aligned}\mathbf{P}\left[\forall i \in [j,k],\ \phi_i(x) \le \frac{ui}{k} + 10 \mid \phi_j(x) = t_1,\ ;\ \phi(x) \in [u-2, u+2]\right]&\\ \le C(k-j)^{-1}\left(\left(\frac{uj}{k} - t_1\right)^2 + (\log(k-j))^2\right)&.\end{aligned} \tag{8.75}$$

Now the challenge lies in estimating the cost of the constraint $\phi_i(y) \le (\frac{ui}{k} + 10)$, on the segment $[\![j,k]\!]$, knowing $\phi(y)$, $\phi_j(y)$ and $\phi_i(x)$, $i \in [\![1,k]\!]$. After conditioning to $\phi_j(y)$ and $(\phi_i(x))_{i\in[\![1,k]\!]}$, note that $(\phi_i(y))_{i\in[\![j,k]\!]}$ is still a process with independent increments. Hence we can apply Lemma 3.3 provided we get to know the expectation and variance of these increments. Let V_i denote the conditional variance of $\phi_i(y)$ knowing $(\phi_r(x))_{r\in[\![0,k]\!]}$. For a sequence f_i (random or deterministic) indexed by the integers, we set

$$\nabla f_i := f_i - f_{i-1}. \tag{8.76}$$

Let T_i measure the correlation between $\nabla\phi_i(x)$ and $\nabla\phi_i(y)$. We have

$$\begin{aligned}\nabla V_i &= \mathbf{E}[(\nabla\phi_i(y))^2] - T_i\mathbf{E}[\nabla\phi_i(x)\nabla\phi_i(y)],\\ T_i &:= \frac{\mathbf{E}[\nabla\phi_i(x)\nabla\phi_i(y)]}{\mathbf{E}[(\nabla\phi_i(y))^2]}.\end{aligned} \tag{8.77}$$

Note that from (8.1) we have $\mathbf{E}[(\nabla\phi_i(y))^2] \ge 1/2$, and thus we deduce from (3.24) that

$$\sum_{i=j+1}^{k} T_i \le 2\int_0^{t_i} P_t(x,y)\mathrm{d}t \le C. \tag{8.78}$$

Also using (3.24) we obtain that for all $i \in [\![j,k]\!]$

$$\left|V_i - V_j - (i-j)\right| \le C. \tag{8.79}$$

The conditional expectation of $\phi_i(y)$, $i \ge j$ given $\phi_j(y)$ and $(\phi_r(x))_{r\in[\![0,k]\!]}$ is given by

$$\mathbf{E}\big[\phi_i(y) - \phi_j(y) \mid (\phi_r(x))_{r\in[\![0,k]\!]}\big] = \sum_{r=j+1}^{k} T_r\nabla\phi_r(x). \tag{8.80}$$

In particular this is smaller (in absolute value) than $C\log(k-j)$ on the event

$$\mathcal{H}(j,N,x)=\mathcal{H}:=\{|\nabla\phi_i(x)|\le\log(k-j),\ \forall i\in[\![j+1,k]\!]\}. \tag{8.81}$$

Let us show that $\mathcal{H}$ is a very likely event. After conditioning with respect to $\phi_j(x)$ and $\phi(x)$ the increments $\nabla\phi_i(x)$ for $i\in[\![i+1,k]\!]$ are still Gaussian variables. They are of variance smaller than 1 (or 2 for $i=k$), because it is smaller than the original variance and the absolute value of their mean satisfies

$$\left|\frac{\mathbf{E}[(\nabla\phi_i(x))^2]}{\mathbf{E}[(\phi(x)-\phi_j(x))^2]}(u-2-t_1)\right|\le C\frac{|k-j|}{\mathbf{E}[(\phi(x)-\phi_j(x))^2]}\le C', \tag{8.82}$$

where the first inequality is a consequence of the restriction (8.74) (and $\mathbf{E}[(\nabla\phi_i(x))^2]\le 1$) and (8.1) asserts that $\mathbf{E}[(\phi(x)-\phi_j(x))^2]$ is close to $k-j$ to obtain the second inequality. Hence we have, uniformly in t_1 satisfying (8.74)

$$\mathbf{P}\left[\mathcal{H}^{\complement}\mid\phi_j(x)=t_1\ ;\ \phi(x)=u-2\right]\le\exp\left(-c(\log(k-j))^2\right). \tag{8.83}$$

