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Shaobo GAN & Dawei YANG

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# MORSE-SMALE SYSTEMS AND HORSESHOES FOR THREE DIMENSIONAL SINGULAR FLOWS

# BY SHAOBO GAN AND DAWEI YANG

ABSTRACT. – We prove that every three-dimensional vector field can be  $C^1$  accumulated by Morse-Smale ones, or by ones with a transverse homoclinic intersection of some hyperbolic periodic orbit. In contrast to the case of diffeomorphisms [14], the main difficulty here is that we need to deal with singularities. We also make progress on another conjecture related to Palis in this paper.

RÉSUMÉ. – Nous montrons que tout champ de vecteurs en dimension trois peut être accumulé en topologie  $C^1$  ou bien par un champ Morse-Smale, ou bien par un champ possédant une intersection homocline transverse associée à une orbite périodique hyperbolique. Contrairement au cas des difféomorphismes [14], la principale difficulté ici consiste à traiter les singularités. Nous progressons également en direction d'une autre conjecture de Palis.

# 1. Introduction

### 1.1. The main result

One of the main subjects in differentiable dynamical systems is to describe the dynamics of "most" dynamical systems. These theories were established in the last century. See [2] for instance. An important progress is due to Peixoto [43]:

THEOREM (Peixoto). – Assume that  $M^2$  is a closed surface. A  $C^1$  vector field on  $M^2$  is  $C^1$  structurally stable iff it is Morse-Smale. Moreover, every vector field can be accumulated by structurally stable ones in the  $C^1$  topology.

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Smale was interested in the generalization of Peixoto's result and he asked whether Morse-Smale vector fields are dense in the space of vector fields. Levinson wrote Smale that one couldn't expect Morse-Smale systems to be dense generally. Essential ideas were contained in Levinson's paper which was inspired by work of Cartwright and Littlewood. See [2, 48]. Smale noticed the point and he constructed his famous horseshoe (for two dimensional diffeomorphisms or three-dimensional vector fields) [47] which shows that the dynamics may be very complicated and Morse-Smale systems would not be dense in the space of diffeomorphisms or vector fields. As in [2, Page 16]: "At that moment the world turned upside down..., and a new life began".

Actually, Poincaré found an important phenomenon in his famous work [44] on celestial mechanics, which was called a "doubly asymptotical solution". Nowadays mathematicians call it a *transverse homoclinic intersection*. Smale found that his horseshoe is closely related to transverse homoclinic intersections. Three classical results are known:

- Poincaré showed that transverse homoclinic intersections can survive under small perturbations. Moreover, if a system has one transverse homoclinic intersection, then it has infinitely many transverse homoclinic intersections [44].
- Birkhoff showed that if a plane system has one transverse homoclinic intersection, then it has infinitely many hyperbolic periodic orbits [5].
- Smale proved that the existence of a transverse homoclinic intersection is equivalent to the existence of a horseshoe [47].

In this paper, a *horseshoe* of a vector field is a hyperbolic set that is topologically equivalent to a suspension of full shift with two symbols. Hence there are two kinds of typical dynamical systems: Morse-Smale systems or systems with a horseshoe. Their dynamical behaviors are quite different:

- The dynamics of Morse-Smale systems is very simple: the chain recurrent set of a Morse-Smale system is a set which consists in finitely many hyperbolic periodic orbits or singularities. The topological entropy is robustly zero.
- The dynamics of a system with a horseshoe is very complicated: its chain recurrent set contains a non-trivial basic set with dense periodic orbits. The topological entropy is robustly positive.

Is there other typical dynamics beyond the above two ones? Palis formulated the idea for diffeomorphisms, and he conjectured that

CONJECTURE (Palis [40, 41, 42]). – Every system can be approximated either by Morse-Smale systems or by systems exhibiting a horseshoe.

In this paper, we manage to prove such kind of results for three dimensional vector fields.

THEOREM A (Main Theorem). – Every three dimensional vector field can be  $C^1$  approximated by Morse-Smale ones or by ones exhibiting a horseshoe. In other words, Morse-Smale vector fields and vector fields with a horseshoe form a  $C^1$  open dense set in the space of three dimensional vector fields. Important progresses have been made for the conjecture of Palis for diffeomorphisms: in the  $C^1$  topology, Pujals-Sambarino [45] proved it for two-dimensional diffeomorphisms (as a corollary of a stronger result); Bonatti-Gan-Wen [9] gave a proof for three-dimensional diffeomorphisms; and finally Crovisier [14] proved the conjecture for *any* dimensional diffeomorphisms.

