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COMPLEX HYPERBOLIC VOLUME AND INTERSECTION OF BOUNDARY DIVISORS IN MODULI SPACES OF POINTED GENUS ZERO CURVES

BY VINCENT KOZIARZ AND DUC-MANH NGUYEN

ABSTRACT. – We show that the complex hyperbolic metrics defined by Deligne-Mostow and Thurston on $\mathcal{M}_{0,n}$ are singular Kähler-Einstein metrics when $\mathcal{M}_{0,n}$ is embedded in the Deligne-Mumford-Knudsen compactification $\overline{\mathcal{M}}_{0,n}$. As a consequence, we obtain a formula computing the volume of $\mathcal{M}_{0,n}$ with respect to these metrics using intersection of boundary divisors of $\overline{\mathcal{M}}_{0,n}$. In the case of rational weights, following an idea of Y. Kawamata, we show that these metrics actually represent the first Chern class of some line bundles on $\overline{\mathcal{M}}_{0,n}$, from which other formulas computing the same volumes are derived.

RÉSUMÉ. – Nous démontrons que les métriques hyperboliques complexes introduites par Deligne-Mostow et Thurston sur l'espace de modules de surfaces de Riemann de genre zéro avec n points marqués $\mathcal{M}_{0,n}$ sont des métriques Kähler-Einstein singulières sur la compactification de Deligne-Mumford-Knudsen $\overline{\mathcal{M}}_{0,n}$. Nous en déduisons des formules calculant le volume de $\mathcal{M}_{0,n}$ muni de ces métriques en fonction des nombres d'intersection des diviseurs de bord de $\overline{\mathcal{M}}_{0,n}$. De plus, lorsque les poids sont tous rationnels, en développant une idée de Y. Kawamata, nous montrons que ces métriques sont aussi des représentants de la première classe de Chern de certains fibrés en droites sur $\overline{\mathcal{M}}_{0,n}$, ce qui nous permet d'obtenir d'autres formules calculant les mêmes volumes.

1. Introduction

Let $n \geq 3$ and $\mathcal{M}_{0,n}$ be the moduli space of Riemann surfaces of genus 0 with n marked points. Let $\mu = (\mu_1, \dots, \mu_n)$ be real weights satisfying $0 < \mu_s < 1$ and $\sum \mu_s = 2$. Following ideas of E. Picard, P. Deligne and G. D. Mostow [5] constructed—for certain rational values of the μ_s's satisfying some integrality conditions—complex hyperbolic lattices which enable in particular to endow $\mathcal{M}_{0,n}$ with a complex hyperbolic metric Ω_μ. The volume of the corresponding orbifolds has been computed by several authors in some special cases when $n = 5$ (see e.g., [24, 20, 19, 13]).

A few years later, W. P. Thurston noticed [22] that for any n-uple of real weights satisfying the two simple conditions above, one can construct naturally a metric completion $\overline{\mathcal{M}}_{0,n}^{\mu}$

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of $(\mathcal{M}_{0,n}, \Omega_\mu)$, which can be endowed with a *cone manifold structure*. He observed in particular that $(\mathcal{M}_{0,n}, \Omega_\mu)$ always has finite volume (see Section 7.4 for our normalization of the metric and the volume element; we will use equally the notation Ω_μ for the metric and its associated Kähler form). In a more recent paper [18], C. T. McMullen proved a Gauss-Bonnet theorem for cone manifolds from which he derived a formula for the volume of $(\mathcal{M}_{0,n}, \Omega_\mu)$.

The main purpose in this paper is to investigate those complex hyperbolic metrics by using ideas coming from complex (algebraic) geometry with an approach in the spirit of Chapter 17 of [6]. We prove in particular that the extension by zero $\tilde{\Omega}_\mu$ of Ω_μ is a well defined closed positive current on the Deligne-Mumford-Knudsen compactification $\overline{\mathcal{M}}_{0,n}$ of $\mathcal{M}_{0,n}$, and that it is actually a *singular Kähler-Einstein metric* on $\overline{\mathcal{M}}_{0,n}$, associated with a boundary divisor that we make explicit. As a consequence, we show that the volume of $\mathcal{M}_{0,n}$ with respect to Ω_μ can be computed from the intersection numbers of boundary divisors in $\overline{\mathcal{M}}_{0,n}$.

In order to state more precisely our main results, we need a few basic facts about $\overline{\mathcal{M}}_{0,n}$ (see *e.g* [7, 16, 15, 1]). The moduli space $\mathcal{M}_{0,n}$ has complex dimension $N := n-3$ and its complement in the smooth variety $\overline{\mathcal{M}}_{0,n}$ is the union of finitely many divisors called *boundary divisors*, or *vital divisors*, each of which uniquely corresponds to a partition of $\{1,\dots,n\}$ into two subsets $I_0 \sqcup I_1$ such that $\min\{|I_0|,|I_1|\} \geq 2$, see [15] for instance. We will denote by $\mathscr{P}$ the set of partitions satisfying this condition. For each partition $\mathcal{S} := \{I_0, I_1\} \in \mathscr{P}$, we denote by $D_{\mathcal{S}}$ the corresponding divisor in $\overline{\mathcal{M}}_{0,n}$. Exchanging I_0 and I_1 if necessary, we will always assume that $\mu_{\mathcal{S}} := \sum_{s\in I_1} \mu_s \leq 1$ (in order to lighten the notation, we do not write explicitly the dependence of the coefficients $\mu_{\mathcal{S}}$ on μ).

For any $s \in \{1,\dots,n\}$, we also define the divisor class ψ_s on $\overline{\mathcal{M}}_{0,n}$ associated to the pullback of the relative cotangent bundle of the universal curve by the section corresponding to the s-th marked point.

Finally, if D is a divisor on $\overline{\mathcal{M}}_{0,n}$, D^N means as usual that we take the N-th self-intersection of D. Our main result concerns the cohomology class of $\tilde{\Omega}_\mu$.

THEOREM 1.1. – *Let $n \geq 4$ and $\mathcal{M}_{0,n}$ be the moduli space of Riemann surfaces of genus* 0 *with n marked points. Let $\mu = (\mu_1,\dots,\mu_n)$ be real weights satisfying* $0 < \mu_s < 1$ *and* $\sum \mu_s = 2$. *Let $D_\mu := \sum_{\mathcal{S}\in\mathscr{P}} \lambda_{\mathcal{S}} D_{\mathcal{S}}$ where*

$$\lambda_{\mathcal{S}} = (|I_1|-1)(\mu_{\mathcal{S}}-1)+1.$$

Let $\tilde{\Omega}_\mu$ be the current on $\overline{\mathcal{M}}_{0,n}$ defined by the extension by zero of Ω_μ. Then $\tilde{\Omega}_\mu$ is the Kähler form of a singular Kähler-Einstein metric for the pair $(\overline{\mathcal{M}}_{0,n}, D_\mu)$, hence $\tilde{\Omega}_\mu$ is a current which represents the same cohomology class as the $\mathbb{R}$-divisor $\frac{1}{N+1}(K_{\overline{\mathcal{M}}_{0,n}} + D_\mu)$, where $K_{\overline{\mathcal{M}}_{0,n}}$ is the canonical divisor of $\overline{\mathcal{M}}_{0,n}$. Moreover, the volume of $(\mathcal{M}_{0,n}, \Omega_\mu)$ satisfies

$$(1)\quad \mathrm{Vol}(\mathcal{M}_{0,n}, \Omega_\mu) := \int_{\mathcal{M}_{0,n}} \Omega_\mu^N = \frac{1}{(N+1)^N}\left(K_{\overline{\mathcal{M}}_{0,n}} + D_\mu\right)^N$$

$$= \frac{1}{(N+1)^N}\left(\sum_{\mathcal{S}}\left(|I_1|-1\right)\left(\mu_{\mathcal{S}} - \frac{|I_1|}{N+2}\right) D_{\mathcal{S}}\right)^N$$

$$= \frac{1}{2^N}\left(-\sum_{s=1}^{n}\mu_s\,\psi_s + \sum_{\mathcal{S}}\mu_{\mathcal{S}}\,D_{\mathcal{S}}\right)^N.$$

REMARK 1.2. – Formula (1) for the volume is not an immediate consequence of the fact that the divisor $K_{\overline{\mathcal{M}}_{0,n}} + D_\mu$ and the current $(N+1)\,\tilde{\Omega}_\mu$ represent the same cohomology class. In particular, it does not make sense in general to compute the power of a current and we need to control the behavior of Ω_μ^N near the boundary of $\overline{\mathcal{M}}_{0,n}$, see Proposition 7.1 for a more precise statement.

- There exists a completely explicit algorithm to calculate the intersection numbers of divisors of the type $D_{\mathcal{S}}$ (see [17] and Appendix A below), but doing the calculation by hand is rather involved. In Appendices B and C we compute $\mathrm{Vol}(\mathcal{M}_{0,5}, \Omega_\mu)$ for all admissible weight vectors μ, and $\mathrm{Vol}(\mathcal{M}_{0,6}, \Omega_\mu)$ for some examples of μ. As another application, the covolume of Deligne-Mostow lattices can be calculated with our formula and the help of a computer program by C. Faber which computes intersections of boundary divisors in $\overline{\mathcal{M}}_{0,n}$. In this way, we recover the results of [18, Table 1].
- In Corollary 7.5, we compare formula (1) with the one in [18, Th. 1.2]. The two approaches are different: in formula (1) we relate the volume to the top self-intersection of the “orbifold” first Chern class of $(\overline{\mathcal{M}}_{0,n}, D_\mu)$, while McMullen relates it to the *cone manifold Euler characteristic* of Thurston’s completion $\overline{\mathcal{M}}_{0,n}^{\mu}$ of $\mathcal{M}_{0,n}$. Note that $\overline{\mathcal{M}}_{0,n}^{\mu}$ does not play any role in our proof. From the perspective of numerical computations, McMullen’s formula is more practical since the Euler characteristic of $\overline{\mathcal{M}}_{0,n}^{\mu}$ can be calculated from a rather simple formula.

Our approach also sheds light on the relation between Thurston’s completion $\overline{\mathcal{M}}_{0,n}^{\mu}$ of $\mathcal{M}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}$. Recall that Thurston identified $\mathcal{M}_{0,n}$ with the space of flat surfaces homeomorphic to the sphere $\mathbb{S}^2$ having n conical singularities with cone angles given by $2\pi(1-\mu_s)$ up to rescaling. A stratum of $\overline{\mathcal{M}}_{0,n}^{\mu}$ consists of flat surfaces which are the limits when some clusters of singularities collapse into points. On the other hand, each stratum of $\overline{\mathcal{M}}_{0,n}$ is encoded by a tree whose vertices are labeled by the subsets in a partition of $\{1,\dots,n\}$. Every point in such a stratum represents a stable curve with several irreducible components. Among those components, there is a particular one that we call *μ-principal* whose definition depends on μ (see Section 5.1). To each stratum S of $\overline{\mathcal{M}}_{0,n}^{\mu}$, we have a corresponding stratum $\tilde{S}$ of $\overline{\mathcal{M}}_{0,n}$ such that, for any flat surface represented by a point in S, the underlying Riemann surface with punctures is isomorphic to the μ-principal component of the stable curves represented by some points in $\tilde{S}$. So in some sense, one can say that $\overline{\mathcal{M}}_{0,n}^{\mu}$ is obtained from $\overline{\mathcal{M}}_{0,n}$ by “contracting” every boundary stratum to its μ-principal factor.

In the case when there is no subset I of $\{1,\dots,n\}$ such that $\sum_{i\in I}\mu_i = 1$, $\overline{\mathcal{M}}_{0,n}^{\mu}$ is actually a compactification of $\mathcal{M}_{0,n}$. In the literature, one can find other compactifications of $\mathcal{M}_{0,n}$ which are different from the Deligne-Mumford-Knudsen one $\overline{\mathcal{M}}_{0,n}$ (see in particular the papers of B. Hasset [12] and D. I. Smyth [21]). These compactifications are contractions of $\overline{\mathscr{M}_{0,n}}$ and are in general singular. Actually, when compact, Thurston completions corresponding to weights μ as above are compactifications considered by Smyth, but for our purpose it is more convenient to work on the smooth model $\overline{\mathscr{M}_{0,n}}$ and we will not insist on this point of view (see also Remark 6.9 below).

By construction, Ω_μ is the curvature of a Hermitian metric on a holomorphic line bundle over $\mathscr{M}_{0,n}$. When all the weights in μ are rational, Y. Kawamata [14] observed that this line bundle admits a natural extension to $\overline{\mathscr{M}_{0,n}}$. It turns out that Ω_μ can be considered as a representative in the sense of currents of the first Chern class of this extended line bundle. It can be shown that the latter is effective. We develop this algebro-geometric approach in Section 8. By constructing explicit sections and determining their zero divisor, we provide other formulas for the volume which avoid metric considerations. Even though at first glance this approach seems to work only in the case of rational weights, by a continuity argument, our formulas are actually valid for all values of μ satisfying the hypothesis of Theorem 1.1. Namely, we get the following

THEOREM 1.3. – *For each* $1 \leq s < s' \leq n$, *define*

$$\lambda(s,s') = \begin{cases} 0 \text{ if } \sum_{k=s}^{s'} \mu_k \leq 1 \text{ or } \sum_{k=s+1}^{s'-1} \mu_k \geq 1, \\ \min\{\mu_s, \mu_{s'}, \sum_{k=s}^{s'} \mu_k - 1, 1 - \sum_{k=s+1}^{s'-1} \mu_k\} \textit{ otherwise,} \end{cases}$$

and

$$\delta_{\mathscr{S}}(s,s') = \begin{cases} 1 \text{ if } \{s,s'\} \subset I_1 \\ 0 \textit{ otherwise.} \end{cases}$$

Then the effective $\mathbb{R}$-*divisor*

$$D_\sigma := \sum_{\mathscr{S}} \sum_{1 \leq s < s' \leq n} \delta_{\mathscr{S}}(s,s')\lambda(s,s')\, D_{\mathscr{S}}$$

represents the same cohomology class as the current $\tilde{\Omega}_\mu$ *(that is the extension of* Ω_μ *by* 0 *to* $\overline{\mathscr{M}_{0,n}}$*), and we have*

$$\operatorname{Vol}(\mathscr{M}_{0,n}, \Omega_\mu) := \int_{\mathscr{M}_{0,n}} \Omega_\mu^N = \frac{1}{(N+1)^N}(K_{\overline{\mathscr{M}_{0,n}}} + D_\mu)^N = D_\sigma^N. \tag{2}$$

In this paper, many objects and quantities depend on the weights μ. However, as we already said for the coefficients $\mu_{\mathscr{S}}$, this dependence will not always appear explicitly but the reader will have to keep it in mind.

REMARK 1.4. – Whenever there exists a partition $\{I_0, I_1\} \in \mathscr{P}$ such that $\sum_{s\in I_0} \mu_s = \sum_{s\in I_1} \mu_s = 1$, the metric completion of Thurston is not compact and our method does not provide directly a formula for the volume of $(\mathscr{M}_{0,n}, \Omega_\mu)$. However, the formulas in Theorem 1.1 remain valid by continuity arguments (as in [18]). For these reasons, we will assume throughout this paper that the sum of the weights for indices in any subset of $\{1,\dots,n\}$ is always different from 1.

Outline

The paper is organized as follows.

1. In Section 2 we collect the necessary background from the paper of Deligne and Mostow [5]. Associated to any weight vector $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n_{>0}$ such that $\mu_1 + \cdots + \mu_n = 2$, we have a rank one local system $\mathbf{L}$ on the punctured sphere $\mathbb{P}^1_{\mathbb{C}} \setminus \{x_1, \dots, x_n\}$ with monodromy $\exp(2\imath\pi\mu_s)$ at x_s, which is equipped with a Hermitian metric. Assuming $\mu_s \notin \mathbb{Z}$ for some $s \in \{1, \dots, n\}$, we have $\dim_{\mathbb{C}} H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \{x_1, \dots, x_s\}, \mathbf{L}) = n - 2$. Up to a multiplicative constant, there exists a unique section ω of the bundle $\Omega^1(\mathbf{L})$ which is holomorphic on $\mathbb{P}^1_{\mathbb{C}} \setminus \{x_1, \dots, x_n\}$, and has valuation $-\mu_s$ at x_s. This section defines a non-zero cohomology class in $H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \{x_1, \dots, x_n\}, \mathbf{L})$. One can obviously move the points $x_1, \dots, x_n$ around, therefore $H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \{x_1, \dots, x_n\}, \mathbf{L})$ and ω give rise to a local system $\mathbf{H}$ of rank $n - 2$ and a holomorphic line bundle $\mathcal{L}$ on $\mathcal{M}_{0,n}$, the fiber of $\mathcal{L}$ over the point $m \simeq (\mathbb{P}^1_{\mathbb{C}}, \{x_1, \dots, x_n\}) \in \mathcal{M}_{0,n}$ being the line generated by ω in $H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \{x_1, \dots, x_n\}, \mathbf{L})$. Projectivizing $\mathbf{H}$, we get a flat $\mathbb{P}^{n-3}_{\mathbb{C}}$-bundle over $\mathcal{M}_{0,n}$, and $\mathcal{L}$ provides us with a multivalued section Ξ_μ of this bundle. The pullback $\widetilde{\Xi}_\mu$ of Ξ_μ to $\widetilde{\mathcal{M}_{0,n}}$ is an étale mapping from $\widetilde{\mathcal{M}_{0,n}}$ to $\mathbb{P}^{n-3}_{\mathbb{C}}$.

   The Hermitian form of $\mathbf{L}$ gives rise to a Hermitian form $((.,.))$ on $H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \{x_1, \dots, x_n\}, \mathbf{L})$. In the case $0 < \mu_s < 1$ for all s, this Hermitian form has signature $(1, n-3)$ and $((\omega, \omega)) > 0$. It follows that the section $\widetilde{\Xi}_\mu$ takes values in the ball $\mathbf{B} := \{\langle v \rangle \in \mathbb{P}^{n-3}_{\mathbb{C}},\ ((v, v)) > 0\} \subset \mathbb{P}^{n-3}_{\mathbb{C}}$ (here we identify $H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \{x_1, \dots, x_n\}, \mathbf{L})$ with $\mathbb{C}^{n-2}$). The pullback of the canonical metric on $\mathbf{B}$ by $\widetilde{\Xi}_\mu$ provides us with a complex hyperbolic metric on $\mathcal{M}_{0,n}$, which will be denoted by Ω_μ. By definition, Ω_μ is also the Chern form of the Hermitian line bundle $(\mathcal{L}, ((.,.)))$.
2. Our goal is to show that Ω_μ is a singular Kähler-Einstein metric on $\overline{\mathcal{M}_{0,n}}$. For this purpose, we first construct trivializing holomorphic sections of $\mathcal{L}$ in the neighborhood of every point $m \in \partial\overline{\mathcal{M}_{0,n}}$. In Section 3, we recall the construction of local coordinates of $\overline{\mathcal{M}_{0,n}}$ near m by plumbing families. In Section 4, we consider the case where m is contained in a stratum of codimension one in $\overline{\mathcal{M}_{0,n}}$, which means that m represents a stable curve having two genus zero components, denoted by C^0 and C^1, joined at a node. In each component, we assign a positive weight to the point corresponding to the node of m such that the weights associated to all the marked points add up to 2. We have on C^i a rank one local system $\mathbf{L}_i$ and a section ω_i of $\Omega^1(\mathbf{L}_i)$ in the same way as we had $\mathbf{L}$ and ω above. The sections ω_0 and ω_1 will be used as data for the construction of a plumbing family representing a neighborhood $\mathcal{U}$ of m in $\overline{\mathcal{M}_{0,n}}$. As a by-product, we get a holomorphic non-vanishing section Φ of $\mathcal{L}$ in $\mathcal{U} \cap \mathcal{M}_{0,n}$. In Section 5, we generalize this construction to the case where m is contained in a stratum of codimension r with $r > 1$.
3. Section 6 is devoted to the proof of a formula for the Hermitian norm of the section Φ (see Proposition 6.1). The idea of the proof is to use the flat metric approach of Thurston. We start by relating the point of views of Deligne-Mostow and Thurston. Each holomorphic section of $\Omega^1(\mathbf{L})$ on $\mathbb{P}^1_{\mathbb{C}} \setminus \{x_1, \dots, x_n\}$ with valuation $-\mu_s$ at x_s

defines a flat metric on $\mathbb{P}^1_{\mathbb{C}}$ with cone singularities at $x_1, \dots, x_n$. Its Hermitian norm with respect to $((.,.))$ is precisely the area of this flat surface. In [22], Thurston introduced a method to compute this area by performing some surgeries on the flat surface, and obtained in particular an alternative proof that the signature of $((.,.))$ is $(1, n-3)$. We will use the same method to compute the Hermitian norm of $\omega' = \Phi(m')$, where Φ is the section of $\mathscr{L}$ mentioned above and $m' \in \mathscr{U} \cap \mathscr{M}_{0,n}$. As a direct consequence, we obtain a rather explicit formula for the metric Ω_μ near the boundary of $\overline{\mathscr{M}}_{0,n}$ (see Proposition 6.2).

4. In Section 7, we recall some basic facts about singular Kähler-Einstein metrics. It follows immediately from Proposition 6.2 that Ω_μ is a singular Kähler-Einstein metric attached to the pair $(\overline{\mathscr{M}}_{0,n}, D_\mu)$. Theorem 1.1 is then a straightforward consequence of this fact. Comparing Ω_μ with the complex hyperbolic metric considered by McMullen in [18], we get Corollary 7.5.
5. In Section 8, following an idea of Kawamata [14], we construct an extension $\hat{\mathscr{L}}$ of $\mathscr{L}$ to $\overline{\mathscr{M}}_{0,n}$ in the case when all weights μ_s are rational. This extension is the pushforward of a rank one locally free sheaf on the universal curve $\overline{\mathscr{C}}_{0,n}$. By construction, Φ extends naturally to a trivializing holomorphic section of $\hat{\mathscr{L}}$ on $\mathscr{U}$, and Ω_μ is a representative (in the sense of currents) of the first Chern class of $\hat{\mathscr{L}}$. This leads to an alternative method to compute the volume of $\mathscr{M}_{0,n}$ with respect to Ω_μ by using sections of $\hat{\mathscr{L}}$ (see Theorem 8.4). Simplifying a construction by Kawamata, we construct some explicit holomorphic global sections of $\hat{\mathscr{L}}$, for which one can easily determine the zero divisor. By the continuity of the volume with respect to μ (which can be derived from Theorem 1.1), we obtain Theorem 1.3. This approach also allows us to calculate $c_1(\hat{\mathscr{L}})$ by the Grothendieck-Riemann-Roch formula and to recover formula of Theorem 1.1.
6. In the appendix we explain an algorithm computing the intersection numbers of boundary divisors, which is necessary if one wants to compute the volumes explicitly. We then give the results for $\mathscr{M}_{0,5}$ and a special case for $\mathscr{M}_{0,6}$ with the aim to help interested readers to see how concrete computations can be carried out.

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2. Background on rank one local systems on the punctured sphere.

In this section, we summarize the settings and some results in [5, Sec. 2,3] relevant to our purpose.

2.1. Cohomology of a rank one local system on the punctured sphere

Let n be a positive integer such that $n \geq 3$. Let us fix the following data:

- $\Sigma = (x_1, x_2, \dots, x_n)$ is a n-uple of distinct points on the sphere $\mathbb{S}^2 \simeq \mathbb{P}^1_{\mathbb{C}}$,
- $\mu = (\mu_1, \dots, \mu_n)$ is a n-uple of positive real numbers such that

$$\mu_1 + \cdots + \mu_n = 2,$$

- $\alpha_i = \exp(2\pi\imath\mu_i), i = 1, \dots, n$,
- **L** is a complex rank one local system on $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$ with monodromy around x_i given by α_i. Note that up to isomorphism **L** is unique.

Using the C^∞-de Rham description, we can identify $H^\bullet(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$ with the cohomology of the de Rham complex of **L**-valued C^∞ differential forms on $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$, and $H^\bullet_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$ with the cohomology of the subcomplex of compactly supported forms.

Let $\mathbf{L}^\vee$ be the dual local system of **L**. This is the local system with monodromy α_i^{-1} around x_i. The Poincaré duality pairing by integration on $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$, that is

$$\begin{array}{ccc} H^i(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) \otimes H^{2-i}_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}^\vee) & \longrightarrow & \mathbb{C} \\ (\alpha, \beta) & \mapsto & \int_{\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma} \alpha \wedge \beta \end{array}$$

is then a perfect pairing.

Proposition 2.1 (Deligne-Mostow). – *If one of the* α_s, $s \in \{1, \dots, n\}$ *is not* 1*, then* $H^i(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$ *and* $H^i_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$ *vanish for* $i \neq 1$*, and*

$$\dim H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) = \dim H^1_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) = n - 2.$$

There are several ways to describe the homology and cohomology of **L** and $\mathbf{L}^\vee$. For instance, one can use a triangulation $\mathscr{T}$ of $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$ to construct chain complexes giving $H^\bullet(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$ and $H_\bullet(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$ as follows: an i-chain with coefficients in **L** is a formal sum $\sum e_\sigma \cdot \sigma$, where σ is an i-simplex of the triangulation, and e_σ is a horizontal section of the restriction of **L** to σ. An **L**-valued i-cochain associates to each i-simplex σ of the triangulation a horizontal section of **L** over σ. Note that the complex of **L**-valued cochains is dual to the complex of chains with coefficients in $\mathbf{L}^\vee$.

The cohomology with compact support $H^\bullet_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$ is also the cohomology of the complex of **L**-valued cochains compactly supported on $\mathscr{T}$. Its dual complex is the complex of locally finite chains with coefficients in $\mathbf{L}^\vee$, the homology of which will be denoted by $H^{\mathrm{lf}}_\bullet(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}^\vee)$.

One can also use currents to define $H^\bullet(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$. For any chain C with coefficients in $\mathbf{L}^\vee$, there exists a unique $\mathbf{L}^\vee$-valued current (C) such that

$$\int_C \omega = \int_{\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma} (C) \wedge \omega$$

for all **L**-valued C^∞ form ω. The map $C \mapsto (C)$ provides the isomorphisms $H_i(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}^\vee) \simeq H^{2-i}_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}^\vee)$ and $H^{\mathrm{lf}}_i(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}^\vee) \simeq H^{2-i}(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}^\vee)$.