If one adds the conditioning to $\phi(y)$ and $\phi_j(y)$ in (8.80) one obtains, for all $(\phi_i(x))_{i\in[\![0,k]\!]}\in\mathcal{H}$,

$$\begin{aligned}\mathbf{E}&\left[\phi_i(y)\mid(\phi_r(x))_{r\in[\![0,k]\!]},\phi_j(y)=t_2\ ;\ \phi(y)=u-2\right]\\&\ge t_2+\left(\frac{V_i-V_j}{V_k-V_j}\right)(u-2-t_2)+\sum_{r=j+1}^{k}T_r\nabla\phi_r(x)\\&\ge t_2+\left(\frac{V_i-V_j}{V_k-V_j}\right)\frac{(k-j)u}{k}-C(\log(k-j)+1),\end{aligned} \tag{8.84}$$

where to obtain the last inequality we used (8.74) and (8.79) and the definition of $\mathcal{H}$. We want to use Lemma 3.3 for the conditioned process, and thus we need to re-center appropriately the considered Gaussian bridge. We set

$$(X_i)_{i=j}^{k}:=\left(\phi_j(y)-\mathbf{E}\left[\phi_i(y)\mid(\phi_r(x))_{r\in[\![0,k]\!]},\phi_j(y)=t_2\ ;\ \phi(y)=u-2\right]\right)_{i=j}^{k}.$$

We note that

$$\begin{aligned}&\frac{iu}{k}-t_2-\left(\frac{V_i-V_j}{V_k-V_j}\right)\frac{(k-j)u}{k}\\&\quad=\left(\frac{ju}{k}-t_2\right)+\frac{u}{k}\left(\frac{(V_i-V_j)(k-j)}{V_k-V_j}-(j-i)\right)\ge\left(\frac{ju}{k}-t_2\right)-C,\end{aligned} \tag{8.85}$$

so that for $i\in[\![j,k]\!]$ and $(\phi_r(x))_{r\in[\![0,k]\!]}\in\mathcal{H}$, the inequality (8.84) implies that

$$\left\{\phi_i(y)\le\frac{iu}{k}+10\right\}\subset\left\{X_i\le\left(\frac{ju}{k}-t_2\right)+C\log(k-j)\right\}. \tag{8.86}$$

Hence applying Lemma 3.3, for $(\phi_r(x))_{r\in[\![0,k]\!]}\in\mathcal{H}$ we obtain that

$$\begin{aligned}&\mathbf{E}[\forall i\in[\![j,k]\!],\ \phi_i(y)\le iuk+10\mid(\phi_r(x))_{r\in[\![0,k]\!]}\ ;\ \phi_j(y)=t_2\ ;\ \phi(y)=u-2]\\&\qquad\le C(k-j)^{-1}\left[\left(\frac{ju}{k}-t_2\right)^2+\log(k-j)^2\right].\end{aligned} \tag{8.87}$$

Using (8.75) and (8.87), we obtain that

$$\begin{aligned}(8.88)\quad \mathbf{P}\Big[\forall i \in [\![j,k]\!], \phi_i(x), \phi_i(y) \le \frac{ui}{k} + 10 \\ \mid \phi_j(x) = t_1, \ ; \ \phi_j(y) = t_2 \ ; \ \phi(x), \phi(y) \in [u-2, u+2]\Big] \\ \le C(k-j)^{-2}\left[\left(\frac{ju}{k} - t_1\right)^2 + C\log(k-j)^2\right]\left[\left(\frac{ju}{k} - t_2\right)^2 + C\log(k-j)^2\right] \\ + \mathbf{P}\left[\mathcal{H}^{\complement} \mid \phi_j(x) = t_1, \phi(x) = u-2\right].\end{aligned}$$

The last term is negligible when compared to the first and taking the maximum over $t_1 + t_2 = 2t$ satisfying (8.74), this concludes the proof of (8.69). □

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Appendix A

Proof of Theorem 2.6

While the notations in Section 2.6 were chosen to match those used in the literature on co-membranes, we think that for the sake of the proof, it will be simpler for the reader to consider the partition function

$$(A.1)\qquad \widetilde{Z}^{\omega,\beta}_{N,h} := \mathbf{E}_N\left[\exp\left(\sum_{x\in\widetilde{\Lambda}_N}(\beta\omega_x - \lambda(\beta) + h)\Delta_x\right)\right],$$

and the associated free-energy $\widetilde{\mathrm{F}}(\beta, h)$. Our task becomes then to prove

$$(A.2)\qquad \exp\left(-h^{-20}\right) \le \widetilde{\mathrm{F}}(\beta, h) \le \exp\left(-|\log h|^{3/2}\right).$$

Like for the proof of Theorem 2.4, we treat the case of the lower and upper bound separately.

