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Giuseppe PIPOLI

*Inverse mean curvature flow in complex hyperbolic space*

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# INVERSE MEAN CURVATURE FLOW IN COMPLEX HYPERBOLIC SPACE

BY GIUSEPPE PIPOLI

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**ABSTRACT.** – We consider the evolution by inverse mean curvature flow of a closed, mean convex and star-shaped hypersurface in the complex hyperbolic space. We prove that the flow is defined for any positive time, the evolving hypersurface stays star-shaped and mean convex. Moreover the induced metric converges, after rescaling, to a conformal multiple of the standard sub-Riemannian metric on the sphere. Finally we show that there exists a family of examples such that the Webster curvature of this sub-Riemannian limit is not constant.

**RÉSUMÉ.** – Nous considérons l'évolution par l'inverse de la courbure moyenne d'une surface étoilée, fermée et à courbure moyenne positive dans l'espace hyperbolique complexe. Nous montrons que le flot est défini pour tout temps positif et que la surface reste étoilée et à courbure moyenne positive. De plus, la métrique induite, après un changement d'échelle, converge vers un multiple conforme de la métrique sous-riemannienne standard sur la sphère de dimension impaire. Nous allons montrer l'existence d'exemples de données initiales telles que cette limite sous-riemannienne n'a pas courbure de Webster constante.

## 1. Introduction

During last years geometric flows of submanifolds of Riemannian manifolds have been studied intensively. In the class of expanding flows, the leading example is the inverse mean curvature flow. In this paper we consider the evolution by inverse mean curvature flow of real hypersurfaces of the complex hyperbolic space  $\mathbb{C}\mathbb{H}^n$ , with  $n \geq 2$ . For any given smooth hypersurface  $F_0 : \mathcal{M} \rightarrow \mathbb{C}\mathbb{H}^n$ , the solution of the inverse mean curvature flow with initial datum  $F_0$  is a one-parameter family of smooth immersion  $F : \mathcal{M} \times [0, T) \rightarrow \mathbb{C}\mathbb{H}^n$  such that

$$(1.1) \quad \begin{cases} \frac{\partial F}{\partial t} = \frac{1}{H} \nu, \\ F(\cdot, 0) = F_0, \end{cases}$$

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where  $H$  is the mean curvature of  $F_t = F(\cdot, t)$  and  $\nu$  is the outward unit normal vector of  $\mathcal{M}_t = F_t(\mathcal{M})$ . It is the main tool used in the celebrated paper of Huisken and Ilmanen in [8] for proving the Penrose inequality.

In this paper we restrict to the class of star-shaped hypersurface in  $\mathbb{C}\mathbb{H}^n$ . The inverse mean curvature flow of star-shaped hypersurfaces has been already studied in different ambient manifolds: for example the Euclidean space [4, 18], the hyperbolic space [6, 9], asymptotic hyperbolic spaces [13], rotationally symmetric spaces [2] and warped products [17, 19]. In any case it was proved that the flow is defined for any positive time and the evolving hypersurface stays star-shaped for all the life of the flow. Inverse mean curvature flow of star-shaped hypersurfaces in rank one symmetric spaces have been considered for different purposes in [11] too.

The geometry of the ambient manifold influences the nature of the limit of the induced metric. In fact Gerhardt [4] and Urbas [18] proved independently that, for any star-shaped hypersurface of the Euclidean space, the limit metric is, up to rescaling, always the standard round metric on the sphere. In [9] P.K. Hung and M.T. Wang showed that, when the ambient manifold is the hyperbolic space, the limit metric is not always round. More precisely it is a conformal multiple of the standard round metric on the sphere and so it is round only in special cases. The case studied in the present paper has some similarities with the previous results, but a new phenomenon appears: even after rescaling, the evolving metric blows up along a direction. Hence the limit metric is no more Riemannian, but only sub-Riemannian: it is defined only on a codimension-one distribution. The main theorem proved is the following.

**THEOREM 1.1.** – *For any  $\mathcal{M}_0$  closed, mean convex and  $\mathbb{S}^1$ -invariant star-shaped hypersurface in  $\mathbb{C}\mathbb{H}^n$  let  $\mathcal{M}_t$  be its evolution by inverse mean curvature flow, let  $g_t$  be the induced metric on  $\mathcal{M}_t$  and  $\theta_t$  the induced contact form. Then:*

- (1)  $\mathcal{M}_t$  is  $\mathbb{S}^1$ -invariant, star-shaped and mean convex for any time  $t$ ;
- (2) the flow is defined for any positive time;
- (3) there is a smooth  $\mathbb{S}^1$ -invariant function  $f$  such that the rescaled induced metric  $\tilde{g}_t = |\mathcal{M}_t|^{-\frac{1}{n}} g_t$  converges to a sub-Riemannian metric  $\tilde{g}_\infty = e^{2f} \sigma_{sR}$  (i.e., a conformal multiple of the standard sub-Riemannian metric on the sphere  $\mathbb{S}^{2n-1}$ ) and the rescaled contact form  $\tilde{\theta} = |\mathcal{M}_t|^{-\frac{1}{n}} \theta_t$  converges to  $\tilde{\theta}_\infty = e^{2f} \hat{\theta}$ , where  $\hat{\theta}$  is the standard contact form on the odd dimensional sphere;
- (4) moreover there are examples of  $\mathcal{M}_0$  such that the limit does not have constant Webster scalar curvature.

After proving that the flow can be extended for any positive time, we have that the volume of  $\mathcal{M}_t$  becomes arbitrary large and then  $\mathcal{M}_t$  “explores” the structure at infinity of the ambient manifold as  $t$  tends to infinity. Hence the convergence to a sub-Riemannian metric is not surprising because the boundary at infinity of  $\mathbb{C}\mathbb{H}^n$  is  $(\mathbb{S}^{2n-1}, \sigma_{sR})$ , the one point compactification of the Heisenberg group of dimension  $2n - 1$  endowed with its standard sub-Riemannian metric. Different initial data explore this structure at infinity in different ways, but our result shows that we remain in the conformal class of  $\sigma_{sR}$ .

Obviously for any finite time  $t$ ,  $\tilde{g}_t$  is a Riemannian metric, but there is a direction in which the metric is blowing up. This special direction is  $J\nu$ , where  $J$  is the complex structure of  $\mathbb{C}\mathbb{H}^n$  and  $\nu$  is the outward unit normal vector field of  $\mathcal{M}_t$ . Note that, since we are considering only submanifolds of codimension one,  $J\nu$  is for sure tangent to  $\mathcal{M}_t$ . One of the main difficulties in generalizing the previous results is to describe the contribution of this special direction. Its presence gives also a new phenomenon not present in the previous literature. The second fundamental form converges to that of an horosphere with an exponential speed but, unlike for example [4, 17, 19], we have that the speed is not the same for any initial datum: very symmetric hypersurfaces converge twice as fast as the generic  $\mathbb{S}^1$ -invariant hypersurface (see Remark 6.6 below for more details).

If we try to study the limit of the sectional curvature of this family of metric  $\tilde{g}_t$ , it always diverges: this is a general behavior when we try to approximate a sub-Riemannian metric with a family of Riemannian metrics. For that reason another notion of curvature is required. We will use in particular the Webster curvature.

It is very easy to find hypersurfaces of  $\mathbb{C}\mathbb{H}^n$  such that  $\tilde{g}_\infty$  has constant Webster curvature. It is the case of the geodesic spheres: as we will see in detail in Section 4, the evolution of a geodesic sphere is a family of geodesic spheres and the function  $f$  is constant. On the other hand, the search for an example with a non-trivial limit is much more challenging. The main tool for studying the roundness of the limit is the following Brown-York like quantity: for any star-shaped hypersurface  $\mathcal{M}$

$$Q(\mathcal{M}) = |\mathcal{M}|^{-1+\frac{1}{n}} \int_{\mathcal{M}} (H - \hat{H}) d\mu,$$

where  $|\mathcal{M}|$  is the volume of  $\mathcal{M}$  and, if  $\rho$  is the radial function defining  $\mathcal{M}$ ,  $\hat{H}$  is the value of the mean curvature of a geodesic sphere of radius  $\rho$  (see (3.8) for the explicit definition).  $Q$  gives a measure of how  $\mathcal{M}$  is far to being a geodesic sphere. It is not a measure in the strict sense because  $Q$  has not a sign and, even if it is zero for geodesic spheres, it is not in general truly the opposite. In the final section of this paper we found the desired non-trivial examples estimating the behavior of  $Q$  along the inverse mean curvature flow.

This paper is organized as follows. In Section 2 we collect some preliminaries and we fix some notations. In Section 3 we compute the main geometric quantities for a star-shaped hypersurface in  $\mathbb{C}\mathbb{H}^n$ , like the induced metric, the second fundamental form and the mean curvature. In Section 4 we have a simple but meaningful example, i.e., the evolution of the geodesic spheres. In Section 5 we estimate the norm of the gradient of the radial function. As consequence we have that the property of being star-shaped and the mean convexity are preserved by the flow. The study of the derivatives of the radial function continues in Sections 6 and 7. In particular we prove that the solution of the flow is defined for any positive time. Section 8 is devoted to the proof of the convergence of the rescaled induced metric to a sub-Riemannian limit. In the last section we conclude the proof of Theorem 1.1 by studying the Webster curvature of the limit metric and giving a family of non-trivial examples.

Finally we would like to announce that the ideas developed in the present paper have been extended in [16] in the case of the next rank one symmetric space, that is the quaternionic hyperbolic space. An analogous of Theorem 1.1 holds in this other setting too.

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## 2. Preliminaries

### 2.1. Riemannian and sub-Riemannian metrics on the sphere

Every hypersurface considered in this paper is closed and star-shaped and so it is an embedding of  $\mathbb{S}^{2n-1}$ , the sphere of dimension  $2n - 1$ , into  $\mathbb{C}\mathbb{H}^n$ . On that sphere we have different “standard” metrics. Let  $\sigma$  be the usual Riemannian metric on  $\mathbb{S}^{2n-1}$  with constant sectional curvature equal to 1. Since the dimension is odd, we can distinguish an important vector field: we can think the sphere embedded in  $(\mathbb{C}^n, J)$ , then, if  $\nu$  is the unit outward normal to  $\mathbb{S}^{2n-1}$ ,  $\xi = J\nu$  is a unit tangent vector field on the sphere. It is often called the *Hopf vector field*, because it can be characterized also as the unit vector field tangent to the fibers of the Hopf fibration:  $\pi : \mathbb{S}^{2n-1} \rightarrow \mathbb{C}\mathbb{P}^{n-1}$ . It allows us to define the *horizontal distribution*

$$(2.1) \quad \mathcal{H} = \{X \in T(\mathbb{S}^{2n-1}) \mid \sigma(X, \xi) = 0\}.$$

The *Berger metrics* is a family of deformations of  $\sigma$  in the direction of  $\xi$ : for any  $\mu > 0$  let  $e_\mu$  be the Riemannian metric defined by

$$\begin{cases} e_\mu(X, Y) = \sigma(X, Y) & \text{for any } X, Y \in \mathcal{H}; \\ e_\mu(X, \xi) = 0 & \text{for any } X \in \mathcal{H}; \\ e_\mu(\xi, \xi) = \mu. \end{cases}$$

When  $\mu$  converges to infinity, the metric  $e_\mu$  degenerates on the directions tangent to  $\xi$ . At the limit we get  $\sigma_{sR}$  the *standard sub-Riemannian metric*, it is defined only on  $\mathcal{H}$ , but, since  $\mathcal{H} + [\mathcal{H}, \mathcal{H}]$  is the whole tangent space,  $\sigma_{sR}$  is enough to define a distance on  $\mathbb{S}^{2n-1}$ , called the *Carnot-Carathéodory distance*. The following comparison between the Levi-Civita connection of  $e_\mu$  and that of  $\sigma = e_1$  will be used below.

LEMMA 2.1. – *Fix a  $\sigma$ -orthonormal basis  $(Y_1, \dots, Y_{2n-1})$  of  $\mathbb{S}^{2n-1}$  such that  $Y_1 = \xi$  and for every  $r$   $Y_{2r+1} = JY_{2r}$ . Let us denote with  $\nabla_e$  ( $\nabla_\sigma$  respectively) the Levi-Civita connection associated to the metric  $e_\mu$  ( $\sigma$  respectively). Then for every  $1 \leq i, j \leq 2n - 1$  we have:*

$$\nabla_e Y_i Y_j - \nabla_\sigma Y_i Y_j = \begin{cases} (1 - \mu)JY_j & \text{if } i = 1 \neq j; \\ (1 - \mu)JY_i & \text{if } j = 1 \neq i; \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* – Obviously  $\nabla_e Y_1 Y_1 = \nabla_\sigma Y_1 Y_1 = 0$  since the Hopf vector field is tangent to the fibers of the Hopf fibration, and they are geodesic for every  $\mu$ . The metric  $e_\mu$  can be seen as the metric on the total space of the canonical variation of parameter  $\mu$  of the Hopf fibration. By Lemma 3 of [14] and Lemma 9.69 in [1] we have:

$$\begin{aligned} \nabla_e Y_i Y_j &= \nabla_\sigma Y_i Y_j && \text{if } i, j \neq 1; \\ \nabla_e Y_i Y_1 &= \mu \nabla_\sigma Y_i Y_1 = -\mu JY_i && \text{if } i \neq 1, \end{aligned}$$

where the last equality comes from an explicit computation. Finally, if  $i \neq 1$ , we get:

$$\nabla_e Y_1 Y_i = \nabla_e Y_i Y_1 + [Y_1, Y_i] = (1 - \mu)J Y_i + \nabla_\sigma Y_1 Y_i. \quad \square$$

NOTATION 2.2. – We introduce the following notation in order to distinguish between derivatives of a function with respect to different metrics. For any given function  $f : \mathbb{S}^{2n-1} \rightarrow \mathbb{R}$ , let  $f_{ij}$  ( $\hat{f}_{ij}$  respectively) be the components of the Hessian of  $f$  with respect to  $\sigma$  ( $e_\mu$  respectively). The value of  $\mu$  will be clear from the context. The indices go up and down with the associated metric: for instance  $\hat{f}_i^k = \hat{f}_{ij} e_\mu^{jk}$ , while  $f_i^k = f_{ij} \sigma^{jk}$ . Analogous notations will be used for higher order derivatives.

