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SUPRA-MAXIMAL REPRESENTATIONS FROM FUNDAMENTAL GROUPS OF PUNCTURED SPHERES TO PSL$(2, \mathbb{R})$

BY BERTRAND DEROIN AND NICOLAS THOLOZAN

ABSTRACT. – We study a particular class of representations from the fundamental groups of punctured spheres $\Sigma_{0,n}$ to the group PSL$(2, \mathbb{R})$, which we call *supra-maximal*. Though most of them are Zariski dense, we show that supra-maximal representations are *totally non hyperbolic*, in the sense that every *simple* closed curve is mapped to an elliptic or parabolic element. They are also shown to be *geometrizable* (apart from the reducible ones) in the following very strong sense : for any element of the Teichmüller space $\mathscr{T}_{0,n}$, there is a unique holomorphic equivariant map with values in the lower half-plane $\mathbb{H}^-$. In the relative character varieties, the components of supra-maximal representations are shown to be compact and symplectomorphic (with respect to the Atiyah-Bott-Goldman symplectic structure) to the complex projective space of dimension $n-3$ equipped with a certain multiple of the Fubini-Study form that we compute explicitly. This generalizes a result of Benedetto-Goldman [3] for the sphere minus four points.

RÉSUMÉ. – Nous étudions une classe particulière de représentations du groupe fondamental des sphères épointées $\Sigma_{0,n}$ dans le groupe PSL$(2, \mathbb{R})$, que nous appelons *supra-maximales*. Bien qu'elles soient pour la plupart Zariski denses, nous montrons qu'elles sont *totalement non hyperboliques*, au sens où l'image de toute courbe fermée *simple* est elliptique ou parabolique. Nous montrons aussi qu'elles sont *géométrisables* (hormis celles qui sont réductibles) en un sens très fort : pour tout élément de l'espace de Teichmüller $\mathscr{T}_{0,n}$, il existe une unique application équivariante holomorphe à valeurs dans le demi-plan inférieur $\mathbb{H}^-$. Nous montrons également que les représentations supra-maximales forment des composantes compactes des variétés de caractère relatives. Munies de la structure symplectique de Atiyah-Bott-Goldman, ces composantes sont symplectomorphes à l'espace projectif complexe de dimension $n-3$ muni d'un multiple de la forme de Fubini-Study que nous calculons explicitement. Cela généralise un résultat de Benedetto-Goldman pour la sphère à quatre trous.

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Introduction

0.1. Overview

Let $\Sigma_{g,n}$ be a surface obtained from a connected oriented closed surface of genus g by removing n points, called the punctures. We assume in the sequel that the Euler characteristic of $\Sigma_{g,n}$ is negative. Throughout the paper, we will denote by $G = \mathrm{PSL}(2,\mathbb{R})$ the group of orientation-preserving isometries of the half-planes $\mathbb{H}^{\pm} = \{z \in \mathbb{C} \mid \pm\,\mathfrak{im}(z) > 0\}$ equipped with the metrics $\frac{dx^2+dy^2}{y^2}$ of curvature -1, where $z = x + iy$. We denote by $\mathrm{Hom}(\pi_1(\Sigma_{g,n}), G)$ the set of representations from the fundamental group of $\Sigma_{g,n}$ to G, and by $\mathrm{Rep}(\pi_1(\Sigma_{g,n}), G) = \mathrm{Hom}(\pi_1(\Sigma_{g,n}), G)/G$ its quotient by the action of G by conjugation. We will call this latter the *character variety*, even though we do not consider the algebraic quotient (in the sense of geometric invariant theory).

A representation $\rho \in \mathrm{Hom}(\pi_1(\Sigma_{g,n}), G)$ determines a flat oriented $\mathbb{R}\mathbf{P}^1$-bundle over $\Sigma_{g,n}$ which, if we forget the flat connection, is encoded up to isomorphism by a class in $H^2(\Sigma_{g,n}, \mathbb{Z})$, called the Euler class, and denoted $\mathbf{eu}(\rho)$. In the closed case, i.e., when $n = 0$, we have $H^2(\Sigma_{g,0}, \mathbb{Z}) \simeq \mathbb{Z}$, so that the Euler class is an integer that satisfies the well-known Milnor-Wood inequality :

$$|\mathbf{eu}(\rho)| \leq |\chi(\Sigma_{g,0})|, \tag{1}$$

as proved by Wood [28], following an earlier work of Milnor [23]. All the integral values in the interval (1) are achieved on $\mathrm{Hom}(\Sigma_{g,0}, G)$. Goldman proved that the level sets of the Euler class are connected [12], and Hitchin that they are indeed diffeomorphic to vector bundles over some symmetric powers of $\Sigma_{g,0}$ [19]. Goldman also proved in his doctoral dissertation that the Euler class is extremal exactly when the representation is the holonomy of a hyperbolic structure on $\Sigma_{g,0}$ [17]. He conjectured more generally that the components of non-zero Euler class are generically made of holonomies of *branched* $\mathbb{H}^{\pm}$-structures on $\Sigma_{g,0}$ with $k = |\chi(\Sigma_{g,0})| - |\mathbf{eu}|$ branch points (see [16], as well as [26] where the problem is discussed).

This paper is the first in a series aiming at studying the analogous picture on the *relative* character varieties when the surface $\Sigma_{g,n}$ is not closed, namely when $n > 0$. We focus here on a particular family of components of the relative character varieties, that we call *supra-maximal*. They occur only on punctured spheres $\Sigma_{0,n}$ for $n \geq 3$, for some particular choices of elliptic/parabolic peripheral conjugacy classes.

We prove that these components are compact, and more precisely that they are symplectomorphic (with respect to the Atiyah-Bott-Goldman symplectic structure) to the complex projective space of dimension $n - 3$, equipped with a certain multiple of the Fubini-Study form that we compute explicitly. This generalizes to any $n \geq 4$ a result obtained by Benedetto-Goldman in the case $n = 4$ [3].

We also prove that the *supra-maximal representations* (i.e., those lying in supra-maximal components) have very special algebraic and geometric properties. First, we prove that they are totally non hyperbolic, namely that no simple closed curve of $\Sigma_{0,n}$ is mapped to a hyperbolic conjugacy class of G. Moreover, we prove that they are geometrizable by $\mathbb{H}^-$-conifolds in a very strong way.

0.2. Volume, relative Euler class and the refined Milnor-Wood inequality

In the closed case, the Euler class is closely related to the *volume* of the representation, classically defined by the integral

$$\mathrm{Vol}(\rho) = \int_{\Sigma_{g,0}} f^*\left(\frac{dx\wedge dy}{y^2}\right) \tag{2}$$

where $f : \widetilde{\Sigma}_{g,0} \to \mathbb{H}^+$ is any ρ-equivariant smooth map. Namely, we have $\mathrm{Vol}(\rho) = -2\pi\,\mathbf{eu}(\rho)$. Burger and Iozzi [5] and Koziarz and Maubon [20] have independently extended the definition of the volume of a representation $\rho : \pi_1(\Sigma_{g,n}) \to G$ to the case of punctured surfaces. (See also Burger-Iozzi-Wienhard [6] for a generalization to representations into Lie groups of Hermitian type.) This volume can be defined as a bounded cohomology class, or more trivially as an integral of the form (2), where the behavior of the equivariant map is constrained in the neighborhood of the cusps: namely, the completion of the metric $f^*\big(\frac{dx^2+dy^2}{y^2}\big)$ in the neighborhood of a cusp is assumed to be a cone, a parabolic cusp, or an annulus with totally geodesic boundary.

The analogous Milnor-Wood inequality

$$|\mathrm{Vol}(\rho)| \le 2\pi|\chi(\Sigma_{g,n})|, \tag{3}$$

holds in this context [6, 20]. It is also proved in [6] that the volume is continuous as a function on $\mathrm{Rep}(\pi_1(\Sigma_{g,n}), G)$ and achieves every value in the interval defined by (3).

The volume heavily depends on the conjugacy class of the peripherals $\rho(c_i)$, where the c_i are elements of $\pi_1(\Sigma_{g,n})$ freely homotopic to positive loops around the punctures. For instance, its reduction modulo 2π equals the sum $-\sum_i R(\rho(c_i))$, where $R(\rho(c_i))$ is the rotation number of $\rho(c_i)$ [6, Theorem 12]. In order to understand better the dependence of the volume on the $\rho(c_i)$, it is convenient to introduce the following function:

$$\theta : G \to \mathbb{R}_+$$

that maps an element $g \in G$ to

— 0 if g is hyperbolic or positive parabolic (i.e., a parabolic that translates the horocycles based at the fixed point of g clockwise),

— 2π if g is negative parabolic (i.e., a parabolic that translates the horocycles based at the fixed point of g counterclockwise) or the identity,

— the value between 0 and 2π of the rotation angle of g when g is elliptic.

We will denote $\theta_i(\rho) = \theta(\rho(c_i))$ and $\Theta(\rho) = \sum_{i=1}^n \theta_i(\rho)$.

Remark 0.1. – The function θ is one among the many ways of lifting the rotation number to a function from G to $\mathbb{R}$. Note however that it is (up to adding a multiple of 2π) the only lift which is continuous in restriction to the set of elliptic elements and upper semi-continuous on the whole group G.

Definition 0.2. – We define the *relative Euler class* of the representation ρ by

$$-\,\mathbf{eu}(\rho) = \frac{1}{2\pi}\big(\mathrm{Vol}(\rho) + \Theta(\rho)\big). \tag{4}$$

By [6, Theorem 12], the relative Euler class is an integer. In fact, it can be shown that it is the genuine Euler class of the flat oriented $\mathbb{R}\mathbf{P}^1$-bundle with monodromy ρ, relative to some explicit trivializations above the curves c_i. When ρ is the holonomy of a $\mathbb{H}^+$-structure on $\Sigma_{g,n}$ with cusps, totally geodesic boundary or cone singularities of angle between 0 and 2π at the punctures, then $-\mathbf{eu}(\rho) = |\chi(\Sigma)|$ by the Gauss-Bonnet formula.

