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STRONGLY INTERACTING BLOW UP BUBBLES FOR THE MASS CRITICAL NONLINEAR SCHRÖDINGER EQUATION

BY YVAN MARTEL AND PIERRE RAPHAËL

ABSTRACT. – We consider the mass critical two dimensional nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + |u|^2 u = 0, \;\; t \in \mathbb{R}, \; x \in \mathbb{R}^2. \tag{NLS}$$

Let Q denote the positive ground state solution of $\Delta Q - Q + Q^3 = 0$. We construct a new class of multi-solitary wave solutions of (NLS) based on Q: given any integer $K \geq 2$, there exists a global (for $t > 0$) solution $u(t)$ that decomposes asymptotically into a sum of solitary waves centered at the vertices of a K-sided regular polygon and concentrating at a logarithmic rate as $t \to +\infty$, so that the solution blows up in infinite time with the rate

$$\|\nabla u(t)\|_{L^2} \sim |\log t| \;\text{ as }\; t \to +\infty.$$

Using the pseudo-conformal symmetry of the (NLS) flow, this yields the first example of solution $v(t)$ of (NLS) blowing up in finite time with a rate strictly above the pseudo-conformal one, namely,

$$\|\nabla v(t)\|_{L^2} \sim \left|\frac{\log|t|}{t}\right| \;\text{ as }\; t \uparrow 0.$$

Such a solution concentrates K bubbles at a point $x_0 \in \mathbb{R}^2$, that is $|v(t)|^2 \rightharpoonup K\|Q\|_{L^2}^2 \delta_{x_0}$ as $t \uparrow 0$. These special behaviors are due to strong interactions between the waves, in contrast with previous works on multi-solitary waves of (NLS) where interactions do not affect the global behavior of the waves.

RÉSUMÉ. – On considère l'équation de Schrödinger non linéaire critique pour la masse en dimension deux

$$i\partial_t u + \Delta u + |u|^2 u = 0, \;\; t \in \mathbb{R}, \; x \in \mathbb{R}^2. \tag{SNL}$$

Soit Q la solution positive et état fondamental de l'équation $\Delta Q - Q + Q^3 = 0$. On construit une nouvelle classe d'ondes solitaires multiples basées sur Q : étant donné un entier $K \geq 2$, il existe une solution globale (pour $t > 0$) $u(t)$ de (SNL) qui se décompose asymptotiquement en une somme d'ondes solitaires centrées sur les sommets d'un polygone régulier et qui se concentrent à un taux logarithmique quand $t \to +\infty$, de sorte que la solution explose en temps infini

$$\|\nabla u(t)\|_{L^2} \sim |\log t| \;\text{ quand }\; t \to +\infty.$$

Comme conséquence de la symétrie pseudo-conforme du flot de (SNL), on obtient le premier exemple d'une solution $v(t)$ de (SNL) qui explose en temps fini avec un taux strictement supérieur au taux

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pseudo-conforme

$$\|\nabla v(t)\|_{L^2} \sim \left|\frac{\log|t|}{t}\right| \quad \text{quand } t \uparrow 0.$$

Cette solution concentre K bulles en un point $x_0 \in \mathbb{R}^2$, c'est-à-dire $|v(t)|^2 \rightharpoonup K\|Q\|_{L^2}^2\delta_{x_0}$ quand $t \uparrow 0$. Ces comportements particuliers sont dus aux interactions fortes entre les ondes solitaires, par opposition avec les résultats précédents sur les ondes solitaires multiples pour (SNL) où les interactions n'affectent pas le comportement global des ondes.

1. Introduction

1.1. General setting

We consider in this paper the mass critical two dimensional non linear Schrödinger equation (NLS)

$$i\partial_t u + \Delta u + |u|^2 u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2. \tag{1.1}$$

It is well-known (see e.g., [7] and the references therein) that for any $u_0 \in H^1(\mathbb{R}^2)$, there exists a unique maximal solution $u \in \mathcal{C}((-T_\star, T^\star), H^1(\mathbb{R}^2))$ of (1.1) with $u(0) = u_0$. Moreover, the following blow up criterion holds

$$T^\star < +\infty \text{ implies } \lim_{t \uparrow T^\star} \|\nabla u(t)\|_{L^2} = +\infty. \tag{1.2}$$

The mass (i.e., the L^2 norm) and the energy E of the solution are conserved by the flow, where

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 - \frac{1}{4}\int_{\mathbb{R}^2} |u|^4.$$

From a variational argument, the unique (up to symmetry) ground state solution to

$$\Delta Q - Q + Q^3 = 0, \ \ Q \in H^1(\mathbb{R}^2), \ \ Q > 0, \ \ Q \text{ is radially symmetric},$$

attains the best constant C in the following Gagliardo-Nirenberg inequality

$$\forall u \in H^1(\mathbb{R}^2), \ \ \|u\|_{L^4}^4 \le C\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2 \tag{1.3}$$

(see [4, 54, 25]). As a consequence, one has

$$\forall u \in H^1(\mathbb{R}^2), \ \ E(u) \ge \frac{1}{2}\|\nabla u\|_{L^2}^2\left(1 - \frac{\|u\|_{L^2}^2}{\|Q\|_{L^2}^2}\right). \tag{1.4}$$

Together with the conservation of mass and energy and the blow up criterion (1.2), this implies the global existence of any solution with initial data $\|u_0\|_2 < \|Q\|_2$. Actually it is now known that in this case, the solution scatters i.e., behaves asymptotically in large time as a solution of the linear equation, see [19, 12] and references therein.

We also know that $\|u\|_{L^2} = \|Q\|_{L^2}$ corresponds to the mass threshold for global existence since the pseudo-conformal symmetry of the (NLS) equation

$$v(t,x) = \frac{1}{|t|}u\left(\frac{1}{|t|}, \frac{x}{|t|}\right)e^{-i\frac{|x|^2}{4|t|}} \tag{1.5}$$

applied to the solitary wave solution $u(t,x) = e^{it}Q(x)$ yields the existence of an explicit single bubble blow up solution $S(t)$ with minimal mass

$$S(t,x) = \frac{1}{|t|}Q\left(\frac{x}{|t|}\right)e^{-i\frac{|x|^2}{4|t|}}e^{\frac{i}{|t|}}, \quad \|S(t)\|_{L^2} = \|Q\|_{L^2}, \quad \|\nabla S(t)\|_{L^2} \underset{t\sim 0^-}{\sim} \frac{1}{|t|}. \tag{1.6}$$

We refer to [7] for more properties of the pseudo-conformal transform. From [35], minimal mass blow up solutions are *classified* in $H^1(\mathbb{R}^2)$:

$$\|u(t)\|_{L^2} = \|Q\|_{L^2} \text{ and } T^* < +\infty \text{ imply } u \equiv S \text{ up to the symmetries of the flow.}$$

Recall also the following well-known general sufficient criterion for finite time blow up: for initial data $u_0 \in \Sigma = H^1 \cap L^2(|x|^2dx)$, the virial identity

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^2}|x|^2|u(t,x)|^2dx = 16E(u_0) \tag{1.7}$$

implies blow up in finite time provided $E(u_0) < 0$ (by (1.4), this implies necessarily $\|u_0\|_{L^2} > \|Q\|_{L^2}$).

1.2. Single bubble blow up dynamics

We focus now on the case of mass slightly above the threshold, that is

$$\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha_0, \quad 0 < \alpha_0 \ll 1. \tag{1.8}$$

We first recall in this context that a large class of finite time blow up solutions was constructed in [6] (see also [22], [43]) as weak perturbation of the minimal mass solution $S(t)$. In particular, these solutions blow up with the pseudo-conformal blow up rate

$$\|\nabla u(t)\|_{L^2} \underset{t\sim T^*}{\sim} \frac{1}{T^*-t}. \tag{1.9}$$

Second, recall that the series of works [49, 37, 38, 52, 36, 40] provides a thorough study of the *stable* blow up dynamics under condition (1.8), corresponding to the so called *log-log* blow up regime

$$\|\nabla u(t)\|_{L^2} \underset{t\sim T^*}{\sim} c^*\sqrt{\frac{\log|\log(T^*-t)|}{T^*-t}}. \tag{1.10}$$

Third, it is proved in [43] (see also [22]) that solutions constructed in [6] are unstable and correspond in some sense to a threshold between the above log-log blow up and scattering.

Finally, recall that under (1.8), a universal gap on the blow up speed was proved in [52]: given a finite time blow up solution satisfying (1.8), either it blows up in the log-log regime (1.10), or it blows up faster than the pseudo-conformal rate

$$\|\nabla u(t)\|_{L^2} \gtrsim \frac{1}{T^*-t}.$$

(See also [1, 2].) However, the existence of solutions blowing up strictly faster than the conformal speed is a long lasting open problem, which is equivalent, by the pseudo-conformal symmetry (1.5), to the existence of global solutions blowing up in infinite time.

1.3. Multi bubbles blow up dynamics

For larger L^2 mass, it is conjectured (see e.g., [39]) that any finite time blow up solution concentrates at the blow up time universal quanta of mass $m_j > 0$ at a finite number of points $x_k \in \mathbb{R}^2$, that is

$$|u(t)|^2 \rightharpoonup \sum_{k=1}^{K} m_k \delta_{x_k} + |u^*|^2 \text{ as } t \uparrow T^*,$$

where $u^* \in L^2$ is a (possibly zero) residual. The first example of multiple point blow up solution is given in [34]: let $K \geq 1$ and let $(x_k)_{1\leq k\leq K}$ be K arbitrary distinct points of $\mathbb{R}^2$, there exists a finite time blow up solution $u(t)$ of (1.1) with

$$\left\| u(t) - \sum_{k=1}^{K} S(t, . - x_k) \right\|_{H^1} \to 0, \quad |u(t)|^2 \rightharpoonup \|Q\|_{L^2}^2 \sum_{k=1}^{K} \delta_{x_k} \text{ as } t \uparrow 0.$$

In particular, $u(t)$ blows up with the pseudo-conformal rate

$$\|\nabla u(t)\|_{L^2} \sim \frac{1}{|t|} \text{ as } t \uparrow 0.$$

Other general constructions of multi bubble blow up are provided by [51, 16] in the context of the log-log regime. Observe that these works deal with *weak* interactions in the sense that the blow up dynamics of each bubble is not perturbed at the main order by the presence of the other (distant) bubbles.

1.4. Main results

In this paper we construct the first example of infinite time blow up solution of (NLS), related to the *strong* interactions of an arbitrary number $K \geq 2$ of bubbles. As a consequence, using the pseudo-conformal transform, we also obtain the first example of solution blowing up in finite time strictly faster than the conformal blow up rate. Such a solution concentrates the K bubbles at one point at the blow up time.

THEOREM 1 (Infinite time blow up). – *Let $K \geq 2$ be an integer. There exists a solution $u \in \mathcal{C}([0, +\infty), \Sigma)$ of* (1.1) *which decomposes asymptotically into a sum of K solitary waves*

$$\left\| u(t) - e^{i\gamma(t)} \sum_{k=1}^{K} \frac{1}{\lambda(t)} Q\left(\frac{. - x_k(t)}{\lambda(t)}\right) \right\|_{H^1} \to 0, \quad \lambda(t) = \frac{1 + o(1)}{\log t} \text{ as } t \to +\infty, \tag{1.11}$$

where the translation parameters $x_k(t)$ converge as $t \to +\infty$ to the vertices of a K-sided regular polygon, and where $\gamma(t)$ is some phase parameter. In particular,

$$\|\nabla u(t)\|_{L^2} = K^{\frac{1}{2}} \|\nabla Q\|_{L^2} (1 + o(1)) \log t \text{ as } t \to +\infty. \tag{1.12}$$

COROLLARY 2 (Finite time collision). – *Let $u(t) \in C([0, +\infty), \Sigma)$ be given by Theorem 1 and let $v \in \mathcal{C}((-\infty, 0), \Sigma)$ be the pseudo conformal transform of $u(t)$ defined by* (1.5). *Then $v(t)$ blows up at $T^* = 0$ with*

$$\|\nabla v(t)\|_{L^2} = K^{\frac{1}{2}} \|\nabla Q\|_{L^2} (1 + o(1)) \left| \frac{\log |t|}{t} \right|, \quad |v|^2 \rightharpoonup K \|Q\|_{L^2}^2 \delta_0 \text{ as } t \uparrow 0. \tag{1.13}$$

Comments on the main results

1. Dynamics with multiple nonlinear objects. – Multiple bubble solutions with *weak interactions* and asymptotically free Galilean motion have been constructed in various settings, both in stable and unstable contexts, see in particular [34, 46, 27, 28, 21, 11, 50, 5, 16]. As a typical example of weakly interacting dynamics, for the nonlinear Schrödinger equations

$$i\partial_t u + \Delta u + |u|^{p-1}u = 0, \quad x \in \mathbb{R}^d, \quad 1 < p < 1 + \frac{4}{d-2}, \tag{1.14}$$

there exist multi solitary wave solutions satisfying for large t,

$$\left\| u(t) - \sum_{k=1}^{K} e^{-i\Gamma_k(t,x)} \omega_k^{\frac{1}{p-1}} Q\left(\omega_k^{\frac{1}{2}}(. - v_k t)\right) \right\|_{H^1} \lesssim e^{-\gamma t}, \ \ \gamma > 0, \tag{1.15}$$

for any given set of parameters $\{v_k, \omega_k\}_k$ with the decoupling condition $v_k \neq v_{k'}$ if $k \neq k'$ (see [28, 11]).

In [21], two different regimes with *strong interactions* related to the two body problem of gravitation are exhibited for the Hartree model (hyperbolic and parabolic asymptotic motions). We also refer to [46, 29] for works related to sharp interaction problems in the setting of the subcritical (gKdV) equation. We thus see the present work as the first intrusion into the study of strongly interacting non radial multi solitary wave motions for (NLS). Note that the solution given by Theorem 1 is a minimal threshold dynamics and its behavior is unstable by perturbation of the data. An important direction of further investigation is the derivation of *stable* strongly interacting multiple bubbles blow up dynamics.

We observe from the proof of Corollary 2 that the K bubbles of the solution collide at the same point at the blow up time providing the first example of collision at blow up for (NLS). Note that the geometry of the trajectories of the blow up points (straight lines from the origin to the egde of the K-sided regular polygon) is an essential feature of these solutions. A related one dimensional mechanism is involved in the derivation of degenerate blow up curves in the context of "type I" blow up for the wave equation, see [44]. For the nonlinear heat equation in one dimension, solutions for which two points of maximum collide at blow up are constructed in [17]. There are also analogies of the present work with the construction of stationary solutions with mass concentrated along specific nonlinear grids, see [47]. In the context of two dimensional incompressible fluid mechanics, special solutions to the vortex point system are studied as a simplified model for dynamics of interacting and possibly colliding vortex, see for example [45] for an overview of these problems.

2. Minimal mass solutions. – The proof of Theorem 1 follows the now standard strategy of constructing *minimal* dynamics by approximate solutions and compactness, initiated in [34] and extended in various ways and contexts by [27, 21, 11, 53]. We combine in a blow up context the approach developed for multibubble flows in [27, 21] and a specific strategy to construct minimal blow up solutions for (NLS) type equations introduced in [53, 20]. A key ingredient of the proof is the precise tuning of the interactions between the waves. In particular, we observe that the K bubbles in (1.11) have the same phase, which is crucial in our analysis. The dynamics of two symmetric bubbles with opposite phase ($\gamma_1 = \gamma_2 + \pi$) is related to the dynamics of a single bubble on a half-plane with Dirichlet boundary condition and it is known in this context that minimal mass blow up at a boundary point (which corresponds to the collision case) does not exist, see [2].

Note that we restrict ourselves to space dimension 2 for simplicity, but similar results hold for the mass critical (NLS) equation in any space dimension with same proof.

For the mass subcritical ($1 < p < 1 + \frac{4}{d}$) and supercritical ($1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$) nonlinear Schrödinger Equations (1.14), we expect, by using a similar approach, the existence of bounded strongly interacting multi solitary waves, with logarithmic relative distances, i.e., non free Galilean motion. Interestingly enough, the existence of such solutions is *ruled out* in the mass critical case by the virial law (1.7). The scaling instability direction of the critical case is excited by the interactions which lead to the infinite time concentration displayed in Theorem 1.

Conversely, solutions such as $u(t)$ in Theorem 1 cannot exist in the sub and supercritical cases. In the subcritical case, it is well-known that all H^1 solutions are bounded in H^1 ([7]). In the supercritical case, any solution in Σ that is global for $t \geq 0$ satisfies

$$\liminf_{t\to+\infty} \|\nabla u(t)\|_{L^2} \lesssim 1.$$

Indeed in this case, the Virial identity $\frac{d^2}{dt^2}\int |x|^2|u|^2 = 4d(p-1)E(u_0) - (\frac{d}{2}(p-1)-2)\int |\nabla u|^2$ integrated twice in time provides the global bound $\int_0^t \int_0^s \|\nabla u(s')\|_{L^2}^2 ds' ds \lesssim t^2$.

Note that the construction of Theorem 1 is performed near $t = +\infty$ (by translation invariance, it is then obvious to obtain a solution on the time interval $[0, +\infty)$). An interesting question is to understand the behavior of such solutions for $t \leq 0$.