If β is a rectifiable proper map from an open, semi-open, or closed interval I to $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$, and $e \in H^0(I, \beta^*\mathbf{L}^\vee)$, we let $(e \cdot \beta)$ be the $\mathbf{L}^\vee$-valued current for which

$$\int (e \cdot \beta) \wedge \omega = \int_I \langle e, \beta^*\omega \rangle.$$

If $\beta : [0,1] \to \mathbb{P}^1_{\mathbb{C}}$ maps 0 and 1 to Σ and $(0,1)$ into $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$, then for any $e \in H^0((0,1), \beta^*\mathbf{L}^\vee)$, $e \cdot \beta$ is a cycle and hence defines an homology class in $H_1^{\mathrm{lf}}(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}^\vee) \simeq H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}^\vee)$.

Let us fix a partition of Σ into two subsets Σ_0 and Σ_1. Let $\mathrm{T}_0, \mathrm{T}_1$ be two trees (graphs with no cycles) where the number of vertices of T_i is $|\Sigma_i|$, and $\beta : \mathrm{T}_0 \sqcup \mathrm{T}_1 \to \mathbb{P}^1_{\mathbb{C}}$ be an embedding such that the vertex set of T_i is mapped to Σ_i. We choose for any open edge a of $\mathrm{T}_0 \sqcup \mathrm{T}_1$ an orientation, and a non vanishing section $e(a) \in H^0(a, \beta^*\mathbf{L}^\vee)$. For each edge a, $e(a) \cdot \beta|_a$ is then a locally finite cycle on $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$, with coefficients in $\mathbf{L}^\vee$. Let $I_0 \sqcup I_1$ be the partition of $\{1, \dots, n\}$ corresponding to the partition $\Sigma = \Sigma_0 \sqcup \Sigma_1$.

Proposition 2.2 ([5], Prop. 2.5.1). – *If $\prod_{i \in I_0} \alpha_i \neq 0$, then the family*

$$\{e(a) \cdot \beta|_a, a \text{ is an edge of } \mathrm{T}_0 \sqcup \mathrm{T}_1\}$$

is a basis of $H_1^{\mathrm{lf}}(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}^\vee)$.

2.2. Sheaf cohomology

Another way to compute the cohomology of $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$ with coefficients in $\mathbf{L}$ is to use the sheaf cohomology. For this purpose, we will identify $\mathbf{L}$ with its sheaf of locally constant sections. Let $j : \mathbb{P}^1_{\mathbb{C}} \setminus \Sigma \to \mathbb{P}^1_{\mathbb{C}}$ be the natural inclusion, and let $j_!\mathbf{L}$ be the extension of $\mathbf{L}$ by 0 to $\mathbb{P}^1_{\mathbb{C}}$. In this setting, $H_c^\bullet(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$ is the cohomology on $\mathbb{P}^1_{\mathbb{C}}$ with coefficients in $j_!\mathbf{L}$. It is by definition, the hypercohomology on $\mathbb{P}^1_{\mathbb{C}}$ of any complex of sheaves $K^\bullet$ with $\mathscr{H}^0(K^\bullet) = j_!\mathbf{L}$, and $\mathscr{H}^i(K^\bullet) = 0$, for $i \neq 0$. On the other hand, if $\mathbf{L}^\bullet$ is a resolution of $\mathbf{L}$, whose components are acyclic for j_* (that is $R^q j_* \mathbf{L}^k = 0$ for $q > 0$), then $H^\bullet(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$ is the hypercohomology on $\mathbb{P}^1_{\mathbb{C}}$ of $j_*\mathbf{L}$. We have the

Proposition 2.3 ([5], Prop. 2.6.1). – *If $\alpha_i \neq 1$ for all $i \in \{1, \dots, n\}$, then we have $H_c^\bullet(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) \simeq H^\bullet(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$.*

The holomorphic $\mathbf{L}$-valued de Rham complex $\Omega^\bullet(\mathbf{L}) : \mathscr{O}(\mathbf{L}) \to \Omega^1(\mathbf{L})$ is a resolution of $\mathbf{L}$ on $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$. Hence, we can interpret $H^\bullet(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$ as the hypercohomology on $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$ of $\Omega^\bullet(\mathbf{L})$. Since $H^q(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \Omega^p(\mathbf{L})) = 0$, for $q > 0$ (because $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$ is Stein), this gives

$$H^\bullet(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) = H^\bullet\Gamma(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \Omega^\bullet(\mathbf{L})).$$

On the other hand, since we have $R^q j_* \Omega^p(\mathbf{L}) = 0$ for $q > 0$, it follows that

$$H^\bullet(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) = \mathbb{H}^\bullet(\mathbb{P}^1_{\mathbb{C}}, j_*\Omega^\bullet(\mathbf{L})).$$

2.3. The de Rham meromorphic description of the cohomology of L

We will describe a section of $\mathcal{O}(\mathbf{L})$ on an open set $U \subset \mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$ as the product of a multivalued function and a multivalued section of **L**. Those objects are defined as follows: U is provided with a base point o, a multivalued section of a sheaf $\mathcal{F}$ on U is actually a section of the pullback of $\mathcal{F}$ on the universal cover $(\hat{U}, \hat{o})$ of (U, o). A section of **L** at o extends to a unique horizontal multivalued section. A multivalued section of $\mathcal{O}$ is uniquely determined by its germ at o.

Fix an $x_s \in \Sigma$, and let z be a local coordinate which identifies a neighborhood of x_s with a disk D in $\mathbb{C}$ centered at $z(x_s) = 0$. Let $\mathrm{D}^* = \mathrm{D} \setminus \{0\}$. If the monodromy of **L** around x_s is $\alpha_s = \exp(2\pi\imath\mu_s)$, then the monodromy of $z^{-\mu_s}$ is the inverse of that of a horizontal section of **L**. Therefore, any section of $\mathcal{O}(\mathbf{L})$ (resp. $\Omega^1(\mathbf{L})$) on D^* can be written as $u = z^{-\mu_s} \cdot e \cdot f$ (resp. $u = z^{-\mu_s} \cdot e \cdot f dz$), where e is a non-zero (horizontal) multivalued section of **L**, and f is a holomorphic function on D^*. We define u to be *meromorphic* at x_s if f is, and define its valuation at x_s to be

$$v_{x_s}(u) = v_{x_s}(f) - \mu_s.$$

Note that these definitions are independent of the choice of the local coordinate.

Let us write $j_*^m\Omega^\bullet(\mathbf{L})$ for the sheaf complex consisting of meromorphic forms in $\Omega^\bullet(\mathbf{L})$. The inclusion of $j_*^m\Omega^\bullet(\mathbf{L})$ into $j_*\Omega^\bullet(\mathbf{L})$ induces an isomorphism on the cohomology sheaves. This implies

$$\mathbb{H}^\bullet(\mathbb{P}^1_{\mathbb{C}}, j_*^m\Omega^\bullet(\mathbf{L})) \simeq \mathbb{H}^\bullet(\mathbb{P}^1_{\mathbb{C}}, j_*\Omega^\bullet(\mathbf{L})) = H^\bullet(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}).$$

Since we have $H^q(\mathbb{P}^1_{\mathbb{C}}, j_*^m\Omega^p(\mathbf{L})) = 0$ for $q > 0$, $\mathbb{H}^\bullet(\mathbb{P}^1_{\mathbb{C}}, j_*^m\Omega^\bullet(\mathbf{L}))$ is simply $H^\bullet\Gamma(\mathbb{P}^1_{\mathbb{C}}, j_*^m\Omega^\bullet(\mathbf{L}))$, that is the cohomology of the complex of **L**-valued forms holomorphic on $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$ and meromorphic at Σ. To sum up, we have

$$H^\bullet(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) \simeq H^\bullet\Gamma(\mathbb{P}^1_{\mathbb{C}}, j_*^m\Omega^\bullet(\mathbf{L})).$$

PROPOSITION 2.4 ([5], Cor. 2.12). – *There is, up to a constant factor, a unique non-zero* $\omega \in \Gamma(\mathbb{P}^1_{\mathbb{C}}, j_*^m\Omega^1(\mathbf{L}))$ *whose valuation at* x_s *is at least* $-\mu_s$. *Actually, we have* $v_{x_s}(\omega) = -\mu_s$, *and* ω *is invertible on* $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$.

If $\infty \notin \Sigma$, *then, up to a constant factor,* $\omega = e \cdot \prod_{x_s \in \Sigma}(z - x_s)^{-\mu_s} dz$, *and if* $\infty \in \Sigma$, *then* $\omega = e \cdot \prod_{x_s \neq \infty}(z - x_s)^{-\mu_s} dz$.

Moreover, we have

PROPOSITION 2.5 ([5], Prop. 2.13). – *Assume that* $\alpha_s \neq 1$ *for all* $s \in \{1, \ldots, n\}$, *that is none of the* μ_s *is an integer, then the cohomology class of the form* ω *in the previous proposition is not zero.*

Let us assume that none of the α_s is 1. Let $[\omega]$ denote the cohomology class of ω in $H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$. Since we have $H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) \simeq H^1_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$ (cf. Proposition 2.3), ω also gives a cohomology class, denoted again by $[\omega]$, in $H^1_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$. Thus, for any locally finite cycle C in $H_1^{\mathrm{lf}}(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}^\vee)$, $\langle [C], [\omega] \rangle$ is well-defined. If C is represented by a compactly supported cycle, then

$$\langle [C], [\omega] \rangle = \int_C \omega.$$

If $C = e'\cdot\beta$ is a cycle where $\beta : [0,1] \to \mathbb{P}^1_{\mathbb{C}}$ is such that $\beta(0), \beta(1) \in \Sigma$, $\beta((0,1)) \subset \mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$, and e' is a horizontal section of $\beta^*\mathbf{L}^\vee$ on $(0,1)$, we can define a finite cycle C' with coefficients in $\mathbf{L}^\vee$ homologous to C as follows: let $x_{s_0} = \beta(0)$, $x_{s_1} = \beta(1)$, and D_i a small disk centered at x_{s_i} such that $\mathrm{D}_i \cap \Sigma = \{x_{s_i}\}$, and $\mathrm{D}_0 \cap \mathrm{D}_1 = \emptyset$. Let C_i, $i = 0,1$, be a circle centered at x_{s_i} and contained in D_i. Let I denote the interval $[0,1]$. Let $y_0 = \beta(\epsilon_0)$ be the first intersection of $\beta(I)$ and C_0, and $y_1 = \beta(1-\epsilon_1)$ be the last intersection of $\beta(I)$ with C_1. We consider y_0 and y_1 as base points of C_0 and C_1 respectively, and parametrize those circles counter-clockwise by the maps $\gamma_i : [0,1] \to C_i$. Let $I' := [\epsilon_0, 1-\epsilon_1]$, and β' be the restriction of β to I'. Let $e'_i := e'(y_i)/(\alpha_{s_i}^{-1} - 1)$. We also denote by e' the unique horizontal section of $\gamma_i^*\mathbf{L}^\vee$ determined by this vector. Consider the 1-chain $e'_i \cdot \gamma_i$ with coefficients in $\mathbf{L}^\vee$. Since the monodromy of $\mathbf{L}^\vee$ at x_{s_i} is $\alpha_{s_i}^{-1}$, we get $d(e'_i\cdot\gamma_i) = e'\cdot\{y_i\}$. Let C' denote the 1-cycle $e'_0\cdot\gamma_0 + e'\cdot\beta' - e'_1\cdot\gamma_1$. One can easily check that $dC' = 0$, and $[C'] = [C] \in H_1^{\mathrm{lf}}(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}^\vee)$. Since C' is compactly supported, we have

$$\langle [C], [\omega] \rangle = \langle [C'], [\omega] \rangle = \int_{C'} \omega.$$

Remark 2.6. – If $\omega = e \cdot \prod_{x_s\in\Sigma}(z-x_s)^{-\mu_s}dz$, where $0 < \mu_s < 1$ for all $s \in \{1,\dots,n\}$, and β is a path from x_{s_1} to x_{s_2} without passing through any point in Σ, then we also have

$$\langle [C], [\omega] \rangle = \langle e', e \rangle \int_\beta \prod_{x_s\in\Sigma} (z-x_s)^{-\mu_s} dz. \tag{3}$$

2.4. Hermitian structure

Since all the α_s have modulus equal to 1, $\mathbf{L}$ admits a horizontal Hermitian metric $(.,.)$. We can use this metric to define a perfect pairing

$$\psi_0 : H^1_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) \otimes_{\mathbb{C}} H^1_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \bar{\mathbf{L}}) \to H^2_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbb{C}) \simeq \mathbb{C},$$

where $\bar{\mathbf{L}}$ is the complex conjugate local system of $\mathbf{L}$. The vector space $H^1_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \bar{\mathbf{L}})$ is the complex conjugate of $H^1_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$. By setting

$$((u,v)) := \frac{-1}{2\pi\imath}\psi_0(u,\bar{v})$$

we get a Hermitian form on $H^1_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$.

A section ω of $j^m_*\Omega^1(\mathbf{L})$ is said to be *of the first kind* if $v_{x_s}(\omega) > -1$ for all $x_s \in \Sigma$. For such a form, we have $|\int_{\mathbb{P}^1_{\mathbb{C}}\setminus\Sigma} \omega \wedge \overline{\omega}| < \infty$. We define $H^{1,0}(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$ to be the vector space of forms of the first kind in $\Gamma(\mathbb{P}^1_{\mathbb{C}}, j^m_*\Omega^1(\mathbf{L}))$, and $H^{0,1}(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$ as the complex conjugate of $H^{1,0}(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \bar{\mathbf{L}})$. The latter is the space of anti-holomorphic $\mathbf{L}$-valued 1-forms, whose complex conjugate is of the first kind. As usual, such a form ω defines a cohomology class $[\omega] \in H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) \simeq H^1_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$.

Proposition 2.7 ([5] Prop. 2.19). – *If ω_1 and ω_2 are in $H^{1,0}(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) \cup H^{0,1}(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$ then*

$$(([\omega_1],[\omega_2])) = \frac{-1}{2\pi\imath}\int_{\mathbb{P}^1_{\mathbb{C}}\setminus\Sigma} \omega_1 \wedge \overline{\omega}_2.$$

Proposition 2.8 ([5] Prop. 2.20). – *Assume that* $0 < \mu_s < 1$ *for all* $s \in \{1, \dots, n\}$*, then the natural map*

$$H^{1,0}(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) \oplus H^{0,1}(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) \to H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) \simeq H^1_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$$

is an isomorphism. The Hermitian form $((.,.))$ *is positive definite on* $H^{1,0}$*, negative definite on* $H^{0,1}$*, and the decomposition is orthogonal. Since* $\dim_{\mathbb{C}} H^{1,0} = 1$*, and* $\dim_{\mathbb{C}} H^{0,1} = n-3$*, the signature of* $((.,.))$ *is* $(1, n-3)$*.*

2.5. Local system and the line bundle $\mathscr{L}$ over $\mathscr{M}_{0,n}$

Recall that $\mathscr{M}_{0,n}$ is the moduli space parametrizing Riemann surfaces of genus zero and n marked points (punctures). Since every Riemann surface of genus zero is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$, we can also view $\mathscr{M}_{0,n}$ as the space of configurations of n distinct points on $\mathbb{P}^1_{\mathbb{C}}$ up to action of $\mathrm{PGL}(2, \mathbb{C})$. If $\Sigma = \{x_1, \dots, x_n\}$, $n \geq 3$, is a set of n points in $\mathbb{P}^1_{\mathbb{C}}$, then up to action of $\mathrm{PGL}(2, \mathbb{C})$, we can always assume that $x_{n-2} = 0, x_{n-1} = 1, x_n = \infty$. Thus, $\mathscr{M}_{0,n}$ can be identified with the subset of $(\mathbb{P}^1_{\mathbb{C}})^{n-3}$ consisting of $(n-3)$-tuples $(x_1, \dots, x_{n-3})$ such that $x_s \neq x_{s'}$ if $s \neq s'$, and $x_s \notin \{0, 1, \infty\}$.

Over $\mathscr{M}_{0,n}$ we have a fibration $\pi : \mathscr{C}_{0,n} \to \mathscr{M}_{0,n}$ whose fiber over a point $m \in \mathscr{M}_{0,n}$ is the n-punctured sphere represented by m. Let $\overline{\mathscr{M}}_{0,n}$ be the Deligne-Mumford-Knudsen compactification of $\mathscr{M}_{0,n}$. We also have a fibration $\pi : \overline{\mathscr{C}}_{0,n} \to \overline{\mathscr{M}}_{0,n}$ extending the projection from $\mathscr{C}_{0,n}$ to $\mathscr{M}_{0,n}$, where $\overline{\mathscr{C}}_{0,n}$ is the *universal curve* which is a compact space containing $\mathscr{C}_{0,n}$ as an open dense subset. It is well known that π is a flat proper morphism, and there exist by construction n sections $\sigma_1 \dots, \sigma_n$ of π such that $\sigma_s(m)$ is the s^{th} marked point on the stable curve $\pi^{-1}(m)$. Note that $\overline{\mathscr{C}}_{0,n}$ is actually isomorphic to $\overline{\mathscr{M}}_{0,n+1}$, and $\overline{\mathscr{M}}_{0,n}$ is a smooth projective variety.

Fix a vector $\mu := (\mu_1, \dots, \mu_n) \in (\mathbb{R}_{>0})^n$ such that $\mu_1 + \dots + \mu_n = 2$, and $\mu_s \notin \mathbb{N}$ for all $s \in \{1, \dots, n\}$. By [5], Section 3.13, there exists a rank one local system $\mathbf{L}^{\mu}$ on $\mathscr{C}_{0,n}$ such that, for any $m \in \mathscr{M}_{0,n}$, the induced local system $\mathbf{L}^{\mu}_m$ on $\pi^{-1}(m) \simeq (\mathbb{P}^1_{\mathbb{C}}, (x_1, \dots, x_n))$ has monodromy given by $\alpha_s = \exp(2\pi\imath\mu_s)$ at each puncture x_s. Since the projection $\pi : \mathscr{C}_{0,n} \to \mathscr{M}_{0,n}$ is locally topologically trivial, setting $\mathbf{H}^{\mu} := R^1\pi_*\mathbf{L}^{\mu}$ we get a local system of rank $n-2$ over $\mathscr{M}_{0,n}$ whose fiber over m is $H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}^{\mu}_m) \simeq H^1_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}^{\mu}_m)$. Associated to this local system is a flat projective space bundle $\mathbb{P}\mathbf{H}^{\mu}$ whose fiber over m is $\mathbb{P}\mathbf{H}^{\mu}_m \simeq \mathbb{P}^{n-3}_{\mathbb{C}}$.

We have seen that for each $m \in \mathscr{M}_{0,n}$, up to a constant factor, there is a unique $\mathbf{L}^{\mu}_m$-valued meromorphic 1-form $\omega_m \in \Gamma(\mathbb{P}^1_{\mathbb{C}}, j^m_*\Omega^1(\mathbf{L}^{\mu}_m))$ such that the valuation of ω_m at the puncture x_s is exactly $-\mu_s$. By Proposition 2.5, we know that ω_m represents a nontrivial cohomology class in $\mathbf{H}^{\mu}_m$. Thus ω_m provides us with a section of the flat projective space bundle $\mathbb{P}\mathbf{H}^{\mu}$. Let us denote this section by Ξ_{μ}.

Since the pull-back of the bundle $\mathbb{P}\mathbf{H}^{\mu}$ to the universal cover $\widetilde{\mathscr{M}_{0,n}}$ is isomorphic to the trivial bundle $\widetilde{\mathscr{M}_{0,n}} \times \mathbb{P}^{n-3}_{\mathbb{C}}$, the section Ξ_{μ} gives rise to a map $\widetilde{\Xi}_{\mu} : \widetilde{\mathscr{M}_{0,n}} \to \mathbb{P}^{n-3}_{\mathbb{C}}$. We have the following crucial result

Proposition 2.9 ([5], Lem. 3.5, Prop. 3.9). – *The section* Ξ_{μ} *is holomorphic, and the map* $\widetilde{\Xi}_{\mu} : \widetilde{\mathscr{M}_{0,n}} \to \mathbb{P}^{n-3}_{\mathbb{C}}$ *is étale.*

A direct consequence of Proposition 2.9 is that we have a holomorphic line bundle $\mathscr{L}$ over $\mathcal{M}_{0,n}$ whose fiber over m is the line $\mathbb{C}\cdot[\omega_m] \subset H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}^{\mu}_m)$.

Assume moreover that $0 < \mu_s < 1$, for all $s \in \{1,\dots,n\}$. We have seen that in this case, the fiber $\mathbf{H}^{\mu}_m \simeq H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}^{\mu}_m)$ of the local system (flat bundle) $\mathbf{H}^{\mu}$ carries a Hermitian form of signature $(1, n-3)$. This Hermitian form then gives rise to a horizontal Hermitian metric on $\mathbf{H}^{\mu}$. Therefore, we have a flat bundle over $\mathcal{M}_{0,n}$ whose fiber over m is the ball $\mathbf{B}_m \subset \mathbb{P}\mathbf{H}^{\mu}_m$ which is defined by

$$\mathbf{B}_m := \{\mathbb{C}\cdot v \in \mathbb{P}\mathbf{H}^{\mu}_m,\ ((v,v))_m > 0\}.$$

By Proposition 2.8, the line $\mathbb{C}\cdot[\omega_m]$ belongs to $\mathbf{B}_m$. Thus, the map $\widetilde{\Xi}_{\mu}$ actually takes values in a fixed ball $\mathbf{B} \subset \mathbb{P}^{n-3}_{\mathbb{C}}$. As a consequence, we see that $\mathscr{L}$ is locally the pull-back by Ξ_{μ} of the restriction of the tautological line bundle of $\mathbb{P}^{n-3}_{\mathbb{C}}$ to $\mathbf{B}$. Remark that $\mathscr{L}$ carries naturally a Hermitian metric induced by the Hermitian metric on $\mathbf{H}^{\mu}$. In the rest of this paper, we will focus on the line bundle $\mathscr{L}$ and its Chern form.

3. Local coordinates at boundary points of $\overline{\mathcal{M}}_{0,n}$

It is well-known that the complement of $\mathcal{M}_{0,n}$ in $\overline{\mathcal{M}}_{0,n}$ is the union of finitely many divisors called *vital divisors*, each of which uniquely corresponds to a partition of $\{1,\dots,n\}$ into two subsets $I_0 \sqcup I_1$ such that $\min\{|I_0|,|I_1|\} \geq 2$. Let $\mathscr{P}$ be the set of partitions satisfying this condition. For each partition $\mathscr{S} := \{I_0, I_1\} \in \mathscr{P}$, we denote by $D_{\mathscr{S}}$ the corresponding divisor in $\overline{\mathcal{M}}_{0,n}$. Here below, we collect some classical facts on those divisors which are relevant for our purpose (see [15]).

(i) The family $\{D_{\mathscr{S}},\ \mathscr{S} \in \mathscr{P}\}$ consists of smooth divisors with normal crossings.

(ii) If $\mathscr{S} = \{I_0, I_1\}$, then $D_{\mathscr{S}}$ is isomorphic to $\overline{\mathcal{M}}_{0,|I_0|+1} \times \overline{\mathcal{M}}_{0,|I_1|+1}$.

(iii) Let $\mathscr{S} = \{I_0, I_1\}$ and $\mathscr{S}' = \{J_0, J_1\}$ be two partitions in $\mathscr{P}$. Then $D_{\mathscr{S}} \cap D_{\mathscr{S}'} = \emptyset$ unless one of the following occurs:

$$I_0 \subset J_0, \quad I_0 \subset J_1, \quad I_1 \subset J_0, \quad I_1 \subset J_1.$$

We first need to describe a neighborhood of a point m in $\partial\mathcal{M}_{0,n}$. Fix a partition $\mathscr{S} = \{I_0, I_1\} \in \mathscr{P}$. Let $n_0 = |I_0|$, and $n_1 = |I_1| = n - n_0$. Without loss of generality, we can assume that $I_0 = \{1,\dots,n_0\}$ and $I_1 = \{n_0+1,\dots,n\}$. From (ii) we know that $D_{\mathscr{S}}$ is isomorphic to $\overline{\mathcal{M}}_{0,n_0+1} \times \overline{\mathcal{M}}_{0,n_1+1}$. Let m be a point in $D_{\mathscr{S}}$. We will only focus on the case when $m \in D_{\mathscr{S}}$ is a generic point, that is the fiber C_m of π over m is a nodal curve having two irreducible components of genus zero intersecting at a simple node.

The normalization of C_m consists of two Riemann surfaces of genus zero denoted by C^0_m and C^1_m, where C^0_m (resp. C^1_m) contains the marked points $x_1,\dots,x_{n_0}$ (resp. $x_{n_0+1},\dots,x_n$). Let $\Sigma_0 := \{x_1,\dots,x_{n_0}\}$ and $\Sigma_1 := \{x_{n_0+1},\dots,x_n\}$. There are two points $\hat{y}_0 \in C^0_m \setminus \Sigma_0$ and $\hat{y}_1 \in C^1_m \setminus \Sigma_1$ that correspond to the unique node of C_m. The marked curves $(C^0_m, (\hat{y}_0, x_1,\dots,x_{n_0}))$ and $(C^1_m, (\hat{y}_1, x_{n_0+1},\dots,x_n))$ represent respectively two points $m_0 \in \mathcal{M}_{0,n_0+1}$ and $m_1 \in \mathcal{M}_{0,n_1+1}$.

We will now describe how one can embed holomorphically a small disk $\mathrm{D} \subset \mathbb{C}$ centered at 0 into $\overline{\mathcal{M}}_{0,n}$ and transversely to $D_{\mathscr{S}}$ such that 0 is mapped to m. For this, let us fix the following data:

- U is neighborhood of $\hat{y}_0$ in C^0_m such that $U \cap \{x_1, \dots, x_{n_0}\} = \emptyset$, $F : U \to \mathbb{C}$ is a coordinate mapping such that $F(\hat{y}_0) = 0$,
- V is neighborhood of $\hat{y}_1$ in C^1_m such that $V \cap \{x_{n_0+1}, \dots, x_n\} = \emptyset$, $G : V \to \mathbb{C}$ is a coordinate mapping such that $G(\hat{y}_1) = 0$.