One could expect this to be the maximal value of $-\mathbf{eu}(\rho)$, as it is in the closed case. Here we will prove the following analog of the Milnor-Wood inequality:

THEOREM 1. – *For every representation $\rho : \pi_1(\Sigma_{g,n}) \to G$, we have the inequality*

$$\inf\big(-|\chi(\Sigma_{g,n})| + l, \lceil \frac{1}{2\pi}\Theta(\rho)\rceil\big) \leq -\mathbf{eu}(\rho) \leq \sup\big(|\chi(\Sigma_{g,n})|, \lfloor \frac{1}{2\pi}\Theta(\rho)\rfloor\big), \tag{5}$$

where l is the number of elliptic/parabolic/identity conjugacy classes among the $\rho(c_i)$'s, with the identity counted twice.

In particular, the relative Euler class satisfies $|\mathbf{eu}(\rho)| \leq |\chi(\Sigma_{g,n})|$ unless $g = 0$, in which case it can take no more than two additional values: $-\mathbf{eu}(\rho) = n - 1$ or n.

Note that the relative Euler class does not distinguish connected components of the character variety (which is connected for surfaces with punctures), mainly because the function $\theta : G \to \mathbb{R}_+$ is only upper semi-continuous while the volume is a continuous function, see [6, Theorem 1]. However, it does distinguish connected components of *relative* character varieties. It can be related to the degree of a line subbundle of the parabolic Higgs bundle describing the representation (see Section 0.6 and [25]).

0.3. Supra-maximal representations

By Theorem 1, the usual Milnor-Wood inequality

$$|\mathbf{eu}(\rho)| \leq |\chi(\Sigma_{g,n})|$$

holds unless $g = 0$, in which case $-\mathbf{eu}(\rho)$ can be equal to $n - 1$ or n. This motivates the following definition:

DEFINITION 0.3. – A representation $\rho : \pi_1(\Sigma_{0,n}) \to G$ is *supra-maximal* if $-\mathbf{eu}(\rho) = n - 1$ or n.

By Theorem 1, such representations only occur when

$$2\pi(n-1) \leq \Theta(\rho) \leq 2\pi n. \tag{6}$$

We will see soon that $-\mathbf{eu}(\rho) = n$ if and only if the representation ρ is trivial (i.e., sending everyone to the identity). In contrast, the set of representations such that $-\mathbf{eu}(\rho) = n-1$ has non empty interior in the set $\mathrm{Hom}(\pi_1(\Sigma_{g,n}), G)$. Such representations can be constructed by considering a "necklace" of negatively oriented triangle groups with appropriate angles (see Subsection 3.2).

Bowditch raised in [4] the question of whether, given a surface group representation, one can find a simple closed curve mapped to an elliptic or parabolic element. The importance of this question in understanding the dynamics of the mapping class group on character varieties was suggested in [13, 14] and recently made precise in [21].

Here, we prove that supra-maximal representations satisfy Bowditch's property in a very strong way:

THEOREM 2. – *Every supra-maximal representation* $\rho : \pi_1(\Sigma_{0,n}) \to G$ *is totally non hyperbolic, namely every element of* $\pi_1(\Sigma_{0,n})$ *which is homotopic to a simple closed curve is mapped by* ρ *to an elliptic or a parabolic isometry of* G.

Totally non hyperbolic representations were already known to exist when $g = 0$ and $n = 4$: Shinpei Baba observed that the representations lying in the compact component of the relative $\mathrm{PSL}(2, \mathbb{R})$-character varieties of the four punctured sphere discovered by Benedetto and Goldman in [3] have this property.

Their existence in genus zero contrasts with the higher genus case: indeed, Gallo, Kapovich and Marden [8, Part A] showed (among other things) that a non elementary representation from the fundamental group of a closed surface with values into $\mathrm{PSL}(2, \mathbb{C})$ always maps a certain element of the fundamental group isotopic to a simple closed curve to a hyperbolic element.

In a different direction, Yang recently found representations of four punctured spheres with parabolic boundaries that do not satisfy Bowditch's property [29].

A consequence of Theorem 2 together with the work of Gueritaud-Kassel [18] is that a supra-maximal representation is "dominated" by any Fuchsian representation (the holonomy of a complete metric of finite volume on $\Sigma_{0,n}$). To be more precise, let $l(g) := \inf_{x \in \mathbb{H}^+} d(x, g(x))$ be the translation length of an element $g \in G$, and for every representation $\rho \in \mathrm{Hom}(\pi_1(\Sigma_{g,n}), G)$, let $L_\rho : \pi_1(\Sigma_{0,n}) \to [0, \infty)$ be defined by $L_\rho(g) = l(\rho(g))$. Then for any couple (j, ρ) formed by a Fuchsian representation j and a supra-maximal representation ρ, we have $L_\rho \leq L_j$. From this we deduce:

COROLLARY 3. – *In the character variety* $\mathrm{Rep}(\pi_1(\Sigma_{0,n}), G)$*, the subset of supra-maximal representations is compact.*

0.4. Compact components in relative character varieties

Let us fix $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in (0, 2\pi)^n$. We denote by $\mathrm{Rep}_{\boldsymbol{\alpha}}(\pi_1(\Sigma_{0,n}), G)$ the set of conjugacy classes of representations such that $\theta_i(\rho) = \alpha_i$. Because the α_i's are different from 0 and 2π, the space $\mathrm{Rep}_{\boldsymbol{\alpha}}(\Sigma_{0,n}, G)$ has the structure of a smooth manifold and carries a natural symplectic form that has been constructed by Goldman [10], building on works of Atiyah and Bott [1]. Let $\mathrm{Rep}_{\boldsymbol{\alpha}}^{SM}(\Sigma_{0,n}, G)$ denote the set of supra-maximal representations in $\mathrm{Rep}_{\boldsymbol{\alpha}}(\Sigma_{0,n}, G)$. Corollary 3 implies that $\mathrm{Rep}_{\boldsymbol{\alpha}}^{SM}(\Sigma_{0,n}, G)$ forms compact connected components of $\mathrm{Rep}_{\boldsymbol{\alpha}}(\Sigma_{0,n}, G)$. We can say more and describe the symplectic geometry of these components:

THEOREM 4. – *If* $2(n-1)\pi < \sum_{i=1}^n \alpha_i < 2n\pi$*, then the space* $\mathrm{Rep}_{\boldsymbol{\alpha}}^{SM}(\Sigma_{0,n}, G)$ *is non-empty and symplectomorphic to* $\mathbb{C}\mathbf{P}^{n-3}$*, with a multiple of the Fubini-Study symplectic form whose total volume is*

$$\frac{(\pi\lambda)^{n-3}}{(n-3)!},$$

where

$$\lambda = \sum_{i=1}^{n} \alpha_i - 2(n-1)\pi.$$

The proof is given in Subsection 3.3. It makes use of a faithful Hamiltonian action of the torus $(\mathbb{R}/\pi\mathbb{Z})^{n-3}$ on $\mathrm{Rep}^{SM}_{\boldsymbol{\alpha}}(\Sigma_{0,n}, G)$, associated to a pair-of-pants decomposition of $\Sigma_{0,n}$. Delzant proved in [7] that compact symplectic manifolds provided with a faithful Hamiltonian action of a torus of half the dimension are classified by the image of their *moment map*, which is a polytope satisfying certain arithmeticity conditions. Here we compute explicitly the Delzant polytope of our action and recognize the one corresponding to a natural action of $(\mathbb{R}/\pi\mathbb{Z})^{n-3}$ on $\mathbb{C}\mathbf{P}^{n-3}$ with a certain multiple of the Fubini-Study symplectic form.

0.5. Geometrization by $\mathbb{H}^-$-conifolds

In Section 4, we show that supra-maximal representations can be geometrized by $\mathbb{H}^-$-structures in a very strong way. In fact, the set of all possible geometrizations by a $\mathbb{H}^-$-structure is a copy of the Teichmüller space $\mathcal{T}_{0,n}$.

THEOREM 5. – *Let $\rho : \pi_1(\Sigma_{0,n}) \to G$ be a supra-maximal representation. Then either ρ is Abelian, or for every $\sigma \in \mathcal{T}_{0,n}$, there exists a unique ρ-equivariant map $\widetilde{\Sigma}_{0,n} \to \mathbb{H}^-$ which is holomorphic with respect to the complex structure σ.*

This property characterizes the supra-maximal representations. For instance, for maximal representations, namely those satisfying $\mathbf{eu}(\rho) = \chi(\Sigma_{g,n})$, the isomonodromic space of conical $\mathbb{H}^+$-structures is discrete, as was proven by Mondello in [24]. In a companion paper, we will adress the problem of geometrizing representations that are merely maximal, using different techniques.

Notice that Theorem 5, together with the help of the Schwarz lemma, gives an alternative proof of the fact that supra-maximal representations are dominated by Fuchsian representations. Also, the proof of Theorem 5 allows to find explicit parametrizations of supra-maximal components by symmetric powers of the Riemann sphere (which are models for the complex projective spaces). These parametrizations transit via the Troyanov uniformization theorem.

0.6. An interpretation of the results via parabolic Higgs bundles

After the first version of this preprint appeared on arXiv, we discussed with Olivier Biquard about the possibility of a proof using the theory of parabolic Higgs bundles. It turned out that such a proof exists. In the recent preprint [25], Gabriele Mondello independently gave a topological description of relative character varieties of punctured surfaces using parabolic Higgs bundles. We pointed out to him that he was in particular recovering the results of the present paper. Let us sketch very briefly the interplay between our results and the parabolic Higgs bundle theory.

