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AN OBSTRUCTION TO SMALL-TIME LOCAL NULL CONTROLLABILITY FOR A VISCOUS BURGERS' EQUATION

BY FRÉDÉRIC MARBACH

ABSTRACT. – In this work, we are interested in the small-time local null controllability for the viscous Burgers' equation $y_t - y_{xx} + yy_x = u(t)$ on a line segment, with null boundary conditions. The second-hand side is a scalar control playing a role similar to that of a pressure. In this setting, the classical Lie bracket necessary condition introduced by Sussmann fails to conclude. However, using a quadratic expansion of our system, we exhibit a second order obstruction to small-time local null controllability. This obstruction holds although the information propagation speed is infinite for the Burgers equation. Our obstruction involves the $H^{-5/4}$ norm of the control. The proof requires the careful derivation of an integral kernel operator and the estimation of residues by means of *weakly singular integral operator* estimates.

RÉSUMÉ. – Nous nous intéressons à la contrôlabilité locale en temps petit pour l'équation de Burgers visqueuse $y_t - y_{xx} + yy_x = u(t)$, posée sur un segment, avec des conditions de Dirichlet nulles au bord. Le terme source au second membre est un contrôle scalaire qui joue un rôle similaire à celui d'une pression. Dans ce contexte, la condition de crochet de Lie nécessaire classique introduite par Sussmann ne permet pas de conclure. Cependant, en utilisant un développement à l'ordre deux du système étudié, nous mettons en lumière une obstruction de nature quadratique à la contrôlabilité locale en temps petit. Cette obstruction tient alors même que la vitesse de propagation de l'information dans cette équation de Burgers est infinie. Elle fait intervenir la norme $H^{-5/4}$ du contrôle. La démonstration nécessite le calcul soigneux du noyau d'un opérateur intégral, ainsi que l'estimation d'opérateurs résiduels à l'aide de la théorie de régularité pour les *opérateurs intégraux faiblement singuliers*.

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1. Introduction

1.1. Description of the system and our main result

For $T > 0$ a small positive time, we consider the line segment $x \in [0, 1]$ and the following one-dimensional viscous Burgers' controlled system:

$$
\begin{cases} y_t - y_{xx} + y y_x = u(t) & \text{in } (0,T) \times (0,1), \\ y(t,0) = 0 & \text{in } (0,T), \\ y(t,1) = 0 & \text{in } (0,T), \\ y(0,x) = y_0(x) & \text{in } (0,1). \end{cases} \tag{1.1}
$$

The scalar control $u \in L^2(0, T)$ plays a role somewhat similar to that of a pressure for multi-dimensional fluid systems. Unlike some other studies, our control term u depends only on time and not on the space variable. It is supported on the whole segment [0, 1]. For any initial data $y_0 \in H^1_0(0, 1)$ and any fixed control $u \in L^2(0, T)$, it can be shown (see Lemma 7 below) that system (1.1) has a unique solution in the space $X_T = L^2((0, T); H^2(0, 1)) \cap H^1((0, T); L^2(0, 1))$. We are interested in the behavior of this system in the vicinity of the null equilibrium state.

Definition 1. – *We say that system* (1.1) *is* small-time locally null controllable *if, for any small time $T > 0$, for any small size of the control $\eta > 0$, there exists a region of size $\delta > 0$ such that:*

$$
\forall y_0 \in H^1_0(0,1) \text{ s.t. } |y_0|_{H^1_0} \leq \delta, \exists u \in L^2(0,T) \text{ s.t. } |u|_2 \leq \eta \text{ and } y(T,\cdot) = 0, \tag{1.2}
$$

where $y \in X_T$ is the solution to system (1.1) *with initial condition y_0 and control u.*

Theorem 1. – *System* (1.1) *is not* small-time locally null controllable. *Indeed, there exist $T, \eta > 0$ such that, for any $\delta > 0$, there exists $y_0 \in H^1_0(0, 1)$ with $|y_0|_{H^1_0} \leq \delta$ such that, for any control $u \in L^2(0, T)$ with $|u|_2 \leq \eta$, the solution $y \in X_T$ to* (1.1) *satisfies $y(T, \cdot) \neq 0$.*

We will see in the sequel that our proof actually provides a stronger result. Indeed, we prove that, for small times and small controls, whatever the small initial data y_0, the state $y(t)$ drifts towards a fixed direction. Of course, this prevents small-time local null controllability as a direct consequence.

1.2. Motivation: small-time obstructions due to non-linearities

Most of the known obstructions to small-time local null controllability for control systems governed by partial differential equations are due to linear features.

1.2.1. *Linear obstructions.* – The most common cause of linear obstruction is the presence, in the evolution equation, of a finite speed of propagation (e.g., for wave or transport systems). As an example, let us consider the following transport control system:

$$\begin{cases} y_t + My_x = 0 & \text{in } (0,T)\times(0,L), \\ y(t,0) = v_0(t) & \text{in } (0,T), \\ y(0,x) = y_0(x) & \text{in } (0,L), \end{cases} \tag{1.3}$$

where $T > 0$ is the total time, $M > 0$ the propagation speed and $L > 0$ the length of the domain. The control is the boundary data v_0. No condition is imposed at $x = 1$ since the characteristics flow out of the domain. For system (1.3), small-time local null controllability cannot hold. Indeed, even if the initial data y_0 is very small, the control is only propagated towards the right at speed M. Thus, if $T < L/M$, controllability does not hold. Of course, if $T \geq L/M$, the characteristics method allows to construct an explicit control to reach any final state y_1 at time T. We modify (1.3) with a small viscosity $\nu > 0$:

$$\begin{cases} y_t - \nu y_{xx} + My_x = 0 & \text{in } (0,T)\times(0,L), \\ y(t,0) = v_0(t) & \text{in } (0,T), \\ y(t,1) = 0 & \text{in } (0,T), \\ y(0,x) = y_0(x) & \text{in } (0,L). \end{cases} \tag{1.4}$$

System (1.4) is small-time globally null controllable, for any $\nu > 0$ (but the cost of controllability explodes as $\nu \to 0$ if T is too small; see [26] for a precise study). Similarly, the underdetermined inviscid system:

$$\begin{cases} y_t + yy_x = 0 & \text{in } (0,T)\times(0,L), \\ y(0,x) = y_0(x) & \text{in } (0,L) \end{cases} \tag{1.5}$$

is not small-time locally null controllable (whatever choice is made as controlled boundary conditions at $x = 0$ and $x = 1$). Indeed, locally, we have $|y| \leq M$ with a small M. However, its viscous counterpart:

$$\begin{cases} y_t - \nu y_{xx} + yy_x = 0 & \text{in } (0,T)\times(0,L), \\ y(t,0) = v_0(t) & \text{in } (0,T), \\ y(t,1) = 0 & \text{in } (0,T), \\ y(0,x) = y_0(x) & \text{in } (0,L) \end{cases} \tag{1.6}$$

is small-time locally null controllable for any $\nu > 0$ (see [36]).

Other linear features not linked to a finite propagation speed can also yield obstructions to small-time local null controllability; we refer to the recent works [5] for the Kolmogorov equation, [8] for Grushin-type equations, or [40] for the heat equation in a specific setting.

1.2.2. *Quadratic obstructions.* – Very few situations are known when the obstruction comes from the non-linearity of the partial differential equation governing the control system.

An example of such a system is the control of a quantum particle in a moving potential well (box). This is a bilinear controllability problem for the Schrödinger equation. For such system, it can be shown that large time controllability holds (see [4] if only the particle needs to be controlled or [6] to control both the particle and the box). For small times, negative results have been obtained by Coron in [22] (when one tries to control both the particle and the position of the box), by Beauchard, Coron and Teissman in [7] for large controls (but smooth potentials) and by Beauchard and Morancey in [9] (under an assumption corresponding to a non-vanishing Lie-bracket condition). This last paper is related to ours since their proof relies on a coercivity estimate involving the H^{-1} norm of the control. This is natural, due to their Lie-bracket condition, as we will see in Paragraph 1.5 (second example). We refer the reader to these papers for more details and surveys on the controllability of Schrödinger equations.

Theorem 1 can be seen as another example of a situation (in the context of fluid dynamics) where small-time local controllability fails despite an infinite propagation speed, because of a non-linear feature of the system. Moreover, the obstruction we obtain here is specific to the infinite dimensional setting and could not be observed on finite dimensional toy models, because the drift we obtain involves a fractional Sobolev norm, which is not possible in finite dimension (see Paragraph 1.5).

1.3. Previous works concerning Burgers' controllability

Let us recall known results concerning the controllability of the viscous Burgers' equation. More generally, we introduce the following system:

$$\begin{cases} y_t - y_{xx} + yy_x = u(t) & \text{in } (0,T)\times(0,1), \\ y(t,0) = v_0(t) & \text{in } (0,T), \\ y(t,1) = v_1(t) & \text{in } (0,T), \\ y(0,x) = y_0(x) & \text{in } (0,1), \end{cases} \tag{1.7}$$

where v_0 and v_1 are seen as additional controls with respect to the single control u of system (1.1). Various settings have been studied (with either one or two boundary controls, with or without u). Once again, here u only depends on t and not on x. Some studies have been carried out with $v_0 = v_1 = 0$ and a source term $u(t,x)\chi_{[a,b]}$ for $0 < a < b < 1$. However, these studies are equivalent to boundary controls thanks to the usual domain extension argument. Up to our knowledge, Theorem 1 is the first result concerning the case without any boundary control and a scalar control u.

Results involving only a single boundary control (either v_0 or v_1 by symmetry) and $u = 0$

In [36], Fursikov and Imanuvilov prove small-time local controllability in the vicinity of trajectories of system (1.7). Their proof relies on Carleman estimates for the parabolic problem obtained by seeing the non-linear term yy_x as a small forcing term.

Global controllability towards steady states of system (1.7) is possible in large time. Such studies have been carried out by Fursikov and Imanuvilov in [35] for large time global controllability towards all steady states, and by Coron in [24] for global null controllability in bounded time (ie. bounded independently on the initial data).

However, small-time global controllability does not hold. The first obstruction was obtained by Diaz in [29]. He gives a restriction for the set of attainable states starting from 0. Indeed, they must lie under some limit state corresponding to an infinite boundary control $v_1 = +\infty$.

Fernández-Cara and Guerrero derived an asymptotic of the minimal null-controllability time $T(r)$ for initial states of H^1 norm lower than r (see [30]). This shows that the system is not small-time globally null controllable.

Result with both boundary controls v_0 and v_1, but $u = 0$. – Guerrero and Imanuvilov prove in [37] that neither small-time null controllability nor bounded time global controllability hold in this context. Hence, controlling the whole boundary does not provide better controllability properties.

Results with all three scalar controls (namely u, v_0 and v_1). – Chapouly has shown in [19] that the system is small-time globally exactly controllable to the trajectories. Her proof relies on the return method and on the fact that the corresponding inviscid Burgers' system is small-time exactly controllable (see [23, Chapter 6] for other examples of this method applied to Euler or Navier-Stokes).

Result with u and v_0, but $v_1 = 0$. – The author proved in [41] that small-time global null controllability holds. Indeed, although a boundary layer appears near the uncontrolled part of the boundary at $x = 1$, a precise estimation of the creation and dissipation of the boundary layer allows to conclude.

Controllability of the inviscid Burgers' equation. – In [2], Ancona and Marson describe the set of attainable states in a pointwise way for the Burgers' equation on the half-line $x \geq 0$ with only one boundary control at $x = 0$. In [38], Horsin describes the set of attainable states for a Burgers' equation on a line segment with two boundary controls. Thorough studies are also carried out in [1] by Adimurthi et al. In [46], Perrollaz studies the controllability of the inviscid Burgers' equation in the context of entropy solutions with the additional control $u(\cdot)$ and two boundary controls.

1.4. A quadratic approximation for the non-linear system

Starting now, we introduce $\varepsilon = T$ to remember that the total allowed time for controllability is small. Moreover, we want to use the well-known scaling trading *small time* with *small viscosity* for viscous fluid equations. Therefore, we introduce, for $t \in (0,1)$ and $x \in (0,1)$, $\tilde{y}(t,x) = \varepsilon y(\varepsilon t, x)$. Hence, $\tilde{y}$ is the solution to:

$$\begin{cases} \tilde{y}_t - \varepsilon \tilde{y}_{xx} + \tilde{y}\tilde{y}_x = \tilde{u}(t) & \text{in } (0,1)\times(0,1), \\ \tilde{y}(t,0) = 0 & \text{in } (0,1), \\ \tilde{y}(t,1) = 0 & \text{in } (0,1), \\ \tilde{y}(0,x) = \tilde{y}_0(x) & \text{in } (0,1), \end{cases} \tag{1.8}$$

where $\tilde{u}(t) = \varepsilon^2 u(\varepsilon t)$ and $\tilde{y}_0 = \varepsilon y_0$. This scaling has already been used in various fluid mechanics controllability contexts (see [20] for Navier-Stokes, [21] for Euler or [41] for Burgers). As we will prove in Section 6, system (1.8) can help us to deduce results for system (1.1). To further simplify the computations in the following sections, let us drop the tilda signs and the initial data. Therefore, we will study the behavior of the following system near $y \equiv 0$:

$$(1.9)\qquad \begin{cases} y_t - \varepsilon y_{xx} + y y_x = u(t) & \text{in } (0,1)\times(0,1),\\ y(t,0) = 0 & \text{in } (0,1),\\ y(t,1) = 0 & \text{in } (0,1),\\ y(0,x) = 0 & \text{in } (0,1). \end{cases}$$

Properties proven on system (1.9) will easily be translated into properties for system (1.1) in Section 6. Moreover, since we are studying local null controllability, both the control u and the state y are small. Thus, if η describes the size of the control as in Definition 1, let us name our control $\eta u(t)$, with u of size $\mathcal{O}(1)$. We expand y as $y = \eta a + \eta^2 b + \mathcal{O}(\eta^3)$, and we compute the associated systems:

$$(1.10)\qquad \begin{cases} a_t - \varepsilon a_{xx} = u(t) & \text{in } (0,1)\times(0,1),\\ a(t,0) = 0 & \text{in } (0,1),\\ a(t,1) = 0 & \text{in } (0,1),\\ a(0,x) = 0 & \text{in } (0,1) \end{cases}$$

and

$$(1.11)\qquad \begin{cases} b_t - \varepsilon b_{xx} = -a a_x & \text{in } (0,1)\times(0,1),\\ b(t,0) = 0 & \text{in } (0,1),\\ b(t,1) = 0 & \text{in } (0,1),\\ b(0,x) = 0 & \text{in } (0,1). \end{cases}$$

System (1.10) is not controllable. Indeed, the right-hand side $u(t)$ can be written as $u(t)\chi_{[0,1]}$ and $\chi_{[0,1]}$ is an even function on the line segment $[0,1]$. Here and in the sequel, we will abusively say that a function ϕ defined on $[0,1]$ is *even* when it satisfies $\phi\left(\frac{1}{2}+x\right) = \phi\left(\frac{1}{2}-x\right)$ for $x \in \left[-\frac{1}{2},\frac{1}{2}\right]$. Similarly, we will say that ϕ is *odd* when $\phi\left(\frac{1}{2}+x\right) = -\phi\left(\frac{1}{2}-x\right)$ for $x \in \left[-\frac{1}{2},\frac{1}{2}\right]$. Thus, the control only acts on even modes of a. In the linearized system (1.10), all odd modes evolve freely. This motivates the second order expansion of our Burgers' system in order to understand its controllability properties using b. Given systems (1.10) and (1.11), we know that a is even and b is odd.

1.5. A finite dimensional counterpart

Systems (1.10) and (1.11) exhibit an interesting structure. Indeed, the first system is fully controllable (if we consider that a lives within the subspace of even functions), while the second system is indirectly controlled through a quadratic form depending on a. Let us introduce the following finite dimensional control system:

$$(1.12)\qquad \begin{cases} \dot{a} = Ma + u(t)m & \text{in } (0,T),\\ \dot{b} = Lb + Q(a,a) & \text{in } (0,T), \end{cases}$$

where the states $a(t), b(t) \in \mathbb{R}^n \times \mathbb{R}^p$, M is an $n\times n$ matrix, m is a fixed vector in $\mathbb{R}^n$ along which the scalar control acts, L is a $p \times p$ matrix and Q is a quadratic function from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^p$. Moreover, we assume that the pair (M,m) satisfies the classical Kalman rank condition (see [23, Theorem 1.16]). Hence, the state a is fully controllable. We consider the small-time null controllability problem for system (1.12). We want to know, if, for any

$T > 0$, for any initial state (a^0, b^0), there exists a control $u : (0, T) \to \mathbb{R}$ such that the solution to (1.12) satisfies $a(T) = 0$ and $b(T) = 0$. As proved in [13] for the case $L = 0$, the answer to this question is always no in finite dimension, whatever M, m, L and Q.

System (1.12) is a particular case of the more general class of control affine systems. Indeed, if we let $x(t) = (a(t), b(t)) \in \mathbb{R}^{n+p}$, we can write system (1.12) as:

$$\dot{x} = f_0(x) + u(t) f_1(x), \tag{1.13}$$

where $f_0(x) = (Ma, Lb + Q(a,a))$ and $f_1(x) = (m, 0)$. The controllability of systems like (1.13) is deeply linked to the iterated Lie brackets of the vector fields f_0 and f_1 (see [23, Section 3.2] for a review).

Let us give a few examples with $n = 3$. We write $a = (a_1, a_2, a_3)$ and we consider the system:

$$\dot{a}_1 = a_2, \quad \dot{a}_2 = a_3, \quad \dot{a}_3 = u. \tag{1.14}$$

Although the strong structure of Equation (1.14) can seem a little artificial, the general case can be reduced to this one. Indeed, up to a translation of the control, controllable systems can always be brought back to this canonical form introduced by Brunovsky in [14] (for a proof, see [52, Theorem 2.2.7]). The resulting system is *flat*. We can express the full state as derivatives of a single scalar function. Indeed, if we let $\theta = a_1$, we have $a_2 = \theta'$, $a_3 = \theta''$ and $u = \theta'''$. If we choose an initial state (a^0, b^0) with $a^0 = 0$, we obtain $\theta(0) = \theta'(0) = \theta''(0) = 0$. Moreover, if we assume that the control u drives the state (a, b) to $(0, 0)$ at time T, we also have $\theta(T) = \theta'(T) = \theta''(T) = 0$. These conditions allow integration by parts without boundary terms.

To keep the examples simple, we choose $p = 1$ (hence $b = b_1 \in \mathbb{R}$) and we let $L = 0$.

First example. – We consider the evolution $\dot{b} = a_2^2 + a_1 a_3$. If the initial state is (a^0, b^0) where $a^0 = 0$, we can compute $b(T) = b^0 + \int_0^T \theta'^2(t) + \theta(t)\theta''(t) \mathrm{d}t = b^0$. Hence, null controllability does not hold since any control driving a from 0 back to 0 has no action on b. This obstruction to controllability is linked to the fact that $\dim \mathcal{L}(0) = 3$, where $\mathcal{L}$ is the Lie algebra generated by f_0 and f_1. The system is locally constrained to evolve within a 3 dimensional manifold of $\mathbb{R}^4$. Indeed, the evolution equation can be rephrased as $\dot{b} = \frac{\mathrm{d}}{\mathrm{d}t}(a_1 a_2)$. Thus, the quantity $b - a_1 a_2$ is a constant (*conservation law* of the system).

Second example. – We consider the evolution $\dot{b} = a_3^2$. Thus, $b(T) = b^0 + \int_0^T \theta''(t)^2 \mathrm{d}t$. This is also an obstruction to null controllability. Indeed, all choices of control will make b increase. In this setting, we recover the well known second order Lie bracket condition discovered by Sussmann (see [50, Proposition 6.3]). Indeed, here, $[f_1, [f_1, f_0]] = (0_{\mathbb{R}^3}, Q(m, m)) = (0_{\mathbb{R}^3}, 1)$. System (1.13) drifts in the direction $[f_1, [f_1, f_0]]$ and the control cannot prevent it because this direction does not belong to the set of the first order controllable directions $(m, 0)$, $(Mm, 0)$ and $(M^2 m, 0)$ (Lie brackets of f_0 and f_1 involving f_1 once and only once).