On the sphere of odd dimension  $\mathbb{S}^1$  acts in the following way:

$$(2.2) \quad \begin{aligned} S : \mathbb{S}^1 \times \mathbb{S}^{2n-1} \subset \mathbb{C} \times \mathbb{C}^n &\rightarrow \mathbb{S}^{2n-1} \subset \mathbb{C}^n \\ (e^{i\vartheta}, z_1, \dots, z_n) &\mapsto (e^{i\vartheta} z_1, \dots, e^{i\vartheta} z_n). \end{aligned}$$

The action is by isometries for any Berger metric. A function  $\varphi : \mathbb{S}^{2n-1} \rightarrow \mathbb{R}$  is said  $\mathbb{S}^1$ -invariant if it is invariant under the action of  $S$ .

LEMMA 2.3. – Let  $\varphi$  be an  $\mathbb{S}^1$ -invariant smooth function. With respect to the basis introduced in the previous lemma, the Hessian of  $\varphi$  with respect to  $e_\mu$  is:

$$\hat{\varphi}_{ij} = \begin{pmatrix} 0 & \mu J Y_j(\varphi) \\ \mu J Y_i(\varphi) & \varphi_{ij} \end{pmatrix},$$

where we are using Notations 2.2 and  $J$  is the complex structure of  $\mathbb{C}^n$  (once again we are considering  $\mathbb{S}^{2n-1}$  embedded in  $\mathbb{C}^n$ ). Taking the trace and the norm of the Hessian, in particular we have:

$$\begin{aligned} \Delta_e \varphi &= \Delta_\sigma \varphi; \\ |\nabla_e^2 \varphi|_e^2 &= |\nabla_\sigma^2 \varphi|_\sigma^2 + 2(\mu - 1)|\nabla_\sigma \varphi|_\sigma^2. \end{aligned}$$

*Proof.* – Since  $\varphi$  is  $\mathbb{S}^1$ -invariant we have that  $\varphi_1 = Y_1(\varphi) = 0$ . From the previous lemma we get:

$$\hat{\varphi}_{11} = Y_1 Y_1(\varphi) - \nabla_e Y_1 Y_1(\varphi) = 0.$$

For every  $i \neq 1$  we can compute:

$$\begin{aligned} \hat{\varphi}_{1i} &= Y_i Y_1(\varphi) - \nabla_e Y_i Y_1(\varphi) = \mu J Y_i(\varphi); \\ \hat{\varphi}_{i1} &= Y_1 Y_i(\varphi) - \nabla_e Y_1 Y_i(\varphi) \\ &= [Y_i, Y_1](\varphi) + Y_i Y_1(\varphi) - \nabla_e Y_1 Y_i(\varphi) \\ &= \hat{\varphi}_{1i}. \end{aligned}$$

Moreover, if both indexes are different from 1, we have:

$$\begin{aligned} \hat{\varphi}_{ij} &= Y_j Y_i(\varphi) - \nabla_e Y_j Y_i(\varphi) \\ &= Y_j Y_i(\varphi) - \nabla_\sigma Y_j Y_i(\varphi) = \varphi_{ij}. \end{aligned}$$

We point out that, as a consequence of the symmetries considered,

$$|\nabla_e \varphi|_e^2 = |\nabla_\sigma \varphi|_\sigma^2.$$

Taking into account this remark, the formulas for the Laplacian and the norm of the Hessian follow after some trivial computations.  $\square$

## 2.2. Webster curvature

The Webster curvature is a notion of CR geometry. In this section we consider only the case of the sphere, what follows can be said with much more generality. We refer to the monograph [3] for all the details. Fix the standard CR structure on  $\mathbb{S}^{2n-1}$ , then  $\mathcal{H}$ , defined in (2.1), is the horizontal distribution of this structure. On  $\mathcal{H}$  we have a complex structure  $J$ . Let  $\pi : T(\mathbb{S}^{2n-1}) \rightarrow \mathcal{H}$  be the canonical projection. For any 1-form  $\theta$  such that  $\ker \theta = \mathcal{H}$ , there exists a vector field  $\xi_\theta$  such that  $\theta(\xi_\theta) = 1$  and  $d\theta(\xi_\theta, \cdot) = 0$ . We can extend  $J$  to the whole tangent space of  $\mathbb{S}^{2n-1}$  requiring that  $J\xi_\theta = 0$ . For every  $X, Y \in \mathcal{H}$  we can define  $G_\theta(X, Y) = d\theta(X, JY)$ . The metric  $g_\theta = \pi G_\theta + \theta^2$  is called Webster metric associated to  $\theta$ .

**THEOREM / DEFINITION 2.4.** – *Let  $\theta, J, \xi_\theta$  and  $g_\theta$  be as before. There exists a unique linear connection  $\nabla^{TW}$  such that  $\nabla^{TW} J = \nabla^{TW} \theta = \nabla^{TW} \xi_\theta = \nabla^{TW} g_\theta = 0$  and with torsion  $T$  of pure type, i.e., for every  $X, Y \in \mathcal{H}$  the torsion satisfies:*

$$\begin{cases} T(X, Y) = d\theta(X, Y)\xi_\theta; \\ T(X, \xi_\theta) \in \mathcal{H} \\ g_\theta(T(X, \xi_\theta), Y) = g_\theta(T(Y, \xi_\theta), X) = -g_\theta(T(JX, \xi_\theta), JY). \end{cases}$$

*This connection  $\nabla^{TW}$  is called the Tanaka-Webster connection associated to  $\theta$ .*

The Webster curvature of  $\theta$  is the curvature defined in the usual way, but using the Tanaka-Webster connection instead of the Levi-Civita connection.

On  $\mathbb{S}^{2n-1}$  we have a standard contact form:  $\hat{\theta}(\cdot) = \sigma(\xi, \cdot)$ . With respect to this form we have:  $\xi_{\hat{\theta}} = \xi$ ,  $G_{\hat{\theta}} = \sigma_{sR}$  and  $g_{\hat{\theta}} = \sigma$ . Obviously the Webster curvature of  $\hat{\theta}$  is constant. A metric of the form  $e^{2f}\sigma_{sR}$  can be thought as the restriction to  $\mathcal{H} \times \mathcal{H}$  of the Webster metric associated to the 1-form  $e^{2f}\hat{\theta}$ . Then we will talk indifferently about the Webster curvature of a sub-Riemannian metric or of a contact form.

Once we defined the appropriate notion of curvature in CR geometry, a natural question is the Yamabe problem for CR manifolds. This problem was solved in a great generality, but in our special case it can be reformulated as follows: what are the functions  $f$  such that the conformal multiple  $e^{2f}\sigma_{sR}$  has constant Webster scalar curvature? The answer is given by the following formula by Jerison and Lee [10]: the Webster scalar curvature of  $e^{2f}\sigma_{sR}$  is constant if and only if there are  $u > 0$ ,  $c > 0$  and  $\zeta \in \mathbb{S}^{2n-1}$  such that

$$(2.3) \quad e^{-2f}(z) = c |\cosh(u) + \sinh(u)z \cdot \bar{\zeta}|^2, \quad \forall z \in \mathbb{S}^{2n-1}.$$

Here we are considering the odd-dimensional sphere immersed in  $\mathbb{C}^n$  and the norm and the product are the usual ones in  $\mathbb{C}^n$ .

Since in this paper we are considering  $\mathbb{S}^1$ -invariant hypersurface, we show that, in presence of this kind of symmetry, the Jerison and Lee's formula becomes much simpler.

**LEMMA 2.5.** – *Let  $f : \mathbb{S}^{2n-1} \rightarrow \mathbb{R}$  be an  $\mathbb{S}^1$ -invariant function. The following are equivalent:*



- (a)  $f$  satisfies (2.3),
- (b)  $f$  is constant.

*Proof.* – If  $f$  is constant, (a) holds trivially. In order to prove the opposite implication, let  $u > 0$  and  $\zeta = (a_j - ib_j)_{j=1,\dots,n} \in \mathbb{S}^{2n-1} \subset \mathbb{C}^n$  be as in Equation (2.3). By hypothesis we have that for any  $z \in \mathbb{S}^{2n-1}$  and for any  $\vartheta = x + iy \in \mathbb{S}^1 \subset \mathbb{C}$  we have

$$(2.4) \quad \left| \cosh(u) + \sinh(u)(e^{i\vartheta} z) \cdot \bar{\zeta} \right|^2 = \left| \cosh(u) + \sinh(u)z \cdot \bar{\zeta} \right|^2.$$

Fix  $z_1 = (1, 0, \dots, 0)$ , then  $e^{i\vartheta} z \cdot \bar{\zeta} = xa_1 - yb_1 + i(xb_1 + ya_1)$ , while  $z \cdot \bar{\zeta} = a_1 + ib_1$ . Specifying (2.4) for  $z_1$  we have, after some trivial computations, that

$$\sinh(u) \cosh(u)a_1 = \sinh(u) \cosh(u)(xa_1 - yb_1), \quad \forall x, y \text{ such that } x^2 + y^2 = 1,$$

then  $\sinh(u) = 0$  or  $a_1 = b_1 = 0$ . For any  $k = 2, \dots, n$  we can repeat the same computations for  $z_k = (\delta_{kj})_{j=1,\dots,n}$  and we get that  $\sinh(u) = 0$  or  $a_k = b_k = 0$ . Since  $\zeta \neq 0$  the only possibility is that  $\sinh(u) = 0$  and then we have the thesis.  $\square$

### 2.3. Complex hyperbolic space

The complex hyperbolic space is the complex analogous of the real hyperbolic space. It can be defined in many equivalent ways, but, for our purpose, it is convenient to introduce polar coordinates. Let  $\mathbb{C}\mathbb{H}^n$  be  $\mathbb{R}^{2n}$  equipped with the *Bergman metric*  $\bar{g}$ :

$$(2.5) \quad \bar{g} = d\rho^2 + \sinh^2(\rho)e_{\cosh^2(\rho)},$$

where  $\rho$  represents the distance from the origin and  $e_{\cosh^2(\rho)}$  is the Berger metric of parameter  $\cosh^2(\rho)$  on  $\mathbb{S}^{2n-1}$ . Note that, since the Berger metric changes with the radius,  $\bar{g}$  is not given by a warped product. The Riemann curvature tensor of this metric has the following explicit expression

$$(2.6) \quad \begin{aligned} \bar{R}(X, Y, Z, W) = & -\bar{g}(X, Z)\bar{g}(Y, W) + \bar{g}(X, W)\bar{g}(Y, Z) \\ & -\bar{g}(X, JZ)\bar{g}(Y, JW) + \bar{g}(X, JW)\bar{g}(Y, JZ) \\ & -2\bar{g}(X, JY)\bar{g}(Z, JW), \end{aligned}$$

where  $J$  is the complex structure of  $\mathbb{C}\mathbb{H}^n$ . In our model it coincides with the usual complex structure of  $\mathbb{R}^{2n}$ . It follows that the sectional curvature of a plane spanned by two orthonormal vectors  $X$  and  $Y$  is given by

$$(2.7) \quad \bar{K}(X \wedge Y) = -1 - 3\bar{g}(X, JY)^2.$$

Then  $-4 \leq \bar{K} \leq -1$  and it is equal to  $-1$  (respectively to  $-4$ ) if and only if  $X$  and  $JY$  are orthogonal (respectively parallel). This property makes the complex hyperbolic space a *complex space form*. Moreover it is Einstein with  $\bar{Ric} = -2(n + 1)\bar{g}$  and symmetric of rank one, then  $\bar{\nabla}\bar{R} = 0$ .

Here and in the following we are using the convention to put a bar over the symbol for geometric quantity of the fixed ambient manifold  $\mathbb{C}\mathbb{H}^n$ , for example  $\bar{\nabla}$  is the Levi-Civita connection of  $\bar{g}$ .

## 2.4. Inverse mean curvature flow

In (1.1) we defined the inverse mean curvature flow. Since we are considering only closed and mean convex initial data, it is well known that the flow (1.1) has a unique smooth solution, at least for small times. The main geometric quantities for an hypersurface are: the induced metric  $g_{ij}$ , its inverse  $g^{ij}$ , the second fundamental form  $h_{ij}$ , the mean curvature  $H = h_{ij}g^{ji}$  and the volume  $|\mathcal{M}_t|$ . They evolve in the following way along the inverse mean curvature flow.