3. Zero energy global solutions. – From the proof of Theorem 1, the solution u has zero energy. In [40], it is proved that any zero energy solution satisfying (1.8) blows up in finite time with the log-log regime. Thus, in the neighborhood of Q, $e^{it}Q$ is the only global zero energy solution. For the critical (gKdV) equation, a similar result holds though in a stronger topology (see [30]). Note that the existence of global in time zero energy solutions is strongly related to Liouville type theorems and to blow up profile, see [48, 37]. For (NLS), the only known examples of global in positive time zero energy solutions so far were the time periodic solutions $e^{it}P$ where P is any solution to the stationary equation $\Delta P - P + P^3 = 0$. Therefore, the existence of such a non trivial global (for positive time) zero energy solution $u(t)$ is surprising. For other works related to minimal mass solutions and their key role in the dynamics of the flow, we refer to [15, 53, 3, 43, 14, 30, 31].

4. Blow up speed for (NLS). – The question of determining all possible blow up rates for solutions of nonlinear dispersive equations is in general intricate. For the (NLS) Equation (1.14) in the mass supercritical-energy subcritical range, a universal sharp upper bound on the blow up rate has been derived in [43] for radial data, but no such bound exists for the mass critical problem. For (NLS) with a double power non linearity of the form $|u|^{p-1}u + |u|^2u$ where $1 < p < 3$, the minimal mass solution has a surprising blow up rate different from the conformal rate, see [26]. For the mass critical (gKdV) equation, solutions arbitrarily close to the solitary wave with arbitrarily fast blow up speed have been constructed in [32]. Recall that constructions of blow up solutions with various blow up rate are also available in the energy critical and super-critical context, see [24, 23, 41, 13, 10, 18]. However, such general constructions seem by now out of reach for the mass critical (NLS) problem. In this context, the derivation of the anomalous blow up speed (1.13), in spite of its rigidity, is an interesting new fact. We will see in the proof how such a blow up rate is related to strong coupling between the solitary waves.

1.5. Notation

Let $\Sigma = H^1 \cap L^2(|x|^2 dx)$. The L^2 scalar product of two complex valued functions $f, g \in L^2(\mathbb{R}^2)$ is denoted by

$$\langle f, g\rangle = \operatorname{Re}\left(\int_{\mathbb{R}^2} f(x)\overline{g}(x)dx\right).$$

In this paper, K is an integer with $K \geq 2$. For brevity, $\sum_k$ denotes $\sum_{k=1}^K$. For $k = 1, \ldots, K$, e_k denotes the unit vector of $\mathbb{R}^2$ corresponding to the complex number $e^{i\frac{2\pi(k-1)}{K}}$. We define the constant $\kappa = \kappa(K)$ by

$$\kappa = \left|1 - e^{i\frac{2\pi}{K}}\right| = (2 - 2\cos(2\pi/K))^{1/2} > 0. \tag{1.16}$$

Recall that we denote by $Q(x) := Q(|x|)$ the unique radial positive ground state of (1.1):

$$Q'' + \frac{Q'}{r} - Q + Q^3 = 0, \quad Q'(0) = 0, \quad \lim_{r\to+\infty} Q(r) = 0. \tag{1.17}$$

It is well-known and easily checked by ODE arguments that for some constant $c_Q > 0$,

$$\text{for all } r > 1, \quad \left|Q(r) - c_Q r^{-\frac{1}{2}} e^{-r}\right| + \left|Q'(r) + c_Q r^{-\frac{1}{2}} e^{-r}\right| \lesssim r^{-\frac{3}{2}} e^{-r}. \tag{1.18}$$

We set

$$I_Q = \int Q^3(x) e^{x_1} dx, \quad x = (x_1, x_2). \tag{1.19}$$

We denote by $\mathcal{Y}$ the set of smooth functions f such that

$$\text{for all } p \in \mathbb{N}, \text{ there exists } q \in \mathbb{N}, \text{ s.t. for all } x \in \mathbb{R}^2 \quad |f^{(p)}(x)| \lesssim |x|^q e^{-|x|}. \tag{1.20}$$

Let Λ be the generator of L^2-scaling in two dimensions:

$$\Lambda f = f + x \cdot \nabla f.$$

The linearization of (1.1) around Q involves the following Schrödinger operators:

$$L_+ := -\Delta + 1 - 3Q^2, \qquad L_- := -\Delta + 1 - Q^2.$$

Denote by $\rho \in \mathcal{Y}$ the unique radial solution H^1 to

$$L_+\rho = \frac{|x|^2}{4} Q \tag{1.21}$$

which satisfies on $\mathbb{R}^2$

$$|\rho(x)| + |\nabla\rho(x)| \lesssim (1 + |x|^3) Q(x). \tag{1.22}$$

We recall the *generalized null space relations* (see [55])

$$\begin{aligned} &L_- Q = 0, \quad L_+(\Lambda Q) = -2Q, \quad L_-(|x|^2 Q) = -4\Lambda Q, \quad L_+\rho = \frac{|x|^2}{4} Q, \\ &L_+(\nabla Q) = 0, \quad L_-(xQ) = -2\nabla Q, \end{aligned} \tag{1.23}$$

which can be checked easily from direct computations using the equation $\Delta Q + Q^3 = Q$ and the definitions of L_+ and L_-.

We also recall the following standard coercivity property under orthogonality conditions (see e.g., [37, 38, 53, 55, 9]): *there exists* $\mu > 0$ *such that, for all* $\eta \in H^1$,

$$(1.24)\quad \langle L_+ \operatorname{Re}\eta, \operatorname{Re}\eta\rangle + \langle L_- \operatorname{Im}\eta, \operatorname{Im}\eta\rangle \geq \mu\|\eta\|_{H^1}^2 - \frac{1}{\mu}\left(\langle \eta, Q\rangle^2 + \langle \eta, |x|^2 Q\rangle^2 + |\langle \eta, xQ\rangle|^2 + \langle \eta, i\rho\rangle^2 + |\langle \eta, i\nabla Q\rangle|^2\right).$$

1.6. Outline of the paper

The main goal of Sect. 2 is to construct a symmetric K-bubble approximate solution to (NLS) and to extract the formal evolution system of the geometrical parameters of the bubbles. The key observation is that this system contains forcing terms due to the nonlinear interactions of the waves, and has a special solution corresponding at the main order to the regime of Theorem 1 (see Sect. 2.2). In Sect. 3, we prove uniform estimates on particular *backwards solutions* of (NLS) related to the special regime of Theorem 1. We proceed in two main steps. First, we control the residue term by energy arguments in the context of multi-bubbles. Second, a careful adjustment of the *final data* yields a uniform control of the geometrical parameters. In Sect. 4, we finish the proof of Theorem 1 by a compactness argument on a suitable sequence of backwards solutions of (NLS) satisfying the uniform estimates of Sect. 3.

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2. Approximate solution

In this section, we first construct a symmetric K-bubble approximate solution to (NLS) and extract the evolution system of the geometrical parameters of the bubbles. This system contains forcing terms due to the nonlinear interactions of the waves. Second, we write explicitly a particular formal solution of this system that will serve as a guideline for the construction of the special solution $u(t)$ of Theorem 1. Third, we state a standard modulation lemma around the approximate solution. Recall that the integer $K \geq 2$ is fixed.

2.1. Approximate solution and nonlinear forcing

Consider a time dependent C^1 function $\vec{p}$ of the form

$$\vec{p} = (\lambda, z, \gamma, \beta, b) \in (0, +\infty)^2 \times \mathbb{R}^3,$$

with $|b| + |\beta| \ll 1$ and $z \gg 1$. We renormalize the flow by considering

$$u(t,x) = \frac{e^{i\gamma(s)}}{\lambda(s)} v(s,y), \quad dt = \lambda^2(s) ds, \quad y = \frac{x}{\lambda(s)}, \tag{2.1}$$

so that

$$i\partial_t u + \Delta u + |u|^2 u = \frac{e^{i\gamma}}{\lambda^3}\left[i\dot{v} + \Delta v - v + |v|^2 v - i\frac{\dot{\lambda}}{\lambda}\Lambda v + (1-\dot{\gamma})v \right] \tag{2.2}$$

($\dot{v}$ denotes derivation with respect to s). We introduce the following $\vec{p}$-modulated ground state solitary waves, for any $k \in \{1, \dots, K\}$,

$$P_k(s,y) = e^{i\Gamma_k(s, y - z_k(s))} Q_{a(z(s))}(y - z_k(s)), \tag{2.3}$$

for

$$\beta_k = \beta \mathrm{e}_k, \quad z_k = z\mathrm{e}_k, \quad \Gamma_k(s,y) = \beta_k \cdot y - \frac{b}{4}|y|^2, \tag{2.4}$$

and where we have fixed

$$Q_a = Q + a\rho, \quad a(z) = -c_a z^{\frac{1}{2}} e^{-\kappa z}, \quad c_a = \begin{cases} \dfrac{\kappa^{\frac{1}{2}} c_Q I_Q}{4\langle \rho, Q\rangle} > 0 & \text{for } K = 2 \\[2ex] \dfrac{\kappa^{\frac{1}{2}} c_Q I_Q}{2\langle \rho, Q\rangle} > 0 & \text{for } K \geq 3. \end{cases} \tag{2.5}$$

Note that the introduction of such modulated Q_a corresponds to the intrinsic instability of the pseudo-conformal blow up regime ($a = 0$ leads to $b(s) = s^{-1}$). Similar exact Q_a (at any order of a) were introduced in [42]. The explicit above choice of $a(z)$ corresponds to direct integration of the nonlinear interactions at the main order, as explained in Sect. 2.2. We also refer to (3.27) in the proof of Lemma 7 where this choice of $a(z)$ leads to an almost conservation of the mass for the approximate solution $\mathbf{P}$ defined below. Note that the different formula for c_a depending on the value of K corresponds to the fact that for $K \geq 3$, each given soliton has exactly two closest neighbor solitons.

Let

$$\mathbf{P}(s,y) = \mathbf{P}(y; (z(s), b(s), \beta(s))) = \sum_k P_k(s,y). \tag{2.6}$$

Then, $\mathbf{P}$ is an approximate solution of the rescaled equation in the following sense.

LEMMA 3 (Leading order approximate flow). – *Let the vectors of modulation equations be*

$$\vec{m}_k^a = \begin{pmatrix} b + \frac{\dot\lambda}{\lambda} \\ \dot z_k - 2\beta_k + \frac{\dot\lambda}{\lambda} z_k \\ \dot\gamma - 1 + |\beta_k|^2 - \frac{\dot\lambda}{\lambda}(\beta_k\cdot z_k) - (\beta_k \cdot \dot z_k) \\ \dot\beta_k - \frac{\dot\lambda}{\lambda}\beta_k + \frac{b}{2}(\dot z_k - 2\beta_k + \frac{\dot\lambda}{\lambda} z_k) \\ \dot b + b^2 - 2b(b + \frac{\dot\lambda}{\lambda}) - a \end{pmatrix}, \qquad \vec{\mathrm{M}} V = \begin{pmatrix} -i\Lambda V \\ -i\nabla V \\ -V \\ -yV \\ \frac{|y|^2}{4} V \end{pmatrix}. \tag{2.7}$$

Then the error to the renormalized flow (2.2) *at* $\mathbf{P}$,

$$\mathcal{E}_{\mathbf{P}} = i\dot{\mathbf{P}} + \Delta\mathbf{P} - \mathbf{P} + |\mathbf{P}|^2\mathbf{P} - i\frac{\dot\lambda}{\lambda}\Lambda\mathbf{P} + (1-\dot\gamma)\mathbf{P} \tag{2.8}$$

decomposes as

$$\mathcal{E}_{\mathbf{P}} = \sum_k \left[e^{i\Gamma_k}\Psi_k\right](y - z_k), \quad \Psi_k = \vec{m}_k^a \cdot \vec{\mathrm{M}} Q_a + i\dot z a'(z)\rho + G_k + \Psi_{Q_a}, \tag{2.9}$$

where

$$\|G_k\|_{L^\infty} \lesssim z^{-\frac12} e^{-\kappa z}, \quad \|\Psi_{Q_a}\|_{L^\infty} \lesssim |a|^2, \tag{2.10}$$

and

(2.11)
$$\left|\langle G_k, iQ_a\rangle + \kappa c_a\langle\rho, Q\rangle b z^{\frac32} e^{-\kappa z}\right| \lesssim (|\beta|^2 z^2 + |b|^2 z^4 + |\beta| z + |b| z) z^{-\frac12} e^{-\kappa z} + z^3 e^{-2\kappa z}.$$

Proof of Lemma 3. – Step 1: Equation for P_k. Let

$$\mathcal{E}_{P_k} = i\dot P_k + \Delta P_k - P_k + |P_k|^2 P_k - i\frac{\dot\lambda}{\lambda}\Lambda P_k + (1-\dot\gamma)P_k.$$

Let $y_{z_k} = y - z_k$. By direct computations

$$\begin{aligned} i\dot P_k &= \Big[e^{i\Gamma_k}\Big(i\dot z a'(z)\rho - (\dot\beta_k\cdot y_{z_k})Q_a + (\dot z_k\cdot\beta_k)Q_a \\ &\quad + \frac{\dot b}{4}|y_{z_k}|^2 Q_a - \frac{b}{2}(\dot z_k \cdot y_{z_k})Q_a - i(\dot z_k\cdot\nabla Q_a)\Big)\Big](y_{z_k}), \\ \Delta P_k &= \Big[e^{i\Gamma_k}\Big(\Delta Q_a - |\beta_k|^2 Q_a - \frac{b^2}{4}|y_{z_k}|^2 Q_a - ibQ_a \\ &\quad + b(\beta_k\cdot y_{z_k})Q_a + 2i(\beta_k\cdot\nabla Q_a) - ib(y_{z_k}\cdot\nabla Q_a)\Big)\Big](y_{z_k}), \\ \Lambda P_k &= \Big[e^{i\Gamma_k}\Big(\Lambda Q_a + i(\beta_k\cdot y_{z_k})Q_a - i\frac{b}{2}|y_{z_k}|^2 Q_a + (y_{z_k}\cdot\nabla Q_a) \\ &\quad + i(z_k\cdot\beta_k)Q_a - i\frac{b}{2}(z_k\cdot y_{z_k})Q_a + (z_k\cdot\nabla Q_a)\Big)\Big](y_{z_k}). \end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{E}_{P_k} = \Big[e^{i\Gamma_k}\Big(& -i(b+\frac{\dot\lambda}{\lambda})\Lambda Q_a - i(\dot z_k - 2\beta_k + \frac{\dot\lambda}{\lambda}z_k)\cdot\nabla Q_a \\ & -(\dot\gamma - 1 + |\beta_k|^2 - \frac{\dot\lambda}{\lambda}(\beta_k\cdot z_k) - (\beta_k\cdot\dot z_k))Q_a \\ & -(\dot\beta_k - \frac{\dot\lambda}{\lambda}\beta_k + \frac{b}{2}(\dot z_k - 2\beta_k + \frac{\dot\lambda}{\lambda}z_k))\cdot y_{z_k}Q_a \\ & + \frac{1}{4}(\dot b + b^2 - 2b(b+\frac{\dot\lambda}{\lambda}))|y_{z_k}|^2 Q_a + i\dot z a'(z)\rho + \Delta Q_a - Q_a + |Q_a|^2 Q_a\Big)\Big](y_{z_k}).\end{aligned}$$

By $\Delta Q - Q + Q^3 = 0$ and the definition of ρ, $L_+\rho = -\Delta\rho + \rho - 3Q^2\rho = \frac{1}{4}|y|^2 Q$ (see (1.21)), we have

$$\Delta Q_a - Q_a + |Q_a|^2 Q_a = -\frac{a}{4}|y|^2 Q_a + \Psi_{Q_a},$$

where

$$\Psi_{Q_a} = |Q_a|^2 Q_a - Q^3 - 3aQ^2\rho + \frac{a^2}{4}|y|^2\rho. \tag{2.12}$$

We have thus obtained the P_k equation

$$\mathcal{E}_{P_k} = \Big[e^{i\Gamma_k}\Big(\vec m_k^a\cdot\vec{\mathrm{M}}Q_a + i\dot z a'(z)\rho + \Psi_{Q_a}\Big)\Big](y - z_k), \tag{2.13}$$

where $\vec m_k^a$ and $\vec{\mathrm{M}}$ are defined in (2.7).

