# ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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*A Floer Fundamental group*

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# A FLOER FUNDAMENTAL GROUP

## BY Jean-François BARRAUD

ABSTRACT. – The main purpose of this paper is to provide a description of the fundamental group of a symplectic manifold in terms of Floer theoretic objects. As an application, we show that when counted with a suitable notion of multiplicity, non degenerate Hamiltonian diffeomorphisms have enough fixed points to generate the fundamental group.

RÉSUMÉ. – L'objet de cet article est de donner une description du groupe fondamental d'une variété symplectique en terme d'objets de la théorie de Floer. À titre d'application, on montre que tous les difféomorphismes hamiltoniens non dégénérés ont, si on les compte avec une notion convenable de multiplicité, suffisamment de points fixes pour engendrer le groupe fondamental.

## 1. Introduction

### 1.1. Presentation of the results

In many ways, the topology of a space influences its geometry, and this is particularly true in symplectic geometry. Having a symplectic interpretation of a topological invariant is a good tool to explore this relationship. The celebrated Floer Homology ([7, 6]) is of course a strong illustration of this phenomenon. Introduced to prove the homological version of the Arnold conjecture ([1]), it quickly became one of the most powerful tools in symplectic geometry.

However, all the techniques derived from the original Floer construction are homological, or at least chain complex based in nature. The notion of cobordism (among moduli spaces) is even at the root of the original ideas of M. Gromov [9] of using pseudo-holomorphic curves to derive invariants in symplectic geometry. The use of local coefficients in Floer complexes allows Floer theory to involve some homotopical invariants, but purely homotopical tools are still missing, and it is the goal of this paper to provide a Floer theoretic interpretation of the fundamental group.

All the objects this construction is based on are still classical Floer theoretic objects, but the essential non Abelian phenomena that make the difference between the fundamental

group and the first homology group are caught by a deeper use of 1-dimensional moduli spaces, and the use of "augmentations".

More precisely, let $(M, \omega)$ be a connected closed monotone symplectic manifold and choose an Hamiltonian function $H$ on $M$, a possibly time dependent almost complex structure $J$ compatible with $\omega$, and a point $\star$ in $M$ to serve as the base point. Recall the Floer trajectories in this setting are (finite energy) maps $u : \mathbb{R} \times S^1 \to M$ satisfying the Floer equation

$$\frac{\partial u}{\partial s}(s, t) + J_t(u(s, t))\frac{\partial u}{\partial t}(s, t) = J_t(u(s, t))X_{H_t}(u(s, t)),$$

where $X_H$ is the Hamiltonian vector field associated to $H$.

Using a cutoff function $\chi$ to turn off the non homogeneous Hamiltonian term on the positive end of the tube (resp. on both ends but preserving it on an annulus of varying modulus) allows to define moduli spaces denoted by $\mathcal{M}(x, \emptyset)$ (resp. $\mathcal{M}(\star, \emptyset)$), which are Floer counterparts of Morse unstable manifolds (see the comments after Definition 2.4). It is a classical result of Floer theory that for a generic set of auxiliary data $(H, J, \star, \chi)$, all these moduli spaces are smooth finite dimensional manifolds.

Similarly to the Morse setting where a loop can be seen as a concatenation of paths associated to unstable manifolds of index 1 critical points, we use the components of the above 1-dimensional moduli spaces to define a notion of Floer loop (see Definition 2.8). These loops come naturally with concatenation and cancelation relations for which they form a group $\mathcal{L}(H, \star)$. The main statement of the paper is then the following theorem:

THEOREM 1.1. – *There is a natural evaluation map that induces a surjective group homomorphism* $\mathcal{L}(H, J, \star) \longrightarrow\!\!\!\!\!\rightarrow \pi_1(M, \star)$.

A description of the relations is also given, but, although they obviously only depend on $H$, $J$, $\star$, and $\chi$, we resort to an auxiliary Morse function to get a finite presentation for them (see Section 4). Nevertheless, we produce explicit relations such that the generated normal subgroup $\mathcal{R}(H, \star)$ satisfies the following statement:

THEOREM 1.2. – *The evaluation map induces a group isomorphism*

$$\mathcal{L}(H, \star)\big/_{\mathcal{R}(H, \star)} \xrightarrow{\ \sim\ } \pi_1(M, \star) .$$

Notice the construction is presented here in the absolute setting, i.e., Hamiltonian fixed points problem, but also makes sense in the relative one, i.e., intersections of a Lagrangian sub-manifold with its deformations under Hamiltonian isotopies problem. Although the latter can be expected to hold the most interesting applications, we choose to focus on the former for the sake of simplicity and to highlight better the main ideas: the generalization to the latter entails exactly the same issues as for the homology and involves no new idea.

Finally, the construction also makes sense in the stable Morse setting (i.e., study of Morse functions that are quadratic at infinity on $M \times \mathbb{R}^N$). Although the corresponding results have their own interest and would deserve a separate discussion, they will only be quickly sketched without proofs in the last section of this paper (see Section 6), rather as an illustration and a simplified finite dimensional model of the Floer setting.

A natural outcome of this construction is an estimate on the number of fixed points of Hamiltonian diffeomorphisms, but not in usual way, since a notion of multiplicity has to be introduced. Indeed, rather than the critical points themselves, the relevant objects required to build loops are their unstable manifolds (called "steps" in the sequel), and while to one critical point corresponds exactly one unstable manifold in the Morse setting, Floer counterparts of unstable manifolds may have several components, which have all to be taken into account.

Counting the number $\nu_J(x)$ (resp. $\nu_J(\star)$) of steps through a given Conley-Zehnder index $1 - n$ fixed point $x$ (resp. $\star$) defines a notion of multiplicity for these points (that depends on the almost complex structure, see Definition 2.10 for more details). We then have the following theorem:

THEOREM 1.3. – *Let $\rho(\pi_1(M))$ denote the minimal number of generators of the fundamental group. Then:*

$$(1) \qquad \nu_J(\star) + \sum_{|y|=1} \nu_J(y) \geq \rho(\pi_1(M)),$$

*where the sum runs over the contractible 1-periodic orbits, or more precisely over the homotopy classes of cappings of such orbits with Conley-Zehnder index $1 - n$.*

REMARK 1. – The number $\nu_J(\star)$ is a sum of contributions of index $-n$ fixed points (see Definition 2.10), so that inequality (1) can be interpreted as a lower bound for the number of particular Floer configurations associated to fixed points with Conley-Zehnder index $-n$ and $1 - n$.

REMARK 2. – This statement should be compared to its Morse analog, namely that for any Morse function $f : M \to \mathbb{R}$, we have

$$(2) \qquad \sharp \mathrm{Crit}_1(f) \geq \rho(\pi_1(M)),$$

where $\mathrm{Crit}_1(f)$ denotes the set of index 1 critical points.

As already mentioned, the construction, and hence the definition of the multiplicities makes sense in the stable Morse and a fortiori Morse settings (moreover, we claim, without proof, that for $C^2$ small Morse functions, Morse and Floer moduli spaces can be identified like in [7], and that multiplicities coincide in this case). For an index 1 Morse critical point $y$, $\nu_J(y)$ is the number of components of its unstable manifold, and hence always evaluates to 1. Similarly, $\nu_J(\star) + 1$ is the number of Morse trajectories through $\star$ and hence evaluates to 1. As a consequence, $\nu_J(\star) = 0$, so that

$$\nu_J(\star) + \sum_{|y|=1} \nu_J(y) = \sharp \mathrm{Crit}_1(f),$$

and (1) appears as a generalization of (2) to the more general Floer setting.

REMARK 3. – There is no hope to avoid multiplicities in (1) as long as they result from a construction that also applies to the stable Morse setting, which is the case of ours.

Indeed, M. Damian showed in [4] that the stable Morse number (which is the minimal number of critical points of a Morse function which is quadratic at infinity on a product $M \times \mathbb{R}^N$) may be strictly smaller than the Morse number (which is the minimal

number of critical points of a Morse function on $M$), and that stable Morse functions may not have enough points to generate the fundamental group (for instance, such functions do exist on manifolds whose fundamental group is $(A_5)^{20}$).

This implies that the multiplicities in (1) are mandatory, and the construction offers a new point of view on this question: although there may not be enough geometric critical points to generate the fundamental group, it explains how the same point can define several generators to overcome this deficit and still recover the fundamental group.

REMARK 4. – The inequality (1) is obviously different in nature from the Morse inequalities derived from the Floer homology, since one may have $\rho(\pi_1(M)) > \beta_1(M)$ (where $\beta_1(M)$ is the first Betti number of $M$). It is also different from the results of K. Ono and A. Pajitnov ([12], see below) and more generally from any result based on the algebraic study of a chain complex that would also apply to the stable Morse setting. Indeed, examples are known of stable Morse functions that have strictly less critical points than the minimal number of generators of the fundamental group.

The role and the control of the contributions of the multiplicities in general is a deep and intriguing question, closely related to the estimation of the minimal number of periodic orbits.

The following theorem ensures the existence of at least one Hamiltonian periodic orbit with Conley-Zehnder index $1 - n$ and non vanishing multiplicity provided the fundamental group is non trivial:

THEOREM 1.4. – *Let $(M, \omega)$ be a monotone symplectic manifold. Suppose $\pi_1(M) \neq \{1\}$. Then every non degenerate Hamiltonian function has to have at least one contractible $1$-periodic orbit of Conley Zehnder index $1 - n$. Moreover, for a generic choice of possibly time dependent almost complex structure, at least one such orbit has non vanishing multiplicity.*

In particular, this result provides at least one index $1 - n$ orbit even if the first homology group of the manifold vanishes, provided the fundamental group is non trivial.

One interesting feature of this theorem is that its proof is essentially geometric, where the usual Floer technics are rather algebraic: it comes down to patching suspensions of 1-dimensional moduli spaces side to side to form a disk. In this sense, although Theorem 1.4 is not strictly speaking a corollary of the Floer interpretation of the fundamental group given in this paper, it derives from the same principal idea, namely that 1-dimensional moduli spaces do contain information that the homology does not catch.

Moreover, the orbit exhibited in this statement has explicitly non vanishing multiplicity, while this is not immediately obvious in other constructions that provide lower bounds on the number of periodic orbits.

**Relation to the Arnold conjecture and other results**

Theorem 1.3 is obviously a variation on the Arnold conjecture. In its non degenerate and strongest form, this conjecture claims that the total number of 1-periodic orbits of a non degenerate Hamiltonian flow can not be less than the minimal number of critical points for a Morse function. A weaker but maybe more convincing and tractable version involves the *stable* Morse number, which is the minimal number of critical points of Morse functions which are quadratic at infinity on products $M \times \mathbb{R}^N$.

This conjecture is closely related to the birth of symplectic geometry itself. A strong breakthrough was achieved by A. Floer who constructed his chain complex to establish the homological version of the Arnold conjecture for compact monotone symplectic manifolds, opening the way to huge efforts by many authors to generalize his original construction.

Until very recently however, work regarding this conjecture was focused on its homological version.

In a recent work [12], K. Ono and A. Pajitnov use the Floer complex with local coefficients to extend these constraints to the Hamiltonian setting. In particular, they show the following

THEOREM 1.5 (K. Ono, A. Pajitnov). – *Suppose $M$ is a weakly monotone symplectic manifold and let $H$ be a Hamiltonian function on it. Then, if they are all non degenerate, the number $p(H)$ of fixed points of the associated Hamiltonian diffeomorphism satisfies*

$$\begin{cases} p(H) \geq 1 & \text{if } |\pi_1(M)| = +\infty, \\ p(H) \geq \delta(\pi_1(M)) & \text{if } |\pi_1(M)| < +\infty, \end{cases}$$

*where $\delta(\pi_1(M))$ is the minimal number of generators of the kernel of the augmentation $\mathbb{Z}[\pi_1(M)] \to \mathbb{Z}$.*

Similarly to the stable Morse setting, the points of view of this theorem and Theorem 1.3 are essentially different: the former focuses on the number of geometric fixed points, while the latter associates possibly several generators to the same geometric orbit to overcome an eventual lack of generators and still recover the fundamental group.

## 1.2. Organization of the paper

In the second section of the paper (the first is this introduction), the main definitions, statements and technical tools are presented. The third section is dedicated to the comparison of Morse and Floer loops, and the proof of Theorem 1.1. The fourth section is devoted to the description of the relations, and the fifth to the proof of the application (Theorem 1.3) and Theorem 1.4. Finally, the last section is a sketch without proofs of the construction in the stable Morse setting.

## 2. Main definitions and statements.

### 2.1. Preliminaries

Let $(M, \omega)$ be a $2n$-dimensional connected compact symplectic manifold without boundary. For technical reasons, $M$ will be supposed to be either

- symplectically aspherical, which means $\omega$ vanishes on the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M)$, or
- monotone, which means $c_1(TM)$ and $\omega$ are proportional by a positive constant on the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M)$.

These assumptions will allow us to easily

- avoid the transversality issues related to the multiply covered negative curves,
- avoid bubbles on 0 and 1-dimensional moduli spaces,
- ensure finiteness of the number of (lifted) orbits of given Conley-Zehnder index.

Given a Hamiltonian function $H : M \times \mathbb{S}^1 \to \mathbb{R}$, we let $X_H$ be the associated Hamiltonian vector field, $\phi_H^t$ its flow, and $\mathscr{P}(H)$ the set of its contractible 1-periodic orbits.

To handle the index computation when $c_1(TM)$ does not vanish on $\pi_2(M)$, we consider the covering $\tilde{\mathscr{P}}(H)$ associated to the group $\pi_2(M)/\ker c_1$. It is obtained from $\mathscr{P}(H)$ by adjoining a capping class to the orbit in the following way:

$$(3) \qquad \tilde{\mathscr{P}}(H) = \{(\gamma, \bar{\gamma}), \gamma \in \mathscr{P}(H), \bar{\gamma} : D \to M, \bar{\gamma}_{|\partial D} = \gamma\}\big/_{\sim}$$

where $D$ is the 2-disk and $(\gamma, \bar{\gamma}) \sim (\gamma', \bar{\gamma}')$ iff $\gamma = \gamma'$ and $\mu_{CZ}(\bar{\gamma}) = \mu_{CZ}(\bar{\gamma}')$ ($\mu_{CZ}$ denoting the Conley-Zehnder index).