Pick a constant $c \in \mathbb{R}_{>0}$ such that the disk $\mathrm{D}_c := \{|z| < c\} \subset \mathbb{C}$ is contained in both $F(U)$ and $G(V)$. For any $t \in \mathbb{C}$ such that $|t| < c^2$, set $C^0_{m,t} := C^0_m \setminus \{p \in U, |F(p)| \leq |t|/c\}$, and $C^1_{m,t} := C^1_m \setminus \{q \in V, |G(q)| \leq |t|/c\}$. Let A_t denote the annulus $\{|t|/c < |z| < c\} \subset \mathrm{D}_c$. We then define a compact Riemann surface by gluing $C^0_{m,t}$ and $C^1_{m,t}$ via the identification: $p \in F^{-1}(A_t)$ is identified with $q \in G^{-1}(A_t)$ if and only if $F(p)G(q) = t$. Let us denote the surface obtained from this construction by $C_{(m,t)}$.

It is easy to see that the marked curve $(C_{(m,t)}, (x_1, \dots, x_n))$ represents a point in $\mathcal{M}_{0,n}$. We thus have a map $\varphi : \mathrm{D}_{c^2} = \{t \in \mathbb{C}, |t| < c^2\} \to \overline{\mathcal{M}}_{0,n}$, which is defined by $\varphi(0) = (C_m, \Sigma)$ and $\varphi(t) = (C_{(m,t)}, \Sigma)$, for $t \neq 0$. This map is well known to be a holomorphic embedding of D_{c^2} into $\overline{\mathcal{M}}_{0,n}$. The construction above is called a *plumbing*, and the image of D_{c^2} by φ is called a *plumbing family* (see [23, Sec. 2]).

Recall that m is identified with (m_0, m_1) by the isomorphism between $D_{\mathcal{S}}$ and $\overline{\mathcal{M}}_{0,n_0+1} \times \overline{\mathcal{M}}_{0,n_1+1}$. Therefore, we can identify a neighborhood $\mathcal{V}$ of m in $D_{\mathcal{S}}$ with a product space $\mathcal{V}_0 \times \mathcal{V}_1$, where $\mathcal{V}_i$ is a neighborhood of m_i in $\mathcal{M}_{0,n_i+1}$. For any $m' = (m'_0, m'_1) \in \mathcal{V}$, let $C_{m'_0}$ and $C_{m'_1}$ be the curves represented by m'_0 and m'_1 respectively. On $C_{m'_i}$ we have a distinguished marked point $\hat{y}'_i$ which corresponds to the node of the curve $C_{m'}$ represented by m'. We can always identify $C_{m'_i}$ with $\mathbb{P}^1_{\mathbb{C}}$ such that $\hat{y}'_i = 0$. In conclusion, we get the following well known result (see [23, Sec. 2], [1, Chap. 11]).

PROPOSITION 3.1. – *Assume that for all $m' = (m'_0, m'_1) \in \mathcal{V}_0 \times \mathcal{V}_1$ we have some plumbing data (U, V, F, G, c) as above, where F and G depend holomorphically on m'. Then there exists a system of holomorphic local coordinates at m which identifies a neighborhood $\mathcal{U}$ of m in $\overline{\mathcal{M}}_{0,n}$ with $\mathcal{V}_0 \times \mathcal{V}_1 \times \mathrm{D}_{c^2}$. The point in $\overline{\mathcal{M}}_{0,n}$ corresponding to (m'_0, m'_1, t) represents the surface obtained by applying the t-plumbing construction to the nodal surface represented by (m'_0, m'_1). In particular, $\mathcal{U} \cap \mathcal{M}_{0,n}$ is identified with $\mathcal{V}_0 \times \mathcal{V}_1 \times \mathrm{D}^*_{c^2}$ in those coordinates.*

4. Sections of $\mathcal{L}$ near the boundary: generic points

In Section 2.5, we defined a holomorphic line bundle $\mathcal{L}$ over $\mathcal{M}_{0,n}$ by providing local trivializations (see Proposition 2.9). In this section, we investigate $\mathcal{L}$ near the boundary of $\overline{\mathcal{M}}_{0,n}$. Our goal is to exhibit holomorphic sections of $\mathcal{L}$ in a neighborhood of every point $m \in \partial\overline{\mathcal{M}}_{0,n}$. Assume that m is a generic point of a divisor $D_{\mathcal{S}}$.

Let $\mathcal{S} = \{I_0, I_1\}, C^0_m, C^1_m, \Sigma_0, \Sigma_1, \hat{y}_0, \hat{y}_1$ be as in the previous section. We will identify C^0_m (resp. C^1_m) with $\mathbb{P}^1_{\mathbb{C}}$ in such a way that $\hat{y}_0 = 0$ and $\infty \not\in \Sigma_0$ (resp. $\hat{y}_1 = 0$, and $\infty \not\in \Sigma_1$). Set $\hat{\Sigma}_i = \Sigma_i \sqcup \{\hat{y}_i\}$, $i = 0, 1$.

Let $\hat{\mu}_0 = \sum_{n_0+1 \leq s \leq n} \mu_s$, $\hat{\mu}_1 = \sum_{1 \leq s \leq n_0} \mu_s$ and $\hat{\alpha}_i = \exp(2\pi\imath\hat{\mu}_i)$. We assume that $\hat{\mu}_0 < 1$. Denote by $\mathbf{L}_i$ the rank one local system on $C^i_m \setminus \hat{\Sigma}_i$ with monodromy α_s at x_s, and

$\hat{\alpha}_i$ at $\hat{y}_i$. Let $\mathbf{e}_i$ be a horizontal multivalued section of $\mathbf{L}_i$. Set

$$\omega_0 = \mathbf{e}_0 \cdot z^{-\hat{\mu}_0} \prod_{1\leq s\leq n_0} (z-x_s)^{-\mu_s} dz \quad \text{and} \quad \omega_1 := \mathbf{e}_1 \cdot z^{-\hat{\mu}_1} \prod_{n_0+1\leq s\leq n} (z-x_s)^{-\mu_s} dz.$$

Observe that ω_i is a well defined section in $\Gamma(C^i_m, j^m_*\Omega^1(\mathbf{L}_i))$.

We are going to construct a plumbing family starting from m and a section of $\mathscr{L}$ over the corresponding (punctured) family. For this, we first need to fix the plumbing data.

LEMMA 4.1. – *Let r be a real number not in $\{-n, n \in \mathbb{N}^*\}$. For any holomorphic function f defined on a disk* D *in* $\mathbb{C}$ *centered at* 0 *and satisfying $f(0) \neq 0$, there exists a coordinate change $z \mapsto w$ preserving* 0 *such that*

$$z^r f(z) dz = w^r dw$$

on a neighborhood of 0 *in* D*, with suitable determinations of z^r and w^r.*

Proof. – Let us fix a determination of z^r. We will look for a coordinate change of the form $w = zh(z)$. It suffices to find a holomorphic function h defined on a neighborhood of 0 such that $h(0) \neq 0$ and

$$z^r h(z)^r \big(h(z) + zh'(z)\big) = z^r f(z) \Leftrightarrow h(z)^r (h(z) + zh'(z)) = f(z),$$

where $h(z)^r$ is a determination defined near $h(0) \neq 0$. Setting $g(z) := h^{r+1}(z)$, we must have

$$g(z) + \frac{1}{r+1} zg'(z) = f(z).$$

Let $f(z) = \sum_{k\geq 0} c_k z^k$, with $c_0 \neq 0$, be the expansion of f at 0. Assuming that g admits an expansion $g(z) = \sum_{k\geq 0} d_k z^k$, we see that the sequence $(d_k)_{k\geq 0}$ must satisfy

$$d_k\Big(1 + \frac{k}{r+1}\Big) = c_k \Leftrightarrow d_k = \frac{r+1}{r+1+k} c_k.$$

In particular, we see that $d_0 = c_0 \neq 0$, and since $r + 1 + k \neq 0$ for any $k \in \mathbb{N}$, the power series $\sum_{k\geq 0} d_k z^k$ is well defined and has the same convergence radius as $\sum_{k\geq 0} c_k z^k$. Thus $g(z)$ is a well defined holomorphic function on D which satisfies $g(0) \neq 0$. It follows that $h(z) := g(z)^{1/(r+1)}$ is well defined in a neighborhood of 0 for any choice of a determination. Then we choose the determination $h(z)^r$ in such a way that $h(0)^{r+1} = g(0)$, and if we define $w^r = (zh(z))^r := z^r h(z)^r$, the lemma is proved. □

Now, choosing a determination for $\prod_{1\leq s\leq n_0}(z - x_s)^{-\mu_s}$ and $\prod_{n_0+1\leq s\leq n}(z - x_s)^{-\mu_s}$ in a neighborhood of 0, we then get two holomorphic functions f and g which do not vanish at 0. Applying Lemma 4.1 to the forms $z^{-\hat{\mu}_0} f(z)dz$ and $z^{-\hat{\mu}_1} g(z)dz$, we see that there exist two holomorphic functions $F : U \to \mathbb{C}$ and $G : V \to \mathbb{C}$, where U and V are some neighborhoods of 0, such that $F(0) = G(0) = 0$, $F'(0) \neq 0$, $G'(0) \neq 0$ and

$$z^{-\hat{\mu}_0} f(z)dz = F^{-\hat{\mu}_0}(z)dF(z), \qquad z^{-\hat{\mu}_1} g(z)dz = G^{-\hat{\mu}_1}(z)dG(z). \tag{4}$$

Let c be a positive real number such that D_c is contained in both $F(U)$ and $G(V)$. We can now use the tuple (F, U, G, V, c) to construct the plumbing family associated to m. For any $t \in \mathrm{D}_{c^2}$, let $C_{(m,t)}$ be the n-punctured sphere obtained by the construction described in the previous section. Recall that $C_{(m,t)}$ is obtained from $C^0_{m,t}$ and $C^1_{m,t}$ by the gluing rule

$w_1 = t/w_0$ in the coordinates $w_0 = F(z)$ and $w_1 = G(z)$. By the definition of F and G, the expressions of ω_0 and ω_1 in those local coordinates are respectively

$$\omega_0 = \mathbf{e}_1 \cdot w_0^{-\hat{\mu}_0} dw_0, \qquad \omega_1 = \mathbf{e}_2 \cdot w_1^{-\hat{\mu}_1} dw_1. \tag{5}$$

LEMMA 4.2. – *There exists a rank one local system* $\mathbf{L}$ *on* $C_{(m,t)} \setminus \Sigma$ *whose restriction to* $C^i_{m,t} \setminus \Sigma_i$ *is* $\mathbf{L}_i$. *We also have a multivalued horizontal section* $\mathbf{e}$ *of* $\mathbf{L}$ *whose restriction to* $C^i_{m,t}$ *is identified with* $\mathbf{e}_i$.

Proof. – We first remark that $C^i_{m,t}$ is biholomorphic to a disk with n_i punctures, and the annulus A_t is homotopy equivalent to the boundary of $C^i_{m,t}$. By definition, the monodromy of $\mathbf{L}_i$ along the boundary of $C^i_{m,t}$ (with the counterclockwise orientation) is given by $\exp(-2\pi\imath\hat{\mu}_i)$. Observe that the transition map identifies a circle homotopic to the boundary of $C^0_{m,t}$ with a circle homotopic to the boundary of $C^1_{m,t}$ with the inverse orientation. Since we have

$$\exp(-2\pi\imath\hat{\mu}_1) = \exp(-2\pi\imath(2-\hat{\mu}_0)) = \exp(2\pi\imath\hat{\mu}_0),$$

the restriction of $\mathbf{L}_0$ on A_t is isomorphic to the restriction of $\mathbf{L}_1$. We can then identify $\mathbf{L}_0$ with $\mathbf{L}_1$ on A_t by setting $\mathbf{e}_0 \simeq \mathbf{e}_1$. Therefore, we have a well defined rank one local system $\mathbf{L}$ on $C_{(m,t)}$ with the desired monodromies at the punctures and a multivalued horizontal section, denoted by $\mathbf{e}$, whose restriction to $C^i_{m,t}$ is $\mathbf{e}_i$. □

LEMMA 4.3. – *There exists a unique* $\mathbf{L}$*-valued meromorphic* 1*-form* $\omega \in \Gamma(C_{(m,t)}, j^m_* \Omega^1(\mathbf{L}))$, *whose restriction to* $C^0_{m,t}$ *is equal to* ω_0. *Its restriction to* $C^1_{m,t}$ *is equal to* $-t^{1-\hat{\mu}_0}\omega_1$ *for some determination of* $t^{\hat{\mu}_0}$.

Proof. – By definition, ω_i is a section of $\Omega^1(\mathbf{L})$ on $C^i_{m,t}$, meromorphic at the punctures. All we need to show is that

$$\omega_0 = -t^{1-\hat{\mu}_0}\omega_1 \text{ on } A_t, \tag{6}$$

the uniqueness being clear by analytic continuation. Using the local coordinates w_0 and w_1, we have (see (5))

$$\omega_0 = \mathbf{e}_0 \cdot w_0^{-\hat{\mu}_0} dw_0 \text{ and } \omega_1 = \mathbf{e}_1 \cdot w_1^{-\hat{\mu}_1} dw_1$$

for some choices of the determinations $w_0^{\hat{\mu}_0}$ and $w_1^{\hat{\mu}_1}$. Recall that the changes of trivializations on A_t satisfy $w_0 \mapsto t/w_1$ and $\mathbf{e}_0 \mapsto \mathbf{e}_1$. Thus

$$\omega_0 = \mathbf{e}_0 \cdot w_0^{-\hat{\mu}_0} dw_0 = \mathbf{e}_1 \cdot (w_1/t)^{\hat{\mu}_0}(-t/w_1^2) dw_1 = -t^{1-\hat{\mu}_0}\mathbf{e}_1 \cdot w_1^{-\hat{\mu}_1} dw_1 = -t^{1-\hat{\mu}_0}\omega_1,$$

where $t^{\hat{\mu}_0}$ is chosen in such a way that $t^{\hat{\mu}_0} = (t/w_1)^{\hat{\mu}_0} w_1^{2-\hat{\mu}_1}$, which is possible since $\hat{\mu}_0 + \hat{\mu}_1 = 2$. Observe that as t completes a turn around 0, the determination of $t^{\hat{\mu}_0}$ is multiplied by $e^{2\imath\pi\hat{\mu}_0}$. □

Let T_0 be an embedded tree in C^0_m whose vertex set consists of n_0 points in $\hat{\Sigma}_0$. Let T_1 be an embedded tree in C^1_m whose vertex set is exactly Σ_1. Let $a_1, \dots, a_{n_0-1}$ denote the edges of T_0, and $b_1, \dots, b_{n_1-1}$ the edges of T_1. Let $\mathbf{e}'_i$ be an $\overline{\mathbf{L}}_i$-multivalued horizontal section on $C^i_m \setminus \hat{\Sigma}_i$.

By Proposition 2.2, we know that $\{\mathbf{e}'_0\cdot a_j,\ j=1,\dots,n_0-1\}$ (resp. $\{\mathbf{e}'_1\cdot b_j,\ j=1,\dots,n_1-1\}$) is a basis of $H_1^{\mathrm{lf}}(C_m^0\setminus\hat{\Sigma}_0,\overline{\mathbf{L}}_0)$ (resp. a basis of $H_1^{\mathrm{lf}}(C_m^1\setminus\hat{\Sigma}_1,\overline{\mathbf{L}}_1)$). Set

$$\eta_j:=\langle[\mathbf{e}'_0\cdot a_j],[\omega_0]\rangle,\quad \xi_j:=\langle[\mathbf{e}'_1\cdot b_j],[\omega_1]\rangle.$$

Since $[\omega_0]$ and $[\omega_1]$ are not zero (see Proposition 2.5), we have

$$\eta:=(\eta_1,\dots,\eta_{n_0-1})\neq 0\in\mathbb{C}^{n_0-1}\text{ and }\xi:=(\xi_1,\dots,\xi_{n_1-1})\neq 0\in\mathbb{C}^{n_1-1}.$$

LEMMA 4.4. – *Let ω be as in Lemma 4.3. Then there exists a basis of $H_1^{\mathrm{lf}}(C_{(m,t)}\setminus\Sigma,\overline{\mathbf{L}})$ such that the coordinates of ω in the dual basis are given by $(\eta,-t^{1-\hat{\mu}_0}\xi)\in\mathbb{C}^{n-2}$.*

Proof. – We first consider the case when T_0 does not contain $\hat{y}_0$. The tree T_i can always be chosen to be contained entirely in $C^i_{m,t}$. It follows that $\{\mathbf{e}'_0\cdot a_j\}$ and $\{\mathbf{e}'_1\cdot b_j\}$ can be considered as homology classes in $H_1^{\mathrm{lf}}(C_{(m,t)}\setminus\Sigma,\overline{\mathbf{L}})$. Moreover, by Proposition 2.2, the union of those classes makes up a basis of $H_1^{\mathrm{lf}}(C_{(m,t)}\setminus\Sigma,\overline{\mathbf{L}})$. Since the restrictions of ω to $C^0_{m,t}$ and $C^1_{m,t}$ are respectively ω_0 and $-t^{1-\hat{\mu}_0}\omega_1$, we get

$$\langle[\mathbf{e}'_0\cdot a_j],[\omega]\rangle=\langle[\mathbf{e}'_0\cdot a_j],[\omega_0]\rangle=\eta_j,\ j=1,\dots,n_0-1,\tag{7}$$

$$\langle[\mathbf{e}'_1\cdot b_j],[\omega]\rangle=\langle[\mathbf{e}'_1\cdot b_j],-t^{1-\hat{\mu}_0}[\omega_1]\rangle=-t^{1-\hat{\mu}_0}\xi_j,\ j=1,\dots,n_1-1.\tag{8}$$

Thus the lemma is proven in this case.

Consider now the case where T_0 contains $\hat{y}_0$. Remark that in this case, there is a point in Σ_0, say x_{n_0}, which is not contained in T_0. Up to a renumbering, we can assume that the set of edges containing $\hat{y}_0$ as an end is $\{a_j,\ j=1,\dots,k\}$, with $k\le n_0-1$. We can also assume that x_j is the other end of a_j, for $j=1,\dots,k$.

Recall that the plumbing construction is carried out in a neighborhood U of $\hat{y}_0$. Let D_0 be an embedded disk in C^0_m that contains U. For $j=1,\dots,k$, let y_j be the first intersection of a_j with $\partial\mathrm{D}_0$, and a'_j be the subarc of a_j from x_j to y_j. Let a''_j denote the boundary of D_0 considered as a loop based at y_j. Since $\hat{\mu}_0\notin\mathbb{N}$, there exists a constant ε such that $[\mathbf{e}'_0\cdot a_j]=[\mathbf{e}'_0\cdot a'_j+\varepsilon\mathbf{e}'_0\cdot a''_j]$ in $H_1^{\mathrm{lf}}(C^0_m\setminus\hat{\Sigma}_0,\overline{\mathbf{L}}_0)$. Therefore,

$$\eta_j=\langle[\mathbf{e}'_0\cdot a'_j+\varepsilon\mathbf{e}'_0\cdot a''_j],[\omega_0]\rangle=\int_{a'_j}(\mathbf{e}'_0,\omega_0)+\varepsilon\int_{a''_j}(\mathbf{e}'_0,\omega_0),\quad j=1,\dots,k.$$

We construct a new tree T in $C_{(m,t)}$ from T_0 and T_1 by removing $a_1,\dots,a_k$ from T_0, and adding the edges c_j joining x_j to some vertex of T_1 for $j=1,\dots,k$. Note that the vertex set of T is $\Sigma\setminus\{x_{n_0}\}$. Let $\mathbf{e}'$ be an $\mathbf{L}$-multivalued horizontal section on $C_{(m,t)}\setminus\Sigma$. Then $\{[\mathbf{e}'\cdot c_1],\dots,[\mathbf{e}'\cdot c_k],[\mathbf{e}'_0\cdot a_{k+1}],\dots,[\mathbf{e}'_0\cdot a_{n_0-1}],[\mathbf{e}'_1\cdot b_1],\dots,[\mathbf{e}'_1\cdot b_{n_1-1}]\}$ is a basis of $H_1^{\mathrm{lf}}(C_{(m,t)}\setminus\Sigma,\overline{\mathbf{L}})$ by Proposition 2.2.

For $j=1,\dots,k$, since a'_j and a''_j are entirely contained in $C^0_{m,t}$, we can consider $\mathbf{e}'_0\cdot a'_j+\varepsilon\mathbf{e}'_0\cdot a''_j$ as elements of $H_1^{\mathrm{lf}}(C_{(m,t)}\setminus\Sigma,\overline{\mathbf{L}})$. Since the union of $\{a'_j,a''_j,c_j,b_1,\dots,b_{n_1-1}\}$ is homotopic to the boundary of an open disk disjoint from Σ, we deduce that $[\mathbf{e}'_0\cdot a'_j+\varepsilon\mathbf{e}'_0\cdot a''_j]$ is a linear combination of $[\mathbf{e}'\cdot c_j]$ and $[\mathbf{e}'_1\cdot b_1],\dots,[\mathbf{e}'_1\cdot b_{n_1-1}]$. Therefore, $\{[\mathbf{e}'_0\cdot a'_1+\varepsilon\mathbf{e}'_0\cdot a''_1],\dots,[\mathbf{e}'_0\cdot a'_k+\varepsilon\mathbf{e}'_0\cdot a''_k],[\mathbf{e}'_0\cdot a_{k+1}],\dots,[\mathbf{e}'_0\cdot a_{n_0-1}],[\mathbf{e}'_1\cdot b_1],\dots,[\mathbf{e}'_1\cdot b_{n_1-1}]\}$ is also a basis of $H_1^{\mathrm{lf}}(C_{(m,t)}\setminus\Sigma,\overline{\mathbf{L}})$. Since we have

$$\langle[\mathbf{e}'_0\cdot a'_j+\varepsilon\mathbf{e}'_0\cdot a''_j],[\omega]\rangle=\int_{a'_j}(\mathbf{e}'_0,\omega)+\varepsilon\int_{a''_j}(\mathbf{e}'_0,\omega)=\int_{a'_j}(\mathbf{e}'_0,\omega_0)+\varepsilon\int_{a''_j}(\mathbf{e}'_0,\omega_0)=\eta_j,$$

for $j = 1, \dots, k$, the coordinates of $[\omega]$ in the dual basis are given by $(\eta, -t^{1-\hat{\mu}_0}\xi)$. $\square$

REMARK 4.5. – In the proof of Lemma 4.4, we could have chosen T_0 such that the node $\hat{y}_0$ is not contained in T_0, and the proof would have been more direct. However, in the next section where we will treat the case where m belongs to several divisors $D_{\mathscr{S}}$, we will be forced to deal with trees containing points corresponding to nodes and we will use a method which is similar to the one above (see Lemma 5.6).

REMARK 4.6. – Let γ be a small loop around 0 in D_{c^2}. The element of the mapping class group $\mathrm{Mod}_{0,n}$ corresponding to γ is a Dehn twist around a closed curve on $\mathbb{P}^1_{\mathbb{C}}$ separating Σ_0 and Σ_1. It can be shown that the monodromy of the local system $\mathbf{H}^\mu$ around such a loop is given by the matrix $\begin{pmatrix} \mathrm{Id}_{n_0-1} & 0 \\ 0 & e^{2\pi\imath(1-\hat{\mu}_0)}\mathrm{Id}_{n_1-1} \end{pmatrix}$ (see [5, Prop. 9.2] for the case $n_1 = 2$).

Recall that we can write $m = (m_0, m_1)$, where $m_i \in \mathcal{M}_{0,n_i+1}$ represents $(C^i_m, \hat{\Sigma}_i)$. Fix a constant $c > 0$. There exist some neighborhoods $\mathcal{V}_i$ of m_i in $\mathcal{M}_{0,n_i+1}$ such that for any $m' = (m'_0, m'_1) \in \mathcal{V}_0 \times \mathcal{V}_1$ and $t \in \mathrm{D}^*_{c^2}$, we can apply the same plumbing construction with parameter t as above to the curve $C_{m'}$. Let $C_{(m',t)}$ denote the resulting surface in $\mathcal{M}_{0,n}$. By Proposition 3.1, this construction identifies $\mathcal{V}_0 \times \mathcal{V}_1 \times \mathrm{D}_{c^2}$ with a neighborhood of m in $\overline{\mathcal{M}}_{0,n}$.

Let $C^i_{m'}$ be the component of $C_{m'}$ containing Σ_i.

We define the sections $\omega_{m'_i} \in \Gamma(C^i_{m'}, j^m_*\Omega^1(\mathbf{L}_i))$ in the same manner as ω_i. Since $C_{(m',t)}$ is defined by the same plumbing construction as $C_{(m,t)}$, by Lemma 4.3 we get an element $\omega_{(m',t)} \in \Gamma(C_{(m',t)}, j^m_*\Omega^1(\mathbf{L}))$ constructed from $\omega_{m'_0}$ and $\omega_{m'_1}$. Since $\omega_{(m',t)}$ is a vector in the fiber of $\mathscr{L}$ over (m', t), the assignment $\Phi : (m', t) \mapsto \omega_{(m',t)}$ is a section of $\mathscr{L}$ on $\mathcal{V}_0 \times \mathcal{V}_1 \times \mathrm{D}^*_{c^2}$.

LEMMA 4.7. – *Φ is a holomorphic section of $\mathscr{L}$ on $\mathcal{V}_0 \times \mathcal{V}_1 \times \mathrm{D}^*_{c^2}$.*

Proof. – To see that Φ is holomorphic section of $\mathscr{L}$, it is enough to show that the pairings of $\omega_{(m',t)}$ with a basis of $H_1^{\mathrm{lf}}(C_{(m',t)} \setminus \Sigma, \overline{\mathbf{L}})$ are holomorphic functions of (m'_0, m'_1, t). Since $\{\eta_j,\ j = 1, \dots, n_0 - 1\}$ and $\{\xi_j,\ j = 1, \dots, n_1 - 1\}$ are holomorphic functions of m'_0 and m'_1 respectively, the lemma is a direct consequence of Lemma 4.4, see also [5, Sec. 3]. $\square$

5. Sections of $\mathscr{L}$ near the boundary: general case

5.1. Principal component

Each point m in $\overline{\mathcal{M}}_{0,n}$ represents a nodal curve C_m with n marked points $(x_1, \dots, x_n)$. Let $C^0_m, \dots, C^r_m$ be the irreducible components of C_m. The topological type of C_m is encoded by a tree $\mathbf{T}$ whose vertex set is in bijection with the set of irreducible components. Each edge of this tree corresponds to a node of C_m.