LEMMA 2.6. – *Since the ambient space is symmetric the following evolution equations hold:*

$$\begin{aligned}
 (1) \quad & \frac{\partial g_{ij}}{\partial t} = \frac{2}{H} h_{ij}, \\
 (2) \quad & \frac{\partial g^{ij}}{\partial t} = -\frac{2}{H} h^{ij}, \\
 (3) \quad & \frac{\partial H}{\partial t} = \frac{\Delta H}{H^2} - 2 \frac{|\nabla H|^2}{H^3} - \frac{|A|^2}{H} - \frac{\bar{R}ic(v, v)}{H}, \\
 (4) \quad & \frac{\partial h_{ij}}{\partial t} = \frac{\Delta h_{ij}}{H^2} - \frac{2}{H^3} \nabla_i H \nabla_j H + \frac{|A|^2}{H^2} h_{ij} - \frac{2}{H} \bar{R}_{i0j0} + \bar{R}ic(v, v) \frac{h_{ij}}{H^2} \\
 & \quad + \frac{1}{H^2} g^{lr} g^{ms} (2\bar{R}_{risj} h_{lm} - \bar{R}_{rmis} h_{jl} - \bar{R}_{rmjs} h_{il}), \\
 (5) \quad & \frac{\partial h_i^j}{\partial t} = \frac{\Delta h_i^j}{H^2} - \frac{2}{H^3} \nabla_i H \nabla_k H g^{kj} + \frac{|A|^2}{H^2} h_i^j - \frac{2}{H} \bar{R}_{i0k0} g^{kj} - 2 \frac{h_i^k h_k^j}{H} \\
 & \quad + \bar{R}ic(v, v) \frac{h_i^j}{H^2} + \frac{1}{H^2} g^{lr} g^{ms} g^{kj} (2\bar{R}_{risk} h_{lm} - \bar{R}_{rmis} h_{kl} - \bar{R}_{rmks} h_{il}), \\
 (6) \quad & \frac{d |\mathcal{M}_t|}{dt} = |\mathcal{M}_t|, \\
 (7) \quad & \frac{\partial v}{\partial t} = \frac{\nabla H}{H^2}.
 \end{aligned}$$

Here and in the following we are using Einstein convention on repeated indices. Moreover the operation of raising/lowering the indices is done with respect to the induced metric: for example  $h_i^j = h_{ik} g^{kj}$ . The proof of this lemma is similar to the computation of the analogous equations for the mean curvature flow which can be found in [7]. Note that, integrating equation (6), we have that the inverse mean curvature flow is an expanding flow, precisely  $|\mathcal{M}_t| = |\mathcal{M}_0| e^t$ .

## 3. Geometry of star-shaped hypersurfaces

All the hypersurfaces considered in this paper are star-shaped and  $\mathbb{S}^1$ -invariant. In this section we compute the main geometric quantities for a generic star-shaped hypersurface in  $\mathbb{C}\mathbb{H}^n$  and then we will specify them in case of symmetries. Let  $F : \mathbb{S}^{2n-1} \rightarrow \mathbb{C}\mathbb{H}^n$  be a smooth star-shaped immersion. Up to an isometry of the ambient space, we can consider that it is star-shaped with respect to the origin. Then  $F$  is defined by its radial function: there exists a smooth function  $\rho : \mathbb{S}^{2n-1} \rightarrow \mathbb{R}^+$  such that in polar coordinates  $\mathcal{M} = F(\mathbb{S}^{2n-1}) = \{(z, \rho(z)) \in \mathbb{C}\mathbb{H}^n \mid z \in \mathbb{S}^{2n-1}\}$ . The hypersurface is said  $\mathbb{S}^1$ -invariant if the associated radial

function  $\rho$  is  $\mathbb{S}^1$ -invariant. With the same proof of Lemma 3.1 of [15] we can prove the following result.

LEMMA 3.1. – *The evolution of an  $\mathbb{S}^1$ -invariant hypersurface stays  $\mathbb{S}^1$ -invariant during the whole duration of the flow.*

Fix any  $(Y_1, \dots, Y_{2n-1})$  tangent basis of the sphere  $\mathbb{S}^{2n-1}$ , for every  $i$  we define  $\rho_i = Y_i(\rho)$  and  $V_i = F_* Y_i \equiv Y_i + \rho_i \frac{\partial}{\partial \rho}$ . Then  $(V_1, \dots, V_{2n-1})$  is a tangent basis of  $\mathcal{M}$ . The induced metric on  $\mathcal{M}$  is  $g = F^* \bar{g}$ , in local coordinates we have

$$(3.1) \quad g_{ij} = \rho_i \rho_j + \sinh^2(\rho) e_{ij},$$

where for short  $e_{ij} = (e_{\cosh^2(\rho)})_{ij}$ . The inverse of the metric therefore is

$$(3.2) \quad g^{ij} = \frac{1}{\sinh^2(\rho)} \left( e^{ij} - \frac{\rho^i \rho^j}{\sinh^2(\rho) + |\nabla_e \rho|_e^2} \right),$$

where  $e^{ij}$  is the inverse of  $e_{ij}$ ,  $\rho^i = \rho_k e^{ki}$  and the gradient and the norm of  $\rho$  are defined with respect to the metric  $e_{\cosh^2(\rho)}$ . In order to simplify the expressions we can fix a function

$\varphi = \varphi(\rho)$  such that  $\frac{d\varphi}{d\rho} = \frac{1}{\sinh(\rho)}$  and introduce  $v = \sqrt{1 + \frac{|\nabla_e \rho|_e^2}{\sinh^2(\rho)}}$ . Since  $\varphi_i = Y_i(\varphi) = \frac{\rho_i}{\sinh(\rho)}$ , we get

$$\begin{aligned} g_{ij} &= \sinh^2(\rho)(\varphi_i \varphi_j + e_{ij}), \\ g^{ij} &= \frac{1}{\sinh^2(\rho)} \left( e^{ij} - \frac{\varphi^i \varphi^j}{v^2} \right), \\ v &= \sqrt{1 + |\nabla_e \varphi|_e^2}. \end{aligned}$$

A unit normal vector is

$$v = \frac{1}{v} \left( \frac{\partial}{\partial \rho} - \frac{\nabla \rho}{\sinh^2(\rho)} \right) = \frac{1}{v} \left( \frac{\partial}{\partial \rho} - \frac{\nabla \varphi}{\sinh(\rho)} \right).$$

Since the metric of the ambient space is not the same in any direction, it is convenient to use the specific basis tangent to  $\mathbb{S}^{2n-1}$  defined in Lemma 2.1. In this way we have

$$(3.3) \quad e_{\cosh^2(\rho)} = \begin{pmatrix} \cosh^2(\rho) & 0 \\ 0 & i d_{2n-2} \end{pmatrix},$$

where  $i d_{2n-2}$  is the identity matrix of order  $2n - 2$ . The contact form is  $\theta(\cdot) = \bar{g}(Jv, \cdot)$ . For an  $\mathbb{S}^1$ -invariant star-shaped hypersurface in coordinates we have:

$$Jv = \frac{1}{v \sinh(\rho)} \left( \frac{Y_1}{\cosh(\rho)} - \sum_{k \neq 1} \varphi_k J Y_k \right),$$

then

$$(3.4) \quad \begin{aligned} \theta_i &= \frac{1}{v \sinh(\rho)} \left( \frac{g_{1i}}{\cosh(\rho)} - \sum_{k \neq 1} \varphi_k \bar{g}(J Y_k, V_i) \right) \\ &= \frac{\sinh(\rho)}{v} \left( \cosh(\rho) \hat{\theta}_i - \sum_{k \neq 1} \varphi_k \delta_{ik^*} \right), \end{aligned}$$

where

$$\delta_{ik^*} = \begin{cases} \delta_{i,k+1} & \text{if } k = 2r, \\ -\delta_{i,k-1} & \text{if } k = 2r + 1. \end{cases}$$

Now we want to compute the second fundamental form of  $\mathcal{M}$ . For each  $i$  and  $j$  let  $h_{ij} = -\bar{g}(\bar{\nabla}_{V_i} V_j, v)$ . Moreover we introduce the following notation: Latin indices  $i, j, \dots$  range from 1 to  $2n - 1$  and are related to components tangent to the sphere, the index 0 represents the radial direction  $\frac{\partial}{\partial \rho}$  and Greek indices  $\alpha, \beta, \dots$  range from 0 to  $2n - 1$ . An explicit computation, together with the fact that for every  $i$   $\frac{\partial}{\partial \rho}(\rho_i) = 0$  and  $\bar{g}(\bar{\nabla}_{Y_i} \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \rho}) = 0$  we get:

$$h_{ij} = \frac{1}{v} \left( \bar{\Gamma}_{ij}^k \rho_k + \rho_i \rho_k \bar{\Gamma}_{0j}^k + \rho_j \rho_k \bar{\Gamma}_{0i}^k - \bar{\Gamma}_{ij}^0 - Y_i(\rho_j) \right).$$

We have that  $\bar{\Gamma}_{ij}^k = \hat{\Gamma}_{ij}^k$ , the Christoffel symbols of the metric  $e_{\cosh^2(\rho)}$ , then

$$Y_i(\rho_j) - \bar{\Gamma}_{ij}^k \rho_k = \hat{\rho}_{ij},$$

where the "hat" is in the sense of Notation 2.2. For short, let  $Y_0$  be  $\frac{\partial}{\partial \rho}$ , then

$$\begin{aligned} \bar{\Gamma}_{ij}^0 &= \frac{1}{2} (Y_i(\bar{g}_{i\alpha}) - Y_\alpha(\bar{g}_{ij}) + Y_j(\bar{g}_{i\alpha})) \bar{g}^{\alpha 0} \\ &= -\frac{1}{2} \frac{\partial}{\partial \rho} (\bar{g}_{ij}) = -\gamma_i \delta_{ij} = \begin{cases} -\sinh(\rho) \cosh(\rho) (\sinh^2(\rho) + \cosh^2(\rho)) \delta_{ij} & \text{if } i = 1, \\ -\sinh(\rho) \cosh(\rho) \delta_{ij} & \text{if } i \neq 1. \end{cases} \end{aligned}$$

Finally

$$\begin{aligned} \bar{\Gamma}_{i0}^k &= \frac{1}{2} \left( Y_i(\bar{g}_{0\alpha}) - Y_\alpha(\bar{g}_{i0}) + \frac{\partial}{\partial \rho} (\bar{g}_{i\alpha}) \right) \bar{g}^{\alpha k} \\ &= \frac{1}{2} \frac{\partial}{\partial \rho} (\bar{g}_{i\alpha}) \bar{g}^{\alpha k} = \eta_i \delta_{ik} = \begin{cases} \frac{\sinh^2(\rho) + \cosh^2(\rho)}{\sinh(\rho) \cosh(\rho)} \delta_{ik} & \text{if } i = 1, \\ \frac{\cosh(\rho)}{\sinh(\rho)} \delta_{ik} & \text{if } i \neq 1. \end{cases} \end{aligned}$$

Note that

$$\hat{\varphi}_{ij} = \frac{1}{\sinh(\rho)} \hat{\rho}_{ij} - \frac{\cosh(\rho)}{\sinh^2(\rho)} \rho_i \rho_j \quad \Leftrightarrow \quad \hat{\rho}_{ij} = \sinh(\rho) \hat{\varphi}_{ij} + \sinh(\rho) \cosh(\rho) \varphi_i \varphi_j.$$

Analogous formulas hold for  $\rho_{ij}$  and  $\varphi_{ij}$  too. Summing up these quantities we get

$$(3.5) \quad \begin{aligned} h_{ij} &= \frac{1}{v} (-\hat{\rho}_{ij} + (\eta_i + \eta_j) \rho_i \rho_j + \gamma_i \delta_{ij}) \\ &= \frac{\sinh(\rho)}{v} \left( -\hat{\varphi}_{ij} + (\sinh(\rho) \eta_i + \sinh(\rho) \eta_j - \cosh(\rho)) \varphi_i \varphi_j + \frac{\gamma_i}{\sinh(\rho)} \delta_{ij} \right). \end{aligned}$$