Notice that this last equality implies that the two cappings also have the same symplectic area: glued along their boundary, $\bar{\gamma}$ and $\bar{\gamma}'$ with the reversed orientation form a sphere $S$ with vanishing first Chern class, and because of our asphericity or monotonicity assumption, $\iint_S \omega = 0$, which means that $\bar{\gamma}$ and $\bar{\gamma}'$ have the same symplectic area. As a consequence, both the Conley-Zehnder index and the symplectic area are well defined for equivalence classes of cappings.

In the sequel, $\tilde{\mathscr{P}}(H)$ will completely replace $\mathscr{P}(H)$ and no explicit reference to the covering will be made anymore. In particular, what we call a Hamiltonian orbit from now on will in fact be a lift of such an orbit to $\tilde{\mathscr{P}}(H)$.

Each element $x$ in $\tilde{\mathscr{P}}(H)$ then has a well defined Conley-Zehnder index $\mu_{CZ}$. For convenience, we shift the Conley-Zehnder index by $n$ and let

$$|x| = \mu_{CZ}(x) + n.$$

The set of orbits $\tilde{\mathscr{P}}(H)$ splits according to this index, and we let

$$\tilde{\mathscr{P}}_k(H) = \{x \in \tilde{\mathscr{P}}(H), |x| = k\}.$$

Given a (possibly time dependent) $\omega$-compatible almost complex structure $J$, we are interested in the Floer moduli spaces and some classical variants of such that we describe below. Recall the Floer equation for a map $u : \mathbb{R} \times \mathbb{S}^1 \to M$ is the following:

$$(4) \qquad \frac{\partial u}{\partial s} + J_t(u)\Big(\frac{\partial u}{\partial t} - X_H(t, u)\Big) = 0.$$

Moreover, we fix once for all a smooth function $\chi : \mathbb{R} \to [0, 1]$ such that

$$\begin{cases} \chi(s) = 1 & \text{if } s \leq -1, \\ \chi(s) = 0 & \text{if } s \geq 0 \end{cases}$$

and use it to cutoff the Hamiltonian term of the Floer equation on one or both ends of the cylinder by considering the equation

$$(F_i) \qquad \frac{\partial u}{\partial s} + J_{\chi_i(s)t}(u)\Big(\frac{\partial u}{\partial t} - \chi_i(s)\, X_H(t, u)\Big) = 0$$

for different functions $\chi_i : \mathbb{R} \to [0, 1]$, $i = 1, 2, 3$ and $i = (4, R)$, derived from $\chi$, namely (see Figure 1)

1. $\chi_1 \equiv 1$ defines the usual Floer equation,
2. $\chi_2(s) = \chi(s)$ defines the "lower capping equation,"
3. $\chi_3(s) = \chi(-s)$ defines the "upper capping equation,"
4. $\chi_{4,R}(s) = \chi(s - R)\chi(-s - R)$ defines "$R$-perturbed sphere equation".

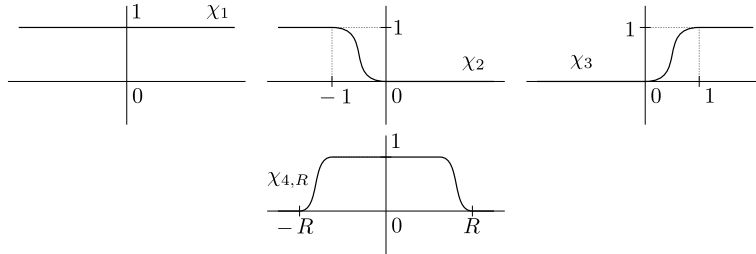

FIGURE 1. Cutoff functions.

In $(F_{4,R})$, $R$ is a real parameter, but notice that for $R \leq 0$, the equation has no Hamiltonian term anymore and does not depend on $R$: $R$ will hence be considered in $[0, +\infty)$.

Recall that the energy of a solution $u$ of this equation is defined as

$$E(u) = \iint \omega(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t} - \chi_i(s) X_H(u))\,ds\,dt = \iint \|\frac{\partial u}{\partial s}\|^2_{\chi_i(s)t}\,ds\,dt,$$

where $\|.\|_t = \omega(., J_t.)$. Solutions of finite energy of this equation have converging ends, either to a point by the classical removal of singularities argument ([2],[10]) if the Hamiltonian term is cut off on this end, or to a Hamiltonian orbit if not ([5]). In the former case, considering the end as a neighborhood of 0 in $\mathbb{C} \setminus \{0\}$, the map $u$ extends holomorphically through 0, and the equations $(F_i)$ above could equivalently be considered as defined on the sphere $\mathbb{CP}^1$ with 2, 1 or no puncture (see for instance [11] for a more uniform description of these equations, or [10] Chapter 8 for the case without punctures, i.e., equation $(F_{4,R})$ with fixed $R$). Anyway, on an end where the Hamiltonian term is cut off, the limit value will be denoted by $u(+\infty)$ or $u(-\infty)$. We abusively but conveniently write that such a trajectory ends at the $\emptyset$ symbol to describe the fact that this limit point is not constrained.

We are interested in the moduli spaces described below and depicted on Figure 2. Let $\star$ be a point in $M$, and $\mathcal{U}$ be the space of smooth maps $u : \mathbb{R} \times \mathbb{S}^1 \to M$ that have finite energy i.e., such that $\iint \|\frac{\partial u}{\partial s}\|^2\,ds\,dt < +\infty$. If $a$ is an oriented disk, let $\bar{a}$ denote the disk with opposite

orientation, and if $b$ is another disk or tube having the same boundary as $a$ , let $\bar{a} \natural b$ denote the gluing of the two.

$$(5) \qquad \overset{\circ}{\mathcal{M}}(y,x) = {}^{\{u \in \mathcal{U}, (F_1), \, \lim_{s \to -\infty} u(s,\cdot)=y \atop \lim_{s \to +\infty} u(s,\cdot)=x}, \text{ and } [y \natural u \natural \bar{x}] = 0\}/_{\mathbb{R}},$$

$$(6) \qquad \overset{\circ}{\mathcal{M}}(y,\emptyset) = \{u \in \mathcal{U}, (F_2), \lim_{s \to -\infty} u(s,\cdot) = y, \text{ and } [y \natural u] = 0\},$$

$$(7) \qquad \overset{\circ}{\mathcal{M}}(\star,x) = {}^{\{u \in \mathcal{U}, (F_3), \, \lim_{s \to -\infty} u(s,\cdot)=\star \atop \lim_{s \to +\infty} u(s,\cdot)=y} \text{ and } [u \natural \bar{x}] = 0\},$$

$$(8) \qquad \overset{\circ}{\mathcal{M}}(\star,\emptyset) = \{(R,u) \in [0,+\infty) \times \mathcal{U}, (F_{4,R}), \lim_{s \to -\infty} u(s,\cdot) = \star \text{ and } [u] = 0\},$$

where the brackets denote classes in $\pi_2(M)/\ker c_1$, and their vanishing expresses the compatibility of the trajectory $u$ with the prescribed lifts of its ends to the covering space $\tilde{\mathcal{P}}(H)$.

Notice that in the last case, the parameter $R$ is allowed to vary, and that the moduli space $\overset{\circ}{\mathcal{M}}(\star,\emptyset)$ is endowed with the map $\overset{\circ}{\mathcal{M}}(\star,\emptyset) \xrightarrow{\pi} [0,+\infty)$ given by $\pi(u,R) = R$.



$$\mathcal{M}(y,x) \qquad \mathcal{M}(y,\varnothing) \qquad \mathcal{M}(\star,x) \qquad \mathcal{M}(\star,\varnothing)$$
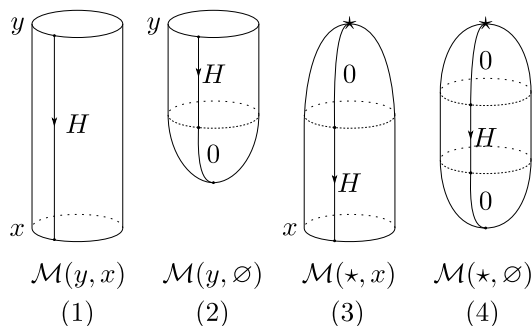$$(1) \qquad\qquad (2) \qquad\qquad (3) \qquad\qquad (4)$$

FIGURE 2. Floer moduli spaces.

The three last types of moduli spaces are used in [13] (in conjunction with a Morse function that we do not use here) to define the PSS homomorphisms and compare Morse and Floer homologies.

Since the elements of $\overset{\circ}{\mathcal{M}}(x,\emptyset)$ for $x \in \tilde{\mathcal{P}}_0(H)$ can be used to define an augmentation on the Floer complex, we use the following terminology:

DEFINITION 2.1. – *Given an index* 0 *Hamiltonian periodic orbit* $x \in \tilde{\mathcal{P}}_0(H)$, *a capping* $\alpha \in \overset{\circ}{\mathcal{M}}(x,\emptyset)$ *is called an "augmentation" of* $x$, *and the couple* $(x,\alpha)$ *an augmented orbit.*

It is well known (see Remark 5 below) that for a generic choice of $(H,J,\star,\chi)$, these moduli spaces are smooth manifolds whose dimension is prescribed by the end constraints and the homotopy class of the tube.

REMARK 5. – The transversality issues for the three first moduli spaces are discussed in [8]. The last moduli space $\overset{\circ}{\mathcal{M}}(\star,\emptyset)$ is somewhat special with this respect, one reason being that for $R = 0$, it involves constant maps, for which the key argument of being "somewhere injective" fails. Transversality for the constant maps is particularly relevant to us since it

implies that such curves are regular for the projection $\mathcal{M}(\star, \emptyset) \to \mathbb{R}$, which in turn means that they can be locally "followed" as $R$ varies (see Proposition 2.6).

The following proposition ensures that constant spheres are indeed regular (for any almost complex structure).

PROPOSITION 2.2. – *Recall the projection* $\overset{\circ}{\mathcal{M}}(\star, \emptyset) \overset{\pi}{\to} [0, +\infty)$. *For* $R = 0$, $\pi^{-1}(R)$ *consists in the single point* $(u_\star, 0)$ *where* $u_\star$ *is the constant map at* $\star$. *This solution is regular, which means that (in the suitable functional spaces) the equation* $(F_{4,0})$ *is a submersion at this point. In particular,* $0$ *is a regular value of* $\pi$.

*Sketch of proof.* – Glossing over the definition of the functional spaces in use, observe that the problem can be reformulated in terms of maps from $\mathbb{CP}^1$ to $M$ in the trivial homology class. For $R = 0$, equation $(F_{4,R})$ simply becomes

$$(9) \qquad Du + J_0(u)\, Du\, i = 0.$$

Points of $\mathcal{M}(\star, \emptyset)$ lying above $R = 0$ are hence $J_0$-holomorphic spheres in the trivial homology class and are therefore constant. The additional condition $u(0) = \star$ implies $\pi^{-1}(0) = \{(u_\star, 0)\}$.

The linearization (with respect to $u$) of the left hand term in (9) at the constant map $u_\star$ leads to a linear operator $L$ defined for maps from $\mathbb{CP}^1$ to a fixed $\mathbb{C}^n = T_\star M$ of the form

$$(10) \qquad L\dot{u} = D\dot{u} + J_0 D\dot{u}\, i,$$

where $J_0 = J_0(\star)$ is constant. The kernel of $L$ consists of the holomorphic spheres in $\mathbb{C}^n$ and hence of the constants. It is therefore $2n$-dimensional and since $2n$ is also the index of $L$, this implies that $L$ is surjective, which easily implies the required submersion property. $\qquad \square$

In particular, under a generic choice of $(H, J, \star, \chi)$, we have :

$$\dim \overset{\circ}{\mathcal{M}}(y, x) = |y| - |x| - 1,$$
$$\dim \overset{\circ}{\mathcal{M}}(y, \emptyset) = |y|,$$
$$\dim \overset{\circ}{\mathcal{M}}(\star, x) = -|x|,$$
$$\dim \overset{\circ}{\mathcal{M}}(\star, \emptyset) = 1.$$

From now on, $(H, J, \star, \chi)$ will be supposed to be chosen so that all these moduli spaces are indeed cut out transversely.

Moreover, all these moduli spaces are compact up to breaking or bubbling off of spheres, and we let

$$\mathcal{M}(x, y) = \overline{\overset{\circ}{\mathcal{M}}(x, y)} \qquad\qquad \mathcal{M}(x, \emptyset) = \overline{\overset{\circ}{\mathcal{M}}(x, \emptyset)}$$
$$\mathcal{M}(\star, y) = \overline{\overset{\circ}{\mathcal{M}}(\star, y)} \qquad\qquad \mathcal{M}(\star, \emptyset) = \overline{\overset{\circ}{\mathcal{M}}(\star, \emptyset)},$$

be the Gromov-Floer compactifications of the previous moduli spaces.

REMARK 6. – Notice however that $\mathcal{M}(\star, \emptyset)$ has a "built-in" (i.e., already present in $\overset{\circ}{\mathcal{M}}(\star, \emptyset)$) boundary component, , that does not come from the Gromov compactification but from the limit case $R = 0$.

In all this paper, only 0 and 1-dimensional moduli spaces will be considered, and no bubbling of sphere can occur on such moduli spaces. This means they will all be compact up to breaking and smooth.

In particular, each 0-dimensional moduli space $_c\mathcal{M}(y,x)$ is compact, and hence finite, and we let

$$\sharp_{\mathrm{abs}}(_c\mathcal{M}(y,x)) = \sum_{\gamma \in _c\mathcal{M}(y,x)} (+1)$$

denote the cardinality of $_c\mathcal{M}(y,x)$.

REMARK 7. – It is usual, when working with pseudo-holomorphic curves or Floer trajectories, to consider the algebraic number $\sharp_{\mathrm{alg}}\ _c\mathcal{M}(x,y)$ of elements in a 0-dimensional moduli space, i.e., to take signs coming from some orientation of the moduli space into account. We stress however that this definition refers to the absolute number, i.e., the sum where each element counts for $+1$.