Comparing with the diffeomorphism case, singularities of vector fields bring more difficulties. This prevents one to use some techniques of diffeomorphisms to singular vector fields, such as Crovisier's central model and Pujals-Sambarino's distortion arguments. By considering the sectional Poincaré maps of the flows, sometimes one can get some (not all) similar properties between *d*-dimensional vector fields and (d - 1)-dimensional diffeomorphisms. But singular vector field displays different dynamics, e.g., the famous *Lorenz attractor* [29]. In the spirit of the Lorenz attractor, *geometric Lorenz attractors* ([1, 17, 18]) were constructed in a theoretical way. Roughly, a geometric Lorenz attractor is a robust attractor of a threedimensional vector field, and it contains a hyperbolic singularity which is accumulated by hyperbolic periodic orbits in a robust way. The return map of a geometric Lorenz attractor has some discontinuous points. This fact gives extra difficulties when one wants to generalize Mañé's classical argument [30] to singular flows <sup>(1)</sup>.

The Lorenz attractor is not hyperbolic because of the existence of a singularity. On the other hand, Mañé [30] showed that a robust attractor of a surface diffeomorphism is hyperbolic. This implies that the dynamics of 3-dimensional singular flows are different from 2-dimensional diffeomorphisms. Morales-Pacifico-Pujals [32, 31, 33] studied geometric Lorenz attractors in an abstract way. They found the right concept, i.e., *singular hyperbolicity*, to describe the weak hyperbolicity of the Lorenz attractor, and they proved that a robust transitive set of a three-dimensional vector field is singular hyperbolic. But the dynamics of singular hyperbolic sets are not as clear as hyperbolic sets. For instance, there is no shadowing lemma of Anosov-Bowen type.

The dynamics of general transitive sets with singularities for three-dimensional vector fields are even more unclear for us than singular hyperbolic sets, even if the transitive sets have some dominated splitting with respect to the linear Poincaré flow. These are the main difficulties that we encounter. For a non-trivial transitive set *without singularities* of a generic any dimensional vector field, one can adapt Crovisier's central model [14] to get a transverse homoclinic intersection of a hyperbolic periodic orbit.

Let us be more precise. Let  $M^d$  be a *d*-dimensional  $C^{\infty}$  compact Riemannian manifold without boundary. Denote by  $\mathcal{K}^1(M^d)$  the space of  $C^1$  vector fields on  $M^d$ . Given  $X \in \mathcal{K}^1(M^d)$ , denote by  $\phi_t = \phi_t^X$  the  $C^1$  flow generated by X and by  $\Phi_t = d\phi_t : TM^d \to TM^d$  the tangent flow on the tangent bundle  $TM^d$ . If  $X(\sigma) = 0$ , then  $\sigma$  is called a *singularity* of X. Other points are called *regular*. Let Sing(X) be the set of singularities of X. For a regular point p, if  $\phi_t(p) = p$  for some t > 0, then p is called *periodic*. Let Per(X) be the set of periodic points of X. If  $x \in Sing(X) \cup Per(X)$ , then x is called a *critical point* of X and Orb(x) is called a *critical orbit* or *critical element* of X.

<sup>&</sup>lt;sup>(1)</sup> Mañé's classical argument was generalized to the case of diffeomorphisms by [45] and to the case of non-singular flow by [4].

For an invariant set  $\Lambda$  and a  $\Phi_t$ -invariant bundle  $E \subset T_{\Lambda}M^d$ , we say that E is *contracting* (w.r.t. the tangent flow  $\Phi_t$ ) if there are constants  $C \ge 1, \lambda > 0$  such that  $\|\Phi_t\|_{E(x)} \| \le Ce^{-\lambda t}$  for every  $x \in \Lambda$  and  $t \ge 0$ ; we say that E is *expanding* if it is contracting for -X.

An invariant set  $\Lambda$  of X is hyperbolic if  $TM^d$  has a continuous  $\Phi_t$ -invariant splitting

$$T_{\Lambda}M^{d} = E^{s} \oplus \langle X \rangle \oplus E^{i}$$

(where the fiber  $\langle X(x) \rangle$  at x is 0-dimensional or 1-dimensional depending on if x is a singularity or not), such that  $E^s$  is contracting and  $E^u$  is expanding. If dim  $E^s$  is independent of  $x \in \Lambda$ , then dim  $E^s$  is called the *index* of  $\Lambda$ .