Let us fix a Riemann surface structure X on $\Sigma_{g,n}$ and denote by D the divisor of the punctures. To any irreducible representation of $\pi_1(\Sigma_{g,n})$ into $\mathrm{SL}(2,\mathbb{R})$ is associated a *stable parabolic Higgs bundle*, namely a holomorphic vector bundle E of rank 2 on Σ_g and a holomorphic section Φ of $K \otimes \mathrm{End}(E) \otimes D$ (where K denotes the canonical bundle of Σ_g), plus some extra data for each cusp which boils down to the data of our $\theta_i(\rho)$. Moreover, the bundle E splits as $L^{1/2} \oplus L^{-1/2}$ and the Higgs field has the form

$$\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix},$$

where $\alpha \in H^0(K \otimes D \otimes L)$ and $\beta \in H^0(K \otimes D \otimes L^{-1})$. Our relative Euler class then coincides with the degree of L. Assume now that L is positive. The classical Higgs bundle proof of the Milnor-Wood inequality (in the case of a closed surface) goes as follows: since the Higgs bundle associated to ρ must be stable, L cannot be invariant by Φ. Therefore, β is non-zero. Since the line bundle $K \otimes D \otimes L^{-1}$ has a holomorphic section, it is non-negative, hence

$$-\mathbf{eu}(\rho) = \deg(L) \leq \deg(KD) = 2g - 2 + n.$$

However, for parabolic Higgs bundles, the stability is a condition on the *parabolic degree* of L, which is defined as $\deg(L) - \frac{1}{2\pi}\sum_{i=1}^n \theta_i(\rho)$. Therefore, in the particular case of the puntured sphere, if $\sum_{i=1}^n \theta_i(\rho) > 2\pi(n-1)$, then the degree of L can "exceptionally" be equal to $n-1$. This will force β to vanish but will not contradict the stability because the parabolic degree of L will still be negative. The vanishing of β implies that the Higgs field is nilpotent, which is reflected by the existence of a ρ-equivariant holomorphic map to $\mathbb{H}^-$. Finally, the stability implies that α does not vanish, and the space of all such Higgs bundles (with fixed values of θ_i) is thus parametrized by $\mathrm{Proj}_{\mathbb{C}}\, H^0(KLD) \simeq \mathbb{C}\mathbf{P}^{n-3}$.

0.7. Acknowledgements

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1. The refined Milnor-Wood inequality

In this section, we establish Theorem 1.

1.1. Reduction to a bound from above

Let us explain first that Theorem 1 is a consequence of the following result, the proof of which will be postponed to the next two subsections.

PROPOSITION 1.1. – *We have*

$$-\mathbf{eu}(\rho) \leq \sup\left(|\chi(\Sigma_{g,n})|, \lfloor \frac{1}{2\pi}\Theta(\rho) \rfloor\right) \leq |\chi(\Sigma_{g,n})| + 2. \tag{7}$$

Moreover, the inequality $-\mathbf{eu}(\rho) > |\chi(\Sigma_{g,n})|$ is possible only when $g = 0$, none of the conjugacy classes of $\rho(c_i)$ are hyperbolic, and the volume is non-positive. Finally, $-\mathbf{eu}(\rho) = |\chi(\Sigma_{0,n})| + 2$ if and only if ρ is the trivial representation.

Let us prove Theorem 1 assuming Proposition 1.1. Let $\overline{\rho}$ be the conjugation of ρ by an orientation-reversing isometry of $\mathbb{H}^+$. We have the formulas

$$\mathrm{Vol}(\overline{\rho}) = -\mathrm{Vol}(\rho)$$

and

$$\Theta(\rho) + \Theta(\overline{\rho}) = 2\pi l,$$

where l is the number of elliptic or parabolic cusps (counting the identity twice). In particular, we deduce

$$-\mathbf{eu}(\overline{\rho}) = \frac{1}{2\pi}\big(\mathrm{Vol}(\overline{\rho}) + \Theta(\overline{\rho})\big) = l + \mathbf{eu}(\rho). \tag{8}$$

Applying (7) to $\overline{\rho}$, we get

$$-\mathbf{eu}(\rho) \geq \inf\big(-|\chi(\Sigma_{g,n})| + l, \lceil \frac{1}{2\pi}\Theta(\rho)\rceil\big),$$

which concludes the proof of Theorem 1 if (7) holds.

The rest of this section is devoted to the proof of Proposition 1.1. We will proceed by induction on the topology of the surface. We thus start by proving the proposition in the case of the pair of pants $\Sigma_{0,3}$.

1.2. The case of the pair of pants

The universal cover $\widetilde{G}$ of the group G acts faithfully on $\widetilde{\mathbb{R}\mathbf{P}^1}$. We denote by m the generator of the covering group that acts positively with respect to the natural orientation of $\mathbb{R}\mathbf{P}^1$, namely $m(x) > x$ for every $x \in \widetilde{\mathbb{R}\mathbf{P}^1}$. On $\widetilde{G}$ there is a well-defined notion of translation number: identifying $\widetilde{\mathbb{R}\mathbf{P}^1}$ with $\mathbb{R}$ in such a way that m is conjugated to the translation $x \mapsto x + 2\pi$, the translation number $T(g)$ of an element $g \in \widetilde{G}$ is the following limit:

$$T(g) = \lim_{k\to\pm\infty} \frac{g^k(x) - x}{k}.$$

It does not depend on $x \in \mathbb{R}$. We refer to [9] for a survey on this notion.

The fundamental group $\pi_1(\Sigma_{0,3})$ is generated by three elements c_1, c_2, c_3 that satisfy the relation $c_1c_2c_3 = 1$, and that correspond to positively oriented loops around the punctures. For each $i = 1, 2, 3$, we denote by $\widetilde{\rho(c_i)}$ the unique lift of $\rho(c_i)$ in $\widetilde{G}$ having translation number $\theta_i(\rho)$. Notice that since $\rho(c_1)\rho(c_2)\rho(c_3) = 1$, there exists $k \in \mathbb{Z}$ such that

$$\widetilde{\rho(c_1)}\widetilde{\rho(c_2)}\widetilde{\rho(c_3)} = m^k.$$

LEMMA 1.2. – *We have* $-\mathbf{eu}(\rho) = k$.

Proof. – This lemma can be seen as a reformulation (in the particular case of $\Sigma_{0,3}$) of Theorem 12 of [6], according to which — given $\tilde{\rho}$ a lift of ρ to $\tilde{G}$ — one has

$$\mathbf{Vol}(\rho) = -\sum_i T\left(\tilde{\rho}(c_i)\right).$$

Indeed, let $\tilde{\rho} : \pi_1(\Sigma_{0,3}) \to \widetilde{G}$ be a lift of the representation ρ. Then for each i there exists some integer k_i such that

$$\widetilde{\rho(c_i)} = \tilde{\rho}(c_i)m^{k_i}.$$

We have the relations

$$k_1 + k_2 + k_3 = k \text{ and } \theta_i(\rho) = T(\tilde{\rho}(c_i)) + k_i.$$

Both come from the fact that m belongs to the center of $\widetilde{G}$. Therefore

$$\sum_i \theta_i(\rho) = \sum_i T(\tilde{\rho}(c_i)) + 2\pi k,$$

and the claim follows from [6, Theorem 12]. □

We now proceed to a case-by-case analysis.

Case of an identity peripheral.– We first consider the case where $\rho(c_i) = \mathrm{Id}$ for some i. Applying a cyclic permutation if necessary, we can assume $\rho(c_3) = \mathrm{Id}$. Thus $\rho(c_2) = \rho(c_1)^{-1}$. We then have $\widetilde{\rho(c_3)} = m$ and

$$\begin{aligned} m^k = \widetilde{\rho(c_1)}\widetilde{\rho(c_2)}\widetilde{\rho(c_3)} &= m \quad \text{if } \rho(c_1), \rho(c_2) \text{ are hyperbolic} \\ &= m^2 \text{ if } \rho(c_1), \rho(c_2) \text{ are elliptic or parabolic} \\ &= m^3 \text{ if } \rho(c_1) = \rho(c_2) = 1. \end{aligned}$$

Notice that in the first case $\Theta(\rho) = 2\pi$, in the second case, $\Theta(\rho) = 4\pi$, and in the third case $\Theta(\rho) = 6\pi$. By Lemma 1.2, we thus have $-\mathbf{eu}(\rho) = \frac{1}{2\pi}\Theta(\rho)$, which proves the inequality (7).

Case of a hyperbolic peripheral.– Assume now that one of the $\rho(c_i)$'s is hyperbolic. Up to cyclic permutation, we can assume that $\rho(c_3)$ is hyperbolic. Notice that the lifts $\widetilde{\rho(c_i)}$ are chosen so that

$$m^{-1}(y) < \widetilde{\rho(c_i)}(y) \leq m(y) \tag{9}$$

for every $y \in \widetilde{\mathbb{R}P^1}$. Moreover, there exist two points $x^\pm \in \widetilde{\mathbb{R}\mathbf{P}^1}$ such that

$$\widetilde{\rho(c_3)}(x^+) > x^+ \text{ and } \widetilde{\rho(c_3)}(x^-) < x^-.$$

From (9), we deduce that

$$m^k(x^+) = \widetilde{\rho(c_1)}\widetilde{\rho(c_2)}\widetilde{\rho(c_3)}(x^+) > m^{-2}(x^+)$$

and similarly

$$m^k(x^-) = \widetilde{\rho(c_1)}\widetilde{\rho(c_2)}\widetilde{\rho(c_3)}(x^-) < m^2(x^-).$$

The integer k hence satisfies $|k| \leq 1$, which implies the proposition in the case one of the $\rho(c_i)$ is hyperbolic. [(1)]

[(1)] Note that there exists a representation having $\rho(c_1), \rho(c_2)$ negative parabolic and $\rho(c_3)$ hyperbolic: such a representation has $-\mathbf{eu}(\rho) = 1$ but $\Theta(\rho) = 4\pi$, showing that Theorem 1 is not sharp when some peripheral is parabolic.

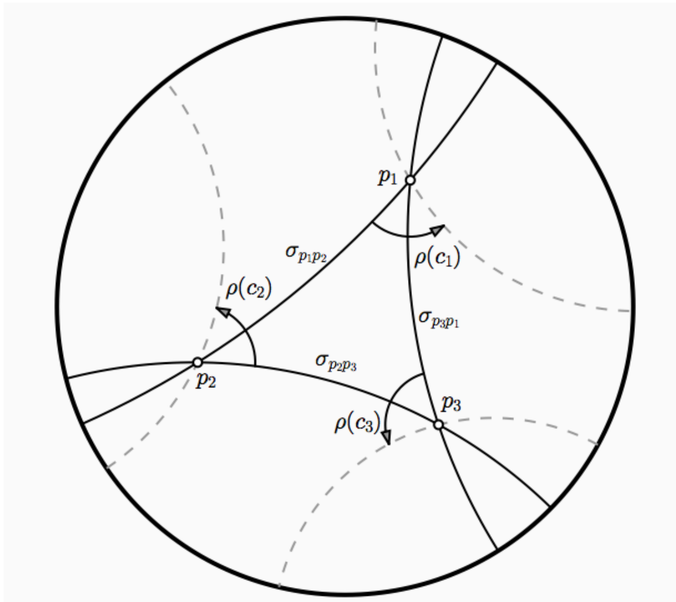

FIGURE 1. A representation with the $\rho(c_i)$ elliptic.