## 2.2. Floer steps and loops

Given a configuration of two consecutive isolated Floer trajectories $(\beta,\alpha) \in _c\mathcal{M}(y,x) \times _c\mathcal{M}(x,\emptyset)$ with $x \in \tilde{\mathcal{P}}_0(H)$ and $y \in \tilde{\mathcal{P}}_1(H) \cup \{\star\}$, the gluing construction ([6], [10]) gives rise to a one dimensional family of trajectories starting with $(\beta,\alpha)$ and ending at some other broken configuration $(\beta',\alpha') \in _c\mathcal{M}(y,x') \times _c\mathcal{M}(x',\emptyset)$. This relation between $(\beta,\alpha)$ and $(\beta',\alpha')$ will be denoted by

$$(11) \qquad\qquad\qquad (\beta,\alpha) \overset{\sharp}{\to} (\beta',\alpha').$$

REMARK 8. – Recall the gluing construction defines an homeomorphism between a neighborhood of the broken configuration $(\alpha,\beta)$ in the compactified moduli space $_c\mathcal{M}(y,\emptyset)$ and $\{(\beta,\alpha)\} \times [0,\epsilon)$ for some $\epsilon > 0$. In particular, this proves that the compactification is a segment and not a circle, and hence that relation (11) necessarily implies that $(\beta,\alpha) \neq (\beta',\alpha')$.

This "move" from one end of a moduli space to another described above in $_c\mathcal{M}(y,\emptyset)$ makes sense for all kinds of configurations, and will be the main ingredient of all the subsequent constructions. It therefore deserves a general definition:

DEFINITION 2.3. – *A Floer step is an oriented connected component with non empty boundary of a 1-dimensional moduli space.*

REMARK 9. – In particular, the same component defines two steps with opposite orientations.

Depending on the type of moduli space under consideration, there are several types of Floer steps. Floer loops will be built out of special steps, called *Floer loop steps*, which are depicted on Figure 4 and specified in the following definition:

DEFINITION 2.4. – *A Floer loop step is a Floer step in some $_c\mathcal{M}(y,\emptyset)$ for $y \in \tilde{\mathcal{P}}_1(H)$ or in $_c\mathcal{M}(\star,\emptyset)$.*

This somewhat abstruse definition is the heart of the construction and deserves some comments.

An enlightening point of view is that of Morse theory. Consider a function $f$ and a Riemannian metric $g$ on a manifold $M$ such that the pair $(f, g)$ is Morse-Smale. Starting with any generic loop in the manifold and pushing it down by the flow of $f$ deforms it into the concatenation of elementary paths, called "Morse steps," that consist in traveling once, in one or the other direction, along the unstable manifold of an index 1 critical point.

It turns out that these steps can be interpreted from the moduli space point of view: let $y$ be an index 1 critical point and $W^u(y)$ its unstable manifold. To a point $p$ in the unstable manifold is associated a path, namely the piece of Morse trajectory from $y$ to $p$, and there is a one to one correspondence between such trajectory pieces and the unstable manifold (see [3] for a detailed presentation of this point of view, and a nice compactification of the unstable manifold derived from it). More precisely, define an "interrupted" Morse trajectory as a solution of the following modified Morse equation

$$(12) \qquad \frac{d\gamma}{ds} = -\chi(s)\nabla f(\gamma(s))$$

where the cutoff function $\chi$ is the same as the one used in $(F_2)$, i.e., a smooth decreasing function such that $\chi(s) = 1$ for $s \leq -1$ and $\chi(s) = 0$ for $s \geq 0$.

Using the same notation as in the Floer setting, let

$$(13) \qquad \overset{\circ}{\mathcal{M}}_{\text{Morse}}(y, \emptyset) = \{\gamma : \mathbb{R} \to M, (12) \text{ and } \lim_{s \to -\infty} \gamma(s) = y\}.$$

This space is naturally endowed with an evaluation map (recall the trajectories are constant for $s \geq 0$ so $\gamma(+\infty) = \gamma(0)$),

$$\overset{\circ}{\mathcal{M}}_{\text{Morse}}(y, \emptyset) \to W^u(y) \subset M$$
$$\gamma \qquad \mapsto \gamma(+\infty),$$

which is one to one and provides an identification between $\overset{\circ}{\mathcal{M}}_{\text{Morse}}(y, \emptyset)$ and $W^u(y)$.
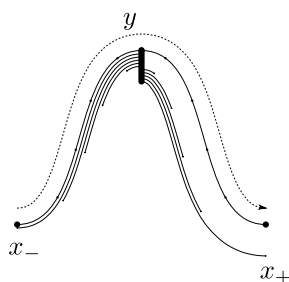


FIGURE 3. The unstable manifold of an index 1 Morse critical point seen as a 1-dimensional moduli space of "interrupted" trajectories.

Moreover, $\overset{\circ}{\mathcal{M}}_{\text{Morse}}(y, \emptyset)$ has a natural compactification $\mathcal{M}_{\text{Morse}}(y, \emptyset)$ as a 1-dimensional segment whose ends are the two "broken" configurations $(\gamma_\pm, \bar{x}_\pm)$ where

– $\gamma_+$ and $\gamma_-$ are the two Morse trajectories rooted at $y$,
– $x_\pm$ is the index 0 critical point such that $x_\pm = \lim_{s \to +\infty} \gamma_\pm$,

– $\bar{x}_\pm$ is the constant solution of (12) at $x_\pm$. It is the one and only one augmentation of $x_\pm$.

The evaluation map extends to this compactification and defines a path running along $W^u(y)$ from $x_-$ to $x_+$, which is the "Morse loop step" associated to $y$.

From this point of view, a Floer loop step through an index 1 periodic orbit is the exact counterpart of a Morse loop step through an index 1 critical point.

REMARK 10. – One noticeable difference between the Morse and Floer settings however, is that the Floer moduli space $\mathcal{M}(y, \emptyset)$ needs not be connected: each connected component can be interpreted as being one "Floer unstable manifold" of the orbit $y$, which hence has to be considered as as many virtually distinct orbits.

REMARK 11. – For orbits $y$ of higher index, the components of the moduli space $\mathcal{M}(y, \emptyset)$ can still be regarded as "Floer unstable manifolds" of $y$. However, there is no control a priori on the topology of such a space: it need not be connected, nor need the connected components be balls.

Similarly, assuming by genericity that $\star$ is not critical for $f$, the Morse counterpart of the space $\mathcal{M}(\star, \emptyset)$ is the collection of segments of the (unique) trajectory passing through $\star$, running from $\star$ down to some arbitrary point $p$ below it along this trajectory. It is in one to one correspondence with (the closure of) the piece of trajectory flowing from $\star$ down to the index 0 critical point below it.
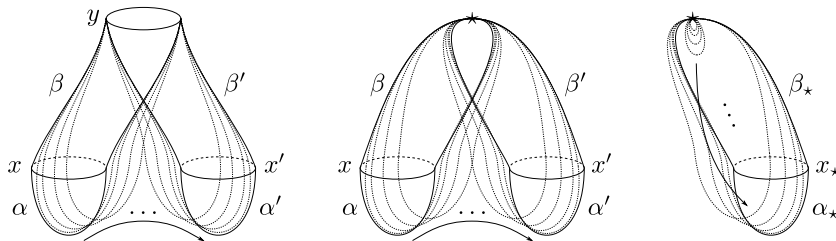


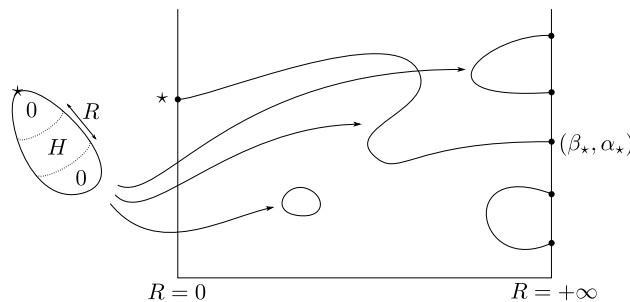FIGURE 4. The three kinds of Floer loop steps.



FIGURE 5. The moduli space $\mathcal{M}(\star, \emptyset)$.

Definitions 2.3 and 2.4 are not very explicit and a more usable description of a step is obtained by specifying its ends:

PROPOSITION 2.5. – *For $y \in \tilde{\mathcal{P}}_1(H)$, a Floer loop step through $y$ is characterized by a quadruple $(\alpha, \beta, \beta', \alpha')$ with $\alpha \in \mathcal{M}(x, \emptyset)$, $\beta \in \mathcal{M}(y, x)$, $\beta' \in \mathcal{M}(y, x')$, and $\alpha' \in \mathcal{M}(x', \emptyset)$ for some $x, x' \in \tilde{\mathcal{P}}_0(H)$ such that*

$$(\beta, \alpha) \xrightarrow{\sharp} (\beta', \alpha').$$

The situation of Floer loop steps through $\star$ is slightly different, as there is one special step that does not look like the others.

Recall that the moduli space $\overset{\circ}{\mathcal{M}}(\star, \emptyset)$ comes with a projection to the non negative reals

$$\overset{\circ}{\mathcal{M}}(\star, \emptyset) \xrightarrow{\pi} [0, +\infty)$$
$$(R, u) \mapsto R.$$

This projection is proper, and extends continuously to a map $\mathcal{M}(\star, \emptyset) \xrightarrow{\pi} [0, +\infty]$ where all the broken configurations lie above $R = +\infty$. Moreover, the gluing construction ensures that exactly one component of $\mathcal{M}(\star, \emptyset)$ ends at each broken configuration.

Observe now that the same holds over $R = 0$: exactly one component of $\mathcal{M}(\star, \emptyset)$ ends at the constant map $(u_\star, 0)$. This is a direct consequence of the regularity of this solution stressed in Proposition 2.2 (surjectivity of $L$ implies that $\pi : \mathcal{M}(\star, \emptyset) \to \mathbb{R}$ is a submersion at $(u_\star, 0)$).

As a consequence, $\mathcal{M}(\star, \emptyset)$ has exactly one connected component that relates $\{\star\}$ to a broken configuration, and all the other components either have no boundary or relate two broken configurations:

$$(14) \qquad \exists! x_\star \in \tilde{\mathcal{P}}_0(H), \exists! (\beta_\star, \alpha_\star) \in \mathcal{M}(\star, x_\star) \times \mathcal{M}(x_\star, \emptyset), \quad (\beta_\star, \alpha_\star) \xrightarrow{\sharp} \star$$

PROPOSITION 2.6. – *There are exactly one orbit $x_\star \in \tilde{\mathcal{P}}_0(H)$ and one pair $(\beta_\star, \alpha_\star) \in \mathcal{M}(\star, x_\star) \times \mathcal{M}(x_\star, \emptyset)$ such that a Floer loop step through $\star$ is*

- *either the special step $\star \xrightarrow{\sharp} (\beta_\star, \alpha_\star)$*
- *or characterized by a quadruple $(\alpha, \beta, \beta', \alpha')$ with $\alpha \in \mathcal{M}(x, \emptyset)$, $\beta \in \mathcal{M}(\star, x)$, $\beta' \in \mathcal{M}(\star, x')$, and $\alpha' \in \mathcal{M}(x', \emptyset)$ for some $x, x' \in \tilde{\mathcal{P}}_0(H)$ such that*

$$(\beta, \alpha) \xrightarrow{\sharp} (\beta', \alpha').$$

REMARK 12. – Considering loop steps entering the second case in the above statement might seem unnatural since, as already mentioned in the Morse setting, only the special step $\star \xrightarrow{\sharp} (\beta_\star, \alpha_\star)$ does exist. In the Floer context however, as well as in the stable Morse setting where examples are much easier to produce (see Section 6 and Figure 20), such steps might exist, and have to be taken into account.

REMARK 13. – Notice there are only finitely many Floer loop steps: there are finitely many periodic orbits, and because of the monotonicity assumption, finitely many lifts of each can have index 0 or 1, and finally, each 0-dimensional moduli space is compact and hence finite.

Notice finally that Floer loop steps are oriented and hence have a start and an end:

DEFINITION 2.7. – *With the notations of Propositions 2.5 and 2.6, a step $(\alpha, \beta, \beta', \alpha')$ is said to start at $\alpha$ and end at $\alpha'$.*

*Similarly, if one of the pairs $(\beta, \alpha)$ or $(\beta', \alpha')$ is replaced by $\star$, the corresponding end is said to be $\star$ itself.*

*Two loop steps are said to be consecutive if the end of the first is the start of the second.*
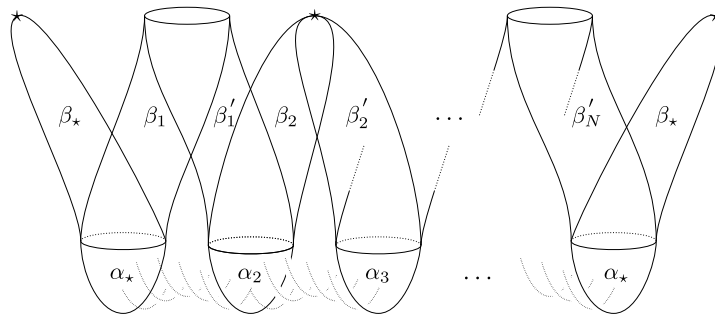


FIGURE 6. A Floer based loop.

DEFINITION 2.8. – *A Floer based loop is a sequence of consecutive Floer loop steps starting and ending at $\star$.*

In other words, a Floer based loop is a sequence

$$(\star, \beta_\star, \alpha_\star), (\alpha_\star, \beta_1, \beta'_1, \alpha_2), (\alpha_2, \beta_2, \beta'_2, \alpha_3), \dots, (\alpha_N, \beta_N, \beta'_N, \alpha_\star), (\alpha_\star, \beta_\star, \star)$$

such that (letting $\alpha_1 = \alpha_{N+1} = \alpha_\star$):

$$\forall i \in \{1, \dots, N+1\}, \quad (\beta_i, \alpha_i) \xrightarrow{\sharp} (\beta'_i, \alpha_{i+1}).$$

Let $\tilde{\mathscr{L}}(H, \star)$ be the set of all Floer based loops. Notice it depends on all the auxiliary data $(H, \star, J, \chi)$ but the dependency on $J$ and $\chi$ is kept implicit to reduce the notation. It carries an obvious concatenation rule that turns it into a semi-group.