Third example. – We consider $\dot{b} = a_2^2$. Thus, $b(T) = b^0 + \int_0^T \theta'^2(t)\mathrm{d}t$. Again, b can only increase. Here, the first *bad* Lie bracket $[f_1,[f_1,f_0]]$ vanishes for $x = 0$. However, we can check that $[f_1,[f_0,[f_0,[f_1,f_0]]]] = (0_{\mathbb{R}^3}, Q(Mm,Mm)) = (0_{\mathbb{R}^3},1)$. Compared with the second example, the increase of b is weaker. Indeed, in the second example, we had $b(T) = b^0 + |u|^2_{H^{-1}(0,T)}$. In this third example, $b(T) = b^0 + |u|^2_{H^{-2}(0,T)}$.

Although these examples may seem caricatural, they reflect the general case. In finite dimension, systems like (1.12) are never small-time controllable. Either because they evolve within a strict manifold, or because some quantity depending on b increases. Moreover, the amount by which b increases is linked to the order of the first bad Lie bracket and can be expressed as a weak norm depending on the control. One of the goals of our work is thus also to investigate the situation in infinite dimension, where Lie brackets are harder to define and compute.

Therefore, the first natural question is to compute the Lie bracket $[f_1,[f_1,f_0]](0)$ for systems (1.10) and (1.11). As we have seen in finite dimension, this Lie bracket is $(0, Q(m,m))$. In our setting, m is the even function $\chi_{[0,1]}$ and $Q(a,a) = -aa_x$. Thus $Q(m,m)$ is null. This can be proved computationally using Fourier series expansions. Let us give a much simpler argument inspired by the formal fact that $\partial_x 1 = 0$. For any $a \in L^2(0,1)$ and any smooth test function ϕ such that $\phi(0) = \phi(1) = 0$, we have:

$$\int_0^1 Q(a,a)\phi = \frac{1}{2}\int_0^1 a^2(x)\phi_x(x)\mathrm{d}x. \tag{1.15}$$

Hence, even if $q := Q(1,1)$ was defined in a very weak sense, (1.15) yields:

$$\langle q,\phi\rangle = \frac{1}{2}\int_0^1 \phi_x = \frac{1}{2}\phi(1) - \frac{1}{2}\phi(0) = 0. \tag{1.16}$$

Since (1.16) is valid for any smooth ϕ null at the boundaries, we conclude that indeed, $q = Q(1,1)$ is null. Therefore, the classical $[f_1,[f_1,f_0]]$ necessary condition by Sussmann does not provide an obstruction to small-time controllability for our system. This also explains why the coercivity property we are going to prove is in a weaker norm than H^{-1}.

1.6. Strategy for the proof

Most of this paper is dedicated to the asymptotic study of systems (1.10) and (1.11) as the viscosity ε tends to zero. In Section 6, we prove that this study is sufficient to conclude about the local null controllability for system (1.1). In order to prove that system (1.1) is not small-time locally null controllable, we intend to exhibit a quantity depending on the state $y(t,\cdot)$ that cannot be controlled. For $\rho \in H^1(0,1)$, we will consider quantities of the form $\langle \rho, y(t,\cdot)\rangle$.

Looking at system (1.11) when ε is very small, we get the idea to consider $\rho(x) = x - \frac{1}{2}$. Indeed, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho(x)b(t,x)\mathrm{d}x = \frac{1}{2}\int_0^1 a^2(t,x)\mathrm{d}x + \frac{\varepsilon}{2}\left(b_x(t,1) - b_x(t,0)\right). \tag{1.17}$$

Formally, if we let $\varepsilon = 0$ in Equation (1.17), it is very encouraging because it shows that the quantity $\langle \rho, b\rangle$ can only increase, whatever the choice of the control. Moreover, since we can compute the amount by which it increases, we have a kind of coercivity and we can hope to be able to use it to overwhelm both residues coming from the fact that $\varepsilon > 0$ as well as residues between the quadratic approximation and the full non-linear system. Sadly, the second term in the right-hand side of Equation (1.17) is hard to handle. However, as a depends linearly on u, and b depends quadratically on a, we expect that we can find a kernel $K^\varepsilon(s_1, s_2)$ such that:

$$\langle \rho, b(1,\cdot)\rangle = \int_0^1 \int_0^1 K^\varepsilon(s_1, s_2) u(s_1) u(s_2) \mathrm{d}s_1 \mathrm{d}s_2. \tag{1.18}$$

Thanks to Equation (1.17), we expect that (1.18) actually defines a positive definite kernel acting on u, allowing us to use its coercivity to overwhelm various residues.

In Section 2, we recall a set of technical well-posedness estimates for heat and Burgers systems.

In Section 3, we show that Formula (1.18) holds and we give an explicit construction of the kernel K^ε. Moreover, we compute formally its limit N as $\varepsilon \to 0$.

In Section 4, we prove that the kernel N is coercive with respect to the $H^{-5/4}(0,1)$ norm of the control u, by recognizing a Riesz potential and a fractional Laplacian.

In Section 5, we use weakly singular integral operator estimates to bound the residues between K^ε and N and thus deduce that K^ε is also coercive, for ε small enough.

In Section 6, we use these results to go back to the controllability of Burgers.

In the appendix, we give a short presentation of the theory of weakly singular integral operators and a sketch of proof of the main estimation lemma we use.

2. Preliminary technical lemmas

In this section, we recall a few useful lemmas and estimates, mostly concerning the heat equation and Burgers equation on a line segment. Throughout this section, ν is a positive viscosity and T a positive time. To lighten the computations, we will use the notation $\lesssim$ to denote inequalities that hold up to a numerical constant. We will not attempt to keep track of these numerical constants. We insist on the fact that these constants do not depend on any parameter (neither the time T, nor the viscosity ν, the control u, or any other unknown).

2.1. Properties of the functional space

We recall the definition given in the introduction and state without proof the following classical lemmas which can be proved using either interpolation theory, Fourier series or Fourier transforms (after extension from $x \in [0,1]$ to $x \in \mathbb{R}$) with respect to time and space.

Definition 2. – *We define the functional space:*

$$X_T = L^2\left((0,T), H^2(0,1)\right) \cap H^1\left((0,T), L^2(0,1)\right). \tag{2.1}$$

We endow the space X_T with the scaling invariant norm:

$$\|z\|_{X_T} := T^{-1/2}\|z\|_2 + T^{-1/2}\|z_{xx}\|_2 + T^{1/2}\|z_t\|_2. \tag{2.2}$$

LEMMA 1. – *$X_T \hookrightarrow \mathcal{C}^0([0,T], H^1(0,1))$. Moreover, for any function $z \in X_T$,*

$$\sup_{t\in[0,T]} |z(t,\cdot)|_{H^1(0,1)} \lesssim \|z\|_{X_T}. \tag{2.3}$$

In particular,

$$\|z\|_\infty \lesssim \|z\|_{X_T}. \tag{2.4}$$

LEMMA 2. – *For any $z \in X_T$, the boundary traces of z_x satisfy:*

$$T^{-1/4}\, |z_x(\cdot,0)|_{H^{1/4}(0,T)} + T^{-1/4}\, |z_x(\cdot,1)|_{H^{1/4}(0,T)} \lesssim \|z\|_{X_T}. \tag{2.5}$$

2.2. Smooth setting for the heat equation

We start by recalling the standard energy estimate in a smooth (strong) setting for the one-dimensional heat equation that will be useful in the sequel.

LEMMA 3. – *Let $f \in L^2((0,T)\times(0,1))$ and $z^0 \in H^1_0(0,1)$. We consider the system:*

$$\begin{cases} z_t - \nu z_{xx} = f & \text{in } (0,T)\times(0,1), \\ z(t,0) = 0 & \text{in } (0,T), \\ z(t,1) = 0 & \text{in } (0,T), \\ z(0,x) = z^0(x) & \text{in } (0,1). \end{cases} \tag{2.6}$$

There is a unique solution $z \in X_T$ to system (2.6). *It satisfies the estimate:*

$$\nu\, \|z_{xx}\|_2 + \sqrt{\nu}\, \|z_x\|_2 + \|z_t\|_2 \lesssim \|f\|_2 + \sqrt{\nu}|z^0_x|_2. \tag{2.7}$$

2.3. Transposition solutions for the heat equation

Let us move on to weaker settings for the heat equation. Moreover, we introduce inhomogeneous boundary data as we will need them in the sequel.

DEFINITION 3. – *Let $f \in (X_T)'$, $v_0, v_1 \in H^{-1/4}(0,T)$ and $z^0 \in H^{-1}(0,1)$. We consider:*

$$\begin{cases} z_t - \nu z_{xx} = f & \text{in } (0,T)\times(0,1), \\ z(t,0) = v_0(t) & \text{in } (0,T), \\ z(t,1) = v_1(t) & \text{in } (0,T), \\ z(0,x) = z^0(x) & \text{in } (0,1). \end{cases} \tag{2.8}$$

We say that $z \in L^2((0,T)\times(0,1))$ is a transposition solution to (2.8) *if, for all $g \in L^2((0,T)\times(0,1))$,*

$$\begin{aligned} \langle z, g\rangle_{L^2,L^2} = \langle f, \varphi\rangle_{(X_T)',X_T} &+ \langle z^0, \varphi(0,\cdot)\rangle_{H^{-1}(0,1),H^1_0(0,1)} \\ &+ \nu\langle v_0, \varphi_x(\cdot,0)\rangle_{H^{-1/4}(0,T),H^{1/4}(0,T)} \\ &- \nu\langle v_1, \varphi_x(\cdot,1)\rangle_{H^{-1/4}(0,T),H^{1/4}(0,T)}, \end{aligned} \tag{2.9}$$

where $\varphi \in X_T$ is the solution to the dual system:

$$\begin{cases} \varphi_t + \nu\varphi_{xx} = -g & \text{in } (0,T)\times(0,1), \\ \varphi(t,0) = 0 & \text{in } (0,T), \\ \varphi(t,1) = 0 & \text{in } (0,T), \\ \varphi(T,x) = 0 & \text{in } (0,1). \end{cases} \tag{2.10}$$

LEMMA 4. – *There exists a unique transposition solution* $z \in L^2((0,T)\times(0,1))$ *to system* (2.8). *Moreover:*

$$\|z\|_2 \lesssim T^{-1/2}\nu^{-1}\left(\|f\|_{(X_T)'} + |z^0|_{H^{-1}}\right) + T^{-1/4}\left(|v_0|_{H^{-1/4}} + |v_1|_{H^{-1/4}}\right). \tag{2.11}$$

Proof. – For any $g \in L^2((0,T)\times(0,1))$, Lemma 3 asserts that system (2.10) admits a unique solution $\varphi \in X_T$ such that $\|\varphi\|_{X_T} \lesssim T^{-1/2}\nu^{-1}\|g\|_{L^2}$. Moreover, thanks to estimates (2.3) and (2.5), the right-hand side of Equation (2.9) defines a continuous linear form on L^2. The Riesz representation theorem therefore proves the existence of a unique $z \in L^2$ satisfying estimate (2.11). □

LEMMA 5. – *Let* $f \in L^2((0,T)\times(0,1))$. *We consider the following heat system:*

$$\begin{cases} z_t - \nu z_{xx} = f_x & \text{in } (0,1)\times(0,1),\\ z(t,0) = 0 & \text{in } (0,1),\\ z(t,1) = 0 & \text{in } (0,1),\\ z(0,x) = 0 & \text{in } (0,1). \end{cases} \tag{2.12}$$

There is a unique solution $z \in L^2((0,T)\times(0,1))$ *to system* (2.12). *Moreover, it satisfies the estimate:*

$$\nu^{1/2}\|z\|_{L^\infty(L^2)} + \nu\|z_x\|_{L^2} \lesssim \|f\|_{L^2}. \tag{2.13}$$

Proof. – For $f \in L^2$, one checks that $f_x \in X_T'$. Hence, we can apply Lemma 4 and system (2.12) has a unique solution $z \in L^2$. In fact, this solution is even smoother. Estimate (2.13) is obtained as usual by multiplying Equation (2.12) by z and integration by parts. □

2.4. Burgers and forced Burgers systems

We move on to Burgers-like systems. For the sake of completeness, we provide a short proof of the existence of a solution to system (1.1) and a precise estimate for forced Burgers-like systems that will be necessary in the sequel.

LEMMA 6. – *Let* $w \in X_T$, $g \in L^2((0,T),H^1(0,1))$ *and* $y^0 \in H^1_0(0,1)$. *We consider* $y \in X_T$ *a solution to the following forced Burgers-like system:*

$$\begin{cases} y_t - \nu y_{xx} = -yy_x + (wy)_x + g_x & \text{in } (0,T)\times(0,1),\\ y(t,0) = 0 & \text{in } (0,T),\\ y(t,1) = 0 & \text{in } (0,T),\\ y(0,x) = y^0(x) & \text{in } (0,1). \end{cases} \tag{2.14}$$

Then,

(2.15)

$$\nu \|y_{xx}\|_2 + \sqrt{\nu}\|y_x\|_2 + \|y_t\|_2 \lesssim \|g_x\|_2 + e^\gamma \|w_x\|_{L^2(L^\infty)} \left(\nu^{-1/2}\|g\|_2 + |y^0|_2^2\right) + \left(1+\sqrt{\gamma}e^\gamma\right)\|w\|_\infty \left(\nu^{-1}\|g\|_2 + \nu^{-1/2}|y^0|_2^2\right) + \left(1+\sqrt{\gamma}e^{6\gamma}\right)e^\gamma \|g\|_{L^2(L^\infty)}\left(\nu^{-3/2}\|g\|_2 + \nu^{-1}|y^0|_2\right) + \left(1+\sqrt{\gamma}e^{6\gamma}\right)\nu^{-1/2}|y^0|_4^2 + \nu^{1/2}|y^0_x|_2,$$

where we introduce $\gamma = \frac{1}{\nu}\|w\|^2_{L^2(L^\infty)}$.

Proof. – L^2 ESTIMATES FOR y AND y_x. — We start by multiplying Equation (2.14) by y, and integrate by parts over $(0,1)$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 y^2 + \nu\int_0^1 y_x^2 = -\int_0^1 wyy_x - \int_0^1 gy_x \leq \frac{2}{2\nu}\int_0^1 w^2y^2 + \frac{\nu}{4}\int_0^1 y_x^2 + \frac{2}{2\nu}\int_0^1 g^2 + \frac{\nu}{4}\int_0^1 y_x^2. \tag{2.16}$$

From (2.16), we deduce:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 y^2 + \nu\int_0^1 y_x^2 \leq \frac{2}{\nu}|w(t,\cdot)|_\infty^2\int_0^1 y^2 + \frac{2}{\nu}\int_0^1 g^2. \tag{2.17}$$

We apply Grönwall's lemma to (2.17) to obtain:

$$\|y\|^2_{L^\infty(L^2)} \leq e^{2\gamma}\left(\frac{2}{\nu}\|g\|_2^2 + |y^0|_2^2\right). \tag{2.18}$$

Plugging (2.18) into (2.17) yields:

$$\nu\|y_x\|_2^2 \leq \left(1+2\gamma e^{2\gamma}\right)\left(\frac{2}{\nu}\|g\|_2^2 + |y^0|_2^2\right). \tag{2.19}$$

L^2 ESTIMATE FOR yy_x. — We repeat a similar technique, multiplying this time Equation (2.14) by y^3. Using the same approach yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 y^4 + 6\nu\int_0^1 y^2y_x^2 \leq \frac{12}{\nu}|w(t,\cdot)|_\infty^2\int_0^1 y^4 + \frac{12}{\nu}|g(t,\cdot)|_\infty^2\int_0^1 y^2. \tag{2.20}$$

We apply Grönwall's lemma to (2.20) to obtain:

$$\|y\|^4_{L^\infty(L^4)} \leq e^{12\gamma}\left(\frac{12}{\nu}\|g\|^2_{L^2(L^\infty)}\|y\|^2_{L^\infty(L^2)} + |y^0|_4^4\right). \tag{2.21}$$

Once again, plugging back estimate (2.21) into (2.20) gives:

$$6\nu\|yy_x\|_2^2 \leq \left(1+12\gamma e^{12\gamma}\right)\left(\frac{12}{\nu}\|g\|^2_{L^2(L^\infty)}\|y\|^2_{L^\infty(L^2)} + |y^0|_4^4\right). \tag{2.22}$$

CONCLUSION. — To conclude the proof, we use Lemma 3, with a source term $f = g_x + w_xy + wy_x - yy_x$. Estimate (2.15) comes from the combination of (2.7) with Equations (2.18), (2.19) and (2.22). □

LEMMA 7. – *For any initial data* $y_0 \in H_0^1(0,1)$ *and any control* $u \in L^2(0,T)$*, system* (1.1) *has a unique solution* $y \in X_T$. *Moreover:*

$$\|y_{xx}\|_2 + \|y_t\|_2 \lesssim |u|_2 + |u|_2^2 + |y^0|_4^2 + |y_x^0|_2, \tag{2.23}$$

$$\|y\|_\infty \leq |y^0|_\infty + |u|_{L^1}. \tag{2.24}$$

Proof. – This type of existence result relies on standard *a priori* estimates and the use of a fixed point theorem. Such techniques are described in [39]. One can also use a semi-group method as in [45]. The quantitative estimate is obtained by applying Lemma 6 with $w = 0$ (hence $\gamma = 0$) and $g(t,x) = xu(t)$. Equation (2.15) yields (2.23). The second estimate (2.24) is a consequence of the maximum principle, which can be applied in this strong setting. □

3. From Burgers to a kernel integral operator

3.1. A general method for evaluating a projection

As we mentionned in the introduction, we are going to consider a projection of the state b against some given profile $\rho(x)$ at the final time $t = 1$. In the sequel, we will abusively use the expression *projection against* ρ to denote the scalar product of a state with ρ. Since a depends linearly on u and b depends quadratically on a, it is natural to look for this projection as a quadratic integral operator acting on our control u. Indeed, let us prove the following result.

LEMMA 8. – *Let* $\rho \in H^{-1}(0,1)$ *and* $\varepsilon > 0$. *There exists a symmetric kernel* K^ε *in* $L^\infty((0,1)^2)$ *such that, for any* $u \in L^2(0,1)$*, the solution to system* (1.10)-(1.11) *satisfies:*

$$\int_0^1 b(1,x)\rho(x)\mathrm{d}x = \iint_{(0,1)^2} K^\varepsilon(s_1,s_2)u(s_1)u(s_2)\mathrm{d}s_1\mathrm{d}s_2. \tag{3.1}$$

Proof. – For $\rho \in H^{-1}(0,1)$, let $\Phi \in L^2((0,1)^2)$ be the transposition solution (as in Definition 3) to:

$$\begin{cases} \Phi_t - \varepsilon\Phi_{xx} = 0 & \text{in } (0,1)\times(0,1), \\ \Phi(t,0) = 0 & \text{in } (0,1), \\ \Phi(t,1) = 0 & \text{in } (0,1), \\ \Phi(0,x) = \rho(x) & \text{in } (0,1). \end{cases} \tag{3.2}$$

Thanks to (1.11) and (2.9), we compute the final time projection as:

$$\begin{aligned} \int_0^1 b(1,x)\rho(x)\mathrm{d}x &= \int_0^1\int_0^1 \Phi(1-t,x)[-aa_x](t,x)\mathrm{d}x\mathrm{d}t \\ &= \frac{1}{2}\int_0^1\int_0^1 \Phi_x(1-t,x)a^2(t,x)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.3}$$

In order to express our projection directly using u, we need to eliminate a from (3.3). This can easily be done using an elementary solution of the heat system. Therefore, we introduce

G the solution to:

$$\begin{cases} G_t - \varepsilon G_{xx} = 0 & \text{in } (0,1)\times(0,1), \\ G(t,0) = 0 & \text{in } (0,1), \\ G(t,1) = 0 & \text{in } (0,1), \\ G(0,x) = 1 & \text{in } (0,1). \end{cases} \tag{3.4}$$

Using the initial condition $a(t=0,\cdot) \equiv 0$ from system (1.10), we can expand a as:

$$a(t,x) = \int_0^t G(t-s,x)u(s)\mathrm{d}s. \tag{3.5}$$

Pluging (3.5) into (3.3) yields:

(3.6)

$$\begin{aligned}\int_0^1 b(1,x)\rho(x)\mathrm{d}x &= \frac{1}{2}\int_0^1\int_0^1 \Phi_x(1-t)\left(\int_0^t G(t-s_1)u(s_1)\mathrm{d}s_1\right)\left(\int_0^t G(t-s_2)u(s_2)\mathrm{d}s_2\right)\mathrm{d}t \\ &= \frac{1}{2}\int_0^1\int_0^1 u(s_1)u(s_2)\left(\int_{s_1\vee s_2}^1\int_0^1 \Phi_x(1-t)G(t-s_1)G(t-s_2)\mathrm{d}t\right)\mathrm{d}s_1\mathrm{d}s_2.\end{aligned}$$

Finally, Equation (3.6) proves (3.1) with:

$$K^{\varepsilon}(s_1,s_2) = \frac{1}{2}\int_{s_1\vee s_2}^1\int_0^1 \Phi_x(1-t,x)G(t-s_1,x)G(t-s_2,x)\mathrm{d}x\mathrm{d}t. \tag{3.7}$$

Thus, we have proved Lemma 8 and we have a very precise description of the kernel that is involved. This kernel depends on the projection profile $\rho(x)$ by means of Φ defined in (3.2). This kernel also depends on the viscosity ε which is involded in the computation of both Φ and of the elementary solution G. Moreover, it is clear that K is a symmetric kernel and since all terms are bounded thanks to the maximum principle, we know that $K \in L^\infty$. In fact, K is even smoother as we will see later on. □

3.2. Choice of a projection profile

As we have seen in the introduction, a natural choice in the low viscosity setting would be $\rho(x) = x - \frac{1}{2}$. We think that our proof could be adapted to work with this profile. However, the computations are tough because it does not satisfy null boundary conditions. Thus, we are going to make a choice which is more intrinsic to the Burgers system.