A.1. The upper bound

We should follow the proof in Section 4 and only indicate the necessary changes. The reader can check that the proof of Proposition 4.1 remains valid for $\widetilde{\mathbf{P}}^{\omega,\beta}_{N,h}$ without changes. The next step to be checked is Proposition 4.4: Replacing $\mathcal{C}_{N_1}(y)$ by

$$(A.3)\qquad \widetilde{\mathcal{C}}_{N_1}(y) := \left\{\exists x \in \widetilde{\Lambda}_{N_1}(y), \sum_{\{z\in\widetilde{\Lambda}_{N_1}(y)\,:\,|z-x|\le(\log N_1)^2\}} \Delta_z \ge (\log N_1)^3\right\},$$

the algebraic manipulations in the proof of Proposition 4.5 allow to prove that (4.38) still holds true with δ_x replaced by Δ_x and $\mathcal{C}_{N_1}$ replaced by $\widetilde{\mathcal{C}}_{N_1}$. The same can be said of

Proposition 4.6 and (4.40). The only point which requires significant care is Proposition 4.7, where we need to prove that when $\|\widehat{\phi}\| \leq r_h$, we have

$$\log \mathbf{E}^{\widehat{\phi}}_{2N_1}\left[e^{4h\sum_{x\in\widetilde{\Lambda}'_{N_1}}\Delta_x - 4\mathbf{1}_{\widetilde{\mathcal{C}}_{N_1}}}\right] \leq e^{-2|\log h|^{3/2}}. \tag{A.4}$$

We have in analogy with (4.58)

$$\log \mathbf{E}^{\widehat{\phi}}_{2N_1}\left[e^{4h\sum_{x\in\widetilde{\Lambda}'_{N_1}}\Delta_x - 4\mathbf{1}_{\widetilde{\mathcal{C}}_{N_1}}}\right] \leq 5N_1^{-2}\max_{x\in\widetilde{\Lambda}'_{N_1}} \mathbf{P}^{\widehat{\phi}}_{2N_1}[\phi_x \leq 0] - \frac{1}{2}\mathbf{P}^{\widehat{\phi}}_{2N_1}[\widetilde{\mathcal{C}}_{N_1}]. \tag{A.5}$$

Setting in analogy with (4.63)

$$u = u(\widehat{\phi}, N_1) := \min_{x\in\widetilde{\Lambda}_N} H^{\widehat{\phi}}_{2N_1}(x). \tag{A.6}$$

We have from (3.8) and in analogy with (4.64)

$$\max_{x\in\widetilde{\Lambda}'_{N_1}} \mathbf{P}^{\widehat{\phi}}_{2N_1}\left(\phi_x \leq 0\right) \leq \begin{cases} 1 & \text{if } u \leq 0, \\ e^{-\frac{u^2}{2V_{N_1}}} & \text{if } u \geq 0. \end{cases} \tag{A.7}$$

To conclude we need a result similar to Proposition 4.8. We prove that under the same assumption, we have

$$\mathbf{P}^{\widehat{\phi}}_{2N_1}[\widetilde{\mathcal{C}}_{N_1}] \geq \begin{cases} \frac{1}{4} & \text{if } u \leq 0, \\ c(\log N^{-1})e^{-\frac{(u+1)^2}{2V'_{N_1}}} & \text{if } u \geq 0. \end{cases} \tag{A.8}$$