For a critical point x, if Orb(x) is a hyperbolic set, then we say that x or Orb(x) is hyperbolic. One can define its index as the index of the hyperbolic set Orb(x).

Recall that a  $C^1$  vector field X is *Morse-Smale* if the non-wandering set  $\Omega(X)$  of X consists of only finitely many hyperbolic critical elements and their stable and unstable manifolds intersect transversely. We use  $\mathcal{M}_{\mathcal{S}}^{\mathcal{S}}$  to denote the set of Morse-Smale vector fields in  $\mathcal{K}^1(M^d)$ . For a hyperbolic periodic orbit  $\gamma$ , define

$$W^{s}(\gamma) = \{x \in M^{d} : \lim_{t \to +\infty} d(\phi_{t}(x), \gamma) = 0\},\$$
  
$$W^{u}(\gamma) = \{x \in M^{d} : \lim_{t \to -\infty} d(\phi_{t}(x), \gamma) = 0\}.$$

We know ([20]) that  $W^s(\gamma)$  and  $W^u(\gamma)$  are submanifolds, which are called the stable and unstable manifolds of  $\gamma$ . If  $W^s(\gamma) \pitchfork W^u(\gamma) \setminus \gamma \neq \emptyset$ , then one says that  $\gamma$  has a *transverse homoclinic orbit*<sup>(2)</sup>. One says that X has a transverse homoclinic orbit if for some hyperbolic periodic orbit  $\gamma$  of X,  $\gamma$  has a transverse homoclinic orbit. Recall that: Birkhoff-Smale theorem asserts that the existence of transverse homoclinic orbits is equivalent to the existence of a horseshoe (non-trivial hyperbolic basic set). We denote:

 $\mathcal{HS} = \{X \in \mathcal{X}^1(M^d) : X \text{ has a transverse homoclinic orbit}\}.$ 

One can restate Theorem A as:

 $\mathcal{MS} \cup \mathcal{HS}$  is open and dense in  $\mathcal{X}^1(M^3)$ .

# 1.2. More on three-dimensional flows

Given a vector field X, let  $\phi_t$  be the flow generated by X. For any  $\varepsilon > 0, \{x_0, x_1, \dots, x_n\}$  is called an  $\varepsilon$ -chain(or  $\varepsilon$ -pseudo-orbit) from  $x_0$  to  $x_n$  if there are  $t_i \ge 1$  such that  $d(\phi_{t_i}(x_i), x_{i+1}) < \varepsilon$  for any  $0 \le i \le n-1$ . For  $x, y \in M^d$ , one says that y is chain attainable from x if for any  $\varepsilon > 0$ , there is an  $\varepsilon$ -chain from x and y. x and y are chain bi-attainable if x is chain attainable from y and y is chain attainable from x. If x is chain attainable from itself, then x is called a chain recurrent point. The set of chain recurrent points is called the chain recurrent set of X, denoted by CR(X). Chain bi-attainability is a closed equivalence relation on CR(X). For each  $x \in CR(X)$ , the equivalence class containing x is called the chain recurrent class of x, denoted by C(x) or C(Orb(x)). These are standard by Conley's theory [12].

Bonatti and Crovisier [6] extended the  $C^1$  connecting lemma to pseudo-orbits. An application of their result gave a useful classification of chain recurrent classes for  $C^1$ -generic vector

<sup>&</sup>lt;sup>(2)</sup> Singularities cannot have transverse homoclinic intersections.

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fields <sup>(3)</sup>: if a chain recurrent class contains a periodic orbit, then it is the *homoclinic class* of this periodic orbit; otherwise, it is called an *aperiodic class*. Here, the *homoclinic class* of a hyperbolic periodic orbit is defined to be the closure of all transverse homoclinic orbits of this periodic orbit.

Hyperbolic periodic orbits may have non-transverse homoclinic intersections, which are called *homoclinic tangencies*. Newhouse [34, 35, 36] studied the bifurcations of homoclinic tangencies crucially, which generate rich dynamics. Newhouse phenomena give typical dynamics beyond uniformly hyperbolic dynamics. There are many results for diffeomorphisms far away from ones with a homoclinic tangency. One can see the introduction of [15].