For any fixed control value $\bar{u} \in \mathbb{R}$, we want to compute the associated steady state $(\bar{a}(x), \bar{b}(x))$ of systems (1.10) and (1.11). Thus, we solve the following system:

$$\begin{cases} -\varepsilon\bar{a}_{xx} = \bar{u} & \text{in } (0,1), \\ -\varepsilon\bar{b}_{xx} = -\bar{a}\bar{a}_x & \text{in } (0,1), \end{cases} \tag{3.8}$$

with boundary conditions $\bar{a}(0) = \bar{a}(1) = \bar{b}(0) = \bar{b}(1) = 0$. Integrating (3.8) with respect to x yields the following family of steady states:

$$\bar{a}(x) = \frac{1}{2\varepsilon}x(1-x)\bar{u} \quad \text{and} \quad \bar{b}(x) = \frac{1}{8\varepsilon^3}\left(\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} - \frac{x}{30}\right)\bar{u}^2. \tag{3.9}$$

Of course, $\bar{b}$ depends quadratically on $\bar{u}$. Thus Equation (3.9) gives the idea of considering:

$$\rho(x) = \frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} - \frac{x}{30}. \tag{3.10}$$

This choice of ρ may seem strange because is has been obtained using an infinite viscosity limit. However, since both ρ and ρ_{xx} satisfy null boundary conditions, the computations of the different kernel residues turn out to be easier. In the sequel, we assume that ρ is defined by (3.10).

3.3. Rough computation of the asymptotic kernel

In this paragraph, we apply Lemma 8 to compute the kernel associated to the choice of ρ given in (3.10). More specifically, we are interested in computing a rough approximation of K^{ε} when $\varepsilon \to 0$. This approximation will serve as a motivation for the following sections. We introduce the asymptotic kernel, defined on the square $(s_1, s_2) \in [0,1]^2$:

$$N(s_1, s_2) := (s_1 + s_2)^{\frac{3}{2}} - |s_1 - s_2|^{\frac{3}{2}}. \tag{3.11}$$

LEMMA 9. – *The following asymptotic expansion holds:*

$$K^{\varepsilon}(s_1, s_2) = \frac{\sqrt{\varepsilon}}{45\sqrt{\pi}} N\,(1 - s_1, 1 - s_2) + \mathcal{O}(\varepsilon), \tag{3.12}$$

in the sense that there exists $C > 0$ *such that, for any* $(s_1, s_2) \in [0,1]^2$ *and* $0 < \varepsilon \leq 1$, *there holds:*

$$\left| K^{\varepsilon}(s_1, s_2) - \frac{\sqrt{\varepsilon}}{45\sqrt{\pi}} N\,(1 - s_1, 1 - s_2) \right| \leq C\varepsilon. \tag{3.13}$$

Proof. – We use, without proof, the following asymptotic expansions for the elementary heat solutions Φ and G (we refer to Lemma 17 and Lemma 18, which prove more detailed asymptotic expansions):

$$\Phi_x(t, x) = \rho_x(x) + \mathcal{O}(\varepsilon), \tag{3.14}$$

$$G(t, x) = \operatorname{erf}\left(\frac{x}{\sqrt{4\varepsilon t}}\right) + \mathcal{O}(\varepsilon). \tag{3.15}$$

Equation (3.15) corresponds to the solution of a heat equation on the real line with an initial data equal to -1 for $x < 0$ and $+1$ for $x > 0$. Thus, it satisfies the boundary condition $G(t, 0) \equiv 0$ and serves as a boundary layer correction. We compute the integrand inside Equation (3.7):

(3.16)

$$\begin{aligned} A^{\varepsilon}(t, s_1, s_2) &:= \frac{1}{2} \int_0^1 \Phi_x(1 - t, x) G(t - s_1, x) G(t - s_2, x) \mathrm{d}x \\ &= \frac{1}{2} \int_0^1 \Phi_x(1 - t, x) \left(G(t - s_1, x) G(t - s_2, x) - 1 \right) \mathrm{d}x \qquad \text{since} \int \Phi_x = 0 \\ &= \int_0^{\frac{1}{2}} \Phi_x(1 - t, x) \left(G(t - s_1, x) G(t - s_2, x) - 1 \right) \mathrm{d}x \qquad \text{by parity.} \end{aligned}$$

Hence, using (3.14) and (3.15):

(3.17)

$$\begin{aligned} A^\varepsilon(t,s_1,s_2) &= \int_0^{\frac{1}{2}} \rho_x(x) \left(\operatorname{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_1)}} \right) \operatorname{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_2)}} \right) - 1 \right) \mathrm{d}x + \mathcal{O}(\varepsilon) \\ &= 2\sqrt{\varepsilon} \int_0^{\frac{1}{4\sqrt{\varepsilon}}} \rho_x\left(2\sqrt{\varepsilon}x\right) \left(\operatorname{erf}\left(\frac{x}{\sqrt{(t-s_1)}} \right) \operatorname{erf}\left(\frac{x}{\sqrt{(t-s_2)}} \right) - 1 \right) \mathrm{d}x + \mathcal{O}(\varepsilon) \\ &= -2\sqrt{\varepsilon}\rho_x(0) \int_0^{+\infty} \left(1 - \operatorname{erf}\left(\frac{x}{\sqrt{(t-s_1)}} \right) \operatorname{erf}\left(\frac{x}{\sqrt{(t-s_2)}} \right) \right) \mathrm{d}x + \mathcal{O}(\varepsilon). \end{aligned}$$

To carry on with the computation, we need the following integral calculus fact. For any $\alpha, \beta > 0$:

$$\int_0^{+\infty} \left(1 - \operatorname{erf}\left(\frac{x}{\sqrt{\alpha}} \right) \operatorname{erf}\left(\frac{x}{\sqrt{\beta}} \right) \right) \mathrm{d}x = \sqrt{\frac{\alpha+\beta}{\pi}}. \tag{3.18}$$

Equality (3.18) can be obtained from an explicit primitive for the integrand. Indeed, for any $X > 0$,

$$\begin{aligned} \int_0^{X} \left(1 - \operatorname{erf}\left(\frac{x}{\sqrt{\alpha}} \right) \operatorname{erf}\left(\frac{x}{\sqrt{\beta}} \right) \right) \mathrm{d}x =& X \left(1 - \operatorname{erf}\left(\frac{X}{\sqrt{\alpha}} \right) \operatorname{erf}\left(\frac{X}{\sqrt{\beta}} \right) \right) \\ & - \sqrt{\frac{\beta}{\pi}} \operatorname{erf}\left(\frac{X}{\sqrt{\alpha}} \right) \exp\left(-\frac{X^2}{\beta} \right) \\ & - \sqrt{\frac{\alpha}{\pi}} \operatorname{erf}\left(\frac{X}{\sqrt{\beta}} \right) \exp\left(-\frac{X^2}{\alpha} \right) \\ & + \sqrt{\frac{\alpha+\beta}{\pi}} \operatorname{erf}\left(\sqrt{\frac{1}{\alpha}+\frac{1}{\beta}} X \right). \end{aligned} \tag{3.19}$$

Equation (3.19) can be checked by differentiation. Taking its limit as $X \to +\infty$ yields (3.18). We return to the computation of the asymptotic kernel as $\varepsilon \to 0$. Using (3.7), (3.17), (3.18) and the value $\rho_x(0) = -\frac{1}{30}$, we obtain:

$$\begin{aligned} K^\varepsilon(s_1,s_2) &= \int_{s_1 \vee s_2}^1 A^\varepsilon(t,s_1,s_2)\mathrm{d}t \\ &= \frac{\sqrt{\varepsilon}}{15\sqrt{\pi}} \int_{s_1 \vee s_2}^1 \sqrt{(t-s_1)+(t-s_2)}\mathrm{d}t + \mathcal{O}(\varepsilon) \\ &= \frac{\sqrt{\varepsilon}}{45\sqrt{\pi}} \cdot \left[(2t-s_1-s_2)^{\frac{3}{2}} \right]_{s_1 \vee s_2}^1 + \mathcal{O}(\varepsilon) \\ &= \frac{\sqrt{\varepsilon}}{45\sqrt{\pi}} N(1-s_1, 1-s_2) + \mathcal{O}(\varepsilon). \end{aligned} \tag{3.20}$$

In Section 5, we prove that this asymptotic formula holds not only punctually, but also as a quadratic operator expansion. Indeed, we estimate the kernel residues between K^ε and $\sqrt{\varepsilon}N$. They turn out to be both small (with respect to ε) and smooth (with respect to the spaces on which they define continuous quadratic forms). □

4. Coercivity of the asymptotic kernel

In this section, our goal is to prove the coercivity of the kernel $N(x, y)$. This is a symmetric real-valued kernel defined on $(0, 1) \times (0, 1)$. Since no confusion is possible, we will use (x, y) instead of (s_1, s_2) for the variables of the kernel to lighten notations of this section. We will prove the following lemma.

LEMMA 10. – *There exists $\gamma > 0$ such that, for any $f \in L^2(0, 1)$:*

$$\int_0^1 \int_0^1 N(x, y) f(x) f(y) \mathrm{d}x \mathrm{d}y \geq \gamma \, \|F\|^2_{H^{-1/4}(0,1)}, \tag{4.1}$$

where F is the primitive of f such that $F(1) = 0$.

4.1. The asymptotic kernel is positive definite

This section uses results and notions from [11]. We will say that a matrix A is *positive semidefinite (psd)* when $\langle Ax|x\rangle \geq 0$ for any $x \in \mathbb{R}^m$. We will say that A is *positive definite* if the inequality is strict for any $x \neq 0$. We will say that A is *conditionally negative semidefinite (cnsd)* when $\langle Ax|x\rangle \leq 0$ for any x such that $\sum x_i = 0$. We will use similar definitions for operators, the condition $\sum x_i = 0$ being translated as $\int f = 0$ for functions.

LEMMA 11. – *For any $f \in L^2(0, 1)$,*

$$\int_0^1 \int_0^1 N(x, y) f(x) f(y) \mathrm{d}x \mathrm{d}y \geq 0. \tag{4.2}$$

Proof. – All necessary arguments can be found in [11, Chapter 3]. Indeed, the kernel $-(x + y)^{3/2}$ is *cnsd.* as is proved in [11, Corollary 2.11]. Moreover, the kernel $|x - y|^{3/2}$ is also *cnsd.* (see [11, Remark 1.10] and [11, Corollary 2.10]). Hence, letting:

$$\psi(x, y) = -(x + y)^{3/2} + |x - y|^{3/2} \tag{4.3}$$

defines a *cnsd.* kernel. Thus, since:

$$N(x, y) = \psi(x, 0) + \psi(y, 0) - \psi(x, y) - \psi(0, 0), \tag{4.4}$$

this kernel is *psd.* thanks to [11, Lemma 2.1]. This proves inequality (4.2). □

Even though it is true that the kernels involved in the proof of Lemma 11 are strictly negative (or positive), we cannot adapt the proof to prove that N is definite. Indeed, Mercer's theorem (which allows us to take the step from matrices to continuous kernels) doesn't preserve strict inequalities. Thus, we have to look for another proof.

4.2. Some insight and facts

Our main insight is that the kernel N is made up of two parts. The most singular one should explain its behavior. Indeed, kernels which can be expressed as a function $r(|x-y|)$ have been extensively studied. For example, [53] and [47] prove asymptotic formulas for the eigenvalues of the $-|x-y|^{3/2}$ part of our kernel:

$$\lambda_n \sim \frac{3\sqrt{2}}{4\pi^2}\left(\frac{1}{n}\right)^{\frac{5}{2}}. \tag{4.5}$$

Moreover, some papers have also studied the eigenvectors of such kernels. For example, in [44], one can find asymptotic developments for eigenvectors of kernels of the form $|x-y|^{-\alpha}$, where $\alpha \in (0,1)$.

Combining the insight that the eigenvectors of N should asymptotically behave like oscillating sinuses and Formula (4.5), we expect that it should be possible to prove Lemma 10 by means of such an asymptotic study. However, we have not been able to prove it using this method. Instead, we give below a proof based on Riesz potentials.

4.3. Highlighting the singular part of the asymptotic kernel

The kernel $N(x,y)$ is rather smooth. In order to prove its coercivity, we will need to isolate its most singular part. In the following lemma, we use integration by parts twice to show that studying the behavior of N is equivalent to studying a more singular kernel. By choosing adequately the primitive, we show that we can also cancel boundary terms.

LEMMA 12. – *Let $f \in L^2(0,1)$ and F be the primitive of f such that $F(1)=0$. Then:*

$$(Nf,f) = \frac{3}{4}\int_0^1\int_0^1 \left((x+y)^{-\frac{1}{2}} + |x-y|^{-\frac{1}{2}}\right) F(x)F(y)\mathrm{d}x\mathrm{d}y. \tag{4.6}$$

Proof. – Let $f \in L^2(0,1)$ and F be the primitive of f such that $F(1)=0$. We start with:

(4.7)

$$\begin{aligned}
-\int_0^1\int_0^1 & |x-y|^{\frac{3}{2}} f(x)f(y)\mathrm{d}x\mathrm{d}y \\
&= -\int_0^1 f(x)\left\{\int_0^x (x-y)^{\frac{3}{2}} f(y)\mathrm{d}y + \int_x^1 (y-x)^{\frac{3}{2}} f(y)\mathrm{d}y\right\}\mathrm{d}x \\
&= F(0)\int_0^1 x^{\frac{3}{2}} f(x)\mathrm{d}x + \frac{3}{2}\int_0^1\int_0^1 |x-y|^{\frac{1}{2}}\,\mathrm{sgn}(y-x) f(x)F(y)\mathrm{d}x\mathrm{d}y \\
&= F(0)\int_0^1 x^{\frac{3}{2}} f(x)\mathrm{d}x + \frac{3}{2}\int_0^1 F(y)\left\{\int_0^y (y-x)^{\frac{1}{2}} f(x)\mathrm{d}x - \int_y^1 (x-y)^{\frac{1}{2}} f(x)\mathrm{d}x\right\}\mathrm{d}y \\
&= F(0)\int_0^1 \left(x^{\frac{3}{2}} f(x) - \frac{3}{2}x^{\frac{1}{2}}F(x)\right)\mathrm{d}x + \frac{3}{4}\int_0^1\int_0^1 |x-y|^{-\frac{1}{2}} F(x)F(y)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

We continue with the other half of the kernel $N(x, y)$:

(4.8)
$$\begin{aligned}\int_0^1 \int_0^1 & (x+y)^{\frac{3}{2}} f(x) f(y) \mathrm{d}x \mathrm{d}y \\ & = -F(0) \int_0^1 x^{\frac{3}{2}} f(x) \mathrm{d}x - \frac{3}{2} \int_0^1 \int_0^1 (x+y)^{\frac{1}{2}} f(x) F(y) \mathrm{d}x \mathrm{d}y \\ & = F(0) \int_0^1 \left(\frac{3}{2} x^{\frac{1}{2}} F(x) - x^{\frac{3}{2}} f(x) \right) \mathrm{d}x + \frac{3}{4} \int_0^1 \int_0^1 (x+y)^{-\frac{1}{2}} F(x) F(y) \mathrm{d}x \mathrm{d}y.\end{aligned}$$

Summing the two previous equalities proves Lemma 12. □

4.4. Riesz potential and fractional Laplacian

In this section, we focus on the most singular part of the kernel. We recognize a Riesz potential of order $\frac{1}{2}$. Using the fractional Laplacian, we can compute the quantity as a usual norm.

LEMMA 13. – *There exists $C > 0$ such that, for any $h \in L^2(0,1)$,*

$$\int_0^1 \int_0^1 |x-y|^{-\frac{1}{2}} h(x) h(y) \mathrm{d}x \mathrm{d}y \geq C \, \|h\|^2_{H^{-1/4}(0,1)} . \tag{4.9}$$

Proof. – We have

$$\begin{aligned}\int_0^1 \int_0^1 |x-y|^{-\frac{1}{2}} h(x) h(y) \mathrm{d}x \mathrm{d}y & = \int_{\mathbb{R}} \int_{\mathbb{R}} |x-y|^{-\frac{1}{2}} h(x) h(y) \mathrm{d}x \mathrm{d}y \\ & = \left((-\Delta)^{-1/4} h, h \right) \\ & = \left((-\Delta)^{-1/8} h, (-\Delta)^{-1/8} h \right) \\ & = \left\| (-\Delta)^{-1/8} h \right\|^2_{L^2} \\ & = \|h\|^2_{\dot{H}^{-1/4}} \\ & \geq \|h\|^2_{H^{-1/4}} .\end{aligned} \tag{4.10}$$

More information on such techniques can be found in [49] or posterior works. □

4.5. Positivity of the smooth part

To conclude the proof of Lemma 10, we show that the *smooth* part of our kernel is of positive type. We could also rely on regularity arguments to prove that its behavior doesn't modify the asymptotic behavior of eigenvectors and eigenvalues of the singular part.

LEMMA 14. – *For any $h \in L^2(0,1)$,*

$$\int_0^1 \int_0^1 (x+y)^{-\frac{1}{2}} h(x) h(y) \mathrm{d}x \mathrm{d}y \geq 0. \tag{4.11}$$

Proof. – We use definitions and theorems found in [11, Chapter 3]. Thanks to [11, Result 1.9, page 69], the kernel given on $(0,1)^2$ by $(x,y) \mapsto x+y$ is *conditionally negative semidefinite (cnsd)*. Hence, using [11, Corollary 2.10, page 78], the kernel given by $(x,y) \mapsto \sqrt{x+y}$ is also *cnsd*. Eventually, [11, exercise 2.21, page 80] proves that the kernel $(x,y) \mapsto 1/\sqrt{x+y}$ is *positive semidefinite*. This means that, for any $n > 0$ and any $c_1, \dots c_n \in \mathbb{R}$ and any $x_1, \dots x_n \in (0,1)$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{c_i c_j}{\sqrt{x_i + x_j}} \geq 0. \tag{4.12}$$

Using Mercer's theorem (see [43]), we deduce that, for any $h \in L^2(0,1)$,

$$\int_0^1 \int_0^1 (x+y)^{-\frac{1}{2}} h(x)h(y) \mathrm{d}x\mathrm{d}y \geq 0. \tag{4.13}$$

□

Combined with Lemma 12 and Lemma 13, Lemma 14 concludes the proof of Lemma 10.

