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A COMPRESSIBLE MULTIFLUID SYSTEM WITH NEW PHYSICAL RELAXATION TERMS

BY Didier BRESCH AND Matthieu HILLAIRET

ABSTRACT. – In this paper, we study the propagation of density-oscillations in solutions to density-dependent compressible Navier Stokes system. As a consequence to this analysis, we derive rigorously a generalization of the one-velocity Baer-Nunziato model for multifluid flows. The derived model includes a new relaxation term, in the PDE that governs the volume fraction of the component fluids, that encodes the change of viscosity and pressure between them.

RÉSUMÉ. – Dans cet article, nous étudions la propagation d'oscillations de densité dans les solutions des équations de Navier-Stokes compressibles fluides à viscosité variable. Nous appliquons cette analyse à la dérivation rigoureuse d'un système de type Baer-Nunziatio pour les écoulements multi-fluide. Le modèle obtenu inclut de nouveaux termes de relaxation dans les équations sur les fractions volumiques des composants du mélange. Ces termes résultent des différences entre les lois de viscosité et de pression dans les différents composants.

1. Introduction

This article is devoted to the rigorous derivation of a multifluid system with one velocity but considering the change of vicosity/pressure between the components. This model generalizes the Baer-Nunziato system with one velocity, already justified rigorously in [3]. Compared with this classical system, it introduces a mixture viscosity in the momentum equation and includes a new relaxation term in the PDEs satisfied by the volume fractions. These new terms highlight a non trivial interaction of the pressure and viscosity jumps in the behavior of the mixture.

If we look at physicists books such as those written by M. Ishii and T. Hibiki (see [9]) or by D. Drew and S.L. Passman (see [5]), we understand well that it is not so easy to choose the averaging process that has to be used to derive appropriate multifluid systems and that formal closure assumptions are often required in the approaches therein. Following [4, 8], we propose in [3] an alternative approach based on first principles exclusively. In this former reference, we consider a mixture in which bubbles of several different viscous compressible fluids coexist. Assuming a simplified behavior of interfaces (mainly a behavior that implies

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continuity of the mechanical quantities such as velocity and normal stresses through interfaces) and also that the different species share the same viscosity/isentropic pressure law, we introduce extended densities and velocities unifying then all the component equations into a single compressible Navier Stokes equation. The coexistence of several fluids in the mixture is then recovered by the fact that the extended density jumps between values inside the range of densities for the different component fluids. Hence, we derive our multifluid system by assuming that the initial density oscillates at small space scale and by computing the associate homogenized system. More precisely, we introduce Young measures to analyze the oscillations of the initial density and we assume that these Young measures concentrate in a finite number of Dirac masses (corresponding to the densities of the component fluids in the mixture). We derive the multifluid system by computing an equation for the concentration of the Dirac masses and their weights in the Young measures. One key difficulty in this method is to prove that finite convex combinations of Dirac masses are preserved through time-evolution by the equation satisfied by Young measures. An intermediate issue is to identify a functional framework in which the compressible Navier Stokes equations are well-posed (including uniqueness of solution) and that enables to consider discontinuous densities. The method we develop in [3, 4, 8] is closely-related to homogenization problem in compressible fluid mechanics. On this topic, the interested reader is also referred to [13, Section 7] where the kinetic equation formulation that we obtain in [3] is proposed in terms of the cumulative distribution function and without characterization of the Young measures which gives the multifluid systems.

In this paper, we extend the previous analysis to the compressible Navier-Stokes equations with density-dependent viscosity in the one-dimensional in space setting. We first study an appropriate notion of solution that encompasses discontinuous densities. We then tackle the homogenization problem for these solutions and the derivation of a multifluid system. From the modeling viewpoint, this density-dependent framework corresponds to a mixture with species having different viscosities. We believe this paper emphasizes the robustness of our method. In particular, we generalize here previous results initiated by A.A. Amosov and A.A. Zlotnik, see [1] and references cited therein, and also results by D. Serre in [14] related to the case with constant viscosity.

We give now a formal statement of our main result. Let assume that the three-dimensional container Ω contains the mixture of two viscous compressible phases described by triplets density/velocity/pressure (ρ_+, u_+, p_+) and (ρ_-, u_-, p_-) respectively. Introducing $(\mu_\pm, \lambda_\pm)$ and $p_\pm$ the respective viscosities and pressure laws of the phases, we obtain that, for $i = +, -$ the triplet is a solution to the compressible Navier Stokes equations

$$\partial_t \rho_i + \mathrm{div}(\rho_i u_i) = 0,$$
$$\partial_t(\rho_i u_i) + \mathrm{div}(\rho_i u_i \otimes u_i) = \mathrm{div}\Sigma_i,$$

on its domain $\mathcal{F}_i(t)$, with the equation of state:

$$\Sigma_i = \mu_i(\nabla u_i + \nabla^\top u_i) + (\lambda_i \mathrm{div} u_i - p_i)\mathbb{I}_3,$$
$$p_i = p_i(\rho_i).$$

Neglecting the properties of the interfaces, so that:

– $\mathcal{F}_+ \cup \mathcal{F}_- \cup (\overline{\mathcal{F}_+} \cap \overline{\mathcal{F}_-}) = \Omega,$

- the phases do not slip one on the other at the interface,
- we have continuity of the normal stresses at the interface,

we have that the extended unknowns

$$\rho = \rho_+ \mathbf{1}_{\mathscr{I}_+} + \rho_- \mathbf{1}_{\mathscr{I}_-}, \quad u = u_+ \mathbf{1}_{\mathscr{I}_+} + u_- \mathbf{1}_{\mathscr{I}_-},$$

satisfy the compressible Navier Stokes equations on the whole container Ω :

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = \operatorname{div}\Sigma. \tag{2}$$

Assuming further that the densities of the different phases range two non-overlapping closed intervals I_+ and I_-, we can complement this system by the equations of states:

$$\Sigma_i = 2\mu(\nabla u + \nabla^\top u) + (l \operatorname{div} u - p)\mathbb{I}_3, \tag{3}$$

$$p = p(\rho), \quad \mu = \mu(\rho), \quad l = l(\rho), \tag{4}$$

where the functions p, μ, l extend the values for the two components:

$$p(\rho) = p_i(\rho), \quad \mu(\rho) = \mu_i, \quad l(\rho) = \lambda_i, \quad \forall\, \rho \in I_i, \qquad i = +, -.$$

In this paper, we restrict to the one-space dimension setting namely:

$$\partial_t \rho + \partial_x(\rho u) = 0, \tag{5}$$

$$\partial_t \rho u + \partial_x(\rho u^2) = \partial_x(\mu(\rho)\partial_x u) - \partial_x p, \tag{6}$$

$$p = p(\rho), \quad \mu = \mu(\rho). \tag{7}$$

We aim to compute a homogenized system for configurations in which any time/space cell of arbitrary small size contains a fraction of phase + and a fraction of phase -. In the bifluid setting, a possible method is to look for two-scale solutions (a kind of WKB expansion) of the following form:

$$\rho(t,x) = \sum_{i=+,-} \theta_i\left(t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) \rho_i^\varepsilon(t,x), \tag{8}$$

$$u(t,x) = u_0\left(t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(t, \frac{t}{\varepsilon}, x, \frac{x}{\varepsilon}\right) + O(\varepsilon^3)\,. \tag{9}$$

Assuming further that

$$\rho_i^\varepsilon(t,x) = \rho_i^0(t,x) + O(\varepsilon), \quad \theta_i(t,\tau,x,y) \in \{0,1\} \quad \text{a.e.}, \tag{10}$$

one obtains (at a formal level) that $(\alpha_\pm, u_0, \rho^0_\pm, \overline{p})$ – where $\alpha_\pm$ denotes the average with respect to the fast variables (τ, y) of $\theta_\pm$ – satisfy the following system:

$$(11)\quad \alpha_+ + \alpha_- = 1,$$

$$(12)\quad \partial_t\alpha_+ + u_0\partial_x\alpha_+ = \frac{\alpha_+\alpha_-}{\alpha_-\mu^0_+ + \alpha_+\mu^0_-}[(p^0_+ - p^0_-) + (\mu^0_- - \mu^0_+)\partial_x u_0],$$

$$(13)\quad \partial_t(\alpha_+\rho^0_+) + \partial_x(\alpha_+\rho^0_+ u_0) = 0,$$

$$(14)\quad \overline{\rho}(\partial_t u_0 + u_0\partial_x u_0) - \partial_x(m\partial_x u_0) + \partial_x\pi = 0,$$

$$(15)\quad \mu^0_+ = \mu(\rho^0_+),\quad \mu^0_- = \mu(\rho^0_-),\quad m = \frac{\mu^0_+\mu^0_-}{\alpha_+\mu^0_- + \alpha_-\mu^0_+},$$

$$(16)\quad p^0_+ = p(\rho^0_+),\quad p^0_- = p(\rho^0_-),\quad \overline{\rho} = \alpha_+\rho^0_+ + \alpha_-\rho^0_-,\quad \pi = \frac{\alpha_+p^0_+\mu^0_- + \alpha_-p^0_-\mu^0_+}{\alpha_+\mu^0_- + \alpha_-\mu^0_+}.$$

We give more details on these computations in Section 3. We remark that, in this two-fluid setting, we obtain a new equation on the fraction α_+ in which the difference on the effective fluxes $F_\pm = -\mu^0_\pm\partial_x u_0 + p^0_\pm$ plays a crucial role. Namely, we obtain

$$\partial_t\alpha_+ + u_0\partial_x\alpha_+ = \frac{\alpha_+\alpha_-}{\alpha_-\mu^0_+ + \alpha_+\mu^0_-}[F_+ - F_-]$$

(we underline that $p^0_\pm$ and $\mu^0_\pm$ are defined by (16)–(15)). In the particular case $\mu(\rho) = \mu =$ cst, the system reduces to the one-dimensional system that has been formally derived by E. Weinan in [17] and rigorously justified by D. Serre in [14]. In that case, the PDE on α_+ simplifies into

$$\partial_t\alpha_+ + u_0\partial_x\alpha_+ = \frac{\alpha_+\alpha_-}{\mu}(p^0_+ - p^0_-).$$

Given these formal computations via a WKB-like analysis, one may conjecture that two-fluid expansions are transported by solutions to (5)–(6)-(7). More generally, in this paper, we consider initial data for (5)–(6)-(7) such that the initial density concentrates on k different values $\rho_1, \ldots \rho_k$ representing k different fluids with volume proportions $\alpha_1, \ldots, \alpha_k$ (meaning that a typical cell contains a proportion α_i of the fluid associated to density ρ_i). We show with our homogenization/Young measures approach that, for small times, the density of the solution (5)–(6)-(7) remains concentrated on k different values and we derive the following multifluid system:

$$(17)\qquad \partial_t\alpha_i + \partial_x(\alpha_i u) = \frac{\alpha_i}{\mu(\rho_i)}f_i,$$

$$(18)\qquad \partial_t\rho_i + u\partial_x\rho_i = -\frac{\rho_i}{\mu(\rho_i)}f_i,$$

$$(19)\qquad \partial_t(\rho u) + \partial_x(\rho u^2) = \partial_x[\mu\partial_x u - p],$$

for $1 \leq i \leq k$ with

$$f_i = \frac{1}{\left[\displaystyle\sum_{j=1}^{k} \frac{\alpha_j}{\mu(\rho_j)}\right]} \left(\partial_x u - \sum_{j=1}^{k} \alpha_j \frac{p(\rho_j)}{\mu(\rho_j)}\right) + p(\rho_i), \quad \text{for } 1 \leq i \leq k,$$

$$\mu = \frac{1}{\displaystyle\sum_{i=1}^{k} \frac{\alpha_i}{\mu(\rho_i)}}, \quad p = \frac{\displaystyle\sum_{i=1}^{k} \frac{\alpha_i p(\rho_i)}{\mu(\rho_i)}}{\displaystyle\sum_{i=1}^{k} \frac{\alpha_i}{\mu(\rho_i)}}.$$

As in the constant viscosity cases this results contains three main difficulties:

– constructing solutions to (5)–(6)-(7) in a framework where uniqueness holds and that enables to consider discontinuous densities,
– computing the homogenized system of (5)–(6)-(7) in terms of Young measures,
– showing that convex combinations of Dirac measures are preserved by time-evolution in solutions to the homogenized system of (5)–(6)-(7).

The outline of the paper is as follows. In the next section, we give rigorous statements of our main results. Then, we give more details on the formal derivation for two-fluid flows via the formal WKB-like analysis mentioned above. In the fourth section, we give the proof of our main results: construction of solutions to (5)–(6)-(7) and computation of the homogenized system by a compactness argument. This homogenized system mixes a momentum equation like (6) with a transport equation satisfied by Young measures. By a weak-strong uniqueness argument, we show that solutions to this transport equation remain concentrated on a finite number of densities if this is the case initially. This last step relies on a fine analysis of measure solutions to some transport equation having a nice structure. For completeness, we detail this analysis in a last independent section. We compare also in an appendix the results that are obtained formally and the one obtained through Young-measures characterization in the bifluid setting.

2. Main results.

In this section, we give more details concerning the assumptions on the equations of state in the system under consideration, the definition of Young measures and the rigorous statements of our main results.

As mentioned in the introduction, we focus on the following Navier-Stokes system with density-dependent viscosity:

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t \rho u + \partial_x(\rho u^2) = \partial_x[\mu \partial_x u] - \partial_x p, \end{array}\right. \quad \text{on } (0, L), \tag{20}$$

completed with the equations of state:

$$p = p(\rho), \quad \mu = \mu(\rho), \tag{21}$$

where p and μ are given and sufficiently smooth: we assume throughout the paper that

$$p \in C^1([0,\infty)), \quad \text{with} \quad p'(s) \geq 0, \qquad \forall\, s \in [0,\infty), \tag{22}$$

$$\mu \in C^1([0,\infty)), \quad \text{with} \quad \mu(s) \geq \mu^0(1+\sqrt{s}), \qquad \forall\, s \in [0,\infty). \tag{23}$$

Here μ^0 is a given strictly positive constant. The method we develop enables to consider pressure laws p that are increasing for large values of s only:

$$p \in C^1([0,\infty)), \quad \text{with} \quad p'(s) \geq 0, \qquad \forall\, s \gg 1, \tag{24}$$

but we give proofs and state our main results with (22) for simplicity. We complement the above PDEs with periodic boundary conditions in x and initial conditions:

$$\rho(0,x) = \rho^0(x), \quad u(0,x) = u^0(x)\,. \tag{25}$$

2.1. Notations and conventions

Throughout the paper, we denote with $\natural$ periodic-function spaces. For instance $L^\infty_\natural$ is the set of bounded L-periodic functions on $\mathbb{R}$. For a Banach space X such as L^p or H^m, we endow $X_\natural$ with the norm:

$$\|u\|_{X_\natural} = \|u_{|(0,L)}\|_{X(0,L)}.$$

We recall that $X_\natural$ is a Banach space endowed with this norm and that a sequence u_n converges toward u in $X_\natural$ for the strong topology (resp. for the weak or the weak-$*$ topology) if and only if u_n converges toward u in $X(-M,M)$ for the strong topology (resp. for the weak or the weak-$*$ topology), whatever the value of $M \in L\mathbb{N}^*$. In particular, if u_n converges toward u in X (endowed with the weak/weak-$*$ topology) then u_n converges toward u in $\mathscr{D}'(\mathbb{R})$.

A key-argument in computing homogenized systems for compressible Navier-Stokes equations is to introduce a tool for describing the lack of (strong-)compactness of a weakly-converging sequence of densities $(\rho_n)_{n\in\mathbb{N}}$. We apply herein the so-called method of Young measures in the spirit of [16]: we identify the density $\rho \in L^\infty_\natural$ with the positive Radon measure $\nu := \nu_\rho$ on $\mathbb{R}\times\mathbb{R}^+$ as defined by:

$$\langle \nu_\rho, B\rangle = \int_{\mathbb{R}} B(x,\rho(x))\mathrm{d}x, \quad \forall\, B \in C_c(\mathbb{R}\times\mathbb{R}^+).$$

As ρ is L-periodic, this measure satisfies:

$$\langle \nu, \tau_k B\rangle = \langle \nu, B\rangle \quad \forall\, B \in C_c(\mathbb{R}\times\mathbb{R}^+), \quad \forall\, k \in \mathbb{Z}, \tag{26}$$

where, given $B \in C_c(\mathbb{R}\times\mathbb{R}^+)$ and $k \in \mathbb{Z}$, we define $\tau_k B(x,\xi) := B(x-kL,\xi)$ for all $(x,\xi) \in \mathbb{R}\times\mathbb{R}^+$. Thus, we introduce the following definition:

Definition 1. – *We call L-periodic Young-measure (or Young-measure for short) any positive Radon measure ν on $\mathbb{R}\times\mathbb{R}^+$ satisfying* (26)*. We denote $\mathscr{Y}_\natural$ the set of L-periodic Young measures.*

The space $\mathscr{Y}_\natural$ is endowed with the topology of the weak convergence, meaning the coarser topology that makes the family of linear mapping $\{\langle\cdot,B\rangle;\ B \in C_c(\mathbb{R}\times\mathbb{R}^+)\}$ continuous.