Raising the second index we have

$$(3.6) h_i^k = -\frac{\hat{\varphi}_{ij} \tilde{e}^{jk}}{v \sinh(\rho)} + \frac{\cosh(\rho)}{v \sinh(\rho)} \delta_i^k + \begin{cases} \frac{\sinh(\rho)}{v \cosh(\rho)} \delta_1^k + \frac{\sinh(\rho)}{v \cosh(\rho)} (\varphi_1)^2 \tilde{e}^{1k} & \text{if } i = 1; \\ \frac{\sinh(\rho)}{v \cosh(\rho)} \tilde{e}_{i1} \tilde{e}^{1k} & \text{if } i \neq 1, \end{cases}$$

where  $\tilde{e}^{ij} = \sinh^2(\rho) g^{ij} = e^{ij} - \frac{\varphi^i \varphi^j}{v^2}$ . Taking the trace of (3.6) we obtain the mean curvature of  $\mathcal{M}$ :

$$\begin{aligned} H &= h_i^i = -\frac{\hat{\varphi}_{ij} \tilde{e}^{ji}}{v \sinh(\rho)} + \frac{\sinh(\rho)}{v \cosh(\rho)} + (2n - 1) \frac{\cosh(\rho)}{v \sinh(\rho)} \\ &\quad + \frac{\sinh(\rho)}{v \cosh(\rho)} (\varphi_1)^2 \tilde{e}^{11} + \frac{\sinh(\rho)}{v \cosh(\rho)} \sum_{i \neq 1} \tilde{e}_{i1} \tilde{e}^{i1} \\ &= -\frac{\hat{\varphi}_{ij} \tilde{e}^{ji}}{v \sinh(\rho)} + \frac{\sinh(\rho)}{v \cosh(\rho)} + (2n - 1) \frac{\cosh(\rho)}{v \sinh(\rho)} \\ &\quad + \frac{\sinh(\rho)}{v \cosh(\rho)} \tilde{e}_{i1} \tilde{e}^{i1} - \sinh(\rho) \cosh(\rho) \frac{\tilde{e}^{11}}{v} \\ &= -\frac{\hat{\varphi}_{ij} \tilde{e}^{ji}}{v \sinh(\rho)} + 2 \frac{\sinh(\rho)}{v \cosh(\rho)} + (2n - 1) \frac{\cosh(\rho)}{v \sinh(\rho)} \\ &\quad - \frac{\sinh(\rho)}{v \cosh(\rho)} \left( 1 - \frac{(\varphi_1)^2}{v^2} \right) \\ (3.7) \quad &= -\frac{\hat{\varphi}_{ij} \tilde{e}^{ji}}{v \sinh(\rho)} + \frac{\hat{H}}{v} + \frac{\sinh(\rho)}{v^3 \cosh(\rho)} (\varphi_1)^2, \end{aligned}$$

where

$$(3.8) \quad \hat{H} = \hat{H}(\rho) = (2n - 1) \frac{\cosh(\rho)}{\sinh(\rho)} + \frac{\sinh(\rho)}{\cosh(\rho)} = \frac{2n \cosh^2(\rho) - 1}{\sinh(\rho) \cosh(\rho)}.$$

If the hypersurface is  $\mathbb{S}^1$ -invariant, these expressions can be simplified because, in this case,  $\rho_1 = \varphi_1 = 0$ . Hence we have:

$$(3.9) \quad h_i^k = -\frac{\hat{\varphi}_{ij} \tilde{e}^{jk}}{v \sinh(\rho)} + \frac{\cosh(\rho)}{v \sinh(\rho)} \delta_i^k + \frac{\sinh(\rho)}{v \cosh(\rho)} \delta_i^1 \delta_1^k$$

$$(3.10) \quad H = -\frac{\hat{\varphi}_{ij} \tilde{e}^{ji}}{v \sinh(\rho)} + \frac{\hat{H}}{v}.$$

The last equation can be also written in a second useful way : let  $\tilde{\sigma}^{ij} = \sigma^{ij} - \frac{\varphi^i \varphi^j}{v^2}$ , then, by Lemma 2.3 we have that

$$\hat{\varphi}_{ij} \tilde{e}^{ij} = \varphi_{ij} \tilde{\sigma}^{ij},$$

hence

$$(3.11) \quad H = -\frac{\varphi_{ij} \tilde{\sigma}^{ji}}{v \sinh(\rho)} + \frac{\hat{H}}{v}.$$

#### 4. The case of geodesic spheres

In this section we specify what we found in the previous one in the case of the geodesic spheres and compute their evolution under inverse mean curvature flow. Let  $\mathcal{M}_0$  be a geodesic sphere, i.e., a star-shaped hypersurface with constant radial function  $\rho = \rho_0$  (of course this function is  $\mathbb{S}^1$ -invariant). From (3.9) we can see that  $\mathcal{M}_0$  has two distinct principal curvatures:  $\lambda(\rho) = \coth(\rho)$  with multiplicity  $2n - 2$  and  $\mu(\rho) = \tanh(\rho) + \coth(\rho) = 2 \coth(2\rho)$  with multiplicity 1 and eigenvector  $\xi$ . It follows that  $H = \hat{H}$ . In particular the mean curvature depends only on the radius and then the evolution of  $\mathcal{M}_0$  by inverse mean curvature flow reduces to an ODE: the evolution on  $\mathcal{M}_0$  is a family of geodesic spheres  $\mathcal{M}_t$  of radius  $\rho(t)$  satisfying

$$\begin{cases} \dot{\rho} = \frac{1}{H} = \frac{\sinh(\rho) \cosh(\rho)}{2n \cosh^2(\rho) - 1}, \\ \rho(0) = \rho_0. \end{cases}$$

Trying to solve this ODE we arrive at an implicit value for  $\rho(t)$ :

$$\cosh(\rho(t)) \sinh^{2n-1}(\rho(t)) = \cosh(\rho_0) \sinh^{2n-1}(\rho_0) e^t.$$

Therefore the solution is defined for any positive time and  $\rho(t) = \frac{t}{2n} + o(1)$  as  $t \rightarrow \infty$ . Moreover  $|\mathcal{M}_t| = |\mathcal{M}_0| e^t$ . Then we get that the rescaled induce metric and the rescaled contact form are:

$$\begin{aligned} \tilde{g}_t &= |\mathcal{M}_t|^{-\frac{1}{n}} g = \frac{\sinh^2(\rho(t))}{|\mathcal{M}_0|^{\frac{1}{n}} e^{\frac{t}{n}}} e_{\cosh^2(\rho(t))}, \\ \tilde{\theta} &= |\mathcal{M}_t|^{-\frac{1}{n}} \theta = \frac{\sinh(\rho(t)) \cosh(\rho(t))}{|\mathcal{M}_0|^{\frac{1}{n}} e^{\frac{t}{n}}} \hat{\theta}. \end{aligned}$$

Obviously  $\frac{\sinh^2(\rho(t))}{e^{\frac{t}{n}}} \rightarrow \frac{1}{4}$ ,  $\frac{\sinh(\rho(t)) \cosh(\rho(t))}{e^{\frac{t}{n}}} \rightarrow \frac{1}{4}$  and  $\cosh^2(\rho(t)) \rightarrow \infty$  as  $t \rightarrow \infty$ . Hence the contact form converges to a constant multiple of the standard contact form on  $\mathbb{S}^{2n-1}$ , then, in particular, the Webster curvature of the limit is constant. Moreover we can see the main new phenomenon of this paper: the rescaled metric does not converge to a Riemannian metric, but to a sub-Riemannian metric defined only on  $\mathcal{H}$ . More precisely  $\tilde{g}_t$  converges to a constant multiple of  $\sigma_{sR}$ .

The following result is useful to bound the evolution of the radial function in the general case.

**LEMMA 4.1.** – *Consider two concentric geodesic spheres in  $\mathbb{C}\mathbb{H}^n$  of radius  $\rho_1(0)$  and  $\rho_2(0)$  respectively. For every  $i = 1, 2$ , let  $\rho_i(t)$  be the evolution by inverse mean curvature flow of initial datum  $\rho_i(0)$ , then there is a positive constant  $c$  depending only on  $n$ ,  $\rho_1(0)$  and  $\rho_2(0)$  such that for every time we have*

$$|\rho_2(t) - \rho_1(t)| \leq c |\rho_2(0) - \rho_1(0)|.$$

*Proof.* – We can suppose that  $\rho_2(0) > \rho_1(0)$  and then this inequality is preserved by the flow. Let us define  $\delta = \delta(t) = \rho_2(t) - \rho_1(t)$ . The function  $\delta$  satisfies

$$\frac{d\delta}{dt} = \frac{1}{(2n-1) \coth(\rho_2) + \tanh(\rho_2)} - \frac{1}{(2n-1) \coth(\rho_1) + \tanh(\rho_1)}.$$

Moreover it is easy to see that  $\frac{d\rho_1}{dt} > c_1$  with  $c_1$  positive constant which depends only on  $\rho_1(0)$ . Then  $\rho_1(t) > c_1t + \rho_1(0)$ . Furthermore trivially  $\tanh(\rho_2) > \tanh(\rho_1) > 0$  and  $\coth(\rho_1) > \coth(\rho_2) > 1$ . It follows that

$$\begin{aligned} \frac{d\delta}{dt} &\leq \frac{1}{2n-1} (\coth(\rho_1) - \coth(\rho_2)) \\ &= \frac{1}{(2n-1) \sinh^2(\tau)} \delta, \quad \text{for some } \tau \in [\rho_1, \rho_2] \\ &\leq \frac{1}{(2n-1) \sinh^2(\rho_1)} \delta \\ &\leq \frac{1}{(2n-1) \sinh^2(c_1t + \rho_1(0))} \delta. \end{aligned}$$

Integrating we have

$$\log(\delta(t)) - \log(\delta(0)) \leq \frac{1}{2n-1} \int_0^t \sinh^{-2}(c_1s + \rho_1(0)) ds < \infty.$$

Then the thesis follows. □

From these properties of the geodesic spheres we can deduce some estimates on the evolution of any star-shaped hypersurface. Let  $\mathcal{M}_0$  be defined by the radial function  $\rho(0)$ ,  $\rho(t)$  its evolution by inverse mean curvature flow and  $\rho_1 = \min_{\mathbb{S}^{2n-1}} \rho(0)$  and  $\rho_2 = \max_{\mathbb{S}^{2n-1}} \rho(0)$ . Then, with the same notation of the previous lemma, we have  $\rho_1(t) \leq \rho(t) \leq \rho_2(t)$  for any time  $t$  the flow is defined. Applying Lemma 4.1 we have that the oscillation of  $\rho(t)$  is bounded by a constant which depends only on the initial datum. Below we will show that the flow is defined for any positive time also for any star-shaped  $\mathbb{S}^1$ -invariant initial datum. It follows that in any case considered we have  $\rho(t) = \frac{t}{2n} + o(1)$  as  $t \rightarrow \infty$ .

### 5. First order estimates

The main result of this section is the proof of part (1) of Theorem 1.1. Moreover we will prove also that the mean curvature converges exponentially to that of an horosphere. The main technical result is the following:

PROPOSITION 5.1. – *There exists a positive constant  $c$  such that*

$$|\nabla_\sigma \varphi|_\sigma^2 \leq ce^{-\frac{t}{n}}.$$

As an immediate geometric consequence we have:

COROLLARY 5.2. – *The evolution of any star-shaped  $\mathbb{S}^1$ -invariant hypersurface stays star-shaped for any time the flow is defined.*

*Proof.* – An hypersurface is star-shaped if and only if  $\frac{\partial}{\partial \rho}$  and  $\nu$  are never orthogonal in  $\mathbb{C}\mathbb{H}^n$ . This means that there exists a positive constant  $c$  such that

$$\bar{g} \left( \frac{\partial}{\partial \rho}, \nu \right) = \frac{1}{v} \geq \frac{1}{c} \quad \Leftrightarrow \quad v^2 \leq c^2.$$

Recalling that  $v^2 = 1 + |\nabla_e \varphi|_e^2$ , the thesis follows from Proposition 5.1 noting that

$$|\nabla_e \varphi|_e^2 = |\nabla_\sigma \varphi|_\sigma^2$$

holds in case of  $\mathbb{S}^1$  invariance. □

The proof of Proposition 5.1 is divided into three steps: first we can prove that  $|\nabla_\sigma \varphi|_\sigma^2$  is bounded, then that it has an exponential decay and, finally, we show that the right exponent is  $\frac{1}{n}$ .

LEMMA 5.3. – *The following estimate holds:*

$$|\nabla_\sigma \varphi|_\sigma^2 \leq \sup_{z \in \mathbb{S}^{2n-1}} |\nabla_\sigma \varphi(z, 0)|_\sigma^2.$$

*Proof.* – Let us define  $\omega = \frac{1}{2} |\nabla_\sigma \varphi|_\sigma^2 = \frac{1}{2} \varphi_k \varphi^k$ . Note that, as a consequence of the  $\mathbb{S}^1$ -invariance,  $\varphi_i e^{ik} = \varphi_i \sigma^{ik}$ , so we do not distinguish between  $\varphi^k$  and  $\hat{\varphi}^k$ .