In this work, we can prove that every *non-trivial* chain recurrent class is a homoclinic class for  $C^1$  generic vector fields which are far away from homoclinic tangencies. Here, a chain recurrent class is called *non-trivial* if it is not reduced to be a critical orbit.

THEOREM B. – There is a dense  $G_{\delta}$  set  $\mathcal{R} \subset \mathcal{X}^1(M^3)$  such that, for every  $X \in \mathcal{R}$ , if X cannot be accumulated by ones with a homoclinic tangency, then every non-trivial chain recurrent class of X is a homoclinic class.

Theorem B is stronger than Theorem A. We will see this point in Section 4.

An important conjecture made by Palis for surface diffeomorphisms is: every twodimensional diffeomorphism can be accumulated either by ones with a homoclinic tangency, or by uniformly hyperbolic ones. This was proved by Pujals-Sambarino [45] in the  $C^1$ topology. For three-dimensional vector fields, as mentioned in [39], excluding homoclinic tangencies and uniform hyperbolic systems, the typical dynamics may include the homoclinic orbits of singularities or Lorenz-like attractors. Arroyo-Rodriguez [4] proved the conjecture of Palis if homoclinic orbits of singularities were involved for three-dimensional vector fields <sup>(4)</sup>. It is still an open problem about the density of Lorenz-like attractors or repellers beyond uniform hyperbolicity and homoclinic bifurcations of periodic orbits (even in the  $C^1$  topology).

Morales-Pacifico-Pujals [31, 33] defined what is "Lorenz-like" in a dynamical way. In [32], Morales-Pacifico gave the notion of singular Axiom A without cycle. Let's be more precise.

We say that a continuous invariant splitting  $T_{\Lambda}M^d = E \oplus F$  w.r.t. the tangent flow over a compact invariant set  $\Lambda$  is a *dominated splitting* with respect to the tangent flow  $\Phi_t$  if there are constants  $C \ge 1, \lambda > 0$ , such that  $\|\Phi_t|_{E(x)}\|\|\Phi_{-t}|_{F(\phi_t(x))}\| \le Ce^{-\lambda t}$  for every  $x \in \Lambda$ and  $t \ge 0$ . For a compact invariant set  $\Lambda$ , we say that  $\Lambda$  admits a *partially hyperbolic splitting* if there is a continuous invariant splitting  $T_{\Lambda}M^d = E^s \oplus E^c \oplus E^u$  w.r.t.  $\Phi_t$ , where  $E^s$  is contracting,  $E^u$  is expanding, and both  $E^s \oplus (E^c \oplus E^u)$  and  $(E^s \oplus E^c) \oplus E^u$  are dominated splittings. In the above definition,  $E^s$  or  $E^u$  is allowed to be trivial.

<sup>&</sup>lt;sup>(3)</sup> Their results are stated for diffeomorphisms. The proof can be adapted to the case of vector fields by a parallel way.

<sup>&</sup>lt;sup>(4)</sup> The precise statement of the work of Arroyo-Rodriguez [4] is: any vector field can be  $C^1$  approximated by one of the following phenomena: uniformly hyperbolicity, a homoclinic tangency and a singular cycle. Even in the non-singular case, it does *not* parallel to the case of diffeomorphisms: on the one hand, the reparametrization problem of the flow gives them difficulties; on the other hand, the diffeomorphisms between sections induced by the flow are only defined locally.

DEFINITION 1.1. – A transitive set  $\Lambda$  of  $X \in \mathcal{X}^1(M^3)$  is called a singular hyperbolic attractor if

1. There is a neighborhood U of  $\Lambda$  such that

$$\Lambda = \bigcap_{t \ge 0} \phi_t(U).$$

- 2.  $\Lambda$  contains a singularity, and every singularity in  $\Lambda$  has index 2.
- 3. A admits a partially hyperbolic splitting  $T_{\Lambda}M^3 = E^s \oplus E^{cu}$ , where dim  $E^s = 1$  and  $E^{cu}$  is area-expanding: there are constants C > 0,  $\lambda > 0$  such that for any  $x \in \Lambda$  and for any  $t \ge 0$ , one has  $|\det(\Phi_t|_{E^{cu}(x)})| \ge Ce^{\lambda t}$ .
- $\Lambda$  is called a singular hyperbolic repeller if it is a singular hyperbolic attractor for -X.