The point m belongs to the intersection of r boundary divisors, each of them being associated with one of the r nodes $p_1, \dots, p_r$ as follows: splitting a node p_j into two points, we get two connected components $C^{(0)}_{m,p_j}$ and $C^{(1)}_{m,p_j}$ from C_m. For $i = 0, 1$, we define the set $I^j_i \subset \{1, \dots, n\}$ as follows: $s \in I^j_i$ if and only if $x_s \in C^{(i)}_{m,p_j}$. Exchanging $C^{(0)}_{m,p_j}$ and

$C^{(1)}_{m,p_j}$ if necessary, we will always assume that $\sum_{s\in I^j_1}\mu_s < 1$. Set $\mathcal{S}_j = \{I^j_0, I^j_1\}$, we have $m \in \bigcap_{j=1}^r D_{\mathcal{S}_j}$.

Let $\Sigma := \{x_1,\dots,x_n\}$. For each component C^j_m, set $\Sigma_j := \Sigma \cap C^j_m$. Note that Σ_j can be empty. We also have on C^j_m some other marked points denoted by $\{y_1,\dots,y_{s_j}\}$ that correspond to nodes of C_m. Set $\hat{\Sigma}_j := \Sigma_j \sqcup \{y_1,\dots,y_{s_j}\}$. We now assign to every point y in $\hat{\Sigma}_j$ a weight $\hat{\mu}(y)$ as follows: if $y = x_s \in \Sigma_j$ then $\hat{\mu}(y) = \mu_s$. If $y \in \{y_1,\dots,y_{s_j}\}$, we have a corresponding node of C_m. Splitting this node into two points, we get two connected components of C_m. Let $\mathring{C}^j_{m,y}$ denote the component that **does not** contain C^j_m. The weight associated to y is then

$$\hat{\mu}(y) := \sum_{x_s\in \mathring{C}^j_{m,y}} \mu_s. \tag{9}$$

Since y corresponds to a node, there exists another marked point y' that is identified with y. Let $C^{j'}_m$ be the irreducible component that contains y'. Since the genus of C_m is zero, we must have $j' \neq j$. As a consequence, we get

$$\hat{\mu}(y') = 2 - \hat{\mu}(y). \tag{10}$$

Let $\hat{\mu}^j$ be the vector recording the weights of the points in $\hat{\Sigma}_j$.

Lemma 5.1. – *We have:*

a) *the sum of the weights in* $\hat{\mu}^j$ *is* 2*,*
b) *there exists a unique component* C^j_m *such that all the weights in* $\hat{\mu}^j$ *are smaller than* 1.

Proof. – The first assertion follows immediately from the definition of the weights at the points corresponding to nodes of C_m. We will prove the second assertion by induction on the number of vertices of $\mathbf{T}$.

If $\mathbf{T}$ has only one vertex, then b) is trivially true. Suppose that $\mathbf{T}$ has $r+1$ vertices, with $r \geq 1$. Pick a component C^j_m corresponding to a leaf of $\mathbf{T}$, that is a vertex which is connected to the rest of $\mathbf{T}$ by only one edge. Suppose that C^j_m satisfies the property of the lemma (that is, all the weights in $\hat{\mu}^j$ are smaller than one). Let us show that C^j_m is the unique component satisfying this condition. Let y be the unique point in $\hat{\Sigma}_j$ that corresponds to a node in C_m. Since the weight $\hat{\mu}(y)$ is less than 1, from a) we have $\sum_{x_s\in C^j_m}\mu_s > 1$.

Consider another irreducible component C^k_m of C_m. There is a point $y_k \in C^k_m$ which corresponds to the node separating C^k_m from C^j_m. Let $\mathring{C}^k_{m,y_k}$ be the component containing C^j_m which is obtained after splitting y_k into two points. By definition, the weight of y_k is

$$\hat{\mu}(y_k) = \sum_{x_s\in \mathring{C}^k_{m,y_k}} \mu_s \geq \sum_{x_s\in C^j_m} \mu_s > 1.$$

Therefore, C^k_m cannot satisfy the condition in b). We can then conclude that C^j_m is the unique component that satisfies this condition.

Assume now that C_m^j does not satisfy the condition of the lemma, which means that $\hat{\mu}(y) > 1$. Let C_m^k be the unique component of C_m that is adjacent to C_m^j, and y' be the point in C_m^k that is identified with y. Note that the weight of y' is given by

$$\hat{\mu}(y') = \sum_{x_s \in C_m^j} \mu_s = 2 - \hat{\mu}(y) < 1.$$

Set $\Sigma' := (\Sigma \setminus \Sigma_j) \sqcup \{y'\}$. We see that each point in Σ' has a weight strictly smaller than 1, and the total weight of the points in Σ' is 2. Let C'_m be the stable curve obtained by removing C_m^j from C_m. Since the tree corresponding to C'_m has a vertex less than $\mathbf{T}$, we can apply the induction hypothesis to conclude that there is a unique component of C'_m that satisfies the desired condition. □

Definition 5.2. – *We call the unique component C_m^j that satisfies the condition that all the weights in $\hat{\mu}^j$ are smaller than* 1 *the μ-*principal component *of C_m.*

In what follows, we will always assume that C_m^0 is the principal component of C_m. Let $\mathbf{v}_j$ be the vertex of $\mathbf{T}$ corresponding to C_m^j. We consider $\mathbf{v}_0$ as the root of $\mathbf{T}$, and set the length of every edge of $\mathbf{T}$ to be one. We define the level L_j of the component C_m^j to be the distance in $\mathbf{T}$ from $\mathbf{v}_j$ to $\mathbf{v}_0$. Observe that we can always choose a numbering of the components of C_m such that $\mathrm{L}_j \leq \mathrm{L}_{j+1}$ for $j = 0, \dots, r-1$.

If C_m^j is not the principal component of C_m, then there is a unique point $\hat{y}_j \in \hat{\Sigma}_j$ which corresponds to the node separating C_m^j from C_m^0. Remark that we have $\hat{\mu}(\hat{y}_j) > 1$, and $\hat{y}_j$ is the unique point in $\hat{\Sigma}_j$ whose weight is greater than 1. We will call $\hat{y}_j$ the *principal node* of C_m^j, and define the *weight* of C_m^j to be $\nu_j = \hat{\mu}(\hat{y}_j) - 1$. The following lemma provides some basic properties of the weights ν_j. Its proof is straightforward from the definition of $\hat{\mu}$ and the fact that $\mathbf{T}$ is a tree.

Lemma 5.3. – *Let C^j be an irreducible component of C_m which is not the principal one. Then we have*

a) $0 < \nu_j < 1$.

b) *Let $\hat{C}_m^j$ be the component containing C_m^j which is obtained by splitting C_m at the principal node of C_m^j, that is the node separating C_m^j from C_m^0. Then we have*

$$\nu_j = 1 - \sum_{x_s \in \hat{C}_m^j} \mu_s.$$

c) *If $\mathbf{v}_k$ is a vertex in the path from $\mathbf{v}_0$ to $\mathbf{v}_j$ and $k \neq j$, then $\nu_k < \nu_j$.*

Remark 5.4. – Every node of C_m is the principal node of a unique component. This is because each node of C_m corresponds to a pair of points $\{y, y'\}$ that are contained in two different components, and we have $\hat{\mu}(y) + \hat{\mu}(y') = 2$ (cf. (10)).

5.2. Construction of sections of $\mathscr{L}$ in a neighborhood of m

Set $k_j := |\hat{\Sigma}_j|$, $j = 0,\dots,r$. For each $j \in \{0,\dots,r\}$, let $\mathbf{L}_j$ be a rank one local system on $C_m^j \setminus \hat{\Sigma}_j$ with monodromy $\exp(2\pi\imath\hat{\mu}(y))$ at any point $y \in \hat{\Sigma}_j$. We will fix an $\mathbf{L}_j$-multivalued horizontal section $\mathbf{e}_j$, and a meromorphic section ω_j of $\Gamma(C_m^j, j_*^m\Omega^1(\mathbf{L}_j))$ with valuation $-\hat{\mu}(y)$ at every point $y \in \hat{\Sigma}_j$.

Let $\{p,q\}$ be a pair of points in the normalization of C_m that correspond to a node, and C_m^j and $C_m^{j'}$ be respectively the components that contain p and q. Using Lemma 4.1, we can find some neighborhoods U of p, and V of q, together with local coordinates F on U, G on V such that

$$\omega_j = \mathbf{e}_j \cdot F^{-\hat{\mu}(p)}dF, \quad \omega_{j'} = \mathbf{e}_{j'} \cdot G^{-\hat{\mu}(q)}dG.$$

Choose a constant $c > 0$ small enough such that for any $t = (t_1,\dots,t_r) \in (\mathrm{D}_{c^2})^r$, the plumbing construction with plumbing data (F,U,G,V) and parameter t_i as above can be carried out at all the nodes simultaneously. For any $j \in \{1,\dots,r\}$, we can assume that t_j is the plumbing parameter at the principal node of C_m^j.

Let $C_{(m,t)}$ denote the resulting surface in $\mathscr{M}_{0,n}$. On $C_{(m,t)}$ we have n marked points $(x_1,\dots,x_n)$ with associated weights $(\mu_1,\dots,\mu_n)$; we will also denote by Σ this finite subset of $C_{(m,t)}$. Let U_m^j be an open subset of C_m^j containing all the points in $\Sigma_j \subset \Sigma$ and disjoint from the regions affected by the plumbing construction. We can consider U_m^j as an open subset of $C_{(m,t)}$. As usual, let $\mathbf{L}$ be a rank one local system on $C_{(m,t)} \setminus \Sigma$, with monodromy $\exp(2\pi\imath\mu_s)$ at x_s.

LEMMA 5.5. – *Let* $\mathbf{j}$ *be the natural embedding of* $C_{(m,t)} \setminus \Sigma$ *into* $C_{(m,t)}$. *Then there exists a unique element* ω *of* $\Gamma(C_{(m,t)}, \mathbf{j}_*^m\Omega^1(\mathbf{L}))$ *such that*

- *the restriction of* ω *to* U_m^0 *is equal to* ω_0,
- *for* $j = 1,\dots,r$, *the restriction of* ω *to* U_m^j *is equal to* $P_j(t)\omega_j$, *where* $P_j(t)$ *is a function of* t *which is defined as follows: let* $0 < i_1 < \dots < i_{L_j} = j$ *be the indices of the vertices of* $\mathbf{T}$ *that are contained in the unique path from* $\mathbf{v}_0$ *to* $\mathbf{v}_j$, *and* ν_{i_s} *is the weight of* $C_m^{i_s}$, *then*

$$P_j(t_1,\dots,t_r) = (-1)^{L_j}\prod_{s=1}^{L_j} t_{i_s}^{\nu_{i_s}}.$$

Proof. – Let $\tilde{C}_m^j$ be the subsurface of C_m which is the union of the components $C_m^0,\dots,C_m^j$. Given $t = (t_1,\dots,t_r)$, we define $\tilde{C}^j_{(m,t_1,\dots,t_j)}$ from $\tilde{C}_m^j$ by applying successively the plumbing constructions with parameter t_i at the principal node of C_m^i, for $i = 1,\dots,j$. Note that $\tilde{C}^r_{(m,t_1,\dots,t_r)} = C_{(m,t)}$. By induction, this lemma is a direct consequence of Lemma 4.3. □

Let us denote by $\omega_{(m,t)}$ the $\mathbf{L}$-valued meromorphic one form given by Lemma 5.5. By construction, $\omega_{(m,t)}$ has valuation $-\mu_s$ at x_s, thus it is an element of the fiber of $\mathscr{L}$ over $C_{(m,t)}$. We would like now to show that the assignment $(m,t) \mapsto \omega_{(m,t)}$ is a holomorphic section of $\mathscr{L}$ on $\mathscr{U} \cap \mathscr{M}_{0,n}$, where $\mathscr{U}$ is a neighborhood of m in $\overline{\mathscr{M}_{0,n}}$.

We first specify an appropriate basis of $H_1^{\mathrm{lf}}(C_{(m,t)} \setminus \Sigma, \overline{\mathbf{L}})$. Let T_0 be an embedded topological tree in C_m^0, whose vertex set is $\hat{\Sigma}_0$ minus one point. For $j = 1,\dots,r$, let T_j be an embedded topological tree in C^j_m whose vertex set is $\hat{\Sigma}_j$ minus the principal node. Let a^j_i, $i = 1,\dots,k_j-2$, denote the edges of T_j. Fix an $\overline{\mathbf{L}}_j$-multivalued horizontal section $\mathbf{e}'_j$ on $C^j_m \setminus \hat{\Sigma}_j$. By Proposition 2.2, the family $\{[\mathbf{e}'_j \cdot a^j_i],\ i = 1,\dots,k_j-2\}$ is a basis of $H^{\mathrm{lf}}_1(C^j_m \setminus \hat{\Sigma}_j, \overline{\mathbf{L}}_j)$. Set

$$\xi^{(j)}_i := \langle [\mathbf{e}'_j \cdot a^j_i], [\omega_j] \rangle = \int_{a^j_i} (\mathbf{e}'_j, \omega_j),\ i = 1,\dots,k_j-2.$$

Let $\xi^{(j)}$ denote the vector $(\xi^{(j)}_1,\dots,\xi^{(j)}_{k_j-2})$. We have

LEMMA 5.6. – *Let ω and P_j, $j = 1,\dots,r$, be as in Lemma 5.5. Then there exists a basis of $H^{\mathrm{lf}}_1(C_{(m,t)} \setminus \Sigma, \overline{\mathbf{L}})$ such that the coordinates of $[\omega]$ in the dual basis are given by $(\xi^{(0)}, P_1(t)\xi^{(1)},\dots,P_r(t)\xi^{(r)}) \in \mathbb{C}^{n-2}$.*

Proof. – Let $\tilde{C}^j_m$ and $\tilde{C}^j_{(m,t_1,\dots,t_j)}$ be as in the proof of Lemma 5.5. Recall that $C_{(m,t)} = \tilde{C}^r_{(m,t_1,\dots,t_r)}$, and $\tilde{C}^j_{(m,t_1,\dots,t_j)}$ is obtained from $\tilde{C}^{j-1}_{(m,t_1,\dots,t_{j-1})}$ and C^j_m by the plumbing construction at the principal node of C^j_m. The lemma then follows from Lemma 4.4 and Lemma 5.5 by induction. □

Each pair $(C^j_m, \hat{\Sigma}_j)$ represents a point m_j in $\mathcal{M}_{0,k_j}$. Hence the point m is contained in a stratum of $\overline{\mathcal{M}}_{0,n}$ which is isomorphic to $\mathcal{M}_{0,k_0} \times \cdots \times \mathcal{M}_{0,k_r}$. Let $\mathcal{V}_j$ be a neighborhood of m_j in $\mathcal{M}_{0,k_j}$ and set $\mathcal{V} := \mathcal{V}_0 \times \cdots \times \mathcal{V}_r$. Let $\mathcal{L}_j$ denote the line bundle over $\mathcal{M}_{0,k_j}$ whose fiber over m_j is $\mathbb{C} \cdot [\omega_j] \subset H^1(C^j_m \setminus \hat{\Sigma}_j, \mathbf{L}_j)$. We extend ω_j to a holomorphic section of $\mathcal{L}_j$ on $\mathcal{V}_j$. Since the plumbing data (F,U,G,V) depend analytically on $(m_0,\dots,m_r)$, the plumbing construction $(m,t) \mapsto C_{(m,t)}$ identifies a neighborhood of m in $\overline{\mathcal{M}}_{0,n}$ with $\mathcal{V} \times (\mathrm{D}_{c^2})^r$.

Let $\omega_{(m,t)}$ denote the $\mathbf{L}$-valued meromorphic one form on $C_{(m,t)}$ defined in Lemma 5.5. The assignment $\Phi : (m,t) \mapsto \omega_{(m,t)}$ provides us with a section of $\mathcal{L}$ on $\mathcal{V} \times (\mathrm{D}^*_{c^2})^r$. Since ω_j is a holomorphic section of $\mathcal{L}_j$, $\xi^{(j)}$ depends analytically on m_j. It follows that the coordinates of $\omega_{(m,t)}$ in a basis of $H^1(C_{(m,t)} \setminus \Sigma, \mathbf{L})$ are given by holomorphic functions of $(m_0,\dots,m_r,t_1,\dots,t_r)$. Thus we have shown

PROPOSITION 5.7. – *The section Φ is holomorphic.*

6. Flat metrics on punctured spheres and Hermitian metric on the line bundle $\mathcal{L}$

6.1. Hermitian norm of the section Φ

Let m be now a point in a stratum $\mathcal{M} := \mathcal{M}_{0,k_0} \times \cdots \times \mathcal{M}_{0,k_r}$ of codimension r in $\overline{\mathcal{M}}_{0,n}$. Let (C_m, Σ) be the stable curve represented by m, and $C^0_m,\dots,C^r_m$ its irreducible components. In what follows we will use the notations of Section 5. Our goal in this section is to prove a formula (cf. (11)) for the Hermitian norm of $\Phi(m,t)$ in $H^1(C_{(m,t)} \setminus \Sigma, \mathbf{L})$, where Φ is the section in Proposition 5.7.

On each irreducible component C^j_m of C_m, we have a finite subset $\hat{\Sigma}_j$ consisting of points in $\Sigma \cap C^j_m$ and the nodes of C^j_m. The pair $(C^j_m, \hat{\Sigma}_j)$ represents a point $m_j \in \mathcal{M}_{0,k_j}$, where $k_j = |\hat{\Sigma}_j|$. We identified a neighborhood of m in $\overline{\mathcal{M}}_{0,n}$ with $\mathcal{V}_0 \times \cdots \times \mathcal{V}_r \times (\mathrm{D}_{c^2})^r$, where

$\mathcal{V}_j \subset \mathcal{M}_{0,k_j}$ is a neighborhood of $m_j := (C^j_m, \hat{\Sigma}_j)$, and c is a positive real constant small enough.

Let $z^j \in \mathbb{C}^{k_j-3}$, $j = 0, \dots, r$, be the coordinates on $\mathcal{V}_j$, and $t = (t_1, \dots, t_r)$ the coordinates on $(\mathrm{D}_{c^2})^r$. In these local coordinates, m is identified with the point $(z^0(m_0), \dots, z^r(m_r), 0 \dots, 0)$, and we have $\mathcal{V} \times (\mathrm{D}^*_{c^2})^r = \mathcal{V} \times (\mathrm{D}_{c^2})^r \cap \mathcal{M}_{0,n}$, where $\mathcal{V} = \mathcal{V}_0 \times \dots \times \mathcal{V}_r$.

Remark that for each $j \in \{1, \dots, r\}$, the subset of $\mathcal{V} \times (\mathrm{D}_{c^2})^r$ defined by $\{t_j = 0\}$ is the intersection of $\mathcal{V} \times (\mathrm{D}_{c^2})^r$ with a boundary divisor $D_{\mathcal{S}_j}$ in $\overline{\mathcal{M}_{0,n}}$. This divisor corresponds to the partition $\mathcal{S}_j = \{I^j_0, I^j_1\}$ of $\{1, \dots, n\}$ that is induced by the splitting of the j-th node of C_m into two points. Thus, we see that the stratum $\mathcal{M}$ of m is precisely the intersection $\bigcap_{1 \le j \le r} D_{\mathcal{S}_j}$.

Recall that $\hat{\mu}^j$ is the vector recording the weights of marked points in C^j_m, and that the component C^0_m of C_m is characterized by the property that all the weights in $\hat{\mu}^0$ are strictly smaller than 1 (see Lemma 5.1). Thus the Hermitian form on $H^1(C^0_m \setminus \hat{\Sigma}_0, \mathbf{L}_0)$ has signature $(1, k_0-3)$. Let us denote this Hermitian form by $((.,.))_0$. Recall that we have defined a section $\Phi : (m,t) \mapsto \omega_{(m,t)}$ of $\mathcal{L}$ on $\mathcal{V} \times (\mathrm{D}^*_{c^2})^r$ (see Proposition 5.7). We will prove the following

Proposition 6.1. – *For $j = 1, \dots, r$, let $P_j(t)$ be as in Lemma 5.5, and $\xi^{(j)} \in \mathbb{C}^{k_j-2}$ be as in Lemma 5.6. For each $j = 1, \dots, r$, there exists a positive definite Hermitian form $((.,.))_j$ on $\mathbb{C}^{k_j-2}$ depending only on μ such that the norm of $[\omega_{(m,t)}]$ in $H^1(C_{(m,t)} \setminus \Sigma, \mathbf{L})$ is given by*

$$((\,[\omega_{(m,t)}], [\omega_{(m,t)}]\,)) = ((\xi^{(0)}, \xi^{(0)}))_0 - \sum_{j=1}^{r} |P_j(t)|^2 ((\xi^{(j)}, \xi^{(j)}))_j. \tag{11}$$

Here we identify $((.,.))_0$ with a Hermitian form on $\mathbb{C}^{k_0-2}$.

As a consequence of Proposition 6.1, we get

Proposition 6.2. – *There exist some neighborhood $\mathcal{V}_j$ of m_j and holomorphic local coordinates $z^j : \mathcal{V}_j \to \mathbb{C}^{k_j-3}$ such that if $m = (z^0, \dots, z^r, \underbrace{0, \dots, 0}_{r})$, and $t = (t_1, \dots, t_r) \in (\mathrm{D}^*_{c^2})^r$, then we have*

$$\|\Phi(m,t)\|^2 = ((\,[\omega_{(m,t)}], [\omega_{(m,t)}]\,)) = 1 - ||z^0||^2 - \sum_{j=1}^{r} |P_j(t)|^2 (1 + ||z^j||^2), \tag{12}$$

and the Chern form of $\mathcal{L}$ on $\mathcal{V} \times (\mathrm{D}^*_{c^2})$ is given by

$$\Omega_\mu := dd^c \log\left(1 - ||z^0||^2 - \sum_{j=1}^{r} |P_j(t)|^2 (1 + ||z^j||^2)\right). \tag{13}$$

In other words, locally at m, Ω_μ is the pullback of the complex hyperbolic metric $dd^c \log(1 - \|w\|^2)$ on $\mathbb{C}^{n-3}$ by the multivalued map

$$(z^0, t_1, \dots, t_r, z^1, \dots, z^r) \mapsto w = (z^0, P_1(t), \dots, P_r(t), P_1(t) z^1, \dots, P_r(t) z^r)$$

(note that even if the map is multivalued, the metric is well defined).

REMARK 6.3. – Recall from Lemma 5.5 that we have $P_j(t) = (-1)^{L_j}\prod_{s=1}^{L_j} t_{i_s}^{\nu_{i_s}}$, where the family of indices $\{i_s,\ s=1,\dots,L_j\}$ records the components of C_m between the principal one, that is, C_m^0, and C_m^j. Since all the exponents ν_{i_s} are positive, the function $|P_j(t)|$ extends by continuity to $(\mathrm{D}_{c^2})^r$. Thus, the function

$$\varphi : (z^0,\dots,z^r,t) \mapsto 1 - ||z^0||^2 - \sum_{j=1}^{r} |P_j(t)|^2(1+||z^j||^2)$$

is a continuous function on $\mathcal{V}\times(\mathrm{D}_{c^2})^r$.

As we will see in the sequel, the Hermitian norm $(([\omega_{(m,t)}],[\omega_{(m,t)}]))$ can be interpreted as the area of the flat surface defined by $\omega_{(m,t)}$. From this viewpoint, the continuity of φ on $(\mathrm{D}_{c^2})^r$ reflects the fact that as t converges to $0 \in (\mathrm{D}_{c^2})^r$, the metric defined by $\omega_{(m,t)}$ "converges" to the metric defined by ω_0 on the principal component of C_m. This convergence in the space of flat metrics is the key point in the construction of the (metric) completion of $\mathcal{M}_{0,n}$ introduced by Thurston [22].

Proof of Prop. 6.2 assuming Prop. 6.1. – Recall that we have a rank one local system $\mathbf{L}_j$ on $C_m^j \setminus \hat{\Sigma}_j$ whose monodromy at the points in $\hat{\Sigma}$ are given by $\hat{\mu}^j$. This local system gives rise to a local system $\mathbf{H}_j$ of rank k_j-2 and a holomorphic line bundle $\mathcal{L}_j$ on $\mathcal{M}_{0,k_j}$. Let Ξ_{μ_j} denote the section of the bundle $\mathbb{P}\mathbf{H}_j$ defined in Proposition 2.9. By construction, ω_j is a vector in the line $\Xi_{\mu_j}(m_j)$. Let $\xi^{(j)} = (\xi_1^{(j)},\dots,\xi_{k_j-2}^{(j)})$ be the coordinates of ω_j in some basis of $H^1(C^j\setminus\hat{\Sigma}_j,\mathbf{L}_j)$. We can choose the basis of $H^1(C_m^j\setminus\hat{\Sigma}_j,\mathbf{L}_j)$ such that

$$((\xi^{(0)},\xi^{(0)}))_0 = -\sum_{i=1}^{k_0-3}|\xi_i^{(0)}|^2 + |\xi_{k_0-2}^{(0)}|^2,$$

and

$$((\xi^{(j)},\xi^{(j)}))_j = \sum_{i=1}^{k_j-2}|\xi_i^{(j)}|^2, \quad \text{for } j=1,\dots,r.$$

Recall that $\omega_j \in H^1(C^j\setminus\hat{\Sigma}_j,\mathbf{L}_j)$ is not trivial so that we can normalize our coordinates in such a way that $\xi_{k_j-2}^{(j)} = 1$. Hence $(\xi_1^{(j)},\dots,\xi_{k_j-3}^{(j)})$ are the coordinates of $\Xi_{\mu_j}(m_j)$ in some local chart of $\mathbb{P}H^1(C_m^j\setminus\hat{\Sigma}_j,\mathbf{L}_j)$. Since Ξ_{μ_j} is étale by Proposition 2.9, we can use $z_i^j = \xi_i^{(j)}$, $i=1,\dots,k_j-3$, to define local coordinates in a neighborhood of m_j. The proposition then follows from (11). □

We will spend the rest of this section to prove Proposition 6.1. For this purpose, we will make use of the flat metric approach introduced by Thurston [22].

6.2. Thurston's coordinates

Let us first recall Thurston's coordinates on the moduli space of flat metrics on the sphere with prescribed cone angles at singularities (see [22, Prop. 3.2]). Fix a vector $(\theta_1,\dots,\theta_n)$, with $0<\theta_s<2\pi$, such that $\theta_1+\cdots+\theta_n = 2\pi(n-2)$. Let M denote a flat surface homeomorphic to $\mathbb{S}^2$ with conical singularities denoted by $x_1,\dots,x_n$, and the cone angle at x_s being θ_s. Let T be a tree whose vertex set consists of $n-1$ points in $\{x_1,\dots,x_n\}$ and all the edges are geodesics (it is not difficult to show that such a tree always exists). Choosing an orientation for every edge of T, then using a developing map, one can associate to each edge of T a complex number (see [22, pp. 525-526]). We then get a vector $Z(M)$ in $\mathbb{C}^{n-2}$ associated with M.