It also carries obvious cancelation rules. More explicitly, if $\sigma = (\alpha, \beta, \beta', \alpha')$ is a Floer loop step, define its inverse $\sigma^{-1}$ to be the same step with the opposite orientation:

$$\sigma^{-1} = (\alpha', \beta', \beta, \alpha).$$

Denote by $\sim$ the associated cancelation rules in $\tilde{\mathscr{L}}(H, \star)$:

$$\sigma_1 \dots \sigma_i \sigma_i^{-1} \dots \sigma_N \sim \sigma_1 \dots \sigma_{i-1} \sigma_{i+1} \dots \sigma_N.$$

The concatenation then endows the quotient space

$$(15) \qquad\qquad \mathscr{L}(H, \star) = {}^{\tilde{\mathscr{L}}(H, \star)}\big/_{\sim}.$$

with a group structure.

A Floer loop step being a one parameter family of tubes, evaluation at the $+\infty$ end of the tube defines a path in $M$ (an arbitrary parameterization can be chosen for each step, since we are only interested in the resulting homotopy class), and induces a map

$$(16) \qquad \tilde{\mathscr{L}}(H, \star) \xrightarrow{\text{ev}} \pi_1(M, \star).$$

This map is compatible with both the concatenation and the cancelation rules and hence induces a group homomorphism

$$(17) \qquad \mathscr{L}(H, \star) \xrightarrow{\text{ev}} \pi_1(M, \star).$$

All the objects involved in Theorem 1.1 are now defined and we recall its statement:

THEOREM 2.9. – *With the above notations, the evaluation map induces a surjective homomorphism*

$$(18) \qquad \mathscr{L}(H, \star) \xrightarrow{\text{ev}} \!\!\!\!\!\!\!\!\! \twoheadrightarrow \pi_1(M, \star) \ .$$

The description of the relations still requires the introduction of further technical ingredients, and we postpone it to Section 4 to focus in the next section on the application to the count, with multiplicity, of Hamiltonian periodic orbits, since it only requires the surjectivity.

## 2.3. Application

DEFINITION 2.10. – *Define the multiplicity of a Hamiltonian orbit $y \in \tilde{\mathscr{P}}_1(H)$ as the number of steps through it, that is*

$$\nu_J(y) = \frac{1}{2} \sum_{x \in \tilde{\mathscr{P}}_0(H)} \sharp_{\text{abs}} \, \mathcal{M}(y, x) \cdot \sharp_{\text{abs}} \, \mathcal{M}(x, \emptyset).$$

*Define the multiplicity of the point $\star$ as the number*

$$\nu_J(\star) = \frac{1}{2} \Big( \sum_{x \in \tilde{\mathscr{P}}_0(H)} \sharp_{\text{abs}} \, \mathcal{M}(\star, x) \cdot \sharp_{\text{abs}} \, \mathcal{M}(x, \emptyset) \Big) - \frac{1}{2}.$$

Notice the counting here is not algebraic but geometric: it is not hard to see that the algebraic count would always be 0.

REMARK 14. – Although they may seem to be $\frac{1}{2}\mathbb{N}$ valued, these numbers are in fact integer valued: as already observed, the gluing construction groups the broken trajectories $(\beta, \alpha)$ from some $y \in \tilde{\mathscr{P}}_1(H)$ to $\emptyset$ in pairs, so there is an even number of such, and the same holds for broken trajectories from $\star$ to $\emptyset$ but for $(\beta_\star, \alpha_\star)$, which proves there is an odd number of such configurations.

REMARK 15. – For $x \in \tilde{\mathscr{P}}_0(H)$, letting $\nu_{J,\star}(x) = \frac{1}{2} \cdot \sharp_{\text{abs}} \, \mathcal{M}(\star, x) \cdot \sharp_{\text{abs}} \, \mathcal{M}(x, \emptyset)$, we have $\nu_J(\star) + 1 = \sum_{x \in \tilde{\mathscr{P}}_0(H)} \nu_{J,\star}(x)$, so that $\nu_J(\star)$ can also be expressed as a sum of ($\frac{1}{2}\mathbb{N}$-valued) "multiplicities" of index 0 periodic orbits.

REMARK 16. – Recall from (13) that all the involved moduli spaces, and hence the notion of multiplicity itself, make sense in the Morse setting. However, the Morse situation is much more constrained, and we know there are exactly two trajectories rooted at each index 1 critical point, and exactly one through $\star$: this implies that in the Morse setting, the multiplicity is always 1 for index 1 critical points and 0 for $\star$.

The following statement is a reformulation of Theorem 1.3 and is a direct corollary of our construction. It will be proven in Section 5.1.

THEOREM 2.11. – *Let $\rho(\pi_1(M))$ be the minimal number of elements in a generating family of $\pi_1(M)$. Then*

$$(19) \qquad\qquad \nu_J(\star) + \sum_{y \in \tilde{\mathscr{P}}_1(H)} \nu_J(y) \geq \rho(\pi_1(M)).$$

*In other words, counted with multiplicities, $\{\star\} \cup \tilde{\mathscr{P}}_1(H)$ contains sufficiently many elements to generate $\pi_1(M)$.*

REMARK 17. – According to Remark 16, the left hand side in (19) in the Morse setting is exactly the number of index 1 critical points, so that in this setting the inequality (19) is nothing but the usual lower estimate of the number of index 1 critical points of a Morse function by the minimal number of generators of the fundamental group $\pi_1(M)$.

REMARK 18. – The term $\nu_J(\star)$ in (19) may be unexpected, since it automatically vanishes in the Morse setting. It is a very natural question to ask how essential it is and if it can be controlled.

The Theorem 1.4, stated in a more precise form below as Theorem 2.12 and proven in Section 5.2, ensures that when $\pi_1(M) \neq \{1\}$, the contribution of the index 1 orbits is at least 1, since it provides at least one such orbit with non vanishing multiplicity.

This lower bound on the number of index 1 orbits may seem rather small, but no better result seems to be known without further assumption on the fundamental group yet. Moreover, the proof itself is very geometric and might be of independent interest: it is a variation, in the usual context of PSS moduli spaces, on the main guiding principle of this paper of using 1-dimensional moduli spaces to catch extra information. We stress however that it is not an application of the construction of the fundamental group, but an illustration that the multiplicities cannot be arbitrary.

THEOREM 2.12. – *Suppose $\pi_1(M) \neq \{1\}$. Let $H$ be a non degenerate Hamiltonian function and $J$ a generic choice of a time dependent almost complex structure $J$ compatible with $\omega$.*

*Then $H$ has at least one contractible $1$-periodic orbit with Conley-Zehnder index $1 - n$ and with non vanishing multiplicity with respect to $J$.*
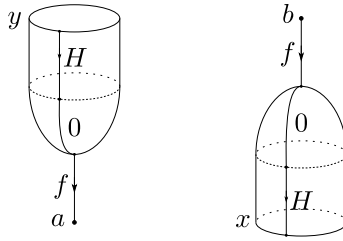
FIGURE 7. Hybrid moduli spaces.

## 2.4. More notations and tools.

2.4.1. *Mixed moduli spaces.* – In addition to the already introduced moduli spaces we will need hybrid Morse-Floer moduli spaces, depicted in Figure 7 and defined below.

Let $f$ be a Morse function and $g$ a Riemannian metric on $M$. By convention, the Morse flow associated to $(f, g)$ is the flow of the negative gradient $-\nabla f$ of $f$ with respect to $g$. Let $\text{Crit}_k(f)$ be the set of index $k$ critical points, and suppose $\text{Crit}_0(f) = \{\star\}$. For $y \in \tilde{\mathscr{P}}_1(H)$ and $a \in \text{Crit}(f)$, we let

$$\overset{\circ}{\mathscr{M}}(y, a) = \{u \in \mathscr{M}(y, \emptyset), u(+\infty) \in W^s(a)\},$$

where $W^s(a)$ is the stable manifold of $a$.

Similarly, for $b \in \text{Crit}_1(f)$ and $x \in \tilde{\mathscr{P}}_0(H)$, we let

$$\overset{\circ}{\mathscr{M}}(b, x) = \{u \in \mathscr{M}(\emptyset, x), u(-\infty) \in W^u(b)\},$$

where $W^u(b)$ is the unstable manifold of $b$.

The couple $(f, g)$ is supposed to be chosen generically, so that all these spaces are cut out transversely. In particular, they have the expected dimensions:

$$\dim \overset{\circ}{\mathscr{M}}(y, a) = |y| - |a| \qquad \dim \overset{\circ}{\mathscr{M}}(b, x) = |b| - |x|,$$

(where the Morse index is also denoted by $|\cdot|$). Moreover, these spaces are compact up to bubbling of spheres and breaking, either at an intermediate Hamiltonian orbit or at an intermediate Morse critical point (see [13]), and the compactifications are denoted by $\mathscr{M}(y, a)$ and $\mathscr{M}(b, x)$. When they are 0 or 1-dimensional, no bubbling can occur on such moduli spaces and they consist of a finite set of points when they are 0-dimensional, and a finite set of circles and segments whose boundary consists in broken configurations when they are 1-dimensional. Finally, recall there is a gluing construction proving every broken configuration does indeed appear on the boundary of a bigger moduli space.

2.4.2. *Crocodile walk.*– We now introduce the main technical tool.

Consider a Hamiltonian orbit $z$ of index 2. Let $B(z)$ be the space of twice broken trajectories from $z$ to $\emptyset$:

$$B(z) = \bigcup_{\substack{|y|=1 \\ |x|=0}} \mathscr{M}(z, y) \times \mathscr{M}(y, x) \times \mathscr{M}(x, \emptyset).$$

For each such trajectory $(\gamma, \beta, \alpha)$ in some $\mathcal{M}(z, y) \times \mathcal{M}(y, x) \times \mathcal{M}(x, \emptyset)$, the gluing construction can take place either at the upper breaking $y$ or at the lower one $x$. Gluing at the upper breaking defines an involution

$$\sharp^\bullet : B(z) \to \quad B(z)$$
$$(\gamma, \beta, \alpha) \mapsto (\gamma', \beta', \alpha),$$

where $(\gamma', \beta')$ is such that $(\gamma, \beta) \overset{\sharp}{\to} (\gamma', \beta')$. Similarly, gluing at the lower breaking defines another involution

$$\sharp_\bullet : B(z) \to \quad B(z)$$
$$(\gamma, \beta, \alpha) \mapsto (\gamma, \beta', \alpha').$$

According to Definition 2.3, upper and lower gluings are both Floer steps, and lower gluings are Floer loop steps.

Iteration of alternately upper and lower gluings then naturally appears as a walk on the space of twice broken trajectories. Moreover, since the intermediate Floer trajectory form a zigzag pattern (see Figure 9), we use the following vocabulary:

DEFINITION 2.13. – *Iteration of alternately upper and lower gluings $\sharp_\bullet \circ \sharp^\bullet \circ \sharp_\bullet \circ \sharp^\bullet \circ \ldots$ will be abbreviated as running a "crocodile walk" on the set $B(z)$ of twice broken trajectories from $z$ to $\emptyset$.*

REMARK 19. – Given a twice broken configuration, the crocodile walk can be started with an upper or a lower gluing: because $\sharp^\bullet$ and $\sharp_\bullet$ are involutions, this only affects the walking direction along the orbit, but not the underlying non-oriented orbit. We consider orbits as oriented however, so through one configuration go exactly two orbits of the crocodile walk, which differ only by the orientation.

REMARK 20. – A more geometric interpretation of the crocodile walk can be given by considering the boundary components of the 2-dimensional moduli space $\mathcal{M}(z, \emptyset)$ (see Figure 8). The set $\mathcal{M}(z, \emptyset) \setminus \overset{\circ}{\mathcal{M}}(z, \emptyset)$ consists in bubbling configurations, which are 2 codimensional, and "boundary components" which are 1 codimensional and consist in broken configurations. The latter components are circles that are either smooth (when they are the product of two smaller moduli spaces without boundary) or have "corners" at twice broken configurations. The crocodile walk consists in moving along such an "angular" boundary component from one corner to the next.

REMARK 21. – Crocodile walks can in fact be defined on any kind of 0-dimensional moduli space of twice broken configurations, like the space of twice broken Floer trajectories between orbits of relative index 3 for instance, or hybrid moduli spaces mixing Floer and Morse trajectories as in the next paragraph.

The crocodile walk is the iteration of a one to one map $(\sharp_\bullet \circ \sharp^\bullet)$ on a finite set, so the orbits all have to be cyclic.

Moreover, if a configuration is reached after an upper (resp. lower) gluing, it has to be left with a lower (resp. upper) one. As a consequence, being cyclic, an orbit has to contain the same number of upper and lower gluings. In particular, it counts an even number of steps.
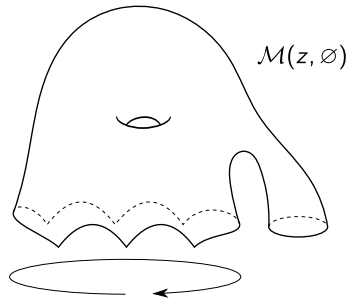
FIGURE 8. The crocodile walk as a way to explore "angular" boundary components of 2-dimensional moduli spaces (here in $\mathcal{M}(z,\emptyset)$ for $z \in \tilde{\mathcal{P}}_2(H)$).

To an orbit of the crocodile walk is not only associated a sequence of twice broken trajectories, but also an abstract polyhedron representing the way the trajectories in the different moduli spaces fit together.
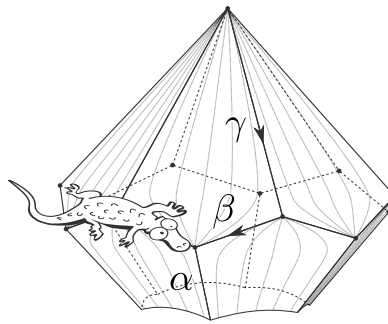


FIGURE 9. An orbit of the crocodile walk on the space of twice broken trajectories from $z \in \tilde{\mathcal{P}}_2(H)$ to $\emptyset$.