Step 2: Equation for $\mathbf{P}$. From step 1 and the definition of $\mathcal{E}_{\mathbf{P}}$ in (2.8), it follows that

$$\mathcal{E}_{\mathbf{P}} = \sum_k \mathcal{E}_{P_k} + |\mathbf{P}|^2\mathbf{P} - \sum_k |P_k|^2 P_k.$$

Observe that

$$|\mathbf{P}|^2\mathbf{P} - \sum_k |P_k|^2 P_k = \sum_{j,k,l} P_k P_j \overline{P}_l - \sum_k |P_k|^2 P_k = \sum_k F_k,$$

with

$$F_k = 2|P_k|^2\sum_{j\neq k} P_j + P_k^2\sum_{j\neq k}\overline{P}_j + \overline{P}_k\sum_{j\neq k,\, l\neq k,\, j\neq l} P_j P_l = \big[e^{i\Gamma_k}G_k\big](y - z_k),$$

where we have set

$$G_k = 2G_k^{(\mathrm{I})} + \overline{G}_k^{(\mathrm{I})} + G_k^{(\mathrm{II})}, \tag{2.14}$$

and

$$\begin{aligned} G_k^{(\mathrm{I})}(y) &= \big[e^{-i\Gamma_k}Q_a^2\big](y)\sum_{j\neq k}\big[e^{i\Gamma_j}Q_a\big](y - (z_j - z_k)),\\ G_k^{(\mathrm{II})}(y) &= \big[e^{-2i\Gamma_k}Q_a\big](y)\sum_{j\neq k,\, l\neq k,\, j\neq l}\Big(\big[e^{i\Gamma_j}Q_a\big](y - (z_j - z_k))\cdot\big[e^{i\Gamma_l}Q_a\big](y - (z_l - z_k))\Big).\end{aligned}$$

Therefore,

$$\mathcal{E}_{\mathbf{P}} = \sum_k \big[e^{i\Gamma_k}\Psi_k\big](y - z_k) \quad\text{where}\quad \Psi_k = \vec m_k^a\cdot\vec{\mathrm{M}}Q_a + i\dot z a'(z)\rho + G_k + \Psi_{Q_a}. \tag{2.15}$$

Step 3: Nonlinear interaction estimates. In order to estimate the various terms in (2.15), we will use the following interaction estimates: let $\omega, \widetilde{\omega} \in \mathbb{R}^2$, $|\omega| \gg 1$, $|\widetilde{\omega}| \gg 1$, let $q \geq 0$, then:

$$\int (1+|y|^q) Q^3(y) Q(y-\omega) dy \lesssim |\omega|^{-\frac{1}{2}} e^{-|\omega|}, \tag{2.16}$$

$$\int (1+|y|^q) Q^2(y) Q(y-\omega) Q(y-\widetilde{\omega}) dy \lesssim e^{-\frac{3}{2}|\omega|} + e^{-\frac{3}{2}|\widetilde{\omega}|}, \tag{2.17}$$

$$\left| \int Q^3(y) Q(y-\omega) dy - c_Q I_Q |\omega|^{-\frac{1}{2}} e^{-|\omega|} \right| \lesssim |\omega|^{-\frac{3}{2}} e^{-|\omega|}, \tag{2.18}$$

with c_Q and I_Q are given by (1.18)–(1.19).

Proof of (2.16). – From (1.18), observe that

$$Q(y)Q(y-\omega) \lesssim (1+|y|)^{-\frac{1}{2}} (1+|y-\omega|)^{-\frac{1}{2}} e^{-|y|} e^{-|\omega|+|y|} \lesssim |\omega|^{-\frac{1}{2}} e^{-|\omega|}. \tag{2.19}$$

Thus,

$$\int (1+|y|^q) Q^3(y) Q(y-\omega) dy \lesssim |\omega|^{-\frac{1}{2}} e^{-|\omega|} \int (1+|y|^q) Q^2(y) dy \lesssim |\omega|^{-\frac{1}{2}} e^{-|\omega|}.$$

Proof of (2.17). – From (2.19),

$$\begin{aligned}\int (1+|y|^q) Q^2(y) Q(y-\omega) Q(y-\widetilde{\omega}) dy &\lesssim \int (1+|y|^q) Q^2(y) Q^{\frac{3}{4}}(y-\omega) Q^{\frac{3}{4}}(y-\widetilde{\omega}) dy \\ &\lesssim e^{-\frac{3}{4}|\omega|} e^{-\frac{3}{4}|\widetilde{\omega}|} \int (1+|y|^q) Q^{\frac{1}{2}}(y) dy \lesssim e^{-\frac{3}{4}|\omega|} e^{-\frac{3}{4}|\widetilde{\omega}|}.\end{aligned}$$

Proof of (2.18). – First, using (1.18),

$$\int_{|y|>\frac{3}{4}|\omega|} Q^3(y) Q(y-\omega) dy \lesssim e^{-\frac{9}{4}|\omega|} \int Q(y-\omega) dy \lesssim e^{-\frac{9}{4}|\omega|}.$$

Second, for $|y| < \frac{3}{4}|\omega|$, we use (1.18) to write

$$\left| Q(y-\omega) - c_Q |y-\omega|^{-\frac{1}{2}} e^{-|y-\omega|} \right| \lesssim |y-\omega|^{-\frac{3}{2}} e^{-|y-\omega|} \lesssim |\omega|^{-\frac{3}{2}} e^{-|\omega|+|y|}.$$

In particular,

$$\left| \int_{|y|<\frac{3}{4}|\omega|} Q^3(y) Q(y-\omega) dy - c_Q \int_{|y|<\frac{3}{4}|\omega|} Q^3(y) |y-\omega|^{-\frac{1}{2}} e^{-|y-\omega|} dy \right| \lesssim |\omega|^{-\frac{3}{2}} e^{-|\omega|}.$$

Still for $|y| < \frac{3}{4}|\omega|$, the expansion

$$|y-\omega|^2 = |\omega|^2 - 2y\cdot\omega + |y|^2$$

implies

$$\left| |y-\omega|^{-\frac{1}{2}} - |\omega|^{-\frac{1}{2}} \right| \lesssim |\omega|^{-\frac{3}{2}} |y|$$

and

$$\left| |y-\omega| - |\omega| + y\cdot\frac{\omega}{|\omega|} \right| \lesssim |\omega|^{-1} |y|^2.$$

Thus,

$$\left| e^{-|y-\omega|} - e^{-|\omega|+y\cdot\frac{\omega}{|\omega|}} \right| \lesssim |\omega|^{-1} |y|^2 \left(e^{-|y-\omega|} + e^{-|\omega|+y\cdot\frac{\omega}{|\omega|}} \right) \lesssim |\omega|^{-1} |y|^2 e^{-|\omega|} e^{|y|}.$$

Therefore,

$$\left| |y-\omega|^{-\frac{1}{2}} e^{-|y-\omega|} - |\omega|^{-\frac{1}{2}} e^{-|\omega| + y\cdot\frac{\omega}{|\omega|}} \right| \lesssim |\omega|^{-\frac{3}{2}}(1+|y|^2) e^{-|\omega|} e^{|y|},$$

and so

$$\left| \int_{|y|<\frac{3}{4}|\omega|} Q^3(y)|y-\omega|^{-\frac{1}{2}} e^{-|y-\omega|} dy - |\omega|^{-\frac{1}{2}} e^{-|\omega|} \int_{|y|<\frac{3}{4}|\omega|} Q^3(y) e^{y\cdot\frac{\omega}{|\omega|}} dy \right|$$
$$\lesssim |\omega|^{-\frac{3}{2}} e^{-|\omega|} \int (1+|y|^2) e^{-2|y|} dy \lesssim |\omega|^{-\frac{3}{2}} e^{-|\omega|}.$$

Also, we see that

$$|\omega|^{-\frac{1}{2}} e^{-|\omega|} \int_{|y|>\frac{3}{4}|\omega|} Q^3(y) e^{y\cdot\frac{\omega}{|\omega|}} dy \lesssim |\omega|^{-\frac{1}{2}} e^{-|\omega|} \int_{|y|>\frac{3}{4}|\omega|} e^{-2|y|} dy \lesssim e^{-2|\omega|}.$$

Since for all $\omega \neq 0$ (see (1.19)),

$$I_Q = \int Q^3(y) e^{y\cdot\frac{\omega}{|\omega|}} dy,$$

we have proved (2.18).

Step 4: Estimates of G_k and Ψ_{Q_a}. We are now in position to prove (2.10) and (2.11). The estimate on Ψ_{Q_a} in (2.10) follows directly from its Definition (2.12). To estimate G_k as in (2.10), we first note that from (1.18), (1.22) and the definition of $a(z)$ in (2.5), we have, for some $q>0$,

$$|Q_a| \lesssim |y|^{-\frac{1}{2}} e^{-|y|} + |a(z)||y|^q e^{-|y|} \lesssim (1+|y|)^{-\frac{1}{2}} e^{-|y|} + z^{\frac{1}{2}} e^{-\kappa z}(1+|y|)^q e^{-|y|}.$$

Moreover, for $j \neq k$, from the definition of κ in (1.16),

$$|z_j - z_k| = z|\mathrm{e}_k - \mathrm{e}_j| \geq \kappa z.$$

From this, it follows easily that for $j \neq k$,

$$|Q_a(y)||Q_a(y-(z_k-z_j))| \lesssim z^{-\frac{1}{2}} e^{-\kappa z},$$

which in light of the explicit Formula (2.14) yields the control of G_k in (2.10).

We now turn to the proof of (2.11) which requires a more careful analysis of the interaction terms. We first compute the main order of the contribution of $G_1^{(\mathrm{I})}$ to $\langle G_1, iQ\rangle$.

For $j = 2,\dots,K$,

$$\langle [e^{-i\Gamma_1} Q_a^2](y)[e^{i\Gamma_j} Q_a](y-(z_j-z_1)), iQ_a\rangle$$
$$= \int Q_a^3(y) Q_a(y - z(\mathrm{e}_j - \mathrm{e}_1)) \sin(\Gamma_j(y - z(\mathrm{e}_j-\mathrm{e}_1)) - \Gamma_1(y)) dy.$$

First, by the decay of ρ (see (1.22)), (2.16) and the definition of $a(z)$ in (2.5), we have

$$\int \left| Q_a^3(y) Q_a(y - z(\mathrm{e}_j-\mathrm{e}_1)) - Q^3(y) Q(y - z(\mathrm{e}_j - \mathrm{e}_1)) \right| dy \lesssim |a| z^{\frac{5}{2}} e^{-\kappa z} \lesssim z^3 e^{-2\kappa z}.$$

Next, note that, since $\Gamma_j = \beta_j \cdot y - \frac{b}{4}|y|^2$,

$$\left| \sin(\Gamma_j(y - z(\mathrm{e}_j - \mathrm{e}_1)) - \Gamma_1(y)) - (\Gamma_j(y - z(\mathrm{e}_j-\mathrm{e}_1)) - \Gamma_1(y)) \right|$$
$$\lesssim |\Gamma_j(y - z(\mathrm{e}_j-\mathrm{e}_1))|^2 + |\Gamma_1(y)|^2 \lesssim |\beta|^2(|y|^2+z^2) + |b|^2(|y|^4+z^4),$$

and

$$\left|(\Gamma_j(y-z(\mathrm{e}_j-\mathrm{e}_1))-\Gamma_1(y))+\frac{b}{4}|z(\mathrm{e}_j-\mathrm{e}_1)|^2\right|\lesssim|\beta|(|y|+z)+|b|(|y|^2+|y|z).$$

Thus, using (2.16),

$$\int Q^3(y)Q(y-z(\mathrm{e}_j-\mathrm{e}_1))\left|\sin(\Gamma_j(y-z(\mathrm{e}_j-\mathrm{e}_1))-\Gamma_1(y))+\frac{b}{4}|z(\mathrm{e}_j-\mathrm{e}_1)|^2\right|dy$$
$$\lesssim(|\beta|^2z^2+|b|^2z^4+|\beta|z+|b|z)z^{-\frac12}e^{-\kappa z}.$$

Therefore, we have proved

$$\left|\langle[e^{i\Gamma_1}Q_a^2](y)[e^{i\Gamma_j}Q_a](y-z(\mathrm{e}_j-\mathrm{e}_1)),iQ_a\rangle+\frac{b}{4}|z(\mathrm{e}_j-\mathrm{e}_1)|^2\int Q^3(y)Q(y-z(\mathrm{e}_j-\mathrm{e}_1))\right|$$
$$\lesssim(|\beta|^2z^2+|b|^2z^4+|\beta|z+|b|z)z^{-\frac12}e^{-\kappa z}+z^3e^{-2\kappa z}.$$

For $j=2$ and $j=K$, we have $|z(\mathrm{e}_j-\mathrm{e}_1))|=\kappa z$, and so using (2.18),

$$\left|\langle[e^{i\Gamma_1}Q_a^2](y)[e^{i\Gamma_j}Q_a](y-z(\mathrm{e}_j-\mathrm{e}_1)),iQ_a\rangle+\frac{b}{4}c_QI_Q\kappa^{\frac32}z^{\frac32}e^{-\kappa z}\right|$$
$$\lesssim(|\beta|^2z^2+|b|^2z^4+|\beta|z+|b|z)z^{-\frac12}e^{-\kappa z}+z^3e^{-2\kappa z}.$$

For $K\geq 4$ and $j=3,\dots,K-1$, we have $|\mathrm{e}_j-\mathrm{e}_1|>\kappa'$, for some $\kappa'>\kappa$. Thus the following bound follows from similar computations

$$\left|\langle[e^{i\Gamma_1}Q_a^2](y)[e^{i\Gamma_j}Q_a](y-z(\mathrm{e}_j-\mathrm{e}_1)),iQ_a\rangle\right|$$
$$\lesssim(|\beta|^2z^2+|b|^2z^4+|\beta|z+|b|z^2)z^{-\frac12}e^{-\kappa' z}+z^3e^{-2\kappa z}.$$

Note that $\langle 2G_1^{(\mathrm{I})}+\overline{G}_1^{(\mathrm{I})},iQ_a\rangle=\langle G_1^{(\mathrm{I})},iQ_a\rangle$. We finally bound the contribution of $G_1^{(\mathrm{II})}$. For $j\neq 1$, $l\neq 1$ and $l\neq j$,

$$\begin{aligned}\langle[e^{-2i\Gamma_k}Q_a](y)&[e^{i\Gamma_j}Q_a](y-(z_j-z_k))[e^{i\Gamma_l}Q_a](y-(z_l-z_k)),iQ_a\rangle\\&=\int Q_a^2(y)Q_a(y-(z_j-z_k))Q_a(y-(z_l-z_k))\\&\quad\times\sin(\Gamma_j(y-(z_j-z_k))+\Gamma_l(y-(z_l-z_k))-2\Gamma_k(y))dy.\end{aligned}$$

By (2.17), the bound on $|a|$ and $|\Gamma_j|\lesssim|\beta|(|y|+z)+|b|(|y|^2+z^2)$, this term is bounded by $(|\beta|z+|b|z^2)e^{-\frac32\kappa z}$.

Gathering these estimates, using the definition of the constant c_a in (2.5) which takes into account the two different cases $K=2$ and $K\geq 2$ (for $K=2$, the soliton P_1 has nonlinear interaction with only one other soliton, while for $K\geq 3$, it has exactly two closest neighboring solitons, P_2 and P_K), we obtain finally

$$|\langle G_k,iQ_a\rangle+\kappa c_a\langle\rho,Q\rangle bz^{\frac32}e^{-\kappa z}|\lesssim(|\beta|^2z^2+|b|^2z^4+|\beta|z+|b|z)z^{-\frac12}e^{-\kappa z}+z^3e^{-2\kappa z},$$

which completes the proof of (2.11). □

2.2. Formal resolution of the modulation system with forcing

From Lemma 3, we derive a simplified modulation system with forcing term and we determine one of its approximate solution that is relevant for the regime of Theorem 1. Moreover, we justify the special choice of function $a(z)$ in (2.5). Formally, i.e., assuming that $\mathbf{P}$ is a solution of (1.1) up to error terms of lower order than the ones in (2.9) (making this rigorous will be the object of the bootstrap estimates in Sect. 4), we have the following bounds ($\vec{m}_k^a$ is defined in (2.7))

$$|\vec{m}_1^a| \lesssim z^{-\frac{1}{2}} e^{-\kappa z}. \tag{2.20}$$

Indeed, (2.20) is obtained from (2.9)–(2.10) by projecting $\mathcal{E}_{\mathbf{P}}$ onto directions related to the generalized null space (1.23) (see Lemma 7 for rigorous computations). To simplify the discussion, we drop the equation of γ, which is not coupled with any other equation and has no influence on the regime. Next, we see that using the first line of $\vec{m}_1^a$, i.e., $|b+\frac{\dot\lambda}{\lambda}| \lesssim z^{-\frac{1}{2}} e^{-\kappa z}$, we can replace $\frac{\dot\lambda}{\lambda}$ by $-b$ in all the other estimates. Similarly, we insert the estimate on $\dot z$ from the second line into the estimate for $\dot\beta$. We obtain the following simplified system

$$|b + \frac{\dot\lambda}{\lambda}| + |\dot z - 2\beta - bz| + |\dot\beta + b\beta| + |\dot b + b^2 - a| \lesssim z^{-\frac{1}{2}} e^{-\kappa z}. \tag{2.21}$$

It is easy to check the following estimates

LEMMA 4. – *Let* $(z_{\text{app}}, \lambda_{\text{app}}(s), \beta_{\text{app}}, b_{\text{app}}(s))$ *be such that*

$$\begin{aligned} &\lambda_{\text{app}}(s) = \log^{-1}(s), && z_{\text{app}}^{-\frac{3}{2}}(s) e^{\kappa z_{\text{app}}(s)} = \frac{\kappa c_a}{2} s^2, \\ &|\beta_{\text{app}}(s)| \lesssim s^{-1}\log^{-\frac{3}{2}}(s), && b_{\text{app}}(s) = s^{-1}\log^{-1}(s). \end{aligned} \tag{2.22}$$

Then,

(2.23)

$$z_{\text{app}}(s) \sim \frac{2}{\kappa}\log(s), \quad |b_{\text{app}} + \frac{\dot\lambda_{\text{app}}}{\lambda_{\text{app}}}| = 0, \quad |\dot z_{\text{app}} - 2\beta_{\text{app}} - b_{\text{app}} z_{\text{app}}| \lesssim s^{-1}\log^{-\frac{1}{2}}(s),$$
$$|a(z_{\text{app}}) + s^{-2}\log^{-1}(s)| \lesssim s^{-2}\log^{-\frac{3}{2}}(s), \qquad |\dot b_{\text{app}} + b_{\text{app}}^2 - a(z_{\text{app}})| \lesssim s^{-2}\log^{-\frac{3}{2}}(s).$$

The above estimates mean that (2.22) is a reasonable guess for the first order asymptotics as $s \to +\infty$ of some particular solutions of (2.21) (we refer to Sect. 3.4 for a rigorous integration of (2.21)). Note that we do not actually determine the main order of $\beta(s)$; to do this, more interaction computations would be necessary. However, since $|\dot\beta + b\beta| \lesssim z^{-\frac{1}{2}} e^{-\kappa z}$, formally, we obtain $|\dot\beta| \lesssim s^{-2}\log^{-2}(s)$, which justifies a bootstrap on $\beta(s)$ of the form $|\beta(s)| \ll s^{-1}\log^{-2}(s)$. Note also that there exist solutions of (2.22) with different asymptotics, corresponding to (NLS) solutions like $v(t)$ of Corollary 2.