5. Exact computation of the kernel and estimation of residues

In this section, we give a detailed and rigorous expansion of the main kernel K^ε. Our goal is to be able to estimate with precision the size and the regularity of all the residues that build up the difference between the asymptotic kernel and the true kernel. As above, we write:

$$K^\varepsilon(s_1, s_2) = \int_{s_1 \vee s_2}^{1} A(t, s_1, s_2) \mathrm{d}t, \quad \text{where} \tag{5.1}$$

$$A(t, s_1, s_2) = \int_0^{\frac{1}{2}} \Phi_x(1-t, x) G(t-s_1, x) G(t-s_2, x) \mathrm{d}x. \tag{5.2}$$

In Equations (5.1) and (5.2), it is implicit that A, Φ_x and G depend on ε. Moreover, in Equation (5.2), we use the fact that G and Φ_x are even to write the integral over $x \in \left(0, \frac{1}{2}\right)$. This breaks the symmetry but will allow us to use a one-sided expansion of G, thereby focusing on its behavior near $x = 0$.

5.1. Regularity of weakly singular integral operators

We know that the asymptotic kernel N is coercive with respect to the $H^{-5/4}$ norm of the control u. Thus, in order for the full kernel to remain coercive for $\varepsilon > 0$, we need to prove that the residues can be bounded with the same norm. In this paragraph, we give conditions on a kernel residue L implying:

$$\forall u \in L^2(0,1), \quad |\langle Lu, u \rangle| \lesssim \|U\|^2_{H^{-1/4}(0,1)}, \tag{5.3}$$

where U is the primitive of u such that $U(0) = 0$. In the following paragraphs, we will check that these conditions are satisfied by our residues. We start with the following lemma, which allows us to express $\langle Lu, u \rangle$ directly as a function of U.

LEMMA 15. – *Let* Γ *be the triangular domain* $\{(x,y)\in(0,1)\times(0,1)$*, s.t.* $x\le y\}$*. Let* $L\in W^{2,1}(\Gamma)$*. We see* L *as the restriction to* Γ *of a symmetric kernel on* $(0,1)\times(0,1)$ *that is smooth on each triangle but not necessarily across the first diagonal. Assume that* $L(\cdot,1)\equiv 0$*. Let* $u\in L^2(0,1)$ *and* U *be the primitive of* u *such that* $U(0)=0$*. Then:*

$$\begin{aligned}(5.4)\qquad \int_\Gamma L(x,y)u(x)u(y)\mathrm{d}x\mathrm{d}y \\ = \int_\Gamma \partial_{12}L(x,y)U(x)U(y)\mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_0^1 \left(\partial_1 L-\partial_2 L\right)(x,x)U^2(x)\mathrm{d}x.\end{aligned}$$

In (5.4)*,* $\partial_1 L$ *and* $\partial_2 L$ *are evaluated on the first diagonal and must thus be computed using points within* Γ*.*

Proof. – We use integration by parts and the boundary conditions $U(0)=0$ and $L(\cdot,1)=0$.

(5.5)

$$\begin{aligned}\int_\Gamma L(x,y)u(x)u(y)\mathrm{d}x\mathrm{d}y &= \int_0^1 u(x)\int_x^1 L(x,y)u(y)\mathrm{d}y\mathrm{d}x \\ &= \int_0^1 u(x)\left([L(x,y)U(y)]_x^1 - \int_x^1 \partial_2 L(x,y)U(y)\mathrm{d}y\right)\mathrm{d}x \\ &= -\int_0^1 L(x,x)U(x)u(x)\mathrm{d}x - \int_0^1 U(y)\int_0^y \partial_2 L(x,y)u(x)\mathrm{d}x \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\{L(x,x)\}\cdot\frac{U^2}{2}(x)\mathrm{d}x \\ &\quad -\int_0^1 U(y)\left([U(x)\partial_2 L(x,y)]_0^y - \int_0^y \partial_{12}L(x,y)U(x)\mathrm{d}x\right)\mathrm{d}y \\ &= \int_\Gamma \partial_{12}L(x,y)U(x)U(y)\mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_0^1 \left(\partial_1 L-\partial_2 L\right)(x,x)U^2(x)\mathrm{d}x.\end{aligned}$$

Equation chain (5.5) concludes the proof of Equation (5.4). □

Equation (5.4) includes a boundary term evaluated on the diagonal, which looks like the L^2 norm of U. This would forbid us to prove any estimate like (5.3). However, all our kernel residues satisfy the condition $\partial_1 L-\partial_2 L=0$ along the diagonal and this term thus vanishes. Hence, our task is to check that the new kernel $\partial_{12}L$ generates a bounded quadratic form on $H^{-1/4}(0,1)$.

LEMMA 16. – *Let* L *be a continuous function defined on* $\Omega=\{(x,y)\in(0,1)\times(0,1)$*, s.t.* $x\neq y\}$*. Assume that there exists* $\kappa>0$ *and* $\frac{1}{2}<\delta\le 1$*, such that, on* Ω*:*

$$(5.6)\qquad |L(x,y)|\le\kappa|x-y|^{-\frac{1}{2}},$$

$$(5.7)\qquad |L(x,y)-L(x',y)|\le\kappa|x-x'|^{\delta}|x-y|^{-\frac{1}{2}-\delta},\quad \textit{for } |x-x'|\le\frac{1}{2}|x-y|,$$

$$(5.8)\qquad |L(x,y)-L(x,y')|\le\kappa|y-y'|^{\delta}|x-y|^{-\frac{1}{2}-\delta},\quad \textit{for } |y-y'|\le\frac{1}{2}|x-y|.$$

Then L defines a continuous quadratic form on $H^{-1/4}(0,1)$. Moreover, there exists a constant $C(\delta)$ depending only on δ (and not on L) such that, for any $U \in L^2(0,1)$:

$$|\langle LU, U\rangle| \leq C(\delta)\kappa |U|^2_{H^{-1/4}(0,1)}. \tag{5.9}$$

This technical lemma is very important for our proof because it gives a quantitative estimate, through κ, of the action of kernels against controls. This lemma can be deduced from the works of Torres [51] and Youssfi [54]. We give a proof skeleton in the appendix. The starting point is to prove that a kernel satisfying estimates (5.6), (5.7) and (5.8) defines a weakly singular integral operator, which is continuous from $H^{-1/4}$ to $H^{+1/4}$. Indeed, such kernels are smoother then standard Calderón-Zygmund operators and it is reasonable to expect that they exhibit some smoothing properties.

We end this section with two useful formulas. Let $a : (0,1)^3 \to \mathbb{R}$ be a function such that $a(t,s_1,s_2) = a(t,s_2,s_1)$. We consider the kernel generated by a:

$$L(s_1,s_2) = \int_{s_1 \vee s_2}^1 a(t,s_1,s_2)\mathrm{d}t. \tag{5.10}$$

Lemma 15 can be applied to such kernels because they satisfy the condition $L(\cdot,1) \equiv 0$. We compute:

$$\partial_1 L(s,s) - \partial_2 L(s,s) = a(s,s,s), \qquad \text{for } s \in (0,1), \tag{5.11}$$

$$\partial_{12} L(s_1,s_2) = -\partial_{s_1} a(s_2,s_1,s_2) + \int_{s_2}^T \partial_{s_1}\partial_{s_2} a(t,s_1,s_2)\mathrm{d}t, \quad \text{for } s_1 < s_2. \tag{5.12}$$

Formulas (5.11) and (5.12) will be used extensively in the following sections. Moreover, as soon as $a(s,s,s) \equiv 0$, we see that the boundary term $\partial_1 L - \partial_2 L$ vanishes.

5.2. Asymptotic expansion of the main kernel

In this section, we make our rough expansions more precise. Therefore we decompose G and Φ using the same first order terms as for the heuristic, but this time we introduce and compute the residues.

5.2.1. *Expansion of the elementary controlled heat solution.* – Recall that we only need to approximate G for $x \in (0,1/2)$. Keeping our approximation introduced in (3.15), we expand G as:

$$G(t,x) = \operatorname{erf}\left(\frac{x}{\sqrt{4\varepsilon t}}\right) + H(t,x), \tag{5.13}$$

where $H \in \mathcal{C}^\infty((0,1)\times(0,1/2))$ is the solution to:

$$\begin{cases} H_t - \varepsilon H_{xx} = 0 & \text{in } (0,1)\times(0,1/2), \\ H(t,0) = 0 & \text{in } (0,1), \\ H_x(t,1/2) = \sigma(\varepsilon t) & \text{in } (0,1), \\ H(0,x) = 0 & \text{in } (0,1/2), \end{cases} \tag{5.14}$$

where the source term σ comes from the boundary condition $G_x(t, 1/2) = 0$ and balances out the trace of the erf() part:

$$\sigma(s) = -\frac{\partial}{\partial x}\left[\operatorname{erf}\left(\frac{x}{\sqrt{4s}}\right)\right]\Big|_{x=\frac{1}{2}} = -\frac{1}{\sqrt{s\pi}}\exp\left(-\frac{1}{16s}\right). \tag{5.15}$$

LEMMA 17. – *Let $0 < \gamma < \frac{1}{16}$. There exists $C(\gamma) > 0$ such that:*

$$\|H_t\|_\infty + \|H_{tx}\|_\infty + \|H_{tt}\|_\infty + \|H_{ttx}\|_\infty \leq C(\gamma)e^{-\gamma/\varepsilon}. \tag{5.16}$$

Proof. – This lemma is due to the exponentially decaying factor within the source term σ defined by (5.15), which allows as many differentiations with respect to x or t as needed to be done. Estimate (5.16) could in fact be derived for further derivatives. Let us give a sketch of proof.

First, $H^{(3)} := H_{ttt}$ is the solution to a similar heat system as (5.14) with the boundary condition $H^{(3)}_x(t, 1/2) = \varepsilon^3\sigma^{(3)}(\varepsilon t)$. We can convert this boundary condition into a source term by writing $H^{(3)}(t, x) = x\varepsilon^3\sigma^{(3)}(\varepsilon t) + \tilde{H}^{(3)}$, where $\tilde{H}^{(3)}$ is now the solution to a heat equation with homogeneous mixed boundary conditions and a source term $-x\varepsilon^4\sigma^{(4)}(\varepsilon t)$. Applying the maximum principle yields an estimate of the form $\|\tilde{H}^{(3)}\|_\infty \leq C(\gamma)e^{-\gamma/\varepsilon}$. Since $\varepsilon H_{ttxx} = H^{(3)}$, we obtain an L^∞ estimate of the same form for H_{ttxx}. By integration with respect to time and space, we obtain (5.16). □

5.2.2. *Expansion of the elementary projection profile heat solution.* – Guided by our rough computations, we decompose $\Phi \in X_1$, the solution to (3.2) as:

$$\Phi(t, x) = \rho(x) + \varepsilon\phi(t, x). \tag{5.17}$$

Thus, we introduce the partial differential equation satisfied by $\phi \in X_1$:

$$\begin{cases} \phi_t - \varepsilon\phi_{xx} = \rho_{xx} & \text{in } (0, 1) \times (0, 1), \\ \phi(t, 0) = 0 & \text{in } (0, 1), \\ \phi(t, 1) = 0 & \text{in } (0, 1), \\ \phi(0, x) = 0 & \text{in } (0, 1). \end{cases} \tag{5.18}$$

LEMMA 18. – *The following estimates hold:*

$$\|\Phi_x\|_\infty \lesssim 1, \tag{5.19}$$

$$\|\phi_x\|_\infty \lesssim 1, \tag{5.20}$$

$$\|\Phi_{tx}\|_\infty = \|\varepsilon\phi_{tx}\|_\infty \lesssim \varepsilon. \tag{5.21}$$

Proof. – Estimates (5.19), (5.20) and (5.21) can be proved using a Fourier series decomposition for heat equations. As an example, let us prove (5.21). We introduce the basis $e_n(x) = \sqrt{2}\sin(n\pi x)$. Since ϕ_t is the solution to a heat equation with initial data $\rho_{xx} \in H^1_0$, we have:

$$\phi_t(t, x) = \sum_{n=1}^{+\infty} e^{-\varepsilon n^2\pi^2 t}\langle\rho_{xx}, e_n\rangle e_n(x). \tag{5.22}$$

Thanks to the choice of ρ in (3.10), we have $\rho_{xx}(0) = \rho_{xx}(1) = 0$. Thus,

$$\langle \rho_{xx}, e_n \rangle = -\frac{1}{n^2\pi^2}\langle \rho_{xxxx}, e_n \rangle = \frac{12\sqrt{2}}{n^3\pi^3}\left((-1)^n - 1\right) = \mathcal{O}\left(\frac{1}{n^3}\right). \tag{5.23}$$

Combining Equations (5.22) and (5.23) yields:

$$\|\phi_{tx}\|_\infty \leq \sum_{n=1}^{+\infty} n\pi |\langle \rho_{xx}, e_n \rangle| \lesssim \sum_{n=1}^{+\infty} \frac{1}{n^2}. \tag{5.24}$$

Equation (5.24) concludes the proof of (5.21). A similar method can be applied to prove (5.19) and (5.20). □

5.2.3. *Five stages expansion of the full kernel.* – Using expansions (5.13) and (5.17), and the fact that $\int \Phi_x = 0$, we break down the generator $A(t, s_1, s_2)$ into 6 smaller kernel generators, A_1 through A_6, defined by:

$$A_1(t,s_1,s_2) = \int_0^{\frac{1}{2}} \rho_x(0)\left(\mathrm{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_1)}}\right)\mathrm{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_2)}}\right) - 1\right)\mathrm{d}x, \tag{5.25}$$

$$A_2(t,s_1,s_2) = \int_0^{\frac{1}{2}} (\rho_x(x) - \rho_x(0))\left(\mathrm{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_1)}}\right)\mathrm{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_2)}}\right) - 1\right)\mathrm{d}x, \tag{5.26}$$

$$A_3(t,s_1,s_2) = \int_0^{\frac{1}{2}} \varepsilon\phi_x(1-t,x)\left(\mathrm{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_1)}}\right)\mathrm{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_2)}}\right) - 1\right)\mathrm{d}x, \tag{5.27}$$

$$A_4(t,s_1,s_2) = \int_0^{\frac{1}{2}} \Phi_x(1-t,x)H(t-s_1,x)\mathrm{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_2)}}\right)\mathrm{d}x, \tag{5.28}$$

$$A_5(t,s_1,s_2) = \int_0^{\frac{1}{2}} \Phi_x(1-t,x)H(t-s_2,x)\cdot\mathrm{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_1)}}\right)\mathrm{d}x, \tag{5.29}$$

$$A_6(t,s_1,s_2) = \int_0^{\frac{1}{2}} \Phi_x(1-t,x)H(t-s_1,x)H(t-s_2,x)\mathrm{d}x. \tag{5.30}$$

It can be checked that A defined in (5.2) is indeed equal to the sum of A_1 through A_6. For each $1 \leq i \leq 6$, we consider the associated kernel generated by A_i:

$$K_i(t,s_1,s_2) = \int_{s_1 \vee s_2}^{T} A_i(t,s_1,s_2)\mathrm{d}t. \tag{5.31}$$

A first remark is that, for each $1 \leq i \leq 6$, $A_i(s,s,s) \equiv 0$ on $(0,1)$. Thus, Equation (5.11) tells us that there will be no boundary term involving $|u|_{H^{-1}}$.

5.2.4. *Proof methodology*. – The six following paragraphs are dedicated to estimates for K_1 through K_6. In order to organize the computations that will be carried out for each of these six kernels, we introduce the notations:

$$T_i(s_1, s_2) = \frac{\partial A_i}{\partial s_1}(t, s_1, s_2)|_{t=s_2}, \tag{5.32}$$

$$Q_i(t, s_1, s_2) = \frac{\partial^2 A_i}{\partial s_1 \partial s_2}(t, s_1, s_2), \tag{5.33}$$

$$R_i(s_1, s_2) = \int_{s_2}^{1} Q_i(t, s_1, s_2)\mathrm{d}t. \tag{5.34}$$

Using Formula (5.12), $\partial_{12} K_i = R_i - T_i$. Therefore, thanks to Lemma 16 and Lemma 15, we need to prove that each T_i and each R_i satisfies the conditions (5.6), (5.7) and (5.8). For a kernel L, we will denote $\kappa(L)$ the associated constant in Lemma 16. In the following paragraphs, we investigate the behavior of $\kappa(\partial_{12} K_i)$ with respect to ε. We end this paragraph with a useful estimation lemma.

LEMMA 19. – *For any $k > 0$ there exists $c_k > 0$ such that, for any $\lambda > 0$, for any $\varepsilon > 0$,*

$$\int_0^{+\infty} x^k \exp\left(-\frac{x^2}{4\varepsilon\lambda}\right) \mathrm{d}x \le c_k\, (\varepsilon\lambda)^{\frac{k+1}{2}}. \tag{5.35}$$

Proof. – Use a change of variables introducing $\tilde{x} = x/\sqrt{4\varepsilon\lambda}$. □

5.3. Handling the first kernel

The kernel K_1 contains the main coercive part of K^ε discovered in Section 3. Starting from its definition in (5.25), we decompose it using a scaling on x:

(5.36)

$$\begin{aligned} A_1(t, s_1, s_2) &= \rho_x(0) \int_0^{\frac{1}{2}} \left(\operatorname{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_1)}}\right) \operatorname{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_2)}}\right) - 1 \right) \mathrm{d}x \\ &= \frac{\sqrt{\varepsilon}}{15} \int_0^{\frac{1}{4\sqrt{\varepsilon}}} \left(1 - \operatorname{erf}\frac{x}{\sqrt{\alpha}} \operatorname{erf}\frac{x}{\sqrt{\beta}}\right) \mathrm{d}x \\ &= \frac{\sqrt{\varepsilon}}{15} \int_0^{+\infty} \left(1 - \operatorname{erf}\frac{x}{\sqrt{\alpha}} \operatorname{erf}\frac{x}{\sqrt{\beta}}\right) \mathrm{d}x - \frac{\sqrt{\varepsilon}}{15} \int_{\frac{1}{4\sqrt{\varepsilon}}}^{+\infty} \left(1 - \operatorname{erf}\frac{x}{\sqrt{\alpha}} \operatorname{erf}\frac{x}{\sqrt{\beta}}\right) \mathrm{d}x, \end{aligned}$$

where we introduce the shorthand notations:

$$\alpha := t - s_1, \tag{5.37}$$

$$\beta := t - s_2, \tag{5.38}$$

which we will also use in the sequel. The first integral in (5.36) gives rise to the main coercive part of the kernel and has already been computed exactly in Lemma 9. The second part in (5.36) is a residue and has to be taken care of. We introduce $\tilde{A}_1$ defined as:

$$\tilde{A}_1(t, s_1, s_2) := \int_{\frac{1}{4\sqrt{\varepsilon}}}^{+\infty} \left(\operatorname{erf}\left(\frac{x}{\sqrt{\alpha}}\right) \operatorname{erf}\left(\frac{x}{\sqrt{\beta}}\right) - 1 \right) \mathrm{d}x. \tag{5.39}$$

Therefore, Equation (5.36) yields:

$$K_1(s_1,s_2) = \frac{\sqrt{\varepsilon}}{45\sqrt{\pi}} N(1-s_1,1-s_2) - \frac{\sqrt{\varepsilon}}{15}\tilde{K}_1(s_1,s_2). \tag{5.40}$$

LEMMA 20. – *There exist $c>0$ and $\gamma>0$ such that, for any $\varepsilon>0$,*

$$\kappa(\partial_{12}\tilde{K}_1) \leq c\cdot \exp\left(-\frac{\gamma}{\varepsilon}\right), \tag{5.41}$$

where $\kappa(\partial_{12}\tilde{K}_1)$ is the constant associated to the weakly singular integral operator $\tilde{K}_1$ in Lemma 16.