For any flat metric (with the same prescribed cone angles at the singularities) close to M, one can also find a geodesic tree isomorphic to T. Hence, we also get an associated vector in $\mathbb{C}^{n-2}$ in the same way. It turns out that this correspondence defines a local chart for the space of flat metrics (with prescribed cone angles) on the sphere. Up to homothety, this space can be identified with $\mathcal{M}_{0,n}$. Therefore, this construction also yields a local coordinate system for $\mathcal{M}_{0,n}$.

Let $m = (\mathbb{P}^1_{\mathbb{C}}, \{x_1, \dots, x_n\}) \in \mathcal{M}_{0,n}$ be the point corresponding to the homothety class of M. Assume that all the cone angles at the singularities are smaller than 2π. In [22], it was proved that the area of M can be expressed as a Hermitian form A of signature $(1, n-3)$ in the coordinates of $Z(M)$, that is

$$\mathrm{Area}(M) = {}^t\overline{Z(M)} \cdot \mathrm{A} \cdot Z(M).$$

Consequently, the induced local chart on $\mathcal{M}_{0,n}$ identifies a neighborhood of m with an open subset in the ball $\mathbf{B} := \{\langle v\rangle,\ {}^t\bar{v}\mathrm{A}v > 0\} \subset \mathbb{P}^{n-3}_{\mathbb{C}}$. By a classical construction, A induces a complex hyperbolic metric on $\mathbf{B}$. Since the area is an invariant of the flat metric, this complex hyperbolic metric is invariant by the coordinate changes. Therefore, we get a well defined complex hyperbolic metric structure on $\mathcal{M}_{0,n}$.

Set $\mu_s := 1-\theta_s/(2\pi)$, and $\mu = (\mu_1, \dots, \mu_n)$. By definition, M is isometric to $(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{g})$, where $\Sigma = \{x_1, \dots, x_n\}$, and $\mathbf{g}$ is a flat metric on $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$ such that each x_s has a neighborhood isometric to an Euclidean cone of angle θ_s. Without loss of generality, we can assume that $\infty \not\in \Sigma$. Remark that $\prod_{1\leq s\leq n} |z-x_s|^{-2\mu_s}|dz|^2$ is a flat metric with the same singularities and the same cone angles as $\mathbf{g}$. Therefore, we must have $\mathbf{g}(z) = \lambda^2 \prod_{1\leq s\leq n} |z-x_s|^{-2\mu_s}|dz|^2$, where λ is a positive real number.

Let $\mathbf{L}$ be the rank one local system on $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$ with monodromy $\exp(2\pi\imath\mu_s)$ at x_s. Choose a horizontal Hermitian metric for $\mathbf{L}$, and let $\mathbf{e}$ be an $\mathbf{L}$-multivalued horizontal section such that the norm of $\mathbf{e}$ is 1. Let

$$\omega = \lambda\mathbf{e} \cdot \prod_{1\leq s\leq n} (z-x_s)^{-\mu_s} dz.$$

Then $\mathbf{g}(z)$ is the metric associated to the $(1,1)$-form $\imath\omega \wedge \overline{\omega}$. Recall that we have a Hermitian form $((.,.))$ on $H^1_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) \simeq H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$ of signature $(1, n-3)$. By Proposition 2.7, we have

$$((\,[\omega], [\omega]\,)) = \imath \int_{\mathbb{P}^1_{\mathbb{C}}\setminus\Sigma} \omega \wedge \overline{\omega} = \mathrm{Area}(M). \tag{14}$$

Fix a base point $p \in \mathbb{P}^1_{\mathbb{C}}\setminus\Sigma$ and consider the universal cover $(\Delta, \tilde{p})$ of $(\mathbb{P}^1_{\mathbb{C}}\setminus\Sigma, p)$. Let f be a determination of the multivalued function $\lambda(z-x_1)^{-\mu_1}\dots(z-x_n)^{-\mu_n}$ in a neighborhood U of p. We also denote by f its pullback to a neighborhood $\tilde{U}$ of $\tilde{p}$. Let φ be a holomorphic function on $\tilde{U}$ such that $f = \varphi'$. Let z be the coordinate on Δ, and set $w = \varphi(z)$. Observe that we have

$$\varphi^* dw = f(z)dz, \text{ and } \varphi^*|dw|^2 = |f(z)|^2|dz|^2,$$

which means that φ realizes an isometry between a neighborhood of $\tilde{p}$ (with the metric $\mathbf{g}$) and an open subset of $\mathbb{C}$ with the standard Euclidean metric $|dw|^2$. In other words, φ is a developing map for $\mathbf{g}$. Therefore, we can extend φ to a locally isometric map from $(\Delta, \tilde{\mathbf{g}})$ to $(\mathbb{C}, |dw|^2)$.

Now let a be an oriented edge of the tree T. The complex number associated to a is given by $\int_{\varphi(\hat{a})} dw = \int_{\hat{a}} f(z)dz$, where $\hat{a}$ is a component of the pre-image of a in Δ. We can consider $\mathbf{e} \cdot a$ as an element of $H_1(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$, therefore, we can write

$$\int_{\hat{a}} f(z)dz = ([\mathbf{e} \cdot a], [\mathbf{e} \cdot f(z)dz]).$$

From Proposition 2.2, we know that the set $\{[\mathbf{e} \cdot a], \quad a \text{ is an edge of T}\}$ is a basis of $H_1(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$. Since the pairing $H_1^{\mathrm{lf}}(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) \otimes H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) \to \mathbb{C}$ is perfect, it follows that the cohomology class of ω is (locally) uniquely determined by the vector $Z(M) \in \mathbb{C}^{n-2}$.

By definition, the hyperbolic metric on $\mathcal{M}_{0,n}$ is the pullback of the complex hyperbolic metric on the ball $\mathbf{B} \subset \mathbb{P}^{n-3}_{\mathbb{C}}$. This metric is defined by the Chern form of the tautological line bundle over $\mathbf{B} \subset \mathbb{P}^{n-3}_{\mathbb{C}}$. Recall that $\mathbb{C} \cdot [\omega]$ is the fiber of $\mathcal{L}$ over m, and $\mathcal{L}$ is actually the pullback of the tautological bundle on $\mathbf{B}$ by the map Ξ_μ (see Proposition 2.9). Thus we have proved the following

Proposition 6.4. – *The Thurston local coordinates on $\mathcal{M}_{0,n}$ are defined by the section Ξ_μ, and the Hermitian form* A *on $\mathbb{C}^{n-2}$ is induced by the Hermitian form* $((.,.))$ *on* $H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$. *Moreover, the complex hyperbolic metric on $\mathcal{M}_{0,n}$ is the one induced by the Chern form of* $(\mathcal{L}, ((.,.)))$.

6.3. Thurston's surgery on flat surfaces

We now describe the *cone adding* construction introduced in [22, pp. 520-521], which is the key idea of the proof that the signature of A is $(1, n-3)$. Let M be a flat surface homeomorphic to the sphere which has n conical singular points as above. Recall that μ_s is the curvature at the cone point x_s. Suppose now that we are given a geodesic arc e on M joining x_i to x_j and $\mu_i + \mu_j < 1$.

We first construct an Euclidean cone whose apex angle is $2\pi(1-\mu_i-\mu_j)$ as follows: Let (ABC) be a triangle in $\mathbb{R}^2$ whose interior angles at A, B, C are given by $((1-\mu_i-\mu_j)\pi, \mu_i\pi, \mu_j\pi)$ respectively, and the length of $\overline{BC}$ is equal to the length of e. Let $(A'B'C')$ be the image of (ABC) by the mirror symmetry. We now glue $\overline{AC}$ to $\overline{A'C'}$, and $\overline{AB}$ to $\overline{A'B'}$ by identifications respecting the order of endpoints. We then obtain a flat surface homeomorphic to a disk, which has a singular point $\hat{x}$ with cone angle $2\pi(1-\mu_i-\mu_j)$ in the interior. The boundary of this disk is the union of two geodesic segments corresponding to $\overline{BC}$ and $\overline{B'C'}$. Let y_i (resp. y_j) denote the identification of B and B' (resp. of C and C'). The interior angles at y_i and y_j are respectively $2\pi\mu_i, 2\pi\mu_j$.

We now slit open M along e and glue the cone constructed above to this surface in such a way that y_i (resp. y_j) is identified with x_i (resp. with x_j). Since e and $\overline{BC}$ have the same length, the gluings are realized by isometries. We thus have a flat surface $\hat{M}$ homeomorphic to $\mathbb{S}^2$. By construction, the cone angles at x_i and x_j in $\hat{M}$ are now equal to 2π, which means that x_i and x_j are regular points in $\hat{M}$. Therefore, $\hat{M}$ has exactly $n-1$ singularities: x_s with $s \notin \{i,j\}$, and $\hat{x}$. Remark that e corresponds to a loop on $\hat{M}$ consisting of two geodesic arcs, we will call e and the corresponding loop the *base* of the added cone. We record here below some key properties of this construction.

- The triangle (ABC) is uniquely determined up to isometry, since its angles are determined by μ_i and μ_j, and the length of $\overline{BC}$ is equal to the length of e. It follows that there exists a positive constant $\kappa(\mu_i,\mu_j)$ such that $\mathrm{Area}((ABC)) = \kappa(\mu_i,\mu_j)|e|^2$, where $|e|$ is the length of e.
- We have
$$\mathrm{Area}(\hat{M}) - \mathrm{Area}(M) = 2\mathrm{Area}((ABC)) = 2\kappa(\mu_i,\mu_j)|e|^2.$$
- The sides of (ABC) can be considered as geodesic segments in $\hat{M}$. Thus, given a developing map of $\hat{M}$, we can associate to those segments the complex numbers $z(\overline{BC}), z(\overline{CA}), z(\overline{AB})$. There exist some complex numbers c_1, c_2 depending only on (μ_1,μ_2) such that
$$z(\overline{AB}) = c_1 z(\overline{BC}), \text{ and } z(\overline{AC}) = c_2 z(\overline{BC}).$$
- We can apply similar constructions to $\hat{M}$ to get other surfaces with less singularities as long as there are two singular points such that the sum of the corresponding curvatures is less than 1.

6.4. Flat surfaces with convex boundary

For our purpose, we will need to consider flat surfaces with boundary. In what follows, by a *flat surface with convex boundary* we will mean a topological surface with boundary M equipped with a flat metric structure with conical singularities satisfying the following property: for any point $x \in \partial M$, there is a neighborhood of x which is isometric to a convex domain in $\mathbb{R}^2$. For such a surface, any path of minimal length (in a fixed homotopy class) joining two points in the interior does not intersect the boundary.

Let Σ denote the set of cone singularities in $\mathrm{int}(M)$. We will also need a generalized notion of homotopy on M. Two arcs $\gamma_0, \gamma_1 : [0,1] \to M$ are said to be *homotopic in $M \setminus \Sigma$ with fixed endpoints* if we have $\gamma_0(0) = \gamma_1(0) = x, \gamma_0(1) = \gamma_1(1) = y$, and there exists a continuous map $H : [0,1]\times[0,1] \to M$ such that $H(.,0) = \gamma_0, H(.,1) = \gamma_1$, $H(0,.) = \{x\}, H(1,.) = \{y\}$, and $H((0,1)\times(0,1)) \subset M \setminus \Sigma$. With this definition, a path with two endpoints in Σ not passing through any other point in Σ may be homotopic to the union of some arcs with endpoints in Σ. Remark that given any developing map for the flat metric, the complex numbers associated to two homotopic paths (that is the difference in $\mathbb{C}$ of the two endpoints) must be the same.

We now suppose that M is a flat surface with convex boundary. Let $\Sigma = \{x_1,\dots,x_n\}$ denote the set of cone points of M, and assume that Σ is contained in the interior of M. All the cone angles θ_s at x_s are supposed to be smaller than 2π, and

$$\sum_{1\le s\le n} (2\pi - \theta_s) < 2\pi. \tag{15}$$

Set $\mu_s = 1 - \theta_s/(2\pi)$. The condition (15) is equivalent to $\mu_1 + \cdots + \mu_s < 1$. Let e be the path of minimal length from x_1 to x_2. Note that e is contained in the interior of M (since M has convex boundary), and e does not pass through any other point in Σ. Let M' be the flat surface obtained by slitting open M along e. One of the boundary components of M' consists of two copies of e, which will be denoted by e_1 and e_2. Since $\mu_1 + \mu_2 < 1$, we can glue an Euclidean cone $\mathbf{C}$ of apex curvature $1 - \mu_1 - \mu_2$ to M' along this boundary component. Let $\hat{M}$ denote the new surface. Remark that $\hat{M}$ also has convex boundary. We consider M' as a subsurface of $\hat{M}$. The singular points x_1, x_2 of M now correspond to two regular points in $\hat{M}$, we denote those points by the same notation. Let $\hat{x}$ be the apex of $\mathbf{C}$, and set $\hat{\Sigma} = \{x_3, \dots, x_n\} \sqcup \{\hat{x}\}$. It is worth noticing that given a path in M which does not cross e, then its image by a developing map for $\hat{M}$ is also the image of a developing map for M. In view of the proof of Proposition 6.1, we will need the following lemma.

Lemma 6.5. – *Let a be a geodesic segment in $\hat{M}$ with endpoints in $\hat{\Sigma}$. We assume that the two endpoints of a are distinct, and a does not contain any point in $\hat{\Sigma}$ in its interior. Then there exists a piecewise geodesic path b in M' connecting two points in Σ, and a constant $\kappa \in \mathbb{C}$ such that, for a fixed choice of the developing map on the universal cover of $\hat{M}$, we have*

$$z(a) = \kappa z(e_1) + z(b),$$

where $z(a), z(e_1), z(b)$ are the complex numbers associated to a, e_1, b respectively.

Proof. – We have two cases:

- Case 1: a does not contain $\hat{x}$. Since $\mathbf{C} \setminus \{\hat{x}\}$ is homeomorphic to a punctured disk, M' is a deformation retract of $\hat{M} \setminus \{\hat{x}\}$. Let b be the image of a by this retraction, then b is homotopic in $\hat{M}$ to a. Thus we have $z(a) = z(b)$.
- Case 2: a contains $\hat{x}$. By assumption, we can consider a as a ray starting from $\hat{x}$ and ending at a point $x_s \in \Sigma$. Let y be the first intersection of a with $\partial\mathbf{C} = e_1 \cup e_2$. Denote by a_0 (resp. a_1) the subsegment of a between $\hat{x}$ and y (resp. between y and x_s). Let e'_1 be the geodesic segment in $\partial\mathbf{C}$ from x_1 to y. Set $b_0 = e'_1 * a_0$ and $b_1 := a_1 * e'_1$. Observe that a is homotopic (in $\hat{M}$) to the path $b_1 * b_0$. Since b_1 does not contain $\hat{x}$, from Case 1, we know that it is homotopic to a piecewise geodesic path b from x_1 to x_s. We thus have $z(a) = z(b_0) + z(b)$. But we have seen that $z(b_0) = \kappa z(e_1)$, where κ is a complex number determined by (μ_1, μ_2). Hence the lemma is proved for this case. □

6.5. Infinite flat metric structures

Let $k \geq 1$ and fix a vector $\mu = (\mu_0, \dots, \mu_k) \in \mathbb{R}^{k+1}_{>0}$ such that $\mu_0 + \cdots + \mu_k = 2$, where $\mu_0 > 1$, but $\mu_i < 1$, for $i = 1, \dots, k$. Let $\Sigma = \{x_0, \dots, x_k\}$ be a set of $k+1$ points in $\mathbb{P}^1_{\mathbb{C}}$ and $\mathbf{L}$ be the rank one local system on $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$ whose monodromy at x_s is $\exp(2\pi\imath\mu_s)$. Fix a horizontal multivalued section $\mathbf{e}$ of $\mathbf{L}$ with Hermitian norm equal to 1. Let ω be a meromorphic section of $\Omega^1(\mathbf{L})$ with valuation $-\mu_s$ at x_s. We can write

$$\omega = \lambda\mathbf{e} \cdot \prod_{0 \leq s \leq k} (z - x_s)^{-\mu_s} dz$$

with $\lambda \in \mathbb{C}^*$. Note that since $\mu_0 > 1$, we have $t\int_{\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma} \omega \wedge \overline{\omega} = \infty$, so ω is not of the first kind.

Let T be an embedded tree in $\mathbb{P}^1_{\mathbb{C}}$ whose vertex set is $\Sigma_0 = \{x_1, \dots, x_k\}$. Let a_j, $j = 1, \dots, k-1$, denote the edges of T. From Proposition 2.2, we know that the family $\{[\mathbf{e} \cdot a_j],\ j = 1, \dots, k-1\}$ is a basis of $H_1^{\mathrm{lf}}(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$. Recall that ω represents a cohomology class in $H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L}) \simeq H^1_c(\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma, \mathbf{L})$. Set $z_j := \langle [\mathbf{e} \cdot a_j], [\omega] \rangle$, and $Z := (z_1, \dots, z_{k-1}) \in \mathbb{C}^{k-1}$. Since the valuations of ω at the endpoints of a_j are all greater than -1, we can write

$$z_j = \lambda \int_{a_j} (z - x_0)^{-\mu_0} \dots (z - x_k)^{-\mu_k} dz.$$

Observe that the $(1,1)$-form $\imath\omega \wedge \overline{\omega}$ defines a flat metric structure on $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$, with conical singularity at x_s for $s = 1, \dots, k$. The cone angle at x_s is $\theta_s = 2\pi(1 - \mu_s)$. Note that this is an infinite metric structure since any geodesic ray cannot reach x_0 in finite time. Let M denote the corresponding flat surface.

Let e_1 be a path of minimal length in M joining x_1 and x_2, such a path must be a geodesic segment which does not contain any singularity in its interior. By assumption, we have $\mu_1 + \mu_2 < 1$. Thus we can add a cone $\mathbf{C}_1$ over e_1 to get a surface with $k-1$ singularities. By construction, the curvature at the new singularity is $\mu_1 + \mu_2$. One can continue adding $k-2$ cones $\mathbf{C}_2, \dots, \mathbf{C}_{k-1}$ to obtain successively the surfaces $M_2, \dots, M_{k-1}$, where M_i has a cone singularity with curvature $\mu_1 + \dots + \mu_{i+1}$ at some point denoted by $\hat{x}_i$, and M_{i+1} is obtained from M_i by adding the cone $\mathbf{C}_{i+1}$ whose base is a geodesic arc, denoted by e_{i+1}, joining $\hat{x}_i$ and x_{i+1}. Note that there exists a positive real constant c_i depending only on $(\mu_1, \dots, \mu_k)$ such that $\mathrm{Area}(\mathbf{C}_i) = c_i|e_i|^2$. Remark also that M_i has $k - i$ singularities, and M_{k-1} is an infinite Euclidean cone with apex angle equal to $2\pi(1 - (\mu_1 + \dots + \mu_k))$.

Choosing a developing map for $M_0 = M$, we get a complex number w_1 associated to e_1. We can extend the developing map of M_0 to get a developing map of $\mathbf{C}_1$. By construction, e_2 is a geodesic ray starting from $\hat{x}_1$ (the apex of $\mathbf{C}_1$), therefore we can extend this developing map to get a complex number w_2 associated to e_2. Continuing this process, we get a vector $W = (w_1, \dots, w_{k-1}) \in \mathbb{C}^{k-1}$, where w_i is the complex number associated to e_i. We have the following lemma, which is implicit in the proof of [22, Prop. 3.3].

Lemma 6.6. – *The complex number w_i is a linear function of Z for $i = 1, \dots, k-1$.*

Proof. – We will prove this lemma by induction. Recall that w_1 is the complex number associated to the geodesic arc e_1 on $M_0 = M$. But this number can be interpreted as the pairing of the homology class $[\mathbf{e} \cdot e_1]$ with $[\omega]$, hence it is a linear function of Z. Note also that by the same argument, the complex number associated to any path in M_0 with endpoints in $\{x_1, \dots, x_k\}$ is a linear function of Z.

Consider the flat surface M_1. As a Riemann surface, M_1 can be identified with $\mathbb{P}^1_{\mathbb{C}}$. Set $\hat{\Sigma}_1 := \{x_0, \hat{x}_1, x_3, \dots, x_k\}$. Let $\mathbf{L}_1$ be the rank one local system on $\mathbb{P}^1_{\mathbb{C}} \setminus \hat{\Sigma}_1$ with monodromy $\exp(2\imath\pi\mu_s)$ at x_s, for $s = 0, 3, \dots, k$, and $\exp(2\imath\pi(\mu_1 + \mu_2))$ at $\hat{x}_1$. The flat metric of M_1 is thus induced by a $\mathbf{L}_1$-valued meromorphic 1-form $\omega_1 \in \Gamma(\mathbb{P}^1_{\mathbb{C}}, \mathbf{j}^m_* \Omega(\mathbf{L}_1))$ with valuation $-\mu_s$ at x_s, for $s = 0, 3, \dots, k$, and $-(\mu_1 + \mu_2)$ at $\hat{x}_1$.

Let T_1 be an embedded tree in M_1 whose vertex set is equal to $\{\hat{x}_1, x_3, \dots, x_k\}$, and whose edges are geodesic segments in M_1. One can construct such a tree by seeking for instance the paths of minimal length joining $\hat{x}_1$ to the other cone points $x_3, \dots, x_k$.

Let $a_1^1, \dots, a_{k-2}^1$ denote the edges of T_1. Fix a multivalued horizontal section $\mathbf{e}_1$ of $\mathbf{L}_1$. Then $\{[\mathbf{e}_1 \cdot a_1^1], \dots, [\mathbf{e}_1 \cdot a_{k-2}^1]\}$ is a basis of $H_1^{\mathrm{lf}}(\mathbb{P}^1_{\mathbb{C}} \setminus \hat{\Sigma}, \mathbf{L}_1) \simeq H_1(\mathbb{P}^1_{\mathbb{C}} \setminus \hat{\Sigma}, \mathbf{L}_1)$.

Set $z_j^1 := \langle [\mathbf{e}_1 \cdot a_j^1], [\omega_1] \rangle$, for $j = 1, \dots, k-2$. Since the complex number associated to any path with endpoints in $\hat{\Sigma}_1 \setminus \{x_0\}$ can be also interpreted as the pairing of $[\omega_1]$ with a homology class in $H_1^{\mathrm{lf}}(\mathbb{P}^1_{\mathbb{C}} \setminus \hat{\Sigma}_1, \mathbf{L}_1)$, it follows that such a number is a linear function of $Z^1 := (z_1^1, \dots, z_{k-2}^1)$. From Lemma 6.5, we deduce that the z_j^1's are linear functions of the vector Z. Therefore, the complex number associated to any path in M_1 with endpoints in $\hat{\Sigma}_1 \setminus \{x_0\}$ is a linear function of Z. In particular w_2 is a linear function of Z. The rest of the proof follows from an induction argument. □

Lemma 6.6 implies that the correspondence $\Psi : \mathbb{C}^{k-1} \to \mathbb{C}^{k-1}, (z_1, \dots, z_{k-1}) \mapsto (w_1, \dots, w_{k-1})$ is a linear map. Our goal now is to show the following.

Proposition 6.7. – *The linear map Ψ is an isomorphism.*

Proof. – Let $\mathcal{L}$ be the holomorphic line bundle over $\mathcal{M}_{0,k+1}$ associated to the weight vector μ (see Section 2.5). To show that Ψ is an isomorphism, we will show that Ψ is injective in a neighborhood of Z. For this, we consider ω as an element in the fiber of the line bundle $\mathcal{L}$ over the point $m = (\mathbb{P}^1_{\mathbb{C}}, \{x_0, x_1, \dots, x_k\}) \in \mathcal{M}_{0,k+1}$, and identify a neighborhood $\mathcal{V}$ of Z in $\mathbb{C}^{k-1}$ with a neighborhood of ω in the total space of $\mathcal{L}$.

We can always assume that $x_0 = \infty, x_1 = 0, x_2 = 1$. A point m' in $\mathcal{M}_{0,k+1}$ close to m corresponds to a tuple $(\mathbb{P}^1_{\mathbb{C}}, \{\infty, 0, 1, x'_3, \dots, x'_k\})$, with x'_i close to x_i. Hence an element of $\mathcal{L}$ close to ω can be written as

$$\omega' = \lambda' \mathbf{e} \cdot z^{-\mu_1}(z-1)^{-\mu_2} \prod_{i=3}^{k} (z - x'_i)^{-\mu_i} dz,$$

where $\lambda' \in \mathbb{C}$ is close to λ, and $\mathbf{e}$ is considered as a horizontal section of $\mathbf{L}$ on the pointed curve represented by m'.

Assume that we have Z' and Z'' in $\mathcal{V}$ such that $\Psi(Z') = \Psi(Z'') = W' = (w'_1, \dots, w'_{k-1})$. Let ω' and ω'' be the points in $\mathcal{L}$ corresponding to Z' and Z''. The projections of ω' and ω'' in $\mathcal{M}_{0,k+1}$ are denoted by m' and m''.

Let M' and M'' denote the flat surfaces defined by ω' and ω''. By definition, the vector W' records the complex numbers associated to the bases of $k-1$ cones added to M' (resp. to M'') to obtain a flat surface M'_{k-1} (resp. M''_{k-1}) with a single singularity. Observe that the surfaces M'_{k-1} and M''_{k-1} are both isometric to a standard infinite Euclidian cone $\mathbf{C}$ with apex angle $2\pi(1 - (\mu_1 + \cdots + \mu_k))$. For the sake of concreteness, $\mathbf{C}$ is defined by the flat metric $|z|^{-2(\mu_1 + \cdots + \mu_k)}|dz|^2$ on $\mathbb{C}$. Note also that $\mathbf{C}$ is also isometric to M_{k-1}.