One knows that the geometric Lorenz attractors as in [17, 18, 1] are singular hyperbolic attractors.  $X \in \mathcal{X}^1(M^3)$  is called *singular Axiom A without cycle* as in [32] if the chain recurrent set of X contains only finitely many chain recurrent classes and if each chain recurrent class is a hyperbolic basic set, or a singular hyperbolic attractor, or a singular hyperbolic repeller.

Singular Axiom A flows include Lorenz-like flows. One ([4, 8, 32]) wonders if singular Axiom A flows and flows with a homoclinic tangency are typical phenomena.

CONJECTURE (Palis: the Conjecture 5.14 of [8]). – Every  $X \in \mathcal{X}^1(M^3)$  can be accumulated either by vector fields with a homoclinic tangency, or by singular Axiom A vector fields without cycle.

We get some progress on this conjecture.

THEOREM C. – There is a dense  $G_{\delta}$  set  $\mathcal{R} \subset \mathcal{X}^{1}(M^{3})$  such that for any  $X \in \mathcal{R}$  and any singularity  $\sigma$  of X, if the chain recurrent class  $C(\sigma)$  is nontrivial and admits a dominated splitting  $T_{C(\sigma)}M^{3} = E \oplus F$  w.r.t. the tangent flow, then  $C(\sigma)$  is a singular hyperbolic attractor or a singular hyperbolic repeller.

Notice that in a joint work with C. Bonatti [10], we proved this result by adding an additional assumption that  $C(\sigma)$  contains a periodic orbit. So, according to this result, to prove the above theorem, we assume that  $C(\sigma)$  contains no periodic orbits. Then we can consider the return map of (singular) cross-sections. After a sequence of perturbations, we will get a contradiction. This is one of the main points of this work.

# 1.3. Entropy of flows

The entropy of a flow is defined to be the entropy of the time-one map of the flow. The definition meets some problems: there are two topological equivalent flows such that one has zero entropy and the other one has positive entropy. This pathology happens because of the existence of singularities. See [37, 50, 49] for references. But using the main theorem of the paper, we can prove:

THEOREM 1.2. – There is a dense open set  $\mathcal{U} \subset \mathcal{X}^1(M^3)$  such that for any  $X \in \mathcal{U}$ , for any Y topologically equivalent to X, one has h(X) = 0 iff h(Y) = 0.

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*Proof.* – By Theorem A, there is a dense open set  $\mathcal{U} \subset \mathcal{K}^1(M^3)$  such that for any  $X \in \mathcal{U}$ , either X is Morse-Smale, or X has a non-trivial hyperbolic basic set. Thus for any  $X \in \mathcal{U}$ , one has

- either h(X) = 0, then X is Morse-Smale, thus for any Y which is topologically equivalent to X, for any point x, both the forward iteration and backward iteration of x with respect to  $\phi_t^Y$  go to a critical element. This feature implies that h(Y) = 0.
- or h(X) > 0, then X has a non-trivial hyperbolic basic set. Since for non-singular equivalent flows X, Y, h(X) > 0 iff h(Y) > 0. We have that h(Y) > 0.

For the relationship between zero-entropy vector fields and Morse-Smale vector fields, one has

THEOREM 1.3. – If a three-dimensional vector field  $X \in \mathcal{X}^1(M^3)$  can be accumulated by  $C^1$  robustly zero-entropy vector fields, then it can be  $C^1$  accumulated by Morse-Smale vector fields.

*Proof.* – By the assumptions, for any  $C^1$  neighborhood  $\mathcal{U}$  of X, there is an open set  $\mathcal{V} \subset \mathcal{U}$  such that every vector field Y in  $\mathcal{V}$  has zero-entropy. By Theorem A, by reducing  $\mathcal{V}$  if necessary, one can assume that for every  $Y \in \mathcal{V}$ , either it is Morse-Smale, or it has a non-trivial hyperbolic basic set. Since  $Y \in \mathcal{V}$  can only have zero-entropy, one has Y is Morse-Smale. This ends the proof.

#### 1.4. Organization of this paper

The proof of the theorems is not short. In particular, for vector fields, they involve more definitions and notations.