Case where none of the $\rho(c_i)$'s is identity or hyperbolic.– In this case each $\rho(c_i)$ has a unique fixed point $p_i \in \mathbb{H} \cup \partial\mathbb{H}$. Either they are distinct, or equal.

We first consider the case where $p_1 = p_2 = p_3$. In this case, the volume is zero. Moreover, if p_i lies in $\mathbb{H}^+$, then $\theta_i(\rho) < 2\pi$ since none of the $\rho(c_i)$ are the identity. In particular $\Theta(\rho) = 2\pi$ or 4π. If p_i lies in $\partial\mathbb{H}^+$, then one of the $\rho(c_i)$ has to be a positive parabolic, since their product is 1. In particular, $\Theta(\rho) = 2\pi$ or 4π as before. So we are done.

Suppose now that the p_i's are distinct. In this case, the image of ρ is a "triangle group". More precisely, for $p, q \in \mathbb{H}^+ \cup \partial\mathbb{H}^+$ distinct, let σ_{pq} be the reflection with respect to the geodesic (pq). We then have the formulas

$$\rho(c_1) = \sigma_{p_3 p_1}\sigma_{p_1 p_2}, \ \rho(c_2) = \sigma_{p_1 p_2}\sigma_{p_2 p_3} \text{ and } \rho(c_3) = \sigma_{p_2 p_3}\sigma_{p_3 p_1}$$

(see Figure 1).

Since none of the $\rho(c_i)$'s is the identity, the triangle $\Delta = p_1 p_2 p_3$ is non degenerate. In particular, ρ is the holonomy of a $\mathbb{H}^+$-structure (if $p_1 p_2 p_3$ is clockwise oriented) or a $\mathbb{H}^-$-structure (if $p_1 p_2 p_3$ is counterclockwise oriented) on the sphere minus three points, obtained by gluing two copies of Δ. In the first case, $\theta_i(\rho)$ is twice the angle of Δ at p_i and the volume of ρ twice the volume of Δ, so by Gauss-Bonnet we get $-\mathbf{eu}(\rho) = 1$. In the second case, $2\pi - \theta_i(\rho)$ is twice the angle of Δ at p_i, and the volume of ρ is $-2\mathbf{Vol}(\Delta)$, so we get $-\mathbf{eu}(\rho) = 2$ in this case.

This concludes the proof of Proposition 1.1 in the case of $\Sigma_{0,3}$.

1.3. The general case

Crucially, our induction will use the following fact:

PROPOSITION 1.3. – *Assume Σ is obtained from a (possibly disconnected) surface Σ' by gluing b with b'^{-1}, where b and b' are two boundary curves of Σ'. Let ρ be a representation of $\pi_1(\Sigma)$ into G and ρ' the restriction of ρ to $\pi_1(\Sigma')$. Then we have the following:*

— *if $\rho(b)$ is the identity, then*

$$-\mathbf{eu}(\rho) = -\mathbf{eu}(\rho') - 2\ ;$$

— *if $\rho(b)$ is parabolic or elliptic, then*

$$-\mathbf{eu}(\rho) = -\mathbf{eu}(\rho') - 1\ ;$$

— *if $\rho(b)$ is hyperbolic, then*

$$-\mathbf{eu}(\rho) = -\mathbf{eu}(\rho').$$

(If Σ' has several connected components, we denote by $\mathbf{eu}(\rho')$ the sum of the Euler classes of the restrictions of ρ to the fundamental group of each connected component.)

Proof. – By additivity of the volume, we have $\mathbf{Vol}(\rho) = \mathbf{Vol}(\rho')$. Therefore,

$$-\mathbf{eu}(\rho') = -\mathbf{eu}(\rho) + \frac{1}{2\pi}\left(\theta(\rho'(b)) + \theta(\rho'(b'))\right).$$

One has $\rho'(b') = \rho'(b)^{-1}$ and therefore

$$\theta(\rho'(b)) + \theta(\rho'(b')) = \begin{cases} 4\pi & \text{if } \rho'(b) \text{ is the identity;} \\ 2\pi & \text{if } \rho'(b) \text{ is elliptic or parabolic;} \\ 0 & \text{if } \rho'(b) \text{ is hyperbolic.} \end{cases} \qquad \square$$

Let us now prove Proposition 1.1 for $\Sigma_{0,n}$ by induction on n.

Proof of Proposition 1.1 in the genus $g = 0$ case. – We decompose $\Sigma = \Sigma_{0,n}$ as the union of $\Sigma' = \Sigma_{0,k+1}$ and $\Sigma'' = \Sigma_{0,n-k+1}$, where some boundary curve b' of $\Sigma_{0,k+1}$ is glued with b''^{-1} for some boundary curve b'' of $\Sigma_{0,n-k+1}$. Denote by ρ' the restriction of ρ to $\pi_1(\Sigma_{0,k+1})$ and by ρ'' the restriction of ρ to $\pi_1(\Sigma_{0,n-k+1})$.

By induction, we have $-\mathbf{eu}(\rho') \le k + 1$ and $-\mathbf{eu}(\rho'') \le n - k + 1$. We can now proceed to a case-by-case study:

— If $-\mathbf{eu}(\rho') = |\chi(\Sigma')| + 2$ and $-\mathbf{eu}(\rho'') = |\chi(\Sigma'')| + 2$, then by, induction hypothesis, both ρ' and ρ'' are trivial. Hence ρ is trivial and $-\mathbf{eu}(\rho) = n$.

— If $-\mathbf{eu}(\rho') = |\chi(\Sigma')| + 2$ and $-\mathbf{eu}(\rho'') = |\chi(\Sigma'')| + 1$ (or the converse) then, by induction hypothesis, ρ' is trivial. Therefore $\rho'(b)$ is the identity. Hence

$$-\mathbf{eu}(\rho) = -\mathbf{eu}(\rho') - \mathbf{eu}(\rho'') - 2 = n - 1.$$

By induction, no boundary curve of $\Sigma_{0,k+1}$ and $\Sigma_{0,n-k+1}$ has hyperbolic image. Hence the same holds for $\Sigma_{0,n}$. Moreover, we have $\lfloor \sum_{i'} \theta_{i'}(\rho') \rfloor = |\chi(\Sigma')| + 2$ and $\lfloor \sum_{i''} \theta_{i''}(\rho'') \rfloor = |\chi(\Sigma'')| + 1$, so

$$\lfloor \sum_i \theta_i(\rho) \rfloor = \lfloor \sum_{i'} \theta_{i'}(\rho') + \sum_{i''} \theta_{i''}(\rho'') - 2 \rfloor = n - 1,$$

proving (7).

— If $-\mathbf{eu}(\rho') = |\chi(\Sigma')| + 1$ and $-\mathbf{eu}(\rho'') = |\chi(\Sigma'')| + 1$, then $\rho'(b)$ is not hyperbolic. If $\rho'(b)$ is the identity, then

$$-\mathbf{eu}(\rho) = -\mathbf{eu}(\rho') - \mathbf{eu}(\rho'') - 2 = n - 2.$$

Otherwise, we have

$$-\mathbf{eu}(\rho) \leq -\mathbf{eu}(\rho') - \mathbf{eu}(\rho'') - 1 = n - 1.$$

Like in the previous case, no boundary curve of Σ is sent to a hyperbolic element. Finally, since $\rho(b'), \rho(b'')$ are not hyperbolic, we get $\theta(\rho(b')) + \theta(\rho(b'')) = 2\pi$, and

$$\begin{aligned}\sum_i \theta_i(\rho) &= \sum_{i'} \theta_{i'}(\rho') + \sum_{i''} \theta_{i''}(\rho'') - 1 \\ &\geq (|\chi(\Sigma')| + 1) + (|\chi(\Sigma'')| + 1) - 1 = |\chi(\Sigma)| + 1,\end{aligned}$$

proving (7).

— If $-\mathbf{eu}(\rho') = |\chi(\Sigma')| + 2$ and $-\mathbf{eu}(\rho'') \leq |\chi(\Sigma'')|$ (or the converse), then $\rho(b')$ is the identity. Hence

$$-\mathbf{eu}(\rho) = -\mathbf{eu}(\rho') - \mathbf{eu}(\rho'') - 2 \leq n - 2.$$

— If $-\mathbf{eu}(\rho') = |\chi(\Sigma')| + 1$ and $-\mathbf{eu}(\rho'') \leq |\chi(\Sigma'')|$ (or the converse), then $\rho(b')$ is not hyperbolic. Hence

$$-\mathbf{eu}(\rho) \leq -\mathbf{eu}(\rho') - \mathbf{eu}(\rho'') - 1 \leq n - 2.$$

— Finally, if $-\mathbf{eu}(\rho') \leq |\chi(\Sigma')|$ and $-\mathbf{eu}(\rho'') \leq |\chi(\Sigma'')|$, then

$$-\mathbf{eu}(\rho) \leq -\mathbf{eu}(\rho') - \mathbf{eu}(\rho'') \leq n - 2. \qquad \square$$

We can now prove Proposition 1.1 in the higher genus case. Note that

$$\frac{1}{2\pi}\Theta(\rho) = \frac{1}{2\pi}\sum_i \theta_i(\rho) \leq n \leq 2g - 2 + n = |\chi(\Sigma_{g,n})|$$

as soon as $g \geq 1$. Therefore, Proposition 1.1 when $g \geq 1$ reduces to the classical Milnor-Wood inequality

$$-\mathbf{eu}(\rho) \leq |\chi(\Sigma_{g,n})|.$$

Proof of Proposition 1.1 if $g > 0$. – We argue by induction on g.