Given $\mathbf{C} \simeq M_{k-1}$, we can recover M from $W = (w_1, \dots, w_{k-1})$ as follows: since $\mathbf{C}_{k-1}$ is a neighborhood of the apex of M_{k-1}, we can choose a developing map for M_{k-1} such that the complex number associated to one of the geodesic segments in the base of $\mathbf{C}_{k-1}$ is w_{k-1}. Cut off the cone $\mathbf{C}_{k-1}$, and glue the two geodesic segments in the base of $\mathbf{C}_{k-1}$, we obtain the flat surface M_{k-2} having two singularities. By construction, the cone $\mathbf{C}_{k-2}$ is a neighborhood of one of these singularities. The complex number w_{k-2} determines the embedding of $\mathbf{C}_{k-2}$ into M_{k-2}. Therefore, we can then continue the cutting-regluing operation to remove the remaining $k-2$ cones and get back to the surface M'. Note that along this process, one needs to keep track of the developing map chosen for $\mathbf{C} \simeq M_{k-1}$.

Clearly, we can recover M' and M'' from W' and W'' in the same way. Since M'_{k-1} and M''_{k-1} are isometric, and $W' = W''$, we can conclude that M' and M'' are isometric. The isometry between M' and M'' induces an isomorphism between m' and m''. Therefore, we have $m' = m''$, which means that ω' and ω'' belong to the same fiber of $\mathscr{L}$. Hence, there is a complex number λ such that $\omega' = \lambda\omega''$, or equivalently $Z' = \lambda Z''$. Since Ψ is a linear map, we have $\Psi(Z') = \lambda\Psi(Z'') \Leftrightarrow W' = \lambda W'$. Recall that by construction, all the coordinates of W' are non-zero, thus we must have $\lambda = 1$, and $Z' = Z''$. The proposition is then proven. $\square$

6.6. Proof of Proposition 6.1: case of codimension one

We now give the proof of Proposition 6.1 in the case $r = 1$, that is m is a generic point in a divisor $D_{\mathscr{S}}$, where $\mathscr{S} = \{I_0, I_1\} \in \mathscr{P}$ (see Section 3).

We can assume that $I_0 = \{1, \dots, n_0\}$ and $I_1 = \{n_0+1, \dots, n\}$. Let C^0_m, C^1_m be the corresponding irreducible components of C_m. For $i = 0, 1$, let $\hat{\mu}_i, \hat{y}_i, \hat{\Sigma}_i, \mathbf{L}_i, \omega_i, \mathrm{T}_i$ be as in Section 4.

Let $((.,.))_0$ be the Hermitian form on $H^1(C^0_m \setminus \hat{\Sigma}_0, \mathbf{L}_0)$. By Proposition 2.8, we know that $((.,.))_0$ has signature $(1, n_0 - 2)$. Let $\xi^{(0)} = (\xi^{(0)}_1, \dots, \xi^{(0)}_{n_0-1}) \in \mathbb{C}^{n_0-1}$ (resp. $\xi^{(1)} = (\xi^{(1)}_1, \dots, \xi^{(1)}_{n_1-1}) \in \mathbb{C}^{n_1-1}$) be the vector recording the pairings of $[\omega_0]$ (resp. $[\omega_1]$) with the basis of $H^{\mathrm{lf}}_1(C^0_m \setminus \hat{\Sigma}_0, \overline{\mathbf{L}}_0)$ associated to T_0 (resp. the basis of $H^{\mathrm{lf}}_1(C^1_m \setminus \hat{\Sigma}_1, \overline{\mathbf{L}}_1)$ associated to T_1). Let $C_{(m,t)}$ be the stable curve obtained from the plumbing construction in Section 4, where $t \in \mathrm{D}_{c^2}$. We need to show the following

Proposition 6.8. – *There exists a positive definite Hermitian form $((.,.))_1$ on $\mathbb{C}^{n_1-1}$ depending only on $(\mu_{n_0+1}, \dots, \mu_n)$ such that, if $\omega_{(m,t)}$ is the element of $H^{1,0}(C_{(m,t)} \setminus \Sigma, \mathbf{L})$ defined in Lemma 4.3, then we have*

$$((([\omega_{(m,t)}], [\omega_{(m,t)}])) = ((\xi^{(0)}, \xi^{(0)}))_0 - |t|^{2(1-\hat{\mu}_0)}((\xi^{(1)}, \xi^{(1)}))_1. \tag{16}$$

Proof. – Let M_0 be the flat surface defined by ω_0 on C^0_m, M_1 the surface defined by $-t^{1-\hat{\mu}_0}\omega_1$ on C^1_m, and M the flat surface defined by $\omega_{(m,t)}$ on $C_{(m,t)}$. Let (F, U, G, V, c) be the plumbing data as in Section 4. Choose a constant $\tau \in (|t|/c, c)$, and let γ_τ be the curve in $C_{(m,t)}$ which corresponds to the set $\{p \in U, |F(p)| = \tau\} \simeq \{q \in V, |G(q)| = |t|/\tau\}$. Since the metric defined by ω_0 in U is the pullback of $|F(z)|^{-2\hat{\mu}_0}|dF(z)|^2$, we deduce that γ_τ is the set of points whose distance in M_0 to $\hat{y}_0$ is $R(\tau) := \frac{\tau^{1-\hat{\mu}_0}}{1-\hat{\mu}_0}$. In particular, γ_τ has constant curvature $1/R(\tau)$ and length equal to $2\pi\tau^{1-\hat{\mu}_0}$.

The curve γ_τ cuts M into two subsurfaces, the one that contains Σ_i is denoted by M^τ_i. The observation above implies that ∂M^τ_1 is convex. Remark that M^τ_i can be viewed as a subsurface of M_i. By definition, M^τ_1 contains n_1 cone singularities corresponding to the points in Σ_1 in its interior. Since the sum of the curvatures at those points is smaller than 1, one can add $n_1 - 1$ cones $\mathbf{C}_1, \dots, \mathbf{C}_{n_1-1}$ to M^τ_1 to get a flat surface $\hat{M}^\tau_1$ having a single singularity with cone angle $2\pi(1-\hat{\mu}_0)$.

Let $W = (w_1, \dots, w_{n_1-1})$ be the vector recording the complex numbers associated to the bases of the cones $\mathbf{C}_1, \dots, \mathbf{C}_{n-1}$. By Proposition 6.4, there is a linear isomorphism Ψ of $\mathbb{C}^{n_1-1}$ such that $W = -t^{1-\hat{\mu}_0}\Psi(\xi^{(1)})$.

Recall that the total area of the added cones is equal to $\sum_{i=1}^{n_1-1} c_i|w_i|^2$, where the c_i's are real positive constants determined by the weight vector $(\mu_{n_0+1}, \dots, \mu_n)$. Therefore, there is a positive definite Hermitian form $((.,.))_1$ on $\mathbb{C}^{n_1-1}$ such that

$$\sum_{i=1}^{n_1-1} \mathrm{Area}(\mathbf{C}_i) = |t|^{2(1-\hat{\mu}_0)}((\xi^{(1)}, \xi^{(1)}))_1.$$

We now remark that $\hat{M}_1^\tau$ is isometric to a subset of the Euclidean cone $\mathbf{C}$ defined by the metric $|z|^{-2\hat{\mu}_0}|dz|^2$ on $\mathbb{C}$. Since $\partial\hat{M}_1^\tau \simeq \gamma_\tau$ has constant curvature $1/R(\tau)$, γ_τ corresponds to the set of points in $\mathbf{C}$ whose distance to the apex is $R(\tau)$. Hence $\hat{M}_1^\tau$ is isometric to the flat surface defined by $|z|^{-2\hat{\mu}_0}|dz|^2$ on the disk D_τ. It follows that $\hat{M}_1^\tau$ is isometric to the flat metric defined by ω_0 on the set $\{p \in U, |F(p)| \leq \tau\}$.

Let $\hat{M}^\tau$ be the flat surface obtained by gluing $\hat{M}_1^\tau$ to M_0^τ along γ_τ. From the argument above, we conclude that $\hat{M}^\tau$ is isometric to M_0. Since $\mathrm{Area}(M) = \mathrm{Area}(\hat{M}^\tau) - \sum_{1\leq i\leq n_1-1}\mathrm{Area}(\mathbf{C}_i) = \mathrm{Area}(M_0) - \sum_{1\leq i\leq n_1}\mathrm{Area}(\mathbf{C}_i)$, we have

$$\begin{aligned}(([\omega_{(m,t)}], [\omega_{(m,t)}])) &:= (([\omega_0], [\omega_0])) - |t|^{2(1-\hat{\mu}_0)}((\xi^{(1)}, \xi^{(1)}))_1 \\ &= ((\xi^{(0)}, \xi^{(0)}))_0 - |t|^{2(1-\hat{\mu}_0)}((\xi^{(1)}, \xi^{(1)}))_1.\end{aligned}$$

□

REMARK 6.9. – In [22], Thurston introduced a completion $\overline{\mathcal{M}_{0,n}}^\mu$ of $\mathcal{M}_{0,n}$ with respect to the complex hyperbolic metric induced by $\mathcal{L}$. The space $\overline{\mathcal{M}_{0,n}}^\mu$ is equipped with a *cone-manifold* structure. In this setting, m corresponds to a point in a stratum of codimension $n_1 - 1$ representing the flat surfaces on which all the cone points in Σ_1 collide. The quantity $1 - \hat{\mu}_0$ can be interpreted as the *scalar cone angle* at m (see [22, Sec. 3]). Note also that if $m = (m_0, m_1)$ with $m_i \in \mathcal{M}_{0,n_i+1}$, then the flat surface corresponding to m is uniquely determined (up to a rescaling) by m_0. Thus for all $m_1' \in \mathcal{M}_{0,n_1+1}$, the point $m' = (m_0, m_1')$ represents the same element of $\overline{\mathcal{M}_{0,n}}^\mu$.

6.7. Proof of Proposition 6.1: general case

Proof. – Let $\tilde{C}^j_{(m,t_1,\dots,t_j)}$ be as in the proof of Lemma 5.5, where $\tilde{C}^r_{(m,t_1,\dots,t_r)} = C_{(m,t)}$. Recall that $\tilde{C}^{j+1}_{(m,t_1,\dots,t_{j+1})}$ is obtained from $\tilde{C}^j_{(m,t_1,\dots,t_j)}$ and C_m^{j+1} by a plumbing construction at the principal node of C_m^{j+1}. Therefore Proposition 6.1 follows from Proposition 6.8 and Lemma 5.5 by induction. □

7. Singular Kähler-Einstein metrics

Our aim now is to explain that the metric constructed in the previous section on $\mathcal{M}_{0,n}$ is actually a singular Kähler-Einstein metric on $\overline{\mathcal{M}_{0,n}}$, and that this fact will enable us to compute the volume of $\mathcal{M}_{0,n}$ endowed with this metric.

7.1. General setting

We recall some basic facts about singular Kähler-Einstein metrics on projective varieties in a simplified setting as we will not need a very high degree of generality (for instance, see [3] and [11] and the references therein for a more general exposition).

In general, we will identify a Hermitian metric on a complex manifold with its associated $(1,1)$-form. Given a divisor D, we will often use the same notation for the $(1,1)$-cohomology class it defines. The support of D will be denoted by $|D|$. By a slight abuse of notation, when D is a $\mathbb{Z}$-divisor, we will denote both the associated holomorphic line bundle and the rank 1 locally free sheaf of holomorphic sections by $\mathcal{O}(D)$.

7.1.1. *Singular metrics on $\mathbb{R}$-line bundles.* – Let X be a complex projective manifold of dimension N and $D = \sum_{i=1}^{k} \lambda_i D_i$ an $\mathbb{R}$-divisor, that is, for each i, D_i is an irreducible and reduced subvariety of codimension 1 and $\lambda_i \in \mathbb{R}^*$. As in the case of a $\mathbb{Z}$-divisor, we can attach to D an "$\mathbb{R}$-line bundle" $\mathcal{O}(D)$. The latter can be endowed with a "metric" which writes locally on a suitable covering (V_j) of X as $e^{-\phi_j}$, where the real functions ϕ_j satisfy compatibility properties analogous to the case of a line bundle in the usual sense (see [9], Section 19.A). The regularity of ϕ_j will be discussed later on, but let us say that they are in L^1_{loc}. In general, abuzing notation, we write $h_D = e^{-\phi_D}$ for this metric. The "curvature" of h_D is the globally defined closed $(1,1)$-current $\imath\Theta(h_D) = \frac{\imath}{2\pi}\partial\bar{\partial}\phi_D = dd^c\phi_D$ and is a representative of the cohomology class $\{D\} \in H^{1,1}(X,\mathbb{R})$. Here $d = \partial + \bar{\partial}$ and $d^c = \frac{1}{2\pi\imath}(\partial - \bar{\partial})$ which are both real operators.

For instance, if we have some section of $\mathcal{O}(D_i)$ whose zero divisor is D_i given by a holomorphic function f_i in local coordinates, then we can take $\phi_D = \sum_{i=1}^{k} \lambda_i \log|f_i|^2$. By the Lelong-Poincaré formula we have $\imath\Theta(h_D) = \sum_{i=1}^{k} \lambda_i [D_i] = [D]$ where $[D_i]$ is the current of integration over D_i. It will be more convenient to choose an arbitrary smooth metric h_0 on $\mathcal{O}(D)$ and to write the previous metric $h_D = e^{-\varphi_D} h_0$ for some function $\varphi_D : X \to [-\infty, +\infty)$ which is smooth on $X \backslash |D|$. If we let Θ_0 be the curvature of h_0 then $[D] = \Theta_0 + dd^c\varphi_D$. In particular, we have $\Theta_0 = -dd^c\varphi_D$ on $X \backslash |D|$.

7.1.2. *Singular Kähler-Einstein metrics and their volume.* – From now on, we will assume that D is a $\mathbb{R}$-divisor with simple normal crossings and that the pair (X, D) is klt (for Kawamata log terminal), which will just mean for us that $\lambda_i < 1$, in particular D is not necessarily effective.

Let us fix a smooth volume form dV on X which is the same as a smooth metric on the anti-canonical line bundle $-K_X := \Lambda^N T_X$. The opposite of the $(1,1)$-form associated to the curvature of this metric, that we will denote in a standard way by $\Theta_{K_X} := dd^c \log(dV)$, is a representative of the first Chern class $c_1(K_X)$.

The following proposition will be our main tool for the proof of Theorem 1.1.

Proposition 7.1. – *Assume moreover that we have a smooth Kähler metric Ω on the restriction of the tangent bundle T_X to $X \backslash |D|$ which satisfies*

(i) $\mathrm{Ric}(\Omega) = -c\,\Omega$ *on* $X \backslash |D|$, *where* $\mathrm{Ric}(\Omega) = -dd^c \log(\Omega^N)$ *is the Ricci form of* Ω *and* c *is a positive real number;*

(ii) *there exists a continuous function* φ *on* X *and smooth on* $X\backslash|D|$ *such that* $\Omega^N = e^{\varphi-\varphi_D} dV$ *on* $X\backslash|D|$, *where* φ_D *is as above.*

Then the extension $\tilde{\Omega}$ *of* Ω *by* 0 *satisfies*

$$\mathrm{Ric}(\tilde{\Omega}) = -c\,\tilde{\Omega} + [D]$$

in the sense of currents, namely $\tilde{\Omega}$ *is a* singular Kähler-Einstein metric *attached to the pair* (X, D) *and in particular we have* $c\{\tilde{\Omega}\} = c_1(K_X + D)$. *Moreover,*

$$\int_{X\backslash|D|} \Omega^N = \frac{1}{c^N}(K_X + D)^N. \tag{17}$$

Condition (i) means that Ω is a Kähler-Einstein metric on $X\backslash|D|$ with negative Einstein constant $-c$ and condition (ii) imposes some control on the behavior of Ω^N at infinity that is, near the support $|D|$ of the boundary divisor. remark that (X, D) being klt precisely implies that Ω^N is integrable near the boundary.

In order to prove Proposition 7.1 we will need the following simple

LEMMA 7.2. – *Let* Ω *be a smooth closed positive* $(1,1)$*-form on* $X\backslash|D|$ *and assume that* Ω *has continuous local potentials on* X, *that is, for any* $x \in X$, *there exists a neighborhood* U *of* x *in* X *and a function* $\varphi_U : U \to \mathbb{R}$ *which is continuous on* U *and smooth on* $U\backslash|D|$ *such that* $\Omega = dd^c\varphi_U$ *on* $X\backslash U$. *Then the extension* $\tilde{\Omega}$ *by* 0 *of* Ω *to* X *is a well defined closed positive current on* X, *and for any* x *and* U, $\tilde{\Omega}|_U = dd^c\varphi_U$ *in the sense of currents.*

Proof. – By assumption, φ_U is psh on $U\backslash|D|$ and by standard arguments (see [10] for instance), it is known that $\varphi_U|_{U\backslash|D|}$ can be extended in a unique way as a psh function on the whole of U. In particular, the extension belongs to $L^1_{\mathrm{loc}}(U)$. But as φ_U is continuous, this extension is actually φ_U. Moreover, still as φ_U is continuous, its Lelong numbers along D vanish hence $\tilde{\Omega}$ is well defined and coincides with $dd^c\varphi_U$ on U. □

Proof of Proposition 7.1. – Set $\Theta := \Theta_{K_X} + \Theta_0$. Note that Θ is a smooth form on X. Since $\Theta_0 = -dd^c\varphi_D$ on $X\backslash|D|$, (i) and (ii) imply that $c\,\Omega = \Theta + dd^c\varphi$ on $X\backslash|D|$ and in particular $\Theta + dd^c\varphi$ is a positive current on $X\backslash|D|$ (we also say that φ is Θ-psh). By the previous lemma (applied locally to $\psi_U + \varphi|_U$ where ψ_U is a local potential of the smooth form Θ), the equality $c\,\tilde{\Omega} = \Theta + dd^c\varphi$ is valid on X that is, $c\,\tilde{\Omega}$ and Θ both are representatives of $c_1(K_X + D)$. We now obtain from (ii) that

$$\mathrm{Ric}(\tilde{\Omega}) = -dd^c\varphi + dd^c\varphi_D + \Theta_0 - \Theta = -c\tilde{\Omega} + [D].$$

In general, if T is a closed positive $(1,1)$-current on X, it is not always possible to define T^N in a reasonable way. However, if $T = \Theta + dd^c\varphi$ with Θ smooth and φ locally bounded then, following the work of Bedford-Taylor [2], one can define a closed (p,p)-current T^p for any $p \geq 1$ and moreover $\{T^p\} = \{T\}^p$ ([8], Cor. 9.3).

In our case where $T = \tilde{\Omega}$ as above, we hence have $\{\tilde{\Omega}^N\} = \frac{1}{c^N}(K_X + D)^N$. Finally, as the wedge product in the sense of Bedford-Taylor puts no mass on pluripolar sets (as a consequence of the Chern-Levine-Nirenberg inequality, see the comment following Proposition A.6.3 in [4]), we conclude that the volume of $X\backslash|D|$ endowed with the smooth metric Ω satisfies equality (17). □

7.2. Singular Kähler-Einstein metrics on $\overline{\mathscr{M}}_{0,n}$

We shall apply now the formalism of the previous section to the situation where $X = \overline{\mathscr{M}}_{0,n}$, and $X\backslash|D| = \mathscr{M}_{0,n}$ (here $N = n - 3$). Recall that we supposed that the sum of the weights for indices in any subset of $\{1, \dots, n\}$ is always different from 1. We defined $D_\mu := \sum_{\mathscr{S}} \lambda_{\mathscr{S}} D_{\mathscr{S}}$ where

$$\lambda_{\mathscr{S}} = (|I_1| - 1)(\mu_{\mathscr{S}} - 1) + 1$$

if $\mathscr{S} = \{I_0, I_1\}$ and, exchanging I_0 and I_1 if necessary, $\mu_{\mathscr{S}} := \sum_{s \in I_1} \mu_s < 1$ ($\mu_{\mathscr{S}} = \hat{\mu}_0$ in the notation of Section 4). Observe that each $\lambda_{\mathscr{S}}$ is smaller than 1. Here and in the sequel, the sums are always taken over all the (unordered) partitions $\mathscr{S} \in \mathscr{P}$, that is, satisfying $\min\{|I_0|, |I_1|\} \geq 2$.

PROPOSITION 7.3. – *The extension by* 0 *of the Chern form* Ω_μ *defined in Proposition 6.2 is a singular Kähler-Einstein metric attached to the pair* $(\overline{\mathscr{M}}_{0,n}, D_\mu)$. *More precisely,* $\mathrm{Ric}(\Omega_\mu) = -(N+1)\,\Omega_\mu + [D_\mu]$ *and*

$$\int_{\mathscr{M}_{0,n}} \Omega_\mu^N = \frac{1}{(N+1)^N} \left(K_{\overline{\mathscr{M}}_{0,n}} + \sum_{\mathscr{S}} \lambda_{\mathscr{S}} D_{\mathscr{S}} \right)^N .$$

Proof. – We will check that assumptions (i) and (ii) of Proposition 7.1 are satisfied.

Let us first recall a few basic facts about complex hyperbolic N-space: it can be seen as the unit ball $\mathbf{B}^N \subset \mathbb{C}^N \subset \mathbb{P}^N_{\mathbb{C}}$ and we can identify its group of biholomorphisms with $\mathrm{PU}(1, N)$. We restrict to $\mathbf{B}^N$ the exact sequence of vector bundles

$$0 \to L \to \underline{\mathbb{C}}^{N+1} \to Q \to 0,$$

where L is the tautological line subbundle of the trivial bundle $\underline{\mathbb{C}}^{N+1} = \mathbb{P}^N_{\mathbb{C}} \times \mathbb{C}^{N+1}$ and Q is the quotient bundle. The group $\mathrm{U}(1, N)$ acts on this exact sequence and preserves the constant Hermitian metric of signature $(1, N)$ on $\underline{\mathbb{C}}^{N+1}$. In restriction to L, this metric is positive definite and hence defines a Hermitian metric on the line bundle L. The Chern form $c_1(L)$ associated with this metric is a positive $(1,1)$-form on $\mathbf{B}^N$, and the corresponding metric has constant holomorphic sectional curvature: it is Kähler-Einstein and $\mathrm{Ric}(c_1(L)) = -(N+1)\, c_1(L)$. That the Einstein constant $-c$ is equal to $-(N+1)$ is due to the fact that on $\mathbf{B}^N$, the tangent bundle is naturally isomorphic to $\mathrm{Hom}(L, Q)$ hence the canonical bundle can be identified with L^{N+1}. As $\mathscr{L}$ is the pullback of L by an immersion, this proves that Ω_μ defines a metric on $\mathscr{M}_{0,n}$ and that $\mathrm{Ric}(\Omega_\mu) = -(N+1)\,\Omega_\mu$ on $\mathscr{M}_{0,n}$, that is, assumption (i) is satisfied with $c = N + 1$.

Proposition 6.2 gives the expression of the metric Ω_μ in local coordinates centered at a point m of $\overline{\mathscr{M}}_{0,n}$. More precisely, recall that locally it is the pullback of the complex hyperbolic metric on $\mathbf{B}^N$ by the multivalued map

$$(z^0, t_1, \dots, t_r, z^1, \dots, z^r) \mapsto (z^0, P_1(t), \dots, P_r(t), P_1(t)z^1, \dots, P_r(t)z^r),$$

where r is the number of vital divisors crossing at m, $z^j \in \mathbb{C}^{k_j - 3}$ and $P_j(t)$ is described in Lemma 5.5.

As the volume form associated with the complex hyperbolic metric on $\mathbf{B}^N$ is

$$\left(\frac{\iota}{2\pi}\right)^N \frac{1}{(1-\|w\|^2)^{N+1}} dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_N \wedge d\bar{w}_N$$

a straightforward computation shows that in the above coordinates

$$\begin{aligned}\Omega_\mu^N &= \left(\frac{\iota}{2\pi}\right)^N \frac{dz^0 \wedge d\bar{z}^0 \wedge \bigwedge_{j=1}^r \nu_j^2 |t_j|^{-2} |P_j(t)|^2 dt_j \wedge d\bar{t}_j \wedge \bigwedge_{j=1}^r |P_j(t)|^{2(k_j-3)} dz^j \wedge d\bar{z}^j}{\left(1-||z^0||^2 - \sum_{j=1}^r |P_j(t)|^2 (1+||z^j||^2)\right)^{N+1}} \\ &= \left(\frac{\iota}{2\pi}\right)^N \frac{\prod_{j=1}^r \left(\nu_j^2 |t_j|^{-2} |P_j(t)|^{2(k_j-2)}\right) \bigwedge_{j=0}^r dz^j \wedge d\bar{z}^j \wedge \bigwedge_{j=1}^r dt_j \wedge d\bar{t}_j}{\left(1-||z^0||^2 - \sum_{j=1}^r |P_j(t)|^2 (1+||z^j||^2)\right)^{N+1}},\end{aligned}$$

where for any $j = 0, \ldots, r$, $dz^j \wedge d\bar{z}^j$ stands for $\bigwedge_{i=1}^{k_j-3} dz_i^j \wedge d\bar{z}_i^j$.

If $r = 0$, that is, if $m \in \mathcal{M}_{0,n}$, then Ω_μ is smooth in a neighborhood of m. Assume now that $r \geq 1$. The divisors $D_{\mathcal{S}_j}$ passing through m are given by $t_j = 0$, $1 \leq j \leq r$, and they correspond to partitions $\mathcal{S}_j = \{I_0^j, I_1^j\}$ (see Section 5). We have to determine the power of $|t_j|^2$ in the numerator of Ω_μ^N. From the combinatorial description in Section 5, we see that $|t_j|^{2\nu_j}$ appears $(|I_1^j| - 1)$ times (which is the dimension of the stratum $\mathcal{M}_{0,|I_1^j|+1}$ plus 1) in total in the product of the $|P_i|^{2(k_i-2)}$ so that the power of $|t_j|^2$ is $(|I_1^j|-1)\nu_j - 1 = -(|I_1^j|-1)(\mu_{\mathcal{S}_j} - 1) - 1 = -\lambda_{\mathcal{S}_j}$ hence

$$\Omega_\mu^N = \left(\frac{\iota}{2\pi}\right)^N \frac{\prod_{j=1}^r \nu_j^2 \bigwedge_{j=1}^r dt_j \wedge d\bar{t}_j \wedge \bigwedge_{j=0}^r dz^j \wedge d\bar{z}^j}{\prod_{j=1}^r |t_j|^{2\lambda_{\mathcal{S}_j}} \left(1-||z^0||^2 - \sum_{j=1}^r |P_j(t)|^2 (1+||z^j||^2)\right)^{N+1}} = e^{\varphi - \varphi_{D_\mu}} dV,$$

where, up to the multiplication by smooth functions,

$$\varphi = -\log\left(1-||z^0||^2 - \sum_{j=1}^r |P_j(t)|^2 (1+||z^j||^2)\right)^{N+1},$$

$\varphi_{D_\mu} = \sum_{j=1}^r \lambda_{\mathcal{S}_j} \log |t_j|^2$ and $dV = \bigwedge_{j=1}^r dt_j \wedge d\bar{t}_j \wedge \bigwedge_{j=0}^r dz^j \wedge d\bar{z}^j$.