We want to compute the evolution equation for  $\omega$ . We start with the evolution of the radial function:

$$\frac{1}{Hv} = \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_i} \frac{\partial x_i}{\partial t} = \frac{\partial \rho}{\partial t} - \frac{\rho^i \rho_i}{Hv \sinh^2(\rho)}.$$

Then

$$(5.1) \quad \frac{\partial \rho}{\partial t} = \frac{1}{Hv} \left( 1 + \frac{|\nabla_\sigma \rho|_\sigma^2}{\sinh^2(\rho)} \right) = \frac{v}{H},$$

and so

$$(5.2) \quad \frac{\partial \varphi}{\partial t} = \frac{1}{\sinh(\rho)} \frac{\partial \rho}{\partial t} = \frac{v}{H \sinh(\rho)} =: \frac{1}{F}$$

holds. From the explicit computation of the mean curvature (3.11) we have

$$(5.3) \quad F = F(\varphi, \varphi_i, \varphi_{ij}) = -\frac{\varphi_{ij} \tilde{\sigma}^{ij}}{v^2} + \frac{\sinh(\rho) \hat{H}}{v^2}.$$

Now we want to compute the evolution equation of  $\omega$ : let  $a^{ij} = -\frac{\partial F}{\partial \varphi_{ij}} = \frac{\tilde{\sigma}^{ij}}{v^2}$ ,  $b^i = \frac{\partial F}{\partial \varphi_i}$  and, for simplicity of notation,  $\nabla = \nabla_\sigma$ , then

$$\begin{aligned} \frac{\partial \omega}{\partial t} &= \varphi^k \nabla_k \frac{\partial \varphi}{\partial t} \\ &= -\frac{1}{F^2} \left( -a^{ij} \varphi_{ijk} \varphi^k + b^i \varphi_{ik} \varphi^k + \frac{\partial F}{\partial \varphi} \varphi_k \varphi^k \right) \\ &= -\frac{1}{F^2} \left( -a^{ij} \varphi_{ijk} \varphi^k + b^i \omega_i + 2 \frac{\partial F}{\partial \varphi} \omega \right). \end{aligned}$$

We can apply the rule for interchanging derivatives:

$$\varphi_{ijk} = \varphi_{kji} + R^m{}_{ijk} \varphi_m,$$

where this time  $R$  is the Riemann curvature tensor of  $\sigma$ , i.e.,  $R_{sijk} = \sigma_{sj} \sigma_{ik} - \sigma_{sk} \sigma_{ij}$ . Since  $a^{ij}$  is symmetric we get:

$$\begin{aligned} -a^{ij} \varphi_{ijk} \varphi^k &= -a^{ij} \varphi_{kji} \varphi^k - a^{ij} (\delta_j^m \sigma_{ik} - \delta_k^m \sigma_{ij}) \varphi_m \varphi^k \\ &= -a^{ij} \omega_{ij} + a^{ij} \varphi_{ik} \varphi_j^k - a^{ij} \varphi_i \varphi_j + 2a_i^i \omega. \end{aligned}$$



The following equality holds:

$$-a^{ij} \varphi_i \varphi_j + 2a_i^i \omega = \frac{4(n-1)}{v^2} \omega \geq 0.$$

Moreover, due to the  $\mathbb{S}^1$ -invariance, we have

$$\frac{\partial}{\partial \rho} \tilde{\sigma} = 0, \quad \frac{\partial}{\partial \rho} v = 0,$$

hence

$$\begin{aligned} \frac{\partial F}{\partial \varphi} &= \frac{\partial F}{\partial \rho} \frac{\partial \rho}{\partial \varphi} = \sinh(\rho) \frac{1}{v^2} \frac{\partial}{\partial \rho} (\sinh(\rho) \hat{H}) \\ (5.4) \quad &= \frac{\sinh^2(\rho)}{v^2 \cosh^2(\rho)} (2n \cosh^2(\rho) + 1) > 0. \end{aligned}$$

Note that  $a^{ij}$  is positive definite. Finally we have that

$$a^{ij} \varphi_{ik} \varphi_j^k = a^{ij} \sigma^{kl} \varphi_{ik} \varphi_{jl} \geq 0$$

because, as showed in [2], if  $A$ ,  $B$  and  $C$  are symmetric matrices, with  $A$  and  $B$  positive definite, then  $tr(ACBC) \geq 0$ . The thesis follows by the maximum principle.  $\square$

Now we can use the previous result to bound the mean curvature. In particular we show that the mean convexity is preserved.

LEMMA 5.4. – *There exist two positive constants  $c_1$  and  $c_2$  depending only on  $n$  and the initial datum such that for any time the flow is defined*

$$0 < c_1 \leq H \leq c_2.$$

*Proof.* – From Lemma 2.6 and the fact that  $|A|^2 \geq \frac{1}{2n-1} H^2$  we can compute

$$\frac{\partial H}{\partial t} \leq \frac{\Delta H}{H^2} - \frac{H}{2n-1} + \frac{2(n+1)}{H}.$$

By the maximum principle, it is easy to show that  $H$  is bounded from above. To prove that  $H$  is bounded from below we define  $\psi = \frac{v}{\sinh(\rho)H} e^{\frac{t}{2n}} = \frac{1}{F} e^{\frac{t}{2n}} = \frac{\partial \varphi}{\partial t} e^{\frac{t}{2n}}$  and we prove that this function is bounded from above. Preceding like in the proof of Lemma 5.3:

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= \frac{\partial}{\partial t} \left( \frac{\partial \varphi}{\partial t} e^{\frac{t}{2n}} \right) \\ &= -\frac{1}{F^2} \left( -a^{ij} \frac{\partial \varphi_{ij}}{\partial t} + b^i \frac{\partial \varphi_i}{\partial t} + \frac{\partial F}{\partial \varphi} \frac{\partial \varphi}{\partial t} \right) e^{\frac{t}{2n}} + \frac{1}{2n} \psi \\ &= -\frac{1}{F^2} \left( -a^{ij} \psi_{ij} + b^i \psi_i + \frac{\partial F}{\partial \varphi} \psi \right) + \frac{1}{2n} \psi. \end{aligned}$$

From (5.4) we have that

$$\frac{\partial F}{\partial \varphi} \geq 2n \frac{\sinh^2(\rho)}{v^2},$$

moreover  $\frac{1}{F^2} = \psi^2 e^{-\frac{t}{n}}$ . By Lemma 5.3  $v^2$  is bounded. Since the function  $\sinh^2(\rho) e^{-\frac{t}{n}}$  is bounded too, we get that

$$(5.5) \quad -\frac{1}{F^2} \frac{\partial F}{\partial \varphi} \psi + \frac{1}{2n} \psi \leq -2n \frac{\sinh^2(\rho)}{v^2} e^{-\frac{t}{n}} \psi^3 + \frac{1}{2n} \psi \leq -c \psi^3 + \frac{1}{2n} \psi,$$

for some positive constant  $c$ . By the maximum principle we deduce that  $\psi$  is bounded. This implies that there is a constant  $c > 0$  such that  $H \geq cv \frac{e^{\frac{1}{2n}}}{\sinh(\rho)}$ . The thesis follows since  $v \geq 1$  by definition and  $\frac{e^{\frac{1}{2n}}}{\sinh(\rho)}$  is bounded.  $\square$

As a consequence we can improve Lemma 5.3.

LEMMA 5.5. – *There are two positive constants  $c$  and  $\gamma$  such that:*

$$|\nabla_{\sigma}\varphi|_{\sigma}^2 \leq ce^{-\gamma t}.$$

*Proof.* – Since we proved that  $H$  is bounded, we have that  $\frac{2}{F^2} \frac{\partial F}{\partial \varphi} \geq \gamma > 0$  for some  $\gamma$ . Proceeding like in the proof of Lemma 5.3 we get the thesis.  $\square$

With the help of this estimate, we can refine the result of Lemma 5.4.

LEMMA 5.6. – *There is a positive constant  $c$  such that:*

$$|H - 2n| \leq ce^{-\gamma t},$$

where  $\gamma$  is the same as in the previous lemma.

*Proof.* – Let  $l_i^j$  be equal to  $h_i^j + \delta_i^j - \delta_i^1 \delta_1^i$ . Explicit computations give:

$$\begin{aligned} |l|^2 &= l_i^j l_j^i = |A|^2 + 2(H - 2n) + 6(n - 1) - 2(h_1^1 - 2); \\ L &= b_i^i = H - 2n + 2(2n - 1). \end{aligned}$$

Obviously we have

$$|l|^2 \geq \frac{L^2}{2n - 1}.$$

Moreover, by (3.9),

$$h_1^1 = \frac{1}{v} \left( \frac{\cosh(\rho)}{\sinh(\rho)} + \frac{\sinh(\rho)}{\cosh(\rho)} \right),$$

hence by Lemma 5.5

$$|h_1^1 - 2| \leq ce^{-\gamma t}.$$

By Lemma 2.6 we get:

$$\begin{aligned} \left( \frac{\partial}{\partial t} - \frac{\Delta}{H^2} \right) (H - 2n) &\leq -\frac{1}{H} (|A|^2 - 2(n + 1)) \\ &= -\frac{1}{H} (|l|^2 - 2(H - 2n) - 4(2n - 1) + 2(h_1^1 - 2)) \\ &\leq -\frac{1}{H} \left( \frac{L^2}{2n - 1} - 2(H - 2n) - 4(2n - 1) \right) + ce^{-\gamma t} \\ (5.6) \qquad &= -\frac{H - 2n}{(2n - 1)H} (H + 2n - 2) + ce^{-\gamma t}. \end{aligned}$$

Hence  $H - 2n \leq ce^{-\gamma' t}$ , where  $\gamma' \leq \gamma$ . Using this information and starting again from (5.6) we have:

$$\left( \frac{\partial}{\partial t} - \frac{\Delta}{H^2} \right) (H - 2n) \leq -\left( \frac{1}{n} + ce^{-\gamma' t} \right) (H - 2n) + ce^{-\gamma t}.$$

Applying again the maximum principle we have the desired estimate from above. On the other side we can consider the evolution of the function  $\psi$  defined in Lemma 5.4. Using what we proved so far, the reaction term (5.5) becomes:

$$-\frac{1}{F^2} \frac{\partial F}{\partial \varphi} \psi + \frac{1}{2n} \psi \leq \frac{\psi}{2n} (1 - 4n^2 (1 + ce^{-\gamma t}) \psi^2).$$

By the maximum principle we have

$$\psi \leq \frac{1}{2n} + ce^{-\gamma t},$$

where the constant  $c$  can be different from above. By the definition of  $\psi$  we get the thesis.  $\square$

Finally we are able to prove the main result of this section.

*Proof of Proposition 5.1.* By the previous lemma and (5.4) we have:

$$\frac{2}{F^2} \frac{\partial F}{\partial \varphi} \geq \frac{4n}{H^2} \geq \frac{1}{n(1 + ce^{-\gamma t})}.$$

With the same computations of the proof of Proposition 5.1, and by the maximum principle we have that  $|\nabla_\sigma \varphi|_\sigma^2 \leq y$ , where  $y$  is the solution of the Cauchy problem:

$$\begin{cases} y' = -\frac{1}{n(1+ce^{-\gamma t})} \\ y(0) = y_0 \geq \sup_{t=0} |\nabla_\sigma \varphi|_\sigma^2. \end{cases}$$

Then  $|\nabla_\sigma \varphi|_\sigma^2 \leq y_0 \left(\frac{1+c}{e^{\gamma t}+c}\right)^{\frac{1}{n\gamma}} \leq ce^{-\frac{t}{n}}$ , for some  $c$ .  $\square$

We finish this section noting that the proofs of Lemma 5.6 can be repeated using  $\gamma = \frac{1}{n}$ , hence we have:

**PROPOSITION 5.7.** – *There is a positive constant  $c$  such that:*

$$|H - 2n| \leq ce^{-\frac{1}{n}t}.$$

### 6. Second order estimates

In this section we collect some important results concerning the second order derivative of the radial function. First of all, we prove that the principal curvatures of the evolving hypersurface stay bounded. As a consequence we have the long time existence of the flow. In the second part of this section we prove that the Hessian of the radial function is bounded. In view of (3.9), this implies the convergence with exponential speed of the second fundamental form to that of the horospheres. In this context a new phenomenon appears: in the cases already known in literature, see for example [4, 5, 17, 19], we always have the same speed of convergence for any initial datum. Here, in the general case, we can expect just half of the speed of the convergence of, for example, the evolution of a geodesic sphere. See Remark 6.6 below for more details.