The inequality (A.4) follows from plugging the estimates (A.7) and (A.8) into (A.5). To prove (A.8) we set

$$x_{\min} := \operatorname*{argmin}_{x\in\widetilde{\Lambda}'_N} H^{\widehat{\phi}}_{2N_1}(x), \tag{A.9}$$

define $\widehat{\Lambda}$ like in (4.72). Using Lemma 4.9 and symmetry, we have

$$\mathbf{P}^{\widehat{\phi}}_{2N_1}\left[\widetilde{\mathcal{C}}_{N_1}\right]\mathbf{P}^{\widehat{\phi}}_{2N_1}\left[\sum_{z\in\widehat{\Lambda}}\Delta_z \geq (\log N_1)^3\right] \leq \mathbf{P}_{2N_1}\left[\sum_{z\in\widehat{\Lambda}}\mathbf{1}_{\{\phi_z\geq u+1/4\}} \geq (\log N_1)^3\right]. \tag{A.10}$$

When $u \leq 0$, we observe that for h sufficiently large (recall that the variance of ϕ_x diverges uniformly in $\widehat{\Lambda}$ when h tends to zero)

$$\mathbf{E}_{2N_1}\left[\sum_{z\in\widehat{\Lambda}}\mathbf{1}_{\{\phi_z\geq u+1/4\}}\right] \geq \frac{1}{3}|\widehat{\Lambda}|, \tag{A.11}$$

and thus we obtain the result using Markov's inequality for $\sum_{z\in\widehat{\Lambda}}\mathbf{1}_{\phi_z<u+1/4}$. For the second case, it is sufficient to observe that

$$\sum_{z\in\widehat{\Lambda}}\mathbf{1}_{\{\phi_z\geq u+1/4\}} \leq \sum_{z\in\widehat{\Lambda}}\mathbf{1}_{[u+1/4,u+7/4]}(\phi_z), \tag{A.12}$$

and thus the proof of Proposition 4.8 adapts (with u replaced by $u+1$) and produces the required bound. □

A.2. The lower bound

We chose to follow, for this part as well, the proof of Theorem 2.4 through Sections 5 to 8. We set

$$\Delta_x^u := \mathbf{1}_{\{\phi_x \ge u\}} \quad \text{and} \quad \Delta_x^{\widehat{\phi},u} := \mathbf{1}_{\{\phi_x + H_N^{m,\widehat{\phi}}(x) \ge u\}}. \tag{A.13}$$

A first observation is that the proof of Proposition 5.3 remains valid when δ is replaced by Δ (as the distribution of the set of contact has no role in the proof), and we thus need only to prove (5.17) with the same choice of parameters (5.16) and δ replaced by Δ.

Moving to Section 6, we see that Proposition 6.1 needs no adaptation. In Proposition 6.2, we need to adapt the definition of $\mathcal{C}_N$ for (6.4) to be valid for Δ, we introduce the change below and let $\widetilde{\mathcal{C}}_N$ denote the alternative event. For Proposition 6.3, the statement (i) needs no change, and in (ii) and (iii) we need to replace $\mathcal{C}_N$ by $\widetilde{\mathcal{C}}_N$ and $\mathcal{B}_N$ by

$$\widetilde{\mathcal{B}}_N := \widetilde{\mathcal{C}}_N \cap \left\{ \widetilde{L}_N \le (\log N)^{\frac{1+\alpha}{2}} \right\}, \text{ where } \widetilde{L}_N := \sum_{x \in \Lambda'_N} \Delta_x^{\widehat{\phi},u}. \tag{A.14}$$

Mutatis mutandis the proof of (5.17) in the modified setup follows, and we can move to Section 7.

The proof of Lemma 7.2 needs some more work. While the proof of (7.26) for Δ follows from standard Gaussian estimates (cf. (3.8)), we need to dwell a bit more on (7.27). The crucial estimate we need to prove is the following, (recall (7.32))

$$\begin{aligned} &\mathbf{P}\big[\forall i \in [\![j(x), k]\!],\ \phi_i(x) \le \gamma(i - j(x)) + 100(\log\log N)\ ;\ \phi(x) + H(x) \ge u\big] \\ &\qquad \le C N^{-2} (\log N)^{\alpha} \left[H(x)^2 + (\log\log N)^2 \right] \exp\left(\gamma H(x) - \frac{\gamma^2}{2} j(x) \right), \end{aligned} \tag{A.15}$$

assuming that $H(x) \le 4u/5$. A first step is to show that

$$\mathbf{P}\big[\phi_x + H(x) \ge u\big] \le C N^{-2} (\log N)^{\alpha+1} \exp\left(\gamma H(x) - \frac{\gamma^2}{2} j(x) \right), \tag{A.16}$$

which follows immediately from Gaussian tail estimates (3.8) just like in the proof of (7.33). Then we must only show that (7.36) is valid for all $t \ge u - H(x)$. This is immediate since due to positive correlation in the Gaussian field, the r.h.s. in (7.36) is a decreasing function of t.