To complete this formal discussion, we justify the choice of $a(z)$ in (2.5) in the regime given by (2.22). Indeed, projecting Ψ_1 onto the direction iQ_a, from (2.11), we obtain at the leading order

$$|\dot z a'(z) - \kappa c_a b z^{\frac{3}{2}} e^{-\kappa z}| \lesssim (|\beta|^2 z^2 + |b|^2 z^4 + |\beta| z + |b| z) z^{-\frac{1}{2}} e^{-\kappa z} + z^3 e^{-2\kappa z}. \tag{2.24}$$

In the regime suggested by (2.22), since $|\beta| \ll |b|z$, we have $bz \sim \dot{z}$ and thus, simplifying $\dot{z}$, we obtain

$$|a'(z) - \kappa c_a z^{\frac{1}{2}} e^{-\kappa z}| \lesssim z^{-\frac{1}{2}} e^{-\kappa z},$$

which justifies the definition (2.5) by integrating in z.

2.3. Modulation of the approximate solution

We state a standard modulation result around $\mathbf{P}$. We restrict ourselves to the case of solutions invariant by the rotation preserving $\mathbf{P}$. Denote by τ_K the rotation of center 0 and angle $\frac{2\pi}{K}$ in $\mathbb{R}^2$. Since Q and ρ are radial, by definition of P_k and β_k, z_k in (2.3) and (2.4), we have for $k \in \{1, \dots, K-1\}$, $P_k(y) = P_{k+1}(\tau_K y)$ and $P_K(y) = P_1(\tau_K y)$. In particular, it follows that $\mathbf{P}(\tau_K y) = \mathbf{P}(y)$, i.e., $\mathbf{P}$ is invariant by the rotation τ_K. Note also that Equation (1.1) is invariant by rotation. In particular, if a solution of (1.1) is invariant by the rotation τ_K at some time, then it is invariant by rotation at any time. In this context, the following modulation result relies on a standard argument based on the Implicit Function Theorem (see e.g., Lemma 2 in [38]) and we omit its proof.

LEMMA 5 (Modulation around $\mathbf{P}$). – *Let I be some time interval. Let $u \in C(I, H^1(\mathbb{R}^2))$ be a solution of* (1.1) *invariant by the rotation τ_K and such that*

$$\sup_{t\in I} \left\| e^{-i\tilde{\gamma}(t)}\tilde{\lambda}(t)u(t,\tilde{\lambda}(t)\,.\,) - \sum_k Q(\,.\, - \mathrm{e}_k\tilde{z}(t)) \right\|_{H^1} < \delta \tag{2.25}$$

for some $\tilde{\lambda}(t) > 0$, $\tilde{\gamma}(t) \in \mathbb{R}^2$, $\tilde{z}(t) > \delta^{-1}$, where $\delta > 0$ is small enough. Then, there exists a C^1 function

$$\vec{p} = (\lambda, z, \gamma, \beta, b) : I \to (0,\infty)^2 \times \mathbb{R}^3,$$

such that, for $\mathbf{P}(t,y) = \mathbf{P}(y; z(t), b(t), \beta(t))$ as defined in (2.6)*, the solution $u(t)$ decomposes on I as*

$$u(t,x) = \frac{e^{i\gamma(t)}}{\lambda(t)}(\mathbf{P} + \varepsilon)(t,y), \quad y = \frac{x}{\lambda(t)}, \tag{2.26}$$

where for all $t \in I$,

$$|b(t)| + |\beta(t)| + \|\varepsilon(t)\|_{H^1} + |z(t)|^{-1} \lesssim \delta, \tag{2.27}$$

and, setting $\varepsilon(t,y) = \left[e^{i\Gamma_1}\eta_1\right](t, y - z_1)$,

$$\langle \eta_1(t), |y|^2 Q\rangle = |\langle \eta_1(t), yQ\rangle| = \langle \eta_1(t), i\rho\rangle = |\langle \eta_1(t), i\nabla Q\rangle| = \langle \eta_1(t), i\Lambda Q\rangle = 0. \tag{2.28}$$

Moreover, ε is also invariant by the rotation τ_K.

Note that the choice of the special orthogonality conditions (2.28) is related to the generalized null space of the linearized equation around Q, (1.23) and to the coercivity property (1.24). See the proof of Lemma 7 for a technical justification of these choices (see also [53]).

3. Backwards uniform estimates

In this section, we prove uniform estimates on particular backwards solutions. The key point is to carefully adjust their final data to obtain uniform estimates corresponding to the special regime of Theorem 1 and Lemma 4.

Let $(\lambda^{in}, z^{in}, b^{in}) \in (0,+\infty)^2 \times \mathbb{R}$ to be chosen with $\lambda^{in} \ll 1$, $z^{in} \gg 1$, $|b^{in}| \ll 1$. Let $u(t)$ for $t \le 0$ be the solution of (1.1) with data (see (2.6))

$$u(0,x) = \frac{1}{\lambda^{in}}\mathbf{P}^{in}\left(\frac{x}{\lambda^{in}}\right) \quad \text{where} \quad \mathbf{P}^{in}(y) = \mathbf{P}(y;(z^{in}, b^{in}, 0)) \tag{3.1}$$

(we arbitrarily fix $\gamma^{in} = \beta^{in} = 0$). Note that $u(0)$ satisfies (2.25) and, by continuity of the solution of (1.1) in H^1, it exists and satisfies (2.25) on some maximal time interval $(t^{\text{mod}}, 0]$, where $t^{\text{mod}} \in [-\infty, 0)$. Note also that by invariance by rotation of Equation (1.1), $u(t)$ is invariant by the rotation τ_K. On $(t^{\text{mod}}, 0]$, we consider $(\vec{p}, \varepsilon)$ the decomposition of u defined from Lemma 5. For $s^{in} \gg 1$, we normalize the rescaled time s as follows, for $t \in (t^{\text{mod}}, 0]$,

$$s = s(t) = s^{in} - \int_t^0 \frac{d\tau}{\lambda^2(\tau)}. \tag{3.2}$$

Observe from (3.1) that

$$\begin{aligned} &\lambda(s^{in}) = \lambda^{in}, \quad b(s^{in}) = b^{in}, \quad z(s^{in}) = z^{in}, \\ &\gamma(s^{in}) = 0, \qquad \beta(s^{in}) = 0, \qquad \varepsilon(s^{in}) \equiv 0. \end{aligned} \tag{3.3}$$

PROPOSITION 6 (Uniform backwards estimates). – *There exists $s_0 > 10$ such that for all $s^{in} > s_0$, there exists a choice of parameters $(\lambda^{in}, z^{in}, b^{in})$ with*

$$\begin{aligned} &\left| \left(\frac{2}{\kappa c_a}\right)^{\frac12} (z^{in})^{-\frac34} e^{\frac{\kappa}{2} z^{in}} - s^{in} \right| < s^{in} \log^{-\frac12}(s^{in}), \\ &\lambda^{in} = \log^{-1}(s^{in}), \quad b^{in} = \left(\frac{2c_a}{\kappa}\right)^{\frac12} (z^{in})^{-\frac14} e^{-\frac{\kappa}{2} z^{in}}, \end{aligned} \tag{3.4}$$

such that the solution u of (1.1) *corresponding to* (3.1) *exists and satisfies* (2.25) *on the rescaled interval of time* $[s_0, s^{in}]$*, the rescaled time s being defined in* (3.2)*. Moreover, the decomposition of u given by Lemma 5 on* $[s_0, s^{in}]$

$$u(s,x) = \frac{e^{i\gamma(s)}}{\lambda(s)}(\mathbf{P} + \varepsilon)(s,y), \quad y = \frac{x}{\lambda(s)},$$

satisfies the following uniform estimates, for all $s \in [s_0, s^{in}]$,

$$\begin{aligned} &\left| z(s) - \frac{2}{\kappa}\log(s) \right| \lesssim \log(\log(s)), \quad \left|\lambda(s) - \log^{-1}(s)\right| \lesssim \log^{-\frac32}(s), \\ &\left|b(s) - s^{-1}\log^{-1}(s)\right| + |\beta(s)| + \|\varepsilon(s)\|_{H^1} \lesssim s^{-1}\log^{-\frac32}(s), \quad |a(s)| \lesssim s^{-2}\log^{-1}(s). \end{aligned} \tag{3.5}$$

The key point in Proposition 6 is that s_0 and the constants in (3.5) are independent of s^{in} as $s^{in} \to +\infty$. Observe that estimates (3.5) match the discussion of Sect. 2.2.

The rest of this section is devoted to the proof of Proposition 6. The proof relies on a bootstrap argument, integration of the differential system of geometrical parameters and energy estimates. We estimate ε by standard energy arguments in the framework of multi-bubble solutions. The particular regime of the geometrical parameters does not create any further difficulty. On the contrary, the special behavior $b(s) \sim s^{-1}\log^{-1}(s)$ simplifies this part of the proof (see step 2 of the proof of Proposition 8). We control the geometrical parameters of the bubbles in the bootstrap regime adjusting the final data $(\lambda^{in}, z^{in}, b^{in})$.

3.1. Bootstrap bounds

The proof of Proposition 6 follows from bootstrapping the following estimates, chosen in view of the formal computations in Sect. 2.2,

$$\begin{aligned} &\left|\left(\frac{2}{\kappa c_a}\right)^{\frac{1}{2}} z^{-\frac{3}{4}} e^{\frac{\kappa}{2}z} - s\right| \leq s\log^{-\frac{1}{2}}(s),\\ &\frac{1}{2}s^{-1}\log^{-1}(s) \leq b(s) \leq 2s^{-1}\log^{-1}(s),\\ &|\beta(s)| \leq s^{-1}\log^{-\frac{3}{2}}(s), \quad \|\varepsilon(s)\|_{H^1} \leq s^{-1}\log^{-\frac{3}{2}}(s). \end{aligned} \tag{3.6}$$

Note that the estimate on z in (3.6) immediately implies that, for s large

$$e^{-\kappa z} \lesssim s^{-2}\log^{-\frac{3}{2}}(s), \quad \left|z(s) - \frac{2}{\kappa}\log(s)\right| \lesssim \log(\log(s)), \quad |a(s)| \lesssim s^{-2}\log^{-1}(s). \tag{3.7}$$

For $s_0 > 10$ to be chosen large enough (independent of s^{in}), and all $s^{in} \gg s_0$, we define

$$s^* = \inf\{\tau \in [s_0, s^{in}];\ (3.6) \text{ holds on } [\tau, s^{in}]\}. \tag{3.8}$$

3.2. Control of the modulation equations

We claim the following bounds on the modulation system $\vec{m}_1^a$ and on the error $\mathcal{E}_{\mathbf{P}}$ given by (2.7), (2.8)–(2.9) in the bootstrap regime (3.6).

LEMMA 7 (Pointwise control of the modulation equations and the error).
The following estimates hold on $[s^*, s^{in}]$.

$$|\vec{m}_1^a(s)| \lesssim s^{-2}\log^{-2}(s). \tag{3.9}$$

$$|\langle \eta_1(s), Q\rangle| \lesssim s^{-2}\log^{-2}(s), \tag{3.10}$$

$$|\dot{z} - bz| \lesssim s^{-1}\log^{-1}(s), \quad |\dot{\beta}| + |\dot{b} - a| \lesssim s^{-2}\log^{-2}(s). \tag{3.11}$$

Moreover, for all $s \in [s^*, s^{in}]$, *for all* $y \in \mathbb{R}^2$,

$$|\mathcal{E}_{\mathbf{P}}(s,y)| + |\nabla \mathcal{E}_{\mathbf{P}}(s,y)| \lesssim s^{-2}\log^{-2}(s) \sum_k Q^{1/2}(y - z_k(s)). \tag{3.12}$$

Recall that $\varepsilon(s,y) = \left[e^{i\Gamma_1}\eta_1\right](s, y - z_1)$.

We see from (3.10) and (3.6) that Q is a special direction for η_1. In step 4 of the proof of Lemma 7, it is controlled directly, thanks to the special choice of a in (2.5) and L^2 norm conservation. Alternatively, one could impose the additional orthogonality condition $\langle \eta_1, Q\rangle = 0$ to (2.28) by modulating the parameter a, at the cost of some other technical difficulties.

Proof of Lemma 7. – The proofs of the first two estimates are to be combined. Since $\varepsilon(s^{in}) \equiv 0$, we can define

$$s^{**} = \inf\{s \in [s^*, s^{in}];\ |\langle \eta_1(\tau), Q\rangle| \leq C^{**}\tau^{-2}\log^{-2}(\tau) \text{ holds on } [s, s^{in}]\},$$

for some constant $C^{**} > 1$ to be chosen large enough. We work on the interval $[s^{**}, s^{in}]$.

Step 1: Equation of ε and change of variable. Let $v = \mathbf{P} + \varepsilon$ in (2.1).

It follows from (2.2), (2.8) that ε satisfies the following equation

$$i\dot{\varepsilon} + \Delta\varepsilon - \varepsilon + \big(|\mathbf{P}+\varepsilon|^2(\mathbf{P}+\varepsilon) - |\mathbf{P}|^2\mathbf{P}\big) - i\frac{\dot{\lambda}}{\lambda}\Lambda\varepsilon + (1-\dot{\gamma})\varepsilon + \mathcal{E}_{\mathbf{P}} = 0. \tag{3.13}$$

Since the orthogonality conditions (2.28) in Lemma 5 are written for η_1, we change the space variable to match the one of the bubble P_1. Recall that η_1 is defined such that $\varepsilon(s,y) = \big[e^{i\Gamma_1}\eta_1\big](s, y - z_1)$. Define similarly $\mathbf{P}_1$ and $\mathcal{E}_{\mathbf{P}_1}$ such that

$$\mathbf{P}(s,y) = \big[e^{i\Gamma_1}\mathbf{P}_1\big](s, y - z_1), \quad \mathcal{E}_{\mathbf{P}}(s,y) = \big[e^{i\Gamma_1}\mathcal{E}_{\mathbf{P}_1}\big](s, y - z_1).$$

We rewrite the equation of ε into the following equation for η_1 (see also step 1 of the proof of Lemma 3)

$$i\dot{\eta}_1 + \Delta\eta_1 - \eta_1 + (|\mathbf{P}_1 + \eta_1|^2(\mathbf{P}_1+\eta_1) - |\mathbf{P}_1|^2\mathbf{P}_1) + \vec{m}_1^0 \cdot \vec{\mathrm{M}}\eta_1 + \mathcal{E}_{\mathbf{P}_1} = 0. \tag{3.14}$$

Step 2: General null space like computation. A standard observation is that Equation (3.14) above contains the first order differential system of the parameters $\vec{p} = (\lambda, z, \gamma, \beta, b)$ through the term $\mathcal{E}_{\mathbf{P}_1}$ (see the definition of Lemma 3) and the orthogonality conditions (2.28). To derive this system, it is enough to project the Equation (3.14) on each direction chosen as orthogonality condition for η_1. We claim a preliminary general computation that will be used in the next step for this derivation. Let $A(y)$ and $B(y)$ be two real-valued functions in $\mathcal{Y}$. We claim the following estimate on $[s^{**}, s^{in}]$

$$\left|\frac{d}{ds}\langle \eta_1, A + iB\rangle - \Big[\langle \eta_1, iL_-A - L_+B\rangle - \langle \vec{m}_1^a \cdot \vec{\mathrm{M}}Q, iA - B\rangle\Big]\right| \lesssim s^{-2}\log^{-2}(s) + s^{-1}|\vec{m}_1^a|. \tag{3.15}$$

We compute from (3.14),

$$\begin{aligned}\frac{d}{ds}\langle \eta_1, A + iB\rangle &= \langle \dot{\eta}_1, A + iB\rangle = \langle i\dot{\eta}_1, iA - B\rangle \\ &= \langle -\Delta\eta_1 + \eta_1 - (2Q^2\eta_1 + Q\overline{\eta}_1), iA - B\rangle \\ &- \langle |\mathbf{P}_1+\eta_1|^2(\mathbf{P}_1+\eta_1) - |\mathbf{P}_1|^2\mathbf{P}_1 - 2Q^2\eta_1 - Q^2\overline{\eta}_1, iA - B\rangle \\ &- \langle \vec{m}_1^0 \cdot \vec{\mathrm{M}}\eta_1, iA - B\rangle - \langle \mathcal{E}_{\mathbf{P}_1}, iA - B\rangle.\end{aligned}$$

First, since A and B are real-valued, we have

$$\langle -\Delta\eta_1 + \eta_1 - (2Q^2\eta_1 + Q\overline{\eta}_1), iA - B\rangle = \langle \eta_1, iL_-A - L_+B\rangle.$$