The fundamental remark underlying the above identification is that, given a bounded sequence $(\rho_n)_{n\in\mathbb{N}} \in L^\infty_\natural$, say $\|\rho_n\|_{L^\infty_\natural} \leq 2C_0$ for all n, we may apply abstract functional analysis arguments (see [16, Theorem 1]) and prove that, up to an extraction that we do not relabel, the sequence $\nu_n = \nu_{\rho_n}$ converges weakly to some Young measure ν. In particular, below we say that a bounded sequence $(\rho_n)_{n\in\mathbb{N}}$ converges to ν in the sense of Young measures if the associated sequence of positive Radon measures $(\nu_{\rho_n})_{n\in\mathbb{N}}$ converges weakly to ν. We emphasize that, as in [16, Corollary 1], a Young measure could be seen equivalently as a L-periodic measurable mapping $\mathbb{R} \to P(\mathbb{R}^+)$ (where $P(\mathbb{R}^+)$ stands for the probability measures on $\mathbb{R}^+$). However, we stick to the point of view of a measure on the product space $\mathbb{R}\times\mathbb{R}^+$ in this paper. This shall enable to compute partial differential equations satisfied by the Young measures.

2.2. Main results

Our first target result is the following theorem:

THEOREM 2. – *Given $\rho^0 \in L^\infty_\natural$ and $u^0 \in H^1_\natural$ satisfying*

$$\underline{\rho^0} := \inf \rho^0(x) > 0, \quad \overline{\rho^0} = \sup \rho^0(x) < \infty, \tag{27}$$

there exists $T_0 > 0$ depending on $\underline{\rho^0}, \overline{\rho^0}, \|u^0\|_{H^1_\natural}$ such that there exists at least one pair (ρ, u) for which:

(HDS)$_a$ *we have the regularity statements:*

$$\rho \in L^\infty((0,T_0); L^\infty_\natural) \cap C([0,T_0]; L^1_\natural), \tag{28}$$

$$u \in L^\infty((0,T_0); H^1_\natural) \cap C([0,T_0]; L^2_\natural), \tag{29}$$

$$z := \mu(\rho)\partial_x u - p(\rho) \in L^2((0,T_0); H^1_\natural); \tag{30}$$

(HDS)$_b$ (ρ,u) *satisfies* (20) *in* $\mathscr{D}'((0,T)\times\mathbb{R})$, *with* p,μ *given by* (21), *and matches initial conditions* (25) *in* $L^2_\natural \times H^1_\natural$*;*

(HDS)$_c$ *we have the following bounds :*

- *for a.e.* $(t,x) \in (0,T_0)\times\mathbb{R}$ *there holds:*

$$\frac{1}{2}\underline{\rho^0} \le \rho(t,x) \le 2\overline{\rho^0}, \tag{31}$$

- *for a.e.* $t \in (0,T_0)$ *there holds (see* (63) *for the definition of* q*):*

$$\int_0^L \left[\frac{1}{2}|u(t,\cdot)|^2 + q(\rho(t,\cdot))\right] + \int_0^t \int_0^L \mu|\partial_x u|^2 \le \int_0^L \left[\frac{1}{2}|u^0|^2 + q(\rho^0)\right], \tag{32}$$

- *there exists a constant K_0 depending only on $\underline{\rho^0}, \overline{\rho^0}$ and $\|u^0\|_{H^1_\natural}$ for which*

$$\sup_{t\in(0,T_0)} \|u(t,\cdot)\|^2_{H^1_\natural} + \int_0^{T_0} \|\partial_x z(t,\cdot)\|^2_{L^2_\natural} \le K_0. \tag{33}$$

We call HD solutions to (20)–(21) the solutions obtained by applying Theorem 2 (after D. Hoff and B. Desjardins who constructed independently such solutions for the constant viscosity case). The scheme of our proof follows classical lines but we write details for reader's convenience:

- first, we obtain classical solutions to a regularized version of our system using the BD entropy procedure. This procedure in one-D is well known since the work in 1968 by Y. Kanel in [10]. In our case, "regularized" only means that we assume the initial data to satisfy further $\rho^0 \in H^1_\natural$;

– second, we prove that the strong solutions are HD-solutions on some time-interval $(0, T_0)$ where T_0 depends only on $\underline{\rho^0}, \overline{\rho^0}, \|u^0\|_{H^1_\natural}$;
– third, we apply a compactness argument showing that a sequence of solutions to the regularized system converges to the solution whose existence is claimed in our theorem. These HD solutions provide us with the solutions on which we justify the homogenized procedure through Young measures.

Enlarging the range of the compactness argument, we also obtain the main result of this paper, namely, the mathematical justification of a generalization of the one-velocity Baer-Nunziato model. More precisely we obtain the following mathematical result:

THEOREM 3. – *Let $T_0 > 0$ and $(\rho_n, u_n)_{n\in\mathbb{N}}$ be a sequence of solutions to* (20)–(21) *on* $(0, T_0)$, *in the sense of Theorem 2, with respective initial data $\rho_n^0 \in L^\infty_\natural$ and $u_n^0 \in H^1_\natural$. Assume that the sequence of initial data satisfies*

- $u_n^0 \rightharpoonup u^0$ *in* $H^1_\natural - w$,
- *there exists a constant $C_0 > 0$ such that $1/C_0 \leq \rho_n^0 \leq C_0$ uniformly,*
- *there exists $(\alpha_i^0, \rho_i^0)_{i=1,\dots,k} \in [L^\infty_\natural]^{2k}$ such that ρ_n^0 converges in the sense of Young measures to ν^0 as defined by:*

$$\langle \nu^0, B\rangle = \int_{\mathbb{R}} \sum_{i=1}^{k} \alpha_i^0(x) B(x, \rho_i^0(x))\mathrm{d}x, \quad \forall\, B \in C_c(\mathbb{R}\times\mathbb{R}^+).$$

Then there exists

$$(\alpha_1, \dots, \alpha_k) \in L^\infty((0, T_0); L^\infty_\natural) \cap C([0, T_0]; L^1_\natural) \text{ with} \tag{34}$$

$$\alpha_i \geq 0, \quad \forall i \in \{1, \dots, k\}, \quad \sum_{i=1}^{k} \alpha_i = 1, \quad a.e.$$

$$(\rho_1, \dots, \rho_k) \in L^\infty((0, T_0); L^\infty_\natural) \cap C([0, T_0]; L^1_\natural) \text{ with} \tag{35}$$

$$C_0/2 \leq \rho_i \leq 2C_0 \text{ a.e. }, \quad \forall i \in \{1, \dots, k\},$$

$$u \in L^\infty((0, T_0); H^1_\natural) \cap C([0, T_0]; H^1_\natural - w); \tag{36}$$

such that, up to the extraction of a subsequence:

- u^n *converges to* u *in* $C([0, T]; L^2_\natural)$,
- $\rho^n(t, \cdot)$ *converges to* ν_t *in the sense of Young measures for a.e.* $t \in (0, T)$, *where:*

$$\langle \nu_t, B\rangle = \int_{\mathbb{R}} \sum_{i=1}^{k} \alpha_i(t, x) B(x, \rho_i(t, x))\mathrm{d}x, \quad \forall\, B \in C_c(\mathbb{R}\times\mathbb{R}^+).$$

Moreover, $((\alpha_i, \rho_i)_{i=1,\dots,k}, u)$ is a L-periodic solution on $(0, T)$ to:

$$\partial_t \alpha_i + \partial_x(\alpha_i u) = \frac{\alpha_i}{\mu(\rho_i)} f_i, \tag{37}$$

$$\partial_t \rho_i + u\partial_x \rho_i = -\frac{\rho_i}{\mu(\rho_i)} f_i, \tag{38}$$

$$\partial_t(\rho u) + \partial_x(\rho u^2) = \partial_x[m\partial_x u - \pi], \tag{39}$$

where :

$$\rho = \sum_{j=1}^{k} \alpha_j \rho_j, \quad m = \left[\sum_{j=1}^{k} \frac{\alpha_j}{\mu(\rho_j)} \right]^{-1}, \quad \pi = m \sum_{j=1}^{k} \alpha_j \frac{p(\rho_j)}{\mu(\rho_j)}. \tag{40}$$

and

$$f_i = \frac{1}{\left[\sum_{j=1}^{k} \frac{\alpha_j}{\mu(\rho_j)} \right]} \left(\partial_x u - \sum_{j=1}^{k} \alpha_j \frac{p(\rho_j)}{\mu(\rho_j)} \right) + p(\rho_i), \tag{41}$$

completed with initial conditions

$$\alpha_i(0,\cdot) = \alpha_i^0 \ in \ L^1_\natural, \tag{42}$$

$$\rho_i(0,\cdot) = \rho_i^0 \ in \ L^1_\natural, \tag{43}$$

$$u(0,\cdot) = u^0 \ in \ H^1_\natural. \tag{44}$$

3. Formal WKB-like analysis for bifluid flows.

In this part, we show how to get the bifluid system using a formal WKB decomposition. We refer the interested reader to [17] and [15] that contain formal discussions in the heat-conducting case or with non-monotone pressure laws.

We assume throughout this section that (ρ, u) is a solution to the compressible Navier Stokes system (5)–(6)-(7) given by the expansion (8)–(9) in which (10) is satisfied.

3.1. General remarks

Let us pick $\beta \in C^1(\mathbb{R})$ and multiply formally the continuity equation by β'. We obtain the classical equation

$$\partial_t \beta(\rho) + \partial_x(\beta(\rho)u) + (\rho\beta'(\rho) - \beta(\rho))\partial_x u = 0.$$

Restricting to functions β satisfying

$$\beta = 1 \text{ on the support of } \rho_+^0, \quad \beta = 0 \text{ on the support of } \rho_-^0,$$

we derive the supplementary equation

$$\partial_t \theta_+ + u\partial_x \theta_+ = 0. \tag{45}$$

Then, we can decompose the derivatives in terms of the slow variables (t, x) and fast variables (τ, y) of θ_+. We obtain two equations when we consider terms which are $O(1/\varepsilon)$ or terms which are $O(1)$:

$$\partial_\tau \theta_+ + u_0 \partial_y \theta_+ = 0 \tag{46}$$

$$\partial_t \theta_+ + u\partial_x \theta_+ = -\frac{(u - u_0)}{\varepsilon} \partial_y \theta_+. \tag{47}$$

The first equation provides the behavior of θ_+ on a cell. This equation is consistent with the assumption that θ_+ is an indicator function. Averaging the second equation, with respect to the fast variable, we derive the following PDE on the averaged quantity $\alpha_+ = \overline{\theta_+}$

$$\partial_t \alpha_+ + \overline{u \partial_x \theta_+} = -\overline{\frac{(u-u_0)}{\varepsilon} \partial_y \theta_+}. \tag{48}$$

We denote temporarily with bars averages on a cell. As this lightens notations a lot, we keep this convention throughout this section only. However, it must not be confused with lower and upper bounds for densities as it has been used in the statement of our Theorem 2 and as it will be in the next section. Remark that there is no vacuum in the mixture so that $\theta_+ + \theta_- = 1$ a.e.. Consequently, we have:

$$\alpha_+ + \alpha_- = 1.$$

Choosing then β such that:

$$\beta = \mathrm{Id} \text{ on the support of } \rho_+^0, \quad \beta = 0 \text{ on the support of } \rho_-^0,$$

we obtain that

$$\partial_t(\theta_+ \rho_+^0) + \partial_x(\theta_+ \rho_+^0 u) = 0.$$

Keeping only the first order terms in the expansion of u and averaging with respect to fast variables, we obtain then:

$$\partial_t(\alpha_+ \rho_+^0) + \partial_x(\rho_+^0 \overline{\theta_+ u_0}) = 0. \tag{49}$$

Now the main objective is to calculate the averaged terms in (48)–(49) and to complete the system with an evolution equation for u_0. To proceed, we need to work on the momentum equation. We distinguish then two cases: the constant viscosity case which yields the system which has been justified recently in [14] (generalized to the multi-dimensional case in [3]) and the density-dependent viscosity case which yields the homogenized system under consideration in this paper.

3.2. The constant viscosity case

Plugging the expansion of u in (48), we obtain at first order in ε:

$$\partial_t \alpha_+ + \overline{u_0 \partial_x \theta_+} = -\overline{u_1 \partial_y \theta_+}. \tag{50}$$

We recall now briefly the different steps that yield the limit system (11)–(15). We interprete the divergent terms (in ε) of the momentum equation. This yields the following cascade of equations:

Order ε^{-2}. – We obtain:

$$\partial_{yy} u_0 = 0.$$

The velocity field u_0 is therefore independent of the fast variable y.

Order ε^{-1}. – We obtain:

$$\rho^0 \left(\partial_\tau u_0 + u_0 \partial_y u_0\right) = 2\mu \partial_{xy} u_0 + \mu \partial_{yy} u_1 - \partial_y p^0, \tag{51}$$

(where we denote $\rho^0 = \theta_+ \rho^0_+ + \theta_- \rho^0_-$ for the density and $p^0 = \theta_+ p^0_+ + \theta_- p^0_-$ for the pressure). As u_0 does not depend on y (and ρ^0 remains far from 0), multiplying this equation by $\partial_\tau u_0$ and integrating on a cell, we obtain $\partial_\tau u_0 = 0$. Therefore, u_0 does not depend on both fast variables. In particular

$$\overline{u_0 \partial_x \theta_+} = u_0 \partial_x \alpha_+, \quad \overline{\theta_+ u_0} = \alpha_+ u_0,$$

and (49) rewrites:

$$\partial_t(\alpha_+ \rho^0_+) + \partial_x(\alpha_+ \rho^0_+ u_0) = 0. \tag{52}$$

Applying then that u_0 does not depend on the fast variables, (51) entails that (we assume here that μ is constant):

$$\mu \partial_{yy} u_1 - \partial_y p^0 = 0, \quad \text{and then,} \quad \mu \partial_y u_1 = p^0 - \overline{p^0}.$$

Multiplying this identity by θ_+ and averaging (we recall that $p^0_+ = p(\rho^0_+)$ and $p^0_- = p(\rho^0_-)$ do not depend of the fast variable) yields:

$$-\overline{u_1 \partial_y \theta_+} = \overline{\theta_+ \partial_y u_1} = \frac{1}{\mu}\overline{\theta_+(p^0 - \overline{p^0})} = \frac{\alpha_+ \alpha_-}{\mu}(p^0_+ - p^0_-).$$

Finally, we obtain the expected equation for the volume fraction:

$$\partial_t \alpha_+ + u_0 \partial_x \theta_+ = \frac{\alpha_+ \alpha_-}{\mu}(p^0_+ - p^0_-). \tag{53}$$

Order ε^0. – In the momentum equation, we have now:

$$\rho^0 \partial_t u_0 + \rho^0 u_0 \partial_x u_0 + \rho^0 \left(\partial_\tau u_1 + u_0 \partial_y u_1\right) = \partial_x \Sigma_0 + \partial_y \Sigma_1, \tag{54}$$

where

$$\Sigma_0 = \mu(\partial_x u_0 + \partial_y u_1) - \sum_{i=+,-} \theta_i \, p(\rho^0_i).$$

On the left-hand side, we recall that

$$\partial_y u_1 = \frac{1}{\mu}\left(\sum_{i=+,-} \theta_i p^0_i - \overline{p}^0\right)$$

and, in terms of the fast variables (τ, y), $\partial_y u_1$ is thus a linear function of $\theta_\pm$ only so that (46) induces that

$$\partial_y(\partial_\tau u_1 + u_0 \partial_y u_1) = 0$$

and consequently (because $\partial_\tau u_1 + u_0 \partial_y u_1$ has average 0 on a cell):

$$\partial_\tau u_1 + u_0 \partial_y u_1 = 0.$$

Taking the average of (54) w.r.t. fast variables, we obtain finally:

$$\overline{\rho} \partial_t u_0 + \overline{\rho} u_0 \partial_x u_0 = \partial_x \overline{\Sigma}_0$$

with

$$\overline{\rho} = \alpha_+ \rho^0_+ + \alpha_- \rho^0_-, \quad \overline{\Sigma}_0 = \mu \partial_x u_0 - \sum_{i=+,-} \alpha_i p^0_i.$$

Combining with (52)–(53), this completes the justification of the bifluid system (11)–(15) in the constant-viscosity case.

3.3. The density-dependent viscosity case

In this second case, we go back to the relation

$$\partial_t \theta_+ + u_0 \partial_x \theta_+ = -u_1 \partial_y \theta_+, \tag{55}$$

that we want to average. We write again the different scales of the momentum equation. We recall that we assume density-dependent viscosity $\mu = \mu(\rho)$. Therefore we can write

$$\mu = \theta_+ \mu_+^\varepsilon + \theta_- \mu_-^\varepsilon,$$

where we assume at first order that $\mu_\pm^\varepsilon \sim \mu_\pm^0$ which does not depend on the fast variables.

Order ε^{-2}. – We have

$$\partial_y [\mu \partial_y u_0] = 0.$$

This implies that

$$\mu \partial_y u_0 = K_0.$$

To determine K_0 we use the equation at order ε^{-1} for θ_+ (and θ_-) (46) that we multiply by μ_+^ε (et μ_-^ε respectively). After combinations, we obtain:

$$\partial_\tau \mu + u_0 \partial_y \mu = 0.$$

Averaging the equation on a cell yields that

$$0 = \overline{u_0 \partial_y \mu} = -\overline{\mu \partial_y u_0} = K_0.$$

Finally, we infer that $\partial_y u_0 = 0$ and therefore u_0 does not depend of the space fast variables.