LEMMA 6.1. – *Let us define the tensor  $M_i^j = Hh_i^j$ , then*

$$\begin{aligned} \frac{\partial M_i^j}{\partial t} &= \frac{\Delta M_i^j}{H^2} - \frac{2}{H^3} \langle \nabla H, \nabla M_i^j \rangle - \frac{2}{H^3} \nabla_i H \nabla_k H g^{kj} - 2 \frac{M_i^k M_k^j}{H^2} - 2 \bar{R}_{i0k0} g^{kj} \\ &\quad + \frac{1}{H^2} g^{qs} g^{kj} (2 \bar{R}_{pisk} M_q^p - \bar{R}_{pqis} M_k^p - \bar{R}_{pqks} M_i^p). \end{aligned}$$

*Proof.* – Some simple computations give that

$$\begin{aligned} \nabla M_i^j &= h_i^j \nabla H + H \nabla h_i^j \\ \Delta M_i^j &= h_i^j \Delta H + H \Delta h_i^j + 2 \langle \nabla H, \nabla h_i^j \rangle \\ &= h_i^j \Delta H + H \Delta h_i^j + \frac{2}{H} \langle \nabla H, \nabla M_i^j \rangle - 2 \frac{h_i^j}{H} |\nabla H|^2. \end{aligned}$$

Using the evolution equation for  $H$  and  $h_i^j$  in Lemma 2.6 we get the thesis. □

COROLLARY 6.2. – *The principal curvatures of the hypersurface evolving by inverse mean curvature flow are uniformly bounded.*

*Proof.* – Since by Lemma 5.4  $H$  is bounded (in particular from below), it is sufficient to prove that the principal curvatures are bounded from above. Let  $m = 2n - 1$  and  $\mu_1 \leq \dots \leq \mu_m$  the eigenvalues of  $M_i^j$ . The trace of  $M_i^j$  is  $\sum_i \mu_i = H^2 > 0$ , then  $\mu_m > 0$  everywhere. To conclude the proof we want to show that  $\mu_m$  is bounded from above. Fix any time  $T^*$  strictly smaller than the maximal time  $T$ . We can find a point  $(x_0, t_0)$  where  $\mu_m$  reaches its maximum in  $\mathbb{S}^{2n-1} \times [0, T^*]$ . At this point we can fix an orthonormal basis which diagonalizes  $M_i^j$ , then we can say that at this point  $\mu_m$  satisfies the same equation for  $M_m^m$  found in Lemma 6.1. The following estimates hold:

$$\begin{aligned} -\frac{2}{H^2} \nabla_m H \nabla_k H g^{km} &= -\frac{2}{H^2} (\nabla_m H)^2 \leq 0, \\ -\bar{R}_{m0j0} g^{jm} &= -\bar{R}_{m0m0} \leq 4, \\ 2g^{qs} g^{km} \bar{R}_{pmsk} M_q^p &= \sum_q 2\mu_q \bar{R}_{qmqm}, \\ g^{qs} g^{km} \bar{R}_{pqms} M_k^p &= g^{qs} g^{km} \bar{R}_{pqks} M_m^p \\ &= \sum_q \mu_m \bar{R}_{qmqm}. \end{aligned}$$

It follows that

$$\begin{aligned} g^{qs} g^{km} (2 \bar{R}_{pmsk} M_q^p - \bar{R}_{pqms} M_k^p - \bar{R}_{pqks} M_m^p) &= 2 \sum_q \bar{R}_{qmqm} (\mu_q - \mu_m) \\ &\leq -8 \sum_q (\mu_q - \mu_m) \\ &= 8(2n - 1)\mu_m - 8H^2. \end{aligned}$$

Putting together these computations we have that at  $(x_0, t_0)$

$$\frac{\partial \mu_m}{\partial t} \leq \frac{\Delta \mu_m}{H^2} - \frac{2}{H^3} \langle \nabla H, \nabla \mu_m \rangle - \frac{2\mu_m}{H^2} (\mu_m - 4(2n - 1)).$$

If  $t_0 \neq 0$  we can deduce that in  $(x_0, t_0)$

$$0 \leq \frac{\partial \mu_m}{\partial t} - \frac{\Delta \mu_m}{H^2} + \frac{2}{H^3} \langle \nabla H, \nabla \mu_m \rangle \leq -\frac{2\mu_m}{H^2} (\mu_m - 4(2n - 1)).$$

Since  $\mu_m$  is positive, it follows that  $\mu_m \leq 4(2n - 1)$ . Hence  $\mu_m$  reaches its maximum at time  $t_0 = 0$  or it is bounded by a constant independent of the choice of  $T^*$ .  $\square$

It follows the uniform parabolicity of Equation (5.1) and a uniform  $C^2$ -estimate for the function  $\rho(\cdot, t)$ . Arguing as in Chapter 2.6 of [4], we can apply the  $C^{2,\alpha}$  estimates of [12] to conclude that the solution of the flow is defined for any positive time and it is smooth, since the initial datum is smooth.

In the sequel we can perform a better analysis giving a uniform estimate on the Hessian of  $\rho$ . We prefer to work with the auxiliary function  $\varphi$ .

PROPOSITION 6.3. – *There is a positive constant  $c$  such that*

$$|\nabla_\sigma^2 \varphi|_\sigma^2 \leq ce^{-\frac{t}{n}}.$$

*Proof.* – The proof is similar to that of Lemma 5.3, but we will use more than once the interchanging rule. Let us define this time  $\omega = \frac{1}{2} |\nabla_\sigma^2 \varphi|_\sigma^2$ , then, recalling Notations 2.2 we have:

$$\begin{aligned} \frac{\partial}{\partial t} \omega &= \varphi^{rs} \nabla_r \nabla_s \frac{\partial}{\partial t} \varphi \\ &= \frac{1}{F^2} \varphi^{rs} \left( a^{ij} \varphi_{ijrs} - b^i \varphi_{irs} - \frac{\partial F}{\partial \varphi} \varphi_{rs} \right) \\ &= \frac{1}{F^2} \left( a^{ij} \varphi^{rs} \varphi_{ijrs} - b^i \varphi^{rs} \varphi_{irs} - 2 \frac{\partial F}{\partial \varphi} \omega \right). \end{aligned}$$

Applying twice the interchanging rule we have:

$$\begin{aligned} a^{ij} \varphi^{rs} \varphi_{rsij} &= a^{ij} \varphi^{rs} (\varphi_{isrj} + (\delta_i^m \sigma_{sr} - \delta_r^m \sigma_{si}) \varphi_{mj}) \\ &= a^{ij} \varphi_{rs} [\nabla_i (\varphi_{jrs} + (\delta_j^m \sigma_{sr} - \delta_s^m \sigma_{rj}) \varphi_m)] \\ &\quad + a^{ij} \varphi_{ij} \Delta \varphi - a^{ij} \varphi_i^r \varphi_{rj} \\ &= a^{ij} \varphi_{rs} \varphi_{ijrs} + 2a^{ij} \varphi_{ij} \Delta \varphi - 2a^{ij} \varphi_i^r \varphi_{rj} \\ &= a^{ij} \omega_{ij} + 2a^{ij} \varphi_{ij} \Delta \varphi - 2a^{ij} \varphi_i^r \varphi_{rj} \\ &\quad - a^{ij} \varphi_i^{rs} \varphi_{jrs}; \end{aligned}$$

while

$$\begin{aligned} -b^i \varphi^{rs} \varphi_{irs} &= -b^i \varphi^{rs} (\varphi_{rsi} + (\delta_i^m \sigma_{rs} - \delta_r^m \sigma_{ir}) \varphi_m) \\ &= -b^i \varphi^{rs} \varphi_{rsi} - b^i \varphi_i \Delta \varphi + b^i \varphi_i^r \varphi_r. \end{aligned}$$

Summing up these quantities we have:

$$\frac{\partial}{\partial t} \omega = \frac{a^{ij}}{F^2} \omega_{ij} - \frac{b^i}{F^2} \omega_i - \frac{2}{F^2} \frac{\partial F}{\partial \varphi} \omega + \frac{R}{F^2},$$

where the remainder term is

$$R = 2a^{ij} \varphi_{ij} \Delta \varphi - 2a^{ij} \varphi_j^r \varphi_{ri} - a^{ij} \varphi_i^{rs} \varphi_{jrs} \\ - b^i \varphi_i \Delta \varphi + b^i \varphi_i^r \varphi_r.$$

From (5.4) and Proposition 5.6 we get

$$-\frac{2}{F^2} \frac{\partial F}{\partial \varphi} = -\frac{2}{H^2} \left( 2n + \frac{1}{\cosh^2(\rho)} \right) \leq -\frac{1}{n} + ce^{-\frac{t}{n}}.$$

Moreover we claim that

$$R \leq c(\omega + 1).$$

In fact we notice that, for  $t$  big enough,  $a^{ij} \geq \frac{1}{2} \sigma^{ij}$ , hence

$$-a^{ij} \varphi_i^{rs} \varphi_{jrs} \leq -\frac{1}{2} |\nabla_\sigma^3 \varphi|_\sigma^2 \leq 0, \\ -2a^{ij} \varphi_i^r \varphi_{rj} \leq c\omega.$$

Obviously  $(\Delta \varphi)^2 \leq 2(2n - 1)\omega$  and, by (3.11),

$$a^{ij} \varphi_{ij} = \frac{\sinh(\rho)}{v} \left( \frac{\hat{H}}{v} - H \right),$$

therefore

$$2a^{ij} \varphi_{ij} \Delta \varphi \leq (a^{ij} \varphi_{ij})^2 + (\Delta \varphi)^2 \leq c(\omega + 1).$$

An explicit computation shows that

$$b^i = -\frac{2F}{v^2} \varphi^i + \frac{\varphi_{rs}}{v^4} \left( \varphi^r \delta^{si} + \varphi^s \delta^{ri} - 2 \frac{\varphi^r \varphi^s \varphi^i}{v^2} \right).$$

It follows that

$$|b^i|_\sigma^2 \leq c(\omega + 1).$$

Using these results we can finally estimate

$$-b^i \varphi_i \Delta \varphi \leq \frac{1}{2} |b^i|_\sigma^2 |\nabla_\sigma \varphi|_\sigma^2 + \frac{1}{2} (\Delta \varphi)^2 \leq c(\omega + 1), \\ b^i \varphi_i^r \varphi_{ri} \leq c(|b^i|_\sigma^2 + |\nabla \varphi|_\sigma^2 + \omega) \leq c(\omega + 1).$$

Summing up what we found we have that

$$\frac{\partial}{\partial t} \omega \leq \frac{a^{ij}}{F^2} \omega_{ij} - \frac{b^i}{F^2} \omega_i + \left( -\frac{1}{n} + ce^{-\frac{t}{n}} \right) \omega + ce^{-\frac{t}{n}}.$$

The thesis follows by the maximum principle.  $\square$

**REMARK 6.4.** – In (5.1) we computed the scalar flow satisfied by the radial function. The flow (1.1) is defined at least as the corresponding flow for  $\rho$ . We computed two different expressions for the mean curvature—(3.10) and (3.11)—each one define, formally, a different scalar flow. However, as consequence of Lemma 2.3, in the special case of  $\mathbb{S}^1$ -invariance the two flows coincide, so, from a technical point of view, the choice of studying the derivatives with respect to  $\sigma$ , and not  $e_{\cosh^2(\rho)}$  is coherent.

A consequence of this proposition is the convergence of the second fundamental form to that of an horosphere.

COROLLARY 6.5. – *There is a positive constant  $c$  such that*

$$\left| h_i^k - \delta_i^k - \delta_i^1 \delta_1^k \right|^2 \leq c e^{-\frac{t}{n}}.$$

Moreover on the horizontal distribution we have a faster convergence: taking the sum over  $i$ ,  $k \neq 1$

$$\left| h_i^k - \delta_i^k \right|^2 \leq c e^{-\frac{2t}{n}}.$$

*Proof.* – By the expression of the second fundamental form (3.9), of the mean curvature (3.10) and Lemma 2.3 we have:

$$\begin{aligned} \left| h_i^k - \delta_i^k - \delta_i^1 \delta_1^k \right|^2 &= \frac{\hat{\varphi}_{ij} \hat{\varphi}_{kr} \tilde{e}^{jk} \tilde{e}^{ri}}{v^2 \sinh^2(\rho)} - \frac{2\hat{\varphi}_{ij} \tilde{e}^{ji}}{v \sinh(\rho)} \left( \frac{\cosh(\rho)}{v \sinh(\rho)} - 1 \right) \\ &\quad + (2n-1) \left( \frac{\cosh(\rho)}{v \sinh(\rho)} - 1 \right)^2 + \left( \frac{\sinh(\rho)}{v \cosh(\rho)} - 1 \right)^2 \\ &\quad + 2 \left( \frac{\cosh(\rho)}{v \sinh(\rho)} - 1 \right) \left( \frac{\sinh(\rho)}{v \cosh(\rho)} - 1 \right) \\ &= \frac{\varphi_{ij} \varphi_{kr} \tilde{\sigma}^{jk} \tilde{\sigma}^{ri}}{v^2 \sinh^2(\rho)} + 2 \frac{|\nabla_\sigma \varphi|_\sigma^2}{v^2} + 2 \left( H - \frac{\hat{H}}{v} \right) \left( \frac{\cosh(\rho)}{v \sinh(\rho)} - 1 \right) \\ (6.1) \quad &\quad + (2n-1) \left( \frac{\cosh(\rho)}{v \sinh(\rho)} - 1 \right)^2 + \left( \frac{\sinh(\rho)}{v \cosh(\rho)} - 1 \right)^2 \\ &\quad + 2 \left( \frac{\cosh(\rho)}{v \sinh(\rho)} - 1 \right) \left( \frac{\sinh(\rho)}{v \cosh(\rho)} - 1 \right). \end{aligned}$$

By Proposition 5.1, Proposition 5.7 and Proposition 6.3 we get that all the terms appearing in the last equality can be bounded by  $c e^{-\frac{2t}{n}}$ , except the “bad” term  $2 \frac{|\nabla_\sigma \varphi|_\sigma^2}{v^2}$  that is just smaller than  $c e^{-\frac{t}{n}}$ . Hence the first estimate is proven. Finally, if we restrict our attention to the horizontal distribution, the same computations give:

$$\begin{aligned} \left| h_i^k - \delta_i^k \right|^2 &= \frac{\varphi_{ij} \varphi_{kr} \tilde{\sigma}^{jk} \tilde{\sigma}^{ri}}{v^2 \sinh^2(\rho)} + 2 \left( H - \frac{\hat{H}}{v} \right) \left( \frac{\cosh(\rho)}{v \sinh(\rho)} - 1 \right) \\ &\quad + (2n-2) \left( \frac{\cosh(\rho)}{v \sinh(\rho)} - 1 \right)^2 \leq c e^{-\frac{2t}{n}}. \square \end{aligned}$$

REMARK 6.6. – For some initial data we are able to find the “fast” convergence of the second fundamental form in the whole tangent space, because  $|\nabla_\sigma \rho|_\sigma^2$  converges to (or is, like in the trivial case of geodesic spheres) zero. In general the estimate found in the previous corollary is optimal. In fact, for the examples that we will discuss in the last section, we have that the gradient of the radial function is just bounded and cannot converge to zero. Hence Proposition 5.1 cannot be improved and the “bad” term in Equation (6.1) decays slower than all others. Moreover Proposition 5.7 says that we don’t see this difference at the level of the mean curvature and we find the optimal speed in any case. The reason is that, like shown in Lemma 2.3, if  $\varphi$  is an  $S^1$ -invariant function, then  $\Delta_e \varphi = \Delta_\sigma \varphi$ .