Finally, by Remark 6.3, φ is continuous and hence assumption (ii) of Proposition 7.1 is also satisfied, which completes the proof. □

7.3. Proof of Theorem 1.1

Having proved Proposition 7.3, we just need to explain how we obtain alternative expressions of $K_{\overline{\mathcal{M}}_{0,n}} + \sum_{\mathcal{S}} \lambda_{\mathcal{S}} D_{\mathcal{S}}$. We first notice that the canonical divisor $K_{\overline{\mathcal{M}}_{0,n}}$ can be expressed in terms of the vital divisors $D_{\mathcal{S}}$. Indeed,

$$K_{\overline{\mathcal{M}}_{0,n}} \sim \psi - 2\delta,$$

where $\psi = \sum_{s=1}^n \psi_s$ is the ψ-divisor class (see [1, p. 335]), $\delta = \sum_{\mathcal{S}} D_{\mathcal{S}}$ is the boundary divisor, and $\sim$ stands for the linear equivalence of divisors (see [1, p. 386]; here we use that $\overline{\mathcal{M}}_{0,n}$ is a fine moduli space and that the Hodge bundle is zero on $\overline{\mathcal{M}}_{0,n}$).

For pairwise distinct $i, j, k \in \{1, \ldots, n\}$, denote by $\delta_{i|jk}$ the divisor in $\overline{\mathcal{M}}_{0,n}$ corresponding to curves with a node separating the i-th marked point from the j-th and k-th marked points. It is well-known (see [25] or [1, Chap. 17]) that for any such choice of i, j, k we have $\psi_i \sim \delta_{i|jk}$ hence

$$\begin{aligned}(n-1)(n-2)\psi \sim \sum_{i,j,k} \delta_{i|jk} &= \sum_{\mathscr{S}} \big(|I_0|\,|I_1|(|I_1|-1) + |I_1|\,|I_0|(|I_0|-1)\big) D_{\mathscr{S}} \\ &= (n-2)\sum_{\mathscr{S}} |I_0|\,|I_1|\, D_{\mathscr{S}}\end{aligned}$$

and substituting ψ in the above expression of $K_{\overline{\mathscr{M}}_{0,n}}$ we get

$$K_{\overline{\mathscr{M}}_{0,n}} \sim_{\mathbb{Q}} \sum_{\mathscr{S}} \frac{(|I_0|-2)(|I_1|-2)-2}{n-1} D_{\mathscr{S}},$$

where $\sim_{\mathbb{Q}}$ stands for the $\mathbb{Q}$-linear equivalence of divisors. Thus, we obtain

$$K_{\overline{\mathscr{M}}_{0,n}} + \sum_{\mathscr{S}} \lambda_{\mathscr{S}} D_{\mathscr{S}} \sim_{\mathbb{Q}} \sum_{\mathscr{S}} \Big(|I_1|-1\Big)\left(\mu_{\mathscr{S}} - \frac{|I_1|}{N+2}\right) D_{\mathscr{S}}. \tag{18}$$

Now, recall that $N = n-3$ and notice that

$$\begin{aligned}2(n-1)\Big(|I_1|-1\Big)\left(\mu_{\mathscr{S}} - \frac{|I_1|}{N+2}\right) &= 2\big(|I_1|-1\big)\big((n-1)\mu_{\mathscr{S}} - |I_1|\big) \\ &= \big(n-2+|I_1|-|I_0|\big)(n-1)\mu_{\mathscr{S}} - 2|I_1|\big(|I_1|-1\big) \\ &= (n-2)(n-1)\mu_{\mathscr{S}} + \big(|I_1|-|I_0|\big)\big(|I_1|+|I_0|-1\big)\mu_{\mathscr{S}} - 2|I_1|\big(|I_1|-1\big) \\ &= (n-2)(n-1)\mu_{\mathscr{S}} + \Big(|I_1|\big(|I_1|-1\big) - |I_0|\big(|I_0|-1\big)\Big)\mu_{\mathscr{S}} - 2|I_1|\big(|I_1|-1\big) \\ &= (n-2)(n-1)\mu_{\mathscr{S}} - |I_0|\big(|I_0|-1\big)\mu_{\mathscr{S}} - (2-\mu_{\mathscr{S}})|I_1|\big(|I_1|-1\big).\end{aligned}$$

By a similar computation as above we get⁽¹⁾

$$\begin{aligned}(n-1)(n-2)\sum_i \mu_i\,\psi_i &\sim \sum_{i,j,k} \mu_i\,\delta_{i|jk} \\ &= \sum_{\mathscr{S}} \Big(|I_1|\big(|I_1|-1\big)\sum_{i\in I_0}\mu_i + |I_0|\big(|I_0|-1\big)\sum_{i\in I_1}\mu_i\Big) D_{\mathscr{S}} \\ &= \sum_{\mathscr{S}} \Big(|I_1|\big(|I_1|-1\big)(2-\mu_{\mathscr{S}}) + |I_0|\big(|I_0|-1\big)\mu_{\mathscr{S}}\Big) D_{\mathscr{S}}\end{aligned}$$

and therefore

$$\frac{2}{(N+1)}\Big(K_{\overline{\mathscr{M}}_{0,n}} + \sum_{\mathscr{S}} \lambda_{\mathscr{S}} D_{\mathscr{S}}\Big) \sim_{\mathbb{Q}} -\sum_s \mu_s\,\psi_s + \sum_{\mathscr{S}} \mu_{\mathscr{S}} D_{\mathscr{S}}. \tag{19}$$

REMARK 7.4. – If we define $\mathscr{P}'$ to be the set of unordered partitions of $\{1,\dots,n\}$ into two non-empty subsets $I_0 \sqcup I_1$ then with the convention $D_{\{s\},\{s\}^c} = -\psi_s$, we get the expression

$$\{\Omega_\mu\} = \frac{1}{2}\sum_{\mathscr{S}\in\mathscr{P}'} \mu_{\mathscr{S}} D_{\mathscr{S}}.$$

Finally, formula (18) and (19) together with Proposition 7.3 imply Theorem 1.1.

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(1) We are grateful to D. Zvonkine for explaining this trick to us.

7.4. Comparison with McMullen's formula

In [18], McMullen proves a Gauss-Bonnet formula for Riemannian cone manifolds. Using in particular the fact that on the unit N-ball there exists a $\mathrm{PU}(1,N)$-invariant metric with constant holomorphic sectional curvature, this formula enables him to calculate the volume of $\mathcal{M}_{0,n}$ endowed with a metric g_μ proportional to Ω_μ (see [18, Th. 1.2]). The ratio Ω_μ/g_μ is computed explicitly below.

If X is a smooth N-ball quotient then by the Hirzebruch proportionality theorem, the total Chern class of X is given by

$$c(X) = \left(1 + \frac{c_1(X)}{N+1}\right)^{N+1} = \left(1 - \frac{c_1(K_X)}{N+1}\right)^{N+1}$$

hence we have the following equality

$$c_1^N(K_X) = (-1)^N (N+1)^{N-1} \chi(X) \tag{20}$$

(where $\chi(X) = c_N(X)$ is the Euler characteristic of X). In general, the above equalities make sense at the level of the $\mathrm{PU}(1,N)$-invariant forms on $\mathbf{B}^N$ which represent the respective cohomology classes.

As the metric g_μ of McMullen is normalized in order to have constant holomorphic sectional curvature -1, we have $\mathrm{Ric}(g_\mu) = -\frac{N+1}{2} g_\mu$ hence, if ω_μ is the Kähler form associated with g_μ, $c_1(X) = -\frac{N+1}{4\pi}\omega_\mu$ and $c_N(X) = (-1)^N \frac{(N+1)}{(4\pi)^N}\omega_\mu^N$, as pullback of $\mathrm{PU}(1,N)$-invariant forms on $\mathbf{B}^N$. As a consequence, the volume we compute and the one computed by McMullen are related by

$$\int_{\mathcal{M}_{0,n}} \Omega_\mu^N = \frac{1}{(4\pi)^N} \int_{\mathcal{M}_{0,n}} \omega_\mu^N = \frac{N!}{(4\pi)^N} \operatorname{vol}(\mathcal{M}_{0,n}, g_\mu).$$

Therefore, Theorem 1.1 and [18, Th. 1.2] imply

Corollary 7.5. – *We have*

$$\left(-\sum_s \mu_s \psi_s + \sum_{\mathcal{S}} \mu_{\mathcal{S}} D_{\mathcal{S}}\right)^N = \frac{(-2)^N}{(N+1)} \sum_{\mathcal{Q}} (-1)^{|\mathcal{Q}|+1} (|\mathcal{Q}|-3)! \prod_{B \in \mathcal{Q}} \max\Big(0, 1 - \sum_{i \in B} \mu_i\Big)^{|B|-1},$$

where $\mathcal{Q}$ ranges over all partitions of the indices $\{1, \ldots, n\}$ into blocks B.

As noticed by McMullen (see [18, Sec. 8]), the sum in the right hand side of the previous formula can be regarded as the *cone manifold Euler characteristic* of the metric completion of $(\mathcal{M}_{0,n}, g_\mu)$. From our point of view, one should interpret $c_1\big(K_{\overline{\mathcal{M}}_{0,n}} + D_\mu\big)$ as the first Chern class of $\overline{\mathcal{M}}_{0,n}^{\mu}$. Therefore, Corollary 7.5 can be viewed as a generalization of formula (20) in the context of complex hyperbolic cone manifolds.

8. A more algebro-geometric approach

8.1. Kawamata's extension

Throughout this section, we will assume that all the weights (μ_s) are rational numbers. If $d \in \mathbb{N}^*$ is such that $d\mu_s \in \mathbb{N}$ for all s, then the local system $\mathbf{L}^{\otimes d}$ on $\mathbb{P}^1_{\mathbb{C}} \setminus \Sigma$ is trivial. Y. Kawamata proves in [14] that the line bundle $\mathcal{L}^{\otimes d}$ has a natural extension to $\overline{\mathcal{M}}_{0,n}$ that we denote abusively by $\hat{\mathcal{L}}^{\otimes d}$ (that is, $\hat{\mathcal{L}}$ is only a $\mathbb{Q}$-divisor). This extension is constructed in the following way: it follows immediately from the description in Section 2.5 that $\mathcal{L}^{\otimes d}$ is isomorphic to $\pi_*\mathcal{O}\big(d(K_{\mathcal{C}_{0,n}/\mathcal{M}_{0,n}} + \sum_s \mu_s \Gamma_s)\big)$, where Γ_s is the divisor given by the s-th section of the universal curve, and $K_{\mathcal{C}_{0,n}/\mathcal{M}_{0,n}} := K_{\mathcal{C}_{0,n}} \otimes K^{\vee}_{\mathcal{M}_{0,n}}$ is the relative canonical bundle of the fibration $\pi_{|\mathcal{C}_{0,n}} : \mathcal{C}_{0,n} \to \mathcal{M}_{0,n}$. Observe that for any $m \in \mathcal{M}_{0,n}$, $\deg\Big(K_{\mathcal{C}_{0,n}/\mathcal{M}_{0,n}}|_{\pi^{-1}(m)}\Big) = \deg\big(K_{\mathbb{P}^1_{\mathbb{C}}}\big) = -2$ and since $\sum_s \mu_s = 2$, the restriction of $\mathcal{O}\big(d(K_{\mathcal{C}_{0,n}/\mathcal{M}_{0,n}} + \sum_s \mu_s \Gamma_s)\big)$ to any fiber of $\pi_{|\mathcal{C}_{0,n}}$ is trivial. Therefore, $\pi_*\mathcal{O}\big(d(K_{\mathcal{C}_{0,n}/\mathcal{M}_{0,n}} + \sum_s \mu_s \Gamma_s)\big)$ is indeed a rank 1 invertible sheaf on $\mathcal{M}_{0,n}$.

The first task is to extend $\mathcal{O}\big(d(K_{\mathcal{C}_{0,n}/\mathcal{M}_{0,n}} + \sum_s \mu_s \Gamma_s)\big)$ to a line bundle on $\overline{\mathcal{C}}_{0,n}$ whose restriction to each fiber of π over $\overline{\mathcal{M}}_{0,n}$ is still trivial. Kawamata remarks that such a natural extension is given by the divisor $d\Lambda$ where

$$\Lambda := K_{\overline{\mathcal{C}}_{0,n}/\overline{\mathcal{M}}_{0,n}} + \sum_s \mu_s \Gamma_s - \sum_{\mathcal{S}} (1-\mu_{\mathcal{S}}) F^1_{\mathcal{S}}$$

and the effective divisor $\sum_{\mathcal{S}}(1-\mu_{\mathcal{S}})F^1_{\mathcal{S}}$ is defined in the following way: for any $\mathcal{S} \in \mathcal{P}$, $\pi^{-1}(D_{\mathcal{S}})$ is a divisor in $\overline{\mathcal{C}}_{0,n}$ with two irreducible components $F^0_{\mathcal{S}}$ and $F^1_{\mathcal{S}}$. Over a generic point of $D_{\mathcal{S}}$, these two components correspond respectively to the two irreducible components of the nodal curve associated with the partition $\mathcal{S} = \{I_0, I_1\}$ (recall that by definition, $\mu_{\mathcal{S}} := \sum_{s\in I_1} \mu_s < 1$).

It is easy to see that the restriction to each fiber $\pi^{-1}(m) \subset \overline{\mathcal{C}}_{0,n}$ of the line bundle associated with the above divisor is indeed trivial for any $m \in \overline{\mathcal{M}}_{0,n}$. It is sufficient to check that its degree is 0 in restriction to each irreducible component of any stable curve $C_m = C^0_m \cup \cdots \cup C^r_m$. First remark that $\pi^{-1}(D_{\mathcal{S}}) = F^0_{\mathcal{S}} + F^1_{\mathcal{S}}$ is trivial in restriction to C_m and so, for any j, $F^1_{\mathcal{S}}|_{C^j_m} = -F^0_{\mathcal{S}}|_{C^j_m}$. As a first consequence, if $F^0_{\mathcal{S}} \cap C^j_m = \emptyset$ or $F^1_{\mathcal{S}} \cap C^j_m = \emptyset$, then $F^1_{\mathcal{S}}|_{C^j_m} = 0$. Moreover, noticing that $1-\mu_{\mathcal{S}} = 1-\sum_{s\in I_1}\mu_s = \sum_{s\in I_0}\mu_s - 1$ for any $\mathcal{S}$, we have (using the notation of Section 5.1)

$$d\Big(K_{\overline{\mathcal{C}}_{0,n}/\overline{\mathcal{M}}_{0,n}} + \sum_s \mu_s \Gamma_s - \sum_{\mathcal{S}}(1-\mu_{\mathcal{S}})F^1_{\mathcal{S}}\Big)_{|C^j_m} = d\Big(K_{\mathbb{P}^1_{\mathbb{C}}} + \sum_{i=1}^{s_j} y_i + \sum_{s\in\Sigma_j} \mu_s x_s + \sum_{i=1}^{s_j}(\hat{\mu}(y_i)-1)y_i\Big),$$

whose degree is indeed equal to 0. Finally, one defines $\hat{\mathcal{L}}^{\otimes d} := \pi_*\mathcal{O}(d\Lambda)$.

REMARK 8.1. – In fact, we also have $\hat{\mathcal{L}}^{\otimes d} = \pi_*\mathcal{O}\big(d\big(K_{\overline{\mathcal{C}}_{0,n}/\overline{\mathcal{M}}_{0,n}} + \sum_s \mu_s \Gamma_s\big)\big)$, but the divisor $d\Lambda$ is more natural, even if less obvious at first glance; for instance, one has $\mathcal{O}(d\Lambda) = \pi^*\hat{\mathcal{L}}^{\otimes d}$.

8.2. Trivializations of Kawamata's extension

In Sections 4 and 5, for each point $m \in \overline{\mathcal{M}_{0,n}}$ we found a neighborhood $\mathcal{U}$ of m in $\overline{\mathcal{M}_{0,n}}$ and we constructed a holomorphic section of $\mathcal{L}$ on $\mathcal{U} \cap \mathcal{M}_{0,n}$ that we denote by $\Phi_{\mathcal{U}}$ or simply by Φ. We can regard $\Phi^{\otimes d}$ as a holomorphic section of $\mathcal{O}\big(d(K_{\mathscr{C}_{0,n}/\mathcal{M}_{0,n}} + \sum_s \mu_s \Gamma_s)\big)$ on $\pi^{-1}(\mathcal{U} \cap \mathcal{M}_{0,n})$. From the description in Section 4, we see immediately that as such, $\Phi^{\otimes d}$ extends as a section of $\mathcal{O}\big(d(K_{\overline{\mathscr{C}_{0,n}}/\overline{\mathcal{M}_{0,n}}} + \sum_s \mu_s \Gamma_s)\big)$ on the whole of $\pi^{-1}(\mathcal{U})$ if m is a generic point of $D_{\mathcal{S}}$. Moreover, it vanishes exactly on $F^1_{\mathcal{S}}$ up to the order $d(1-\mu_{\mathcal{S}})$, that is, it is a non-vanishing holomorphic section of the extension $\mathcal{O}(d\Lambda)$ on $\pi^{-1}(\mathcal{U})$, hence providing a local trivialization of the line bundle $\mathcal{O}(d\Lambda)$ and so a trivialization of $\widehat{\mathcal{L}}^{\otimes d}$ on $\mathcal{U}$. In the same way, it can be proven that $\Phi^{\otimes d}$ provides a local trivialization of $\widehat{\mathcal{L}}^{\otimes d}$ near any point m of $\overline{\mathcal{M}_{0,n}}$ but we omit the proof since we only need to consider trivializations near generic points of $\partial\overline{\mathcal{M}_{0,n}}$.

Recall that the line bundle $\mathcal{L}$ is equipped with a metric coming from the Hermitian form $((.,.))$ defined in Section 2.4. It is important to note that by Proposition 6.2 and Remark 6.3, the induced metric on $\mathcal{L}^{\otimes d}$, whose curvature on $\mathcal{M}_{0,n}$ is $d\Omega_\mu$, extends as a continuous metric on $\widehat{\mathcal{L}}^{\otimes d}$. Thus, by Lemma 7.2, the extension by 0 of Ω_μ is a representative of $c_1(\widehat{\mathcal{L}}) = \frac{1}{d} c_1(\widehat{\mathcal{L}}^{\otimes d})$. Summing up, we get the

Proposition 8.2. – *Assume that* $0 < \mu_s < 1, \mu_s \in \mathbb{Q}$ *for all* $s \in \{1, \dots, n\}$. *Let* $d \in \mathbb{N}$ *be a positive integer such that* $d\mu_s \in \mathbb{N}$ *for all s. Then the push-forward* $\widehat{\mathcal{L}}^{\otimes d}$ *of the Kawamata line bundle* $\mathcal{O}(d\Lambda)$ *is an extension of* $\mathcal{L}^{\otimes d}$ *over* $\overline{\mathcal{M}_{0,n}}$. *If* $m \in \overline{\mathcal{M}_{0,n}}$ *is contained in a stratum of codimension r, with a neighborhood identified with* $\mathcal{V} \times (\mathrm{D}_{c^2})^r$, *then the section* $\Phi^{\otimes d}$ *on* $\mathcal{V} \times (\mathrm{D}^*_{c^2})^r$ *defined in Section 5 extends naturally to a nowhere vanishing section of* $\widehat{\mathcal{L}}^{\otimes d}$ *in* $\mathcal{V} \times (\mathrm{D}_{c^2})^r$. *Moreover, the extension by* 0 *of* Ω_μ *is a representative of* $c_1(\widehat{\mathcal{L}})$.

Remark 8.3. – One can also prove that the restriction of $\widehat{\mathcal{L}}^{\otimes d}$ to the stratum of m is the pull-back of the d-tensor power of the induced line bundle on the μ–principal factor $\overline{\mathcal{M}_{0,k_0}}$ (see Section 5 for the definition of μ-principal component/factor).

Actually, the above extension of $\Phi^{\otimes d}$ can be described in more *concrete* terms. For this, let us give an alternative description of a plumbing family. Let m be a generic point of some divisor $D_{\mathcal{S}}$ with $\mathcal{S} = \{I_0, I_1\}$, and let $C^0_m = (\mathbb{P}^1_{\mathbb{C}}, (0, (x_s)_{s\in I_0})$ and $C^1_m = (\mathbb{P}^1_{\mathbb{C}}, (0, (y_s)_{s\in I_1})$ (here we denote the marked points on C^1_m by y_s rather than x_s) where we use the conventions of Section 4. Consider the family $\overline{\mathcal{C}}$ of rational curves above a disk D centered at 0 which is described (in inhomogeneous coordinates) by $\varpi : \overline{\mathcal{C}} = \big\{(x, y, t) \in \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \times \mathrm{D},\ xy = t\big\} \to \mathrm{D}$, $(x, y, t) \mapsto t$ (note that the fibers are all smooth except the one above 0 which is a nodal curve with $(0, 0)$ as only node). In this setting, $\Gamma_s = \{x_s\} \times \mathbb{P}^1_{\mathbb{C}} \times \mathrm{D} \cap \overline{\mathcal{C}}$, for $s \in I_0$, and $\Gamma_s = \mathbb{P}^1_{\mathbb{C}} \times \{y_s\} \times \mathrm{D} \cap \overline{\mathcal{C}}$, for $s \in I_1$. Remark however that in general, this family is not isomorphic to the one described in the course of Section 4, where we used additional changes of coordinates F and G. Let $p_i : \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \times \mathrm{D} \to \mathbb{P}^1_{\mathbb{C}}$, $i = 0, 1$, be the natural projection onto the $(i+1)$-th factor $\mathbb{P}^1_{\mathbb{C}}$. Define on $\mathbb{P}^1_{\mathbb{C}}$ two sections

$$\omega_0 := \frac{(dz)^{\otimes d(2-\mu_{\mathscr{S}})}}{\prod_{s\in I_0}(z-x_s)^{2d\mu_s}} \in \Gamma\Big(\mathbb{P}^1_{\mathbb{C}}, d\big((2-\mu_{\mathscr{S}})K_{\mathbb{P}^1_{\mathbb{C}}} + 2\sum_{s\in I_0}\mu_s x_s\big)\Big)$$

and

$$\omega_1 := \frac{(dz)^{\otimes d\mu_{\mathscr{S}}}}{\prod_{s\in I_1}(z-y_s)^{2d\mu_s}} \in \Gamma\Big(\mathbb{P}^1_{\mathbb{C}}, d\big(\mu_{\mathscr{S}}K_{\mathbb{P}^1_{\mathbb{C}}} + 2\sum_{s\in I_1}\mu_s y_s\big)\Big).$$

Then $\omega = p_0^*\omega_0 \otimes p_1^*\omega_1$ induces a section of $\mathcal{O}\big(2d(K_{\overline{\mathcal{C}}_m/\mathrm{D}} + \sum_s \mu_s\Gamma_s)\big)$ on $\overline{\mathcal{C}}$. Near the point $(0,0,0) \in \overline{\mathcal{C}}$, in the coordinates (x,y), a trivialization of $K_{\overline{\mathcal{C}}/\mathrm{D}}$ is provided by $\kappa = \frac{1}{2}\left(\frac{dx}{x} - \frac{dy}{y}\right) = \frac{dx}{x} = -\frac{dy}{y}$. Since we have

$$\omega = \frac{(dx)^{\otimes d(2-\mu_{\mathscr{S}})} \otimes (dy)^{\otimes d\mu_{\mathscr{S}}}}{\prod_{s\in I_0}(x-x_s)^{2d\mu_s}\prod_{s\in I_1}(y-y_s)^{2d\mu_s}},$$

the section induced by ω (in restriction to $\overline{\mathcal{C}}$) is given by

$$(-1)^{d\mu_{\mathscr{S}}}\frac{x^{d(2-\mu_{\mathscr{S}})}y^{d\mu_{\mathscr{S}}}}{\prod_{s\in I_0}(x-x_s)^{2d\mu_s}\prod_{s\in I_1}(y-y_s)^{2d\mu_s}}\kappa^{\otimes 2d}.$$

If we factorize by $(-xy)^{d\mu_{\mathscr{S}}} = (-t)^{d\mu_{\mathscr{S}}}$ and take the "square root", we find

$$\tau_m := \frac{x^{d(1-\mu_{\mathscr{S}})}}{\prod_{s\in I_0}(x-x_s)^{d\mu_s}\prod_{s\in I_1}(y-y_s)^{d\mu_s}}\kappa^{\otimes d}.$$

Since τ_m does not vanish outside of the nodal curve $C_m = \varpi^{-1}(0)$, and vanishes to order $d(1-\mu_{\mathscr{S}})$ on the component C^1_m, we conclude that as a section of $\mathcal{O}(d\Lambda)$ on $\overline{\mathcal{C}}_m$, τ_m is equal to $\Phi^{\otimes d}$ up to the multiplication by an invertible function on D. It will be more convenient below to use the coordinates (x,t) (even if those are only coordinates away from $x=0$) in which

$$\tau_m = \frac{\prod_{s\in I_1}(-x_s)^{d\mu_s}}{\prod_{s\in I_0}(x-x_s)^{d\mu_s}\prod_{s\in I_1}(x-tx_s)^{d\mu_s}}(dx)^{\otimes d}$$

and where we used the notation $x_s := 1/y_s$ if $s \in I_1$.

8.3. Other formulas for the volume and proof of Theorem 1.3

As a direct consequence of the discussion in Section 8.1, we get

THEOREM 8.4. – *Under the assumptions of Proposition 8.2, let σ be a global section of $\hat{\mathscr{L}}^{\otimes d}$ over $\overline{\mathscr{M}}_{0,n}$ and let us define $D_\sigma := \frac{1}{d}\mathrm{div}(\sigma)$ where $\mathrm{div}(\sigma)$ is the divisor of σ. Then*

$$\int_{\mathscr{M}_{0,n}} \Omega_\mu^N = \frac{1}{(N+1)^N}(K_{\overline{\mathscr{M}}_{0,n}} + D_\mu)^N = D_\sigma^N.$$

In what follows, we will construct explicitly a holomorphic section σ of $\hat{\mathcal{L}}^{\otimes d}$ and determine the corresponding divisor D_σ. Our goal is to prove Theorem 1.3.