$g = 1$. The surface $\Sigma_{1,n}$ is obtained from $\Sigma_{0,n+2}$ by gluing together b and b'^{-1}, for two boundary curves b and b'. Denote by ρ' the restriction of ρ to $\pi_1(\Sigma_{0,n+2})$. If $-\mathbf{eu}(\rho') = n + 2$, then ρ' is trivial, hence ρ is trivial and $-\mathbf{eu}(\rho) = n$. If $-\mathbf{eu}(\rho') = n + 1$, then $\rho'(b)$ is not hyperbolic. Hence

$$-\mathbf{eu}(\rho) \leq -\mathbf{eu}(\rho') - 1 = n.$$

Finally, if $-\mathbf{eu}(\rho') \leq n$ then $-\mathbf{eu}(\rho) \leq n$.

$g \geq 2$. The surface $\Sigma_{g,n}$ is obtained from $\Sigma_{g-1,n+2}$ by gluing together two boundary curves b and b'. Denote by ρ' the restriction of ρ to $\pi_1(\Sigma_{g-1,n+2})$. By induction hypothesis, we have

$$-\mathbf{eu}(\rho') \leq 2(g-1) - 2 + n + 2 = 2g - 2 + n,$$

hence

$$-\mathbf{eu}(\rho) \leq -\mathbf{eu}(\rho') \leq 2g - 2 + n. \qquad \square$$

2. Supra-maximal representations

2.1. Definition and examples

Supra-maximal representations are representations whose Euler class violates the classical Milnor-Wood inequality. It happens only in the following situation:

Definition 2.1. – A representation $\rho : \pi_1(\Sigma_{g,n}) \to G$ is called *supra-maximal* if $g = 0$ and $-\mathbf{eu}(\rho) = n - 1$ or $-\mathbf{eu}(\rho) = n$.

As we saw in Proposition 1.1, supra-maximal representations have Euler class $n-1$, except for the trivial representation which has Euler class n.

2.2. Supra-maximal representations are "totally non hyperbolic"

A first important fact about supra-maximal representations is that they send every simple closed curve to a non hyperbolic element.

Proposition 2.2. – *Let $\rho : \pi_1(\Sigma_{0,n}) \to G$ be a representation of Euler class $-\mathbf{eu}(\rho) = n-1$ or n. Then, for any element γ in $\pi_1(\Sigma_{0,n})$ freely homotopic to a simple closed curve, $\rho(\gamma)$ is not hyperbolic.*

Proof. – Let γ be a simple closed curve in $\Sigma_{0,n}$. If γ is freely homotopic to a boundary curve, then $\rho(\gamma)$ is not hyperbolic, as part of Proposition 1.1. Othewise, γ cuts $\Sigma_{0,n}$ into two surfaces Σ' and Σ''. We saw in the demonstration of Proposition 1.1 that the restrictions of ρ to $\pi_1(\Sigma')$ and $\pi_1(\Sigma'')$ are both supra-maximal, and therefore that the images of the boundary curves of Σ' by ρ (and in particular $\rho(\gamma)$) are non hyperbolic. $\square$

Remark 2.3. – Shinpei Baba observed that the representations lying in the Benedetto-Goldman compact components [3] are totally elliptic. This remark gave us the idea that supra-maximal representations should form compact components of relative character varieties.

2.3. Domination

As a corollary, we obtain that ρ is dominated by any Fuchsian representation:

Corollary 2.4. – *Let $\rho : \pi_1(\Sigma_{0,n}) \to G$ be a representation of relative Euler class $n-1$ or n. Then, for any Fuchsian representation $j : \pi_1(\Sigma_{0,n}) \to G$, there exists a 1-Lipschitz (j,ρ)-equivariant map from $\mathbb{H}^+$ to $\mathbb{H}^+$.*

Proof. – According to the work of Guéritaud-Kassel [18], in order to obtain the conclusion, it is enough to know that

$$L_\rho(\gamma) \leq L_j(\gamma)$$

for every simple closed curve γ. This is obviously true since, by Proposition 2.2, $L_\rho(\gamma) = 0$ for every simple closed curve γ. □

As a consequence, we obtain that the length spectrum L_ρ of any supra-maximal representation is bounded by the following interesting function

$$C(\gamma) = \inf\{L_j(\gamma), j : \pi_1(\Sigma_{0,n}) \to G \text{ Fuchsian}\}.$$

It is invariant by conjugation and by the braid group, and measures a certain complexity of the corresponding element of $\pi_1(\Sigma_{0,n})$. It would be interesting to understand this function in more details. See Basmajian [2] for related results.

2.4. Compactness of the space of supra-maximal representations

Corollary 2.4 provides a uniform control on all supra-maximal representations, from which we can prove that the space of supra-maximal representations is compact.

Let $\mathrm{Hom}(\Sigma_{0,n}, G)$ denote the space of representations of $\pi_1(\Sigma_{0,n})$ into G and $\mathrm{Rep}(\Sigma_{0,n}, G)$ its quotient by the action of G by conjugation. The natural topology of $\mathrm{Hom}(\Sigma_{0,n}, G)$ induces a non-Hausdorff topology on $\mathrm{Rep}(\Sigma_{0,n}, G)$.

Proposition 2.5. – *The space of supra-maximal representations is a compact subset of* $\mathrm{Rep}(\Sigma_{0,n}, G)$.

Proof. – Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of supra-maximal representations. We fix a Fuchsian representation j. By Corollary 2.4, we can find a sequence of 1-Lipschitz maps $f_n : \mathbb{H}^+ \to \mathbb{H}^+$ such that f_n is (j,ρ_n)-equivariant. Up to conjugating each ρ_n and composing each f_n by an isometry of $\mathbb{H}^+$, we can assume that each f_n fixes a given base point. By Ascoli's theorem, up to extracting a subsequence, f_n converges uniformly on every compact set to a 1-Lipschitz map f_∞. This map f_∞ is (j,ρ_∞)-equivariant for some representation ρ_∞ and we have

$$\rho_n \underset{n\to+\infty}{\to} \rho_\infty$$

in $\mathrm{Rep}(\Sigma_{0,n}, G)$.

Finally, since the function $-\mathbf{eu} : \mathrm{Rep}(\Sigma_{0,n}, G) \to \mathbb{R}$ is upper semi-continuous, the limit ρ_∞ is still supra-maximal. We have thus proved that the set

$$\{\rho \in \mathrm{Rep}(\Sigma_{0,n}, G) \to \mathbb{R} \mid -\mathbf{eu}(\rho) \geq n-1\}$$

is sequentially compact, hence compact. □

2.5. Compact components in relative character varieties

Recall that $\mathrm{Rep}_{\boldsymbol{\alpha}}^{SM}(\Sigma_{0,n}, G)$ denotes the set of supra-maximal representations in the corresponding relative character variety $\mathrm{Rep}_{\boldsymbol{\alpha}}(\Sigma_{0,n}, G)$.

PROPOSITION 2.6. – *If* $2(n-1)\pi < \sum_{i=1}^{n} \alpha_i < 2n\pi$, *then* $\mathrm{Rep}_{\boldsymbol{\alpha}}^{SM}(\Sigma_{0,n}, G)$ *forms a non-empty compact connected component of* $\mathrm{Rep}_{\boldsymbol{\alpha}}(\Sigma_{0,n}, G)$.

Proof. – Since $\mathrm{Rep}_{\boldsymbol{\alpha}}(\Sigma_{0,n}, G)$ is closed in $\mathrm{Rep}(\Sigma_{0,n}, G)$, its intersection with the set of supra-maximal representations is compact by Proposition 2.5. Since none of the α_i is equal to 0 or 2π, the Euler class is continuous in restriction to $\mathrm{Rep}_{\boldsymbol{\alpha}}(\Sigma_{0,n}, G)$. Since it takes integral values, the subset of supra-maximal representations is a union of connected components.

It remains to prove that $\mathrm{Rep}_{\boldsymbol{\alpha}}^{SM}(\Sigma_{0,n}, G)$ is non-empty and connected. We postpone this to Section 3.3. □

3. Symplectic geometry of supra-maximal components

3.1. The Goldman symplectic structure on (relative) character varieties

Goldman constructed in [10] a natural symplectic structure on the character variety $\chi_G(\Sigma)$ of the fundamental group of a closed connected oriented surface Σ into G, and in fact into any semi-simple Lie group. Moreover, he found in [11] a duality between conjugacy invariant functions on G and certain "twisting" deformations of representations.

More precisely, let $F : G \to \mathbb{R}$ be a function invariant by conjugation. Recall that there is a natural non-degenerate bilinear form κ_G on the Lie algebra $\mathfrak{g}$ which is invariant under the adjoint action of G: the Killing form. At a point $g \in G$ where F is $\mathcal{C}^1$, we define $\delta_g F$ as the vector in $\mathfrak{g}$ such that

$$\mathrm{d}F_g(g \cdot v) = \kappa_G(\delta_g F, v)$$

for all $v \in \mathfrak{g}$. Because F is invariant by conjugation, $\delta_g F$ is centralized by g.

Now, let $\rho : \pi_1(\Sigma) \to G$ be a representation. Let b denote a simple closed curve in Σ which is not homotopic to a boundary curve. If b is separating, then it cuts Σ into two surfaces Σ' and Σ'' and we can write

$$\pi_1(\Sigma) = \pi_1(\Sigma') * \pi_1(\Sigma'')/b' \sim b''.$$

If b is not separating, cutting along b gives a compact surface Σ' and we can write

$$\pi_1(\Sigma) = \pi_1(\Sigma') * \langle u \rangle / b_{\mathrm{left}} \sim u b_{\mathrm{right}} u^{-1}.$$

Since $\rho(b)$ centralizes $\delta_{\rho(b)} F$, we can "twist" the representation ρ along b and define a representation $\Phi_{F,b,t}(\rho)$ by

$$\begin{aligned} \Phi_{F,b,t}(\rho) : \gamma \in \pi_1(\Sigma') &\mapsto \rho(\gamma) \\ \gamma \in \pi_1(\Sigma'') &\mapsto \exp(t\delta_{\rho(b)}F)\rho(\gamma)\exp(-t\delta_{\rho(b)}F) \end{aligned}$$

if b is separating and

$$\begin{aligned} \Phi_{F,b,t}(\rho) : \gamma \in \pi_1(\Sigma') &\mapsto \rho(\gamma) \\ u &\mapsto u \exp(t\delta_{\rho(b)}F) \end{aligned}$$

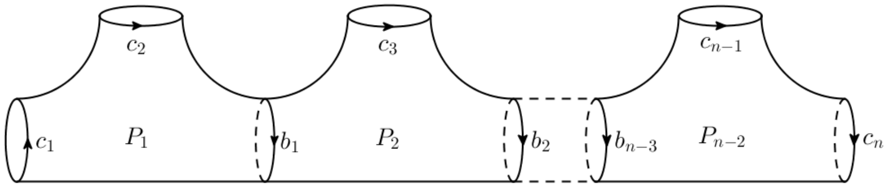

FIGURE 2. A pair-of-pants decomposition of $\Sigma_{0,n}$.

if b is non-separating. On can prove that $\Phi_{F,b,t}$ induces a well-defined flow on the character variety of Σ.