An orbit $W$ of the crocodile walk is a sequence

$$((\gamma_1,\beta_1,\alpha_1),(\gamma_2,\beta_1',\alpha_1),(\gamma_2,\beta_2,\alpha_2),\ldots,(\gamma_N,\beta_N,\alpha_N))$$

such that $(\gamma_1,\beta_1,\alpha_1) = (\gamma_N,\beta_N,\alpha_N)$ and

(20) $$(\gamma_k,\beta_k) \stackrel{\sharp}{\to} (\gamma_{k+1},\beta_k') \quad \text{and} \quad (\beta_k',\alpha_k) \stackrel{\sharp}{\to} (\beta_{k+1},\alpha_{k+1}).$$

LEMMA 2.14. – *Let $W$ be an orbit of the crocodile walk like above. There exists an abstract disk $\Delta(W)$ endowed with a continuous map $\Delta(W) \stackrel{\text{ev}}{\to} M$ whose restriction to the boundary is the concatenation of evaluation of the Floer steps $(\beta_1',\alpha_1) \stackrel{\sharp}{\to} (\beta_2,\alpha_2),\ldots,(\beta_{N-1}',\alpha_{N-1}) \stackrel{\sharp}{\to} (\beta_N,\alpha_N)$.*

*Proof.* – Let $\mathcal{M}_k^\bullet$ (resp. $\mathcal{M}_{\bullet k}$) be an abstract copy of the component of the moduli space relating $(\gamma_k,\beta_k)$ to $(\gamma_{k+1},\beta_k')$(resp. $(\beta_k',\alpha_k)$ to $(\beta_{k+1},\alpha_{k+1})$). Let $\Sigma\mathcal{M}_k^\bullet$ be its suspension: it is the suspension of a segment and hence can be identified with the standard diamond.

Recall that before compactification, the evaluation along the real line $\mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{S}^1$ defines a map

$$\overset{\circ}{_{\mathcal{C}}\mathcal{M}_k^\bullet} \times \mathbb{R} \overset{\text{ev}}{\longrightarrow} M.$$

Since the action is strictly decreasing along the Floer trajectories, it can be used to define a parameterization of the trajectories, and to define a continuous map

$$\overset{\circ}{_{\mathcal{C}}\mathcal{M}_k^\bullet} \times [-1, 1] \overset{\text{ev}}{\longrightarrow} M$$

that extends continuously to the compactification and descends to the suspension

$$\Sigma \,_{\mathcal{C}}\mathcal{M}_k^\bullet \overset{\text{ev}}{\longrightarrow} M.$$
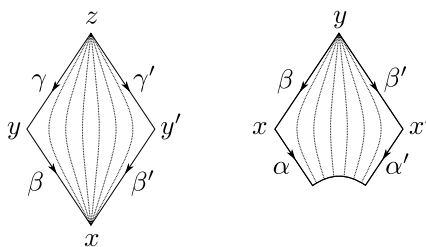


FIGURE 10. Steps suspensions.

We think of $\Sigma \,_{\mathcal{C}}\mathcal{M}_k^\bullet$ as a diamond (see Figure 10), and on the four sides, the evaluation map is the action-normalized evaluation along the broken trajectories $(\gamma_k, \beta_k)$ on the left and $(\gamma_{k+1}, \beta'_k)$ on the right.

A similar construction can also be achieved for the $_{\mathcal{C}}\mathcal{M}_{\bullet k}$ spaces. The lower end of the trajectories is not constrained however, and the suspension should be replaced by the half suspension $\Sigma' \,_{\mathcal{C}}\mathcal{M}_{\bullet k} = {_{\mathcal{C}}\mathcal{M}_{\bullet k} \times [-1, 1]}\big/_{_{\mathcal{C}}\mathcal{M}_{\bullet k} \times \{1\}}$. We think of this as a truncated diamond, or a pentagon (see Figure 10). It is endowed with an evaluation map whose restriction

- to the upper left side (i.e., $[0, 1] \times \{(\beta'_k, \alpha_k)\}$) is $\beta'_k$;
- to the lower left side (i.e., $[-1, 0] \times \{(\beta'_k, \alpha_k)\}$) is $\alpha_k$;
- to the upper right side (i.e., $[0, 1] \times \{(\beta_{k+1}, \alpha_{k+1})\}$) is $\beta_{k+1}$;
- to the lower right side (i.e., $[-1, 0] \times \{(\beta_{k+1}, \alpha_{k+1})\}$) is $\alpha_{k+1}$;
- to the bottom side (i.e., $\{-1\} \times \,_{\mathcal{C}}\mathcal{M}_{\bullet k}$) is the evaluation at the center of the augmentations $\text{ev}(u) = u(+\infty)$.

We identify all these diamonds and pentagons along their shared sides in the order of the gluings appearing in the orbit $W$ (see Figure 11). Formally, we let

$$(21) \qquad \Delta(W) = \Big( \bigsqcup_{k=1}^{N} \Sigma \,_{\mathcal{C}}\mathcal{M}_k^\bullet \sqcup \Sigma' \,_{\mathcal{C}}\mathcal{M}_{\bullet k} \Big)\Big/_\sim$$

where $\sim$ is the identification, for each $k$ of

- the upper right side of $\Sigma \,_{\mathcal{C}}\mathcal{M}_k^\bullet$ with the upper left side of $\Sigma \,_{\mathcal{C}}\mathcal{M}_{k+1}^\bullet$;
- the lower right side of $\Sigma \,_{\mathcal{C}}\mathcal{M}_k^\bullet$ with the upper left side of $\Sigma' \,_{\mathcal{C}}\mathcal{M}_{\bullet k+1}$;
- the lower left side of $\Sigma \,_{\mathcal{C}}\mathcal{M}_{k+1}^\bullet$ with the upper right side of $\Sigma' \,_{\mathcal{C}}\mathcal{M}_{\bullet k+1}$.
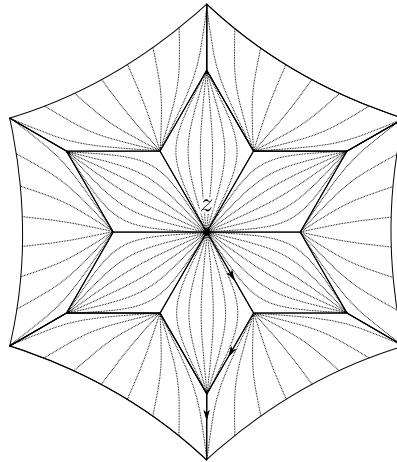
FIGURE 11. The disk $\Delta(W)$.

The resulting 2-dimensional polyhedron $\Delta(W)$ is a disk. Moreover, since it is compatible with all the identifications, the evaluation map descends to $\Delta(W)$ and defines a continuous map

(22) $$\Delta(W) \xrightarrow{\text{ev}} M.$$

and has the desired behavior on the boundary. □

REMARK 22. – Reversing the orientation of $W$ reverses the orientation of the associated disk.

REMARK 23. – Regarding the crocodile walk orbit $W$ as a boundary component of a 2-dimensional moduli space, the disk $\Delta(W)$ is essentially the same as the half suspension of this boundary component.

This geometric point of view does not avoid the above description however, since the structure of the disk and in particular the behavior of the evaluation on its boundary is crucial to our construction.

2.4.3. *Hybrid walks*. – As already observed, the "crocodile walk" can in fact be run on many kinds of moduli spaces, in particular on a hybrid moduli space mixing Morse trajectories rooted at an index 1 critical point of our Morse function $f$ and Floer tubes.

Let $b \in \mathrm{Crit}_1(f)$, let $\{\gamma_-, \gamma_+\} = {}_c\mathcal{M}(b, \star)$ be the two Morse trajectories rooted at $b$ (recall $\mathrm{Crit}_0(f) = \{\star\}$). Let

$$B(b) = \bigcup_{y \in \tilde{\mathscr{P}}_1(H) \cup \{\star\}} {}_c\mathcal{M}(b, y) \times \partial\, {}_c\mathcal{M}(y, \emptyset).$$

This space plays the role of twice broken trajectories, but as already observed, the space $\partial\, {}_c\mathcal{M}(\star, \emptyset)$ has one (and only one) point which is not a breaking: $B(b)$ splits as

the union $B(b) = B'(b) \cup B_\star(b)$ of the set of twice broken trajectories

$$B'(b) = \bigcup_{\substack{y \in \tilde{\mathscr{P}}_1(H) \cup \{\star\} \\ x \in \tilde{\mathscr{P}}_0(H)}} {}_{\mathscr{C}}\mathscr{M}(b, y) \times {}_{\mathscr{C}}\mathscr{M}(y, x) \times {}_{\mathscr{C}}\mathscr{M}(x, \emptyset),$$

and the two special isolated configurations that are not broken twice:

$$B_\star(b) = {}_{\mathscr{C}}\mathscr{M}(b, \star) \times \{\star\} = \{(\gamma_-, \star), (\gamma_+, \star)\},$$

where $\star$ is seen as the constant sphere in ${}_{\mathscr{C}}\mathscr{M}(\star, \emptyset)$.

Upper and lower gluings can be performed on $B'(b)$, but have to be replaced by the relevant Floer steps on $B_\star(b)$, and we let

(23)
$$\sharp^\bullet(\gamma_\pm, \star) = (\gamma_\mp, \star),$$
$$\sharp_\bullet(\gamma_\pm, \star) = (\gamma_\pm, \beta_\star, \alpha_\star).$$

If the latter was already discussed, observe the former is rather a Morse step. To see it as a Floer step, consider the moduli space of solutions $(u, R)$ of $(F_{4,R})$ such that $u(-\infty) \in W^u(b)$, but restrict attention to the boundary component given by $R = 0$: the configurations $(\gamma_\pm, \star)$, regarded as such configurations that underwent a Morse breaking, are related by the moduli space obtained by gluing at the Morse breaking and preserving the $R = 0$ condition.

Defined in this way, the maps $\sharp^\bullet$ and $\sharp_\bullet$ form two involutions on $B(b)$ again, and iterated composition of alternately $\sharp^\bullet$ and $\sharp_\bullet$ defines a walk, still called a crocodile walk, whose orbits are all cyclic.

REMARK 24. – Notice for later use that the steps used in the definition of $\sharp_\bullet$ are all Floer loop steps.
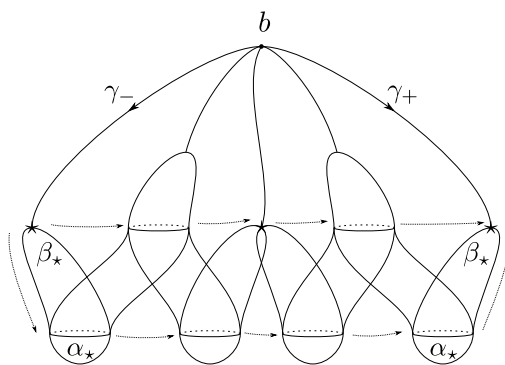


FIGURE 12. The orbit $W_\star(b)$ of the crocodile walk associated to an index 1 Morse critical point $b$.

DEFINITION 2.15. – *The orbit of $(\gamma_-, \star)$ starting with a lower gluing will be denoted by $W_\star(b)$.*

It is a cyclic sequence of the following form

$$(\gamma^-, \star), (\gamma^-, \beta_\star, \alpha_\star), (\gamma_1, \beta_1, \alpha_1), \ldots (\gamma_N, \beta_N, \alpha_N), (\gamma^+, \beta_\star, \alpha_\star), (\gamma^+, \star),$$

where

- $N$ is even (the orbits being cyclic, they have to count the same number of upper and lower steps, and hence an even number of elements),
- for $1 \le i \le N$
  - $\gamma_i \in \mathcal{M}(b, y)$ for some $y \in \{\star\} \cup \tilde{\mathcal{P}}_1(H)$,
  - $\beta_i \in \mathcal{M}(y, x)$ for some $x \in \tilde{\mathcal{P}}_0(H)$,
  - $\alpha_i \in \mathcal{M}(x, \emptyset)$,
- for all $i$ with $0 \le i < N/2$:

$$(\gamma_{2i}, \beta_{2i}) \overset{\sharp}{\to} (\gamma_{2i+1}, \beta_{2i+1}) \qquad \text{and} \qquad \alpha_{2i} = \alpha_{2i+1}$$

$$\gamma_{2i+1} = \gamma_{2i+2} \quad \text{and} \quad (\beta_{2i+1}, \alpha_{2i+1}) \overset{\sharp}{\to} (\beta_{2i+2}, \alpha_{2i+2}).$$

(with the convention $(\gamma_0, \beta_0) = (\gamma_-, \beta_\star)$ and $(\gamma_{N+1}, \beta_{N+1}) = (\gamma_+, \beta_\star)$).

In particular (recall Remark 24), the sequence of lower steps

(24) $$(\star, \beta_\star, \alpha_\star), \ldots (\alpha_{2i-1}, \beta_{2i-1}, \beta_{2i}, \alpha_{2i}), \ldots (\alpha_\star, \beta_\star, \star)$$

form a Floer loop.
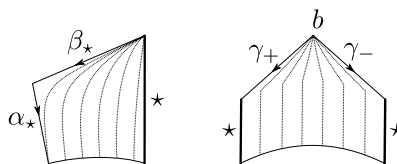


FIGURE 13. Half suspensions of the steps $(\beta_\star, \alpha_\star) \overset{\sharp}{\to} \star$ and $(\gamma_+, \star) \overset{\sharp}{\to} (\gamma_-, \star)$.
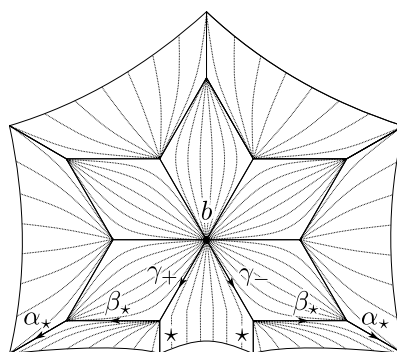


FIGURE 14. The disk $\Delta(W_\star(b))$.

The construction of the polyhedron $\Delta(W_\star(b))$ still makes sense for this special orbit: exactly two new kinds of moduli spaces have to be taken into account, namely the ones

associated to the steps

$$(\gamma_\pm, \star) \overset{\sharp}{\to} (\gamma_\pm, \beta_\star, \alpha_\star) \quad \text{and} \quad (\gamma_+, \star) \overset{\sharp}{\to} (\gamma_-, \star).$$

In both cases the bottom ends of the configurations are free and the half suspension of the relevant moduli space component is endowed with a continuous evaluation map.