Proof. – Recalling notations (5.32), (5.33) and (5.34), we compute:

$$\tilde{T}_1(s_1,s_2) = \left(\partial_{s_1}\tilde{A}_1\right)|_{t=s_2} = \frac{1}{\sqrt{\pi}}\Delta^{-3/2}\int_{\frac{1}{4\sqrt{\varepsilon}}}^{+\infty} x\exp\left(-\frac{x^2}{\Delta}\right)\mathrm{d}x, \tag{5.42}$$

$$\tilde{Q}_1(t,s_1,s_2) = \partial_{s_1}\partial_{s_2}\tilde{A}_1(t,s_1,s_2) = \frac{1}{\pi}(\alpha\beta)^{-3/2}\int_{\frac{1}{4\sqrt{\varepsilon}}}^{+\infty} x^2\exp\left(-x^2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\right)\mathrm{d}x, \tag{5.43}$$

$$\tilde{R}_1(s_1,s_2) = \int_{s_2}^{1}\tilde{Q}_1(t,s_1,s_2) = \frac{1}{\pi}\int_{s_2}^{1}(\alpha\beta)^{-3/2}\int_{\frac{1}{4\sqrt{\varepsilon}}}^{+\infty} x^2\exp\left(-x^2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\right)\mathrm{d}x\mathrm{d}t, \tag{5.44}$$

where we introduce $\Delta = s_2 - s_1$, that will also be used in the sequel. We claim that both $\tilde{T}_1$ and $\tilde{R}_1$ are $\mathcal{C}^\infty$ kernels on $(0,1)\times(0,1)$. Moreover, all their derivatives are bounded by $e^{-\gamma/\varepsilon}$ for any $\gamma < 1/16$, thanks to the exponential terms in (5.42) and (5.44). We omit the detailed computations in order to focus on the tougher kernels. □

5.4. Handling the second kernel

Using the definition of ρ given in (3.10), we rewrite A_2 defined in (5.26) as:

$$\begin{aligned} A_2(t,s_1,s_2) &= \int_0^{\frac{1}{2}} (\rho_x(x)-\rho_x(0))\,\mathrm{erf}\left(\frac{x}{\sqrt{4\varepsilon\alpha}}\right)\mathrm{erf}\left(\frac{x}{\sqrt{4\varepsilon\beta}}\right)\mathrm{d}x \\ &= \int_0^{\frac{1}{2}} x^2(x-1)^2\mathrm{erf}\left(\frac{x}{\sqrt{4\varepsilon\alpha}}\right)\mathrm{erf}\left(\frac{x}{\sqrt{4\varepsilon\beta}}\right)\mathrm{d}x. \end{aligned} \tag{5.45}$$

First part. – Remembering that $\mathrm{erf}(+\infty)=1$, we consider the first order derivative:

$$T_2(s_1,s_2) = \left(\partial_{s_1}A_2\right)|_{t=s_2} = \frac{1}{2\sqrt{\pi\varepsilon}}\Delta^{-3/2}\int_0^{\frac{1}{2}} x^3(x-1)^2\exp\left(-\frac{x^2}{4\varepsilon\Delta}\right)\mathrm{d}x. \tag{5.46}$$

Using Lemma 19 and differentiating gives:

$$\begin{aligned} |T_2(s_1,s_2)| &\lesssim \varepsilon^{3/2}\Delta^{1/2}, \\ \left|\partial_{s_1}T_2(s_1,s_2)\right| &\lesssim \varepsilon^{3/2}\Delta^{-1/2}, \\ \left|\partial_{s_2}T_2(s_1,s_2)\right| &\lesssim \varepsilon^{3/2}\Delta^{-1/2}. \end{aligned} \tag{5.47}$$

Estimates (5.47) prove that $\kappa(T_2) \lesssim \varepsilon^{3/2}$. In fact, T_2 is smoother than the weakly singular integral operators studied in Lemma 16, since such operators allow degeneracy like $\Delta^{-1/2}$ along the diagonal. Moreover, we proved that T_2 is Lipschitz continuous, whereas Lemma 16 only requires $\mathcal{C}^p$ with $p > \frac{1}{2}$.

Second part. – Now we consider the second order derivative. Let us compute:

$$Q_2(t,s_1,s_2) = \partial_{s_1}\partial_{s_2} A_2(t,s_1,s_2) = \frac{1}{4\pi\varepsilon}(\alpha\beta)^{-3/2}\int_0^{\frac{1}{2}} x^4(x-1)^2 \exp\left(-\frac{x^2}{4\varepsilon}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\right)\mathrm{d}x. \tag{5.48}$$

Thanks to Lemma 19, we estimate the size of Q_2:

$$|Q_2(t,s_1,s_2)| \lesssim \varepsilon^{3/2}(\alpha\beta)^{-3/2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^{-5/2} = \frac{\varepsilon^{3/2}\alpha\beta}{(\alpha+\beta)^{5/2}}. \tag{5.49}$$

Writing $\alpha = \Delta + \tau$ and $\beta = \tau$, we can estimate:

$$|R_2(s_1,s_2)| = \left|\int_{s_2}^1 Q_2(t,s_1,s_2)\mathrm{d}t\right| \lesssim \varepsilon^{3/2}\int_0^1 \frac{\tau(\Delta+\tau)}{(\Delta+2\tau)^{5/2}}\mathrm{d}\tau \lesssim \varepsilon^{3/2}\Delta^{-1/2}. \tag{5.50}$$

We should now move on to computing $\partial_{s_1} R_2$ and $\partial_{s_2} R_2$, to establish the missing estimates on R_2. However, the computations associated to R_2 are very similar to the ones that we carry out for R_3. Since R_3 is a little harder, we skip the proof for R_2 and refer the reader to the proof of R_3, which is fully detailed in the next paragraph. Therefore, we claim that:

$$\kappa(\partial_{12}K_2) \lesssim \varepsilon^{3/2}. \tag{5.51}$$

5.5. Handling the third kernel

In this section, we consider:

$$A_3(t,s_1,s_2) = \varepsilon\int_0^{\frac{1}{2}} \phi_x(1-t,x)\left(\operatorname{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_1)}}\right)\operatorname{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_2)}}\right)-1\right)\mathrm{d}x. \tag{5.27}$$

First part. – Remembering that $\operatorname{erf}(+\infty) = 1$, we consider the first order derivative:

$$T_3(s_1,s_2) := \left(\partial_{s_1} A_3\right)|_{t=s_2} = \frac{\sqrt{\varepsilon}}{2\sqrt{\pi}}\Delta^{-3/2}\int_0^{\frac{1}{2}} \phi_x(1-s_2,x)\cdot x\exp\left(-\frac{x^2}{4\varepsilon\Delta}\right)\mathrm{d}x. \tag{5.52}$$

Thanks to Lemma 18 and Lemma 19, we have:

$$|T_3(s_1,s_2)| \lesssim \varepsilon^{1/2}\Delta^{-3/2}\|\phi_x\|_\infty\cdot\int_0^{\frac{1}{2}} x\exp\left(-\frac{x^2}{4\varepsilon\Delta}\right)\mathrm{d}x \lesssim \varepsilon^{3/2}\Delta^{-1/2}. \tag{5.53}$$

Moreover,

$$\begin{aligned} \left|\partial_{s_1} T_3(s_1,s_2)\right| \lesssim\ & \varepsilon^{1/2}\Delta^{-5/2}\|\phi_x\|_\infty\cdot\int_0^{\frac{1}{2}} x\exp\left(-\frac{x^2}{4\varepsilon\Delta}\right)\mathrm{d}x \\ & + \varepsilon^{1/2}\Delta^{-3/2}\|\phi_x\|_\infty\cdot\int_0^{\frac{1}{2}} \frac{x^3}{4\varepsilon\Delta^2}\exp\left(-\frac{x^2}{4\varepsilon\Delta}\right)\mathrm{d}x \\ \lesssim\ & \varepsilon^{3/2}\Delta^{-3/2}. \end{aligned} \tag{5.54}$$

and

(5.55)
$$\begin{aligned}\left|\partial_{s_2} T_3(s_1,s_2)\right| \lesssim\ & \varepsilon^{1/2}\Delta^{-3/2}\left\|\phi_{xt}\right\|_\infty \cdot \int_0^{\frac12} x\exp\left(-\frac{x^2}{4\varepsilon\Delta}\right)\mathrm{d}x \\ & + \varepsilon^{1/2}\Delta^{-5/2}\left\|\phi_{x}\right\|_\infty \cdot \int_0^{\frac12} x\exp\left(-\frac{x^2}{4\varepsilon\Delta}\right)\mathrm{d}x \\ & + \varepsilon^{1/2}\Delta^{-3/2}\left\|\phi_{x}\right\|_\infty \cdot \int_0^{\frac12} \frac{x^3}{4\varepsilon\Delta^2}\exp\left(-\frac{x^2}{4\varepsilon\Delta}\right)\mathrm{d}x \\ \lesssim\ & \varepsilon^{3/2}\Delta^{-3/2}.\end{aligned}$$

Putting together estimates (5.53), (5.54) and (5.55) proves that $\kappa(T_3) \lesssim \varepsilon^{3/2}$.

Second part. – Let us move on to the second order derivative part. We compute:

(5.56)
$$Q_3(t,s_1,s_2) = \partial_{s_1}\partial_{s_2} A_3 = \frac{1}{4\pi}(\alpha\beta)^{-3/2}\int_0^{\frac12} x^2\phi_x(1-t,x)\exp\left(-\frac{x^2}{4\varepsilon}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\right)\mathrm{d}x.$$

Combining Lemma 19 and Lemma 18 yields:

$$|Q_3(t,s_1,s_2)| \lesssim \frac{\varepsilon^{3/2}}{(\alpha+\beta)^{3/2}}. \tag{5.57}$$

Writing $\alpha = \Delta + \tau$ and $\beta = \tau$, we can estimate:

$$|R_3(s_1,s_2)| = \left|\int_{s_2}^1 Q_3(t,s_1,s_2)\mathrm{d}t\right| \lesssim \int_0^1 \left(\frac{\varepsilon}{\Delta+2\tau}\right)^{3/2}\mathrm{d}\tau \lesssim \varepsilon^{3/2}\Delta^{-1/2}. \tag{5.58}$$

Now we will prove similar estimates for the first order derivatives of R_3. Differentiating Equation (5.56) with respect to s_1 (or similarly α) yields:

(5.59)
$$\begin{aligned}\partial_{s_1} Q_3(t,s_1,s_2) =& \frac{3}{8\pi}\alpha^{-5/2}\beta^{-3/2}\int_0^{\frac12} x^2\phi_x(1-t,x)\exp\left(-\frac{x^2}{4\varepsilon}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\right)\mathrm{d}x \\ & - \frac{1}{16\pi\varepsilon}(\alpha\beta)^{-3/2}\frac{1}{\alpha^2}\int_0^{\frac12} x^4\phi_x(1-t,x)\exp\left(-\frac{x^2}{4\varepsilon}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\right)\mathrm{d}x.\end{aligned}$$

Combining Lemma 19 and Lemma 18 gives:

(5.60)
$$\left|\partial_{s_1} Q_3(t,s_1,s_2)\right| \lesssim \alpha^{-5/2}\beta^{-3/2}\frac{\varepsilon^{3/2}}{\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^{3/2}} + \alpha^{-7/2}\beta^{-3/2}\frac{\varepsilon^{3/2}}{\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^{5/2}} \lesssim \varepsilon^{3/2}\alpha^{-5/2}.$$

Integration with respect to t yields an estimate of $\partial_{s_1} R_3$:

$$\left|\partial_{s_1} R_3(s_1,s_2)\right| \lesssim \int_{s_2}^1 \left|\partial_{s_1} Q_3(t,s_1,s_2)\right|\mathrm{d}t \lesssim \varepsilon^{3/2}\int_{s_2}^1 \frac{\mathrm{d}t}{\alpha^{5/2}} \lesssim \varepsilon^{3/2}\Delta^{-3/2}. \tag{5.61}$$

From this, we deduce that:

$$|R_3(s_1,s_2) - R_3(\tilde{s_1},s_2)| \lesssim \varepsilon^{3/2}\Delta^{-3/2}\left|s_1-\tilde{s_1}\right|. \tag{5.62}$$

Eventually, we finish with the regularity of R_3 with respect to s_2. We compute the difference for $s_1 < s_2 < \tilde{s_2}$ with $\tilde{s_2} - s_2 \leq \frac{1}{2}(s_2 - s_1)$:

(5.63)
$$\begin{aligned}|R_3(s_1,s_2) - R_3(s_1,\tilde{s_2})| &= \left|\int_{s_2}^1 Q_3(t,s_1,s_2)\mathrm{d}t - \int_{\tilde{s_2}}^1 Q_3(t,s_1,\tilde{s_2})\mathrm{d}t\right| \\ &= \left|\int_{s_2}^{\tilde{s_2}} Q_3(t,s_1,s_2)\mathrm{d}t - \int_{\tilde{s_2}}^1 \left(Q_3(t,s_1,\tilde{s_2}) - Q_3(t,s_1,s_2)\right)\mathrm{d}t\right| \\ &\leq \int_{s_2}^{\tilde{s_2}} \frac{\varepsilon^{3/2}}{\Delta^{3/2}}\mathrm{d}t + \left|\int_{\tilde{s_2}}^1 \int_{s_2}^{\tilde{s_2}} \partial_{s_2} Q_3(t,s_1,s)\mathrm{d}s\mathrm{d}t\right| \\ &\leq \frac{\varepsilon^{3/2}}{\Delta^{3/2}}|s_2 - \tilde{s_2}| + \int_{s_2}^{\tilde{s_2}}\int_{\tilde{s_2}}^1 \left|\partial_{s_2} Q_3(t,s_1,s)\right| \mathrm{d}t\mathrm{d}s.\end{aligned}$$

The first term is already in the correct form. We need to work on the second term. Proceeding as above, differentiating Equation (5.56) with respect to s_2 (or similarly β), then combining Lemma 19 and Lemma 18 gives:

$$\left|\partial_{s_2} Q_3(t,s_1,s)\right| \lesssim \varepsilon^{3/2}\frac{1}{t-s}\frac{1}{(t-s+t-s_1)^{3/2}}. \tag{5.64}$$

We compute:

(5.65)
$$\begin{aligned}\int_{s_2}^{\tilde{s_2}}\int_{\tilde{s_2}}^1 \left|\partial_{s_2} Q_3(t,s_1,s)\right| \mathrm{d}t\mathrm{d}s &\leq \varepsilon^{3/2}\int_{s_2}^{\tilde{s_2}}\int_{\tilde{s_2}}^1 \frac{1}{t-s}\frac{1}{(t-s_1)^{3/2}}\mathrm{d}t\mathrm{d}s \\ &\leq \varepsilon^{3/2}\Delta^{-3/2}\int_{s_2}^{\tilde{s_2}}\int_{\tilde{s_2}}^1 \frac{\mathrm{d}t}{t-s}\mathrm{d}s \\ &\leq \varepsilon^{3/2}\Delta^{-3/2}\int_{s_2}^{\tilde{s_2}} |\ln(\tilde{s_2} - s)|\,\mathrm{d}s \\ &\leq \varepsilon^{3/2}\Delta^{-3/2}|s_2 - \tilde{s_2}|\left(1 + \ln|s_2 - \tilde{s_2}|\right).\end{aligned}$$

This last estimate does not give Lipschitz regularity, but it does provide Hölder $\mathcal{C}^p$ regularity for any exponent $p < 1$, which is enough. Together, estimates (5.58), (5.62) and (5.65) prove that $\kappa(R_3) \lesssim \varepsilon^{3/2}$.

5.6. Handling the fourth kernel

In this section, we consider:

$$A_4(t,s_1,s_2) = \int_0^{\frac{1}{2}} \Phi_x(1-t,x)H(t-s_1,x)\operatorname{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_2)}}\right)\mathrm{d}x. \tag{5.28}$$

First part. – We consider the first order derivative:

(5.66)
$$\begin{aligned}T_4(s_1,s_2) &= \left(\partial_{s_1} A_4\right)|_{t=s_2} \\ &= \int_0^{\frac{1}{2}} \Phi_x(1-s_2,x)H_t(s_2-s_1,x)\mathrm{d}x,\end{aligned}$$

where we used the fact that $\mathrm{erf}(+\infty) = 1$. The following estimates are straightforward:

$$|T_4(s_1, s_2)| \le \|\Phi_x\|_\infty \|H_t\|_\infty, \tag{5.67}$$

$$|T_4(s_1, s_2) - T_4(\tilde{s}_1, s_2)| \le |s_1 - \tilde{s}_1| \cdot \|\Phi_x\|_\infty \|H_{tt}\|_\infty, \tag{5.68}$$

$$|T_4(s_1, s_2) - T_4(s_1, \tilde{s}_2)| \le |s_2 - \tilde{s}_2| \cdot \|\Phi_x\|_\infty \|H_{tt}\|_\infty + |s_2 - \tilde{s}_2| \cdot \|\Phi_{tx}\|_\infty \|H_t\|_\infty. \tag{5.69}$$

Second part. – We move on to the second order derivative part. We compute:

$$\begin{aligned} Q_4(t, s_1, s_2) &= \partial_{s_1}\partial_{s_2} A_4(t, s_1, s_2) \\ &= -\frac{1}{2\sqrt{\pi\varepsilon}}\beta^{-3/2}\int_0^{\frac{1}{2}} x\Phi_x(1-t, x)H_t(\alpha, x)\exp\left(-\frac{x^2}{4\varepsilon\beta}\right)\mathrm{d}x. \end{aligned} \tag{5.70}$$

Since $H_t(t, 0) \equiv 0$, $|H_t(t, x)| \le x\,\|H_{tx}\|_\infty$. Using Lemma 19, we obtain:

$$\begin{aligned} |Q_4(t, s_1, s_2)| &\lesssim \varepsilon^{-1/2}\beta^{-3/2}\|H_{tx}\|_\infty \|\Phi_x\|_\infty \int_0^{\frac{1}{2}} x^2 \exp\left(-\frac{x^2}{4\varepsilon\beta}\right)\mathrm{d}x \\ &\lesssim \varepsilon\,\|H_{tx}\|_\infty \|\Phi_x\|_\infty. \end{aligned} \tag{5.71}$$

By integration over $t \in (s_2, 1)$, we obtain:

$$|R_4(s_1, s_2)| \lesssim \varepsilon\,\|H_{tx}\|_\infty \|\Phi_x\|_\infty. \tag{5.72}$$

Now we establish the regularity of Q_4 with respect to s_1. Differentiating Equation (5.70) with respect to s_1 (or α), and applying the same techniques yields the estimate:

$$\left|\partial_{s_1} Q_4(t, s_1, s_2)\right| \lesssim \varepsilon\,\|H_{ttx}\|_\infty \|\Phi_x\|_\infty. \tag{5.73}$$

This proves that:

$$|R_4(s_1, s_2) - R_4(\tilde{s_1}, s_2)| \lesssim \varepsilon\,\|H_{ttx}\|_\infty \|\Phi_x\|_\infty \cdot |s_1 - \tilde{s_1}|. \tag{5.74}$$

Finally, we consider the regularity of Q_4 with respect to s_2. We know that:

$$|R_4(s_1, s_2) - R_4(s_1, \tilde{s_2})| \le \int_{s_2}^{\tilde{s_2}} |Q_4(t, s_1, s_2)|\,\mathrm{d}t + \int_{s_2}^{\tilde{s_2}}\int_{\tilde{s_2}}^{1} \left|\partial_{s_2} Q_4(t, s_1, s)\right|\mathrm{d}t\,\mathrm{d}s. \tag{5.75}$$

This first part obviously gives rise to a Lipschitz estimate. As for the second part, we compute $\partial_{s_2} Q_4$ by differentiating (5.70) with respect to β. We obtain

(5.76)

$$\begin{aligned} \partial_{s_2} Q_4(t, s_1, s)(t, s_1, s) = &-\frac{3}{4\sqrt{\pi\varepsilon}}\beta^{-5/2}\int_0^{\frac{1}{2}} x\Phi_x(t, x)H_t(\alpha, x)\exp\left(-\frac{x^2}{4\varepsilon\beta}\right)\mathrm{d}x \\ &+ \frac{1}{8\sqrt{\pi}}\varepsilon^{-3/2}\beta^{-7/2}\int_0^{\frac{1}{2}} x^3\Phi_x(t, x)H_t(\alpha, x)\exp\left(-\frac{x^2}{4\varepsilon\beta}\right)\mathrm{d}x. \end{aligned}$$

Similar estimates yield:

$$\left|\partial_{s_2} Q_4(t, s_1, s)\right| \lesssim \varepsilon\,\|H_{tx}\|_\infty \|\Phi_x\|_\infty \cdot \frac{1}{t-s}. \tag{5.77}$$