THEOREM 3.1 (Goldman). – *Let F be a function of class $\mathcal{C}^1$ on G invariant by conjugation. If b is a simple closed curve in Σ, denote by $\mathbf{F}_b$ the function on $\chi_G(\Sigma)$ defined by*

$$\mathbf{F}_b(\rho) = F(\rho(b)).$$

Then $\mathbf{F}_b$ is $\mathcal{C}^1$ and its Hamiltonian flow (with respect to the Goldman symplectic form) is the flow $\left(\Phi_{F,b,t}\right)_{t\in\mathbb{R}}$.

This generalizes to relative character varieties of surfaces with boundary (see for instance [15]).

3.2. A Hamiltonian action of $(\mathbb{R}/\pi\mathbb{Z})^{n-3}$

We consider the decomposition of $\Sigma_{0,n}$ into pairs of pants given by Figure 2.

If ρ is a representation of $\pi_1(\Sigma_{0,n})$ into G, we note $\beta_i(\rho) = \theta(\rho(b_i))$. We also set $\bar{\alpha}_i = 2\pi - \alpha_i$.

LEMMA 3.2. – *If $\rho \in \mathrm{Rep}^{SM}_{\boldsymbol{\alpha}}(\Sigma_{0,n}, G)$, then, for all $1 \leq i \leq n-3$, we have*

$$\sum_{k=1}^{i+1} \bar{\alpha}_k \leq \beta_i(\rho) \leq 2\pi - \sum_{k=i+2}^{n} \bar{\alpha}_k.$$

In particular, $\rho(b_i)$ is elliptic.

Proof. – The curve b_i cuts $\Sigma_{0,n}$ into two surfaces $\Sigma'_{0,i+2}$ and $\Sigma''_{0,n-i}$. Let ρ' and ρ'' denote respectively the restrictions of ρ to $\pi_1(\Sigma')$ and $\pi_1(\Sigma'')$. Since none of the α_i is equal to 2π, neither ρ' nor ρ'' is trivial. Since ρ is supra-maximal, ρ' and ρ'' are also supra-maximal. Applying Proposition 1.1 to ρ', we get

$$\beta_i(\rho) + \sum_{k=1}^{i+1} \alpha_k \geq 2\pi(i+1).$$

Applying Proposition 1.1 to ρ'', we get

$$\bar{\beta}_i(\rho) + \sum_{k=i+2}^{n} \alpha_k \geq 2\pi(n-i-1),$$

where $\bar{\beta}_i(\rho) = \theta(\rho(b_i)^{-1})$.

If $\rho(b_i)$ were the identity, then we would get

$$-\mathbf{eu}(\rho) = -\mathbf{eu}(\rho') - \mathbf{eu}(\rho'') - 2 = |\chi(\Sigma_{0,n})|$$

and ρ would not be supra-maximal.

Therefore, $\bar{\beta}_i(\rho) = 2\pi - \beta_i(\rho)$ and we get

$$\sum_{k=1}^{i+1} \bar{\alpha}_k \leq \beta_i(\rho) \leq 2\pi - \sum_{k=i+2}^{n} \bar{\alpha}_k.$$

In particular, $\rho(b_i)$ is elliptic. □

Since $\rho(b_i)$ is elliptic for all $\rho \in \mathrm{Rep}_\alpha^{SM}(\Sigma_{0,n}, G)$, The functions β_i are $n-3$ well-defined smooth functions on $\mathrm{Rep}_\alpha^{SM}(\Sigma_{0,n}, G)$.

Proposition 3.3. – *The Hamiltonian flow associated to the function β_i is π-periodic.*

Proof. – Recall that the function θ on G associates to an elliptic element g the rotation angle of g in $\mathbb{H}^+$. The function θ is invariant by conjugation. Let us compute its gradient (with respect to the Killing metric, which is a non degenerate pseudo-Riemannian metric on G) at the point

$$g_0 = \begin{pmatrix} \cos(\theta_0/2) & -\sin(\theta_0/2) \\ \sin(\theta_0/2) & \cos(\theta_0/2) \end{pmatrix}.$$

We can write

$$\theta(g) = f(\mathrm{Tr}(g)),$$

where f satisfies

$$f(2\cos(x/2)) = x.$$

For $u \in \mathfrak{sl}(2,\mathbb{R})$, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f\left(\mathrm{Tr}(g_0 \exp(tu))\right) &= f'\left(\mathrm{Tr}(g_0)\right) \mathrm{Tr}(g_0 u) \\ &= f'(2\cos(\theta_0/2))\mathrm{Tr}\left((g_0 - \mathrm{Tr}(g_0)\mathrm{I}_2)u\right). \end{aligned}$$

Since $g_0 - \mathrm{Tr}(g_0)\mathrm{I}_2 \in \mathfrak{sl}(2,\mathbb{R})$, we deduce that

$$\begin{aligned} \delta_{g_0}\theta &= f'(2\cos(\theta_0/2))(g_0 - \mathrm{Tr}(g_0)\mathrm{I}_2) \\ &= \begin{pmatrix} 0 & -f'(2\cos(\theta_0/2))\sin(\theta_0/2) \\ f'(2\cos(\theta_0/2))\sin(\theta_0/2) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

since $-f'(2\cos(\theta_0/2))\sin(\theta_0/2) = \frac{\mathrm{d}}{\mathrm{d}x}_{|x=\theta_0} f(2\cos(x/2)) = 1$.

Therefore, the flow

$$t \mapsto \exp(t\delta_{g_0}\theta)$$

is π-periodic.

Since every elliptic element is conjugated to some g_0, we obtain, thanks to Theorem 3.1, that the Hamiltonian flow

$$\Phi_i = \Phi_{\theta, b_i}$$

associated to the function $\beta_i : \rho \mapsto \theta(\rho(b_i))$ on $\mathrm{Rep}^{SM}_{\boldsymbol{\alpha}}(\Sigma_{0,n}, G)$ is π-periodic. □

Since the curves b_i are pairwise disjoint, the functions β_i Poisson commute and their Hamiltonian flows together provide an action of $(\mathbb{R}/\pi\mathbb{Z})^{n-3}$ on $\mathrm{Rep}_{\boldsymbol{\alpha}}(\Sigma_{0,n}, G)$.

3.3. The Delzant polytope of the Hamiltonian action

In this subsection we prove Theorem 4.

By the work of Delzant [7], in order to understand the symplectic structure of the manifold $\mathrm{Rep}^{SM}_{\boldsymbol{\alpha}}(\Sigma_{0,n}, G)$, it is essentially enough to understand the image of the *moment map*:

$$\begin{aligned} \boldsymbol{\beta} : \mathrm{Rep}^{SM}_{\alpha}(\Sigma_{0,n}, G) &\to \mathbb{R}^{n-3} \\ \rho &\mapsto (\beta_1(\rho), \dots, \beta_{n-3}(\rho)) \,. \end{aligned}$$

More precisely, Delzant proved the following:

THEOREM 3.4 (Delzant [7]). – *Let (M, ω) and (M', ω') be two compact symplectic manifolds of dimension $2(n-3)$ provided with a Hamiltonian action of $(\mathbb{R}/\pi\mathbb{Z})^{n-3}$. Assume that the moment maps $\boldsymbol{\beta}$ and $\boldsymbol{\beta}'$ of these actions have the same image (up to translation). Then there exists a symplectomorphism $\varphi : (M, \omega) \to (M', \omega')$ that conjugates the actions of $(\mathbb{R}/\pi\mathbb{Z})^{n-3}$.*

LEMMA 3.5. – *The moment map $\boldsymbol{\beta}$ of the Hamiltonian action of $\mathbb{R}/\pi\mathbb{Z}$ described in Subsection 3.2 satisfies the following $n-2$ affine inequalities:*

$$\beta_1 \geq \bar{\alpha}_1 + \bar{\alpha}_2, \tag{10}$$

$$\beta_i - \beta_{i-1} \geq \bar{\alpha}_{i+1}, \quad 2 \leq i \leq n-3, \tag{11}$$

$$\beta_{n-3} \leq 2\pi - \bar{\alpha}_{n-1} - \bar{\alpha}_n. \tag{12}$$

Conversely, if $(x_1, \dots, x_{n-3}) \in \mathbb{R}_+^{n-3}$ satisfies the inequalities (10), (11), (12) *(when substituting x_i to β_i), then the set of supra-maximal representations ρ satisfying $\beta_i(\rho) = x_i$ is non-empty and connected.*

Proof. – Let $P_1, \dots, P_{n-2}$ denote the pants in the pair-of-pants decomposition given in Figure 2. Let ρ be a representation in $\mathrm{Rep}^{SM}_{\boldsymbol{\alpha}}(\Sigma_{0,n}, G)$. Then, as in the proof of Proposition 1.1, the restriction of ρ to $\pi_1(P_i)$ is supra-maximal for every i. In particular, the sum of the rotation angles of the images of the boundary curves of P_i (with the proper choice of orientation) is at least 4π. Applying this to each P_i gives the required inequalities.