The proof of Proposition 6.2 adapts verbatim, defining $\widetilde{\mathcal{C}}_N := \mathcal{D}_N \cap \widetilde{\mathcal{C}}'_N$ where $\widetilde{\mathcal{C}}'_N$ is defined as in (7.42) with δ replaced by Δ.

Moving to Section 8, we remark that the proof of (6.8) needs no other changes than replacing δ by Δ and $\mathcal{C}_N$ by $\widetilde{\mathcal{C}}_N$. To prove our modified version (6.9), we must prove (8.4) and (8.5) with δ, L and $\mathcal{B}$ replaced by Δ, $\widetilde{L}$ and $\widetilde{\mathcal{B}}$ respectively (and the proof adapt by performing only these elementary changes).

The proof of Lemma 8.1 requires no specific adaptation as we have already checked all the necessary estimates. For (8.5), note that we have similarly to (8.28)

$$\mathbf{E}\left[\widetilde{L}_N \mathbf{1}_{\widetilde{\mathcal{B}}_N} \right] \ge \mathbf{E}\left[\widetilde{L}'_N \right] - \sqrt{\mathbf{E}\left[(\widetilde{L}'_N)^2 \right]} \sqrt{\mathbf{P}\left[\widetilde{\mathcal{B}}_N^{\complement} \right]}, \tag{A.17}$$

where $\widetilde{L}'_N$ is a lower bound for $\widetilde{L}_N$ defined as follows

$$\text{(A.18)}\qquad \begin{aligned} \Delta'_x &:= \mathbf{1}_{\left\{(\phi(x)-u+H(x))\in[0,2],\ \forall i\in[\![1,k]\!],\ \phi_i(x)\le \frac{ui}{k}+10\right\}},\\ \widetilde{L}'_N &:= \sum_{x\in\Lambda'_N} \Delta'_x. \end{aligned}$$

Now the reader can check that changing $[-1,1]$ by $[0,2]$ (which is the only difference between $\widetilde{L}'_N$ and L'_N) does not affect the proof of Lemma 8.3 and 8.4. □

Appendix B

Estimates on heat-kernels and random walks

B.1. Proof of Lemma 3.1

To estimate the Green Function of the massive field we use a bit of potential theory. We let a denote the potential Kernel of Δ in $\mathbb{Z}^2$ i.e.,

$$\text{(B.1)}\qquad a(x) := \lim_{T\to\infty} \int_0^T (P_t(0,0) - P_t(x,0))\,\mathrm{d}t.$$

From [45, Theorem 4.4.4] we have

$$\text{(B.2)}\qquad a(x) := \frac{1}{2\pi}\log|x| + O(1).$$

Set $a(x,y) := a(x-y)$. Now recall that X is a continuous time random-walk on $\mathbb{Z}^2$ with generator Δ and that P^x denotes its law when the initial condition is $x\in\mathbb{Z}^2$, and τ_A denote the hitting time of A. Let T_m be a Geometric variable of mean m^{-2} which is independent of X.

By adapting the proof of [45, Proposition 4.6.2(b)] we obtain that

$$\text{(B.3)}\qquad \begin{aligned} G^{m,*}(x,y) &= E^x\left[a\left(X_{\tau_{\partial\Lambda_N}\wedge T_m}, y\right)\right] - a(x,y),\\ G^m(x,y) &= E^x\left[a\left(X_{T_m}, y\right)\right] - a(x,y). \end{aligned}$$

Considering the case $y = x$ and when there is no boundary, it is not difficult to see that

$$\text{(B.4)}\qquad G^m(x,x) = E^x\left[a\left(X_{T_m}, x\right)\right] := -\frac{1}{2\pi}\log m + O(1).$$

In the case $x = y$ with boundary, this is more delicate. On one side it is easy to deduce from (B.3) that for some appropriate $C > 0$,

$$\text{(B.5)}\qquad \frac{1}{2\pi}\log\left(\min\left(d(x,\partial\Lambda_N), m^{-1}\right)\right) - C \le G^{m,*}(x,x) \le -\frac{1}{2\pi}\log m + C.$$

What remain to prove is that $\frac{1}{2\pi}\log d(x,\partial\Lambda_N)$ is an upper bound (which is a concern only if $d(x,\Lambda_N)\le m^{-1}$).