In the former however, one side is not associated to a broken trajectory but to the constant one $\star$ and the half suspension is seen as having 4 sides. The evaluation map restricts to (see Figure 13)

- $\beta_{\star|\mathbb{R}}$ and $\alpha_{\star|\mathbb{R}}$ (suitably rescaled using the action) on the broken side;
- the constant path $\{\star\}$ on the "non broken" side;
- the evaluation in $M$ of the Floer step $(\beta_\star, \alpha_\star) \overset{\sharp}{\to} \star$ on the bottom.

In the latter, the half suspension can again be represented by a pentagon and the evaluation map restricts to (see Figure 13)

- $\gamma_+$ and $\gamma_-$ on the upper left and right sides;
- the constant trajectory $\star$ on the lower left and right sides;
- the concatenation $\gamma_+ \cdot \gamma_-$ on the bottom side.

The gluing construction used in (21) adapts straightforwardly to the 3 special steps and results in a disk endowed with a continuous evaluation map to $M$

$$(25) \qquad \Delta(W_\star(b)) \overset{\text{ev}}{\to} M.$$

The restriction of the evaluation map to the boundary is the concatenation of the trajectories $\gamma_+$ and $\gamma_-$ and of the Floer loop formed by the lower steps used in the crocodile walk.

## 3. Generation of the fundamental group

In this section, homomorphisms from the group of Floer loops to that of Morse loops and vice versa are constructed, in order to prove Theorem 2.9, and to later study the relations.

In Section 3.1, we use the classical operation of pushing arbitrary loops down by the flow in order to turn Floer loops into Morse loops. This operation itself is not required for the proof of Theorem 2.9, but it is reinterpreted purely in terms of moduli spaces which makes it compatible with Floer theory. This is used in Section 3.2 to define a similar operation in the reverse direction, turning Morse loops into Floer loops in the same homotopy class. Finally, Section 3.3 gathers the proof of Theorem 2.9, which immediately follows from the possibility of deforming Morse loops into (homotopical) Floer ones, since the result is well known in the Morse setting.

Let $f$ be a Morse function having a single minimum at $\star$, and $g$ a Riemannian metric on $M$ such that the pair $(f, g)$ is Morse Smale, and all the relevant hybrid moduli spaces are cut out transversely.

Recall the Morse version of the Definitions 2.4 and 2.8: each choice of orientation on the unstable manifold of each index 1 Morse critical point defines a path we call a Morse step (notice that since $f$ has a single minimum, all the steps are in fact loops). Picking an arbitrary orientation for each such point $b$ allows to represent the associated Morse steps algebraically

as $b^{\pm}$, and hence to identify the group of Morse loops $\mathscr{L}(f, \star)$ to the free group generated by $\mathrm{Crit}_1(f)$.

### 3.1. From Floer to Morse loops

LEMMA 3.1. – *There exists a group homomorphism $\mathscr{L}(H, \star) \overset{\phi}{\to} \mathscr{L}(f, \star)$ making the following diagram commutative:*

$$(26) \qquad \begin{array}{ccc} \mathscr{L}(H, \star) & \overset{\mathrm{ev}}{\longrightarrow} & \pi_1(M, \star) \ , \\ \Big\downarrow \phi & & \Big\| \mathrm{Id} \\ \mathscr{L}(f, \star) & \overset{\mathrm{ev}}{\longrightarrow} & \pi_1(M, \star) \end{array}$$

*i.e., such that*

$$\forall w \in \mathscr{L}(H, \star), \quad \mathrm{ev}(\phi(w)) \sim \mathrm{ev}(w) \ \text{in} \ \pi_1(M, \star).$$

*Proof.* – Pushing a generic topological loop $\gamma$ down by the flow of the Morse function $f$ deforms it into a Morse loop $\varphi_f^{+\infty}(\gamma)$, i.e., a word in the index 1 critical points. Here generic means that the loop avoids the stable manifolds of all the index $k \geq 2$ Morse critical points. Notice that the evaluation of the Floer steps form a finite collection of 1-dimensional segments in $M$, and the stable manifolds of index $k \geq 2$ critical points of $f$ are codimension $k \geq 2$ submanifolds. Therefore, for a generic (and even open dense) choice of $(f, g)$, Floer loops and such unstable manifolds do not meet, and we get a well defined map

$$(27) \qquad \begin{array}{ccc} \tilde{\mathscr{L}}(H, \star) & \overset{\phi}{\to} & \mathscr{L}(f, \star) \\ \gamma & \mapsto & \varphi_f^{+\infty}(\mathrm{ev}(\gamma)). \end{array}$$

This map is obviously compatible both with the concatenation and cancelation rules, and hence induces a group homomorphism

$$(28) \qquad \mathscr{L}(H, \star) \overset{\phi}{\to} \mathscr{L}(f, \star).$$

Finally, $\phi$ is defined using a deformation and hence preserves the homotopy class, which means that the diagram (28) is commutative. $\qquad\square$

Since the second row of (26) is onto, Theorem 2.9 comes down to proving that any Morse loop can be deformed into a Floer loop. Unfortunately, this deformation can not be obtained like $\phi$ by pushing a loop down by a flow, since there is no such thing as a Floer flow on the loop space.

However, a reinterpretation of $\phi$ in terms of moduli spaces and crocodile walks can be given, allowing to generalize this definition to the Floer setting and obtain a map in the reverse direction. This reinterpretation is quickly sketched below, to serve as an introduction for the reverse construction and to stress that the two constructions are essentially the same, but will not be discussed in details and could be skipped by the reader. The construction in the reverse direction on the other hand, for which all the relevant technical material was already introduced in Section 2.4.3, will be discussed in the next section.

Consider a Floer loop step $\sigma = (\alpha, \beta, \beta', \alpha')$ through some $y \in \tilde{\mathscr{P}}_1(H)$. From our genericity assumption, the Morse flow line $\gamma_\alpha$ (resp. $\gamma_{\alpha'}$) passing through the center $\alpha(+\infty)$ (resp. $\alpha'(+\infty)$) of $\alpha$ (resp. $\alpha'$) ends at $\star$. Denote by $\bar{\alpha}$ (resp. $\bar{\alpha}'$) the configuration obtained

by appending to $\alpha$ (resp. $\alpha'$) the piece of trajectory $\gamma_\alpha$ (resp. $\gamma_{\alpha'}$) running from $\alpha(+\infty)$ (resp. $\alpha'(+\infty)$) down to $\star$.

A crocodile walk can be run on the space of configurations consisting of

- a trajectory from $y$ to some $x \in \mathrm{Crit}_1(f) \cup \tilde{\mathcal{P}}_0(H)$;
- a trajectory from $x$ to $\star$;
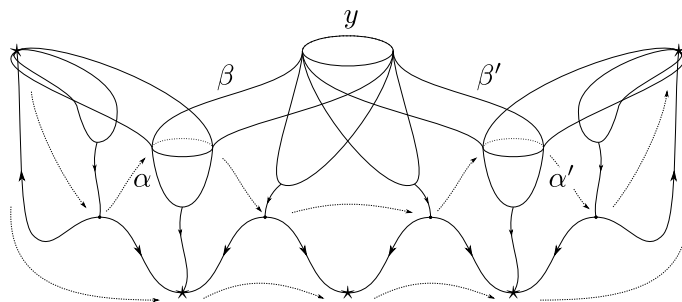- the (trivial!) Morse trajectory $\star \in \mathcal{M}_{\mathrm{Morse}}(\star, \emptyset)$.



FIGURE 15. From Floer to Morse loops.

Starting with the configuration $(\beta, \bar\alpha, \star)$, the first upper step consists in gluing $\beta$ and $\bar\alpha$. The other end of the associated component of $\mathcal{M}(y, \star)$ is a configuration broken either at an index 0 Hamiltonian orbit $x$, or at an index 1 Morse critical point $b$ (see Figure 15).

In the former case, the new configuration has to be $(\beta', \bar\alpha', \star)$ (simply forget what happened to the Morse flow line and consider the definition of a Floer loop step).

In the latter, the lower part of the configuration is a Morse trajectory $\gamma_\pm \in \mathcal{M}(b, \star) = \{\gamma_-, \gamma_+\}$. The next (lower) step consists in replacing $\gamma_\pm$ by $\gamma_\mp$ (recall from the comments on Definition 2.8 that this can be interpreted as a step along the Morse moduli space $\mathcal{M}_{\mathrm{Morse}}(b, \emptyset)$). The next upper step is then a gluing at $b$, and the same alternative holds again.

After a finite number of iterations of this process, the configuration $(\beta', \bar\alpha', \star)$ has to be reached (from an upper step). Similarly to (23), moduli spaces involving interrupted Morse trajectories give rise to the following special steps

$$\sharp_\bullet(\beta', \bar\alpha', \star) = (\beta', \alpha'),$$
$$\sharp^\bullet(\beta', \alpha') = (\beta, \alpha),$$
$$\sharp_\bullet(\beta, \alpha) = (\beta, \bar\alpha, \star)$$

that close the walk orbit.

Let $W_\sigma$ be the orbit of the crocodile walk described above. The lower non special steps in this orbit form a sequence of consecutive Morse steps $\phi(\sigma)$.

Repeating this process for all the Floer loop steps $\sigma_i$ in a Floer loop $\gamma = (\sigma_1, \ldots, \sigma_N)$ (including the first and last ones $\star \xrightarrow{\sharp} (\beta_\star, \alpha_\star)$ and $(\beta_\star, \alpha_\star) \xrightarrow{\sharp} \star$ for which it still makes sense), we get a sequence $\phi(\gamma) = \phi(\sigma_1)\ldots\phi(\sigma_N)$ which is a Morse loop. This defines a map $\mathcal{L}(H, \star) \to \mathcal{L}(f, \star)$ which is a group homomorphism, and it is a straightforward observation that this map is the same as (28).

Finally, observe that all the disks $\Delta(W_{\sigma_i})$ patch side to side to form a disk endowed with an evaluation map realizing a homotopy from $\mathrm{ev}(\gamma)$ to $\mathrm{ev}(\phi(\gamma))$.

## 3.2. From Morse to Floer loops.

LEMMA 3.2. – *There exists a group homomorphism* $\mathcal{L}(f, \star) \xrightarrow{\psi} \mathcal{L}(H, \star)$ *making the following diagram commutative:*

(29)
$$
\begin{array}{ccc}
\mathcal{L}(f, \star) & \xrightarrow{\ \mathrm{ev}\ } & \pi_1(M, \star) \\[2pt]
{\scriptstyle\psi}\big\downarrow & & \big\|{\scriptstyle\mathrm{Id}} \\[2pt]
\mathcal{L}(H, \star) & \xrightarrow{\ \mathrm{ev}\ } & \pi_1(M, \star),
\end{array}
$$

*i.e., such that*
$$ \forall w \in \mathcal{L}(f, \star), \quad \mathrm{ev}(\psi(w)) \sim \mathrm{ev}(w) \ \text{ in } \ \pi_1(M, \star). $$

*Proof.* – Let $b$ be an index 1 critical point of $f$ and $(\gamma_b^-, \gamma_b^+)$ be the two Morse trajectories from $b$ to $\star$.

Recall that the crocodile walk on the space
$$ B(b) = \bigcup_{y \in \tilde{\mathcal{P}}_1(H) \cup \{\star\}} \mathcal{M}(b, y) \times \partial \mathcal{M}(y, \emptyset) $$

was described in Section 2.4.3. In particular, using the notations introduced there, it has a special orbit $W_\star(b)$ (see Figure 12) of the form

$$ (\gamma_b^-, \star), (\gamma_b^-, \beta_\star, \alpha_\star), (\gamma_1, \beta_1, \alpha_1), \ldots (\gamma_N, \beta_N, \alpha_N), (\gamma_b^+, \beta_\star, \alpha_\star), (\gamma_b^+, \star). $$

Recall from (24) that the lower steps in this orbit form a Floer loop. Denoting it by $\psi(b)$, we have
$$ \psi(b) = \big((\star, \beta_\star, \alpha_\star), \ldots (\alpha_{2i-1}, \beta_{2i-1}, \beta_{2i}, \alpha_{2i}), \ldots (\alpha_\star, \beta_\star, \star)\big), $$

and we get a map
$$ \tilde{\mathcal{L}}(f, \star) \xrightarrow{\psi} \mathcal{L}(H, \star), $$

which is obviously compatible with both the concatenation and cancelation rules, and hence induces a group homomorphism

(30)
$$ \mathcal{L}(f, \star) \xrightarrow{\psi} \mathcal{L}(H, \star). $$

Finally, the homotopy is provided by the disk $\Delta(W_\star(b))$ and the evaluation map (25): its restriction to the boundary is the concatenation of the Morse loop $\gamma^{-1}$ and the Floer loop $\psi(b)$.   □

## 3.3. Proof of Theorem 2.9.

*Proof of Theorem 2.9.* – Theorem 2.9 is now a straightforward corollary of Lemma 3.2: since the map on the first line of (29) is onto, so has to be the map on the second.   □

## 4. Relations and fundamental groups

It is natural to ask for a Floer theoretic interpretation of the relations. It is the object of this section to provide a family of generators of $\ker(\mathscr{L}(H, \star) \xrightarrow{\text{ev}} \pi_1(M, \star))$ that can be expressed in terms of Floer and PSS moduli spaces.

REMARK 25. – Although the subgroup of relations obviously only depends on $(H, J, \star, \chi)$, the proposed generators will depend on the choice of an additional auxiliary Morse function (and metric). Being able to a priori select a finite family that would generate the relations and depend on $(H, J, \star, \chi)$ only would be more satisfactory but is unfortunately unclear.

Moreover, resorting to a Morse function may seem to weaken the construction since Morse functions already give full access to the fundamental group. It should be observed however, that the Morse function is used in a different way from the usual one here: it is used to define hybrid moduli spaces, mixing Morse and Floer objects, and the present description of the relations depicts how the Morse relations have to be transported from the Morse to the Floer setting by some configurations of 1-dimensional hybrid moduli spaces, and hence may gather some non trivial information.

### 4.1. Floer-Morse-Floer relations

Given a Floer loop $\gamma \in \mathscr{L}(H, \star)$, observe that the evaluations of $\psi(\phi(\gamma))$ and $\gamma$ are homotopic (since both $\phi$ and $\psi$ preserve the homotopy class), so that $\gamma^{-1}\psi(\phi(\gamma))$ is always a relation.