Conversely, let $x_1, \dots, x_{n-3}$ satisfy the inequalities of Lemma 3.5. Fix $2 \leq i \leq n-3$. Then we have $4\pi \leq x_i + (2\pi - x_{i-1}) + \alpha_{i+1} < 6\pi$. Therefore, there exists a supra-maximal representation ρ_i of $\pi_1(P_i)$ sending the boundary curves $\bar{b}_{i-1}$, b_i and c_{i+1} respectively to rotations of angle $2\pi - x_{i-1}$, x_i and α_{i+1}. Moreover, this representation is unique up to conjugation. It is obtained by considering a hyperbolic triangle $p_1 p_2 p_3$ oriented clockwise with angles $\pi - x_{i-1}/2$, $x_i/2$ and $\alpha_{i+1}/2$, and setting $\rho_i(\bar{b}_{i-1}) = \sigma_{p_3 p_1}\sigma_{p_1 p_2}$, $\rho_i(b_i) = \sigma_{p_1 p_2}\sigma_{p_2 p_3}$ and $\rho_i(c_{i+1}) = \sigma_{p_2 p_3}\sigma_{p_3 p_1}$ (cf Figure 1).

Similarly, there is a representation ρ_1 (resp. ρ_{n-2}) of $\pi_1(P_1)$ (resp. $\pi_1(P_{n-2})$) satisfying

$$(\theta(\rho_1(c_1)), \theta(\rho_1(c_2)), \theta(\rho_1(b_1))) = (\alpha_1, \alpha_2, x_1)$$

(resp.

$$(\theta(\rho_{n-2}(c_1)), \theta(\rho_{n-2}(c_2)), \theta(\rho_{n-2}(b_1))) = (2\pi - x_{n-3}, \alpha_{n-1}, \alpha_n)\ ).$$

Now, there is a way to conjugate the ρ_i so that they can be glued together to form a supra-maximal representation ρ satisfying $\theta(\rho(c_i)) = \alpha_i$ and $\theta(\rho(b_i)) = x_i$. More precisely, one can choose g_1 in G, and then recursively choose $g_{i+1} \in G$ such that

$$g_{i+1}\rho_{i+1}(\bar{b}_i)^{-1}g_{i+1}^{-1} = g_i\rho_i(b_i)g_i^{-1}.$$

(This is possible because both $\rho_{i+1}(\bar{b}_i)^{-1}$ and $\rho_i(b_i)$ are rotations of angle x_i.) There exists a representation ρ whose restriction to each $\pi_1(P_i)$ gives $\mathrm{Ad}_{g_i} \circ \rho_i(b_i)$. Since the restriction of ρ to each $\pi_1(P_i)$ is supra-maximal, and since $\rho(b_i)$ is never trivial, the representation ρ itself is supra-maximal.

Finally, two choices of g_{i+1} coincide up to left multiplication by an element of the centralizer of $\rho_{i+1}(\bar{b}_i)$ which is connected. Therefore the space of all choices of the $(g_1, \dots, g_{n-2})$ is connected. Since any supra-maximal representation ρ in $\boldsymbol{\beta}^{-1}(x_1, \dots, x_{n-3})$ is obtained by such a gluing, it follows that the fiber $\boldsymbol{\beta}^{-1}(x_1, \dots, x_{n-3})$ is connected. □

It remains to identify the polytope defined by the equalities (10), (11), (12) to the Delzant polytope of a certain torus action on $\mathbb{C}\mathbf{P}^{n-3}$.

Recall that $\mathbb{C}\mathbf{P}^{n-3}$ carries a natural Kähler form ω_{FS}. There are $n-3$ natural commuting Hamiltonian actions $r_1, \dots, r_{n-3}$ of $\mathbb{R}/\pi\mathbb{Z}$ on $\mathbb{C}\mathbf{P}^{n-3}$, given by

$$r_k(\theta) \cdot [z_0, \dots, z_{n-3}] = [z_0, \dots, z_{k-1}, e^{2i\theta}z_k, z_{k+1}, \dots, z_n].$$

With a convenient scaling of ω_{FS}, a moment map of the action r_k with respect to the Fubini-Study symplectic form is the function

$$\begin{array}{llcl} \mu_k : & \mathbb{C}\mathbf{P}^{n-3} & \to & \mathbb{R} \\ & [z_0, \dots, z_{n-3}] & \mapsto & \frac{|z_k|^2}{\sum_{j=0}^{n-3}|z_j|^2} \end{array}.$$

(See [7, Example p.317].)

The image of the moment map $^- = (\mu_1, \dots, \mu_{n-3})$ is the symplex

$$\{(x_1, \dots, x_{n-3}) \in \mathbb{R}_+^{n-3} \mid x_1 + \dots + x_{n-3} \leq 1\}.$$

Though this is not exactly the same simplex as the image of the moment map $\boldsymbol{\beta}$, it is identical up to translation, dilation, and a linear transformation in $\mathrm{SL}(n, \mathbb{Z})$.

To be more precise, let us set $\lambda = 2\pi - \sum_{j=1}^{n} \bar{\alpha}_i$. Let us define an action r'_k of $\mathbb{R}/\pi\mathbb{Z}$ on $\mathbb{C}\mathbf{P}^{n-3}$ by

$$r'_k(\theta) \cdot [z_0, \dots, z_{n-3}] = [e^{2i\theta}z_0, \dots, e^{2i\theta}z_k, z_{k+1}, \dots, z_n].$$

Then the function

$$\mu'_k = \lambda \sum_{j=1}^{k} \mu_j + \sum_{j=1}^{k+1} \alpha_j$$

is a moment map for the action r'_k with respect to the symplectic form $\lambda\omega_{FS}$. The actions $r'_1, \dots, r'_{n-3}$ still commute and provide a new action of $(\mathbb{R}/\pi\mathbb{Z})^{n-3}$ with moment map

$$\bar{}' = (\mu'_1, \dots, \mu'_{n-3}).$$

Given the affine relation between $\bar{}$ and $\bar{}'$, one sees that the image of $\bar{}'$ is the set of vectors $(x_1, \dots, x_{n-3})$ in $\mathbb{R}^{n-3}$ satisfying

$$\begin{aligned} x_1 &\geq \bar{\alpha}_1 + \bar{\alpha}_2, \\ x_j &\geq x_{j-1} + \bar{\alpha}_{j+1}, \quad 2 \leq j \leq n-3, \\ x_{n-3} &\leq \lambda + \sum_{j=1}^{n-2} \bar{\alpha}_j = 2\pi - \bar{\alpha}_{n-1} - \bar{\alpha}_n. \end{aligned}$$

These are exactly the inequalities (10), (11), (12). By Delzant's theorem, it follows that $\mathrm{Rep}^{SM}_{\boldsymbol{\alpha}}(\Sigma_{0,n}, G)$ is isomorphic (as a symplectic manifold with a Hamiltonian action of $(\mathbb{R}/\pi\mathbb{Z})^{n-3}$) to $(\mathbb{C}\mathbf{P}^{n-3}, \lambda\omega_{FS})$ with the action $(r'_1, \dots, r'_{n-3})$.

In particular, the symplectic volume of $\mathrm{Rep}^{SM}_{\boldsymbol{\alpha}}(\Sigma_{0,n}, G)$ is equal to

$$(\lambda)^{n-3} \int_{\mathbb{C}\mathbf{P}^{n-3}} \omega_{FS}^{n-3}.$$

Using the Hamiltonian action $(r_1, \dots, r_{n-3})$ of $(\mathbb{R}/\pi\mathbb{Z})^{n-3}$, we see that

$$\begin{aligned} \int_{\mathbb{C}\mathbf{P}^{n-3}} \omega_{FS}^{n-3} &= \pi^{n-3} \int_{x_i \geq 0, \sum x_i \leq 1} \mathrm{d}x_1 \dots \mathrm{d}x_{n-3} \\ &= \frac{\pi^{n-3}}{(n-3)!}. \end{aligned}$$

Thus

$$\int_{\mathrm{Rep}^{SM}_{\boldsymbol{\alpha}}(\Sigma_{0,n}, G)} \omega_{\mathrm{Goldman}}^{n-3} = \frac{(\pi\lambda)^{n-3}}{(n-3)!}.$$

This ends the proof of Theorem 4.

4. Geometrization of supra-maximal representations

In this section we prove Theorem 5. We fix the numbers $\alpha_1, \dots, \alpha_n \in (0, 2\pi)$ such that

$$2\pi(n-1) < \sum_i \alpha_i < 2\pi n.$$

Recall the following uniformization theorem, independently proved by McOwen and Troyanov:

THEOREM 4.1 ([22, 27]). – *Let Σ be a compact Riemann surface of genus g, and $D = \sum_i \kappa_i p_i$ be a divisor on Σ with coefficients $\kappa_i \in (-\infty, 2\pi]$. Assume that its degree $\kappa(D) = \sum_i \kappa_i$ satisfies*

$$\kappa > 2\pi\chi(\Sigma).$$

Then there exists a unique conformal metric g_D on $\Sigma \setminus Supp(D)$ having curvature -1, and whose completion at each p_i is either a cone of angle $\theta_i = 2\pi - \kappa_i$ if $\kappa_i < 2\pi$, or a parabolic cusp if $\kappa_i = 2\pi$. Here $Supp(D)$ is the union of the p_i's.

In this result, the local curvatures κ_i's are not necessarily integers. Suppose that $\Sigma = \mathbb{C}\mathbf{P}^1$ is the Riemann sphere, and let $p_1, \dots, p_n$ be distinct points on $\mathbb{C}\mathbf{P}^1$. Assume now that $Q \in \mathrm{Sym}^{n-3}(\mathbb{C}\mathbf{P}^1)$, and let

$$D := \sum_{i=1}^{n} \alpha_i p_i - 2\pi Q.$$

We have $\sum_i \alpha_i - 2\pi(n-3) > 2\pi\chi(\mathbb{C}\mathbf{P}^1)$ so Troyanov's theorem yields a conformal metric g_D on $\mathbb{C}\mathbf{P}^1 \setminus \mathrm{supp}(D)$. Its completion at the p_i's are cones of angle congruent to $\overline{\alpha_i}$ modulo $2\pi\mathbb{Z}$ (depending if some q_j's coalesce with p_i), whereas at the points of $\mathrm{supp}(Q)$ they are cones of angle a multiple of 2π (it can be a large multiple if some of the q_j coalesce). In particular, there exists an orientation preserving conformal (hence holomorphic) map $f : \mathbb{C}\mathbf{P}^1 \setminus \widetilde{\{p_1, \dots, p_n\}} \to \mathbb{H}^-$ such that

$$g_D = f^*\Big(\frac{dx^2 + dy^2}{y^2}\Big). \tag{13}$$

This map f is unique up to post-composition by an orientation preserving isometry of $\mathbb{H}^-$, namely by an element of G. In particular, the map f is equivariant with respect to a representation $\rho : \pi_1(\mathbb{C}\mathbf{P}^1 \setminus \{p_1, \dots, p_n\}) \to G$, which is well-defined up to conjugacy by an element of G. This representation is called the holonomy of g_D.