Order ε^{-1}. – With the same arguments as previously, we compute:

$$\rho^0 \partial_\tau u_0 = \partial_y [\mu^0 (\partial_y u_1 + \partial_x u_0)] - \partial_y p^0.$$

Because $\partial_\tau u_0$ is constant, this yields $\partial_\tau u_0 = 0$ after multiplication by $\partial_\tau u_0$ and integration on a cell. Hence, u_0 does not depend on both fast variables again and we obtain (52). The above equation then reduces to:

$$0 = \partial_y [\mu^0 (\partial_y u_1 + \partial_x u_0)] - \partial_y p^0.$$

This entails:

$$\mu^0 \partial_y u_1 + (\mu^0 - \overline{\mu^0}) \partial_x u_0 - (p^0 - \overline{p}^0) = K_1. \tag{56}$$

With this writing of (56), we have $K_1 = \overline{\mu^0 \partial_y u_1}$. To calculate K_1, we proceed as previously: we multiply (55) by μ_+^0 (and its equivalent for θ_- by μ_-^0). After combinations, this entails:

$$\partial_t \mu^0 + u_0 \partial_x \mu^0 + u_1 \partial_y \mu^0 = \sum_{i=\pm} \theta_i \left(\partial_t + u_0 \partial_x\right) \mu_i^0.$$

Averaging with respect to the fast variable, we obtain:

$$\begin{aligned}\overline{u_1 \partial_y \mu^0} &= \sum_{i=\pm} \alpha_i \,(\partial_t + u_0\partial_x)\, \mu_i^0 - (\partial_t + u_0\partial_x)\, \overline{\mu^0}\\ &= -\sum_{i=\pm} \mu_i^0 \,(\partial_t + u_0\partial_x)\, \alpha_i\\ &= (\mu_-^0 - \mu_+^0)\,(\partial_t + u_0\partial_x)\,\alpha_+,\end{aligned}$$

and

$$K_1 = \overline{\mu^0 \partial_y u_1} = (\mu_+^0 - \mu_-^0)\,(\partial_t + u_0\partial_x)\,\alpha_+. \tag{57}$$

We proceed with computing $\overline{u_1 \partial_y \theta_+}$. By multiplying (56) with θ_+, we obtain:

$$\begin{aligned}-\overline{u_1 \partial_y \theta_+} &= \overline{\theta_+ \partial_y u_1}\\ &= \overline{\frac{\theta_+}{\mu^0}(p^0 - \overline{p^0}) - \frac{\theta_+}{\mu^0}(\mu^0 - \overline{\mu^0})\partial_x u_0 + \frac{\theta_+}{\mu^0} K_1}.\end{aligned}$$

On the right-hand side, we have:

$$\begin{aligned}\overline{\frac{\theta_+}{\mu^0}(p^0 - \bar{p}^0)} &= \frac{\alpha_+}{\mu_+^0} p_+^0 - \frac{\alpha_+}{\mu_+^0}\left(\alpha_+ p_+^0 + \alpha_- p_-^0\right)\\ &= \frac{\alpha_+\alpha_-}{\mu_+^0}(p_+^0 - p_-^0),\end{aligned}$$

and also:

$$\overline{\frac{\theta_+}{\mu^0} K_1} = \frac{\alpha_+}{\mu_+^0}(\mu_+^0 - \mu_-^0)\,(\partial_t + u_0\partial_x)\,\alpha_+,$$

and finally :

$$\overline{\frac{\theta_+}{\mu^0}(\mu^0 - \bar{\mu}^0)\partial_x u_0} = \partial_x u_0 \frac{\alpha_+\alpha_-}{\mu_+^0}(\mu_+^0 - \mu_-^0).$$

Combining these identities yields:

$$-\overline{u_1 \partial_y \theta_+} = \frac{\alpha_+\alpha_-}{\mu_+^0}\left((p_+^0 - p_-^0) - \partial_x u_0(\mu_+^0 - \mu_-^0)\right) + \alpha_+ \left(1 - \frac{\mu_-^0}{\mu_+^0}\right)(\partial_t + u_0\partial_x)\,\alpha_+.$$

Therefore we derive the following equation on α_+ :

$$\left(1 + \alpha_+ \left(\frac{\mu_-^0}{\mu_+^0} - 1\right)\right)(\partial_t \alpha_+ + u_0 \partial_x \alpha_+) = \frac{\alpha_+\alpha_-}{\mu_+^0}\left((p_+^0 - p_-^0) - \partial_x u_0(\mu_+^0 - \mu_-^0)\right), \tag{58}$$

which may be rewritten as:

$$\partial_t \alpha_+ + u_0 \partial_x \alpha_+ = \frac{\alpha_+\alpha_-}{\alpha_+\mu_-^0 + \alpha_-\mu_+^0}\left((p_+^0 - p_-^0) - \partial_x u_0(\mu_+^0 - \mu_-^0)\right). \tag{59}$$

To compute an evolution equation for u^0, we write the ε^0 order of the momentum equation as in the previous case. We remark again that, thanks to (56), the quantity $\partial_y u_1$ depends on the fast variable only through $\theta_\pm$ so that after averaging, we obtain:

$$\overline{\rho}\partial_t u_0 + \overline{\rho} u_0 \partial_x u_0 = \partial_x \overline{\Sigma}_0,$$

with

$$\overline{\rho} = \alpha_+ \rho^0_+ + \alpha_- \rho^0_-, \quad \overline{\Sigma}_0 = \mu^0 \partial_x u_0 - \sum_{i=+,-} \alpha_i p^0_i + \overline{\mu^0 \partial_y u_1}.$$

Combining (57) and (59), we have:

$$\overline{\mu^0 \partial_y u_1} = \frac{\alpha_+ \alpha_- (\mu^0_+ - \mu^0_-)}{\alpha_+ \mu^0_- + \alpha_- \mu^0_+} \left((p^0_+ - p^0_-) - \partial_x u_0 (\mu^0_+ - \mu^0_-) \right),$$

so that, after tedious but straightforward algebraic combinations (using many times that $\alpha_+ + \alpha_- = 1$), we obtain:

$$\overline{\Sigma}_0 = \frac{\mu^0_+ \mu^0_-}{\alpha_+ \mu^0_- + \alpha_- \mu^0_+} \partial_x u_0 - \frac{\alpha_+ p^0_+ \mu^0_- + \alpha_- p^0_- \mu^0_+}{\alpha_+ \mu^0_- + \alpha_- \mu^0_+}.$$

This completes the justification of the bifluid system (11)–(15) mentioned in the introduction.

4. Proofs of main results.

In this section, we complete the proofs of our main results Theorem 2 and Theorem 3. To obtain Theorem 2, we first construct a Cauchy theory for classical solutions to density-dependent compressible Navier-Stokes equations. This existence result is quite similar to previous analysis on the whole line [12, 7]. Hence, we shall only recall the main energy estimates underlying the proofs and skip technicalities (see Section 4.1). Then, we remark that the classical solutions are HD-solutions on a small time interval. This second argument reduces again to a priori estimates (see Section 4.2). Finally, to extend the Cauchy theory to discontinuous initial densities, we approximate the target initial data with a sequence of smooth initial data. The previous construction yields an associated sequence of HD-solutions. It remains then to prove that this sequence of HD-solutions converges to a HD-solution matching the target initial data. It turns out that this compactness argument is a particular case of Theorem 3 (when $k = 1$). So, the end of the proof is embedded in the proof of Theorem 3 (see Section 4.3).

4.1. Strong solution theory

By adapting the arguments of [12] to our periodic framework, we have the following existence theorem:

THEOREM 4. – *Given* $\rho^0 \in H^1_\natural$ *and* $u^0 \in H^1_\natural$ *satisfying*

$$\underline{\rho^0} := \inf \rho^0(x) > 0,$$

there exists a unique pair (ρ, u) *such that:*

(CS)$_a$ *we have the regularity statement*

$$\rho \in C([0,\infty); H^1_\natural) \text{ with } \rho > 0, \tag{60}$$

$$u \in C([0,\infty); H^1_\natural) \cap L^2_{loc}((0,\infty); H^2_\natural)\,; \tag{61}$$

(CS)$_b$ (ρ, u) *satisfies* (20) *a.e. in* $(0,\infty) \times \mathbb{R}$ *with* p, μ *given by* (21) *;*

(CS)$_c$ (ρ, u) *matches initial conditions* (25) *a.e..*

Proof. – This theorem is similar to previous results on the whole line [12, 7]. We sketch the main steps of the proof for completeness and recall the energy estimates that underly the construction of solutions.

Local existence of solutions is obtained by a classical fixed-point argument so that the only difficulty lies in proving these solutions are global. As the local-in-time theory yields a time of existence depending only on

$$\mathcal{E}(0) := \underline{\rho^0} + \|\rho^0\|_{H^1_\natural} + \|u^0\|_{H^1_\natural},$$

we aim to obtain a local-in-time uniform bound on $\mathcal{E}(t)$ for the associated solution (ρ, u). This solution is defined a priori on a non-extendable time interval $[0, T_*)$.

Step 1. Dissipation of energy. – First, with classical arguments, we obtain

$$\int_0^L \left[\frac{\rho(t,x)|u(t,x)|^2}{2}\mathrm{d}x + q(\rho(t,x))\right] + \int_0^t \int_0^L \mu(t,x)|\partial_x u(s,x)|^2 \mathrm{d}x\mathrm{d}t = \int_0^L \left[\frac{\rho^0(x)|u^0(x)|^2}{2}\mathrm{d}x + q(\rho^0(x))\right]\mathrm{d}x \tag{62}$$

for all $t \in [0, T_*)$ where q is defined by:

$$q(z) = z\partial_z^{-1}\left\{\frac{p(z)}{z^2}\right\}. \tag{63}$$

Remark. – In the definition of q, we denote with ∂_z^{-1} a primitive of $p(z)/z^2$. We remark that we assume neither that p vanishes in the origin (though completely natural) nor that p is superlinear in the origin (which is classically assumed). So, this primitive has to be constructed with care in order to ensure that q is positive globally. For this, we remark that $z \mapsto p(z)$ is fixed up to a constant in the sense that we may change this equation of state with a constant and keep the property (ρ, u) solution to (20)–(21). According to this principle, we may assume that $p(1) = 0$ in order to set:

$$q(z) = z\left[\int_1^z \frac{p(s)\mathrm{d}s}{s^2} + 1\right].$$

Under the assumption that p is increasing, we have then that $q(z) > 0$ for $z \in [0, \infty)$.

Step 2. BD entropy. – We control now the growth of the H^1–norm of ρ. Namely, we adapt to our periodic case the BD-entropy method which may be found in its simplest form in [2] for instance. So, we introduce $\varphi \in C^1(0, \infty)$ defined by

$$\varphi(z) = \int_1^z \frac{\mu(s)}{s^2}\mathrm{d}s, \quad \forall\, z \in (0, \infty).$$

Note that for a nonlinear function φ_1 of the density, we have

$$\partial_t\varphi_1(\rho) + u\partial_x\varphi_1(\rho) + \varphi_1'(\rho)\rho\partial_x u = 0.$$

Thus differentiating with respect to space yields:

$$\partial_t\partial_x(\varphi_1(\rho)) + \partial_x(u\partial_x\varphi_1(\rho)) + \partial_x(\varphi_1'(\rho)\rho\partial_x u) = 0.$$

Let us now choose $\varphi_1(\rho) = \int_1^\rho \mu(s)/s\mathrm{d}s$, then we get from the definition of φ

$$\partial_t(\rho\partial_x\varphi(\rho)) + \partial_x(\rho u\partial_x\varphi(\rho)) + \partial_x(\mu(\rho)\partial_x u) = 0.$$

Adding the relation to the momentum equation gives

$$\partial_t(\rho(u+\partial_x\varphi(\rho)))+\partial_x(\rho u(u+\partial_x\varphi(\rho)))+\partial_x p(\rho)=0. \tag{64}$$

In what follows, we denote $\varphi_x := \partial_x\varphi(\rho(x))$ to be distinguished with $z\to\varphi'(z)$ the simple derivative of the above defined function φ. We keep subscript x to denote partial derivatives w.r.t. space variable (we have thus $\partial_x u = u_x$). Testing the Equation (64) with $u+\varphi_x$ yields finally:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^L\left\{\rho\,|u+\varphi_x|^2+q(\rho)\right\}\right]+\int_0^L p'\varphi'|\rho_x|^2=0\,. \tag{65}$$

As $p'\varphi'\geq 0$ we conclude that

$$\int_0^L\left\{\rho\,|u+\varphi_x|^2+q(\rho)\right\}\leq C_0,\quad \forall\, t\geq 0\,.$$

Hence:

$$\int_0^L|\sqrt{\rho}\varphi'(\rho)\rho_x|^2\leq C_0,\quad \forall\, t\geq 0. \tag{66}$$

As the continuity equation implies the conservation of the mean of ρ on $(0,L)$ we derive that, setting $f\in C^1(0,\infty)$ any primitive of $z\mapsto\mu(z)/z^{3/2}$, there holds:

$$\|f(\rho(t,\cdot))\|_{L^\infty_\sharp}\leq C_0,\quad \forall\, t\geq 0\,.$$

In particular, our assumption (23) on μ enforces that $f(z)$ diverges when $z\to 0$ or $z\to\infty$. Hence, we obtain from the control above that

$$\|\rho(t,\cdot)\|_{L^\infty_\sharp}+\|\rho^{-1}(t,\cdot)\|_{L^\infty_\sharp}\leq C_0,\quad \forall\, t\geq 0, \tag{67}$$

and, plugging this inequality into (66) (and applying again that the mean of ρ is constant with time so that the $\|\partial_x\rho\|_{L^2_\sharp}$ controls the H^1-norm of ρ), we get:

$$\|\rho(t,\cdot)\|_{H^1_\sharp}\leq C_0,\quad \forall\, t\geq 0\,. \tag{68}$$

From the BD-entropy argument we developed up to now, we obtain global-in-time control on the ρ in the H^1-norm and in the $L^\infty-$norm from above and from below.

Step 3. Regularity. – Finally, we obtain propagation of the H^1 regularity for u. We note that the momentum equation is satisfied in $L^2_{\mathrm{loc}}((0,T_*)\times\mathbb{R})$, so that we can multiply the momentum equation by u_{xx} on $(0,t)\times(0,L)$ for arbitrary $t>0$. This yields:

$$\int_0^t\int_0^L\rho(\partial_t u+u\partial_x u)u_{xx}=\int_0^t\int_0^L\partial_x\left(\mu\partial_x u-p_x\right)u_{xx}.$$

On the left-hand side, *via* a standard approximation argument (up to project on a finite number of Fourier-modes and let then the number of modes go to infinity) we have:

$$\begin{aligned}&\int_0^t\int_0^L\rho(\partial_t u+u\partial_x u)u_{xx}\\&\qquad=-\frac{1}{2}\left[\int_0^L\rho(s,x)|u_x(s,x)|^2\mathrm{d}x\right]_{s=0}^{s=t}-\int_0^t\int_0^L\left(\rho_x(\partial_t u+u\partial_x u)u_x+\rho|u_x|^2u_x\right).\end{aligned}$$

Plugging this identity in the previous one, we infer that:

$$\frac{1}{2}\left[\int_0^L \rho(s,x)|u_x(s,x)|^2\mathrm{d}x\right]_{s=0}^{s=t} + \int_0^t\int_0^L \mu|u_{xx}|^2$$
$$= \int_0^t\int_0^L \left[(p_x - \mu_x u_x)u_{xx} - \left(\rho_x\,(\partial_t u + u\partial_x u) + \rho|u_x|^2\right)u_x\right].$$

On the right-hand side, we have, due to the previous controls on ρ in L^∞ and H^1 norms:

$$\left|\int_0^t\int_0^L (p_x - \mu_x u_x)u_{xx}\right| \le \frac{C}{\mu^0\varepsilon}\int_0^t \left(\|p\|^2_{H^1_\natural} + \|\mu\|^2_{H^1_\natural}\|u_x\|^2_{L^2_\natural}\right) + \varepsilon\int_0^t\int_0^L \mu|u_{xx}|^2,$$
$$\le \frac{C_0}{\varepsilon}\int_0^t\left(1 + \|u_x\|^2_{L^2_\natural}\right) + \varepsilon\int_0^t\int_0^L \mu|u_{xx}|^2.$$

Then, we replace

$$\rho_x\,(\partial_t u + u\partial_x u) = \frac{\rho_x}{\rho}\,[\mu u_{xx} + \mu_x u_x - p_x]$$

so that for a.e. $s \in (0,t)$:

$$\left|\int_0^L \rho_x\,(\partial_t u + u\partial_x u)\,u_x\right| \le \|\mu\|^{\frac{1}{2}}_{L^\infty_\natural}\|\rho^{-1}\|_{L^\infty_\natural}\|\rho\|_{H^1_\natural}\|u_x\|_{L^\infty_\natural}\left(\int_0^L \mu|u_{xx}|^2\right)^{\frac{1}{2}}$$
$$+ \|\rho^{-1}\|_{L^\infty_\natural}\|\rho\|_{H^1_\natural}\|\mu_x\|_{L^2_\natural}\|u_x\|^2_{L^\infty_\natural}$$
$$+ \|\rho^{-1}\|_{L^\infty_\natural}\|\rho\|_{H^1_\natural}\|p_x\|_{L^2_\natural}\|u_x\|_{L^\infty_\natural}$$
$$\le \frac{C_0}{\varepsilon}\left(1 + \|u_x\|^2_{L^2_\natural}\right) + \varepsilon\int_0^L \mu|u_{xx}|^2,$$

where we applied the previous controls on ρ in the H^1 and L^∞ norms, and that, for an absolute constant C, there holds:

$$\|u_x\|_{L^\infty_\natural} \le C\|u_x\|^{\frac{1}{2}}_{L^2_\natural}\|u_{xx}\|^{\frac{1}{2}}_{L^2_\natural}.$$

This entails again:

$$\left|\int_0^t\int_0^L \rho_x\,(\partial_t u + u\partial_x u)\,u_x\right| \le \frac{C_0}{\varepsilon}\int_0^t\left(1 + \|u_x\|^2_{L^2_\natural}\right) + \varepsilon\int_0^t\int_0^L \mu|u_{xx}|^2.$$