DEFINITION 4.1. – *Define the set of "Floer-Morse-Floer relations" as*

$$R_{\text{FMF}}(H) = \{\gamma^{-1}\psi(\phi(\gamma)), \ \gamma \in \mathscr{L}(H, \star)\}.$$
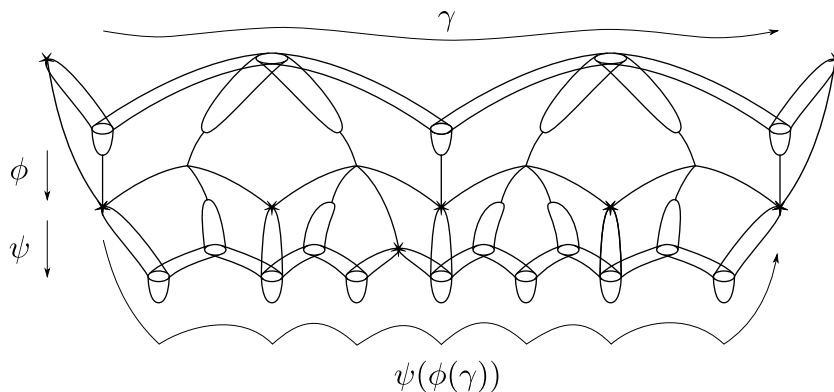


FIGURE 16. A relation in $R_{\text{FMF}}(H)$.

REMARK 26. – The notation $R_{\text{FMF}}(H)$ only highlights the dependency on $H$ but this set depends in fact on all the auxiliary data $(H, J, f, g, \star, \chi)$.

REMARK 27. – The set $R_{\mathrm{FMF}}(H)$ is not finite since there is one relation for each Floer loop. However, it is induced by the substitution rule at the Floer loop steps level

$$\sigma \to \psi(\phi(\sigma))$$

which is finite.

Since $\phi$ and $\psi$ are described in terms of crocodile walk, so can these relations. Glossing over the moduli spaces involving $\star$, consider a Floer loop step $\sigma$ through some $y_0 \in \tilde{\mathscr{P}}_1(H)$. The configurations consisting of

  – a trajectory $\delta$ from $y_0$ to some $z \in \mathrm{Crit}_1(f) \cup \tilde{\mathscr{P}}_0(H)$,
  – a trajectory $\gamma$ from $z$ to some $y \in \{\star\} \cup \tilde{\mathscr{P}}_1(H)$,
  – a trajectory $\beta$ from $y$ to some $x \in \tilde{\mathscr{P}}_0(H)$,
  – a trajectory $\alpha \in {}_{\mathscr{\circ}}\mathscr{M}(x, \emptyset)$,

are broken three times and hence present 3 levels where to perform a gluing (or more generally a step). The relation is obtained by running the crocodile walk on the two lower gluings "from $\star$ to $\star$," then performing one upper gluing, and repeating this process.

## 4.2. Relations associated to Morse 2-cells

Given an index 2 Morse critical point $c$ of $f$, let $\rho_c$ be the relation in $\mathscr{L}(f, \star)$ given by the boundary of the associated 2 cell and define:

$$(31) \qquad\qquad R_{\mathrm{M2}}(f) = \{\rho_c, \ c \in \mathrm{Crit}_2(f)\},$$

$$(32) \qquad\qquad R_{\mathrm{M2}}(H) = \{\psi(\rho_c), \ c \in \mathrm{Crit}_2(f)\}.$$

REMARK 28. – The notation $R_{\mathrm{M2}}(H)$ only highlights the dependency on $H$ but this set depends in fact on all the auxiliary data $(H, J, f, g, \star, \chi)$.
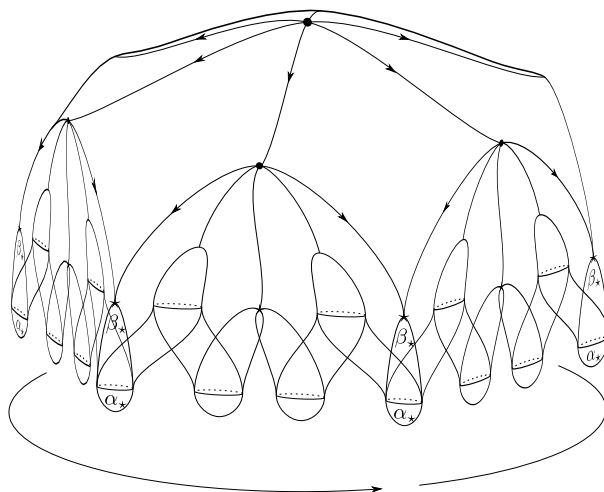


FIGURE 17. A Floer relation associated to a Morse 2-cell.

REMARK 29. – For all $\rho \in R_{\mathrm{M2}}(H)$ we have $\mathrm{ev}(\rho) = 1$ in $\pi_1(M, \star)$, so that $R_{\mathrm{M2}}(H)$ is indeed a collection of relations.

REMARK 30. – These relations can also be described in terms of crocodile walks. Glossing over the moduli spaces involving $\star$ again, consider an index 2 Morse critical point $c$ and the configurations consisting of

 – a trajectory $\delta$ from $c$ to some $z \in \mathrm{Crit}_1(f)$;
 – a trajectory $\gamma$ from $z$ to some $y \in \tilde{\mathcal{P}}_1(H) \cup \mathrm{Crit}_0(f)$;
 – a trajectory $\beta$ from $y$ to some $x \in \tilde{\mathcal{P}}_0(H)$;
 – a trajectory $\alpha \in \mathcal{M}(x, \emptyset)$.

The relation associated to $c$ can be obtained using the same algorithm as discussed previously, i.e., running the crocodile walk on the two lower levels "from $\star$ to $\star$," then performing one upper step and repeating this process.

## 4.3. Fundamental group

We can finally define the subgroup of relations:

DEFINITION 4.2. – *Denote by $\mathcal{R}(H, \star)$ the normal subgroup of $\mathcal{L}(H, \star)$ generated by $R_{\mathrm{FMF}}(H)$ and $R_{\mathrm{M2}}(H)$:*

$$\mathcal{R}(H, \star) = <R_{\mathrm{FMF}}(H), R_{\mathrm{M2}}(H)>.$$

REMARK 31. – The group $\mathcal{R}(H, \star)$ obviously depends on $(H, J, \star, \chi)$, but it is a consequence of Theorem 4.4 that it does not depend on $(f, g)$.

DEFINITION 4.3. – *The Floer fundamental group associated to $(H, J, \star, \chi)$ is defined as the group*

$$\pi_1(H, \star) = \mathcal{L}(H, \star) / \mathcal{R}(H, \star).$$

REMARK 32. – The group should be denoted as $\pi_1(H, J, \star, \chi)$ to emphasize its dependency on all the auxiliary data but it is kept implicit to reduce notations.

In the same way, let $\mathcal{R}(f, \star) = <R_{\mathrm{M2}}(f)>$ be the normal subgroup of $\mathcal{L}(f, \star)$ generated by the boundary of Morse 2-cells, $\pi_1(f, \star) := \mathcal{L}(f, \star) / \mathcal{R}(f, \star)$ and recall the well known fact that $\pi_1(f, \star) \simeq \pi_1(M, \star)$.

THEOREM 4.4. – *The evaluation induces a group isomorphism*

$$\pi_1(H, \star) \xrightarrow{\ \mathrm{ev}\ } \pi_1(M, \star).$$

*The maps $\phi$ and $\psi$ also induce isomorphisms which are inverse one of the other*

$$\pi_1(H, \star) \underset{\psi}{\overset{\phi}{\rightleftarrows}} \pi_1(f, \star).$$

*Proof.* –   1. Compatibility with the relations for $\psi$: observe that $\mathcal{R}(f, \star) = <R_{\mathrm{M2}}(f)>$ and $\psi(<R_{\mathrm{M2}}(f)>) \subset <\psi(R_{\mathrm{M2}}(f))>$. Since $\psi(R_{\mathrm{M2}}(f)) = R_{\mathrm{M2}}(H)$ and $<R_{\mathrm{M2}}(H)> \subset \mathcal{R}(H, \star)$, we have

$$\psi(\mathcal{R}(f, \star)) \subset \mathcal{R}(H, \star).$$

2. Compatibility with the relations for $\phi$: Similarly, $\phi(\mathcal{R}(H, \star)) \subset {<}\phi(R_{\mathrm{FMF}}(H)) \cup \phi(R_{\mathrm{M2}}(H)){>}$. But for $\rho \in R_{\mathrm{FMF}}(H)) \cup R_{\mathrm{M2}}(H)$, we have $\mathrm{ev}(\phi(\rho)) = \mathrm{ev}(\rho) = 1$, so that

$$\phi(\mathcal{R}(H, \star)) \subset \mathcal{R}(f, \star).$$

3. The fact that $\phi \circ \psi = \mathrm{Id}_{\pi_1(f, \star)}$: This follows directly from $\mathrm{ev} \circ \phi \circ \psi = \mathrm{ev}$. In particular this implies surjectivity of $\phi$ and injectivity of $\psi$.

4. The fact that $\psi \circ \phi = \mathrm{Id}_{\pi_1(H, \star)}$: This is built in the definition of the relations $R_{\mathrm{FMF}}(H)$: for $\gamma \in \mathcal{L}(H, \star)$, we have $\gamma^{-1}\psi(\phi(\gamma)) \in R_{\mathrm{FMF}}(H)$, so that $\gamma = \psi(\phi(\gamma))$ in $\pi_1(H, \star)$. This implies injectivity of $\phi$ and surjectivity of $\psi$.

5. The fact that $\ker\big(\mathrm{ev} : \mathcal{L}(H, \star) \to \pi_1(M, \star)\big) = \mathcal{R}(H, \star)$: The relation $\mathcal{R}(H, \star) \subset \ker \mathrm{ev}$ is obvious since this is true for all the generators of $\mathcal{R}(H, \star)$. Conversely, let $\gamma \in \mathcal{L}(H, \star)$ such that $\mathrm{ev}(\gamma) = 1$ in $\pi_1(M, \star)$. Then $\mathrm{ev}(\phi(\gamma)) = 1$ so that $\phi(\gamma) \in \mathcal{R}(f, \star)$. As a consequence

$$\psi(\phi(\gamma)) \in \psi(\mathcal{R}(f, \star)) \subset \mathcal{R}(H, \star).$$

Finally, since $\gamma^{-1}\psi(\phi(\gamma)) \in \mathcal{R}(H, \star)$, we have $\gamma \in \mathcal{R}(H, \star)$.

This ends the proof that $\pi_1(H, \star) \xrightarrow{\mathrm{ev}} \pi_1(M, \star)$ is injective, and hence an isomorphism since it was already proven to be surjective. $\qquad\square$

## 5. Application and proof of Theorem 2.12.

### 5.1. Generating $\pi_1(M)$ with steps

The Theorem 2.11 is a direct consequence of a weaker version of Theorem 2.9 where Floer loops are replaced by Floer steps.

*Proof of Theorem 2.11.* — Fix a generic set of data $(H, J, \star, \chi, f, g)$ where $\star$ is the single minimum of the Morse function $f$. Let $\sigma_\star$ denote the special step $\star \xrightarrow{\sharp} (\beta_\star, \alpha_\star)$. Let $\mathcal{S}(H)$ be the free group generated by all the Floer loop steps but the special one.

Recall that the map $\phi$ was defined at the step level:

$$(33) \qquad\qquad \mathcal{S}(H) \xrightarrow{\ \phi\ } \mathcal{L}(f, \star) \xrightarrow{\ \mathrm{ev}\ } \pi_1(M, \star)$$

(notice that although Floer loop steps evaluate as free paths in $M$ and not necessarily as based loops, they are still pushed down into Morse based loops by $\phi$ because the Morse function was chosen to have only one index 0 critical point).

Notice that the left hand side of (19) is nothing but the number of generators of $\mathcal{S}(H)$, so that Theorem 2.11 reduces to proving that in (33), $\mathrm{ev} \circ \phi$ is onto.

Observe now that in a loop $w \in \tilde{\mathcal{L}}(H, \star)$, the only occurrences of $\sigma_\star$ and $\sigma_\star^{-1}$ are:

- $\sigma_\star$ at the beginning of $w$,
- $\sigma_\star^{-1}$ at the end of $w$,
- possible pairs $(\sigma_\star^{-1}\sigma_\star)$ within $w$.

In particular, this means that removing $\sigma_\star$ and $\sigma_\star^{-1}$ at the ends of the loops defines an injective group homomorphism

$$\mathscr{L}(H, \star) \overset{\tau}{\hookrightarrow} \mathscr{S}(H).$$

We end up with the following commutative diagram:

(34)
$$
\begin{array}{ccccc}
\mathscr{L}(H, \star) & \overset{\phi}{\longrightarrow} & \mathscr{L}(f, \star) & \overset{\mathrm{ev}}{\longrightarrow} & \pi_1(M, \star) \\
\Big\downarrow{\scriptstyle \tau} & & \Big\downarrow{\scriptstyle \tau'} & & \Big\downarrow{\scriptstyle \tau''} \\
\mathscr{S}(H) & \overset{\phi}{\longrightarrow} & \mathscr{L}(f, \star) & \overset{\mathrm{ev}}{\longrightarrow} & \pi_1(M, \star),
\end{array}
$$

where $\tau'$ and $\tau''$ are the conjugation by $\phi(\sigma_\star^{-1})$ and $\mathrm{ev}(\phi(\sigma_\star^{-1}))$ respectively. In particular, surjectivity of the composition of the maps appearing on the first row implies that of the second. $\qquad\square$

## 5.2. Proof of Theorem 2.12.

In this section, we want to prove Theorem 2.12, namely that if $\pi_1(M) \neq \{1\}$, then every non-degenerate Hamiltonian $H$ should have at least one contractible 1-periodic orbit of index 1 (i.e., Conley-Zehnder index $1 - n$) with non vanishing multiplicity.

This is not a consequence of the above construction, but uses similar ideas arranged slightly differently: it is based on a variant of the crocodile walk to patch (suspensions of) 1-dimensional PSS moduli spaces together and fill any Morse loop with a disk when there are no index 1 Hamiltonian orbits.