Lemma 4.2. – *ρ belongs to* $\mathrm{Rep}_\alpha^{SM}(\pi_1(\mathbb{C}\mathbf{P}^1 \setminus \{p_1, \dots, p_n\}), G)$.

Proof. – By construction $\theta_i(\rho) = \alpha_i$. We get

$$-\mathbf{eu}(\rho) = \frac{1}{2\pi}\Big(-\mathrm{Vol}(g_D) + \sum \alpha_i\Big) = n - 1,$$

by Gauss-Bonnet formula. □

Remark 4.3. – We chose f with values in $\mathbb{H}^-$ so that $\mathrm{Vol}(\rho) = -\mathrm{Vol}(g_D)$.

Let $\mathcal{M}_{0,n}$ and $\mathcal{T}_{0,n}$ respectively denote the moduli space and the Teichmüller space of $\Sigma_{0,n}$. Those spaces are complex manifolds that can be described in the following way: $\mathcal{M}_{0,n}$ is identified with the set of tuples $(p_1, \dots, p_{n-3}) \in (\mathbb{C}\mathbf{P}^1 \setminus \{0, 1, \infty\})^{n-3}$ of distincts points, and $\mathcal{T}_{0,n}$ is the universal cover of $\mathcal{M}_{0,n}$. In this description, the conformal structure corresponding to the tuple $(p_1, \dots, p_{n-3})$ is $\mathbb{C}\mathbf{P}^1 \setminus \{p_1, \dots, p_n\}$, where $p_{n-2} = 0$, $p_{n-1} = 1$ and $p_n = \infty$. An element of $\mathcal{T}_{0,n}$ is the data of a tuple $(p_1, \dots, p_{n-3})$ as before, plus an isotopy class of diffeomorphisms $\varphi : \Sigma_{0,n} \to \mathbb{C}\mathbf{P}^1 \setminus \{p_1, \dots, p_n\}$.

Lemma 4.2 enables to define a map

$$H : \mathcal{T}_{0,n} \times \mathrm{Sym}^{n-3}(\mathbb{C}\mathbf{P}^1) \to \mathcal{T}_{0,n} \times \mathrm{Rep}_\alpha^{SM}(\pi_1(\Sigma_{0,n}), G),$$

by the formula

$$H(p_1, \dots, p_n, [\varphi], Q) = (p_1, \dots, p_n, [\rho \circ \varphi_*]) \tag{14}$$

where $[\rho]$ is the holonomy of the metric g_D on $\mathbb{C}\mathbf{P}^1 \setminus \mathrm{Supp}(D)$, with $D = \sum_i \alpha_i p_i - 2\pi Q$. As explained above, the holonomy is well defined on the fundamental group of $\mathbb{C}\mathbf{P}^1 \setminus \{p_1, \dots, p_n\}$.

Lemma 4.4. – *H is injective.*

Proof. – Assume that we are given distinct points $p_1, \ldots, p_n$, and two elements $Q, Q' \in \mathrm{Sym}^{n-3}(\mathbb{C}\mathbf{P}^1)$ such that

$$H(p_1, \ldots, p_n, Q) = H(p_1, \ldots, p_n, Q').$$

If $D = \sum \alpha_i p_i - 2\pi Q$ and $D' = \sum \alpha_i p_i - 2\pi Q$, we can find developing maps f and f' of g_D and $g_{D'}$ respectively that are equivariant with respect to the same representation $\rho : \pi_1(\mathbb{C}\mathbf{P}^1 \setminus \{p_1, \ldots, p_n\}) \to G$.

Consider the following function

$$\delta = u \circ d(f, f'),$$

where $d(.,.)$ is the hyperbolic distance in $\mathbb{H}$, and where $u = \frac{\cosh - 1}{2}$. The function δ descends to a function defined on $\mathbb{C}\mathbf{P}^1 \setminus \{p_1, \ldots, p_n\}$. Notice that, by Schwarz lemma, f and f' are 1-Lipschitz maps with respect to the uniformizing metric of $\mathbb{P}^1 \setminus \{p_1, \ldots, p_n\}$, so that f and f' have the same limit in each connected component of a preimage of a cusp: the fixed point of the image by ρ of the stabilizer of the component. In particular, the function δ tends to 0 at each of the cusps p_i.

Assume by contradiction that δ is not identically zero. The set of zero values is then discrete by holomorphicity of $f - f'$. Identifying holomorphically the upper half-plane $\mathbb{H}$ with the unit disk $\mathbb{D}$, we have

$$\log \delta = \log |f - f'|^2 - \log(1 - |f|^2) - \log(1 - |f'|^2).$$

In particular $\log \delta$ is strictly subharmonic everywhere, except at the points where f and f' have zero derivative. This contradicts the maximum principle.

The lemma follows. □

Proposition 4.5. – *H is continuous.*

Proof. – Assume that

$$(p_1^k, \ldots, p_n^k, [\varphi^k], Q^k) \underset{k \to \infty}{\longrightarrow} (p_1, \ldots, p_n, [\varphi], Q).$$

One can choose the diffeomorphisms $\varphi_k : \Sigma_{0,n} \to \mathbb{P}^1 \setminus \{p_1^k, \ldots, p_n^k\}$ in such a way that they converge uniformly on compact subsets of $\Sigma_{0,n}$ to φ in the smooth topology.

The family of metrics $g_k := (\varphi^k)^* g_{D_k}$ is bounded on each compact set of $\Sigma_{0,n} \setminus \varphi^{-1}(\mathrm{Supp}(Q))$, by the Schwarz lemma. Since it satisfies an elliptic PDE, it is bounded in the smooth topology on compact sets. In particular, to prove our claim, it suffices to prove that if a subsequence g_{k_j} converges in the smooth topology to some metric g_∞ on compact subsets of $\Sigma_{0,n} \setminus \varphi^{-1}(Q)$, then in fact $g_\infty = \varphi^* g_D$. Indeed, this will prove that g_k converges to $\varphi^* g_D$ in the smooth topology, and in particular, that the holonomies ρ_k of $(\varphi^k)^* g_{D_k}$ tend to the holonomy ρ of $\varphi^* g_D$ when k tends to infinity.

So let us assume in the sequel that (g_k) converges to g_∞ when k tends to infinity, and let us prove that $g_\infty = \varphi^* g_D$. By the Schwarz lemma again, the metric g_∞ is bounded by $\varphi^* g_P$, where g_P is the Poincaré metric on $\mathbb{C}\mathbf{P}^1 \setminus \big(\{p_1, \ldots, p_n\} \cup \mathrm{supp}(Q)\big)$. In particular, at each point q of $\varphi^{-1}(\mathrm{Supp}(Q)) \cup \{1, \ldots, n\}$ it admits a completion isometric to a cone. Denote by $\kappa_\infty(q)$ the curvature of g_∞ at such a point. By the uniqueness part of Troyanov uniformization theorem, it suffices to prove that $\kappa_\infty(q)$ is the coefficient of q in the divisor D.

Burger, Iozzi and Wienhard proved that the volume of a representation depends continuously on this representation (see [6, Theorem 1]). In particular, we have the following property:

$$\int_{\Sigma_{0,n}} \mathrm{vol}(g_\infty) = \mathrm{Vol}(\rho_\infty) = \lim_{k\to\infty} \mathrm{Vol}(\rho_k) = \lim_{k\to\infty} \int_{\Sigma_{0,n}} \mathrm{vol}(g_k), \tag{15}$$

where $\mathrm{vol}(g)$ stands for the volume form of g on $\Sigma_{0,n}$.

Let $U \subset \overline{\Sigma}_{0,n}$ be any open set with smooth boundary not containing points of $\varphi^{-1}(\mathrm{Supp}(D))$ on its boundary. The uniform convergence of g_k to g on compact subsets of $\Sigma_{0,n} \setminus \varphi^{-1}(\mathrm{Supp}(D))$, together with Fatou's lemma, show that

$$\int_U \mathrm{vol}(g_\infty) \leq \liminf_{k\to\infty} \int_U \mathrm{vol}(g_k). \tag{16}$$

Any strict inequality in (16) would contradict the conservation of volume (15). So

$$\int_U \mathrm{vol}(g_\infty) = \lim_{k\to\infty} \int_U \mathrm{vol}(g_k).$$

Applying Gauss Bonnet to the metrics g_k on U, and using the uniform convergence of g_k to g_∞, we get

$$\sum_{q\in U\cap\varphi^{-1}(\mathrm{Supp}(D))} \kappa_\infty(q) = \lim_{k\to\infty} \sum_{q\in U\cap\varphi^{-1}(\mathrm{Supp}(D))} \kappa_k(q).$$

This concludes the proof that $\kappa_\infty(q)$ is the coefficient of D at the point q. □

Corollary 4.6. – *H is a homeomorphism. In particular, for $\tau \in \mathcal{T}_{0,n}$, the map*

$$H(\tau,\cdot) : Sym^{n-3}(\mathbb{C}\mathbf{P}^1) \simeq \mathbb{C}\mathbf{P}^{n-3} \to \mathrm{Rep}_\alpha^{SM}(\pi_1(\Sigma_{0,n}), G)$$

is a homeomorphism.

Proof. – The map H is a continuous, proper, injective map between connected manifolds of the same dimension. So the result is a consequence of the invariance of domain theorem. □

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