Second, note that

$$\begin{aligned}&|\mathbf{P}_1+\eta_1|^2(\mathbf{P}_1+\eta_1) - |\mathbf{P}_1|^2\mathbf{P}_1 - 2Q^2\eta_1 - Q^2\overline{\eta}_1 \\ &= 2(|\mathbf{P}_1|^2 - Q^2)\eta_1 + (\mathbf{P}_1^2 - Q^2)\overline{\eta}_1 + 2\mathbf{P}_1|\eta_1|^2 + \overline{\mathbf{P}}_1\eta_1^2 + 2\mathbf{P}_1|\eta_1|^2 + |\eta_1|^2\eta_1,\end{aligned}$$

and recall the expression of $\mathbf{P}_1$

$$\mathbf{P}_1 = Q + a\rho + \sum_{k=2}^{K} e^{i(\Gamma_k(y-(z_k-z_1))-\Gamma_1(y))} Q_a(y-(z_k-z_1)).$$

Therefore, using $A, B \in \mathcal{Y}$, (3.6)–(3.7) and $|z_k - z_j| \geq \kappa z$, for $k \neq j$, we have, for some $q > 0$,

$$|\langle(|\mathbf{P}_1|^2 - Q^2)\eta_1, iA - B\rangle| + |\langle(\mathbf{P}_1^2 - Q^2)\overline{\eta}_1, iA - B\rangle| \lesssim (|a| + z^q e^{-\kappa z})\|\eta_1\|_{L^2} \lesssim s^{-3}\log^q(s).$$

Next,

$$|\langle 2\mathbf{P}_1|\eta_1|^2 + \overline{\mathbf{P}}_1\eta_1^2 + 2\mathbf{P}_1|\eta_1|^2, iA - B\rangle| \lesssim \|\varepsilon\|_{L^2}^2 \lesssim s^{-2}\log^{-3}(s),$$
$$|\langle|\eta_1|^2\eta_1, iA - B\rangle| \lesssim \langle|\varepsilon|^3, |A| + |B|\rangle \lesssim \|\varepsilon\|_{H^1}^3 \lesssim s^{-3}\log^{-\frac{9}{2}}(s).$$

Third, we have, using (3.6)–(3.7), integration by parts,

$$|\langle\vec{m}_1^0 \cdot \vec{\mathrm{M}}\eta_1, iA - B\rangle| \lesssim \left|\langle\vec{m}_1^a \cdot \vec{\mathrm{M}}\eta_1, iA - B\rangle\right| + \left|\langle(\vec{m}_1^a - \vec{m}_1^0) \cdot \vec{\mathrm{M}}\eta_1, iA - B\rangle\right|$$
$$\lesssim s^{-1}\log^{-\frac{3}{2}}(s)|\vec{m}_1^a| + s^{-3}\log^{-\frac{5}{2}}(s).$$

Finally, we claim the following estimate, which is enough to complete the proof of (3.15).

$$\left|\langle\mathcal{E}_{\mathbf{P}_1}, iA - B\rangle - \langle\vec{m}_1^a \cdot \vec{\mathrm{M}}Q, iA - B\rangle\right| \lesssim s^{-2}\log^{-2}(s) + s^{-1}|\vec{m}_1^a|. \tag{3.16}$$

Indeed, recall the expression of $\mathcal{E}_{\mathbf{P}_1}$ (from (2.8)–(2.9))

$$\begin{aligned}\mathcal{E}_{\mathbf{P}_1} &= \Psi_1 + \sum_{k=2}^{K} e^{i(\Gamma_k(y-(z_k-z_1))-\Gamma_1(y))}\Psi_k(y-(z_k-z_1)).\\ &= \vec{m}_1^a \cdot \vec{\mathrm{M}}Q_a + i\dot{z}a'(z)\rho + G_1 + \Psi_{Q_a} + \sum_{k=2}^{K} e^{i(\Gamma_k(y-(z_k-z_1))-\Gamma_1(y))}\Psi_k(y-(z_k-z_1)).\end{aligned}$$

First, since $Q_a = Q + a\rho$, by (3.6)–(3.7),

$$\left|\langle\vec{m}_1^a \cdot \vec{\mathrm{M}}(Q_a - Q), iA - B\rangle\right| \lesssim |a||\vec{m}_1^a| \lesssim s^{-2}\log^{-1}(s)|\vec{m}_1^a|.$$

Second, from (3.6)–(3.7),

$$\begin{aligned}|\langle\dot{z}a'(z)\rho, iA - B\rangle| &\lesssim |a'(z)||\dot{z}| \lesssim s^{-2}\log^{-1}(s)(|\vec{m}_1^a| + |\beta| + |\frac{\dot{\lambda}}{\lambda}|z)\\ &\lesssim s^{-2}\log^{-1}(s)((z+1)|\vec{m}_1^a| + |\beta| + |b|z)\\ &\lesssim s^{-2}|\vec{m}_1^a| + s^{-3}\log^{-1}(s).\end{aligned} \tag{3.17}$$

Third, from (2.10) and (3.6)–(3.7),

$$|\langle G_1, iA - B\rangle| \lesssim \|G_1\|_{L^\infty} \lesssim z^{-\frac{1}{2}}e^{-\kappa z} \lesssim s^{-2}\log^{-2}(s).$$

Fourth, from (2.10) and (3.6)–(3.7),

$$|\langle\Psi_{Q_a}, iA - B\rangle| \lesssim \|\Psi_{Q_a}\|_{L^\infty} \lesssim |a|^2 \lesssim s^{-4}\log^{-2}(s).$$

Last, since $A, B \in \mathcal{Y}$, for $k \geq 2$, we have

$$|\langle e^{i(\Gamma_k(y-(z_k-z_1))-\Gamma_1(y))}(\vec{m}_k^a \cdot \vec{\mathrm{M}}Q_a(.-(z_k-z_1))), iA - B\rangle| \lesssim s^{-1}|\vec{m}_k^a|,$$

and, proceeding as before for the other terms in Ψ_k, we obtain

$$|\langle e^{i(\Gamma_k(y-(z_k-z_1))-\Gamma_1(y))}\Psi_k(y-(z_k-z_1)), iA - B\rangle| \lesssim s^{-1}|\vec{m}_k^a| + s^{-2}\log^{-2}(s).$$

The proof of (3.16) (and thus the one of (3.15)) is complete.

Step 3: Modulation differential system. We now use the specific orthogonality conditions chosen in (2.28) and the general Formula (3.15) to control the modulation vector $\vec{m}_1^a$ as in (3.9), i.e., to justify the first order differential system of the modulation parameters. Before proceeding with these computations, we briefly describe the general scheme used to obtain a control at order $s^{-2}\log^{-2}(s)$ as in (3.9). We also refer to [53] for similar computations. Each orthogonality condition required on η_1 in (2.28) is of the form $\langle \eta_1, A + iB\rangle = 0$, where A and B are real-valued Schwartz functions. Inserting such relation into (3.15), one obtains

$$\left|\langle \vec{m}_1^a \cdot \vec{\mathrm{M}}Q, iA - B\rangle\right| \lesssim |\langle \eta_1, iL_-A - L_+B\rangle| + s^{-2}\log^{-2}(s) + s^{-1}\left|\vec{m}_1^a\right|. \tag{3.18}$$

Using the five orthogonality conditions in (2.28) and non degeneracy conditions (related to the choice of the orthogonality conditions and to the definition of the vector $\vec{\mathrm{M}}$), (3.18) implies directly an estimate of the vector $\vec{m}_1^a$ of the form

$$\left|\vec{m}_1^a\right| \lesssim \|\eta_1\|_{L^2} + s^{-2}\log^{-2}(s) + s^{-1}\left|\vec{m}_1^a\right|,$$

and thus, for s large, and using (3.6), $\left|\vec{m}_1^a\right| \lesssim s^{-1}\log^{-3/2}(s)$. However, this is not sharp enough for our needs (compare with the size of a in (3.7)). Following [53], we will see that the special choice of orthogonality conditions in (2.28) and the relations (1.23) lead to cancelations of the first order terms in η_1 in the right-hand side of (3.18), and thus prove the desired estimate (3.9). Now, we check one by one these special cancelations.

<u>$\langle \eta_1, |y|^2Q\rangle = 0$.</u> Using the first orthogonality condition of (2.28), namely $\langle \eta_1, |y|^2Q\rangle = 0$, we obtain (3.18) with $A = |y|^2Q$ and $B = 0$, that is

$$\left|\langle \vec{m}_1^a \cdot \vec{\mathrm{M}}Q, i|y|^2Q\rangle\right| \lesssim \left|\langle \eta_1, iL_-(|y|^2Q)\rangle\right| + s^{-2}\log^{-2}(s) + s^{-1}\left|\vec{m}_1^a\right|.$$

Since $L_-(|y|^2Q) = -4\Lambda Q$ (by explicit computation), by the fifth orthogonality relation in (2.28), one has $\langle \eta_1, iL_-(|y|^2Q)\rangle = -4\langle \eta_1, i\Lambda Q\rangle = 0$. Moreover, by the definitions of $\vec{m}_1^a$ and $\vec{\mathrm{M}}$ in Lemma 3, and parity properties, we have

$$\langle \vec{m}_1^a \cdot \vec{\mathrm{M}}Q, i|y|^2Q\rangle = c_1\left(b + \frac{\dot{\lambda}}{\lambda}\right),$$

where $c_1 = -\langle \Lambda Q, |y|^2Q\rangle \neq 0$. Therefore, we obtain

$$\left|b + \frac{\dot{\lambda}}{\lambda}\right| \lesssim s^{-2}\log^{-2}(s) + s^{-1}|\vec{m}_1^a|. \tag{3.19}$$

We see that the desired cancelation has been obtained in the estimate of $|b + \frac{\dot{\lambda}}{\lambda}|$.

The computations corresponding to the other four orthogonality relations in (2.28) are perfomed in a similar way using (3.18) and (1.23). We briefly sketch them.

$\underline{\langle \eta_1, yQ\rangle = 0}$. Let $A = yQ$ and $B = 0$. Since $L_-(yQ) = -2\nabla Q$, $\langle \eta_1, i\nabla Q\rangle = 0$ and $\langle \vec{m}_1^a \cdot \vec{\mathrm{M}}Q, iyQ\rangle = c_2(\dot z - 2\beta + \frac{\dot\lambda}{\lambda}z)$, where $c_2 = -\langle \partial_{y_1}Q, y_1Q\rangle \neq 0$, we obtain similarly

$$\left|\dot z - 2\beta + \frac{\dot\lambda}{\lambda}z\right| \lesssim s^{-2}\log^{-2}(s) + s^{-1}|\vec{m}_1^a|. \tag{3.20}$$

$\underline{\langle \eta_1, i\rho\rangle = 0}$. Let $A = 0$ and $B = \rho$. Recall that $L_+\rho = |y|^2Q$ (this is the definition of ρ) and $\langle \eta_1, |y|^2Q\rangle = 0$ (from (2.28)). Since

$$\langle \vec{m}_1^a \cdot \vec{\mathrm{M}}Q, |y|^2Q\rangle = c_3(\dot\gamma - 1 + \beta^2 - \frac{\dot\lambda}{\lambda}\beta z - \beta\dot z) + c_4(\dot b + b^2 - 2b(b + \frac{\dot\lambda}{\lambda}) - a),$$

where $c_3, c_4 \neq 0$, we obtain, for some c,

$$\left|(\dot\gamma - 1 + \beta^2 - \frac{\dot\lambda}{\lambda}\beta z - \beta\dot z) + c(\dot b + b^2 - 2b(b + \frac{\dot\lambda}{\lambda}) - a)\right| \lesssim s^{-2}\log^{-2}(s) + s^{-1}|\vec{m}_1^a|. \tag{3.21}$$

$\underline{\langle \eta_1, i\nabla Q\rangle = 0}$. Let $A = 0$ and $B = \nabla Q$. Since $L_+\nabla Q = 0$, and $\langle \vec{m}_1^a \cdot \vec{\mathrm{M}}Q, \nabla Q\rangle = c_5\left(\dot\beta - \frac{\dot\lambda}{\lambda}\beta + \frac{b}{2}(\dot z - 2\beta + \frac{\dot\lambda}{\lambda}z)\right)$, where $c_5 = -\langle y_1Q, \partial_{y_1}Q\rangle \neq 0$, we obtain

$$\left|\dot\beta - \frac{\dot\lambda}{\lambda}\beta + \frac{b}{2}(\dot z - 2\beta + \frac{\dot\lambda}{\lambda}z)\right| \lesssim s^{-2}\log^{-2}(s) + s^{-1}|\vec{m}_1^a|. \tag{3.22}$$

$\underline{\langle \eta_1, i\Lambda Q\rangle = 0}$. Let $A = 0$ and $B = \Lambda Q$ in (3.18). Note that $L_+(\Lambda Q) = -2Q$, and by the definition of s^{**}, $|\langle \eta_1, Q\rangle| \lesssim C^{**}s^{-2}\log^{-2}(s)$ (observe that we do not use an exact cancelation here, but the fact that $\langle \eta_1, Q\rangle$ is a special direction, controlled directed using mass properties in the next step). Moreover, $\langle \vec{m}_1^a \cdot \vec{\mathrm{M}}Q, \Lambda Q\rangle = c_6(\dot b + b^2 - 2b(b + \frac{\dot\lambda}{\lambda}) - a)$, where $c_6 = \frac{1}{4}\langle |y|^2Q, \Lambda Q\rangle \neq 0$, so that we obtain from (3.18)

$$\left|\dot b + b^2 - 2b(b + \frac{\dot\lambda}{\lambda}) - a\right| \lesssim C^{**}s^{-2}\log^{-2}(s) + s^{-1}|\vec{m}_1^a|. \tag{3.23}$$

Combining (3.19)–(3.23), we have proved, for all $s \in [s^{**}, s^{in}]$,

$$|\vec{m}_1^a(s)| \lesssim C^{**}s^{-2}\log^{-2}(s) + s^{-1}|\vec{m}_1^a|,$$

and thus, for s large enough,

$$|\vec{m}_1^a(s)| \lesssim C^{**}s^{-2}\log^{-2}(s). \tag{3.24}$$

Step 4: Minimal mass property of the approximate solution. The proof of the degeneracy estimate (3.10) relies on the following minimal mass property for the ansatz $\mathbf{P}$ under the bootstrap assumptions (3.6):

$$\left|\|\mathbf{P}(s)\|_{L^2} - \|\mathbf{P}^{in}\|_{L^2}\right| \lesssim s^{-2}\log^{-2}(s). \tag{3.25}$$

Note that the implicit constant on the right-hand side does not depend on C^{**}. By the Definition (2.9) of $\mathcal{E}_{\mathbf{P}}$, we have

$$\frac{1}{2}\frac{d}{ds}\|\mathbf{P}\|_{L^2}^2 = \langle i\dot{\mathbf{P}}, i\mathbf{P}\rangle = \langle \mathcal{E}_{\mathbf{P}}, i\mathbf{P}\rangle.$$

In view of the formula for $\mathcal{E}_{\mathbf{P}}$ (2.9), and the definition of $\mathbf{P} = \sum_j P_j$, (3.25) follows by integration of the following estimate: for all $j, k \in \{1, \dots, K\}$,

$$|\langle [e^{i\Gamma_k}\Psi_k](y - z_k), i[e^{i\Gamma_j}Q_a](y - z_j)\rangle| \lesssim s^{-3}\log^{-2}(s). \tag{3.26}$$

Proof of (3.26). – We start by proving (3.26) in the case $j = k = 1$. From (2.9):

$\langle [e^{i\Gamma_k}\Psi_k](y-z_k), i[e^{i\Gamma_k}Q_a](y-z_k)\rangle = \langle \Psi_1, iQ_a\rangle = \langle \vec{m}_1^a\cdot\vec{\mathrm{M}}Q_a + i\dot{z}a'(z)\rho + G_1 + \Psi_{Q_a}, iQ_a\rangle.$

Note that $\langle \vec{\mathrm{M}}Q, iQ\rangle = 0$. Thus, by (3.24), (3.6)–(3.7),

$$|\langle \vec{m}_1^a\cdot\vec{\mathrm{M}}Q_a, iQ_a\rangle| \lesssim |a||\vec{m}_1^a| \lesssim C^{**}s^{-4}\log^{-3}(s) \lesssim s^{-3}\log^{-3}(s).$$

Next, we claim the following estimate, which justifies the special choice of $a(z)$ done in (2.5) (see also Sect. 2.2)

$$|\langle i\dot{z}a'(z)\rho + G_1, iQ_a\rangle| \lesssim s^{-3}\log^{-2}(s). \tag{3.27}$$

Indeed, first by (3.6)–(3.7) and (2.11),

$$|\langle G_1, iQ_a\rangle + \kappa c_a\langle\rho, Q\rangle bz^{\frac{3}{2}}e^{-\kappa z}| \lesssim s^{-3}\log^{-2}(s). \tag{3.28}$$

Second, we note that by (3.24) and (3.6)–(3.7),

$$|\dot{z} - bz| \lesssim s^{-1}\log^{-1}(s), \tag{3.29}$$

and that by the definition of $a(z)$ in (2.5),

$$|a'(z) - c_a\kappa z^{\frac{1}{2}}e^{-\kappa z}| \lesssim z^{-\frac{1}{2}}e^{-\kappa z} \lesssim s^{-2}\log^{-2}(s). \tag{3.30}$$

Gathering (3.28)–(3.30), we obtain (3.27). Finally, since Q_a and Ψ_{Q_a} given by (2.12) are real-valued, we have the cancelation

$$\langle \Psi_{Q_a}, iQ_a\rangle = 0.$$

The collection of above estimates concludes the proof of (3.26) for $j = k = 1$.