We have similarly:

$$\left|\int_0^t\int_0^L \rho|u_x|^2 u_x\right| \le \frac{C_0}{2\varepsilon}\int_0^t (1 + \|u_x\|^4_{L^2_\natural}) + \varepsilon\int_0^t\int_0^L \mu|u_{xx}|^2.$$

Combining all these computations in our first identity, and choosing ε sufficiently small, yields:

$$\frac{1}{2}\left[\int_0^L \rho(s,x)|u_x(s,x)|^2\mathrm{d}x\right]_{s=0}^{s=t} + \frac{1}{2}\int_0^t\int_0^L \mu|\partial_x u_x|^2 \le C_0\int_0^t\left(1 + \|u_x\|^4_{L^2_\natural}\right).$$

Applying a standard Gronwall inequality and recalling the dissipation of energy estimate, we obtain then that:

$$\sup_{t\in(0,T_*)}\int_0^L |u_x(t,x)|^2 \mathrm{d}x \le C_0(1+t)\exp(C_0), \quad \forall\, t\ge 0.$$

This ends the proof. □

Remarks

1. In case p merely satisfies (24), Equation (65) induces that for a constant $C_{p\mu}>0$ there holds:

(69)
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^L \left\{\rho\,|u+\varphi_x|^2+q(\rho)\right\}\right]\le C_{p\mu}\int_0^L \rho|\varphi_x|^2\,.$$

Hence, recalling that the total energy of the solution remains uniformy bounded with time, we obtain, by applying a Gronwall lemma, that there exists a positive constant C_0 depending only on initial data, for which:

$$\int_0^L \left\{\rho\,|u+\varphi_x|^2+q(\rho)\right\}\le C_0C_{p\mu}(1+t)\exp(2C_{p\mu}t)\quad \forall\, t\ge 0\,.$$

2. Recently, B. Haspot has extended the range of viscosity laws that provide global existence of strong solution for the compressible Navier-Stokes equation with density-dependent viscosity if initially the density is far from vacuum. His nice idea is to remark that the equation on $v=u+\partial_x\varphi(\rho)$ contains a damping term if we replace the pressure term in terms of the v and u. More precisely, we get the equation

(70)
$$\partial_t(\rho v)+\partial_x(\rho u v)+\frac{p'(\rho)\rho^2}{\mu(\rho)}v=\frac{p'(\rho)\rho^2}{\mu(\rho)}u.$$

Thus for $p(s)=as^\gamma$ ($\gamma>1$) if we assume $\mu(s)\le C+Cp(s)$ for all $s\ge 0$ then he first proves that v is $L^\infty(0,T;L^\infty)$ and then, coming back to the mass equation, that $1/\rho$ belongs to L^∞. This allows him to extend a local in time result to a global one. In conclusion, our homogenized result may for instance be extended to the shallow-water system where $\mu(\rho)=\rho$ and $p(\rho)=a\rho^2$.

4.2. Uniform estimates

Prior to establishing Theorem 2, we show in this section that the global strong solutions of the previous section, that we construct for initial data $(\rho^0,u^0)\in H^1_\natural\times H^1_\natural$, are indeed HD solutions:

LEMMA 5. – *Given* $(\rho^0,u^0)\in H^1_\natural\times H^1_\natural$, *the unique global strong solution* (ρ,u) *to* (20)-(21)-(25) *satisfies* ***(HDS)***$_a$, ***(HDS)***$_b$ *and* ***(HDS)***$_c$ *of Theorem 2 on a time interval* $(0,T_0)$ *with* T_0 *depending only on* $\underline{\rho^0}$, $\overline{\rho^0}$, $\|u^0\|_{H^1_\natural}$.

The remainder of this subsection is devoted to the proof of this lemma. We note that this completes the proof of Theorem 2 in the case where ρ^0 satisfies the further property $\rho^0\in H^1_\natural$.

So, let $(\rho^0, u^0) \in H^1_\natural \times H^1_\natural$ and (ρ, u) the associated global strong solution given by Theorem 4. Clearly, **(CS)**$_a$ (resp. **(CS)**$_b$) induces that **(HDS)**$_a$ (resp. **(HDS)**$_b$) holds on arbitrary time-interval $(0, T_0)$. For instance, one may remark that

$$\partial_x z = \mu \partial_{xx} u + \partial_x \mu \partial_x u - \partial_x p \in L^2(0, T_0; L^2_\natural),$$

as, by construction, we have

$$(\mu, \partial_{xx} u) \in L^\infty((0, T_0) \times \mathbb{R}) \times L^2(0, T_0; L^2_\natural),$$
$$(\partial_x \mu, \partial_x u) \in L^\infty(0, T_0; L^2_\natural) \times L^2(0, T_0; L^\infty_\natural).$$

We remind also that this solution satisfies the dissipation energy estimate (62). Hence, denoting by

$$\mathcal{E}^c_0 := \int_0^L \left[\frac{\overline{\rho^0} |u^0(x)|^2}{2} \mathrm{d}x + \max_{[\underline{\rho^0}, \overline{\rho^0}]} q(z) \right] \mathrm{d}x$$

we have that, for arbitrary $T_0 > 0$:

$$\sup_{t \in (0, T_0)} \left[\frac{1}{2} \int_0^L \rho(t,x) |u(t,x)|^2 \mathrm{d}x + \int_0^t \int_0^L \mu(s,x) |\partial_x u(s,x)| \mathrm{d}s \mathrm{d}x \right] \leq \mathcal{E}^0_c. \tag{71}$$

The only point is thus to obtain the bounds (31) and (33). Note also that thanks to the regularity **(CS)**$_a$, these conditions are indeed satisfied but for a sufficiently small $\tilde{T}_0$ only. The actual difficulty is thus to prove that we may choose $\tilde{T}_0 = T_0$ depending only on $\underline{\rho^0}, \overline{\rho^0}, \|u^0\|_{H^1_\natural}$. For this purpose, in what follows, we pick a positive time $\tilde{T}_0$ for which (31) and (33) are satisfied by (ρ, u) on $[0, \tilde{T}_0]$ for a well chosen K^0. We show then, that, if we assume $\tilde{T}_0 < T_0$, for some T_0 to be constructed with the expected dependencies, we obtain a better bound for (ρ, u). By a standard connectedness argument [(1)], we obtain then that we may choose $\tilde{T}_0 = T_0$.

For the computations below, we introduce the following notations:

- we introduce the function $\kappa = p/\mu$;
- given $\beta \in C([0, \infty))$ (mainly $\beta = p, \mu$ or κ) we denote
$$K^0_\beta = \max\{\beta(z), z \in [\underline{\rho^0}/2, 2\overline{\rho^0}]\} ;$$
- $K^0_u = 144 \left(\frac{1}{\mu^0} + \overline{\rho^0} \right) \left[\| \sqrt{\mu(\rho^0)} \partial_x u_0 - \kappa(\rho^0) \|^2_{L^2_\natural} + 1 + L |K^0_\kappa|^2 \right].$

We remark that K^0_u does depend only on $\overline{\rho^0}, \underline{\rho^0}, \|u^0\|_{H^1_\natural}$. It will play the role of K^0 in our proof.

According to the method of proof we described above, we assume from now on that $\tilde{T}_0 > 0$ is chosen and fixed such that we have the a priori bounds:

$$\frac{\underline{\rho^0}}{2} \leq \rho(t,x) \leq 2\overline{\rho^0} \quad \text{on } (0, \tilde{T}_0) \times \mathbb{R} \tag{72}$$

(1) Given the regularity statements **(CS)**$_a$ the following quantities are continuous functions of time-variable $t \in [0, \infty)$:

$$\min_{[0,L]} \rho(t,x), \quad \max_{[0,L]} \rho(t,x), \quad \int_0^t \|\partial_x z\|^2_{L^2_\natural}, \quad \sup_{(0,t)} \|u\|_{H^1_\natural}.$$

$$\sup_{(0,\tilde{T}_0)} \|u\|^2_{H^1_\natural} + \int_0^{\tilde{T}_0} \|\partial_x z\|^2_{L^2_\natural} \mathrm{d}s \leq K^0_u. \tag{73}$$

We state first the following lemma:

PROPOSITION 6. – *Let K^0_d denote the quantity:*

$$K^0_d = \frac{1}{\mu^0}\left(\sqrt{L} + \frac{1}{\sqrt{L}}\right)\left(2K^0_\mu \mathcal{E}^c_0 + 2L|K^0_p|^2 + K^0_u\right)^{\frac{1}{2}} + \frac{K^0_p}{\mu^0},$$

(see (23) *for the definition of μ^0). Then, K^0_d depends only on $\underline{\rho^0}, \overline{\rho^0}, \|u^0\|_{H^1_\natural}$ and, if $\tilde{T}_0 < 1$, there holds*

$$\int_0^{\tilde{T}_0} \|\partial_x u\|_{L^\infty_\natural} \leq |\tilde{T}_0|^{\frac{1}{2}} K^0_d. \tag{74}$$

Proof. – We recall first the classical embedding $H^1_\natural \subset L^\infty_\natural$ with the embedding inequality:

$$\|v\|_{L^\infty_\natural} \leq \left(\sqrt{L} + \frac{1}{\sqrt{L}}\right)\|v\|_{H^1_\natural}, \quad \forall\, v \in H^1_\natural,$$

Let now $T \leq \tilde{T}_0$. Due to (73), we have

$$\int_0^T \int_0^L |\partial_x z|^2 \leq K^0_u.$$

Then, by construction, there holds:

$$|z|^2 \leq 2\left(|\mu|^2|\partial_x u|^2 + |p|^2\right).$$

Consequently, recalling (71), we obain:

$$\begin{aligned}\int_0^T \int_0^L |z|^2 &\leq 2\int_0^T \int_0^L |\mu|^2|\partial_x u|^2 + 2\int_0^T \int_0^L |p|^2 \\ &\leq 2K^0_\mu \int_0^T \int_0^L \mu|\partial_x u|^2 + 2TL|K^0_p|^2 \\ &\leq 2K^0_\mu \mathcal{E}^c_0 + 2TL|K^0_p|^2.\end{aligned}$$

Finally, we have:

$$\int_0^T \|z\|^2_{L^\infty_\natural} \leq \left(\sqrt{L} + \frac{1}{\sqrt{L}}\right)^2 \left(2K^0_\mu \mathcal{E}^c_0 + 2TL|K^0_p|^2 + K^0_u\right)$$

and thus

$$\int_0^T \|z\|_{L^\infty_\natural} \leq \sqrt{T}\left(\sqrt{L} + \frac{1}{\sqrt{L}}\right)\left(2K^0_\mu \mathcal{E}^c_0 + 2TL|K^0_p|^2 + K^0_u\right)^{\frac{1}{2}}.$$

Then, we remark that

$$\partial_x u = \frac{z+p}{\mu}, \quad \text{so that (with the bound (23)),} \quad |\partial_x u| \leq \frac{1}{\mu^0}|z| + \frac{p}{\mu^0}$$

and :

$$\int_0^T \|\partial_x u\|_{L^\infty_\natural} \leq \frac{\sqrt{T}}{\mu^0}\left(\sqrt{L} + \frac{1}{\sqrt{L}}\right)\left(2K^0_\mu \mathcal{E}^c_0 + 2TL|K^0_p|^2 + K^0_u\right)^{\frac{1}{2}} + \frac{T}{\mu^0}K^0_p.$$

Hence, under the further restriction $T < 1$, we obtain:

$$\int_0^T \|\partial_x u\|_{L^\infty_\natural} \le \frac{\sqrt{T}}{\mu^0}\left[\left(\sqrt{L}+\frac{1}{\sqrt{L}}\right)\left(2K^0_\mu\,\mathcal{E}^c_0 + 2L|K^0_p|^2 + K^0_u\right)^{\frac{1}{2}} + K^0_p\right],$$

which yields the expected result setting $T = \tilde{T}_0$. □

We now consider the continuity equation and derive bounds for ρ:

PROPOSITION 7. – *There exists T^ρ_0 depending only on $\underline{\rho^0}, \overline{\rho^0}, \|u^0\|_{H^1_\natural}$ for which, if we assume that $\tilde{T}_0 < T^\rho_0$ then, there holds:*

$$\frac{2}{3}\underline{\rho^0} < \rho(t,x) < \frac{3}{2}\overline{\rho^0} \quad \forall\,(t,x) \in (0,\tilde{T}_0)\times\mathbb{R}.$$

Proof. – By standard arguments, we have that, for arbitrary $m \in [1,\infty[\cup]-\infty,-1[$ there holds:

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^L |\rho|^m\right] + \frac{1}{m}\int_0^L u\partial_x|\rho|^m = -\int_0^L |\rho|^m\partial_x u$$

so that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^L |\rho|^m\right] \le |m-1|\int_0^L |\rho|^m|\partial_x u| \le 2|m|\|\partial_x u\|_{L^\infty_\natural}\int_0^L |\rho|^m.$$

Consequently, there holds:

$$\left(\int_0^L |\rho|^m\right)^{\frac{1}{m}} \le \left(\int_0^L |\rho^0|^m\right)^{\frac{1}{m}} \exp\left(2\int_0^T \|\partial_x u\|_{L^\infty_\natural}\right).$$

In the limit $|m| \to \infty$ we thus have

$$\|\rho(t,\cdot)\|_{L^\infty_\natural} \le \overline{\rho^0}\exp\left(2\int_0^T \|\partial_x u\|_{L^\infty_\natural}\right),$$

$$\||\rho(t,\cdot)|^{-1}\|_{L^\infty_\natural} \le \frac{1}{\underline{\rho^0}}\exp\left(2\int_0^T \|\partial_x u\|_{L^\infty_\natural}\right).$$

Setting

$$T^\rho_0 := \min\left(\frac{1}{2}, \left|\frac{1}{2K^0_d}\ln\left(\frac{3}{2}\right)\right|^2\right)$$

(which has then the expected dependencies, see the definition of K^0_d), and assuming $\tilde{T}_0 < T^\rho_0 < 1$, we apply Lemma 6 on $(0,\tilde{T}_0)$ and obtain:

$$\underline{\rho^0}\exp\left(-2|\tilde{T}_0|^{\frac{1}{2}}K^0_d\right) \le \rho(t,x) \le \overline{\rho^0}\exp\left(2|\tilde{T}_0|^{\frac{1}{2}}K^0_d\right),$$

where

$$\exp(2|\tilde{T}_0|^{\frac{1}{2}}K^0_d) \le \exp(2|T^\rho_0|^{\frac{1}{2}}K_d) \le \frac{3}{2}.$$ □

We conclude with deriving estimates for u and z :

PROPOSITION 8. – *There exists $T_0^u > 0$ depending only on $\underline{\rho^0}, \overline{\rho^0}, \|u^0\|_{H^1_\natural}$ for which, if we assume that $\tilde{T}_0 < T_0^u$, there holds:*

$$\sup_{t\in[0,\tilde{T}_0]} \|\partial_x u\|^2_{L^2_\natural} + \int_0^{\tilde{T}_0} |\partial_x z|^2 \mathrm{d}s \leq \left(\frac{16}{\mu^0} + 144\overline{\rho^0}\right)\left[\|\sqrt{\mu(\rho^0)}\partial_x u_0 - \kappa(\rho^0)\|^2_{L^2_\natural} + 1 + L|K^0_\kappa|^2\right].$$

Proof. – The proof of this result is based on the use of a suitable multiplier for the momentum equation:

$$\rho(\partial_t u + u\partial_x u) = \partial_x\left[\mu\partial_x u - p\right]$$

which holds in $L^2_{\mathrm{loc}}((0,\infty)\times\mathbb{R})$. Precisely, we introduce the following conventions:

- the operator $\mathbb{E}$ corresponds to the mean of an L-periodic L^1–function;
- the operator ∂_x^{-1} corresponds to the periodic mean-free primitive of an L-periodic function of mean 0. It maps $H^m_\natural$ into $H^{m+1}_\natural$ for arbitrary $m \in \mathbb{N}$ and admits a straightforward density extension for $m \in \mathbb{Z}\setminus\mathbb{N}$ (when $m \in \mathbb{Z}\setminus\mathbb{N}$, $H^m_\natural$ stands for the dual of the subspace of $H^{|m|}_\natural$ containing all functions with mean zero);
- throughout the proof, C_0 is a constant which depends only on $\underline{\rho^0}, \overline{\rho^0}$ and $\|u^0\|_{H^1_\natural}$. It may vary between lines.