Note that the Green Function with Dirichlet boundary condition is an increasing function of the domain and a decreasing function of m. Hence to obtain an upper bound on $G^{m,*}$, we can compare it with the variance of the massless free field in the half plane $\mathbb{Z}_+\times\mathbb{Z}$ at the point

$$\text{(B.6)}\qquad x_d := (d(x,\partial\Lambda_N), 0),$$

that is given by

$$E^{x_d}\left[a\left(X_{\tau_{\{0\}\times\mathbb{Z}}}, x_d\right)\right] \geq G^{m,*}(x,x). \tag{B.7}$$

Now note that $\tau_{\{0\}\times\mathbb{Z}}$ is simply the hitting time of zero by one dimensional simple random walk starting from $d(x, \partial\Lambda_N)$. Hence

$$P^{x_d}[\tau_{\{0\}\times\mathbb{Z}} \geq t] \leq C d(x,\partial\Lambda_N) t^{-1/2}.$$

As the second coordinate of $X_{\tau_{\{0\}\times\mathbb{Z}}}$ is simply the value of an independent random walk evaluated at τ we get that for some constant C' all $u > 0$

$$P^{x_d}[|X_{\tau_{\{0\}\times\mathbb{Z}}} - x_d| \geq u] \leq C\frac{d(x,\partial\Lambda_N)}{u}. \tag{B.8}$$

This tail estimate, together with (B.2) is sufficient to conclude that

$$E^{x_d}\left[a\left(X_{\tau_{\{0\}\times\mathbb{Z}}}, x_d\right)\right] \leq \frac{1}{2\pi}\log d(x,\partial\Lambda_N) + C. \tag{B.9}$$

□

B.2. Proof of Lemma 3.2

Proof. – Let us start with (i). The first inequality in (3.23) can be deduced from [45, Theorem 2.3.6] which is a fine estimate for $P_t(x,x) - P_t(x,y)$ in discrete time.

For the second one, we notice that we can reduce the problem to proving that for any $u, v \in [0, N]$

$$\left(p_t^*(u,u) + p_t^*(v,v) - 2p_t^*(u,v)\right) \leq \frac{C|u-v|^2}{t^{3/2}}, \tag{B.10}$$

where p_t^* is the heat-kernel associated with the simple random-walk on $[\![0, N]\!]$ with Dirichlet boundary condition. Indeed if x and y differ by only one coordinate, say $x_1 = y_1$ we can factorize the l.h.s of (3.23) by the common coordinate and obtain

$$\begin{aligned} &p_t^*(x_1,x_1)\left[p_t^*(x_2,x_2) + p_t^*(y_2,y_2) - 2p_t^*(x_2,y_2)\right] \\ &\qquad \leq \frac{C}{\sqrt{t}}\left[p_t^*(x_2,x_2) + p_t^*(y_2,y_2) - 2p_t^*(x_2,y_2)\right]. \end{aligned} \tag{B.11}$$

If the two coordinates of x and y differ, then if we let φ be a field with covariance function P_t^*, the l.h.s of (3.23) can be rewritten as

$$\mathbb{E}[(\varphi_x - \varphi_y)^2] \leq 2\left(\mathbb{E}[(\varphi_x - \varphi_z)^2] + \mathbb{E}[(\varphi_y - \varphi_z)^2]\right) \tag{B.12}$$

and we reduce to the first case by choosing $z = (x_1, y_2)$.

Now, by Fourier decomposition of the kernel, we have

$$\left(p_t^*(u,u) + p_t^*(v,v) - 2p_t^*(u,v)\right) = \frac{2}{N}\sum_{i=1}^{N-1} e^{-\lambda_i t}\left[\sin\left(\frac{i\pi u}{N}\right) - \sin\left(\frac{i\pi v}{N}\right)\right]^2, \tag{B.13}$$

where $\lambda_i := 2\left(1 - \cos\left(\frac{i\pi}{N}\right)\right)$. The sum can obviously be bounded by

$$\frac{C|v-u|^2}{N^3}\sum_{i=1}^{N-1} e^{-\lambda_i t} i^2. \tag{B.14}$$

It is a simple exercise to show that this sum is of order $N^3 t^{-3/2}$.