- 1. In Section 2, we give various kinds of definitions of flows associated to a vector field X. Liao defined these flows in a very abstract way. In fact, all these flows have their geometric meanings. We will deal with dominated splittings for the tangent flow and the linear Poincaré flow. For the estimations stated in this section (which is crucial for singular flow), Liao had very original ideas by a sequence of papers. We restate some of them and give the proof by ourselves. The proof is more intuitive.
- 2. In Section 3, we will study the generic properties which are implied by connecting lemmas and by the ergodic closing lemma. Bonatti and Crovisier [6] gave a  $C^1$  connecting lemma for pseudo-orbits which helps us to obtain generic results for chain recurrent classes. We notice that Lyapunov stable chain recurrent classes with a critical element are robust for generic vector fields. The proof is not difficult, but it opens a new door: Lyapunov stable chain recurrent class will survive under generic small perturbations.
- 3. In Section 4 we first give the proof of Theorem A by assuming Theorem B. Then we reduce the proofs of Theorem B and Theorem C to several sub-results.
- 4. In Section 5, we prove that the Lyapunov stable chain recurrent class admits a partially hyperbolic splitting  $E^{ss} \oplus E^{cu}$  and every singularity in the chain recurrent class is Lorenz-like. In this section, the main novelty of this paper is that we use some uniform estimation on vector fields away from homoclinic tangencies and a suitable application of Liao's shadowing lemma.

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5. In Section 6, we prove that every nontrivial partially hyperbolic chain recurrent class with singularities contains periodic orbits for generic three-dimensional vector fields. We notice that it contains a periodic orbit iff it is singular hyperbolic. When the chain recurrent class is not singular hyperbolic, it is not singular hyperbolic robustly. Then by a sequence of perturbations, the continuation of the chain recurrent class intersects the closure of the basin of some sink. This implies that the continuation of the chain recurrent class is not Lyapunov stable. We can get a contradiction by Lemma 3.15. The difficulty we encounter is similar to the case of one-dimensional endomorphisms with singularities, where "singularities" means that the points where the endomorphisms fail to be a local diffeomorphisms. For flows, for every central unstable curve in the cross-section, in principle we will know that its length will grow near the local stable manifold of the singularities. But when it is cut by local stable manifold of singularities, its image under the return map will be disconnected. This facts make the dynamics unclear. We have a good control in this section for this phenomenon.

# 2. Flows associated to a vector field and dominated splittings

#### 2.1. Tangent flow, linear Poincaré flow and their extensions

Given  $X \in \mathcal{X}^1(M^d)$ , X generates a  $C^1$  flow  $\phi_t : M^d \to M^d$ , and the *tangent flow*  $\Phi_t = d\phi_t : TM^d \to TM^d$ . Denote by  $\pi : TM^d \to M^d$  the bundle projection.

Denote the normal bundle of X by

$$_{\mathcal{O}}\mathcal{N} = _{\mathcal{O}}\mathcal{N}^X = \bigcup_{x \in M^d \setminus \operatorname{Sing}(X)} _{\mathcal{O}}\mathcal{N}_x,$$

where  $\mathcal{N}_x$  is the orthogonal complement of the flow direction X(x), i.e.,

$$\mathcal{N}_x = \{ v \in TM^d : v \perp X(x) \}.$$

Given  $x \in M^d \setminus \text{Sing}(X)$  and  $v \in \mathcal{N}_x$ ,  $\psi_t(v)$  is the orthogonal projection of  $\Phi_t(v)$ on  $\mathcal{N}_{\phi_t(x)}$  along the flow direction, i.e.,

$$\psi_t(v) = \Phi_t(v) - \frac{\langle \Phi_t(v), X(\phi_t(x)) \rangle}{|X(\phi_t(x))|^2} X(\phi_t(x)),$$

where  $\langle \cdot, \cdot \rangle$  is the inner product on  $T_x M$  given by the Riemannian metric.

By the definition,  $\|\psi_t\|$  is uniformly bounded for t in any bounded interval although it is just defined on the regular set which is not compact in general.

This flow can also be defined in a more general way by Liao [28]. Li, Gan and Wen [23] used the terminology of "extended linear Poincaré flow". For every point  $x \in M^d$ , one can define the sphere fiber at x by

$$S_x M^d = \{ v : v \in T_x M^d, |v| = 1 \}.$$

The sphere bundle  $SM^d = \bigcup_{x \in M^d} S_x M^d$  is compact. One can define the *unit tangent flow* 

$$\Phi^I_t: SM^d \to SM^d$$

as

$$\Phi_t^I(v) = \frac{\Phi_t(v)}{|\Phi_t(v)|}$$

for any  $v \in SM^d$ .