We now prove (3.26) in the case $k = 1$ and $j \in \{2, \dots, K\}$. Note that

$$\begin{aligned}&|\langle [e^{i\Gamma_k}\Psi_k](y-z_k), i[e^{i\Gamma_j}Q_a](y-z_j)\rangle|\\ &\quad = |\langle \vec{m}_1^a\cdot\vec{\mathrm{M}}Q_a + i\dot{z}a'(z)\rho + G_1 + \Psi_{Q_a}, ie^{i(\Gamma_j(y-(z_j-z_k))-\Gamma_k(y))}Q_a(y-(z_j-z_k))\rangle|.\end{aligned}$$

First, by (3.24), for some $q > 0$,

$$\begin{aligned}&|\langle \vec{m}_1^a\cdot\vec{\mathrm{M}}Q_a, ie^{i(\Gamma_j(y-(z_j-z_k))-\Gamma_k)}Q_a(y-(z_j-z_k))\rangle|\\ &\qquad\qquad \lesssim |\vec{m}_1^a|z^q e^{-\kappa z} \lesssim C^{**}s^{-4}\log^q(s) \lesssim s^{-3}\log^{-2}(s).\end{aligned}$$

Second, using similar arguments, for some $q > 0$,

$$|\langle i\dot{z}a'(z)\rho + G_1 + \Psi_{Q_a}, ie^{i(\Gamma_j(y-(z_j-z_k))-\Gamma_k)}Q_a(y-(z_j-z_k))\rangle| \lesssim s^{-4}\log^q(s).$$

The collection of above estimates concludes the proof of (3.25).

Step 5: Proof of (3.10). The conservation of mass for the solution u and (3.1) implies:

$$\|u(s)\|_{L^2} = \|u(s^{in})\|_{L^2} = \|\mathbf{P}^{in}\|_{L^2}.$$

By (2.26),

$$\langle \varepsilon(s), \mathbf{P}\rangle = \frac{1}{2}\left(\|u(s)\|_{L^2}^2 - \|\mathbf{P}(s)\|_{L^2}^2 - \|\varepsilon(s)\|_{L^2}^2\right).$$

Therefore, using (3.6)–(3.7) and (3.25), we obtain

$$|\langle \varepsilon(s), \mathbf{P}\rangle| \lesssim s^{-2}\log^{-2}(s).$$

Now, we use the symmetry $\langle \varepsilon, P_k\rangle = \langle \varepsilon, P_j\rangle = K^{-1}\langle \varepsilon(s), \mathbf{P}\rangle$ for all $j,k \in \{1,\dots,K\}$. Moreover, by (3.6)–(3.7),

$$\langle \varepsilon(s), P_1\rangle = \langle \eta_1, Q_a\rangle = \langle \eta_1, Q\rangle + O(|a|\|\varepsilon\|_{L^2}) = \langle \eta_1, Q\rangle + O(s^{-3}\log^{-\frac{5}{2}}(s)).$$

Gathering this information, we obtain $|\langle \eta_1, Q\rangle| \lesssim s^{-2}\log^{-2}(s)$, i.e., estimate (3.10). In particular, choosing C^{**} large enough, we have $s^{**} = s^*$.

Step 6: Conclusion. The estimate (3.11) is a direct consequence of (3.9) and (3.6)–(3.7). We now turn to the proof of (3.12). Using (3.9), (3.6)–(3.7) and (1.22),

$$|\dot z a'(z)\rho| \lesssim Q^{\frac12}(|b|z + s^{-1}\log^{-1}(s))s^{-2}\log^{-1}(s) \lesssim Q^{\frac12}s^{-3}\log^{-1}(s).$$

By (3.9),

$$|\vec m_1^a \cdot \vec{\mathrm{M}} Q_a| \lesssim Q^{\frac12}s^{-2}\log^{-2}(s).$$

Next, by the definition of G_k in (2.14), the decay $|\rho| \lesssim Q^{\frac78}$ (see (1.22)) and $|\mathrm{e}_k - \mathrm{e}_1| \geq \kappa$ for $k \neq 1$,

$$\begin{aligned}|G_1| &\lesssim Q^{\frac12}\sum_{k=2}^{K}\left(Q^{\frac54}(y)Q(y - z(\mathrm{e}_k - \mathrm{e}_1)) + |a|Q^{\frac54}(y)Q^{\frac78}(y - z(\mathrm{e}_k - \mathrm{e}_1))\right)\\ &\lesssim Q^{\frac12}(z^{-\frac12}e^{-\kappa z} + s^{-2}e^{-\frac{\kappa}{2}z}) \lesssim Q^{\frac12}s^{-2}\log^{-2}(s).\end{aligned}$$

Finally, by the definition of Ψ_{Q_a} in (2.12),

$$|\Psi_{Q_a}| \lesssim Q^{\frac12}|a|^2 \lesssim Q^{\frac12}s^{-4}\log^{-2}(s).$$

The same estimates hold for $\nabla\mathcal{E}$, which finishes the proof of (3.12). □

3.3. Energy functional

Consider the nonlinear energy functional for ε

$$\mathbf{H}(s,\varepsilon) = \frac12\int\left(|\nabla\varepsilon|^2 + |\varepsilon|^2 - \frac12\left(|\mathbf{P}+\varepsilon|^4 - |\mathbf{P}|^4 - 4|\mathbf{P}|^2\,\mathrm{Re}\,(\varepsilon\overline{\mathbf{P}})\right)\right).$$

Pick a smooth function $\chi : [0,+\infty) \to [0,\infty)$, non increasing, with $\chi \equiv 1$ on $[0,\frac{1}{10}]$, $\chi \equiv 0$ on $[\frac18,+\infty)$. We define the localized momentum:

$$\mathbf{J} = \sum_k J_k,\quad J_k(s,\varepsilon) = b\,\mathrm{Im}\int (z_k\cdot\nabla\varepsilon)\bar\varepsilon\chi_k,\quad \chi_k(s,y) = \chi\left(\log^{-1}(s)|y - z_k(s)|\right).$$

Finally, set

$$\mathbf{F}(s,\varepsilon) = \mathbf{H}(s,\varepsilon) - \mathbf{J}(s,\varepsilon).$$

The functional F is coercive in ε at the main order and it is an almost conserved quantity for the problem.

PROPOSITION 8 (Coercivity and time control of the energy functional).
For all $s \in [s^*, s^{in}]$,

$$\mathbf{F}(s,\varepsilon(s)) \gtrsim \|\varepsilon(s)\|_{H^1}^2 + O(s^{-4}\log^{-4}(s)), \tag{3.31}$$

and

$$\left|\frac{d}{ds}[\mathbf{F}(s,\varepsilon(s))]\right| \lesssim s^{-2}\log^{-2}(s)\|\varepsilon(s)\|_{H^1} + s^{-1}\log^{-1}(s)\|\varepsilon(s)\|_{H^1}^2. \tag{3.32}$$

Proof of Proposition 8. – Step 1: Coercivity. The proof of the coercivity (3.31) is a standard consequence of the coercivity property (1.24) around one solitary wave with the orthogonality properties (2.28), (3.10), and an elementary localization argument. Hence we briefly sketch the argument. First, using the coercivity property (1.24) and the orthogonality properties (2.28), (3.10) and localization arguments, we have

$$\mathbf{H}(s,\varepsilon) \gtrsim \|\varepsilon\|_{H^1}^2 + O(s^{-4}\log^{-4}(s)). \tag{3.33}$$

Note that the error term $O(s^{-4}\log^{-4}(s))$ is due to the fact that the bound (3.10) replaces a true orthogonality $\langle \eta_1(s), Q\rangle = 0$. We refer to the proof of Lemma 4.1 in Appendix B of [33] for a similar proof. Second, we note that by (3.6)–(3.7), $|\mathbf{J}(s,\varepsilon)| \lesssim |b|z\|\varepsilon\|_{H^1}^2 \lesssim s^{-1}\|\varepsilon\|_{H^1}^2$, and (3.31) follows.

Step 2: Variation of the energy. We estimate the time variation of the functional $\mathbf{H}$ and claim: for all $s \in [s^*, s^{in}]$,

$$\left|\frac{d}{ds}[\mathbf{H}(s,\varepsilon(s))] - \sum_k \langle \dot z_k\cdot\nabla P_k, 2|\varepsilon|^2 P_k + \varepsilon^2\overline{P}_k\rangle\right| \lesssim s^{-2}\log^{-2}(s)\|\varepsilon(s)\|_{H^1} + s^{-1}\log^{-1}(s)\|\varepsilon(s)\|_{H^1}^2. \tag{3.34}$$

The time derivative of $s \mapsto H(s,\varepsilon(s))$ splits into two parts

$$\frac{d}{ds}[\mathbf{H}(s,\varepsilon(s))] = D_s\mathbf{H}(s,\varepsilon(s)) + \langle D_\varepsilon \mathbf{H}(s,\varepsilon(s)), \dot\varepsilon_s\rangle,$$

where D_s denotes differentiation of $\mathbf{H}$ with respect to s and D_ε denotes differentiation of $\mathbf{H}$ with respect to ε. First compute:

$$D_s\mathbf{H} = -\langle \dot{\mathbf{P}}, |\mathbf{P}+\varepsilon|^2(\mathbf{P}+\varepsilon) - |\mathbf{P}^2|\mathbf{P} - (2\varepsilon|\mathbf{P}|^2 + \overline{\varepsilon}\mathbf{P}^2)\rangle.$$

Observe that by the definition of P_k in (2.3),

$$\dot P_k = -\dot z_k\cdot\nabla P_k + i(\dot\beta_k\cdot(y-z_k) - \tfrac{\dot b}{4}|y-z_k|^2)P_k + \dot z a'(z)\rho_k \quad \text{where} \quad \rho_k = \left[e^{i\Gamma_k}\rho\right](y-z_k).$$

By (3.11), (3.6)–(3.7) and (2.5),

$$|\dot\beta_k| + |\dot b| + |\dot z a'(z)| \lesssim s^{-2}\log^{-2}(s).$$

Since

$$\int \left||\mathbf{P}+\varepsilon|^2(\mathbf{P}+\varepsilon) - |\mathbf{P}^2|\mathbf{P} - (2\varepsilon|\mathbf{P}|^2 + \overline{\varepsilon}\mathbf{P}^2)\right| \lesssim \|\varepsilon\|_{H^1}^2,$$

we obtain

$$\left|\left\langle i(\dot\beta_k\cdot(y-z_k) - \tfrac{\dot b}{4}|y-z_k|^2)P_k + \dot z a'(z)\rho_k, |\mathbf{P}+\varepsilon|^2(\mathbf{P}+\varepsilon) - |\mathbf{P}^2|\mathbf{P} - (2\varepsilon|\mathbf{P}|^2 + \overline{\varepsilon}\mathbf{P}^2)\right\rangle\right| \lesssim s^{-2}\log^{-2}(s)\|\varepsilon\|_{H^1}^2.$$

Next, note that

$$|\mathbf{P}+\varepsilon|^2(\mathbf{P}+\varepsilon) - |\mathbf{P}^2|\mathbf{P} - (2\varepsilon|\mathbf{P}|^2 + \overline{\varepsilon}\mathbf{P}^2) = 2|\varepsilon|^2\mathbf{P} + \varepsilon^2\overline{\mathbf{P}} + |\varepsilon|^2\varepsilon.$$

By (3.11) and (3.6)–(3.7), $|\dot z| \lesssim s^{-1}$ and thus by (3.6)–(3.7),

$$\left|\langle \dot z_k\cdot\nabla P_k, |\varepsilon|^3\rangle\right| \lesssim s^{-1}\|\varepsilon\|_{H^1}^3 \lesssim s^{-2}\log^{-\frac{3}{2}}(s)\|\varepsilon\|_{H^1}^2.$$

For $j \neq k$, since $e^{-\kappa z} \lesssim s^{-2}$ by (3.6)–(3.7) and the decay properties of P_k, P_j,

$$\left|\langle \dot{z}_k \cdot \nabla P_k, 2|\varepsilon|^2 P_j + \varepsilon^2 \overline{P}_j\rangle\right| \lesssim |s|^{-3}\|\varepsilon\|_{H^1}^2.$$

Gathering these computations, we have obtained

$$D_s\mathbf{H}(s,\varepsilon) = \sum_k \langle \dot{z}_k \cdot \nabla P_k, 2|\varepsilon|^2 P_k + \varepsilon^2 \overline{P}_k\rangle + O(s^{-2}\|\varepsilon\|_{H^1}^2). \tag{3.35}$$

Second,

$$D_\varepsilon\mathbf{H}(s,\varepsilon) = -\Delta\varepsilon + \varepsilon - \left(|\mathbf{P}+\varepsilon|^2(\mathbf{P}+\varepsilon) - |\mathbf{P}|^2\mathbf{P}\right),$$

so that the Equation (3.13) of ε rewrites

$$i\dot{\varepsilon} - D_\varepsilon\mathbf{H}(s,\varepsilon) - i\frac{\dot{\lambda}}{\lambda}\Lambda\varepsilon + (1-\dot{\gamma})\varepsilon + \mathcal{E}_{\mathbf{P}} = 0.$$

In particular,

$$\begin{aligned}\langle D_\varepsilon\mathbf{H}(s,\varepsilon), \dot{\varepsilon}\rangle &= \langle iD_\varepsilon\mathbf{H}(s,\varepsilon), i\dot{\varepsilon}\rangle \\ &= \frac{\dot{\lambda}}{\lambda}\langle D_\varepsilon\mathbf{H}(s,\varepsilon), \Lambda\varepsilon\rangle - (1-\dot{\gamma})\langle iD_\varepsilon\mathbf{H}(s,\varepsilon), \varepsilon\rangle - \langle iD_\varepsilon\mathbf{H}(s,\varepsilon), \mathcal{E}_{\mathbf{P}}\rangle.\end{aligned}$$

We recall that

$$\langle -\Delta\varepsilon, \Lambda\varepsilon\rangle = \|\nabla\varepsilon\|^2, \quad \langle \varepsilon, \Lambda\varepsilon\rangle = 0, \quad \langle |\varepsilon|^2\varepsilon, \Lambda\varepsilon\rangle = \frac{1}{2}\int |\varepsilon|^4,$$

and thus, using also (3.6)–(3.7), (1.3), and the decay properties of Q,

$$|\langle D_\varepsilon\mathbf{H}(s,\varepsilon), \Lambda\varepsilon\rangle| \lesssim \|\varepsilon\|_{H^1}^2 + \|\varepsilon\|_{H^1}^4 \lesssim \|\varepsilon\|_{H^1}^2.$$

In particular, from (3.9) and (3.6)–(3.7), we deduce

$$\left|\frac{\dot{\lambda}}{\lambda}\langle D_\varepsilon\mathbf{H}(s,\varepsilon), \Lambda\varepsilon\rangle\right| \lesssim s^{-1}\log^{-1}(s)\|\varepsilon\|_{H^1}^2.$$

Note that the estimate on b in (3.6)–(3.7) implies $|b| \lesssim s^{-1}\log^{-1}(s) \ll s^{-1}$ which avoids the use of virial localized identities (as in [53, 20]) to control the above term. By (3.9) and (3.6)–(3.7), we estimate

$$|(1-\dot{\gamma})\langle iD_\varepsilon\mathbf{H}(s,\varepsilon), \varepsilon\rangle| \lesssim s^{-2}\|\varepsilon\|_{H^1}^2.$$

Finally, integrating by parts, using (3.12) and (3.6)–(3.7), we have

$$|\langle iD_\varepsilon\mathbf{H}(s,\varepsilon), \mathcal{E}_{\mathbf{P}}\rangle| \lesssim \langle |\nabla\varepsilon|, |\nabla\mathcal{E}_{\mathbf{P}}|\rangle + \langle |\varepsilon| + |\varepsilon|^3, |\mathcal{E}_{\mathbf{P}}|\rangle \lesssim s^{-2}\log^{-2}(s)\|\varepsilon\|_{H^1}.$$

The collection of above estimates finishes the proof of (3.34).