Then, we let $T \in (0,\tilde{T}_0)$ and we set, with $\kappa(\rho) = p(\rho)/\mu(\rho)$:

$$v = \partial_t u - \partial_x^{-1}\left[\partial_t\kappa - \mathbb{E}[\partial_t\kappa]\right] \quad \text{on } (0,T).$$

We recall that thanks to the continuity equation, there holds, for arbitrary $\beta \in C^1([0,\infty))$

$$\partial_t\beta(\rho) + \partial_x(\beta(\rho)u) + (\beta'(\rho)\rho - \beta(\rho))\partial_x u = 0. \tag{75}$$

Hence we have that κ, μ, p belong to the space $H^1(0,T;L^2_\natural)\cap C([0,T];C_\natural)\cap L^2(0,T;H^1_\natural)$. Consequently, $v \in L^2((0,T);L^2_\natural)$ and we have then:

$$\int_0^T\int_0^L \rho(\partial_t u + u\partial_x u)v = \int_0^T\int_0^L \partial_x\left[\mu\partial_x u - p\right]v. \tag{76}$$

On the right-hand side, we note that we can approximate u by projecting on Fourier series with a finite number of terms. This yields a sequence u^N converging to u in $H^1(0,T;L^2_\natural)\cap C([0,T];H^1_\natural)\cap L^2(0,T;H^2_\natural)$. Furthermore, the extension of ∂_x^{-1} to negative Sobolev spaces yields that:

$$v = \partial_x^{-1}\left[\partial_t\partial_x u - (\partial_t\kappa - \mathbb{E}[\partial_t\kappa])\right].$$

Hence the following formal integrations by parts that are valid for L-periodic trigonometric polynoms (with $v^N = \partial_t u^N - \partial_x^{-1}\left[\partial_t \kappa - \mathbb{E}[\partial_t \kappa]\right]$)

$$\begin{aligned}\int_0^L \partial_x \left[\mu \partial_x u^N - p\right] v^N &= -\int_0^L \left[\mu \partial_x u^N - p\right] \partial_x v^N \\ &= -\int_0^L \mu \left[\partial_x u^N - \kappa\right] \partial_t \left[\partial_x u^N - \kappa\right] - \int_0^L \left[\mu \partial_x u^N - p\right] \mathbb{E}[\partial_t \kappa] \\ &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \mu \left|\partial_x u^N - \kappa\right|^2 + \frac{1}{2}\int_0^L \partial_t \mu \left|\partial_x u^N - \kappa\right|^2 \\ &\quad - \mathbb{E}[\mu \partial_x u^N - p] \int_0^L \partial_t \kappa,\end{aligned}$$

extend into:

$$\begin{aligned}\int_0^T \int_0^L \partial_x \left[\mu \partial_x u - p\right] v = -\frac{1}{2}\left[\int_0^L \mu \left|\partial_x u - \kappa\right|^2\right]_0^T &+ \frac{1}{2}\int_0^T \int_0^L \partial_t \mu \left|\partial_x u - \kappa\right|^2 \\ &- \int_0^T \mathbb{E}[\mu \partial_x u - p] \int_0^L \partial_t \kappa.\end{aligned}$$

This simplifies the RHS of (76), whereas, on the left-hand side, we have:

$$\begin{aligned} LHS &= \int_0^T \int_0^L \rho \left(\partial_t u + u \partial_x u\right)\left(\partial_t u - \partial_x^{-1}[\partial_t \kappa - \mathbb{E}[\partial_t \kappa]]\right) \\ &\geq \frac{1}{2}\int_0^T \int_0^L \rho \left|\partial_t u + u \partial_x u\right|^2 - \int_0^T \int_0^L \rho |u \partial_x u|^2 - \int_0^T \int_0^L \rho \left|\partial_x^{-1}[\partial_t \kappa - \mathbb{E}[\partial_t \kappa]]\right|^2 .\end{aligned}$$

Finally, (76) reduces to [(2)]:

$$\begin{aligned}(77)\quad & \frac{1}{2}\left[\int_0^L \mu \left|\partial_x u - \kappa\right|^2\right]_{t=T} + \frac{1}{4\overline{\overline{\rho^0}}}\int_0^T \int_0^L \left|\partial_x \left[\mu \partial_x u - p\right]\right|^2 \\ &\leq \frac{1}{2}\left[\int_0^L \mu(\rho^0) \left|\partial_x u^0 - \kappa(\rho^0)\right|^2\right] + \frac{1}{2}\int_0^T \int_0^L \partial_t \mu \left|\partial_x u - \kappa\right|^2 - \int_0^T \mathbb{E}[z] \int_0^L \partial_t \kappa \\ &\quad + \int_0^T \int_0^L \rho |u \partial_x u|^2 + \int_0^T \int_0^L \rho \left|\partial_x^{-1}[\partial_t \kappa - \mathbb{E}[\partial_t \kappa]]\right|^2, \\ &\leq \frac{1}{2}\left[\int_0^L \mu(\rho^0) \left|\partial_x u^0 - \kappa(\rho^0)\right|^2\right] + \frac{1}{2} I_1 - I_2 + I_3 + I_4.\end{aligned}$$

We bound now I_1, I_2, I_3, I_4.

(2) Note that $\rho(\partial_t u + u \partial_x u) = \partial_x[\mu \partial_x u - p]$ and $\rho \leq 2\overline{\rho^0}$.

Applying (75) with $\beta = 1/\mu$, we have first:

$$\begin{aligned} I_1 &= -\int_0^T \int_0^L \partial_t \left[\frac{1}{\mu}\right] |\mu \partial_x u - p|^2 \\ &= \int_0^T \int_0^L \partial_x \left[\frac{u}{\mu}\right] |\mu \partial_x u - p|^2 - \int_0^T \int_0^L \frac{(\mu'\rho + \mu)}{\mu^2} \partial_x u \, |\mu \partial_x u - p|^2 \\ &= -2 \int_0^T \int_0^L \frac{u}{\mu} [\mu \partial_x u - p] \partial_x [\mu \partial_x u - p] - \int_0^T \int_0^L \frac{(\mu'\rho + \mu)}{\mu^2} \partial_x u \, |\mu \partial_x u - p|^2 . \end{aligned}$$

Recalling that thanks to (71):

$$\|u\|^2_{L^2_\natural} = \int_0^L |u|^2 \leq \frac{4}{\underline{\rho^0}} \int_0^L \frac{\rho |u|^2}{2} \leq \frac{4 \mathcal{E}^0_c}{\underline{\rho^0}},$$

we obtain that, for arbitrary small ε:

$$\begin{aligned} |I_1| &\leq C_0 \int_0^T \left[\|u\|_{L^\infty_\natural} \|z\|_{L^2_\natural} \|\partial_x z\|_{L^2_\natural} + \|\partial_x u\|_{L^\infty_\natural} \|z\|^2_{L^2_\natural} \right] \\ &\leq \frac{1}{8\underline{\rho^0}} \int_0^T \|\partial_x z\|^2_{L^2_\natural} + C_0 \int_0^T \|z\|^2_{L^2_\natural} \left[\|u\|^2_{L^2_\natural} + \|\partial_x u\|^2_{L^2_\natural} + \|\partial_x u\|_{L^\infty_\natural} \right]. \end{aligned}$$

Rewriting z in terms of $\partial_x u$ and $p(\rho), \mu(\rho)$, we obtain finally that

$$|I_1| \leq C_0 \int_0^T \left(\|\partial_x u\|^2_{L^2_\natural} + \|\partial_x u\|_{L^\infty_\natural} + 1 \right) \int_0^L \mu |\partial_x u - \kappa|^2 + \frac{1}{8\underline{\rho^0}} \int_0^T \|\partial_x z\|^2_{L^2_\natural}. \tag{78}$$

Concerning $I_2 = \int_0^T \int_0^L \mathbb{E}[z] \partial_t \kappa$, we have, applying (75):

$$\partial_t \kappa + \partial_x(\kappa u) + (\kappa'\rho - \kappa) \partial_x u = 0,$$

so that

$$\int_0^L \partial_t \kappa = -\int_0^L (\kappa'\rho - \kappa) \partial_x u$$

and consequently, with the same arguments as above:

$$|I_2| \leq \int_0^T C_0 \|z\|_{L^2_\natural} \|\partial_x u\|_{L^2_\natural} \leq C_0 \int_0^T \|\partial_x u\|^2_{L^2_\natural} + C_0 \int_0^T \int_0^L \mu |\partial_x u - \kappa|^2. \tag{79}$$

Concerning $I_3 = \int_0^T \int_0^L \rho |u \partial_x u|^2$, we proceed as previously:

$$\begin{aligned} |I_3| &\leq C_0 \int_0^T \|u\|^2_{L^\infty_\natural} \int_0^L |\partial_x u|^2 \\ &\leq C_0 \int_0^T \left(1 + \|\partial_x u\|^2_{L^2_\natural} \right) \int_0^L |\partial_x u|^2. \end{aligned}$$

Finally, expressing $\partial_x u$ in terms of z and functions of ρ, there still exists a constant C_0 for which:

$$|I_3| \leq C_0 \int_0^T (1 + \|\partial_x u\|^2_{L^2_\natural}) \int_0^L \mu |\partial_x u - \kappa|^2 + C_0 \int_0^T (1 + \|\partial_x u\|^2_{L^2_\natural}). \tag{80}$$

Then, for I_4, we note as previously that:

$$\partial_t \kappa - \mathbb{E}[\partial_t \kappa] = -\partial_x(\kappa u) - [(\kappa'\rho - \kappa)\partial_x u - \mathbb{E}[(\kappa'\rho - \kappa)\partial_x u]].$$

Consequently, there holds:

$$\partial_x^{-1}\left[\partial_t\kappa-\mathbb{E}[\partial_t\kappa]\right]=-\left[\kappa u-\mathbb{E}[\kappa u]\right]-w,$$

where

$$w=\partial_x^{-1}\left[(\kappa'\rho-\kappa)\partial_x u-\mathbb{E}[(\kappa'\rho-\kappa)\partial_x u]\right].$$

A classical Poincaré-Wirtinger inequality yields that:

$$\|\partial_x^{-1}\left[\partial_t\kappa-\mathbb{E}[\partial_t\kappa]\right]\|^2_{L^2_\natural}\le C_0[\|u\|^2_{L^2_\natural}+\|\partial_x u\|^2_{L^2_\natural}].$$

Hence $I_4=\int_0^T\rho|\partial_x^{-1}\left[\partial_t\kappa-\mathbb{E}[\partial_t\kappa]\right]|^2$ satisfies:

$$|I_4|\le\int_0^T C_0\left(1+\|\partial_x u\|^2_{L^2_\natural}\right). \tag{81}$$

Combining the computations (78)–(81) of I_1, I_2, I_3, I_4, we obtain finally that (77) reads:

$$\begin{aligned}\left[\int_0^L\mu|\partial_x u-\kappa|^2\right]_{t=T}&+\frac{1}{8\underline{\rho^0}}\int_0^T\int_0^L|\partial_x z|^2\\&\le\left[\int_0^L\mu(\rho^0)\left|\partial_x u^0-\kappa(\rho^0)\right|^2\right]+\int_0^T f(t)\int_0^L\mu|\partial_x u-\kappa|^2+\int_0^T g(t),\end{aligned} \tag{82}$$

where

$$f(t)=C_0\left(1+\|\partial_x u\|^2_{L^2_\natural}+\|\partial_x u\|_{L^\infty_\natural}\right)$$

and

$$g(t)=C_0\left(1+\|\partial_x u\|^2_{L^2_\natural}\right).$$

We have that (we may assume $T<1$ without restriction so that (74) holds true):

$$\begin{aligned}\int_0^T f(t)\mathrm{d}t&\le C_0\left(T+\int_0^T\|\partial_x u\|_{L^\infty(0,L)}+\|\partial_x u\|^2_{L^2_\natural}\right)\\&\le C_0\left(T(1+K_u^0)+\sqrt{T}K_d^0\right).\end{aligned}$$

Consequently, there exists $T_0^u<1$ depending only on $\underline{\rho^0},\overline{\rho^0},\|u^0\|_{H^1_\natural}$ such that:

$$\exp\left(\int_0^{T_0^u}f(t)\mathrm{d}t\right)\le 2.$$

Similarly we have:

$$\int_0^T g(t)\mathrm{d}t\le C_0T\left(1+K_u^0\right).$$

Hence, restricting the size of T_0^u if necessary, but keeping the same dependencies, we have that, for $T<T_0^u$:

$$\int_0^T g(t)\mathrm{d}t\le\left[\int_0^L\mu(\rho^0)\left|\partial_x u^0-\kappa(\rho^0)\right|^2+1\right].$$

Finally, by a standard application of the Gronwall lemma, we obtain then that, for arbitrary $T < T_0^u$, there holds:

$$\sup_{t\in[0,T]} \|\sqrt{\mu}(\partial_x u - \kappa)\|^2_{L^2_\natural} \leq 8\left(\|\sqrt{\mu(\rho^0)}\partial_x u_0 - \kappa(\rho^0)\|^2_{L^2_\natural} + 1\right).$$

Consequently:

$$\sup_{t\in[0,T]} \|\partial_x u\|^2_{L^2_\natural} \leq \frac{16}{\mu^0}\left[\|\sqrt{\mu(\rho^0)}\partial_x u_0 - \kappa(\rho^0)\|^2_{L^2_\natural} + 1 + L|K_p^0|^2\right],$$

and we also have:

$$\begin{aligned}\frac{1}{8\overline{\rho^0}}\int_0^T |\partial_x z|^2 \mathrm{d}s &\leq 2\left[\int_0^L \mu(\rho^0)\left|\partial_x u^0 - \kappa(\rho^0)\right|^2\right] \\ &\quad + 8\int_0^T f(t)\mathrm{d}t\left(\|\sqrt{\mu(\rho^0)}(\partial_x u_0 - \kappa(\rho^0))\|^2_{L^2_\natural} + 1\right) \\ &\leq 18\left(\|\sqrt{\mu(\rho^0)}(\partial_x u_0 - \kappa(\rho^0))\|^2_{L^2_\natural} + 1\right).\end{aligned}$$

Finally, we have indeed, that, for arbitrary $T \in [0, T_0^u)$ there holds:

$$\begin{aligned}&\sup_{t\in[0,T]} \|\partial_x u\|^2_{L^2_\natural} + \int_0^T |\partial_x z|^2 \mathrm{d}s \\ &\qquad \leq \left(\frac{16}{\mu^0} + 144\overline{\rho^0}\right)\left[\|\sqrt{\mu(\rho^0)}\partial_x u_0 - \kappa(\rho^0)\|^2_{L^2_\natural} + 1 + L|K_\kappa^0|^2\right]. \qquad \square\end{aligned}$$

Combining Proposition 7 and Proposition 8, we obtain finally, that, for $T_0 = \min(1, T_0^\rho, T_0^u)/2$ we have $(\mathbf{HDS})_c$ with

$$K_0 = 144\left(\frac{1}{\mu^0} + \overline{\rho^0}\right)\left[\|\sqrt{\mu(\rho^0)}\partial_x u_0 - \kappa(\rho^0)\|^2_{L^2_\natural} + 1 + L|K_\kappa^0|^2\right].$$

This completes the proof of Lemma 5.

4.3. Compactness arguments

In this last section, we complete the proof of our main results: Theorem 2 and Theorem 3. We first justify that we may only prove Theorem 3.

Indeed, to complete the proof of Theorem 2, we remark that, given an initial data $(\rho^0, u^0) \in L^\infty_\natural \times H^1_\natural$ we may approximate this initial data by a sequence $(\rho_n^0, u_n^0) \in [L^\infty_\natural \cap H^1_\natural]^2$ satisfying

$$\underline{\rho^0} \leq \rho_n^0 \leq \overline{\rho^0} \quad \|u_n^0\|_{H^1_\natural} \leq \|u^0\|_{H^1_\natural}, \quad \forall n \in \mathbb{N}, \tag{83}$$

and

$$\rho_n^0 \to \rho^0 \text{ in } L^1_\natural \quad u_n^0 \to u^0 \text{ in } H^1_\natural. \tag{84}$$

This can be done by a standard mollifying/projection argument. Then, Lemma 5 entails that there exists $T_0 > 0$ independent of $n \in \mathbb{N}$ for which there exists a HD solution (ρ_n, u_n) to (20)–(21) on $(0, T_0)$ associated with initial data (ρ_n^0, u_n^0). It remains then to prove that we can extract a subsequence of these HD solutions that converges to an HD solution to (20)–(21) on $(0, T_0)$ associated with initial data (ρ^0, u^0).

We remark then that the sequence of solutions (ρ_n, u_n) satisfies the assumptions of Theorem 3 with $k = 1$. Indeed, we already have the required bounds thanks to Lemma 5, and, in the limit $n \to \infty$, we note that the property $\rho_n^0 \to \rho^0$ in $L^1_\natural$ implies that

$$\nu_{\rho_n} \rightharpoonup \nu_\rho \quad \text{in} \quad \mathscr{Y}_\natural - w* .$$

Consequently, applying Theorem 3 implies that, up to the extraction of a subsequence (ρ_n, u_n) converges to a solution to the multifluid system (37)–(41) with $k = 1$ species. But, in the case $k = 1$ the multifluid system reduces to the classical compressible Navier-Stokes system (5)–(7). So, by a first application of Theorem 3 in this particular case, where the sequence of HD solutions (ρ_n, u_n) are constructed by application of Lemma 5, we extend the range of initial data for which we have local existence of HD solution and obtain Theorem 2.

The remainder of this section is devoted to the proof of Theorem 3. In what follows we consider a sequence of HD solutions (ρ_n, u_n) on a possibly small time-interval $(0, T_0)$ and associated with initial data $(\rho_n^0, u_n^0) \in L^\infty_\natural \times H^1_\natural$ satisfying:

$$\frac{1}{C_0} \leq \rho_n^0 \leq C_0 \quad \|u_n^0\|_{H^1_\natural} \leq C_0, \quad \forall\, n \in \mathbb{N}. \tag{85}$$

Thanks to these uniform bounds **(HDS)**$_c$ yields that $(\rho_n)_{n\in\mathbb{N}}$, and $(u_n)_{n\in\mathbb{N}}$, are bounded respectively in $L^\infty(0, T_0; L^\infty_\natural)$ and $L^\infty(0, T_0; H^1_\natural)$. We have also that the sequence $(z_n)_{n\in\mathbb{N}}$ defined by:

$$z_n := \mu_n \partial_x u_n - p_n, \quad \forall\, n \in \mathbb{N},$$

is bounded in $L^2(0, T_0; H^1_\natural)$.