Given a compact invariant set  $\Lambda$  of X, denote by

$$\widetilde{\Lambda} = \text{Closure}\left(\bigcup_{x \in \Lambda \setminus \text{Sing}(X)} \frac{X(x)}{|X(x)|}\right)$$

in  $SM^d$ . Thus the essential difference between  $\widetilde{\Lambda}$  and  $\Lambda$  is on the singularities. We have more information on  $\widetilde{\Lambda}$ : it tells us how regular points in  $\Lambda$  accumulate singularities.

For any  $x \in M^d$ , and any two orthogonal vectors  $v_1, v_2 \in T_x M^d$ , if  $|v_1| \neq 0$ , one can define

$$\chi_t(v_1, v_2) = (\Phi_t(v_1), \Phi_t(v_2) - \frac{\langle \Phi_t(v_1), \Phi_t(v_2) \rangle}{|\Phi_t(v_1)|^2} \Phi_t(v_1)).$$

By definition the two components of  $\chi_t$  are still orthogonal. If one denotes

$$\chi_t = (\operatorname{proj}_1(\chi_t), \operatorname{proj}_2(\chi_t))$$

then for any regular point  $x \in M^d$  and any vector  $v \in \mathcal{N}_x$ , one has

$$\psi_t(v) = \operatorname{proj}_2 \chi_t(X(x), v)$$

One can normalize the first component of  $\chi_t$ : for any  $x \in M^d$ , and any two orthogonal vectors  $v_1, v_2 \in T_x M^d$ , if  $|v_1| = 1$ , one can define

$$\chi_t^{\#}(v_1, v_2) = (\Phi_t^I(v_1), \ \Phi_t(v_2) - \frac{\langle \Phi_t(v_1), \Phi_t(v_2) \rangle}{|\Phi_t(v_1)|^2} \Phi_t(v_1)).$$

 $\chi_t^{\#}$  is also a continuous flow, and for any regular point x and any  $v \in \mathcal{N}_x$ , one has

$$\chi_t^{\#}(\frac{X(x)}{|X(x)|}, v) = (\Phi_t^I(\frac{X(x)}{|X(x)|}), \ \psi_t(v)).$$

By the continuity of  $\chi_t$ , one can extend the definition of  $\psi_t$  "to singularities": for any  $u \in \widetilde{\Lambda}$ , one defines  $\widetilde{\mathcal{N}}_u = \{v \in T_{\pi(u)}M : \langle u, v \rangle = 0\}$ .  $\widetilde{\mathcal{N}}$  is a (d-1)-dimensional vector bundle on the base space  $\widetilde{\Lambda}$  (for a formal discussion, see [23]). For  $u \in \widetilde{\Lambda}$  and  $v \in \widetilde{\mathcal{N}}_u$ , one can define  $\widetilde{\psi}_t(v) = \operatorname{proj}_2 \chi_t(u, v)$ . By the definition we know that  $\operatorname{proj}_2 \chi_t$  is a continuous flow defined on  $\widetilde{\Lambda}$ . Thus,  $\widetilde{\psi}_t$  can be viewed as a compactification of  $\psi_t$ .

## 2.2. Scaled linear Poincaré flow $\psi_t^*$ and Liao's estimations

The results in this subsection are in the spirit of Liao [24, 27, 28]. One can find the scaled linear Poincaré flow  $\psi_t^*$  in Liao's work. Liao used his "canonical equations" to give some uniform estimations in a more analytical way. In this work, we first use the notations  $\mathcal{P}$  and  $\mathcal{P}^*$  to study the local sectional dynamics to understand Liao's powerful tools in a more geometrical way. Although all the estimations (especially Lemma 2.5) cannot be seen in Liao's original work, we still call them Liao's estimations.

For our purpose, we need another flow  $\psi_t^* : \mathcal{N} \to \mathcal{N}$  (called *scaled linear Poincaré flow*). Given  $x \in M^d \setminus \operatorname{Sing}(X)$ , and  $v \in \mathcal{N}_x$ ,

$$\psi_t^*(v) = \frac{|X(x)|}{|X(\phi_t(x))|} \psi_t(v) = \frac{\psi_t(v)}{\|\Phi_t\|_{(X(x))}\|}.$$

In our case, this scaled linear Poincaré flow will help us to overcome some difficulties produced by singularities since it gives uniform estimations on some non-compact sets.