## 7. Higher order estimates

Following the same procedure as in the previous section, we can show that the spatial derivative of any order of  $\varphi$  has an exponential decay.

NOTATION 7.1. – 1) In order to avoid confusion with the meaning of the indices, in this section we use the following notation: capital letters count the number of derivations (for example  $\nabla^K \varphi$  is the  $K$ -th derivative of the function  $\varphi$ ), while lowercase letters are indices representing a direction.

2) In the next proof we will use also a well established notation: given two tensors  $S$  and  $T$ , we write  $S * T$  for any linear combination formed by contraction on  $S$  and  $T$  by  $\sigma$ .

PROPOSITION 7.2. – *For any integer  $K$ , there is a positive constant  $c$  which depends only on  $K$ ,  $n$ , and  $\mathcal{M}_0$ , such that*

$$|\nabla_\sigma^K \varphi|_\sigma^2 \leq c e^{-\frac{t}{n}}.$$

*Proof.* – The proof follows the same strategy as in Proposition 6.3. Fix  $K$  and let  $\omega = \frac{1}{2} |\nabla_\sigma^K \varphi|_\sigma^2$ . Applying a finite number of times the interchanging rule for derivative we have:

$$\begin{aligned} \frac{\partial}{\partial t} \omega &= \nabla^{i_1} \dots \nabla^{i_K} \nabla_{i_1} \dots \nabla_{i_K} \frac{\partial}{\partial t} \varphi \\ &= \frac{a^{ij}}{F^2} \omega_{ij} - \frac{b^i}{F^2} \omega_i - \frac{2}{F^2} \frac{\partial F}{\partial \rho} \omega \\ &\quad - \frac{2a^{ij}}{F^2} \nabla_i \nabla_{i_1} \dots \nabla_{i_K} \varphi \nabla_j \nabla^{i_1} \dots \nabla^{i_K} \varphi \\ &\quad + \frac{1}{F^2} (a * \nabla^K \varphi * \nabla^K \varphi) + \frac{1}{F^2} (b * \nabla^K \varphi * \nabla^{K-1} \varphi), \end{aligned}$$

Arguing as in the proof of Proposition 6.3, and supposing by induction that the thesis holds for  $K - 1$ , we can show that in this case too we have:

$$\frac{\partial}{\partial t} \omega \leq \frac{a^{ij}}{F^2} \omega_{ij} - \frac{b^i}{F^2} \omega_i + \left( -\frac{1}{n} + c e^{-\frac{t}{n}} \right) \omega + c e^{-\frac{t}{n}}.$$

The thesis follows once again as an application of the maximum principle.  $\square$

An immediate consequence of Proposition 7.2 is the following.

COROLLARY 7.3. – *For any integer  $K$  there is a positive constant  $c$  which depends only on  $K$ ,  $n$ , and  $\mathcal{M}_0$ , such that*

$$|\nabla_\sigma^K \rho|_\sigma^2 \leq c.$$



8. Convergence of the rescaled metric and contact form

In this section we prove part (3) of Theorem 1.1 by studying the limit of the rescaled metric and of the rescaled contact form.

THEOREM 8.1. – *There is a smooth  $\mathbb{S}^1$ -invariant function  $f$  such that the metric  $\tilde{g}_t = |\mathcal{M}_t|^{-\frac{1}{n}} g_t$  converges to a sub-Riemannian metric  $\tilde{g}_\infty = e^{2f} \sigma_{sR}$  and the contact form  $\tilde{\theta} = |\mathcal{M}_t|^{-\frac{1}{n}} \theta_t$  converges to the conformal multiple  $\tilde{\theta}_\infty = e^{2f} \hat{\theta}$ .*

*Proof.* – For every time  $t$ , let  $\tilde{\rho}_t$  be the radius of a geodesic sphere  $B_{\tilde{\rho}_t}$  such that  $|\mathcal{M}_t| = |B_{\tilde{\rho}_t}|$  and define  $f_t = \rho(x, t) - \tilde{\rho}_t$ . The mean curvature of  $B_{\tilde{\rho}_t}$  is  $\tilde{H} = \hat{H}(\tilde{\rho})$ , then  $\frac{\partial \tilde{\rho}}{\partial t} = \tilde{H}^{-1}$  and  $\tilde{\rho} = \frac{t}{2n} + o(1)$  as  $t \rightarrow \infty$ .

We recall that  $g_{ij} = \sinh^2(\rho) (\varphi_i \varphi_j + e_{ij})$ . Obviously  $e_{\cosh^2(\rho)} \rightarrow \sigma_{sR}$  as  $t \rightarrow \infty$ . By Proposition 5.1 we have that each  $\varphi_i$  is going to zero, hence

$$\begin{aligned} \lim_{t \rightarrow \infty} \tilde{g}_t &= \frac{1}{|\mathcal{M}_0|^{\frac{1}{n}}} \left( \lim_{t \rightarrow \infty} \sinh^2(\rho) e^{-\frac{t}{n}} \right) \sigma_{sR} \\ &= \gamma \left( \lim_{t \rightarrow \infty} e^{2f_t} \right) \sigma_{sR}, \end{aligned}$$

for some positive constant  $\gamma$ .

In analogous way, recalling the expression of the contact form (3.4), we get

$$\begin{aligned} \lim_{t \rightarrow \infty} \tilde{\theta}_t &= \frac{1}{|\mathcal{M}_0|^{\frac{1}{n}}} \left( \lim_{t \rightarrow \infty} \sinh(\rho) \cosh(\rho) e^{-\frac{t}{n}} \right) \hat{\theta} \\ &= \gamma \left( \lim_{t \rightarrow \infty} e^{2f_t} \right) \hat{\theta}, \end{aligned}$$

where  $\gamma$  is the same constant appearing in the limit of  $\tilde{g}_t$ .

The thesis follows if we can prove that  $f_t$  converges, as  $t$  goes to infinity, to a smooth function  $f_\infty$ , then  $f$  will be a multiple of such  $f_\infty$ . Since for any integer  $K$ ,  $|\nabla_\sigma^K f_t|_\sigma^2 = |\nabla_\sigma^K \rho|_\sigma^2$  and it is uniformly bounded by Corollary 7.3, we have to show that there exists a positive constant  $c$  such that  $\left| \frac{\partial f_t}{\partial t} \right| \leq c e^{-\frac{t}{n}}$ . By (5.1), we have that

$$\frac{\partial f_t}{\partial t} = \frac{\partial \rho}{\partial t} - \frac{d \tilde{\rho}_t}{dt} = \frac{v}{H} - \frac{1}{\tilde{H}}.$$

Using triangle inequality we have

$$\left| \frac{\partial f_t}{\partial t} \right| \leq \frac{1}{H} |v - 1| + \frac{1}{H \tilde{H}} (|H - 2n| + |\tilde{H} - 2n|).$$

Since we know that  $H$ ,  $\tilde{H}$  and  $v$  are bounded and positive, the desired estimate follows by Proposition 5.1 and Proposition 5.7. □

### 9. The curvature of the limit metric

In this section we conclude the proof of Theorem 1.1 by showing that  $\tilde{\theta}_\infty = e^{2f} \hat{\theta}$  (or equivalently  $\tilde{g}_\infty = e^{2f} \sigma_{sR}$ ) does not have necessarily constant Webster scalar curvature. We recall that, as shown in Lemma 2.5, since we are considering only  $\mathbb{S}^1$ -invariant hypersurfaces, Jerison and Lee's Formula (2.3) can be simplified and we have to check just that some of such functions  $f$  are not constant.

The construction of the required hypersurface is inspired by the solution of the analogous problem in the real hyperbolic space [9]. However it is well known that in  $\mathbb{C}\mathbb{H}^n$  there are no totally umbilical hypersurfaces, so the trace-free part of the second fundamental form cannot have the same strong meaning that it has in the case of hyperbolic space: see Propositions 3 and 5 of [9]. So we defined a Brown-York type quantity on hypersurfaces which gives a measure of how the hypersurface is far from being a geodesic sphere. For any star-shaped hypersurface  $\mathcal{M}$  we define

$$(9.1) \quad Q(\mathcal{M}) = |\mathcal{M}|^{-1+\frac{1}{n}} \int_{\mathcal{M}} (H - \hat{H}) d\mu,$$

where  $\hat{H}$  was defined in (3.8).  $Q$  is not a true measure, because we do not know its sign: it is trivially zero when  $\mathcal{M}$  is a geodesic sphere, but in general it is not truly the opposite. One of the main properties of  $Q$  is the following.

**PROPOSITION 9.1.** – *Let  $\tilde{\mathcal{M}}^\tau$  be a family of hypersurfaces in  $\mathbb{C}\mathbb{H}^n$  that are radial graph of the functions  $\tilde{\rho}(z, \tau) = \tau + f(z) + o(1)$ , for some fixed  $\mathbb{S}^1$ -invariant function  $f : \mathbb{S}^{2n-1} \rightarrow \mathbb{R}$ . Then*

$$\lim_{\tau \rightarrow \infty} Q(\tilde{\mathcal{M}}^\tau) = \left( \int_{\mathbb{S}^{2n-1}} e^{2nf} d\sigma \right)^{-1+\frac{1}{n}} \int_{\mathbb{S}^{2n-1}} e^{2nf} \left( e^{-f} \Delta_\sigma e^{-f} - n |\nabla_\sigma e^{-f}|_\sigma^2 \right) d\sigma.$$

Moreover if  $\lim_{\tau \rightarrow \infty} Q(\tilde{\mathcal{M}}^\tau) \neq 0$ , then  $e^{2f} \sigma_{sR}$  - the limit of the rescaled metric on  $\tilde{\mathcal{M}}^\tau$  - does not have constant Webster curvature.

*Proof.* – First of all, note that, since we are considering only  $\mathbb{S}^1$ -invariant hypersurfaces, the contribution of the special direction is ruled out and then we can consider the usual Riemannian Laplacian, the gradient and the volume form associated to  $\sigma$  even if we know that the limit metric is sub-Riemannian. From the expression of the mean curvature of a star-shaped hypersurface (3.11) we have that

$$H - \hat{H} = -\frac{\varphi_{ij} \tilde{\sigma}^{ij}}{v \sinh(\rho)} + \hat{H} \left( \frac{1}{v} - 1 \right) = -\frac{\varphi_{ij} \tilde{\sigma}^{ij}}{v \sinh(\rho)} - \frac{\hat{H}}{v(v+1)} |\nabla_\sigma \varphi|_\sigma^2.$$

Since  $\tilde{\rho}_1 = f_1 = 0$  for every  $j$ , we can compute:

$$(9.2) \quad \left\{ \begin{aligned} v^2 &= 1 + \frac{1}{\sinh^2(\tilde{\rho})} |\nabla_\sigma \tilde{\rho}|_\sigma^2 \\ &= 1 + O(e^{-\tau}); \\ \hat{H} &= 2n + O(e^{-\tau}); \\ \tilde{\sigma}^{ij} &= \sigma^{ij} - \frac{\tilde{\rho}^i \tilde{\rho}^j}{v^2 \sinh^2(\tilde{\rho})} \\ &= \sigma^{ij} + O(e^{-\tau}); \\ \varphi_i &= \frac{\tilde{\rho}_i}{\sinh(\tilde{\rho})} = \frac{e^f}{\sinh(\tilde{\rho})} \nabla_i e^{-f}; \\ \varphi_{ij} &= \frac{1}{\sinh(\tilde{\rho})} \left( \tilde{\rho}_{ij} - \frac{\cosh(\tilde{\rho})}{\sinh(\tilde{\rho})} \tilde{\rho}_i \tilde{\rho}_j \right) \\ &= \frac{1}{\sinh(\tilde{\rho})} (f_{ij} - f_i f_j + o(1)) \\ &= -\frac{e^f}{\sinh(\tilde{\rho})} \left( \nabla_{ij}^2 e^{-f} + o(1) \right). \end{aligned} \right.$$

It follows that

$$\begin{aligned} \lim_{\tau \rightarrow \infty} Q(\tilde{\mathcal{M}}^\tau) &= \lim_{\tau \rightarrow \infty} \left[ \left( \int \sinh^{2n-1}(\tilde{\rho}) \cosh(\tilde{\rho}) d\mu_\tau \right)^{-1+\frac{1}{n}} \right. \\ &\quad \left. * \int \left( \sinh^{2n-1}(\tilde{\rho}) \cosh(\tilde{\rho}) \left( -\frac{\varphi_{ij} \tilde{\sigma}^{ij}}{v \sinh(\tilde{\rho})} - \frac{\hat{H}}{v(v+1)}(v^2-1) \right) \right) d\mu_\tau \right] \\ &= \left( \int_{\mathbb{S}^{2n-1}} e^{2nf} d\sigma \right)^{-1+\frac{1}{n}} \int_{\mathbb{S}^{2n-1}} e^{2nf} \left( e^{-f} \Delta_\sigma e^{-f} - n |\nabla_\sigma e^{-f}|_\sigma^2 \right) d\sigma. \end{aligned}$$

This formula shows that if  $\lim_{\tau \rightarrow \infty} Q(\tilde{\mathcal{M}}^\tau) \neq 0$ , then  $e^{-f} \Delta_\sigma e^{-f} - n |\nabla_\sigma e^{-f}|_\sigma^2 \neq 0$  and so  $f$  cannot be constant. Lemma 2.5 tells us that the limit metric  $e^{2f} \sigma_{sR}$  does not have constant Webster curvature. Finally note that the opposite is not true because  $e^{-f} \Delta_\sigma e^{-f} - n |\nabla_\sigma e^{-f}|_\sigma^2$  does not have necessarily a sign.  $\square$

If we compare  $Q$  with the modified Hawking mass studied for the real hyperbolic case in [9],  $Q$  has the disadvantage that it works only with  $\mathbb{S}^1$ -invariant data and it does not characterize the constant curvature limit. However Proposition 9.1 suggests that the study of the asymptotic behavior of  $Q$  is enough to find a family of initial data such that the limit of the rescaled metric does not have constant Webster curvature. In order to complete this goal we need to study the evolution equation of  $Q$ .