As a consequence of the bounds on $(\rho_n)_{n\in\mathbb{N}}$, for arbitrary $\beta \in C^1(\mathbb{R})$ the sequence $\beta_n = \beta(\rho_n)$ is also bounded in $L^\infty(0, T_0; L^\infty_\natural)$. We apply this property in particular for the sequences defined as follows:

$$\mu_n := \mu(\rho_n), \quad p_n := p(\rho_n), \quad \kappa_n := \kappa(\rho_n), \quad \forall\, n \in \mathbb{N}.$$

Lemma 6 implies then that $(\partial_x u_n)_{n\in\mathbb{N}}$ is bounded in $L^1(0, T_0; L^\infty_\natural) \cap L^\infty(0, T_0; L^2_\natural)$.

As a first application of these uniform bounds, we obtain (up to the extraction of a subsequence that we do not relabel for conciseness):

$$\begin{aligned}
&\rho_n \rightharpoonup \rho, \quad p_n \rightharpoonup p^\infty, \quad \mu_n \rightharpoonup \mu^\infty, \quad \kappa_n \rightharpoonup \kappa^\infty \text{ in } L^\infty(0, T_0; L^\infty_\natural) - w*,\\
&u_n \rightharpoonup u \text{ in } L^\infty(0, T_0; H^1_\natural) - w*, \text{ with } \partial_x u \in L^1(0, T_0; L^\infty_\natural),\\
&z_n \rightharpoonup z^\infty \text{ in } L^2(0, T_0; H^1_\natural) - w.
\end{aligned}$$

Furthermore, introducing:

$$\underline{\rho}_\infty := \liminf \left(\inf_{(0,T_0)\times\mathbb{R}} \rho_n \right)_{n\in\mathbb{N}}, \quad \overline{\rho}_\infty := \limsup \left(\sup_{(0,T_0)\times\mathbb{R}} \rho_n \right)_{n\in\mathbb{N}},$$

classical weak convergence arguments also yield that:

- for a.e. $(t, x) \in (0, T_0) \times \mathbb{R}$ there holds:

$$\underline{\rho}_\infty \leq \rho(t, x) \leq \overline{\rho}_\infty, \tag{86}$$

- there exists a constant K_0 depending only on C_0 and $\liminf_{n\in\mathbb{N}} \|u_n^0\|_{H^1_\natural}$ for which

$$\sup_{t\in(0,T_0)} \|u(t,\cdot)\|_{H^1_\natural} + \int_0^{T_0} \|z^\infty(t,\cdot)\|_{H^1_\natural} \le K_0. \tag{87}$$

4.3.1. *Convergence of momentum equation.* – We want now to pass to the limit in the momentum equation satisfied by ρ_n and u_n. To this end, we first obtain strong-compactness for two quantities. We have:

LEMMA 9. – *Up to the extraction of a subsequence, we have that*

$$u_n \to u \textit{ in } C([0,T_0];L^2_\natural).$$

Proof. – We already have that u_n is bounded in $C([0,T_0];L^2_\natural) \cap L^\infty(0,T_0;H^1_\natural)$, where $H^1_\natural \subset L^2_\natural$ is compact. Furthermore, we have from the momentum equation that:

$$\partial_t u_n = -u_n \partial_x u_n + \frac{1}{\rho_n}\partial_x z_n.$$

Consequently:

$$\begin{aligned}\|\partial_t u_n\|_{L^2(0,T_0;L^2_\natural)} \\ &\le \|u_n\|_{L^\infty(0,T_0;L^\infty_\natural)}\|u_n\|_{L^2(0,T_0;H^1_\natural)} + \||\rho_n|^{-1}\|_{L^\infty(0,T_0;L^\infty_\natural)}\|\partial_x z_n\|_{L^2(0,T_0;L^2_\natural)}.\end{aligned}$$

But the bounds claimed above and the embedding $H^1_\natural \subset L^\infty_\natural$ yield that the right-hand side of this inequality is bounded uniformly in $n\in\mathbb{N}$. Consequently, we have that $(u_n)_{n\in\mathbb{N}}$ is also uniformly equicontinuous in $C([0,T_0];L^2_\natural)$ and, by an Ascoli argument, we may extract a strongly converging subsequence. □

Remark. – We can then prove that $\rho^n|u^n|^2 \to \rho|u|^2$ (in $L^2(0,T_0;L^2_\natural) - w$ for instance) and, if ρ^0_n converges strongly to ρ^0, we may "pass to the limit" in the dissipation estimate satisfied by (ρ^n,u^n). This yields that:

$$\int_0^L \left[\frac{\rho(t,\cdot)|u(t,\cdot)|^2}{2} + q^\infty\right] + \int_0^t\int_0^L \mu|\partial_x u|^2 \le \int_0^L \left[\frac{\rho^0|u^0|^2}{2} + q^0\right] \text{ a.e. in } (0,T_0)\,, \tag{88}$$

where $q^0 = q(\rho^0)$.

Also, we state the equivalent result to the viscous-flux lemma that was crucial to the proof by P.-L. Lions [11] and by E. Feireisl, A. Novotný and H. Petzeltová [6] to obtain existence of global weak solutions to compressible Navier-Stokes systems:

LEMMA 10. – *Let $\beta \in C^1(0,\infty)$ then, up to the extraction of a subsequence, we have that*

$$\beta(\rho_n) \rightharpoonup \beta^\infty \textit{ in } L^\infty(0,T_0;L^\infty_\natural) - w*,$$
$$\beta(\rho_n)z_n \rightharpoonup \beta^\infty z^\infty \textit{ in } L^2(0,T_0;L^2_\natural) - w.$$

Proof. – Under the assumptions of this lemma (and keeping the conventions of the previous section for the operator ∂_x^{-1}), we set:

$$\beta_n = \beta(\rho_n), \quad w_n = \partial_x^{-1}\left[\beta_n - \mathbb{E}[\beta_n]\right].$$

Then, β_n and w_n are bounded respectively in $C([0,T_0];L^2_\natural) \cap L^\infty(0,T_0;L^\infty_\natural)$ and $C([0,T_0];H^1_\natural)$. In particular, we may extract a subsequence s.t. β_n and $\beta_n z_n$ converge respectively in $L^\infty(0,T_0;L^\infty_\natural) - w*$ and $L^2(0,T_0;L^2_\natural) - w$. We denote β^∞ the weak-$*$ limit of β_n.

We have, as previously:

$$\partial_t\beta_n = -\partial_x(\beta_n u_n) - (\beta'(\rho_n)\rho_n - \beta_n)\partial_x u_n \in L^\infty(0,T_0;H^{-1}_\natural).$$

This implies that

$$\partial_t w_n = -\left(\beta_n u_n - \mathbb{E}[\beta_n u_n]\right) - \partial_x^{-1}\left[(\beta'(\rho_n)\rho_n - \beta_n)\partial_x u_n - \mathbb{E}[(\beta'(\rho_n)\rho_n - \beta_n)\partial_x u_n]\right].$$

In particular, by applying the Poincaré-Wirtinger inequality (in the same spirit as for the computation of I_4 on page 279), we obtain that $\partial_t w_n$ is bounded in $L^\infty(0,T_0;L^2_\natural)$. We may then apply an Ascoli argument again and extract a subsequence we do not relabel such that:

$$w_n \to w^\infty = \partial_x^{-1}\left[\beta^\infty - \mathbb{E}[\beta^\infty]\right] \text{ in } C([0,T_0];L^2_\natural)$$

(as this is the only possible limit). The computation of $\partial_t\beta_n$ implies also that:

$$\partial_t\mathbb{E}[\beta_n] = -\mathbb{E}[(\beta'(\rho_n)\rho_n - \beta_n)\partial_x u_n] \in L^\infty(0,T_0).$$

We have thus that $\mathbb{E}[\beta_n]$ is bounded in $W^{1,\infty}(0,T_0)$ and we may extract also a subsequence for which $\mathbb{E}[\beta_n] \to \mathbb{E}[\beta^\infty]$ in $C([0,T_0])$.

For any $n\in\mathbb{N}$ and $\varphi\in C_c^\infty((0,T_0)\times\mathbb{R})$ we have then:

$$\int_0^{T_0}\int_{\mathbb{R}}\beta_n z_n\varphi = \int_0^{T_0}\int_{\mathbb{R}}\partial_x w_n z_n\varphi + \int_0^{T_0}\int_{\mathbb{R}}\mathbb{E}[\beta_n]z_n\varphi.$$

On the one hand, we have:

$$\begin{aligned}\int_0^{T_0}\int_{\mathbb{R}}\mathbb{E}[\beta_n](t)z_n(t,x)\varphi(t,x)\mathrm{d}x\mathrm{d}t \quad &= \quad \int_0^{T_0}\left[\mathbb{E}[\beta_n](t)\int_{\mathbb{R}}z_n(t,x)\varphi(t,x)\mathrm{d}x\right]\mathrm{d}t \\ &\xrightarrow[n\to\infty]{} \int_0^{T_0}[\mathbb{E}[\beta^\infty](t)\int_{\mathbb{R}}z^\infty(t,x)\varphi(t,x)\mathrm{d}x]\mathrm{d}t\end{aligned}$$

due to the strong convergence of $(\mathbb{E}(\beta_n))_{n\in\mathbb{N}}$ in $C([0,T_0])$ and the weak convergence of $(\int_{\mathbb{R}} z_n\varphi)_{n\in\mathbb{N}}$ in $L^2(0,T_0)$.

On the other hand, there holds:

$$\int_0^{T_0}\int_{\mathbb{R}}\partial_x w_n z_n\varphi = -\int_0^{T_0}\int_{\mathbb{R}}w_n\varphi\partial_x z_n - \int_0^{T_0}\int_{\mathbb{R}}w_n z_n\partial_x\varphi.$$

Combining the strong convergence of w_n in $C([0,T_0];L^2_\natural)$ and the weak convergence of z_n in $L^2(0,T_0;H^1_\natural)$ yields:

$$\int_0^{T_0}\int_{\mathbb{R}}\partial_x w_n z_n\varphi \xrightarrow[n\to\infty]{} -\int_0^{T_0}\int_{\mathbb{R}}w^\infty\varphi\partial_x z^\infty - \int_0^{T_0}\int_{\mathbb{R}}w^\infty z^\infty\partial_x\varphi.$$

These computations entail finally that

$$\begin{aligned}\lim_{n\to\infty}\int_0^{T_0}\int_{\mathbb{R}}\beta_n z_n\varphi &= -\int_0^{T_0}\int_{\mathbb{R}}w^\infty\varphi\partial_x z^\infty - \int_0^{T_0}\int_{\mathbb{R}}w^\infty z^\infty\partial_x\varphi + \int_0^{T_0}\int_0^L\mathbb{E}[\beta^\infty]w^\infty\varphi \\ &= \int_0^{T_0}\int_{\mathbb{R}}\beta^\infty z^\infty\varphi.\end{aligned}$$

This completes the proof. □

We can now pass to the limit in the equations satisfied by (ρ_n, u_n).

PROPOSITION 11. – *We have in* $\mathscr{D}'((0,T_0)\times\mathbb{R})$:

$$\partial_t(\rho u) + \partial_x(\rho u^2) = \partial_x\left[m^\infty(\partial_x u - \kappa^\infty)\right],$$

where

$$m^\infty = \left[\lim \frac{1}{\mu(\rho_n)}\right]^{-1}.$$

Proof. – We recall that, for a given $n \in \mathbb{N}$ there holds:

$$\partial_t(\rho_n u_n) + \partial_x(\rho_n |u_n|^2) = \partial_x z_n.$$

Combining the weak convergences of $(\rho_n)_{n\in\mathbb{N}}$ in $L^\infty(0,T_0;L^\infty_\natural)-w*$ together with the strong convergence of $(u_n)_{n\in\mathbb{N}}$ in $C([0,T_0];L^2_\natural)$, we obtain that

$$\rho_n u_n \rightharpoonup \rho u \text{ in } L^2(0,T_0;L^2_\natural) - w,$$
$$\rho_n |u_n|^2 \rightharpoonup \rho u^2 \text{ in } L^2(0,T_0;L^1_\natural) - w.$$

This enables to pass to the limit in the left-hand side of the momentum equation:

$$\partial_t(\rho_n u_n) + \partial_x(\rho_n|u_n|^2) \rightharpoonup \partial_t(\rho u) + \partial_x(\rho u^2) \text{ in } \mathscr{D}'((0,T_0)\times\mathbb{R}).$$

On the right-hand side we have that $z_n \rightharpoonup z^\infty$ so that:

$$\partial_t(\rho u) + \partial_x(\rho u^2) = \partial_x z^\infty.$$

It remains to compute z in terms of ρ and u. We have, for fixed $n \in \mathbb{N}$:

$$\partial_x u_n = \frac{z_n}{\mu_n} + \kappa(\rho_n).$$

We pass to the limit in this identity (in $L^2(0,T_0;L^2_\natural) - w$ for instance), and apply Lemma 10. This yields:

$$\partial_x u = \lim\left[\frac{1}{\mu(\rho_n)}\right] z^\infty + \kappa^\infty \quad \text{or equivalently} \quad z^\infty = m^\infty\left(\partial_x u - \kappa^\infty\right).$$

This ends the proof of this proposition. □

As classical in these compactness arguments, the main difficulty now is to find a relation between μ^∞, κ^∞ and ρ. In full generality, this is not possible: the operators "lim" and the operator "composition by a continuous function β" do not commute. To analyze more precisely the commutators, we apply Young-measure theory.

4.3.2. *Convergence of the densities in the sense of Young measures.*– For a given $n \in \mathbb{N}$ we introduce the Young measure ν_n as defined by:

$$\langle \nu_n, B\rangle = \int_{\mathbb{R}} B(x, \rho_n(t,x))\mathrm{d}x \quad \forall\, B \in C_c(\mathbb{R}\times\mathbb{R}^+).$$

We recall that, by assumption, we have that $\nu_n(0,\cdot)$ converges in $\mathscr{Y}_\natural$ to ν^0 the Young measure associated to the weights $(\alpha_i^0)_{i=1,\dots,k}$ and densities $(\rho_i^0)_{i=1,\dots,k}$ meaning that:

$$\langle \nu^0, B\rangle = \int_{\mathbb{R}} \sum_{i=1}^k \alpha_i^0(x) B(x, \rho_i^0(x))\mathrm{d}x \quad \forall\, B \in C_c(\mathbb{R}\times\mathbb{R}^+).$$

From the regularity $\rho_n \in C([0,T_0]; L^1_\natural)$ we also have that $\nu_n \in C([0,T_0]; \mathscr{Y}_\natural - w*)$. The main result of this section reads:

Proposition 12. – *There exists a subsequence we do not relabel such that $\nu_n \rightharpoonup \nu$ in $C([0,T_0]; \mathscr{Y}_\natural - w*)$. Furthermore, ν is a solution to:*

$$\partial_t \nu + \partial_x(\nu u) - \left(\partial_\xi\left(\xi \frac{\nu}{\mu(\xi)}\right) + \frac{\nu}{\mu(\xi)}\right) z^\infty - \left(\partial_\xi\left(\frac{\xi p(\xi)\nu}{\mu(\xi)}\right) + \frac{\nu p(\xi)}{\mu(\xi)}\right) = 0, \tag{89}$$

in $\mathscr{D}'((0,T_0)\times\mathbb{R}\times\mathbb{R}^+)$ with initial condition:

$$\nu(0,\cdot) = \nu^0. \tag{90}$$

Proof. – First, we obtain that $(\nu_n)_{n\in\mathbb{N}}$ is compact in $C([0,T_0]; \mathscr{Y}_\natural - w*)$. To this end, we note that the vector space engendered by tensorized functions $B = \varphi\otimes\beta$ is dense in $C_c(K)$ for arbitrary $K \Subset \mathbb{R}\times\mathbb{R}^+$. So, we prove that for arbitrary $B \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+)$, satisfying the further property that B is tensorized, the sequence $(\langle \nu_n, B\rangle)_{n\in\mathbb{N}}$ is compact in $C([0,T_0])$. As $C_c(K)$ is separable for an increasing sequence of compact subsets $K \Subset \mathbb{R}$ covering $\mathbb{R}\times\mathbb{R}^+$, this shall entail the expected property.

Let $\beta \in C_c^\infty(\mathbb{R}^+)$ and $\varphi \in C_c^\infty(\mathbb{R})$. As already remarked above, we have that

$$(\langle \nu_n, \varphi\otimes\beta\rangle)_{n\in\mathbb{N}} \text{ is bounded in } C([0,T_0]). \tag{91}$$

Furthermore, because $\rho_n \in L^\infty(0,T_0; L^\infty_\natural)$ with $(u, \partial_x u) \in C([0,T_0]; L^2_\natural)$, classical arguments imply that ρ_n is a solution to the continuity equation in the sense of renormalized solutions. Hence, we have that $\beta_n := \beta(\rho_n)$ satisfies:

$$\beta_n \in C([0,T_0]; L^1_\natural) \cap L^\infty(0,T_0; L^\infty_\natural),$$

with

$$\begin{cases} \partial_t \beta_n + \partial_x(\beta_n u_n) + (\beta'(\rho_n)\rho_n - \beta(\rho_n))\partial_x u_n = 0, & \text{in } \mathscr{D}'((0,T_0)\times\mathbb{R}), \\ \beta_n(0,\cdot) = \beta(\rho_n^0), & \text{in } L^\infty_\natural. \end{cases} \tag{92}$$

In particular, for arbitrary $\psi \in C_c^\infty(0,T_0)$, we may use as test-function in this equation the tensorized $\psi(t)\otimes\varphi(x)$. This entails that

$$-\int_0^{T_0} \langle \nu_n, \varphi\otimes\beta\rangle \partial_t\psi - \int_0^{T_0}\left[\int_{\mathbb{R}} \beta_n u_n \partial_x\varphi\right]\psi + \int_0^{T_0}\int_{\mathbb{R}}\left[(\beta_n'\rho_n - \beta_n)\partial_x u_n \varphi\right]\psi = 0.$$

Combining the uniform bounds on ρ_n and β_n in $L^\infty(0,T_0;L^\infty_\natural)$ with the uniform bounds on u_n in $L^\infty(0,T_0;H^1_\natural)$ we obtain from the above equality that

$$\partial_t\langle \nu_n,\varphi\otimes\beta\rangle \quad \text{is bounded in } L^2(0,T_0). \tag{93}$$

Combining (91) and (93) we apply an ascoli argument yielding that, up to the extraction of a subsequence, $(\langle \nu_n,\varphi\otimes\beta\rangle)_{n\in\mathbb{N}}$ converges in $C([0,T_0])$.