For (ii) we can just use large deviations estimates (see, e.g., below in the proof of (iv)) for $|x-y| \ge C\sqrt{t\log t}$ with C chosen sufficiently large, and use the Local Central Limit Theorem [45, Theorem 2.1.1] to cover the case $|x-y| \le C\sqrt{t\log t}$. The inequality (3.25) is obtained by expanding the integral and replacing $P_t(x,y)$ by the bound obtained above.

For (iii) we can compare to the half-plane case where $x = x_d$ (recall that from the argument presented before (B.6) this gives an upper bound). In that case we have

$$P_t^*(x,x) = P[X_t = 0; \forall s \in [0,t],\ X_s \le d], \tag{B.15}$$

where X is the simple random walk on $\mathbb{Z}^2$ starting from zero. By a reflexion argument we have

$$P[X_t = 0; \forall s \in [0,t],\ X_s < d] = P[X_t = 0] - P[X_t = 2d] = P_t(0,0) - P_t(0, 2d\mathbf{e}_1). \tag{B.16}$$

The later quantity can be estimated with [45, Theorem 2.3.6], and shown to be smaller than $2d^2/t^2$. For (iv), for simplicity let us first consider the case $x = y$ we have

$$\begin{aligned} P_t(x,x) - P_t^*(x,x) &= P[X_t = 0; \exists s \in [0,t],\ X_s + x \in \partial\Lambda_N] \\ &\le P\left[X_t = 0; \max_{s\in[0,t]} |X_s| \ge d\right]. \end{aligned} \tag{B.17}$$

Summing over the faces of the square and using the reflexion principle, the right-hand side is smaller than

$$4P\left[X_t = 0; \max X_s^{(1)} \ge d\right] \le 4P_t(2d\mathbf{e}_1). \tag{B.18}$$

The later quantity can be estimated with the LCLT [45, Theorem 2.3.6], when $\frac{d^2}{t} \le 100\log d$. For smaller values of t, it is sufficient to get an upper bound on the probability $P[X_t^{(1)} \ge d]$. Recalling that

$$\log E\left[e^{\lambda X_t^{(1)}}\right] = (\cosh(\lambda) - 1)t, \tag{B.19}$$

the standard Chernov bound computation yields

$$\log P[X_t^{(1)} \ge d] \le -d\log\left[(d/t) + \sqrt{1 + (d/t)^2}\right] + t\left(\sqrt{(d/t)^2 + 1} - 1\right), \tag{B.20}$$

which yields the right bound in all cases. □

B.3. Proof of Lemma 3.3

Let V_i denote the variance of X_i (without conditioning), $\nabla V_i = (V_i - V_{i-1})$ and set $V := V_k$ ($V \in [k/2, k]$). After conditioning to $X_k := 0$, the process $(X_i)_{i=1}^k$ remains Gaussian and centered but the covariance structure is given by

$$\mathbf{E}\left[X_i X_j \mid X_k = 0\right] = \frac{V_i(V - V_j)}{V} \qquad 0 \le i \le j \le k. \tag{B.21}$$

We denote by $\widetilde{\mathbf{P}}$ the law of the conditioned process. We can couple this process with a Brownian Motion conditioned to $B_V = 0$: a centered Brownian bridge $(B_t)_{t\in[0,V]}$, by setting $X_i := B_{V_i}$. Note that we have (by applying standard reflexion argument at the first hitting time of x)

$$\widetilde{\mathbf{P}}\left[\max_{t\in[0,V]} B_t \ge x\right] = e^{-\frac{2x^2}{V}}. \tag{B.22}$$

As the max of B is larger than that of X this gives the lower bound. To prove (i), by monotonicity, we can restrict the proof to the case $x \geq (\log k)$. To estimate the difference between (B.22) and the probability we have to estimate, we let B^i denote the Brownian bridges formed by B between the X_i,

$$(B^i_s)_{s\in[V_{i-1},V_i]} := B_s - \frac{(s-V_{i-1})B_{V_{i-1}} + (V_i - s)B_{V_i}}{V_i - V_{i-1}}.$$