LEMMA 9.2. – For any star-shaped  $\mathcal{M}_0$  the following evolution equation holds:

$$\begin{aligned} \frac{\partial Q(\mathcal{M}_t)}{\partial t} &= \frac{1}{n} Q(\mathcal{M}_t) + |\mathcal{M}_t|^{-1+\frac{1}{n}} \int \left( \left( \frac{2n-1}{\sinh^2(\rho)} - \frac{1}{\cosh^2(\rho)} \right) \frac{v}{H} \right) d\mu \\ &\quad - |\mathcal{M}_t|^{-1+\frac{1}{n}} \int \left( \frac{1}{H} (|A|^2 - 2(n+1)) \right) d\mu. \end{aligned}$$

Proof. – Since  $\hat{H} = (2n-1) \frac{\cosh(\rho)}{\sinh(\rho)} + \frac{\sinh(\rho)}{\cosh(\rho)}$  and  $\frac{\partial \rho}{\partial t} = \frac{v}{H}$ , it follows easily that

$$\frac{\partial \hat{H}}{\partial t} = \frac{v}{H} \left( \frac{1}{\cosh^2(\rho)} - \frac{2n-1}{\sinh^2(\rho)} \right).$$

The thesis follows using this computation, the evolution of  $H$  in Lemma 2.6 and the fact that

$$\frac{\Delta H}{H^2} - 2 \frac{|\nabla H|^2}{H^3} = -\Delta \left( \frac{1}{H} \right),$$

hence its integral vanishes.  $\square$

Now we want to show that if  $Q$  decreases, it decreases very slowly.

**PROPOSITION 9.3.** – *Let  $\mathcal{M}_t$  an  $\mathbb{S}^1$ -invariant star-shaped hypersurface of  $\mathbb{C}\mathbb{H}^n$  evolving by inverse mean curvature flow. There is a positive constant  $c$  which depends only on  $n$  and  $\mathcal{M}_0$  such that*

$$\frac{\partial Q(\mathcal{M}_t)}{\partial t} \geq -ce^{-\frac{t}{n}}.$$

*Proof.* – By (3.9), (3.10) and (3.11) we can compute:

$$\begin{aligned} |A|^2 - 2(n+1) &= h_i^k h_k^i - 2(n+1) \\ &= \frac{\hat{\varphi}_{ij} \hat{\varphi}_{kh} \tilde{e}^{jk} \tilde{e}^{hi}}{v^2 \sinh^2(\rho)} - \frac{2 \cosh(\rho)}{v^2 \sinh^2(\rho)} \hat{\varphi}_{ij} \tilde{e}^{ji} - 2(n+1) \\ &\quad (2n-1) \frac{\cosh^2(\rho)}{v^2 \sinh^2(\rho)} + \frac{\sinh^2(\rho)}{v^2 \cosh^2(\rho)} + \frac{2}{v^2} \\ &= \frac{\hat{\varphi}_{ij} \hat{\varphi}_{kh} \tilde{e}^{jk} \tilde{e}^{hi}}{v^2 \sinh^2(\rho)} - 2 \frac{\cosh(\rho)}{v \sinh(\rho)} \left( H - \hat{H} + \hat{H} \left( 1 - \frac{1}{v} \right) \right) \\ &\quad + \frac{2n-1}{v^2 \sinh^2(\rho)} - \frac{1}{v^2 \cosh^2(\rho)} - 2(n+1) \left( 1 - \frac{1}{v^2} \right). \end{aligned}$$

Moreover, by Lemma 2.3

$$\hat{\varphi}_{ij} \hat{\varphi}_{kh} \tilde{e}^{jk} \tilde{e}^{hi} = \varphi_{ij} \varphi_{kh} \tilde{\sigma}^{jk} \tilde{\sigma}^{hi} + 2 \sinh^2(\rho) |\nabla_\sigma \varphi|_\sigma^2.$$

Then, by Lemma 9.2 we have:

$$\begin{aligned} |\mathcal{M}_t|^{1-\frac{1}{n}} \frac{\partial Q}{\partial t} &= \int \left( \frac{1}{n} - \frac{2 \cosh(\rho)}{vH \sinh(\rho)} \right) (H - \hat{H}) d\mu - \int \frac{\varphi_{ij} \varphi_{kh} \tilde{\sigma}^{jk} \tilde{\sigma}^{ih}}{v^2 H \sinh^2(\rho)} d\mu \\ &\quad + \int \frac{1}{H} \left( \frac{2n-1}{\sinh^2(\rho)} - \frac{1}{\cosh^2(\rho)} \right) \left( v - \frac{1}{v^2} \right) d\mu \\ &\quad + \int \frac{|\nabla_\sigma \varphi|_\sigma^2}{v^2 H} \left( 2n - \frac{2 \cosh(\rho)}{1+v \sinh(\rho)} \hat{H} \right) d\mu. \end{aligned}$$

By Proposition 6.3, Proposition 5.1, Proposition 5.7 and the fact that the radius grows like  $\frac{t}{2n}$ , we can estimate all the terms in the evolution of  $Q$  in the following way

$$\begin{aligned} \left| \frac{1}{n} - \frac{2 \cosh(\rho)}{vH \sinh(\rho)} \right| &\leq \left| \frac{1}{n} - \frac{2}{H} \right| + \frac{2}{H} \left| 1 - \frac{\cosh(\rho)}{\sinh(\rho)} \right| + \frac{2 \cosh(\rho)}{H \sinh(\rho)} \left| 1 - \frac{1}{v} \right| \\ &\leq ce^{-\frac{t}{n}}; \\ \left| H - \hat{H} \right| &\leq |H - 2n| + |2n - \hat{H}| \leq ce^{-\frac{t}{n}}; \end{aligned}$$

$$\left| \frac{\varphi_{ij}\varphi_{kh}\tilde{\sigma}^{jk}\tilde{\sigma}^{ih}}{v^2 H \sinh^2(\rho)} \right| \leq ce^{-\frac{2t}{n}};$$

$$1 - \frac{1}{v^2} \leq ce^{-\frac{t}{n}};$$

hence:

$$\left( \frac{2n-1}{\sinh^2(\rho)} - \frac{1}{\cosh^2(\rho)} \right) \left( v - \frac{1}{v^2} \right) \geq -\frac{1}{\cosh^2(\rho)} \frac{v^2+v+1}{v^2(v+1)} |\nabla_\sigma \varphi|_\sigma^2 \geq -ce^{-\frac{2t}{n}}.$$

Finally,

$$\left| 2n - \frac{2}{1+v} \frac{\cosh(\rho)}{\sinh(\rho)} \hat{H} \right| \leq \left| 2n - \hat{H} \right| + \hat{H} \left| \frac{\cosh(\rho)}{\sinh(\rho)} - 1 \right| + \hat{H} \frac{\cosh(\rho)}{\sinh(\rho)} \left| 1 - \frac{2}{1+v} \right| \leq ce^{-\frac{t}{n}}.$$

Therefore

$$\frac{\partial Q}{\partial t} \geq -c |\mathcal{M}_t|^{-1+\frac{1}{n}} \int e^{-\frac{2t}{n}} d\mu \geq -ce^{-\frac{t}{n}}. \quad \square$$

Now, following the strategy of [9] we can complete the proof of Theorem 1.1

**PROPOSITION 9.4.** – *There is an  $\mathcal{M}_0$  such that the rescaled induced metric  $\tilde{g}_\infty$  does not have constant Webster curvature.*

*Proof.* – Fix a positive constant  $c_0$  and let  $f : \mathbb{S}^{2n-1} \rightarrow \mathbb{R}$  be an  $\mathbb{S}^1$ -invariant function such that

$$\left( \int_{\mathbb{S}^{2n-1}} e^{2nf} d\sigma \right)^{-1+\frac{1}{n}} \int_{\mathbb{S}^{2n-1}} e^{2nf} \left( e^{-f} \Delta_\sigma e^{-f} - n |\nabla_\sigma e^{-f}|^2 \right) d\sigma \geq 4c_0.$$

Consider the family of  $\mathbb{S}^1$ -invariant star-shaped hypersurfaces  $\widetilde{\mathcal{M}}^\tau$  defined by the radial function  $\tilde{\rho}^\tau(z) = \tau + f(z)$ . We can fix a  $\tau$  big enough such that  $\widetilde{\mathcal{M}}^\tau$  is mean convex, and, by Proposition 9.1,  $Q(\widetilde{\mathcal{M}}^\tau) \geq 2c_0$ . Let  $\mathcal{M}_t^\tau$  be the evolution by inverse mean curvature flow of such  $\widetilde{\mathcal{M}}^\tau$ . We want to estimate the evolution of  $Q(\mathcal{M}_t^\tau)$ . The constant  $c$  appearing in Proposition 9.3 depends on  $n$  and the initial datum, hence on  $n, f$  and  $\tau$ . In view of (9.2), in our case it can be written as

$$c = \tilde{c}e^{-2\tau},$$

where  $\tilde{c}$  depends only on  $n$  and  $f$ . It follows that, up to increase even more the parameter  $\tau$ , Proposition 9.3 ensures that

$$\lim_{t \rightarrow \infty} Q(\mathcal{M}_t^\tau) \geq c_0 > 0.$$

The thesis follows from Proposition 9.1. □

We finish by noting that we can find an  $\mathbb{S}^1$ -invariant function  $f$  such that

$$\left( \int_{\mathbb{S}^{2n-1}} e^{2nf} d\sigma \right)^{-1+\frac{1}{n}} \int_{\mathbb{S}^{2n-1}} e^{2nf} \left( e^{-f} \Delta_\sigma e^{-f} - n |\nabla_\sigma e^{-f}|_\sigma^2 \right) d\sigma$$

is large as desired. We will show that such an example exists in  $\mathbb{S}^3$ , but an analogous construction holds also in higher dimension.

EXAMPLE 9.5. – Consider  $\mathbb{S}^3$  immersed in  $\mathbb{C}^2$ . Let  $(z_1, z_2)$  be its complex coordinates and  $\zeta = |z_2|^2 - |z_1|^2$ . Note that  $\zeta$  is an  $\mathbb{S}^1$ -invariant function on  $\mathbb{S}^3$ . For every  $k \in \mathbb{N}$ , let us define

$$f_k : (z_1, z_2) \in \mathbb{S}^3 \mapsto k\zeta \in \mathbb{R}.$$

Some explicit computations show that

$$e^{4f_k} \left( e^{-f_k} \Delta_\sigma e^{-f_k} - 2|\nabla_\sigma e^{-f_k}|_\sigma^2 \right) = 4ke^{2f_k} (k\zeta^2 + 2\zeta - k).$$

It follows that there exists a constant  $\gamma$  independent on  $k$  such that

$$\begin{aligned} Q_k &:= \left( \int_{\mathbb{S}^3} e^{4f_k} d\sigma \right)^{-\frac{1}{2}} \int_{\mathbb{S}^3} e^{4f_k} \left( e^{-f_k} \Delta_\sigma e^{-f_k} - 2|\nabla_\sigma e^{-f_k}|_\sigma^2 \right) d\sigma \\ &= \gamma k \left( \int_{-1}^1 \sqrt{1-\zeta^2} e^{4k\zeta} d\zeta \right)^{-\frac{1}{2}} \int_{-1}^1 \sqrt{1-\zeta^2} (k\zeta^2 + 2\zeta - k) e^{2k\zeta} d\zeta \\ &= \gamma k \left( \frac{\pi}{4k} I_1(4k) \right)^{-\frac{1}{2}} \frac{\pi}{4k} I_2(2k), \end{aligned}$$

where  $I_p(x)$  is the modified Bessel function of the first kind. As  $x$  goes to infinity we have the asymptotic expansion:

$$I_p(x) \sim \frac{e^x}{\sqrt{2\pi x}}.$$

It follows that

$$Q_k \sim \gamma' k^{\frac{1}{4}} \text{ as } k \rightarrow \infty,$$

for some constant  $\gamma'$  independent on  $k$ .

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