Let $H$ be a non degenerate Hamiltonian, and pick a triple $(J, f, g)$ where $J$ is an (possibly time dependent) almost complex structure compatible with $\omega$, $f$ a Morse function with a single minimum denoted by $\star$ and $g$ a Riemannian metric such that $(H, J, \star, \chi, f, g)$ satisfies our transversality assumptions. We pick coherent orientations on all the 0 and 1-dimensional moduli spaces $_c\mathscr{M}(b, x)$ for $b \in \mathrm{Crit}_{0,1}(f)$ and $x \in \tilde{\mathscr{P}}_0(H) \cup \mathrm{Crit}_0(f)$.

Suppose $H$ has no index 1 orbit, or more precisely that it has no index 1 orbit with non vanishing multiplicity: this means there are no Floer trajectories from an index 1 to an index 0 orbit that admit at least one augmentation. For convenience, let

$$\tilde{\mathscr{P}}_0(H)^* = \left\{ x \in \tilde{\mathscr{P}}_0(H), \ _c\mathscr{M}(x, \emptyset) \neq \emptyset \right\}.$$

Our assumption can then be written as:

$$\forall y \in \tilde{\mathscr{P}}_1(H), \forall x \in \tilde{\mathscr{P}}_0(H)^*, \ \ _c\mathscr{M}(y, x) = \emptyset.$$

Let $b$ be an index 1 Morse critical point, such that the unstable manifold of $b$ defines a non trivial loop $\gamma$ in $M$, and let $\gamma_-$ and $\gamma_+$ be the two Morse flow lines rooted at $b$. For convenience, we consider $\gamma$ as based at $b$ and let:

$$\gamma = \gamma_+ \cdot \gamma_-^{-1}.$$

For $x \in \tilde{\mathscr{P}}_0(H)^*$ consider the space

$$B(b) = \{\gamma_-, \gamma_+\} \times {_c\mathscr{M}}(\star, x).$$

Since $H$ has no index 1 orbit related to $x$ by a Floer trajectory, $B(b)$ is the set of all broken hybrid trajectories from $b$ to $x$.

In particular, gluing $\gamma_\pm$ with a trajectory $\beta \in {}_c\mathcal{M}(\star, x)$ defines a 1-dimensional family of trajectories from $b$ to $x$ whose other end has to be of the same form. This defines a one to one correspondence:

$$B(b) \overset{\sigma}{\to} B(b)$$

$$(\gamma_\epsilon, \beta) \mapsto (\gamma_{\epsilon'}, \beta') \text{ such that } (\gamma_\epsilon, \beta) \overset{\sharp}{\to} (\gamma_{\epsilon'}, \beta').$$

Permuting $\gamma_-$ and $\gamma_+$ defines another one to one correspondence

$$B(b) \overset{\tau}{\to} B(b)$$

$$(\gamma_\pm, \beta) \mapsto (\gamma_\mp, \beta).$$

Notice both $\sigma$ and $\tau$ reverse the orientation.

Consider now an orbit of $\rho = \tau \circ \sigma$. It has to be cyclic, and is a sequence

$$(\gamma_{\epsilon_1}, \beta_1), \ldots, (\gamma_{\epsilon_k}, \beta_k),$$

(with $\epsilon_i = \pm 1$) such that $(\gamma_{\epsilon_i}, \beta_i) \overset{\sharp}{\to} (\gamma_{-\epsilon_{i+1}}, \beta_{i+1})$, with the convention that $(\gamma_{\epsilon_{k+1}}, \beta_{k+1}) = (\gamma_{\epsilon_1}, \beta_1)$.

To each gluing, is associated a 1-dimensional space, and we let $\Sigma_i$ be its suspension. It is a diamond, endowed with an evaluation map to $M$ that coincides with

- $\gamma_{\epsilon_i}$ on the upper left edge,
- $\beta_i$ on the lower left edge,
- $\gamma_{-\epsilon_{i+1}}$ on the upper right edge,
- $\beta_{i+1}$ on the lower right edge.

Gluing all these diamonds side by side along the lower edges provides a disk, endowed with a continuous evaluation map to $M$, whose restriction to the boundary is

$$\gamma_{\epsilon_1}^{-1} \gamma_{-\epsilon_2} \gamma_{\epsilon_2}^{-1} \cdots \gamma_{-\epsilon_k} \gamma_{\epsilon_k}^{-1} \gamma_{-\epsilon_1}.$$

This loop is therefore trivial, but $\gamma_{-\epsilon_i} \gamma_{\epsilon_i}^{-1} = \gamma^{-\epsilon_i}$ so $\gamma^{\sum \epsilon_i} = 1$. Moreover, the orientation of the couple $(\gamma_{\epsilon_i}, \beta_i)$ is constant with respect to $i$ (because one moves from one to the next by two gluings and the orientation is reversed by each gluing) and it can be supposed to be positive without loss of generality. This means that $\epsilon_i = \epsilon(\beta_i)$ for all $i$ and hence $\sum \epsilon_i = \sum \epsilon(\beta_i)$ (where $\epsilon(\beta_i)$ is the orientation of $\beta_i$). As a consequence, we get

$$\gamma^{\sum \epsilon(\beta_i)} \sim 1 \text{ in } \pi_1(M, \star).$$

Observe now that the orbits of $\rho$ induce a partition of ${}_c\mathcal{M}(\star, x)$, so repeating this for all the orbits $O_1, \ldots, O_N$ of $\rho$, we derive

$$\gamma^{\sum_{O_1} \epsilon(\beta_i)} \cdots \gamma^{\sum_{O_N} \epsilon(\beta_i)} = \gamma^{\sum_{\beta \in {}_c\mathcal{M}(\star, x)} \epsilon(\beta)} = \gamma^{n_x} \sim 1 \text{ in } \pi_1(M, \star),$$

where $n_x = \sharp_{\mathrm{alg}} \, {}_c\mathcal{M}(\star, x)$ is the algebraic number of elements in ${}_c\mathcal{M}(\star, x)$ (i.e., the sum of signs $\pm 1$ associated to each element in ${}_c\mathcal{M}(\star, x)$ according to a choice of coherent orientations). Recall this number is the component along $x$ of the image of $\star$ under the PSS homomorphism from the Morse to the Floer complex (using $\mathbb{Z}$ coefficients):

$$PSS_{MF}(\star) = \sum_{x \in \tilde{\mathcal{P}}_0(H)} n_x x.$$

Let $PSS_{FM}$ be the PSS homomorphism from the Floer to the Morse complex. Since $PSS_{FM} \circ PSS_{MF}$ induces the identity in homology, we have

$$\sum_{x \in \bar{\mathscr{P}}_0(H)} n_x \, m_x = 1,$$

where $m_x = \sharp_{\text{alg}}(_{_\circ}\mathscr{M}(x, \emptyset))$. In particular we also have $\sum_{x \in \bar{\mathscr{P}}_0(H)^*} n_x \, m_x = 1$.

As a consequence we have

$$\gamma = \gamma^{\sum_{x \in \bar{\mathscr{P}}_0(H)^*} n_x m_x} = 1 \text{ in } \pi_1(M, \star).$$

This is a contradiction, since we supposed $\gamma$ was non trivial. This ends the proof of Theorem 2.12.

## 6. Stable Morse version

To some extent, a stable Morse function can be considered as a simplified finite dimensional model for the action functional on the free loop space. This section is devoted to a quick sketch of the analog of the main construction in the stable Morse setting. Although it would deserve a dedicated discussion, it is only addressed here to shed some light on the phenomena encountered along the construction that do not appear in the usual Morse setting, like the existence of several steps through the same critical point or of steps through $\star$. Therefore, we limit ourselves to defining the relevant moduli spaces, and leave all the proofs to the reader.

### 6.1. Setting

Let $M$ be a smooth closed manifold of dimension $n$, $\star$ a point in $M$, $N_\pm$ be two integers, $N = N_+ + N_-$ and $H$ a Morse function on $M \times \mathbb{R}^N$ that is quadratic at infinity with signature $(N_+, N_-)$. Namely, we suppose that there is a compact set $K$ such that $\forall (m, u, v) \in (M \times \mathbb{R}^{N+} \times \mathbb{R}^{N-}) \setminus K$, $H(m, u, v) = \|u\|^2 - \|v\|^2$.

For convenience, the Morse index will be shifted by $N_-$ and we let, for a critical point $x$ of $H$:

$$|x| = \mu(x) - N_-$$

where $\mu$ denotes the usual Morse index.

We also pick a Riemannian metric $g$ on $M \times \mathbb{R}^N$ and denote by $\phi^t$ the associated negative gradient flow of $H$.

### 6.2. Moduli spaces

For $x, y \in \text{Crit}(H)$ the usual space of trajectories from $y$ to $x$ can be described as

$$_{_\circ}\mathscr{M}(y, x) = \left( W^u(y) \cap W^s(x) \right)_{/\mathbb{R}}.$$

The counterpart of the "augmentations" required for the construction are now trajectories "hitting $M \times \mathbb{R}^{N+}$," namely

$$_{_\circ}\mathscr{M}(y, \emptyset) = W^u(y) \cap \left( M \times \mathbb{R}^{N+} \times \{0\} \right),$$

and the counterpart of the evaluation map $u \mapsto u(+\infty)$ is the projection $M \times \mathbb{R}^{N+} \xrightarrow{\pi_+} M$:

$$\mathcal{M}(y, \emptyset) \xrightarrow{\text{ev}} M$$
$$(m, u, 0) \mapsto m.$$

Similarly, the spaces

$$\mathcal{M}(\star, x) = \left(\{\star\} \times \mathbb{R}^{N-}\right) \cap W^s(x),$$
$$\mathcal{M}(\star, \emptyset) = \{(p, R) \in \{\star\} \times \mathbb{R}^{N-} \times ]0, +\infty[, \phi^R(p) \in M \times \mathbb{R}^{N+}\}$$

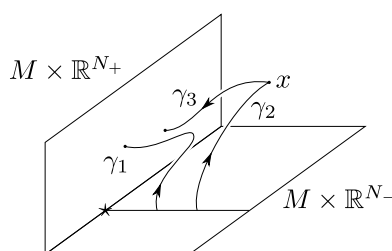are the counterparts of the spaces that were denoted by the same notations in the Floer setting.



FIGURE 18. Augmentations and other configurations in the stable Morse setting: $\gamma_1 \in \mathcal{M}(\star, \emptyset)$, $\gamma_2 \in \mathcal{M}(\star, x)$, $\gamma_3 \in \mathcal{M}(x, \emptyset)$.

The triple $(H, g, \star)$ is supposed to be chosen generically so that all the considered moduli spaces are cut out transversely. In this situation, they are all smooth manifolds of dimension:

$$\dim \mathcal{M}(y, x) = |y| - |x| - 1,$$
$$\dim \mathcal{M}(y, \emptyset) = |y|,$$
$$\dim \mathcal{M}(\star, x) = -|x|,$$
$$\dim \mathcal{M}(\star, \emptyset) = 1.$$

Moreover, they are compact up to breaking at intermediate critical points (although $M \times \mathbb{R}^N$ is not compact), and the gluing construction also makes sense in this setting.

Notice in particular that $\mathcal{M}(\star, \emptyset)$ still has a projection $\pi$ to $[0, +\infty]$, and that $\pi^{-1}(\{0\})$ consists of exactly one point, namely $\star$ itself, since $\pi^{-1}(\{0\}) = (\{\star\} \times \mathbb{R}^{N-}) \cap (M \times \mathbb{R}^{N+}) = \{(\star, 0)\}$.

With these notations, the definitions given in the Floer setting make sense literally and give rise to suitable notions of "stable Morse steps and loops" and to the associated group $\mathcal{L}(H, \star)$.

Picking now a Morse function $f$ on $M$ having a single minimum at $\star$ and a metric on $M$, one can consider the following hybrid moduli spaces (see Figure 19):

$$\mathcal{M}(b, x) = \pi_-^{-1}(W_f^u(b)) \cap W^s(x),$$
$$\mathcal{M}(x, b) = W^u(x) \cap \pi_+^{-1}(W_f^s(b)),$$

where $W_f^u$ and $W_f^s$ denote the stable and unstable manifolds with respect to the negative gradient of $f$ in $M$, and $\pi_\pm$ are the projections $M \times \mathbb{R}^{N_\pm} \xrightarrow{\pi_\pm} M$.
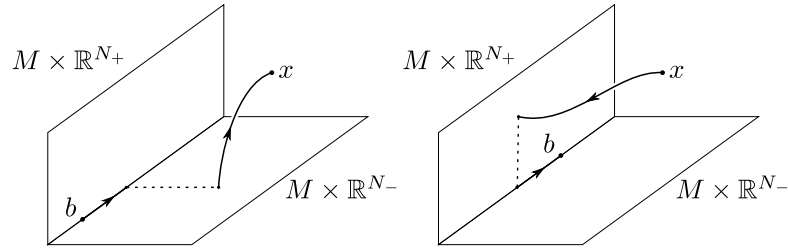


FIGURE 19. Hybrid trajectories between Morse and "stable Morse" critical points.

Using these hybrid moduli spaces, the proof of the following statement follows literally that of its Floer analog and is left to the reader:

THEOREM 6.1. – *The map $\mathcal{L}(H, \star) \xrightarrow{\text{ev}} \pi_1(M, \star)$ is onto.*

### 6.3. Multiplicities

Since the stable Morse situation is much easier to handle than the Floer one, it is now not hard to give examples where several steps are associated to the same index 1 critical point or where there is more than one step going through $\star$. The Figure 20 illustrates the former phenomenon, and the latter is similar.
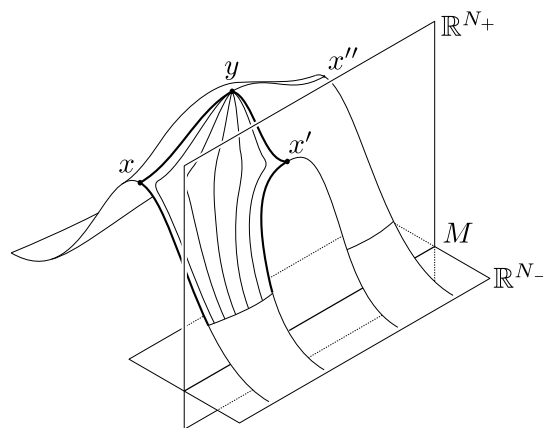


FIGURE 20. Two steps through the same index 1 critical point $y$: one involves $x$ and $x'$, the other $x'$ and $x''$.

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