We prove now that the limit Young measure ν satisfies (89)–(90). We already have the initial condition (90) by assumption. Then, we again remark that, by a density argument, it is sufficient to prove that (89) is satisfied when tested against tensorized test-functions $\psi\otimes\varphi\otimes\beta$. Given $\varphi\in C^\infty_c(\mathbb{R})$ and $\beta\in C^\infty_c(\mathbb{R})$ we may reproduce the computations above and obtain that, for arbitrary $\psi\in C^\infty_c(0,T_0)$, we have:

$$-\int_0^{T_0}\langle \nu_n,\varphi\otimes\beta\rangle\partial_t\psi-\int_0^{T_0}\left[\int_{\mathbb{R}}\beta_n u_n\partial_x\varphi\right]\psi+\int_0^{T_0}\left[\int_{\mathbb{R}}(\beta_n'\rho_n-\beta_n)\partial_x u_n\varphi\right]\psi=0.$$

To pass to the limit in this equation, we first remark that $\langle \nu_n,\varphi\otimes\beta\rangle$ converges to $\langle \nu,\varphi\otimes\beta\rangle$ in $C([0,T_0])$. Consequently, we have:

$$\int_0^{T_0}\langle \nu_n,\varphi\otimes\beta\rangle\partial_t\psi \xrightarrow[n\to\infty]{} \int_0^{T_0}\langle \nu,\varphi\otimes\beta\rangle\partial_t\psi.$$

Then, we recall that, up to the extraction of a subsequence, we have $\beta_n\rightharpoonup\beta^\infty$ in $L^2(0,T_0;L^2_\natural)$ (see Lemma 10). Consequently, we can multiply the weak convergence of β_n with the strong convergence of u_n in $C([0,T_0];L^2_\natural)$. This implies that:

$$\int_0^{T_0}\left[\int_{\mathbb{R}}\beta_n u_n\partial_x\varphi\right]\psi \xrightarrow[n\to\infty]{} \int_0^{T_0}\left[\int_{\mathbb{R}}\beta^\infty u\partial_x\varphi\right]\psi.$$

Finally, we rewrite the last integral:

$$\int_0^{T_0}\left[\int_{\mathbb{R}}(\beta_n'\rho_n-\beta_n)\partial_x u_n\varphi\right]\psi=\int_0^{T_0}\left[\int_{\mathbb{R}}\frac{(\beta_n'\rho_n-\beta_n)}{\mu_n}z_n\varphi\right]\psi+\int_0^{T_0}\left[\int_{\mathbb{R}}\frac{(\beta_n'\rho_n-\beta_n)}{\mu_n}p_n\varphi\right]\psi.$$

In the first integral, we rewrite

$$\frac{(\beta_n'\rho_n-\beta_n)}{\mu_n}=f_\beta(\rho_n)\text{ with }f_\beta(\xi)=\frac{\beta'(\xi)\xi-\beta(\xi)}{\mu(\xi)}$$

and apply Lemma 10. This entails that:

$$\int_0^{T_0}\left[\int_{\mathbb{R}}\frac{(\beta_n'\rho_n-\beta_n)}{\mu_n}z_n\varphi\right]\psi=\int_0^{T_0}\left[\int_{\mathbb{R}}f_\beta^\infty z^\infty\varphi\right]\psi.$$

As for the second integral, we remark that

$$\frac{(\beta_n'\rho_n-\beta_n)}{\mu_n}p_n=g_\beta(\rho_n)\text{ with }g_\beta(\xi)=\frac{\beta'(\xi)\xi-\beta(\xi)}{\mu(\xi)}p(\xi).$$

Consequently, we apply the weak-convergence of ρ_n in terms of young measures. This yields that

$$\int_0^{T_0}\left[\int_{\mathbb{R}}\frac{(\beta_n'\rho_n-\beta_n)}{\mu_n}p_n\varphi\right]\psi \xrightarrow[n\to\infty]{} \int_0^{T_0}\langle \nu,\varphi\otimes g_\beta\rangle\psi.$$

In the limit $n \to \infty$, we obtain finally:

(94)
$$-\int_0^{T_0} \langle \nu, \varphi \otimes \beta \rangle \partial_t \psi - \int_0^{T_0} \left[\int_{\mathbb{R}} \beta^\infty u \partial_x \varphi \right] \psi = \int_0^{T_0} \left[\int_{\mathbb{R}} f_\beta^\infty z^\infty \varphi \right] \psi + \int_0^{T_0} \langle \nu, g_\beta \otimes \varphi \rangle \psi.$$

On the left-hand side, we note that $u \in L^\infty(0, T_0; H^1_\natural) \subset L^\infty(0, T_0; C_\natural)$. Consequently, we may rewrite:

$$\int_0^{T_0} \left[\int_{\mathbb{R}} \beta^\infty u \partial_x \varphi \right] \psi = \int_0^{T_0} \langle \nu, [u \partial_x \varphi] \otimes \beta \rangle \psi.$$

By applying the explicit formula for f_β, we rewrite also:

$$\int_0^{T_0} \left[\int_{\mathbb{R}} f_\beta^\infty z^\infty \varphi \right] \psi = \int_0^{T_0} \left[\int_{\mathbb{R}} \langle \nu, \varphi \otimes \frac{\xi \partial_\xi \beta(\xi) - \beta(\xi)}{\mu(\xi)} \rangle z^\infty \right] \psi.$$

We may rewrite similarly the term g_β in the last integral. This enables to rewrite (94) in tems of the duality-pairing of distributions on $(0, T_0) \times \mathbb{R} \times \mathbb{R}^+$:

$$\begin{aligned} &- \langle \nu, \partial_t \psi \otimes \varphi \otimes \beta \rangle - \langle \nu, u \partial_x (\psi \otimes \varphi \otimes \beta) \rangle \\ &\qquad = \langle \nu, z^\infty \frac{\xi}{\mu} \partial_\xi (\psi \otimes \varphi \otimes \beta) \rangle + - \langle \nu, \frac{z^\infty}{\mu} \psi \otimes \varphi \otimes \beta \rangle \\ &\qquad\qquad \langle \partial_\xi \left[\frac{\xi p \nu}{\mu} \right], \frac{\xi p}{\mu} \partial_\xi (\psi \otimes \varphi \otimes \beta) \rangle - \langle \nu, \frac{p}{\mu} \psi \otimes \varphi \otimes \beta \rangle. \end{aligned}$$

We recognize here the weak-form of (89) for tensorized test-functions. This ends the proof. □

To end the proof of Theorem 2 we remark that (89)–(90) enters the framework of Section 5. Indeed, we rewrite (89) as

(95)
$$\partial_t \nu + \partial_x (\nu u_x) + \partial_\xi (\nu u_\xi) + g \nu = 0,$$

with (u_x, u_ξ, g) a vector-field depending a priori on $(t, x, \xi) \in (0, T) \times \mathbb{R} \times \mathbb{R}^+$ and being L-periodic w.r.t. $x-$variable.

Thanks to the computations above, we have that $u_x = u \in C([0, T_0]; C(\mathbb{R} \times \mathbb{R}^+))$ satisfies:

$$\partial_x u_x = \frac{z^\infty}{m^\infty} + \kappa^\infty \in L^1(0, T_0; L^\infty(\mathbb{R} \times \mathbb{R}^+)), \quad \partial_\xi u_x = 0\,.$$

We also have that (note that $z^\infty \in L^2(0, T_0; H^1_\natural) \subset L^1(0, T_0; C_\natural)$):

$$u_\xi(t, x, \xi) = -\left(\frac{\xi}{\mu(\xi)} z^\infty(t, x) + \frac{\xi p(\xi)}{\mu(\xi)} \right) \in L^1(0, T_0; C(\mathbb{R} \times \mathbb{R}^+))$$

such that:

$$\partial_\xi u_\xi \in L^1(0, T_0; C(\mathbb{R} \times \mathbb{R}^+));$$

$$\int_0^T \int_0^L \sup_{\xi \in [0, M]} |\partial_x u_\xi(t, x, \xi)| \mathrm{d}\xi \mathrm{d}x < \infty, \quad \forall\, M > 0.$$

Finally, we have that:

$$g(t, x, \xi) = -\left(\frac{z^\infty(t, x)}{\mu(\xi)} + \frac{p(\xi)}{\mu(\xi)} \right) \in L^1(0, T_0; C(\mathbb{R} \times \mathbb{R}^+)),$$

satisfies:

$$\partial_\xi g \in L^1(0, T_0; C(\mathbb{R} \times \mathbb{R}^+)),$$

$$\int_0^T \int_0^L \sup_{\xi \in [0,M]} |\partial_x g(t, x, \xi)| \mathrm{d}\xi \mathrm{d}x < \infty, \quad \forall M > 0.$$

Equation (95) with the above assumptions on the data (u_x, u_ξ, g) is analyzed in Section 5. Hence, Lemma 13 and Lemma 14 ensure that ν is the unique solution to (89)–(90) and that it writes as a convex combination of k Dirac measures. Plugging formally $\nu = \sum_{i=1}^k \alpha_i \delta_{\xi=\rho_i}$ in (89)–(90) we get that the (α_i, ρ_i) are solutions of the expected pde system. We note that these equations are actually satisfied by construction (see the proof of Lemma 14). This ends the proof of Theorem 3.

5. Young measures identification and transport equation

In this section we consider periodic Young-measures solution to the transport equation:

$$\partial_t \nu + \mathrm{div}(u\nu) + g\nu = 0, \tag{96}$$

in $\mathscr{D}'((0, T) \times \mathbb{R} \times \mathbb{R}^+)$, with initial condition:

$$\nu(0, \cdot) = \nu^0 \in \mathscr{Y}_\natural. \tag{97}$$

For legibility, we turn to notations (x_1, x_2) for space variables and $u = (u_1, u_2)$ for velocity-fields. Throughout this section, we assume that this velocity-field satisfies:

- $u := u(t, x_1, x_2)$ and $g := g(t, x_1, x_2)$ are L-periodic w.r.t. x_1-variable,
- $u_1 \in L^1(0, T; C(\mathbb{R} \times \mathbb{R}^+))$ with:

$$\partial_1 u_1 \in L^1((0, T); C(\mathbb{R} \times \mathbb{R}^+)); \tag{98}$$

$$\partial_2 u_1 = 0 \quad \text{a.e.} \tag{99}$$

- $u_2 \in L^1((0, T); C(\mathbb{R} \times \mathbb{R}^+))$ with:

$$\int_0^T \int_0^L \sup_{x_2 \in [0,M]} |\partial_1 u_2(t, x_1, x_2)| \mathrm{d}x_1 \mathrm{d}t < \infty, \quad \forall M > 0, \tag{100}$$

$$\partial_2 u_2 \in L^1((0, T); C(\mathbb{R} \times \mathbb{R}^+)). \tag{101}$$

As for the source term g, we assume that

- $g \in L^1(0, T; C(\mathbb{R} \times \mathbb{R}^+))$ with

$$\int_0^T \int_0^L \sup_{x_2 \in [0,M]} |\partial_1 g(t, x_1, x_2)| \mathrm{d}x_1 \mathrm{d}t < \infty, \quad \forall M > 0, \tag{102}$$

$$\partial_2 g \in L^1((0, T); C(\mathbb{R} \times \mathbb{R}^+)). \tag{103}$$

We first obtain a uniqueness result:

LEMMA 13. – *Given* $C_0 > 0$, *for arbitrary* $\nu^0 \in \mathcal{Y}_\natural$ *such that* $Supp(\nu^0) \subset \mathbb{R}\times(1/C_0, C_0)$, *there exists* $T_* \leq T$ *such that* (96)–(97) *admits at most one solution* $\nu \in C([0,T_*]; \mathcal{Y}_\natural - w*)$ *with support in* $\mathbb{R}\times(1/2C_0, 2C_0)$.

Proof. – We provide a proof with a duality-regularization argument. By difference, we assume that ν satisfies :

- ν is a continuous function on $[0, T_*]$ with values in periodic real-valued Radon measures on $\mathbb{R}\times\mathbb{R}^+$ (endowed with the weak topology);
- ν_t has support in $\mathbb{R}\times(1/2C_0, 2C_0)$ for arbitrary $t \in (0,T)$ and $\nu_{|_{t=0}} = 0$;
- for arbitrary $\varphi \in C_c^\infty((0,T)\times\mathbb{R}\times\mathbb{R}^+)$ we have:

$$\int_0^T \langle \nu, \partial_t\varphi + u\cdot\nabla\varphi - g\varphi\rangle = 0\,.$$

Our aim is then to prove that ν vanishes globally on $[0,T]$.

First, by a standard regularization argument, we have that, for arbitrary $t \in [0,T]$ and $\varphi \in W^{1,1}([0,t]; C_c^1(\mathbb{R}\times\mathbb{R}^+))$ there holds:

$$\langle \nu_t, \varphi(t,\cdot)\rangle = -\int_0^t \langle \nu_s, \partial_t\varphi + u\cdot\nabla\varphi - g\varphi\rangle \mathrm{d}s. \tag{104}$$

We also remark that characteristics associated with u are well-defined at least locally in time. Indeed, given $s_0 \in (0,T)$ and $(x_1^0, x_2^0) \in \mathbb{R}\times(1/C_0, C_0)$ a characteristics on $(0,T)$ passing through (x_1^0, x_2^0) in s_0 is a solution to

$$\begin{cases} \dot{x}_1 = u_1(t,x_1,x_2), \\ \dot{x}_2 = u_2(t,x_1,x_2), \end{cases} \qquad \begin{cases} x_1(s_0) = x_1^0, \\ x_2(s_0) = x_2^0. \end{cases}$$

Because of our assumptions on u_1 we have that the equation on x_1 actually reads:

$$\dot{x}_1 = u_1(t,x_1),$$

where $\partial_1 u_1 \in L^1(0,T; L^\infty_\natural)$ so that this ODE enters the framework of the Cauchy-Lipschitz theorem. Then $t \mapsto x_1(t)$ is fixed and we may remark that u_2 is also Lipschitz with respect to x_2 yielding local existence and uniqueness of solutions to the equation on x_2. Classical Gronwall-like argument also imply that there exists $T_* < T$, depending only on u and C_0, for which any characteristics crossing $\mathbb{R}\times(1/2C_0, 2C_0)$ between 0 and T_* remains in $\mathbb{R}\times(0,4C_0)$.

Then, we introduce mollified velocities and source term $(u^\varepsilon, g^\varepsilon)_{\varepsilon>0}$ obtained by convolution with tensorized mollifiers $(\rho_\varepsilon)_{\varepsilon>0}$. Given the assumed regularity on u and g we have that $(u^\varepsilon, g^\varepsilon) \in L^1(0,T; C^1(\mathbb{R}\times\mathbb{R}^+))$. We shall use the following convergence afterwards:

- we have the classical convergences

$$\|u_2^\varepsilon - u_2\|_{L^1(0,T;L^\infty(\mathbb{R}\times(0,4C_0)))} + \|g^\varepsilon - g\|_{L^1(0,T;L^\infty(\mathbb{R}\times(0,4C_0))} = 0; \tag{105}$$

- thanks to (98)–(99) we have $\nabla u_1 \in L^1(0,T; L^\infty(\mathbb{R}\times(0,4C_0)))$ and:

$$\|u_1^\varepsilon - u_1\|_{L^1(0,T;L^\infty(\mathbb{R}\times(0,4C_0)))} \leq C\varepsilon; \tag{106}$$

- applying (100)–(102) in the computations of $\partial_2 g^\varepsilon$ and $\partial_2 u_2^\varepsilon$ and ∇u_1 we obtain the uniform bounds:

(107)
$$\|\partial_2 u_2^\varepsilon\|_{L^1(0,T;L^\infty(\mathbb{R}\times(0,4C_0)))} + \|\partial_2 g^\varepsilon\|_{L^1(0,T;L^\infty(\mathbb{R}\times(0,4C_0)))} + \|\nabla u_1^\varepsilon\|_{L^1(0,T;L^\infty(\mathbb{R}\times(0,4C_0)))} \leq C;$$

– applying (100)–(102) in the computations of $\partial_1 g^\varepsilon$ and $\partial_1 u_2^\varepsilon$ we obtain the estimates:

$$\|\partial_1 u_2^\varepsilon\|_{L^1(0,T;L^\infty(\mathbb{R}\times(0,4C_0)))} + \|\partial_1 g^\varepsilon\|_{L^1(0,T;L^\infty(\mathbb{R}\times(0,4C_0)))} \leq \frac{C}{\sqrt{\varepsilon}}. \tag{108}$$

Let now $t \in (0, T_*)$ and $\varphi^\natural \in C_c^1(\mathbb{R}\times\mathbb{R}^+)$ with support in $\mathbb{R}\times(1/2C_0, 2C_0)$ we construct now φ^ε solution to

$$\partial_t \varphi^\varepsilon + u^\varepsilon \cdot \nabla\varphi^\varepsilon = g^\varepsilon \varphi^\varepsilon \text{ on } (0,t)\times\mathbb{R}\times\mathbb{R}^+,$$
$$\varphi(t,\cdot) = \varphi^\natural \text{ on } \mathbb{R}\times\mathbb{R}^+.$$