Step 3: Variation of the localized momentum. We now claim: for all $s \in [s^*, s^{in}]$,

$$\begin{aligned}&\left|\frac{d}{ds}[\mathbf{J}(s,\varepsilon(s))] - b\sum_k \langle z_k \cdot \nabla P_k, 2|\varepsilon|^2 P_k + \varepsilon^2\overline{P}_k\rangle\right| \\ &\qquad\qquad \lesssim s^{-2}\log^{-2}(s)\|\varepsilon(s)\|_{H^1} + s^{-1}\log^{-1}(s)\|\varepsilon(s)\|_{H^1}^2.\end{aligned} \tag{3.36}$$

Indeed, we compute, for any k,

$$\begin{aligned}\frac{d}{ds}[J_k(s,\varepsilon(s))] &= \dot{b}\,\mathrm{Im}\int (z_k \cdot \nabla\varepsilon)\bar{\varepsilon}\chi_k + b\,\mathrm{Im}\int (\dot{z}_k \cdot \nabla\varepsilon)\bar{\varepsilon}\chi_k + b\,\mathrm{Im}\int (z_k \cdot \nabla\varepsilon)\bar{\varepsilon}\dot{\chi}_k \\ &\quad + b\langle i\dot{\varepsilon}, z_k \cdot (2\chi_k\nabla\varepsilon + \varepsilon\nabla\chi_k)\rangle.\end{aligned}$$

By (3.9) and (3.6)–(3.7), we have

$$\left|\dot{b}\,\mathrm{Im}\int (z_k\cdot\nabla\varepsilon)\bar{\varepsilon}\chi_k\right| + \left|b\,\mathrm{Im}\int (\dot{z}_k\cdot\nabla\varepsilon)\bar{\varepsilon}\chi_k\right| \lesssim s^{-2}\|\varepsilon\|_{H^1}^2.$$

Note that by direct computations, (3.9) and (3.6)–(3.7),

$$|\dot{\chi}_k| \lesssim (s^{-1}\log^{-1}(s)|y-z_k| + |\dot{z}_k|)\log^{-1}(s)|\chi'(\log^{-1}(s)(y-z_k(s)))| \lesssim s^{-1}\log^{-1}(s)$$

and so, by (3.6)–(3.7),

$$\left|b\,\mathrm{Im}\int (z_k\cdot\nabla\varepsilon)\bar{\varepsilon}\dot{\chi}_k\right| \lesssim s^{-2}\log^{-2}(s)\|\varepsilon\|_{H^1}^2.$$

Now, we use the Equation (3.13) of ε to estimate $b\langle i\dot{\varepsilon}, z_k\cdot(2\chi_k\nabla\varepsilon+\varepsilon\nabla\chi_k)\rangle$. By integration by parts, we check the following

$$\langle\Delta\varepsilon, 2(z_k\cdot\nabla\varepsilon)\chi_k\rangle = \int |\nabla\varepsilon|^2(z_k\cdot\nabla\chi_k) - 2\langle(\nabla\varepsilon\cdot\nabla\chi_k),(z_k\cdot\nabla\varepsilon)\rangle,$$

$$\langle\Delta\varepsilon, \varepsilon(z_k\cdot\nabla\chi_k)\rangle = -\int |\nabla\varepsilon|^2(z_k\cdot\nabla\chi_k) + \frac{1}{2}\int |\varepsilon|^2(z_k\cdot\nabla(\Delta\chi_k)).$$

Thus,

$$\langle\Delta\varepsilon, z_k\cdot(2\chi_k\nabla\varepsilon+\varepsilon\nabla\chi_k)\rangle = -2\langle(\nabla\varepsilon\cdot\nabla\chi_k),(z_k\cdot\nabla\varepsilon)\rangle + \frac{1}{2}\int |\varepsilon|^2(z_k\cdot\nabla(\Delta\chi_k)).$$

By (3.6)–(3.7), $|b|\lesssim s^{-1}\log^{-1}(s)$ and $|z_k|\lesssim \log(s)$. Moreover, $|\nabla\chi_k|\lesssim\log^{-1}(s)$. Therefore,

$$|b\langle(\nabla\varepsilon\cdot\nabla\chi_k),(z_k\cdot\nabla\varepsilon)\rangle| \lesssim s^{-1}\log^{-1}(s)\|\varepsilon\|_{H^1}^2.$$

Similarly, by $|\nabla(\Delta\chi_k)|\lesssim\log^{-3}(s)$, we obtain

$$\left|b\int |\varepsilon|^2(z_k\cdot\nabla(\Delta\chi_k))\right| \lesssim s^{-1}\log^{-3}(s)\|\varepsilon\|_{H^1}^2.$$

In conclusion, for term $\Delta\varepsilon$ in the equation of ε, we obtain

$$|b\langle\Delta\varepsilon, z_k\cdot(2\chi_k\nabla\varepsilon+\varepsilon\nabla\chi_k)\rangle| \lesssim s^{-1}\log^{-1}(s)\|\varepsilon\|_{H^1}^2.$$

For the mass term in the equation of ε, we simply check by integration by parts that

$$\langle\varepsilon, z_k\cdot(2\chi_k\nabla\varepsilon+\varepsilon\nabla\chi_k)\rangle = 0.$$

We also check that

$$\langle i\Lambda\varepsilon, z_k\cdot(2\chi_k\nabla\varepsilon+\varepsilon\nabla\chi_k)\rangle = 2\langle i\varepsilon,(z_k\cdot\nabla\varepsilon)\chi_k\rangle + \langle i(y\cdot\nabla\varepsilon), \varepsilon(z_k\cdot\nabla\chi_k)\rangle,$$

and thus, by (3.6)–(3.7),

$$|b|\left|\frac{\dot{\lambda}}{\lambda}\right||\langle i\Lambda\varepsilon, z_k\cdot(2\chi_k\nabla\varepsilon+\varepsilon\nabla\chi_k)\rangle| \lesssim s^{-2}\log^{-1}(s)\|\varepsilon\|_{H^1}^2.$$

Next, from (3.12),

$$|b\langle\mathcal{E}_{\mathbf{P}}, z_k\cdot(2\chi_k\nabla\varepsilon+\varepsilon\nabla\chi_k)\rangle| \lesssim s^{-3}\log^{-2}(s)\|\varepsilon\|_{H^1}.$$

Now, we only have to deal with the term

$$b\langle|\mathbf{P}+\varepsilon|^2(\mathbf{P}+\varepsilon)-|\mathbf{P}|^2\mathbf{P}, z_k\cdot(2\chi_k\nabla\varepsilon+\varepsilon\nabla\chi_k)\rangle.$$

Recall that $|\mathbf{P}+\varepsilon|^2(\mathbf{P}+\varepsilon)-|\mathbf{P}^2|\mathbf{P} = (2\varepsilon|\mathbf{P}|^2+\overline{\varepsilon}\mathbf{P}^2)+2|\varepsilon|^2\mathbf{P}+\varepsilon^2\overline{\mathbf{P}}+|\varepsilon|^2\varepsilon$. First, by (3.6)–(3.7), it is clear that

$$\left|b\langle 2|\varepsilon|^2\mathbf{P}+\varepsilon^2\overline{\mathbf{P}}+|\varepsilon|^2\varepsilon, z_k\cdot(2\chi_k\nabla\varepsilon+\varepsilon\nabla\chi_k)\rangle\right| \lesssim s^{-1}\|\varepsilon\|_{H^1}^3 \lesssim s^{-2}\log^{-\frac{3}{2}}(s)\|\varepsilon\|_{H^1}^2.$$

Second, since $|b|\lesssim s^{-1}\log^{-1}(s)$, $|z_k|\lesssim \log(s)$ and $|\nabla\chi_k|\lesssim\log^{-1}(s)$,

$$\left|b\langle 2\varepsilon|\mathbf{P}|^2+\overline{\varepsilon}\mathbf{P}^2, \varepsilon(z_k\cdot\nabla\chi_k)\rangle\right| \lesssim s^{-1}\log^{-1}(s)\|\varepsilon\|_{H^1}^2.$$

Third, by the decay property of Q and the definition of χ_k,

$$\left|b\left\langle 2\varepsilon\Big(|\mathbf{P}|^2-\sum_j|P_j|^2\Big)+\overline{\varepsilon}\Big(\mathbf{P}^2-\sum_j P_j^2\Big), (z_k\cdot\nabla\varepsilon)\chi_k\right\rangle\right| \lesssim s^{-2}\|\varepsilon\|_{H^1}^2,$$

and, for $j\neq k$,

$$\left|b\left\langle 2\varepsilon|P_j|^2+\overline{\varepsilon}P_j^2, (z_k\cdot\nabla\varepsilon)\chi_k\right\rangle\right| \lesssim s^{-2}\|\varepsilon\|_{H^1}^2.$$

Finally, we compute by integration by parts,

$$\begin{aligned}\langle 2\varepsilon|P_k|^2+\overline{\varepsilon}P_k^2, (z_k\cdot\nabla\varepsilon)\chi_k\rangle &= -\langle z_k\cdot\nabla P_k, 2|\varepsilon|^2P_k+\varepsilon^2\overline{P}_k\rangle\\ &\quad -\frac{1}{2}\operatorname{Re}\left(\int\left(2|\varepsilon|^2P_k^2+\varepsilon^2|P_k|^2\right)(z_k\cdot\nabla\chi_k)\right).\end{aligned}$$

As before,

$$\left|b\operatorname{Re}\left(\int\left(2|\varepsilon|^2P_k^2+\varepsilon^2|P_k|^2\right)(z_k\cdot\nabla\chi_k)\right)\right| \lesssim s^{-1}\log^{-1}(s)\|\varepsilon\|_{H^1}^2.$$

The collection of above bounds concludes the proof of (3.36).

Step 4: Conclusion. Recall that, by (3.11), $|\dot z_k - bz_k|\lesssim s^{-1}\log^{-1}(s)$, and so

$$\left|\langle(\dot z_k-bz_k)\cdot\nabla P_k, 2|\varepsilon|^2P_k+\varepsilon^2\overline{P}_k\rangle\right| \lesssim s^{-1}\log^{-1}(s)\|\varepsilon\|_{H^1}^2,$$

and (3.32) now follows from (3.34), (3.36). This concludes the proof of Proposition 8. □

3.4. End of the proof of Proposition 6

We close the bootstrap estimates (3.6) and prove (3.5).

Step 1: Closing the estimates in ε. By (3.32) in Proposition 8 and then (3.6)–(3.7), we have

$$\left|\frac{d}{ds}[\mathbf{F}(s,\varepsilon(s))]\right| \lesssim s^{-2}\log^{-2}(s)\|\varepsilon\|_{H^1}+s^{-1}\log^{-1}(s)\|\varepsilon\|_{H^1}^2 \lesssim s^{-3}\log^{-\frac{7}{2}}(s).$$

Thus, by integration on $[s,s^{in}]$ for any $s\in[s^*,s^{in}]$, using $\varepsilon(s^{in})=0$ (see (3.3)), we obtain

$$|\mathbf{F}(s,\varepsilon(s))|\lesssim s^{-2}\log^{-\frac{7}{2}}(s).$$

By (3.31) in Proposition 8, we obtain

$$\|\varepsilon(s)\|_{H^1}^2\lesssim s^{-2}\log^{-\frac{7}{2}}(s).$$

Therefore, for s_0 large enough, for all $s\in[s^*,s^{in}]$,

$$\|\varepsilon(s)\|_{H^1}^2\leq\frac{1}{2}s^{-2}\log^{-3}(s),$$

which strictly improves the estimate on $\|\varepsilon\|_{H^1}^2$ in (3.6).

Step 2: Closing the parameter estimates. First, note that from (3.11), $|\dot{\beta}| \lesssim s^{-2}\log^{-2}(s)$. Together with the choice $\beta(s^{in}) = \beta^{in} = 0$ (see (3.4)), direct integration in time gives, for all $s \in [s^*, s^{in}]$, $|\beta(s)| \lesssim s^{-1}\log^{-2}(s)$. For s_0 large enough, we obtain, for all $s \in [s^*, s^{in}]$,

$$|\beta(s)| < \frac{1}{2}s^{-1}\log^{-\frac{3}{2}}(s),$$

which strictly improves the estimate on $\beta(s)$ in (3.6).

Second, recall from (3.11), (3.7) and the definition of $a(z)$ in (2.5), for all $s \in [s^*, s^{in}]$,

$$\left|\dot{b} + c_a z^{\frac{1}{2}} e^{-\kappa z}\right| \lesssim s^{-2}\log^{-2}(s), \quad \left|\dot{z}z^{-1} - b\right| \lesssim s^{-1}\log^{-2}(s).$$

Since $|\dot{b}| \lesssim s^{-2}\log^{-1}(s)$ and $|\dot{z}z^{-1}| \lesssim s^{-1}\log^{-1}(s)$, it follows that

$$\left|\dot{b}b + c_a \dot{z} z^{-\frac{1}{2}} e^{-\kappa z}\right| \lesssim s^{-3}\log^{-3}(s).$$

Integrating on $[s, s^{in}]$ for any $s \in [s^*, s^{in})$, using the special relation between b^{in} and z^{in} fixed in (3.4)

$$b^2(s^{in}) = \frac{2c_a}{\kappa} z^{-\frac{1}{2}}(s^{in}) e^{-\kappa z(s^{in})}, \quad b(s^{in}) > 0,$$

we obtain

(3.37)
$$\left|b^2 - \frac{2c_a}{\kappa} z^{-\frac{1}{2}} e^{-\kappa z}\right| \lesssim s^{-2}\log^{-3}(s) + \int_s^{s^{in}} \left|\dot{z} z^{-\frac{3}{2}} e^{-\kappa z}\right| ds' \lesssim s^{-2}\log^{-3}(s), \quad b(s) > 0.$$

From (3.6)–(3.7) and (3.7), we have

$$\left|\frac{2c_a}{\kappa} z^{-\frac{1}{2}} e^{-\kappa z} - s^{-2}\log^{-2}(s)\right| \lesssim s^{-2}\log^{-\frac{5}{2}} s.$$

Therefore, the following estimate on $b(s)$ follows from (3.37)

$$\left|b^2 - s^{-2}\log^{-2}(s)\right| \lesssim s^{-2}\log^{-\frac{5}{2}}(s).$$

This implies, for all $s \in [s^*, s^{in}]$,

$$\left|b - s^{-1}\log^{-1}(s)\right| \lesssim s^{-1}\log^{-\frac{3}{2}}(s), \tag{3.38}$$

which strictly improves the estimate on $b(s)$ in (3.6).

Finally, we address the estimate on $z(s)$. From (3.37), (3.6)–(3.7) and (3.11), we have

$$\left|b - \left(\frac{2c_a}{\kappa}\right)^{\frac{1}{2}} z^{-\frac{1}{4}} e^{-\frac{\kappa}{2}z}\right| + \left|\dot{z}z^{-1} - \left(\frac{2c_a}{\kappa}\right)^{\frac{1}{2}} z^{-\frac{1}{4}} e^{-\frac{\kappa}{2}z}\right| \lesssim s^{-1}\log^{-2}(s). \tag{3.39}$$

Using $z \lesssim \log^{-1}(s)$, we obtain

$$\left|\frac{d}{ds}\left(z^{-\frac{3}{4}} e^{\frac{\kappa}{2}z}\right) - \left(\frac{\kappa c_a}{2}\right)^{\frac{1}{2}}\right| \lesssim \log^{-1}(s) + \left|\dot{z} z^{-\frac{7}{4}} e^{\frac{\kappa}{2}z}\right| \lesssim \log^{-1}(s). \tag{3.40}$$

We need to adjust the initial choice of $z(s^{in}) = z^{in}$ through a topological argument (see [11] for a similar argument). We define ζ and ξ the following two functions on $[s^*, s^{in}]$

$$\zeta(s) = \left(\frac{2}{\kappa c_a}\right)^{\frac{1}{2}} z^{-\frac{3}{4}} e^{\frac{\kappa}{2}z}, \quad \xi(s) = (\zeta(s) - s)^2 s^{-2}\log(s). \tag{3.41}$$

Then, (3.40) writes

$$|\dot{\zeta}(s)-1|\lesssim \log^{-1}(s). \tag{3.42}$$

According to (3.6), our objective is to prove that there exists a suitable choice of

$$\zeta(s^{in})=\zeta^{in}\in[s^{in}-s^{in}\log^{-\frac12}(s^{in}),s^{in}+s^{in}\log^{-\frac12}(s^{in})],$$

so that $s^*=s_0$. Assume for the sake of contradiction that for all $\zeta^\sharp\in[-1,1]$, the choice

$$\zeta^{in}=s^{in}+\zeta^\sharp s^{in}\log^{-\frac12}(s^{in})$$

leads to $s^*=s^*(\zeta^\sharp)\in(s_0,s^{in})$. Since all estimates in (3.6) except the one on $z(s)$ have been strictly improved on $[s^*,s^{in}]$, it follows from $s^*(\zeta^\sharp)\in(s_0,s^{in})$ and continuity that

$$|\zeta(s^*(\zeta^\sharp))-s^*|=s^*\log^{-\frac12}s^* \quad \text{i.e.,}\quad \zeta(s^*(\zeta^\sharp))=s^*\pm s^*\log^{-\frac12}s^*.$$

We need a transversality condition to reach a contradiction. We compute:

$$\dot{\xi}(s)=2(\zeta(s)-s)(\dot{\zeta}(s)-1)s^{-2}\log(s)-(\zeta(s)-s)^2(2s^{-3}\log(s)-s^{-3}).$$

At $s=s^*$, this gives

$$|\dot{\xi}(s^*)+2(s^*)^{-1}|\lesssim (s^*)^{-1}\log^{-\frac12}(s^*).$$

Thus, for s_0 large enough,

$$\dot{\xi}(s^*)<-(s^*)^{-1}. \tag{3.43}$$

Define the function Φ by

$$\Phi\,:\,\zeta^\sharp\in[-1,1]\mapsto(\zeta(s^*)-s^*)(s^*)^{-1}\log^{\frac12}(s^*)\in\{-1,1\}.$$

A standard consequence of the transversality property (3.43) is the continuity of the function $\zeta^\sharp\in[-1,1]\mapsto s^*(\zeta^\sharp)$. In particular, the function Φ is also continuous from $[-1,1]$ to $\{1,-1\}$. Moreover, for $\zeta^\sharp=-1$ and $\zeta^\sharp=1$, $\xi(s^*)=1$ and $\dot{\xi}(s^*)<0$ by (3.43) and so in these cases $s^*=s^{in}$. Thus, $\Phi(-1)=-1$ and $\Phi(1)=1$, but this is in contradiction with the continuity.

In conclusion, there exists at least a choice of

$$\zeta(s^{in})=\zeta^{in}\in(s^{in}-s^{in}\log^{-\frac12}(s^{in}),s^{in}+s^{in}\log^{-\frac12}(s^{in}))$$

such that $s^*=s_0$.

Step 3: Conclusion. To finish proving (3.5), we only have to prove the estimate on $\lambda(s)$. From (3.9) and (3.38), we obtain

$$\left|\frac{\dot{\lambda}}{\lambda}+s^{-1}\log^{-1}(s)\right|\lesssim s^{-1}\log^{-\frac32}(s).$$

By integration on $[s,s^{in}]$, for any $s\in[s_0,s^{in})$, using the value $\lambda(s^{in})=\lambda^{in}=\log^{-1}(s^{in})$ (see (3.4)), we have

$$|\log(\lambda(s))+\log(\log(s))|\lesssim \log^{-\frac12}(s),$$

and thus

$$\left|\lambda(s)-\log^{-1}(s)\right|\lesssim \log^{-\frac32}(s). \tag{3.44}$$

This concludes the proof of Proposition 6.