Using a union bound we have

$$\begin{aligned}\widetilde{\mathbf{P}}\left[\max_{i\in[\![1,k-1]\!]} X_i \leq x\right] &\leq \widetilde{\mathbf{P}}\left[\max_{t\in[0,V]} B_t \leq 2x\right] + \sum_{i=1}^{k} \widetilde{\mathbf{P}}\left[\max_{s\in[V_i,V_{i+1}]} B^i_s \geq x\right] \\ &= \left(1 - e^{\frac{-8x^2}{V}}\right) + \sum_{i=1}^{k} \exp\left(-\frac{2x^2}{\nabla V_i}\right),\end{aligned} \tag{B.23}$$

where in the last line we used (B.22) for B and B^i. This is smaller than Cx^2/k for some well chosen C. □

Appendix C
Proof of Proposition 2.2

We use Proposition 5.1 to prove the lower bound in the asymptotic, and then briefly explain how to obtain a matching upper bound. First note that using (5.1) for $\beta = 0$ and $u = 0$, we obtain

$$\textsc{f}(h) \geq \lim_{N\to\infty} \frac{1}{N^2} \log \mathbf{E}^m_N \left[e^{\sum_{x\in\widetilde{\Lambda}_N} h\delta_x}\right] - f(m). \tag{C.1}$$

Now, using Jensen's inequality we have

$$\frac{1}{N^2} \log \mathbf{E}^m_N \left[e^{\sum_{x\in\widetilde{\Lambda}_N} h\delta_x}\right] \geq \frac{h}{N^2}\mathbf{E}^m_N\left[\sum_{x\in\widetilde{\Lambda}_N} \delta_x\right] \geq hP[\mathcal{N}(\sigma_m) \in [-1,1]], \tag{C.2}$$

where $\sigma_m := \sqrt{G^m(x,x)}$ denotes the standard deviation of the infinite volume massive free field and $\mathcal{N}(\sigma_m)$ is a centered normal variable with standard deviation σ_m. As the variance grows when m tends to zero we obtain that for arbitrary $\varepsilon > 0$ for $m \leq m_\varepsilon$ we have

$$\textsc{f}(h) \geq (1-\varepsilon)\frac{2h}{\sqrt{2\pi}\sigma_m} - f(m). \tag{C.3}$$

Using the above inequality for $m = \frac{\sqrt{h}}{|\log h|}$, using (3.21) to estimate σ_m and (5.6) for $f(m)$ we obtain that for any ε, for $h \leq h_\varepsilon$ sufficiently small we have

$$\textsc{f}(h) \geq hP[\sigma_m \mathcal{N} \in [-1,1]] - f(m) \geq \frac{2h}{\sqrt{(1/2)|\log h|}}(1-\varepsilon). \tag{C.4}$$

Concerning the upper bound, we can show as in [35, Equation (2.20)] that $\sup_{\widehat{\phi}} Z^{\widehat{\phi}}_{N,h}$ is a sub-multiplicative function and thus that we have for every $N \geq 1$,

$$\textsc{f}(h) \leq \sup_{\widehat{\phi}} \frac{1}{N^2} \log Z^{\widehat{\phi}}_{N,h}. \tag{C.5}$$

We use this inequality for

$$N = h^{-1/2}|\log h|^{-1}.$$

In that case, using the bound $e^x \le 1 + xe^K$, valid for $x \in [0, K]$, for the exponential in the partition function gives

$$Z_{N,h}^{\widehat{\phi}} \le 1 + e^{(\log h)^{-2}} h \mathbf{E}_N^{\widehat{\phi}} \left[\sum_{x \in \widetilde{\Lambda}_N} \delta_x \right] \le 1 + e^{(\log h)^{-2}} h \mathbf{E}_N \left[\sum_{x \in \widetilde{\Lambda}_N} \delta_x \right], \tag{C.6}$$

where in the last inequality we used the fact that the probability for a Gaussian of a given variance to be in $[-1, 1]$ is maximized if its mean is equal to zero. Using (3.22) then to estimate the probability, it is a simple exercise to check that for any $\varepsilon > 0$ and N large enough, we have

$$\mathbf{E}_N \left[\sum_{x \in \widetilde{\Lambda}_N} \delta_x \right] \le \frac{(1+\varepsilon)2N^2}{\sqrt{\log N}}. \tag{C.7}$$

Combining all these inequalities, we obtain that for h sufficiently small

$$\textsc{f}(h) \le \frac{(1+2\varepsilon)2\sqrt{2}h}{\sqrt{|\log h|}}. \tag{C.8}$$

□

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