Classical results on convection equations yield that φ^ε has the requested regularity to be a test-function in (104) for ε sufficiently small. In particular, the convergence of the flow associated with u^ε towards the flow associated with u ensures that, for ε sufficiently small, $\varphi^\varepsilon(s,\cdot)$ has compact support in $\mathbb{R}\times(0,4C_0)$ for any $s\in[0,t]$. Consequently, we have

$$\langle \nu_t, \varphi^\natural\rangle = -\int_0^t \langle \nu_s, (u-u^\varepsilon)\cdot\nabla\varphi^\varepsilon - (g-g^\varepsilon)\varphi^\varepsilon\rangle \mathrm{d}s.$$

This entails that:

$$|\langle \nu_t, \varphi^\natural\rangle| \leq C\,[I_1 + I_2 + I_3],$$

where :

$$I_1 = \int_0^t \|(u_1 - u_1^\varepsilon)\partial_1\varphi^\varepsilon\|_{L^\infty(\mathbb{R}\times(0,4C_0))},$$
$$I_2 = \int_0^t \|(u_2 - u_2^\varepsilon)\partial_2\varphi^\varepsilon\|_{L^\infty(\mathbb{R}\times(0,4C_0))},$$
$$I_3 = \int_0^t \|(g - g^\varepsilon)\varphi^\varepsilon\|_{L^\infty(\mathbb{R}\times(0,4C_0))}.$$

Concerning I_3 at first, we apply classical maximum-principle arguments yielding that, for any $s \in (0,t)$:

$$\|\varphi^\varepsilon(s,\cdot)\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} \leq \|\varphi^\natural\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} \exp\left(\int_0^t \|g^\varepsilon\|_{L^\infty(\mathbb{R}\times(0,4C_0))}\right).$$

Due to the convergence of g^ε towards g we obtain that φ^ε is uniformly bounded independently of ε and that

$$|I_3| \leq C\int_0^t \|g - g^\varepsilon\|_{L^\infty(\mathbb{R}\times(0,4C_0))} \to 0 \quad \text{when } \varepsilon\to 0.$$

Then, we differentiate the transport equation for φ^ε w.r.t. x_2. As $\partial_2 u_1^\varepsilon = 0$, we obtain that $\varphi_2^\varepsilon = \partial_2\varphi^\varepsilon$ satisfies:

$$\partial_t \varphi_2^\varepsilon + u^\varepsilon\cdot\nabla\varphi_2^\varepsilon = \partial_2 g^\varepsilon \varphi^\varepsilon + g^\varepsilon\varphi_2^\varepsilon - \partial_2 u_2^\varepsilon \varphi_2^\varepsilon \text{ on } (0,t)\times\mathbb{R}\times\mathbb{R}^+,$$
$$\varphi_2^\varepsilon(t,\cdot) = \partial_2\varphi^\natural \text{ on } \mathbb{R}\times\mathbb{R}_+.$$

Refering again to a maximum principle argument for transport equations, we obtain that, for any $s \in (0,t)$:

$$\|\varphi_2^\varepsilon(s,\cdot)\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} \leq \left(\|\partial_2\varphi^\natural\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} + \int_0^t \|\partial_2 g^\varepsilon\varphi^\varepsilon\|_{L^\infty(\mathbb{R}\times(0,4C_0))}\right) \exp\left(\int_0^t \|g^\varepsilon\|_{L^\infty(\mathbb{R}\times(0,4C_0))} + \|\partial_2 u_2\|_{L^\infty(\mathbb{R}\times(0,4C_0))}\right).$$

Applying the uniform bound on φ^ε together with (107) we get:

$$\|\varphi_2^\varepsilon(s,\cdot)\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} \leq C, \quad \forall\, s \in (0,t).$$

Combining this remark with the convergence (105) we obtain then:

$$\begin{aligned}|I_2| &\leq C \sup_{(0,t)} \|\varphi_2^\varepsilon\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} \int_0^t \|(u_2^\varepsilon - u_2)\|_{L^\infty(\mathbb{R}\times(0,4C_0))}\\ &\leq C\int_0^t \|(u_2^\varepsilon - u_2)\|_{L^\infty(\mathbb{R}\times(0,4C_0))} \to 0 \quad \text{when } \varepsilon \to 0.\end{aligned}$$

Finally, to compute I_1 we differentiate the transport equation for φ^ε w.r.t. x_1. We obtain that $\varphi_1^\varepsilon = \partial_1\varphi^\varepsilon$ satisfies:

$$\begin{aligned}&\partial_t\varphi_1^\varepsilon + u^\varepsilon\cdot\nabla\varphi_1^\varepsilon = \partial_1 g^\varepsilon\varphi^\varepsilon + g^\varepsilon\varphi_1^\varepsilon - \partial_1 u_2^\varepsilon\varphi_2^\varepsilon - \partial_1 u_1^\varepsilon\varphi_1^\varepsilon \text{ on } (0,t)\times\mathbb{R}\times\mathbb{R}^+,\\ &\varphi_1^\varepsilon(t,\cdot) = \partial_1\varphi^\natural \text{ on } \mathbb{R}\times\mathbb{R}^+.\end{aligned}$$

Again, this yields that, for any $s \in (0,t)$:

$$\|\varphi_1^\varepsilon(s,\cdot)\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} \leq \left(\|\partial_1\varphi^\natural\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} + \int_0^t (\|\partial_1 g^\varepsilon\varphi^\varepsilon\|_{L^\infty(\mathbb{R}\times(0,4C_0))} + \|\partial_1 u_2^\varepsilon\varphi_2^\varepsilon\|_{L^\infty(\mathbb{R}\times(0,4C_0))})\right) \exp\left(\int_0^t \|g^\varepsilon\|_{L^\infty(\mathbb{R}\times(0,4C_0))} + \|\partial_1 u_1^\varepsilon\|_{L^\infty(\mathbb{R}\times(0,4C_0))}\right).$$

Applying the uniform bound on φ^ε and φ_2^ε with (107) and (108) we conclude that

$$\sup_{s\in(0,t)} \|\varphi_1^\varepsilon(s,\cdot)\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} \leq \frac{C}{\sqrt{\varepsilon}}.$$

Combining this remark with the convergence (106) we obtain then:

$$\begin{aligned}|I_1| &\leq C \sup_{(0,t)} \|\varphi_1^\varepsilon\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)} \int_0^t \|(u_1^\varepsilon - u_1)\|_{L^\infty(\mathbb{R}\times(0,4C_0))}\\ &\leq C\sqrt{\varepsilon} \to 0 \quad \text{when } \varepsilon \to 0.\end{aligned}$$

Finally, we have $\langle \nu_t, \varphi^\natural\rangle = 0$ whatever the value of $\varphi^\natural$. As ν_t has compact support in $(1/2C_0, 2C_0)$ we conclude that $\nu_t = 0$ globally. □

We then construct solutions for initial data which are convex combinations of Dirac measures. Namely, we assume that there exists $(\alpha_i^0, \rho_i^0)_{i=1,\dots,k} \in [L^\infty_\natural]^{2k}$ satisfying:

$$0 \le \alpha_i^0(x) \le 1 \quad \sum_{i=0}^{k} \alpha_i^0(x) = 1 \quad \text{a.e. in } \mathbb{R}, \tag{109}$$

$$\frac{1}{C_0} \le \rho_i^0(x) \le C_0 \quad \text{a.e. in } \mathbb{R}, \tag{110}$$

and we consider the initial data for (96) that reads:

$$\nu^0 = \sum_{i=1}^{k} \alpha_i^0(x)\delta_{\xi=\rho_i^0(x)}. \tag{111}$$

We show that we can construct a solution to (96) with the same structure (under the above assumptions on the velocity u and g). Namely, there holds:

Lemma 14. – *Let* (109)–(110)-(111) *hold true. There exists* $T_0 < T$ *and*

$$(\alpha_i, \rho_i) \in L^\infty((0, T_0) \times \mathbb{R}) \cap C([0, T_0]; L^1_\natural)$$

satisfying

$$0 \le \alpha_i(t, x) \le 1 \quad \sum_{i=0}^{k} \alpha_i(t, x) = 1 \quad a.e. \tag{112}$$

$$\frac{1}{2C_0} \le \rho_i(t, x) \le 2C_0 \quad a.e., \tag{113}$$

such that $\nu = \sum_{i=1}^{k} \alpha_i \delta_{\xi=\rho_i} \in C([0, T_0]; \mathscr{Y}_\natural)$ *is a solution to* (96)–(97).

Proof. – The proof is straightforward. Let $\nu = \sum_{i=1}^{k} \alpha_i \delta_{\xi=\rho_i}$ with (α_i, ρ_i) as in the statement of the theorem. We have thus that, for arbitrary $\phi \in C_c(\mathbb{R} \times [0, M])$, there holds:

$$\langle \nu_t, \phi\rangle = \int_0^L \sum_{i=1}^{k} \alpha_i(t, x)\phi(x, \rho_i(t, x))\mathrm{d}x. \tag{114}$$

Hence $\langle \nu, \phi\rangle \in C([0, T_0])$ with $|\langle \nu_t, \phi\rangle| \le L\|\phi\|_{L^\infty_\natural}$ so that we have indeed $\nu \in C([0, T_0]; \mathscr{Y}_\natural)$.

Then, applying a classical density argument, we obtain that ν is charaterized by its action on tensorized test-functions $(x, \xi) \mapsto \psi(x)\beta(\xi)$. Plugging $\psi \otimes \beta$ as test-function in (96)–(97), we obtain the following equations:

$$\partial_t \sum_{i=1}^{k} \alpha_i \beta(\rho_i) + \partial_1 \sum_{i=1}^{k} \alpha_i \beta(\rho_i) u_1(\cdot, \cdot, \rho_i) = \sum_{i=1}^{k} \alpha_i u_2(\cdot, \cdot, \rho_i)\beta'(\rho_i) - \sum_{i=1}^{k} \alpha_i g(\cdot, \cdot, \rho_i)\beta(\rho_i).$$

$$\sum_{i=1}^{k} \alpha_i \beta(\rho_i)_{|_{t=0}} = \sum_{i=1}^{k} \alpha_i^0 \beta(\rho_i^0).$$

Finally, we obtain that ν is a solution to (96)–(97) if the (α_i, ρ_i) satisfy simultaneously:

$$\partial_t \alpha_i + \partial_1 (\alpha_i u_1) + \alpha_i g(\cdot, \cdot, \rho_i) = 0 \tag{115}$$

$$[\alpha_i]_{|_{t=0}} = \alpha_i^0 \tag{116}$$

and

$$\partial_t \rho_i + u_1 \partial_1 \rho_i - u_2(\cdot,\cdot,\rho_i) = 0 \tag{117}$$

$$[\rho_i]_{|t=0} = \rho_i^0. \tag{118}$$

Remark that we introduced that u_1 does not depend on ρ_i (by assumption). The existence of a solution

$$(\alpha_i, \rho_i) \in L^\infty((0,T_0)\times\mathbb{R}) \cap C([0,T_0]; L^1_\natural)$$

to this system satisfying (112)–(113) follows from a straightforward adaptation of Di Perna-Lions arguments in the spirit of [3, Lemma 2]. □

Appendix

Formal calculation versus Young measure method

Let us compare in this appendix the system obtained through a formal WKB method and the system derived using kinetic formulation and characterization of the Young measures. With the Young measure method in the two-fluid setting, we get the following equation on α_+:

$$\partial_t \alpha_+ + u\partial_x \alpha_+ + \alpha_+ \partial_x u = \frac{\alpha_+}{\mu_+}\left[\frac{1}{\frac{\alpha_+}{\mu_+} + \frac{\alpha_-}{\mu_-}}\left(\partial_x u - \left(\alpha_+ \frac{p_+}{\mu_+} + \alpha_- \frac{p_-}{\mu_-}\right)\right) + p_+\right].$$

Thus we get the following equation

$$\partial_t \alpha_+ + u\partial_x \alpha_+ = \kappa_1 \partial_x u + \kappa_2$$

with

$$\begin{aligned}
\kappa_1 &= \frac{\alpha_+}{\mu_+}\frac{1}{\frac{\alpha_+}{\mu_+} + \frac{\alpha_-}{\mu_-}} - \alpha_+ \\
&= \frac{\alpha_+}{\mu_+}\frac{1 - \mu_+(\frac{\alpha_+}{\mu_+} + \frac{\alpha_-}{\mu_-})}{\frac{\alpha_+}{\mu_+} + \frac{\alpha_-}{\mu_-}} \\
&= \frac{\alpha_+\alpha_-}{\mu_+}\frac{1 - \frac{\mu_+}{\mu_-}}{\frac{\alpha_+}{\mu_+} + \frac{\alpha_-}{\mu_-}} = \frac{\alpha_+\alpha_-(\mu_- - \mu_+)}{\alpha_+\mu_+ + \alpha_-\mu_-}
\end{aligned}$$

and

$$\begin{aligned}
\kappa_2 &= \frac{\alpha_+}{\mu_+}\left[p_+ - \frac{1}{\frac{\alpha_+}{\mu_+} + \frac{\alpha_-}{\mu_-}}\left(\alpha_+ \frac{p_+}{\mu_+} + \alpha_- \frac{p_-}{\mu_-}\right)\right] \\
&= \frac{\alpha_+\alpha_-}{\mu_+\mu_-}\frac{1}{\frac{\alpha_+}{\mu_+} + \frac{\alpha_-}{\mu_-}}(p_+ - p_-) \\
&= \frac{\alpha_+\alpha_-}{\alpha_-\mu_+ + \alpha_+\mu_-}(p_+ - p_-).
\end{aligned}$$

This reads

$$\partial_t \alpha_+ + u\partial_x \alpha_+ = \frac{\alpha_+\alpha_-}{\alpha_-\mu_+ + \alpha_+\mu_-}\left[(p_+ - p_-) + (\mu_- - \mu_+)\partial_x u\right].$$

As for the momentum equation, we obtain :

$$\partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x(m^\infty \partial_x u) + \partial_x \pi^\infty = 0,$$

where

$$m^\infty = \frac{1}{\frac{\alpha_+}{\mu_+} + \frac{\alpha_-}{\mu_-}} = \frac{\mu_+\mu_-}{\alpha_+\mu_- + \alpha_-\mu_+}$$

and $\pi^\infty = m^\infty \kappa^\infty$ with

$$\kappa^\infty = \alpha_+ \frac{p_+}{\mu_+} + \alpha_- \frac{p_-}{\mu_-}$$

and thus :

$$\pi^\infty = m^\infty \kappa^\infty = \frac{\alpha_+ p_+ \mu_- + \alpha_- p_- \mu_+}{\alpha_+\mu_- + \alpha_-\mu_+}.$$

This is, up to the notations, the system obtained using the WKB method.

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BIBLIOGRAPHY

[1] A. A. Amosov, A. A. Zlotnik, On the error of quasi-averaging of the equations of motion of a viscous barotropic medium with rapidly oscillating data, *Comp. Maths. Math. Phys.* **36** (1996), 1415–1428.

[2] D. Bresch, B. Desjardins, Sur la théorie globale des équations de Navier-Stokes compressibles, in *Actes des 33es journées Équations aux dérivées partielles, Evian*, 2006.

[3] D. Bresch, M. Hillairet, Note on the derivation of multi-component flow systems, *Proc. Amer. Math. Soc.* **143** (2015), 3429–3443.

[4] D. Bresch, X. Huang, A multi-fluid compressible system as the limit of weak solutions of the isentropic compressible Navier-Stokes equations, *Arch. Ration. Mech. Anal.* **201** (2011), 647–680.

[5] D. A. Drew, S. L. Passman, *Theory of multicomponent fluids*, Applied Mathematical Sciences **135**, Springer, 1999.

[6] E. Feireisl, A. Novotný, H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations, *J. Math. Fluid Mech.* **3** (2001), 358–392.

[7] B. Haspot, Existence of global strong solution for the compressible Navier-Stokes equations with degenerate viscosity coefficients in 1D, *Math. Nachr.* **291** (2018), 2188–2203.

[8] M. Hillairet, Propagation of density-oscillations in solutions to the barotropic compressible Navier-Stokes system, *J. Math. Fluid Mech.* **9** (2007), 343–376.

[9] M. Ishii, T. Hibiki, *Thermo-fluid dynamics of two-phase flow*, Springer, 2006.

[10] J. I. Kanel′, A model system of equations for the one-dimensional motion of a gas, *J. Diff. Eq.* **4** (1968), 374–380.

[11] P.-L. Lions, *Mathematical topics in fluid mechanics. Vol. 2*, Oxford Lecture Series in Mathematics and its Applications, The Clarendon Press Univ. Press, 1998.

[12] A. Mellet, A. Vasseur, Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations, *SIAM J. Math. Anal.* **39** (2007/08), 1344–1365.

[13] P. Plotnikov, J. Sokołowski, *Compressible Navier-Stokes equations*, Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) **73**, Birkhäuser, 2012.

[14] D. Serre, Variations de grande amplitude pour la densité d'un fluide visqueux compressible, *Phys. D* **48** (1991), 113–128.

[15] D. Serre, Asymptotics of homogeneous oscillations in a compressible viscous fluid, *Bol. Soc. Brasil. Mat. (N.S.)* **32** (2001), 435–442.

[16] L. Tartar, The compensated compactness method applied to systems of conservation laws, in *Systems of nonlinear partial differential equations (Oxford, 1982)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **111**, Reidel, 1983, 263–285.

[17] E. Weinan, Propagation of oscillations in the solutions of 1-D compressible fluid equations, *Comm. Partial Differential Equations* **17** (1992), 347–370.

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