4. Compactness arguments

The objective of this section is to finish the construction of Theorem 1 by passing to the limit on a sequence of solutions given by Proposition 6.

4.1. Construction of a sequence of backwards solutions

We claim the following consequence of Proposition 6.

LEMMA 9. – *There exist* $t_0 > 1$ *and a sequence of solutions* $u_n \in \mathcal{C}([t_0 - T_n, 0], \Sigma)$ *of* (1.1), *where*

$$T_n \to +\infty \quad \text{as } n \to +\infty, \tag{4.1}$$

satisfying the following estimates, for all $t \in [t_0 - T_n, 0]$,

(4.2)

$$\left| z_n(t) - \frac{2}{\kappa}\log(t+T_n)\right| \lesssim \log(\log(t+T_n)), \quad \left|\lambda_n(t) - \log^{-1}(t+T_n)\right| \lesssim \log^{-\frac{3}{2}}(t+T_n),$$

$$\left|b_n(t) - (t+T_n)^{-1}\log^{-3}(t+T_n)\right| + |\beta_n(t)| + \|\varepsilon_n(t)\|_{H^1} \lesssim (t+T_n)^{-1}\log^{-\frac{7}{2}}(t+T_n),$$

$$|a_n(t)| \lesssim (t+T_n)^{-2}\log^{-1}(t+T_n), \quad \varepsilon_n(0) \equiv 0,$$

where $(\lambda_n, z_n, \gamma_n, \beta_n, b_n)$ *are the parameters of the decomposition of* u_n *given by Lemma 5, that is*

$$u_n(t,x) = \frac{e^{i\gamma_n(t)}}{\lambda_n(t)}\left(\sum_k \left[e^{i\Gamma_{k,n}} Q_{a_n}\right]\left(\frac{x}{\lambda_n(t)} - z_n(t)\mathrm{e}_k\right) + \varepsilon_n\left(t, \frac{x}{\lambda_n(t)}\right)\right), \tag{4.3}$$

with $\Gamma_{k,n}(t,y) = \beta_n(t)(\mathrm{e}_k \cdot y) - \frac{b_n(t)}{4}|y|^2$ *and* $Q_{a_n} = Q + a_n\rho$. *Moreover, for all* $t \in [t_0 - T_n, 0]$,

$$\int |u_n(t,x)|^2|x|^2 dx \lesssim 1. \tag{4.4}$$

Proof. – Applying Proposition 6 with $s^{in} = n$ for any large n, there exists a solution $u_n(t)$ of (1.1) defined on the time interval $[-T_n, 0]$ where

$$T_n = \int_{s_0}^{n} \lambda_n^2(s)ds,$$

and whose decomposition satisfies the uniform estimates (3.5) on $[-T_n, 0]$. First, we see that (4.1) follows directly from the estimate on $\lambda_n(s)$ in (3.5).

Proof of (4.2). From (3.5) and the definition of the rescaled time s (see (3.2)), for any $s \in [s_0, n]$, we have

$$t(s) + T_n = \int_{s_0}^{s} \lambda_n^2(s')ds' \quad \text{where} \quad |\lambda_n^2(s) - \log^{-2}(s)| \lesssim \log^{-\frac{5}{2}}(s).$$

Fix $\bar{s}_0 > s_0$ large enough independent of n so that, for all $\bar{s}_0 < s < n$,

$$\frac{1}{2}s\log^{-2}(s) \le \int_{s_0}^{s} \lambda_n^2(s')ds' = s\log^{-2}(s) + O(s\log^{-\frac{5}{2}}(s)) \le \frac{3}{2}s\log^{-2}(s).$$

Fix $t_0 = \frac{3}{2}\bar{s}_0\log^{-2}(\bar{s}_0)$. Then, for all $t \in [t_0 - T_n, 0]$,

$$t + T_n = s\log^{-2}(s)\left(1 + O(\log^{-\frac{1}{2}}(s))\right) \ge \frac{1}{2}s\log^{-2}(s),$$

and

$$s = (t+T_n)\log^2(t+T_n)\left(1+O(\log^{-\frac{1}{2}}(t+T_n))\right).$$

Thus, estimates (4.2) are direct consequences of (3.5).

Proof of (4.4). – From (4.3) and $\varepsilon_n(0)\equiv 0$, we have $u_n(0)\in\Sigma$. It is then standard (see e.g., [7], Proposition 6.5.1) that $u_n\in\mathcal{C}([t_0-T_n,0],\Sigma)$. We claim the following preliminary estimates. Fix $A=\frac{16}{\kappa}\geq 8$. For any $k\in\{1,\dots,K\}$, for all $t\in[t_0-T_n,0]$,

$$\frac{1}{\lambda_n^2(t)}\int\left|Q_{a_n}\left(\frac{x}{\lambda_n(t)}-z_n(t)\mathrm{e}_k\right)\right|^2|x|^2dx\lesssim 1, \tag{4.5}$$

$$\frac{1}{\lambda_n^2(t)}\int_{|x|>A}\left|\nabla_x\left(\left[e^{i\Gamma_{k,n}}Q_{a_n}\right]\left(\frac{x}{\lambda_n(t)}-z_n(t)\mathrm{e}_k\right)\right)\right|^2dx\lesssim(t+T_n)^{-4}. \tag{4.6}$$

Indeed, (4.5) follows from a change of variable and the decay properties of Q and ρ,

$$\frac{1}{\lambda_n^2(t)}\int\left|Q_{a_n}\left(\frac{x}{\lambda_n(t)}-z_n(t)\mathrm{e}_k\right)\right|^2|x|^2dx=\int|Q_{a_n}(y)|^2\,|\lambda_n(t)y+\lambda_n(t)z_n(t)\mathrm{e}_k|^2dy\lesssim 1,$$

where we have used from (4.2),

$$\lambda_n(t)z_n(t)\lesssim 1. \tag{4.7}$$

To show (4.6), we see first that by (4.2),

$$\left|\nabla\left[e^{i\Gamma_{k,n}}Q_{a_n}\right](y)\right|^2\lesssim|\nabla Q_{a_n}(y)|^2+\left(|\beta_n|^2+b_n^2|y|^2\right)Q_{a_n}^2(y)\lesssim e^{-\frac{3}{2}|y|}.$$

Thus, by change of variable (using $A\geq 8$),

$$\begin{aligned}\frac{1}{\lambda_n^2(t)}&\int_{|x|>A}\left|\nabla_x\left(\left[e^{i\Gamma_{k,n}}Q_{a_n}\right]\left(\frac{x}{\lambda_n(t)}-z_n(t)\mathrm{e}_k\right)\right)\right|^2dx\\&=\frac{1}{\lambda_n^2(t)}\int_{|y+z_n(t)\mathrm{e}_k|>A/\lambda_n(t)}\left|\nabla\left[e^{i\Gamma_{k,n}}Q_{a_n}\right](y)\right|^2dy\\&\lesssim\log^2(t+T_n)\int_{|y|>\frac{A}{2}\log(t+T_n)}e^{-\frac{3}{2}|y|}dy\lesssim(t+T_n)^{-\frac{A}{2}}=(t+T_n)^{-4},\end{aligned}$$

where we have used from (4.2) (possibly taking a larger t_0),

$$|y+z_n(t)\mathrm{e}_k|>\frac{A}{\lambda_n(t)}\quad\Rightarrow\quad|y|>\frac{A}{\lambda_n(t)}-|z_n(t)|>\left(\frac{3A}{4}-\frac{4}{\kappa}\right)\log(t+T_n)\geq\frac{A}{2}\log(t+T_n).$$

Thus (4.6) is proved. Observe that (4.5)–(4.6) and (4.2) imply

$$\|xu_n(0)\|_{L^2}\lesssim 1,\quad\|\nabla u_n(t)\|_{L^2(|x|>A)}\lesssim(t+T_n)^{-1}\log^{-\frac{5}{2}}(t+T_n). \tag{4.8}$$

Define $\varphi:\mathbb{R}^2\to[0,1]$ by $\varphi(x)=(|x|-A)^2$ for $|x|>A$ and $\varphi(x)=0$ otherwise. By elementary computations,

$$\frac{d}{dt}\int|u_n|^2\varphi=2\,\mathrm{Im}\int(\nabla\varphi\cdot\nabla u_n)\overline{u}_n=4\int_{|x|>A}\left(\frac{x}{|x|}\cdot\nabla u_n\right)\overline{u}_n\varphi^{\frac{1}{2}}.$$

Thus, by (4.8),

$$\left|\frac{d}{dt}\int|u_n|^2\varphi\right|\lesssim\left(\int|u_n|^2\varphi\right)^{\frac{1}{2}}\left(\int_{|x|>A}|\nabla u_n(t)|^2\right)^{\frac{1}{2}}\lesssim(t+T_n)^{-1}\log^{-\frac{5}{2}}(t+T_n)\left(\int|u_n|^2\varphi\right)^{\frac{1}{2}}.$$

By integration and (4.8), the following uniform bound holds on $[t_0 - T_n, 0]$,

$$\int |u_n(t,x)|^2 \varphi(x) dx \lesssim 1 \text{ and thus } \int |u_n(t,x)|^2 x^2 dx \lesssim 1,$$

which finishes the proof of (4.4). □

4.2. Compactness argument

By (4.2)–(4.4), the sequence $(u_n(t_0 - T_n))$ is bounded in Σ. Therefore, there exists a subsequence of (u_n) (still denoted by (u_n)) and $u_0 \in \Sigma$ such that

$$u_n(t_0 - T_n) \rightharpoonup u_0 \quad \text{weakly in } H^1(\mathbb{R}^2),$$
$$u_n(t_0 - T_n) \to u_0 \quad \text{in } H^\sigma(\mathbb{R}^2) \text{ for } 0 \le \sigma < 1, \text{ as } n \to +\infty.$$

Let u be the solution of (1.1) corresponding to $u(t_0) = u_0$. By the local Cauchy theory for (1.1) (see [7] and [8]) and the properties of the sequence $u_n(t)$ (recall that $T_n \to \infty$), it follows that $u \in \mathcal{C}([t_0, +\infty), \Sigma)$. Moreover, for all $0 \le \sigma < 1$, for all $t \in [t_0, +\infty)$,

$$u_n(t - T_n) \to u(t) \quad \text{in } H^\sigma.$$

By weak convergence in H^1, $u(t)$ satisfies (2.25) for all $t \ge t_0$. Moreover, the decomposition $(\vec{p}, \varepsilon)$ of u satisfies, for all $t \ge t_0$,

$$\vec{p}_n(t - T_n) \to \vec{p}(t), \quad \varepsilon_n(t - T_n) \to \varepsilon(t) \text{ in } H^\sigma, \quad \varepsilon_n(t - T_n) \rightharpoonup \varepsilon(t) \text{ in } H^1 \tag{4.9}$$

(see e.g., [37], Claim p. 598). In particular, for all $t \in [t_0, +\infty)$, $u(t)$ decomposes as

$$u(t,x) = \frac{e^{i\gamma(t)}}{\lambda(t)} \left(\sum_k \left[e^{i\Gamma_k} Q_a \right] \left(\frac{x - \lambda(t) z(t) \mathrm{e}_k}{\lambda(t)} \right) + \varepsilon\left(t, \frac{x}{\lambda(t)} \right) \right), \tag{4.10}$$

where $\Gamma_k(t,y) = \beta(t)(\mathrm{e}_k \cdot y) - \frac{b(t)}{4}|y|^2$ and

$$\left| z(t) - \frac{2}{\kappa} \log(t) \right| \lesssim \log(\log(t)), \quad \left| \lambda(t) - \log^{-1}(t) \right| \lesssim \log^{-\frac{3}{2}}(t),$$
$$\left| b(t) - t^{-1}\log^{-3}(t) \right| + |\beta(t)| + \|\varepsilon(t)\|_{H^1} \lesssim t^{-1} \log^{-\frac{7}{2}}(t), \quad |a(t)| \lesssim t^{-2} \log^{-1}(t), \tag{4.11}$$
$$\int |u(t,x)|^2 |x|^2 dx \lesssim 1.$$

Note that by (4.11), we have for all $k \in \{1, \ldots, K\}$,

$$x_k(t) = \lambda(t) z(t) \mathrm{e}_k \to \frac{2}{\kappa} \mathrm{e}_k, \quad \text{with} \quad \left| x_k(t) - \frac{2}{\kappa} \mathrm{e}_k \right| \lesssim \frac{\log(\log(t))}{\log(t)}.$$

Since $\lambda^{-1}(t) \|\varepsilon(t)\|_{H^1} \lesssim t^{-1} \log^{-\frac{5}{2}}(t)$ and, by (4.10) and (4.11),

$$\lambda^{-1}(t) \left\| e^{i\Gamma_k} Q_a - Q \right\|_{H^1} \lesssim \lambda^{-1}(t)(|\beta(t)| + |b(t)| + |a(t)|) \lesssim t^{-1} \log^{-2}(t), \tag{4.12}$$

we obtain the following stronger form of (1.11)

$$\left\| u(t) - e^{i\gamma(t)} \sum_k \frac{1}{\lambda(t)} Q\left(\frac{\cdot - x_k(t)}{\lambda(t)} \right) \right\|_{H^1} \lesssim t^{-1} \log^{-2}(t). \tag{4.13}$$

Next, since for $j \neq k$, for some q,

$$\lambda^{-2}(t) \int \left| \nabla Q\left(y - z(t) \mathrm{e}_k \right) \cdot \nabla Q\left(y - z(t) \mathrm{e}_j \right) \right| dy \lesssim |z|^q e^{-\kappa z} \lesssim t^{-1},$$

we also obtain (1.12). As a final remark, note that by global existence and uniform bound in Σ, the virial identity (1.7) implies the rigidity $E(u) = 0$. This concludes the proof of Theorem 1.

4.3. Proof of Corollary 2

For $-t_0^{-1} < t < 0$, we set

$$\tilde{z}(t) = z(|t|^{-1}), \quad \tilde{\lambda}(t) = |t|\lambda(|t|^{-1}), \quad \tilde{a}(t) = a(|t|^{-1}), \quad \tilde{b}(t) = b(|t|^{-1}),$$

$$\tilde{\gamma}(t) = \gamma(|t|^{-1}), \quad \tilde{\beta}(t) = \beta(|t|^{-1}), \quad \tilde{\varepsilon}(t) = \varepsilon(|t|^{-1}), \quad \tilde{\Gamma}_k(t,y) = \tilde{\beta}(t)(\mathrm{e}_k \cdot y) - \frac{\tilde{b}(t)}{4}|y|^2,$$

so that from (4.11),

$$\begin{aligned} &\left|\tilde{z}(t) - \frac{2}{\kappa}|\log|t||\right| \lesssim \log|\log|t||, \quad \left|\tilde{\lambda}(t) - |t||\log|t||^{-1}\right| \lesssim |\log|t||^{-\frac{3}{2}}, \\ &\left|\tilde{b}(t) - |t||\log|t||^{-3}\right| + |\tilde{\beta}(t)| + \|\tilde{\varepsilon}(t)\|_{H^1} \lesssim |t||\log|t||^{-\frac{7}{2}}, \quad |\tilde{a}(t)| \lesssim |t||\log|t||^{-1}. \end{aligned} \tag{4.14}$$

We see from (4.10) that the pseudo-conformal transform $v(t)$ of $u(t)$ as defined in (1.5) satisfies

$$v(t,x) = e^{-i\frac{|x|^2}{4|t|}} w(t,x), \quad w(t,x) = \frac{e^{i\tilde{\gamma}(t)}}{\tilde{\lambda}(t)} \left(\sum_k \left[e^{i\tilde{\Gamma}_k} Q_{\tilde{a}} \right] \left(\frac{x}{\tilde{\lambda}(t)} - \tilde{z}(t)\mathrm{e}_k \right) + \tilde{\varepsilon}\left(t, \frac{x}{\tilde{\lambda}(t)}\right) \right).$$

Note in particular that $\tilde{\lambda}(t)\tilde{z}(t) \sim \frac{2}{\kappa}|t|$ as $t \uparrow 0$. From this, it follows that

$$|v(t,x)|^2 \rightharpoonup K\|Q\|_{L^2}^2 \delta_0 \quad \text{as } t \uparrow 0.$$

Finally, since $\nabla v(t,x) = e^{-i\frac{|x|^2}{4|t|}} \left(\nabla w - i\frac{x}{2|t|} w \right)(t,x)$, and as $t \uparrow 0$,

$$\frac{1}{|t|^2} \int |x|^2 |w(t,x)|^2 dx \lesssim \left| \frac{\tilde{\lambda}(t)}{t} \right|^2 \int \Big| \sum_k \left[e^{i\tilde{\Gamma}_k} Q_{\tilde{a}} \right](y - \tilde{z}(t)\mathrm{e}_k) + \tilde{\varepsilon}(t,y) \Big|^2 |y|^2 dy \lesssim 1,$$

$$\int |\nabla w(t,x)|^2 dx \sim K\|\nabla Q\|_{L^2}^2 |t|^{-2} |\log|t||^2,$$

we obtain (1.13). Note that $\int |x|^2|v(t,x)|^2 \lesssim t^2$ implies by (1.7) that $\int |x|^2|v(t,x)|^2 = t^2E(v)$. Thus, $E(v) > 0$.

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