In [14], Y. Kawamata constructs global sections of $\hat{\mathcal{L}}^{\otimes d}$ over $\overline{\mathcal{M}_{0,n}}$. For our purpose, we present here a slight variation of those sections: define $J = (j_1, j'_1, \dots, j_d, j'_d) \in \mathbb{N}^{2d}$ by

$$j_i = s \ \text{ if } \ d\sum_{k=1}^{s-1}\mu_k < i \le d\sum_{k=1}^{s}\mu_k,$$

$$j'_i = s \ \text{ if } \ d\sum_{k=1}^{s-1}\mu_k < d + i \le d\sum_{k=1}^{s}\mu_k,$$

for all $1 \le i \le d$, where by convention $\sum_{k=1}^{0}\mu_k = 0$. The only two important points in the definition of J are that (1) each s appears $d\mu_s$ times in J, and (2) for any i, $j_i \ne j'_i$.

For each $1 \le s < s' \le n$, we define

$$\lambda(s,s') = \frac{1}{d}\#\{i,\ (j_i, j'_i) = (s,s')\}$$

that is the number of times the pair (s,s') occurs as (j_i, j'_i) divided by d. Alternatively, with our choice of J,

$$\lambda(s,s') = \begin{cases} 0 \ \text{ if } \sum_{k=s}^{s'}\mu_k \le 1 \text{ or } \sum_{k=s+1}^{s'-1}\mu_k \ge 1, \\ \min\{\mu_s, \mu_{s'}, \sum_{k=s}^{s'}\mu_k - 1, 1 - \sum_{k=s+1}^{s'-1}\mu_k\} \ \text{ otherwise.} \end{cases} \tag{21}$$

Let $\{x_1, \dots, x_n\}$ be n distinct points on $\mathbb{P}^1_{\mathbb{C}}$. For any pair (j, j') of distinct elements of $\{1, \dots, n\}$, we denote by $\omega_{j,j'}$ the unique non-vanishing rational 1-form on $\mathbb{P}^1_{\mathbb{C}}$ with simple poles at x_j and $x_{j'}$, and satisfying $\mathrm{res}_{x_j} = 1$, $\mathrm{res}_{x_{j'}} = -1$. If the points $x_j, x_{j'}$ are in $\mathbb{P}^1_{\mathbb{C}}\backslash\{\infty\}$ then

$$\omega_{j,j'} = (x_j - x_{j'})\frac{dz}{(z - x_j)(z - x_{j'})}.$$

Finally, let us define

$$\omega_J := \prod_{i=1}^{d}\omega_{j_i,j'_i} \in \Gamma\Big(\mathbb{P}^1_{\mathbb{C}}, d\big(K_{\mathbb{P}^1_{\mathbb{C}}} + \sum_{s=1}^{n}\mu_s x_s\big)\Big).$$

Remark that ω_J is invariant by the action of $\mathrm{PGL}(2,\mathbb{C})$, thus it gives rise to a well-defined non-vanishing section of $\mathcal{L}^{\otimes d}$ on $\mathcal{M}_{0,n}$. This section extends to the whole $\overline{\mathcal{M}_{0,n}}$ as a section σ of $\hat{\mathcal{L}}^{\otimes d}$. We are now going to determine its zero divisor $\mathrm{div}(\sigma)$, whose support must be contained in the boundary divisor of $\overline{\mathcal{M}_{0,n}}$, by using the above trivializations τ_m. Let us fix $\mathcal{S} = \{I_0, I_1\}$ as before. In the notation of Section 8.2, and using the coordinates (x,t) for the universal family above a small disk D transverse to $D_{\mathcal{S}}$ at a generic point m, the section ω_J writes

$$\omega_J = \frac{\prod_{j_i,j'_i\in I_0}(x_{j_i} - x_{j'_i})\prod_{j_i\in I_0,j'_i\in I_1}(x_{j_i} - tx_{j'_i})\prod_{j_i\in I_1,j'_i\in I_0}(tx_{j_i} - x_{j'_i})\prod_{j_i,j'_i\in I_1}t(x_{j_i} - x_{j'_i})}{\prod_{s\in I_0}(x - x_s)^{d\mu_s}\prod_{s\in I_1}(x - tx_s)^{d\mu_s}}(dx)^{\otimes d},$$

that is,

$$\omega_J = t^{\#\{i,\ j_i,j'_i\in I_1\}}f(t)\tau_m = t^{d\sum_{1\le s<s'\le n}\delta_{\mathcal{S}}(s,s')\lambda(s,s')}f(t)\tau_m,$$

where f is an invertible function on D and $\delta_{\mathcal{S}}(s,s') = \begin{cases} 1 \text{ if } \{s,s'\} \subset I_1 \\ 0 \text{ otherwise} \end{cases}$. As a consequence, we obtain the

Proposition 8.5. – *The section ω_J extends as a global section σ of $\hat{\mathscr{L}}^{\otimes d}$ so that*

$$\hat{\mathscr{L}} \sim_{\mathbb{Q}} D_\sigma := \frac{1}{d}\mathrm{div}(\sigma) = \sum_{\mathcal{S}} \sum_{1\leq s<s'\leq n} \delta_{\mathcal{S}}(s,s')\lambda(s,s')\, D_{\mathcal{S}}. \tag{22}$$

Notice that σ and D_σ depend on the multi-indices J. By choosing other multi-indices J satisfying conditions (1) and (2) above, we would obtain other divisors to which $\hat{\mathscr{L}}$ is $\mathbb{Q}$-linearly equivalent.

Proof of Theorem 1.3

Proof. – Applying Theorem 8.4 with D_σ given by (22), we see that Theorem 1.3 is proved if the weights in μ are all rational. If not, one can approximate them by rational numbers in such a way that the numbers $\delta_{\mathcal{S}}(s,s')$ remain unchanged. From (21) we see that $\lambda(s,s')$ depends continuously on μ. Thus D_σ depends continuously on μ. Moreover, from Theorem 1.1, we know that the total volume of $\mathcal{M}_{0,n}$ with respect to Ω_μ depends also continuously on μ. Thus by continuity with respect to μ, Theorem 1.3 is shown in full generality, that is for all μ satisfying the hypothesis of Theorem 1.1. ☐

8.4. Another look at Theorem 1.1

As a final remark, we would like to show now that the point of view adopted in this section also provides an alternative way to find the expression of $\{\Omega_\mu\}$ obtained in the proof of the main theorem.

Here, as before, we have to assume the weights μ_s to be rational, multiply them by a positive integer d in such a way that the numbers $d\mu_s$ are integers, and consider $\hat{\mathscr{L}}^{\otimes d}$. The general case of real weights then follows again by continuity arguments. However, as the reader can easily check, the computations can be made directly as if $\hat{\mathscr{L}}$ was actually a line bundle.

In Section 8.3 we exhibited sections of $\hat{\mathscr{L}}$ whose zero divisor provides representatives of $c_1(\hat{\mathscr{L}})$ which is equal to $\{\Omega_\mu\}$. As $\hat{\mathscr{L}}$ is the pushforward of a line bundle on the universal curve, it is also natural to use the Grothendieck-Riemann-Roch formula to compute $c_1(\hat{\mathscr{L}})$.

Again, we refer to [1] or [25] for the basic material. Let us define Δ as the codimension 2 subvariety of $\overline{\mathscr{C}}_{0,n}$ consisting of the nodes of the singular fibers of the projection $\pi : \overline{\mathscr{C}}_{0,n} \to \overline{\mathcal{M}}_{0,n}$ and $K = K_{\overline{\mathscr{C}}_{0,n}/\overline{\mathcal{M}}_{0,n}}(\sum_s \Gamma_s)$. The Todd class of π is given by

$$\mathrm{td}(\pi) = 1 - \frac{1}{2}\Big(K - \sum_s \Gamma_s\Big) + \frac{1}{12}\Big(K^2 + \sum_s \Gamma_s^2 + \Delta\Big) + \cdots$$

and recall that $\hat{\mathscr{L}} = \pi_*\mathcal{O}(\Lambda)$ where

$$\Lambda = K_{\overline{\mathscr{C}}_{0,n}/\overline{\mathcal{M}}_{0,n}} + \sum_s \mu_s \Gamma_s - \sum_{\mathcal{S}}(1-\mu_{\mathcal{S}})F^1_{\mathcal{S}}.$$

Notice that $R^1\pi_*\,\mathcal{O}(\Lambda)=0$ as $\mathcal{O}(\Lambda)$ is trivial along the fibers of π, hence by the Grothendieck-Riemann-Roch formula, a representative of $c_1(\hat{\mathscr{L}})$ is

$$\frac{1}{2}\pi_*\left\{\left(K+\sum_s(\mu_s-1)\Gamma_s+\sum_{\mathscr{S}}(\mu_{\mathscr{S}}-1)F^1_{\mathscr{S}}\right)\left(\sum_s\mu_s\Gamma_s+\sum_{\mathscr{S}}(\mu_{\mathscr{S}}-1)F^1_{\mathscr{S}}\right)\right\}$$

because $\frac{1}{12}\pi_*\left(K^2+\sum_s\Gamma_s^2+\Delta\right)$ represents the first Chern class of the Hodge bundle, which is zero on $\overline{\mathscr{M}}_{0,n}$. Now, it is well known that for any s and any $s'\neq s$,

$$K\cdot\Gamma_s=0\ ,\quad \Gamma_s\cdot\Gamma_{s'}=0\ ,\quad \pi_*\big(\Gamma_s^2\big)=-\psi_s$$

and straightforward computations show that

$$\begin{aligned}
\pi_*(K\cdot F^1_{\mathscr{S}})&=(|I_1|-1)\,D_{\mathscr{S}},\\
\pi_*(F^1_{\mathscr{S}}\cdot F^1_{\mathscr{S}'})&=\begin{cases}0 & \text{if } \mathscr{S}\neq\mathscr{S}'\\ -D_{\mathscr{S}} & \text{if } \mathscr{S}=\mathscr{S}',\end{cases}\\
\text{and } \pi_*(F^1_{\mathscr{S}}\cdot\Gamma_s)&=\begin{cases}0 & \text{if } s\notin I_1\\ D_{\mathscr{S}} & \text{if } s\in I_1,\end{cases}
\end{aligned}$$

for any s, $\mathscr{S}$ and $\mathscr{S}'$. Therefore,

$$c_1(\hat{\mathscr{L}})=\frac{1}{2}\left(-\sum_s\mu_s(\mu_s-1)\psi_s+\sum_{\mathscr{S}}\mu_{\mathscr{S}}(\mu_{\mathscr{S}}-1)D_{\mathscr{S}}\right).$$

Finally, by a slight variation of the computation in Section 7.3 we obtain

$$\begin{aligned}
(n-2)\sum_i\mu_i(2-\mu_i)\,\psi_i&=\sum_{i,j,k}\mu_i(\mu_j+\mu_k)\,\psi_i\\
&\sim\sum_{i,j,k}\mu_i(\mu_j+\mu_k)\,\delta_{i|jk}\\
&=\sum_{\mathscr{S}}\Big(\big(|I_1|-1\big)\sum_{j\in I_1}\mu_j\sum_{i\in I_0}\mu_i+\big(|I_0|-1\big)\sum_{j\in I_0}\mu_j\sum_{i\in I_1}\mu_i\Big)D_{\mathscr{S}}\\
&=(n-2)\sum_{\mathscr{S}}\mu_{\mathscr{S}}(2-\mu_{\mathscr{S}})D_{\mathscr{S}},
\end{aligned}$$

which implies

$$c_1(\hat{\mathscr{L}})=\frac{1}{2}\left(-\sum_s\mu_s\psi_s+\sum_{\mathscr{S}}\mu_{\mathscr{S}}D_{\mathscr{S}}\right)$$

as expected.

Appendix A

Intersection theory on $\overline{\mathscr{M}}_{0,n}$

In this section we describe an algorithm to compute the intersection numbers of vital divisors in $\overline{\mathscr{M}}_{0,n}$. This algorithm is well known to experts in the field and can be found in [17]. We include it here only for the sake of completeness. We are grateful to D. Zvonkine for having explained it to us.

Intersections of vital divisors in $\overline{\mathcal{M}}_{0,n}$ will produce formal sums of trees whose vertices are labeled by subsets in a partition of $\{1,\dots,n\}$. At every vertex, the sum of the cardinal of the corresponding subset and the number of edges containing it must be at least 3. Such a tree corresponds to a stratum of $\overline{\mathcal{M}}_{0,n}$. Note that we allow $\emptyset$ to be part of a partition. A vital divisor $D_{\mathcal{S}}$, where $\mathcal{S} = \{I_0, I_1\}$ is a partition of $\{1,\dots,n\}$ such that $\min\{|I_0|,|I_1|\} \geq 2$, corresponds to a tree with two vertices labeled by I_0 and I_1.

Here below we will give the rule to compute the intersection of a divisor $D_{\mathcal{S}}$ with a tree T as above. Recursively, this allows us to compute any product $D_{\mathcal{S}_1} \cdot \cdots \cdot D_{\mathcal{S}_{n-3}}$. We first color the vertices of T with respect to the partition $\{I_0, I_1\}$ as follows: the vertices labeled by subsets contained in I_0 are given the red color, those labeled by subsets contained in I_1 are given the blue color. The vertices corresponding to subsets which are not contained in I_0 nor in I_1 are given the black color. Finally, the vertices corresponding to the empty set are given the white color. We have three cases:

Case 1. – There is more than one black vertex. In this case the intersection is empty, we get 0.

Case 2. – There is exactly one black vertex. If there is an edge in T which connects a red vertex and a blue one then we get 0. Otherwise the black vertex separates the red vertices from the blue ones. We subdivide the subset corresponding to the black vertex into two subsets: one is contained in I_0, the other in I_1. We then replace this vertex of T by an edge whose ends are labeled by the two subsets above. We color the new vertices using the same rule. There is a unique configuration such that the new edge separates the red vertices from the blue ones. The intersection is then given by this new tree.

Case 3. – There are no black vertices. We will say that a vertex or an edge of T separates the red vertices from the blue ones if it is contained in any path joining a red vertex to a blue one. We have several subcases:

(a) There are no edges and no vertices that separate the red vertices from the blue ones. In this case the intersection is 0.

(b) There are no edges that separate the red vertices from the blue ones, but there is a vertex A that satisfies this property. Note that A is then unique. We first notice that all the leaves of T must be either red or blue. Thus we can subdivide the set of edges incident to A into two subsets: E' is the set of edges that are contained in some paths joining A to a red leaf, E'' is the set of edges that are contained in some paths joining A to a blue leaf. That $\{E', E''\}$ is a partition of the set of edges incident to A is a consequence of the hypothesis that A separates the red vertices from the blue ones.

    We form a new tree by splitting A into two vertices A', A'' connected by an edge, where A' is attached to all the edges in E', and A'' is attached to all the edges in E''. We associate to A' the subset $A \cap I_0$, and to A'' the subset $A \cap I_1$. In more concrete terms, if A is red then $A' = A$, $A'' = \emptyset$, if A is blue then $A' = \emptyset$, $A'' = A$, if $A = \emptyset$ then $A' = A'' = \emptyset$. This new tree is the intersection of $D_{\mathcal{S}}$ and T. Notice that it is necessary stable because otherwise, there would exist an edge separating the red vertices from the blue ones.

(c) There is an edge e that separates the red vertices from the blue ones. In this case this edge must be unique. Let A and B denote the ends of e. By a slight abuse of notation we will also denote by A and B the corresponding subsets of $\{1,\dots,n\}$. Note that A and B can be empty.

Let $\hat{A}$ be the union of the indices contained in A and the edges incident to A. We pick a pair $\{a_1,a_2\}$ in $\hat{A}$ such that $e \notin \{a_1,a_2\}$. Consider all the partitions of $\hat{A}$ into two subsets $\{\hat{A}_1,\hat{A}_2\}$ such that $e \in \hat{A}_1$, $\{a_1,a_2\} \subset \hat{A}_2$, and $\min\{|\hat{A}_1|,|\hat{A}_2|\} \geq 2$. For any such partition, we remove the vertex A from T and construct a new tree from T as follows: form two new vertices A_1 and A_2, attach A_i to all the edges in $\hat{A}_i$ and add a new edge connecting A_1 and A_2. The new vertex A_i is associated with the set of indices in $\{1,\dots,n\} \cap \hat{A}_i$. Let Σ_A denote the formal sum of all the trees obtained this way.

We apply the same to B, and let Σ_B denote the corresponding formal sum. The intersection of $D_{\mathcal{S}}$ with T is then equal to $-(\Sigma_A + \Sigma_B)$.

The intersection number $D_{\mathcal{S}_1} \cdot \dots \cdot D_{\mathcal{S}_{n-3}}$ is then the sum of all the coefficients of the trees in the final formal sum obtained from this algorithm.

Using this algorithm, we can compute the intersection numbers of vital divisors in $\overline{\mathcal{M}}_{0,5}$ and $\overline{\mathcal{M}}_{0,6}$. As $\mathcal{S} = \{I_0, I_1\}$ is of course determined by either I_0 or I_1, we denote below $D_{\mathcal{S}}$ by D_{I_0} or D_{I_1}.

Case $\overline{\mathcal{M}}_{0,5}$. – We have

$$D_{ij} \cdot D_{ij} = -1, \quad D_{ij} \cdot D_{jk} = 0, \quad D_{ij} \cdot D_{k\ell} = 1.$$

Case $\overline{\mathcal{M}}_{0,6}$. – Recall that $D_I \cdot D_J = 0$ if neither J nor J^c is contained in I or in I^c. The intersections which do not vanish due to this simple rule are recorded here below:

$$\begin{array}{lll}
D_{ij} \cdot D_{ij} \cdot D_{ij} = 1, & D_{ij} \cdot D_{ij} \cdot D_{ijk} = 0, & D_{ij} \cdot D_{ij} \cdot D_{k\ell} = -1, \\
D_{ij} \cdot D_{ijk} \cdot D_{ijk} = -1, & D_{ij} \cdot D_{ijk} \cdot D_{j'k'} = 1, & \\
D_{ijk} \cdot D_{ijk} \cdot D_{ijk} = 2, & & \\
D_{ij} \cdot D_{k\ell} \cdot D_{k'\ell'} = 1. & & .
\end{array}$$

Appendix B

Computation of the volume in $\mathcal{M}_{0,5}$

Here we compute the volume of $\mathcal{M}_{0,5}$ with respect to Ω_μ using the results of Section 8. We may assume that $1 > \mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4 \geq \mu_5 > 0$. Note that in any case, $\mu_2 + \mu_4 \leq 1$ since $\sum \mu_s = 2$. As a consequence, only the following can happen:

- $\mu_2 + \mu_3 \leq 1$ and
  ◊ $\mu_1 + \mu_5 \geq 1$:

$$D_\sigma = (1-\mu_1)D_{13} + (1-\mu_1)D_{14} + (1-\mu_1)D_{25} \text{ and } \int_{\mathcal{M}_{0,5}} \Omega_\mu^2 = (1-\mu_1)^2.$$

◊ $\mu_1+\mu_4\geq 1, \mu_1+\mu_5\leq 1$:

$$D_\sigma=(1-\mu_1)D_{13}+\mu_5 D_{14}+(1-\mu_1-\mu_5)D_{24}+\mu_5 D_{25} \text{ and}$$

$$\int_{\overline{\mathcal{M}}_{0,5}}\Omega_\mu^2=(1-\mu_1)^2-(1-\mu_1-\mu_5)^2.$$

◊ $\mu_1+\mu_3\geq 1, \mu_1+\mu_4\leq 1$:

$$D_\sigma=(1-\mu_1)D_{13}+(1-\mu_2-\mu_3)D_{14}+(1-\mu_1-\mu_5)D_{24}+\mu_5 D_{25} \text{ and}$$

$$\int_{\overline{\mathcal{M}}_{0,5}}\Omega_\mu^2=(1-\mu_1)^2-(1-\mu_1-\mu_4)^2-(1-\mu_1-\mu_5)^2.$$

◊ $\mu_1+\mu_2\geq 1, \mu_1+\mu_3\leq 1$:

$$D_\sigma=\mu_3 D_{13}+(1-\mu_2-\mu_3)D_{14}+(1-\mu_1-\mu_5)D_{24}+\mu_5 D_{25} \text{ and}$$

$$\int_{\overline{\mathcal{M}}_{0,5}}\Omega_\mu^2=2\mu_3\mu_5-(1-\mu_1-\mu_4)^2-(1-\mu_2-\mu_4)^2.$$

◊ $\mu_1+\mu_2\leq 1$:

$$D_\sigma=(1-\mu_4-\mu_5)D_{13}+(1-\mu_2-\mu_3)D_{14}+(1-\mu_1-\mu_5)D_{24} + (1-\mu_3-\mu_4)D_{25}+(1-\mu_1-\mu_2)D_{35}$$

$$\text{and } \int_{\overline{\mathcal{M}}_{0,5}}\Omega_\mu^2=2\sum_{i=1}^{5}(1-\mu_{i-1}-\mu_i)(1-\mu_i-\mu_{i+1})-\sum_{i=1}^{5}(1-\mu_i-\mu_{i+1})^2.$$

- $\mu_2+\mu_3\geq 1$ and $\mu_1+\mu_4\leq 1$:

$$D_\sigma=(\mu_4+\mu_5)D_{13}+\mu_4 D_{24}+\mu_5 D_{25} \text{ and } \int_{\overline{\mathcal{M}}_{0,5}}\Omega_\mu^2=2\mu_4\mu_5.$$

All the formulas are obtained as a straightforward application of Theorem 8.4. However, one can prove after some more (tedious) computations that if $\mu_2+\mu_3\leq 1$ and $\mu_1+\mu_{s-1}\geq 1$, $\mu_1+\mu_s\leq 1$ for some $2\leq s\leq 6$ (which happens for all but the last exceptional case) then

$$\int_{\overline{\mathcal{M}}_{0,5}}\Omega_\mu^2=(1-\mu_1)^2-\sum_{i=s}^{5}(1-\mu_1-\mu_i)^2.$$

Appendix C

An example in $\overline{\mathcal{M}}_{0,6}$

The fact that Ω_μ is a representative of the first Chern class of the Kawamata line bundle $\hat{\mathcal{L}}$ can be exploited to simplify the evaluation of $\int_{\overline{\mathcal{M}}_{0,n}}\Omega_\mu^{n-3}$ in certain cases, especially when the weight vector μ has some symmetry. To illustrate this observation, let us consider the family of weights $\mu=(\alpha,\alpha,\alpha,\beta,\beta,\beta)$, with $0<\beta\leq\alpha$ and $\alpha+\beta=2/3$. Assuming that α and β are both rational, we can find $d\in\mathbb{N}^*$ such that $d\alpha\in 2\mathbb{N}$ and $d\beta\in 2\mathbb{N}$. Define a section σ of the Kawamata line bundle by

$$\sigma=\frac{(x_1-x_2)^{d\frac{\alpha}{2}}(x_2-x_3)^{d\frac{\alpha}{2}}(x_3-x_1)^{d\frac{\alpha}{2}}(x_4-x_5)^{d\frac{\beta}{2}}(x_5-x_6)^{d\frac{\beta}{2}}(x_6-x_4)^{d\frac{\beta}{2}}}{(z-x_1)^{d\alpha}(z-x_2)^{d\alpha}(z-x_3)^{d\alpha}(z-x_4)^{d\beta}(z-x_5)^{d\beta}(z-x_6)^{d\beta}}dz^{\otimes d}.$$

We will use the following equality (which is a consequence of Theorem 8.4) $\int_{\mathcal{M}_{0,6}} \Omega_\mu^3 = \left(\frac{\mathrm{div}(\sigma)}{d}\right)^3$ to compute the volume of $\mathcal{M}_{0,6}$ with respect to Ω_μ.

In what follows, for any subset $I \subset \{1, \dots, 6\}$ such that $2 \leq |I| \leq 4$, D_I is the boundary divisor of $\overline{\mathcal{M}}_{0,6}$ corresponding to the partition $\{I, I^c\}$. In particular, any boundary divisor of $\overline{\mathcal{M}}_{0,6}$ can be written as D_I with $|I| \leq 3$. Set

$$A_1 = D_{123}, \quad A_2 = \sum_{1 \leq i \leq 3} \sum_{4 \leq j < k \leq 6} D_{ijk}, \quad B = \sum_{1 \leq i < j \leq 3} D_{ij}, \quad C = \sum_{4 \leq i < j \leq 6} D_{ij}.$$

Applying the algorithm described in Appendix A, we get the following

$$\begin{cases} A_1^3 = 2, \quad A_2^3 = 18, \qquad B^3 = 3, \quad C^3 = 3, \\ A_1 A_2 = 0, \\ A_1^2 B = A_1^2 C = -3, \qquad A_1 B^2 = A_1 C^2 = 0, \\ A_2^2 B = A_2^2 C = -9, \qquad A_2 B^2 = A_2 C^2 = 0, \\ B^2 C = B C^2 = -9, \\ A_1 B C = A_2 B C = 9. \end{cases}$$

We have two cases

- Case I: $0 < \beta \leq \frac{1}{6} \Leftrightarrow \alpha \geq \frac{1}{2}$. We have $\frac{\mathrm{div}(\sigma)}{d} = \frac{3\beta}{2} A_1 + \frac{\beta}{2} A_2 + \frac{3\beta}{2} B + \frac{\beta}{2} C$. Therefore
$$\left(\frac{\mathrm{div}(\sigma)}{d}\right)^3 = (3A_1 + A_2 + 3B + C)^3 \left(\frac{\beta}{2}\right)^3 = 48 \times \frac{\beta^3}{8} = 6\beta^3.$$
- Case II: $\frac{1}{6} \leq \beta \leq \frac{1}{3} \Leftrightarrow \frac{1}{3} \leq \alpha \leq \frac{1}{2}$. We have $\frac{\mathrm{div}(\sigma)}{d} = \frac{3\beta}{2} A_1 + \frac{\beta}{2} A_2 + \frac{\alpha}{2} B + \frac{\beta}{2} C$. It follows
$$\left(\frac{\mathrm{div}(\sigma)}{d}\right)^3 = \frac{3}{8}((\alpha - 3\beta)^3 + 16\beta^3) = 6\beta^3 - 3(2\beta - \frac{1}{3})^3.$$

To sum up, we have

$$\int_{\mathcal{M}_{0,6}} \Omega_\mu^3 = 6\beta^3 - 3(\max\{2\beta - \tfrac{1}{3}, 0\})^3, \text{ for all } \beta \in (0, \tfrac{1}{3}].$$

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