Therefore:

$$\begin{aligned}\int_{s_2}^{\tilde{s_2}} \int_{\tilde{s_2}}^{1} \left|\partial_{s_2} Q_4(t,s_1,s)\right| \mathrm{d}t\mathrm{d}s &\lesssim \varepsilon \left\|H_{tx}\right\|_\infty \left\|\Phi_x\right\|_\infty \cdot \int_{s_2}^{\tilde{s_2}} \int_{\tilde{s_2}}^{1} \frac{\mathrm{d}t\mathrm{d}s}{t-s} \\ &\lesssim \varepsilon \left\|H_{tx}\right\|_\infty \left\|\Phi_x\right\|_\infty \cdot \int_{s_2}^{\tilde{s_2}} \left|\ln(\tilde{s_2}-s)\right| \mathrm{d}s \\ &\lesssim \varepsilon \left\|H_{tx}\right\|_\infty \left\|\Phi_x\right\|_\infty \cdot \left|\tilde{s_2}-s_2\right| \left(1+\ln\left|\tilde{s_2}-s_2\right|\right).\end{aligned} \tag{5.78}$$

Therefore, for any fixed $p < 1$, we have:

$$\left|R_4(s_1,s_2) - R_4(s_1,\tilde{s_2})\right| \lesssim \varepsilon \left\|H_{tx}\right\|_\infty \left\|\Phi_x\right\|_\infty \cdot \left|\tilde{s_2}-s_2\right|^p. \tag{5.79}$$

Thanks to Lemma 17 and Lemma 18, this proves that, for any $\gamma < \frac{1}{16}$,

$$\kappa(\partial_{12} K_4) \lesssim \exp\left(-\frac{\gamma}{\varepsilon}\right). \tag{5.80}$$

5.7. Handling the fifth kernel

Recall that A_5 was defined by:

$$A_5(t,s_1,s_2) = \int_0^{\frac12} \Phi_x(1-t,x) H(t-s_2,x) \mathrm{erf}\left(\frac{x}{\sqrt{4\varepsilon(t-s_1)}}\right) \mathrm{d}x. \tag{5.29}$$

First part. – The first order derivative T_5 is null. Indeed,

(5.81)

$$\begin{aligned}T_5(s_1,s_2) &= \left(\partial_{s_1} A_5\right)|_{t=s_2} \\ &= \frac{1}{2\sqrt{\pi\varepsilon}} \int_0^{\frac12} \Phi_x(1-s_2,x) H(0,x) \cdot \frac{x}{(s_2-s_1)^{\frac32}} \exp\left(-\frac{x^2}{4\varepsilon(s_2-s_1)}\right) \mathrm{d}x = 0.\end{aligned}$$

Second part. – We consider the second order derivative:

(5.82)

$$Q_5(t,s_1,s_2) = \partial_{s_2}\partial_{s_1} A_5(t,s_1,s_2) = -\frac{1}{2\sqrt{\pi\varepsilon}} \alpha^{-3/2} \int_0^{\frac12} x \Phi_x(t,x) H_t(\beta,x) \exp\left(-\frac{x^2}{4\varepsilon\alpha}\right) \mathrm{d}x.$$

Since $H_t(t,0) \equiv 0$, $|H_t(t,x)| \le x \left\|H_{tx}\right\|_\infty$. Using Lemma 19, we obtain:

$$\begin{aligned}\left|Q_5(t,s_1,s_2)\right| &\lesssim \varepsilon^{-1/2}\alpha^{-3/2} \left\|H_{tx}\right\|_\infty \left\|\Phi_x\right\|_\infty \int_0^{\frac12} x^2 \exp\left(-\frac{x^2}{4\varepsilon\alpha}\right) \mathrm{d}x \\ &\lesssim \varepsilon \left\|H_{tx}\right\|_\infty \left\|\Phi_x\right\|_\infty.\end{aligned} \tag{5.83}$$

By integration over $t \in (s_2, 1)$, we obtain:

$$\left|R_5(s_1,s_2)\right| \lesssim \varepsilon \left\|H_{tx}\right\|_\infty \left\|\Phi_x\right\|_\infty. \tag{5.84}$$

Differentiating (5.82) with respect to α and proceeding likewise yields:

$$\left|\partial_{s_1} Q_5(t,s_1,s_2)\right| \lesssim \varepsilon \left\|H_{tx}\right\|_\infty \left\|\Phi_x\right\|_\infty \cdot \frac{1}{\alpha}. \tag{5.85}$$

Thus,

$$\left|R_5(s_1,s_2) - R_5(\tilde{s_1},s_2)\right| \lesssim \varepsilon \left\|H_{tx}\right\|_\infty \left\|\Phi_x\right\|_\infty \cdot \Delta^{-1} \left|\tilde{s_1}-s_1\right|. \tag{5.86}$$

Differentiation with respect to β is even easier and gives:

$$\left|\partial_{s_2} Q_5(t, s_1, s_2)\right| \lesssim \varepsilon \, \|H_{ttx}\|_\infty \, \|\Phi_x\|_\infty \, , \tag{5.87}$$

from which we easily conclude that R_5 is Lipschitz with respect to s_2.

Thanks to Lemma 17 and Lemma 18, this proves that, for any $\gamma < \frac{1}{16}$,

$$\kappa(\partial_{12} K_5) \lesssim \exp\left(-\frac{\gamma}{\varepsilon}\right). \tag{5.88}$$

5.8. Handling the sixth kernel

Recall that A_6 was defined by:

$$A_6(t, s_1, s_2) = \int_0^{\frac{1}{2}} \Phi_x(1-t, x) H(t - s_1, x) H(t - s_2, x) \mathrm{d}x. \tag{5.30}$$

First part. – The first order derivative T_6 is null. Indeed:

$$T_6(s_1, s_2) = \left(\partial_{s_1} A_6\right)|_{t=s_2} = \int_0^{\frac{1}{2}} \Phi_x(0, x) H_t(s_2 - s_1, x) H(0, x) \mathrm{d}x = 0. \tag{5.89}$$

Second part. – We consider the second order derivative:

$$Q_6(t, s_1, s_2) = \partial_{s_2}\partial_{s_1} A_6(t, s_1, s_2) = \int_0^{\frac{1}{2}} \Phi_x(1 - s_2, x) H_t(t - s_1, x) H_t(t - s_2, x) \mathrm{d}x. \tag{5.90}$$

For any $t \in (0, 1)$, we estimate:

$$\begin{aligned}
|Q_6(t, s_1, s_2)| &\leq \|\Phi_x\|_\infty \, \|H_t\|_\infty^2 \, , \\
|Q_6(t, s_1, s_2) - Q_6(t, \tilde{s}_1, s_2)| &\leq |s_1 - \tilde{s}_1| \cdot \|\Phi_x\|_\infty \, \|H_{tt}\|_\infty \, \|H_t\|_\infty \, , \\
|Q_6(t, s_1, s_2) - Q_6(t, s_1, \tilde{s}_2)| &\leq |s_2 - \tilde{s}_2| \cdot \|\Phi_x\|_\infty \, \|H_t\|_\infty \, \|H_{tt}\|_\infty \\
&\quad + |s_2 - \tilde{s}_2| \cdot \|\Phi_{tx}\|_\infty \, \|H_t\|_\infty^2 \, .
\end{aligned} \tag{5.91}$$

Hence, we can extend these estimates to:

$$R_6(s_1, s_2) = \int_{s_2}^1 Q_6(t, s_1, s_2) \mathrm{d}t. \tag{5.92}$$

The only non immediate extension is:

(5.93)

$$\begin{aligned}
|R_6(s_1, s_2) - R_6(s_1, \tilde{s}_2)| &\leq \int_{s_2}^1 |Q_6(t, s_1, s_2) - Q_6(t, s_1, \tilde{s}_2)| \, \mathrm{d}t + \int_{s_2}^{\tilde{s}_2} |Q_6(t, s_1, \tilde{s}_2)| \, \mathrm{d}t \\
&\leq |s_2 - \tilde{s}_2| \, \big(\|\Phi_x\|_\infty \, \|H_t\|_\infty \, \|H_{tt}\|_\infty \\
&\qquad + \|\Phi_{tx}\|_\infty \, \|H_t\|_\infty^2 + \|\Phi_x\|_\infty \, \|H_t\|_\infty^2\big)
\end{aligned}$$

Thanks to Lemma 17 and Lemma 18, this proves that, for any $\gamma < \frac{1}{16}$,

$$\kappa(\partial_{12} K_6) \lesssim \exp\left(-\frac{\gamma}{\varepsilon}\right). \tag{5.94}$$

5.9. Conclusion of the kernel expansion

LEMMA 21. – *There exists* $\varepsilon_1 > 0$ *and* $k_1 > 0$ *such that, for any* $0 < \varepsilon \leq \varepsilon_1$ *and any* $u \in L^2(0,1)$*,*

$$\langle K^\varepsilon u, u \rangle \geq k_1 \sqrt{\varepsilon} |U|^2_{H^{-1/4}}. \tag{5.95}$$

Proof. – Thanks to the previous paragraphs, we have shown that $K^\varepsilon = \frac{\sqrt{\varepsilon}}{45\sqrt{\pi}} N + R$, where $R = \tilde{K}_1 + K_2 + K_3 + K_4 + K_5 + K_6$ is such that $\kappa(\partial_{12} R) \lesssim \varepsilon^{3/2}$. From Lemma 16, we deduce that there exists C_0 such that, for any $u \in L^2(0,1)$, $|\langle Ru, u\rangle| \leq C_0 \varepsilon^{3/2} |U|^2_{H^{-1/4}}$. Moreover, thanks to Lemma 10, there exists c_0 such that $\langle Nu, u\rangle \geq c_0 |U|^2_{H^{-1/4}}$. Hence, for any $k_1 < c_0/(45\sqrt{\pi})$, (5.95) holds for ε small enough. □

Equation (5.95) gives a weak coercivity, both because the norm involved is related to the $H^{-5/4}$ norm on the control u, and because the coercivity constant $k_1\sqrt{\varepsilon}$ decays when $\varepsilon \to 0$. However, this is enough to overcome the remaining higher order residues, as we prove in the following section.

6. Back to the full Burgers non-linear system

In this section, we conclude the proof of our main result, by bridging the gap between the quadratic approximation studied in Sections 3-5 and the initial non-linear Burgers system.

6.1. Preliminary estimates

6.1.1. *Estimating the first order term.* – We decompose the first order term a (defined by system (1.10)) as $a(t,x) = U(t) + \tilde{a}(t,x)$, where U is the primitive of u such that $U(0) = 0$ and $\tilde{a}$ is the solution to:

$$\begin{cases} \tilde{a}_t - \varepsilon \tilde{a}_{xx} = 0 & \text{in } (0,1)\times(0,1), \\ \tilde{a}(t,0) = -U(t) & \text{in } (0,1), \\ \tilde{a}(t,1) = -U(t) & \text{in } (0,1), \\ \tilde{a}(0,x) = 0 & \text{in } (0,1). \end{cases} \tag{6.1}$$

LEMMA 22. – *The following estimates hold:*

$$\|\tilde{a}\|_2 \lesssim |U|_{H^{-1/4}}, \tag{6.2}$$

$$\|a\|_\infty + \|\tilde{a}\|_\infty \lesssim |u|_2, \tag{6.3}$$

$$\varepsilon \|a_x\|_{L^2(L^\infty)} \lesssim |u|_2. \tag{6.4}$$

Proof. – The first inequality (6.2) is a direct application of estimate (2.11) from Lemma 4. The second inequality is a consequence of the maximum principle. Indeed, thanks to Equation (6.1), $\|\tilde{a}\|_\infty$ is smaller than $|U|_\infty$. Since $a = U + \tilde{a}$, $\|a\|_\infty$ is smaller than $2\,|U|_\infty$. Estimate (6.3) follows because $|U|_\infty \leq |u|_2$. The third inequality stems from Lemma 3. Since

a is even, $a_x(\cdot, 1/2) \equiv 0$. Thus:

$$\begin{aligned}
\|a_x\|^2_{L^2(L^\infty)} &= \int_0^1 \left(\sup_{x\in(0,1)} |a_x(t,x)| \right)^2 \mathrm{d}t \\
&= \int_0^1 \left(\sup_{x\in(0,1)} \left| \int_{\frac{1}{2}}^x a_{xx}(t,x')\mathrm{d}x' \right| \right)^2 \mathrm{d}t \\
&\leq \int_0^1 \int_0^1 a_{xx}^2(t,x')\mathrm{d}x'\mathrm{d}t.
\end{aligned} \tag{6.5}$$

Combined with (2.7), this proves (6.4). □

6.1.2. *Estimating the second order term*

LEMMA 23. – *The following estimates hold:*

$$\varepsilon^{1/2} \|b\|_{L^\infty(L^2)} + \varepsilon \|b_x\|_{L^2} \lesssim |u|_{L^2} \cdot |U|_{H^{-1/4}}, \tag{6.6}$$

$$\varepsilon^{3/2} \|b\|_\infty \lesssim |u|_2^2, \tag{6.7}$$

$$\varepsilon^{3/2} \|b_x\|_{L^2(L^\infty)} \lesssim |u|_2^2. \tag{6.8}$$

Proof. – For the first inequality, we write:

$$-aa_x = -a\tilde{a}_x = -\frac{\mathrm{d}}{\mathrm{d}x}\left[a\tilde{a} - \frac{1}{2}\tilde{a}^2 \right]. \tag{6.9}$$

The term under the derivative is estimated in L^2 as:

$$\left\| a\tilde{a} - \frac{1}{2}\tilde{a}^2 \right\|_{L^2} \leq \|\tilde{a}\|_{L^2} \cdot (\|a\|_\infty + \|\tilde{a}\|_\infty) \lesssim |u|_{L^2} \cdot |U|_{H^{-1/4}}, \tag{6.10}$$

where the last inequality comes from Lemma 22. Thus, we can apply Lemma 5 to prove (6.6).

For the second and third inequalities, thanks to Lemma 3, $\|a_x\|_2 \lesssim \varepsilon^{-1/2}|u|_2$. Moreover, thanks to Lemma 22, $\|a\|_\infty \lesssim |u|_2$. Thus, $\|aa_x\|_2 \lesssim \varepsilon^{-1/2}|u|_2^2$. We can apply Lemma 3 to show that $\|b\|_{X_1} \lesssim \varepsilon^{-3/2}|u|_2^2$. Inequality (6.7) follows from the injection $X_1 \hookrightarrow L^\infty$ (see (2.4) from Lemma 1). Moreover, since $\int_0^1 b_x(t,x)\mathrm{d}x = b(t,1) - b(t,0) = 0$ for any $t \in (0,1)$, the mean value of $b_x(t,\cdot)$ is 0. Thus, $|b_x(t,\cdot)|_\infty \leq |b_{xx}(t,\cdot)|_2$. Hence, $\|b_x\|_{L^2(L^\infty)} \leq \|b_{xx}\|_2$. This proves estimate (6.8). □

6.2. Non-linear residue

Let us expand y as $a + b + r$, where r is the solution to:

$$\begin{cases}
r_t - \varepsilon r_{xx} = -rr_x - [(a+b)r]_x - \left[ab + \frac{1}{2}b^2 \right]_x & \text{in } (0,1)\times(0,1), \\
r(t,0) = 0 & \text{in } (0,1), \\
r(t,1) = 0 & \text{in } (0,1), \\
r(0,x) = 0 & \text{in } (0,1).
\end{cases} \tag{6.11}$$

LEMMA 24. – *System* (6.11) *admits a unique solution* $r \in X_1$. *Moreover, under the assumption:*

$$|u|_2 \leq \varepsilon^{3/2}, \tag{6.12}$$

the following estimate holds:

$$\|r\|_2 + \|r_t\|_2 \lesssim \varepsilon^{-3/2} |u|_2^2 |U|_{H^{-1/4}}. \tag{6.13}$$

Proof. – The existence of $r \in X_1$ can be deduced directly from the equality $r = y - a - b$. To prove the estimate, we will use Lemma 6 with a null initial data, $w = -(a+b)$ and $g = -ab - \frac{1}{2}b^2$. To apply estimate (2.15), we start by computing the norms of w and g that we need. We start with $w = -(a+b)$. Combining (6.3), (6.7) and (6.12) gives:

$$\|w\|_\infty \leq \|a\|_\infty + \|b\|_\infty \lesssim |u|_2 + \varepsilon^{-3/2} |u|_2^2 \lesssim |u|_2. \tag{6.14}$$

In particular, (6.14) and (6.12) yield:

$$\gamma = \frac{1}{\varepsilon} \|w\|_{L^2(L^\infty)}^2 \leq \frac{1}{\varepsilon} \|w\|_\infty^2 \leq \frac{1}{\varepsilon} |u|_2^2 \lesssim 1. \tag{6.15}$$

Finally, combining (6.4) and (6.8):

$$\|w_x\|_{L^2(L^\infty)} \leq \|a_x\|_{L^2(L^\infty)} + \|b_x\|_{L^2(L^\infty)} \lesssim \varepsilon^{-1} |u|_2 + \varepsilon^{-3/2} |u|_2^2 \lesssim \varepsilon^{-1} |u|_2. \tag{6.16}$$

We move on to $g = -ab - \frac{1}{2}b^2$. Combining (6.3), (6.6), (6.7) and (6.12) gives:

$$\begin{aligned} \|g\|_2 &\leq (\|a\|_\infty + \|b\|_\infty) \|b\|_2 \\ &\leq \left(|u|_2 + \varepsilon^{-3/2} |u|_2^2\right) \varepsilon^{-1/2} |u|_2 |U|_{H^{-1/4}} \\ &\leq \varepsilon^{-1/2} |u|_2^2 |U|_{H^{-1/4}}. \end{aligned} \tag{6.17}$$

Combining (6.3), (6.6), (6.7) and (6.12), we obtain:

$$\begin{aligned} \|g\|_{L^2(L^\infty)} &\leq (\|a\|_\infty + \|b\|_\infty) \cdot \|b\|_{L^2(L^\infty)} \\ &\leq (\|a\|_\infty + \|b\|_\infty) \cdot \|b_x\|_2 \\ &\lesssim \varepsilon^{-1} |u|_2^2 |U|_{H^{-1/4}}. \end{aligned} \tag{6.18}$$

Lastly, mixing (6.3), (6.4), (6.6), (6.7) and (6.12) gives:

$$\begin{aligned} \|g_x\|_2 &\leq \|a_x\|_{L^2(L^\infty)} \|b\|_{L^\infty(L^2)} + \|a\|_\infty \|b_x\|_2 + \|b\|_\infty \|b_x\|_2 \\ &\lesssim \varepsilon^{-3/2} |u|_2^2 |U|_{H^{-1/4}} + \varepsilon^{-1} |u|_2^2 |U|_{H^{-1/4}} + \varepsilon^{-5/2} |u|_2^3 |U|_{H^{-1/4}} \\ &\lesssim \varepsilon^{-3/2} |u|_2^2 |U|_{H^{-1/4}}. \end{aligned} \tag{6.19}$$

Eventually, plugging estimates (6.14)-(6.19) into the main estimation (2.15), yields:

$$\|r_t\|_2 \lesssim \varepsilon^{-3/2} |u|_2^2 |U|_{H^{-1/4}}. \tag{6.20}$$

From (6.20) and the initial condition $r(0,\cdot) = 0$, we conclude (6.13). □

LEMMA 25. – *Under the assumption* (6.12)*, we have:*

$$|\langle \rho, r(1,\cdot) \rangle| \lesssim \varepsilon^{-3/2} |u|_2^2 |U|_{H^{-1/4}}^2. \tag{6.21}$$

Proof. – Similarly as in Lemma 8, we compute the final time projection for Equation (6.11) as:

(6.22)
$$\begin{aligned}\langle \rho, r(1,\cdot)\rangle &= \int_0^1 \int_0^1 \Phi_x \left[ab + \frac{1}{2}b^2 + (a+b)r + \frac{1}{2}r^2\right] \\ &= \int_0^1 \int_0^1 \Phi_x(1-t,x)U(t)r(t,x)\mathrm{d}x\mathrm{d}t + \int_0^1 \int_0^1 \Phi_x \left[\frac{1}{2}b^2 + (\tilde{a}+b)r + \frac{1}{2}r^2\right].\end{aligned}$$

We used the fact that $\int_0^1 \Phi_x ab = 0$. We rewrite the first term as:

$$\int_0^1 U(t) \int_0^1 \Phi_x(1-t,x)r(t,x)\mathrm{d}x\mathrm{d}t = \langle U, v\rangle_{H^{-1},H_0^1}, \tag{6.23}$$

where we introduce $v(t) = \int_0^1 \Phi_x(t,x)r(t,x)\mathrm{d}x$ for $t \in (0,1)$. Since $\Phi(0,\cdot) \equiv 0$ and $r(0,\cdot) \equiv 0$, $v(0) = v(1) = 0$. Now we compute its H_0^1 norm:

$$\begin{aligned}\int_0^1 v_t(t)^2 \mathrm{d}t &= \int_0^1 \left(\int_0^1 \Phi_{tx}(1-t,x)r(t,x) + \Phi_x(1-t,x)r_t(t,x)\mathrm{d}x\right)^2 \mathrm{d}t \\ &\le 2\int_0^1 \int_0^1 \Phi_{tx}^2 r^2 + \Phi_x^2 r_t^2 \\ &\le 2\left(\|\Phi_{tx}\|_\infty^2 \|r\|_2^2 + \|\Phi_x\|_\infty^2 \|r_t\|_2^2\right) \\ &\lesssim \varepsilon^2 \|r\|_2^2 + \|r_t\|_2^2 \lesssim \|r_t\|_2^2,\end{aligned} \tag{6.24}$$

where we used estimates (5.19) and (5.21) to estimate Φ. Hence:

$$\begin{aligned}|\langle \rho, r(1,\cdot)\rangle| &\overset{(6.22)\text{ and }(6.23)}{\le} \left|\langle U, v\rangle_{H^{-1},H_0^1}\right| + \left|\int_0^1 \int_0^1 \Phi_x \left(\frac{1}{2}b^2 + (\tilde{a}+b)r + \frac{1}{2}r^2\right)\right| \\ &\overset{(6.24)}{\lesssim} |U|_{H^{-1}} \|r_t\|_2 + \|\Phi_x\|_\infty \left(\|b\|_2^2 + \|\tilde{a}\|_2 \|r\|_2 + \|r\|_2^2\right).\end{aligned} \tag{6.25}$$

From (5.19), we know that $\|\Phi_x\|_\infty \lesssim 1$. Moreover, $|U|_{H^{-1}} \lesssim |U|_{H^{-1/4}}$. Thanks to (6.2), (6.6), (6.13) and (6.12), we conclude from (6.25) that $|\langle \rho, r(1,\cdot)\rangle| \lesssim \varepsilon^{-3/2} |u|_2^2 |U|_{H^{-1/4}}^2$. □

6.3. A first drifting result concerning reachability from zero

The null reachability problem consists in computing the set of states that can be reached in time T, starting from the initial state $y^0 = 0$. Of course, when dealing with viscous equations like (1.1), one may only hope to reach sufficiently smooth states (see [27] and [42] for recent developments concerning the null reachability problem for the one-dimensional heat equation). Here, we prove that, if the control time T is too small, the state *drifts* towards the direction $+\rho$ as a result of the action of the control, whatever control is chosen.

THEOREM 2. – *There exist $T_*, k_* > 0$ such that, for any $0 < T < T_*$ and any $u \in L^2(0,T)$ such that $|u|_{L^2(0,T)} \le 1$, the solution $y \in X_T$ to system (1.1) starting from the null initial condition $y(0,x) \equiv 0$ satisfies:*

$$\langle \rho, y(T,\cdot)\rangle \ge k_* |U|_{H^{-1/4}(0,T)}^2, \tag{6.26}$$

where U, as above, is the primitive of u such that $U(0) = 0$.

Proof. – We are going to use the scaling argument introduced in Paragraph 1.4. Thus, from now on, we reintroduce the tilde signs for functions defined on the scaled time interval $(0,1)$. From Lemma 21, we know that, for $\varepsilon < \varepsilon_1$, $\langle K^\varepsilon \tilde{u}, \tilde{u}\rangle \geq k_1\sqrt{\varepsilon}|\tilde{U}|^2_{H^{-1/4}}$. From Lemma 25, we know that there exists c_2 such that, as soon as $|\tilde{u}|_2 \leq \varepsilon^{3/2}$, $|\langle \rho, r(1,\cdot)\rangle| \leq c_2\varepsilon^{3/2}|\tilde{U}|^2_{H^{-1/4}}$. Hence, if we consider $\tilde{y}$ the solution to (1.9), write $\tilde{y} = a + b + r$, for any $0 < k_* < k_1$, there exists $\varepsilon_2 > 0$ such that, for $\varepsilon < \varepsilon_2$, $\langle \rho, \tilde{y}(1,\cdot)\rangle \geq k_*\sqrt{\varepsilon}|\tilde{U}|^2_{H^{-1/4}}$. Recalling that $\tilde{u}(t) = \varepsilon^2 u(\varepsilon t)$ and $\tilde{y}(t,x) = \varepsilon y(\varepsilon t, x)$, we obtain:

$$\langle \rho, y(\varepsilon,\cdot)\rangle = \left\langle \frac{1}{\varepsilon}\tilde{y}(1,\cdot), \rho\right\rangle \geq k_*\varepsilon^{-1/2}|\tilde{U}|^2_{H^{-1/4}(0,1)} \geq k_*|U|^2_{H^{-1/4}(0,\varepsilon)}, \tag{6.27}$$

under the assumption:

$$|\tilde{u}|_{L^2(0,1)} \leq \varepsilon^{3/2} \quad \Leftrightarrow \quad |u|_{L^2(0,\varepsilon)} \leq 1. \tag{6.28}$$

Theorem 2 follows from (6.27) and (6.28) with $T_* = \varepsilon_2$. Equivalence (6.28) is obtained via a direct change of variable. To establish (6.27), one can compute the weak $H^{-1/4}$ norms using Fourier transforms. □

6.4. Persistence of projections in absence of control

In the absence of control, the projection of the state against any fixed profile $\mu \in L^2(0,1)$ remains almost constant in small time.

LEMMA 26. – *Let $T > 0$, $\mu \in L^2(0,1)$ and $y^0 \in H^1_0(0,1) \cap H^2(0,1)$. Assume that $|y^0|_{H^2} \leq 1$. Consider $y \in X_T$ the solution to system* (1.1) *with initial data y^0 and null control $(u = 0)$. Then,*

$$\langle \mu, y(T,\cdot)\rangle = \langle \mu, y^0\rangle + \mathcal{O}\left(T^{1/2}|\mu|_2|y^0|_{H^2}\right). \tag{6.29}$$

Proof. – We decompose $y = y^0 + z$, where z is the solution to:

$$\begin{cases} z_t - z_{xx} + zz_x = (y^0 z)_x + y^0_{xx} - y^0 y^0_x & \text{in } (0,T)\times(0,1),\\ z(t,0) = 0 & \text{in } (0,T),\\ z(t,1) = 0 & \text{in } (0,T),\\ z(0,x) = 0 & \text{in } (0,1). \end{cases} \tag{6.30}$$

We apply Lemma 6 with $w(t,x) = y^0(x)$ and $g(t,x) = y^0_x - \frac{1}{2}(y^0)^2$ to system (6.30). Estimate (2.15) tells us that $\|z_t\|_2 \lesssim |y^0|_{H^2}$. Here, we need the assumption that $|y^0|_{H^2} \leq C$, where C is any fixed constant, in order to avoid propagating non-linear estimates (involving exponentials). Since $z(0,x) \equiv 0$, we can write:

$$|\langle \mu, z(T,\cdot)\rangle| = \left|\int_0^T \int_0^1 z_t\mu\right| \leq T^{1/2}\,\|z_t\|_2\,|\mu|_2. \tag{6.31}$$

The conclusion (6.29) follows from (6.31). □

6.5. Proof of Theorem 1

To conclude the proof of Theorem 1, we consider an initial data of the form $y^\delta := \delta\rho$, where $\delta > 0$ can be picked as small as we need and ρ is defined in (3.10). For $T > 0$, $u \in L^2(0,T)$ and $\delta > 0$, we consider $y \in X_T$, the solution to system (1.1) with initial data y^δ and control u. To isolate the different contributions, we decompose y as $\bar{y} + y^u + z$, where:

$$\begin{cases} \bar{y}_t - \bar{y}_{xx} + \bar{y}\bar{y}_x = 0 & \text{in } (0,T)\times(0,1), \\ \bar{y}(t,0) = 0 & \text{in } (0,T), \\ \bar{y}(t,1) = 0 & \text{in } (0,T), \\ \bar{y}(0,x) = y^\delta & \text{in } (0,1), \end{cases} \tag{6.32}$$

$$\begin{cases} y^u_t - y^u_{xx} + y^u y^u_x = u(t) & \text{in } (0,T)\times(0,1), \\ y^u(t,0) = 0 & \text{in } (0,T), \\ y^u(t,1) = 0 & \text{in } (0,T), \\ y^u(0,x) = 0 & \text{in } (0,1), \end{cases} \tag{6.33}$$

$$\begin{cases} z_t - z_{xx} + zz_x = -[(\bar{y} + y^u)z]_x - [\bar{y}y^u]_x & \text{in } (0,T)\times(0,1), \\ z(t,0) = 0 & \text{in } (0,T), \\ z(t,1) = 0 & \text{in } (0,T), \\ z(0,x) = 0 & \text{in } (0,1). \end{cases} \tag{6.34}$$

First, we apply Lemma 7 to system (6.32). Estimates (2.23) and (2.24) yield:

$$\begin{aligned} \|\bar{y}_{xx}\|_2 + \|\bar{y}_x\|_2 + \|\bar{y}_t\|_2 &\lesssim \delta, \\ \|\bar{y}\|_\infty \leq |y^0|_\infty &\lesssim \delta. \end{aligned} \tag{6.35}$$

Similarly, we apply Lemma 7 to system (6.33). If we assume that $|u|_2 \leq 1$ and $T \leq 1$, we obtain:

$$\begin{aligned} \|y^u_{xx}\|_2 + \|y^u_x\|_2 + \|y^u_t\|_2 &\lesssim |u|_2\,, \\ \|y^u\|_\infty &\leq |u|_2\,. \end{aligned} \tag{6.36}$$

Next, we look at system (6.34). We apply Lemma 6 with $w = -(\bar{y} + y^u)$, $g = -\bar{y}y^u$ and a null initial data. Combining (6.35) and (6.36) yields the necessary estimates:

$$\|g\|_2 + \|g_x\|_2 + \|g\|_{L^2(L^\infty)} \lesssim \delta\,|u|_2\,, \tag{6.37}$$

$$\|w\|_\infty + \|w\|_{L^2(L^\infty)}\,\|w\|_{L^2(L^\infty)} \lesssim \delta + |u|_2\,. \tag{6.38}$$

Hence, (6.38) yields $\gamma \lesssim 1$. Therefore, plugging (6.37) and (6.38) into (2.15) gives:

$$\|z_{xx}\|_2 + \|z_t\|_2 \lesssim \delta\,|u|_2\,. \tag{6.39}$$

Once again, we use the initial condition $z(0,\cdot) \equiv 0$ and (6.39) to compute:

$$|\langle \rho, z(T,\cdot)\rangle| = \left| \int_0^T \int_0^1 z_t \rho \right| \lesssim T^{1/2}\delta\,|u|_2\,. \tag{6.40}$$

Let $T_* > 0$ be as defined in Theorem 2. Assuming $T \leq T_*$, we combine (6.26), (6.29) and (6.40) to obtain:

$$\langle y(T,\cdot), \rho\rangle \geq \delta|\rho|_2^2 + k_* |U|_{H^{-1/4}}^2 + \mathcal{O}\left(T^{1/2}\delta(1+|u|_2)\right). \tag{6.41}$$

From (6.41), we deduce that $\langle \rho, y(T,\cdot)\rangle > 0$ as soon as T is small enough and under the assumption $|u|_2 \leq 1$. Thus, we have proved Theorem 1 with $\eta = 1$.

7. Conclusion

7.1. Remarks on related systems

System (1.1) is posed with null Dirichlet boundary conditions. One can wonder what happens for other standard boundary conditions. In fact, for both periodic boundary conditions $y(t,0) = y(t,1)$ and for null Neumann boundary conditions $y_x(t,0) = y_x(t,1) = 0$, one checks that the associated controlled Burgers systems are not small-time locally null controllable either. The only controllable direction is the constant state 1 which satisfies the boundary conditions. For any given initial data and control, the same decomposition $y = \bar{y} + y^u + z$ can be used. Moreover, in this setting, $y^u(t,x) = U(t)$. This implies that any projection $\langle y, \rho\rangle$ is almost equal to $\langle \bar{y}, \rho\rangle$ for small times, small controls and directions ρ such that $\langle 1, \rho\rangle = 0$. The associated systems are hence not small-time locally null controllable.

The Hopf-Cole transform is a standard tool to study the Burgers equation. It has already been used to obtain control results (see [37], [41] and the references therein). Here, applying this transformation yields a new result on the small-time local controllability of the bilinear heat equation. Consider the system:

$$\begin{cases} z_t - z_{xx} = v(t)\mu(x)z & \text{in } (0,T)\times(0,1), \\ z_x(t,0) = 0 & \text{in } (0,T), \\ z_x(t,1) = 0 & \text{in } (0,T), \\ z(0,x) = z^0(x) & \text{in } (0,1), \end{cases} \tag{7.1}$$

where $\mu(x) = x$. This bilinear control system is, formally, close to the bilinear Schrödinger systems mentioned in the introduction. This system is studied in the vicinity of the equilibrium state $z_{\text{eq}}(x) \equiv 1$. We introduce the following definition:

DEFINITION 4. – *We say that system* (7.1) *is small-time locally controllable to constants near $z_{\text{eq}} = 1$ if, for any time $T > 0$, for any $\eta > 0$, there exists $\delta > 0$ such that, for any $z^0 \in H^2(0,1)$ with $z_x^0(0) = z_x^0(1) = 0$ and $|z^0 - 1|_{H^2} \leq \delta$, there exists a control $v \in L^2(0,T)$ such that $|v|_{L^2} \leq \eta$ and $z_x(T,\cdot) = 0$, where z is the associated solution to* (7.1).

THEOREM 3. – *System* (7.1) *is not small-time locally controllable to constants near* $z_{\text{eq}} = 1$.

Proof. – Small-time local null controllability of (1.1) and small-time local controllability to constants near $z_{\text{eq}} = 1$ of (7.1) are equivalent notions thanks to the Hopf-Cole transform. If one knows a trajectory z of (7.1), one defines:

$$y(t,x) := -2\frac{z_x(t,x)}{z(t,x)} \quad \text{and} \quad u(t) := -2v(t) \tag{7.2}$$

to obtain a trajectory y of (1.1). Reciprocally, one sets

(7.3)
$$z(t,x) := \exp\left(-\frac{1}{2}\int_0^t y_x(t',0)\mathrm{d}t'\right)\exp\left(-\frac{1}{2}\int_0^x y(t,x')\mathrm{d}x'\right) \quad \text{and} \quad v(t) := -\frac{1}{2}u(t)$$

to build a trajectory of (7.1) from a trajectory of Burgers. The details are left to the reader. □

7.2. Perspectives for quadratic obstructions

We expect that the methodology followed in this paper can be used for a wide variety of nonlinear systems involving a single scalar control. Indeed, when studying small-time local controllability for some formal system $\dot{y} = F(y, u(t))$, the first step is always to consider the linearized system, $\dot{a} = \partial_y F(0)a + \partial_u F(0)u$. When this system is controllable, fixed point or inverse mapping theorems often allow us to deduce that the non-linear system is small-time locally controllable. When the linearized system is not controllable, we can decompose the state y as $a + b$, where the (linear) component a is controllable and the second component b is indirectly controlled through a quadratic source term involving a (and/or, sometimes, u).

What our proof demonstrates, is that it is possible, even for infinite dimensional systems, to express projections of the second order part b as kernels acting on the control. The careful study of these kernels can then lead either to negative results (like it is the case here, because we prove a coercivity lemma), or to positive results (if the kernel is found to have both positive and negative eigenvalues, we can hope to prove that the system can be driven in the two opposite directions).

The coercivity used in this paper, although it involves a weak $H^{-5/4}$ norm of the control u, is in fact pretty strong. Indeed, it was obtained for any small $u \in L^2$. It would have been sufficient to prove the coercivity of the kernel K^ε on the strict subspace:

(7.4) $$\mathcal{V}_\varepsilon = \left\{u \in L^2(0,1), \quad a(t=1,\cdot) \equiv 0, \quad \text{where } a \text{ is the solution to system (1.10)}\right\}.$$

For other systems, it may be easier (or necessary) to restrict the study of the integral operator K^ε to the subspace $\mathcal{V}_\varepsilon$ in order to obtain a conclusion.

As a perspective, an example of such an open problem is the small-time controllability of the non-linear Korteweg de Vries equation for critical domains. Indeed, in [48], Rosier proved that the KdV equation was small-time locally controllable for non critical domains using the linearized system. Then in [25], Coron and Crépeau proved that, for the first critical length, small-time local controllability holds thanks to a third order expansion. In [17] and [18], Cerpa then Cerpa and Crépeau proved that large time local controllability holds for all critical lengths. It remains an open question to know whether small-time local controllability holds for the second critical length. Maybe our method could be adapted to this setting or inspire a new proof.

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Appendix

Weakly singular integral operators

This appendix is devoted to an explanation of Lemma 16. Although a full proof would exceed the scope of this article, we provide here a brief overview of a general method introduced by Torres in [51] to study the regularization properties of weakly singular integral operators. Our presentation is also inspired by a posterior work of Youssfi, who states a very closely related lemma in [54, Remark 6.a].

Let $n \geq 1$. Singular integral operators on $\mathbb{R}^n$ have been extensively studied since the seminal works of Calderón and Zygmund (see [16] and [15]). These integral operators are defined by the singularity of their kernel along the diagonal by an estimate of the form:

$$|K(x,y)| \leq C\,|x-y|^{-n}\,. \tag{A.1}$$

In estimate (A.1), the exponent $-n$ is critical. Indeed, the margins of such kernels are almost in L^1_{loc}. Here, we are interested in a class of integral operators for which the singularity along the diagonal is weaker. Thus, we expect that they exhibit better smoothing properties. Throughout this section, we denote $\Omega = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n, x \neq y\}$.

DEFINITION 5 (Weakly singular integral operator). – *Let* $0 < s < 1$ *and* $0 < \delta \leq 1$. *Consider a kernel* K*, continuous on* Ω*, satisfying:*

$$|K(x,y)| \leq \kappa\,|x-y|^{-n+s}\,, \tag{A.2}$$

$$\left|K(x',y) - K(x,y)\right| \leq \kappa\left|x'-x\right|^{\delta}|x-y|^{-n+s-\delta}\,, \quad \textit{for } \left|x'-x\right| \leq \frac{1}{2}\,|x-y|\,, \tag{A.3}$$

$$\left|K(x,y') - K(x,y)\right| \leq \kappa\left|y'-y\right|^{\delta}|x-y|^{-n+s-\delta}\,, \quad \textit{for } \left|y'-y\right| \leq \frac{1}{2}\,|x-y|\,. \tag{A.4}$$

We introduce the associated integral operator T_K*, continuous from* $\mathscr{D}(\mathbb{R}^n)$ *to* $\mathscr{D}'(\mathbb{R}^n)$*, by defining:*

$$\forall f \in \mathscr{D}(\mathbb{R}^n)\,, \forall x \in \mathbb{R}^n, \;\; T_K(f)(x) = \int K(x,y) f(y)\mathrm{d}y. \tag{A.5}$$

Under these assumptions, we write $T_K \in \mathrm{WSIO}(s,\delta)$.

Definition 5 can be extended for $s \geq 1$. Conditions (A.2), (A.3) and (A.4) must then be extended to the derivatives $\partial_x^\alpha \partial_y^\beta K$ for $\alpha + \beta \leq s$. We restrict ourselves to the simpler setting $0 < s < 1$ as it is sufficient for our study. We define the operator T_K from its kernel K (as this is the case for our applications). Proceeding the other way around is possible but would require more care in the sequel (namely, the so-called *weak boundedness property* to ensure that (A.5) holds; see [54]).

A.1. Atomic and molecular decompositions for Triebel-Lizorkin spaces

We recall the definitions of classical functional spaces involved in this appendix. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ be such that $\varphi(\xi) = 0$ for $|\xi| \geq 1$ and $\varphi(\xi) = 1$ for $|\xi| \leq \frac{1}{2}$. We introduce $\psi(\xi) = \varphi(\xi/2) - \varphi(\xi)$. Hence, $\psi \in \mathscr{S}(\mathbb{R}^n)$ and is supported in the annulus $\{\frac{1}{2} \leq |\xi| \leq 2\}$. We will denote $\dot{\Delta}_j$ and $\dot{S}_j$ the convolution operators with symbols $\psi(2^{-j}\xi)$ and $\varphi(2^{-j}\xi)$.

Definition 6 (Homogeneous Besov space). – *For $\alpha \in \mathbb{R}$, $1 \le p, q \le \infty$, the homogeneous Besov space $\dot{B}_p^{\alpha,q}$ is defined by the finiteness of the norm (with standard modification for $q = \infty$):*

$$\|f\|_{\dot{B}_p^{\alpha,q}} = \left(\sum_{j \in \mathbb{Z}} 2^{\alpha q j} \left\| \dot{\Delta}_j f \right\|_p^q \right)^{1/q} . \tag{A.6}$$

Definition 7 (Homogeneous Triebel-Lizorkin space). – *For $\alpha \in \mathbb{R}$, $1 \le p, q < \infty$, the homogeneous Triebel-Lizorkin space $\dot{F}_p^{\alpha,q}$ is defined by the finiteness of the norm:*

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \left\| \left(\sum_{j \in \mathbb{Z}} 2^{\alpha q j} |\dot{\Delta}_j f|^q \right)^{1/q} \right\|_p . \tag{A.7}$$

Frazier and Jawerth introduced *atoms* and *molecules* both in the context of Besov spaces ([31]) and Triebel-Lizorkin spaces ([32] and [33]). They proved that the norms on these spaces are then translated into sequential norms on the sequence of coefficients of the decomposition. A linear operator will be continuous between two Triebel-Lizorkin spaces if and only if it maps smooth atoms of the first to smooth molecules of the second. The following definitions are borrowed from [51]. For simplicity, we restrict them to the case $1 \le p, q \le +\infty$.

Definition 8 (Smooth atom). – *Let $\alpha \in \mathbb{R}$ and Q be a dyadic cube in $\mathbb{R}^n$ of side length ℓ_Q. A smooth α-atom, associated with the cube Q is a function $a \in \mathcal{D}(\mathbb{R}^n)$ satisfying:*

$$\operatorname{supp}(a) \subset 3Q, \tag{A.8}$$

$$\int x^\gamma a(x) \mathrm{d}x = 0, \quad \forall \gamma \in \mathbb{Z}^n \text{ s.t. } |\gamma| \le \max\{0, [-\alpha]\}, \tag{A.9}$$

$$|\partial_x^\gamma a(x)| \le \ell_Q^{-|\gamma|}, \quad \forall \gamma \in \mathbb{Z}^n \text{ s.t. } |\gamma| \le \max\{0, [\alpha]\} + 1. \tag{A.10}$$

In condition (A.8), $3Q$ denotes the cube with same center as Q but a tripled side length. Multiple normalization choices are possible for condition (A.10). We choose to only include the decay corresponding to the regularity of the atom. This choice only impacts the formula to compute the size of a function from its decomposition on atoms. We have the following representation theorem:

Lemma 27 (Theorem 5.11, [34]). – *Let $\alpha \in \mathbb{R}$, $1 \le p, q < \infty$. Let $f \in \dot{F}_p^{\alpha,q}$. There exists a sequence of reals $(s_Q)_{Q \in \mathcal{Q}}$ indexed by the set $\mathcal{Q}$ of dyadic cubes of $\mathbb{R}^n$ and a sequence of atoms $(a_Q)_{Q \in \mathcal{Q}}$ such that $f = \sum_Q s_Q a_Q$. Moreover, there exists a constant C independent on f such that:*

$$\left\| \left(\sum_Q \ell_Q^{-\alpha q} |s_Q|^q |\chi_Q(x)|^q \right)^{1/q} \right\|_p \le C \|f\|_{\dot{F}_p^{\alpha,q}} . \tag{A.11}$$

The reciprocal inequality to (A.11) is true even for a wider class of functions, the class of molecules.

DEFINITION 9 (Smooth molecule). – *Let $\alpha \in \mathbb{R}$, $M > n$ and $\alpha - [\alpha] < \delta \leq 1$. Let Q be a dyadic cube in $\mathbb{R}^n$ of side length ℓ_Q and center x_Q. A (δ, M) smooth α-molecule associated with Q is a function m satisfying:*

$$|m(x)| \leq \left(1 + \ell_Q^{-1}\left|x - x_Q\right|\right)^{-\max\{M, M-\alpha\}}, \tag{A.12}$$

$$\int x^\gamma m(x)\mathrm{d}x = 0, \quad \forall \gamma \in \mathbb{Z}^n \text{ s.t. } |\gamma| \leq [-\alpha], \tag{A.13}$$

$$|\partial_x^\gamma m(x)| \leq \ell_Q^{-|\gamma|}\left(1 + \ell_Q^{-1}\left|x - x_Q\right|\right)^{-M}, \quad \forall \gamma \in \mathbb{Z}^n \text{ s.t. } |\gamma| \leq [\alpha] \tag{A.14}$$

and an additional Hölder regularity estimate for all $\gamma \in \mathbb{Z}^n$ such that $|\gamma| = [\alpha]$:

$$\left|\partial_x^\gamma m(x) - \partial_x^\gamma m(x')\right| \leq \ell_Q^{-|\gamma|-\delta}\left|x - x'\right|^\delta \sup_{|z|\leq|x-x'|}\left(1 + \ell_Q^{-1}\left|z - (x - x_Q)\right|\right)^{-M}. \tag{A.15}$$

In the definition of a molecule, conditions (A.14) and (A.15) are void by convention if $\alpha < 0$. When $\alpha \geq 0$, condition (A.14) implies (A.12). When $\alpha > 0$, condition (A.13) is void. We have:

LEMMA 28 (Theorem 5.18, [34]). – *Let $\alpha \in \mathbb{R}$, $M > n$ and $\alpha - [\alpha] < \delta \leq 1$. Consider a sequence of reals $(s_Q)_{Q\in\mathcal{Q}}$ indexed by the set $\mathcal{Q}$ of dyadic cubes of $\mathbb{R}^n$ and a sequence of (δ, M) smooth α-molecules $(m_Q)_{Q\in\mathcal{Q}}$. Let $f = \sum_Q s_Q m_Q$. There exists a constant C independent on f such that:*

$$\|f\|_{\dot{F}_p^{\alpha,q}} \leq C\left\|\left(\sum_Q \ell_Q^{-\alpha q}|s_Q|^q|\chi_Q(x)|^q\right)^{1/q}\right\|_p. \tag{A.16}$$

A.2. Circumventing the null average condition

When dealing with singular integral operators, difficulties arise when $T(1) \neq 0$. Most regularity results involve some regularity condition on $T(1)$ (see, for example the early paper [28]). To circumvent this difficulty when handling weakly singular integral operators, we will write $T_K = \tilde{T}_K + \pi$ where $\tilde{T}_K$ satisfies the same regularity estimates as T_K but is such that $\tilde{T}_K(1) = 0$ and π is defined as a paraproduct, for which we can get direct smoothing estimates in the appropriate spaces. For two functions f, g, we introduce the following paraproduct π, inspired by ideas of J.-M. Bony (see the seminal work [12], the nice introduction to paraproducts [10] for a quick overview or [3, Section 2.6.1] for a complete detailed presentation):

$$\pi_g(f) = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j(g)\dot{S}_{j-2}(f). \tag{A.17}$$

LEMMA 29 (Lemma 4, [54]). – *Let $0 < s < \delta \leq 1$ and $T_K \in \mathrm{WSIO}(s, \delta)$. Then, $T_K(1) \in \dot{B}_\infty^{s,\infty}$. Moreover, there exists $C = C(s, \delta)$ such that: $\|T_K(1)\|_{\dot{B}_\infty^{s,\infty}} \leq C\kappa(T_K)$ where $\kappa(T_K)$ is the constant associated to T_K in Definition 5.*

LEMMA 30 (Remark 2, [54]). – *Let $1 \leq p, q < \infty$, $t < 0$ and $s \in \mathbb{R}$. There exists $C = C(p, q, t, s)$ such that, for any $b \in \dot{B}_\infty^{s,\infty}$, π_b is continuous from $\dot{F}_p^{t,q}$ to $\dot{F}_p^{t+s,q}$ and the following estimate holds:*

$$\forall f \in \dot{F}_p^{t,q}, \|\pi_b(f)\|_{\dot{F}_p^{t+s,q}} \leq C\,\|b\|_{\dot{B}_\infty^{s,\infty}}\,\|f\|_{\dot{F}_p^{t,q}}. \tag{A.18}$$

LEMMA 31 (Lemma 2, [54]). – *Let* $0 < s < 1$ *and* $0 < \delta \leq 1$. *Take* $b \in \dot{B}^{s,\infty}_{\infty}$. *Then, the operator* $\pi_b \in \mathrm{WSIO}(s,\delta)$. *Moreover, there exists a constant* $C(s)$ *independent of b such that,* $\kappa(\pi_b) \leq C(s)\|b\|_{\dot{B}^{s,\infty}_{\infty}}$, *where* $\kappa(\pi_b)$ *is the constant in Definition 5 associated to the operator* π_b.

Combining these lemmas allows us to circumvent the $T(1) = 0$ condition. Indeed:

LEMMA 32. – *Let* $0 < s < \delta \leq 1$ *and* $1 \leq p, q < \infty$. *Let* $t \in \mathbb{R}$ *be such that* $-s < t < 0$. *There exists a constant* C *such that, for* $T_K \in \mathrm{WSIO}(s,\delta)$, T_K *is continuous from* $\dot{F}^{t,q}_p$ *into* $\dot{F}^{t+s,q}_p$ *and we have:*

$$\forall f \in \dot{F}^{t,q}_p, \quad \|T_K(f)\|_{\dot{F}^{t+s,q}_p} \leq C\kappa(T_K)\,\|f\|_{\dot{F}^{t,q}_p}, \tag{A.19}$$

where $\kappa(T_K)$ *is the constant associated to* T_K *in Definition 5.*

Proof. – Let $T_K \in \mathrm{WSIO}(s,\delta)$. Thanks to Lemma 29, $T_K(1) \in \dot{B}^{s,\infty}_{\infty}$ and $\|T_K(1)\|_{\dot{B}^{s,\infty}_{\infty}} \lesssim \kappa(T_K)$. Thanks to Lemma 31, $\pi_{T_K(1)} \in \mathrm{WSIO}(s,\delta)$ and $\kappa(\pi_{T_K(1)}) \lesssim \kappa(T_K)$. Hence, we can define $\tilde{T}_K := T_K - \pi_{T_K(1)}$ and $\tilde{T}_K \in \mathrm{WSIO}(s,\delta)$, with a constant $\kappa(\tilde{T}_K) \lesssim \kappa(T_K)$. Moreover, since $\pi_b(1) = b$ for any b, $\tilde{T}_K(1) = 0$. Thanks to Lemma 30, proving the continuity of $\tilde{T}_K$ is sufficient to obtain (A.19).

Let a_Q be a smooth t-atom. We consider $m_Q = \tilde{T}_K(a_Q)$. The next step is to prove that m_Q is almost a (δ, M) smooth $(t+s)$-molecule, with $M = n + s - \delta > n$. As noted above, since $t + s > 0$, we only need to check (A.14) and (A.15). Indeed, lengthy computations and the essential condition $\tilde{T}_K(1) = 0$ provide the existence of a constant D independent on the atom a_Q such that:

$$\left|m_Q(x)\right| \leq D\ell_Q^s \left(1 + \ell_Q^{-1}\left|x - x_Q\right|\right)^{-M}, \tag{A.20}$$

$$\left|m_Q(x) - m_Q(x')\right| \leq D\ell_Q^s \ell_Q^{-\delta}\left|x - x'\right|^{\delta} \sup_{|z|\leq|x-x'|}\left(1 + \ell_Q^{-1}\left|z - (x - x_Q)\right|\right)^{-M}. \tag{A.21}$$

Hence $\tilde{m}_Q := D^{-1}\ell_Q^{-s} m_Q$ is a molecule. For techniques used to prove (A.20) and (A.21), we refer the reader to [51] and [54]. To conclude the proof, we use Lemma 27 and Lemma 28. For $f \in \dot{F}^{t,q}_p$, we write $f(x) = \sum_Q s_Q a_Q(x)$ and each $\tilde{m}_Q = D^{-1}\ell_Q^{-s} T_K(a_Q)$ is a molecule. Thus,

$$\begin{aligned}
\|T_K(f)\|_{\dot{F}^{t+s,q}_p} &= \left\| \sum_Q (D\ell_Q^s s_Q)\cdot m_Q(x) \right\|_{\dot{F}^{t+s,q}_p} \\
&\lesssim \left\| \left(\sum_Q \ell_Q^{-(t+s)q} D^q \ell_Q^{sq} |s_Q|^q |\chi_Q(x)|^q \right)^{1/q} \right\|_p \\
&\lesssim \left\| \left(\sum_Q \ell_Q^{-tq} |s_Q|^q |\chi_Q(x)|^q \right)^{1/q} \right\|_p \\
&\lesssim \|f\|_{\dot{F}^{t,q}_p}.
\end{aligned} \tag{A.22}$$

Equation (A.22) concludes the proof. □

Triebel-Lizorkin spaces offer a natural framework for atomic and molecular decompositions. Of course, setting $p = q = 2$ in the results above also yields results for the more classical homogeneous Sobolev spaces $\dot{H}^{\alpha}$. Thus, Lemma 32 tells us that operators of WSIO(s, δ) continuously map $\dot{H}^{t}$ into $\dot{H}^{t+s}$ for $-s < t < 0$. In particular, this is valid for $s = 1/2$ and $t = -1/4$.

A.3. Kernels defined on bounded domains

Most results involving singular integral operators concern kernels defined on the full space $\mathbb{R}^n \times \mathbb{R}^n$. Here, for finite time controllability, we need to adapt these results to a setting where the kernels are defined on squares, eg. $[0,1] \times [0,1]$. Atoms and molecules are localized functions. Thus, it would be possible to carry on the same proof as above for bounded domains, provided that the analogs of the representation Lemmas 27 and 28 exist for Triebel-Lizorkin spaces on bounded domains. In this paragraph, we give another approach, which consists in proving that a kernel defined on a bounded domain can be extended while satisfying the same estimates.

LEMMA 33. – *Let $n = 1$, $0 < s < 1$ and $0 < \delta \le 1$. Consider a kernel K, defined and continuous on $\Omega_1 = \{(x, y) \in [0,1]^2, x \neq y\}$, satisfying:*

$$|K(x, y)| \le \kappa\, |x - y|^{-1+s}, \tag{A.23}$$

$$\left|K(x', y) - K(x, y)\right| \le \kappa \left|x' - x\right|^{\delta} |x - y|^{-1+s-\delta}, \quad \textit{for } \left|x' - x\right| \le \frac{1}{2}|x - y|, \tag{A.24}$$

$$\left|K(x, y') - K(x, y)\right| \le \kappa \left|y' - y\right|^{\delta} |x - y|^{-1+s-\delta}, \quad \textit{for } \left|y' - y\right| \le \frac{1}{2}|x - y|. \tag{A.25}$$

Then there exists a kernel $\bar{K}$ on $\mathbb{R} \times \mathbb{R}$, continuous on Ω, such that:

- *$\bar{K}$ is an extension of K: $\bar{K}_{|\Omega_1} = K$,*
- *$\bar{K}$ is a weakly singular integral operator of type (s, δ) on Ω,*
- *$\bar{K}$ is associated a constant $\kappa(\bar{K}) \le C\kappa(K)$, where C is independent of K, s and δ.*

Proof. – We start by defining $\bar{K}(x, y)$ on the infinite strip $-1 < y - x < 1$. For $(x, y) \in \Omega_1$, we set $\bar{K}(x, y) = K(x, y)$. Outside of the initial square, we extend by continuity the values taken on the sides of the square and we choose an extension that is constant along all diagonal lines. Therefore, we define $\bar{K}(x, y)$ as:

$$\begin{aligned} K(1 + x - y, 1) &\quad \text{for} \quad 1 \le y,\ 0 < y - x < 1, \\ K(0, y - x) &\quad \text{for} \quad x \le 0,\ 0 < y - x < 1, \\ K(1, 1 + y - x) &\quad \text{for} \quad 1 \le x,\ 0 < x - y < 1, \\ K(x - y, 0) &\quad \text{for} \quad y \le 0,\ 0 < x - y < 1. \end{aligned} \tag{A.26}$$

Outside of the strip, we set:

$$\begin{aligned} \bar{K}(x, y) &= K(0,1)|x - y|^{-1+s}, \quad \text{for } y - x \ge 1, \\ \bar{K}(x, y) &= K(1,0)|x - y|^{-1+s}, \quad \text{for } x - y \ge 1. \end{aligned} \tag{A.27}$$

This completes the definition of $\bar{K}$ on Ω. By construction, one checks that $\bar{K}$ is continuous on Ω. By construction, $\bar{K}$ also satisfies (A.23) on Ω_1, on the whole strip $-1 \le y - x \le 1$

thanks to (A.27) and on the half spaces $y-x \geq 1$ and $y-x \leq -1$ thanks to the decay chosen in (A.27).

The Hölder regularity estimates (A.24) and (A.25) are a little tougher. By symmetry, one only needs to prove, for example, (A.24) on the half place $\mathscr{H} = \{(x,y) \in \mathbb{R}\times\mathbb{R}, \quad y-x > 0\}$. We write $\mathscr{H} = \tilde{\mathscr{H}} \cup \mathscr{H}_1 \cup \mathscr{H}_- \cup \mathscr{H}_+$, where:

$$\begin{aligned} \tilde{\mathscr{H}} &= \{(x,y) \in \mathscr{H}, \quad y-x > 1\}, \\ \mathscr{H}_1 &= \{(x,y) \in \mathscr{H}, \quad 0 \leq x \text{ and } y \leq 1\}, \\ \mathscr{H}_+ &= \{(x,y) \in \mathscr{H}, \quad y-x \leq 1 \text{ and } 1 < y\}, \\ \mathscr{H}_- &= \{(x,y) \in \mathscr{H}, \quad y-x \leq 1 \text{ and } x < 0\}. \end{aligned} \tag{A.28}$$

Let $(x,y) \in \mathscr{H}$ and $(x',y) \in \mathscr{H}$ with $|x-x'| \leq \frac{1}{2}|x-y|$. If both points belong to the same subdomain, then the Hölder regularity estimate in the x direction for $\bar{K}$ is a direct consequence either of (A.27) on $\tilde{\mathscr{H}}$, of (A.26) on $\mathscr{H}_\pm$ and of the hypothesis on K on $\mathscr{H}_1$. If the two points belong to different subdomains, we use a triangular inequality involving a point at the boundary separating the two subdomains. As an example of such a situation, if $x < 0 < x'$ and $y < x+1$, then $(x,y) \in \mathscr{H}_-$ and $(x',y) \in \mathscr{H}_1$. We have:

$$\begin{aligned} \left|\bar{K}(x,y) - \bar{K}(x',y)\right| &= \left|K(0,y-x) - K(x',y)\right| \\ &\leq |K(0,y-x) - K(0,y)| + \left|K(0,y) - K(x',y)\right| \\ &\leq \kappa|x|^\delta|x-y|^{-1+s-\delta} + \kappa|x'|^\delta|x'-y|^{-1+s-\delta} \\ &\leq 5\kappa|x-x'|^\delta|x-y|^{-1+s-\delta}. \end{aligned} \tag{A.29}$$

The last inequality comes from the fact that $|x'|, |x| \leq |x-x'|$ and $|x'-y|^{-1+s-\delta} \leq 4|x-y|^{-1+s-\delta}$ for $|x-x'| \leq \frac{1}{2}|x-y|$. The details of the other situations are left to the reader. □

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