LEMMA 2.1. – For any  $\tau > 0$ , there is  $C_{\tau} > 0$  such that for any  $t \in [-\tau, \tau]$ ,

 $\|\psi_t^*\| \leq C_{\tau},$ 

where  $\|\psi_t^*\| = \sup\{|\psi_t^*(v)| : v \in \mathcal{N} \text{ and } |v| = 1\}.$ 

*Proof.* – For  $t \in [-\tau, \tau]$ , first we know that  $\Phi_t$  is uniformly bounded from 0 and  $\infty$ ; from the definition of the linear Poincaré flow, we know that  $\psi_t$  is uniformly bounded. Thus,  $\psi_t^*(x) = \psi_t(x)/||\Phi_t|_{\leq X(x)>}||$  is uniformly bounded.

For each  $\beta > 0$ , one can define the normal manifold  $N_x(\beta)$  of x as the following:

$$N_x(\beta) = \exp_x(\mathcal{N}_x(\beta)),$$

where  $\mathcal{N}_x(\beta) = \{v \in \mathcal{N}_x : |v| \le \beta\}$ . Take  $\beta_* > 0$  small enough such that for any  $x \in M$ , exp<sub>x</sub> is a diffeomorphism from  $\mathcal{N}_x(\beta_*)$  to its image  $N_x(\beta_*)$ .

To study the dynamics in a small neighborhood of a periodic orbit of a vector field, Poincaré defined the sectional return map of a cross section of a periodic point. By generalizing this idea to every regular point, one can define the sectional Poincaré map between any two cross sections at any two points in the same regular orbit. For our convenience, we define the sectional Poincaré map in the normal bundle.

Given T > 0 and  $x \in M^d \setminus \text{Sing}(X)$ , the flow  $\phi_t$  defines a local holonomy map  $P_{x,\phi_T(x)}$ from  $N_x(\beta_*)$  to  $N_{\phi_T(x)}(\beta_*)$  in a small neighborhood of x. Hence its lift map in the normal bundle gives a map  $\mathcal{P}_{x,\phi_T(x)} : U \to \mathcal{N}_{\phi_T(x)}(\beta_*)$ , where U is a small neighborhood of xin  $\mathcal{N}_x(\beta_*)$  and  $\mathcal{P}_{x,\phi_T(x)} = \exp_{\phi_T(x)}^{-1} \circ P_{x,\phi_T(x)} \circ \exp_x$ . Note that when T' > T > 0, the domain of  $\mathcal{P}_{x,\phi_{T'}(x)}$  is contained in the domain of  $\mathcal{P}_{x,\phi_T(x)}$ . Usually the size of U depends on the orbit of x: if x is very close to a singularity, then U should be very small. But after scaling, we have the following uniform estimation for the relative size of U.

Recall the definition of the linear Poincaré flow. By the geometrical meanings of  $\mathcal{P}$  and  $\psi_t$ , we have

LEMMA 2.2. – For any regular point x and any time t, we have when y = 0,

$$D_{y} \mathcal{P}_{x,\phi_t(x)} = \psi_t(x).$$

*Proof.* – Since the derivative of  $\exp_x(y)$  at y = 0 is the identity map:  $D_0(\exp_x) = \text{Id}$ , by the relationship between  $P_{x,\phi_t(x)}$  and  $\mathcal{P}_{x,\phi_t(x)}$ , we only have to show that  $D_0P_{x,\phi_t(x)} = \psi_t(x)$ . Assume that  $P_{x,\phi_t(x)}(z) = \phi_{\tau(z)}(z)$ , where  $\tau(x) = t$ . Note that  $\tau(z)$  is a  $C^1$  function of z. According to the chain rule, we have that

$$DP_{x,\phi_t(x)}(z) = \Phi_{\tau(z)}(z) + X(\phi_{\tau(z)}(z))D\tau(z).$$

Given  $y \in \mathcal{N}_x$ , write  $\Phi_t(x)(y) = \psi_t(x)(y) + rX(\phi_t(x))$ . At z = x, we have

$$DP_{x,\phi_t(x)}(x)(y) = \Phi_t(x)(y) + X(\phi_t(x))D\tau(x)(y) = \psi_t(x)(y) + (r + D\tau(x)(y))X(\phi_t(x)).$$

Since  $DP_{x,\phi_t(x)}(x)(y) \in \mathcal{N}_{\phi_t(x)}$ , we get  $DP_{x,\phi_t(x)}(x)(y) = \psi_t(x)(y)$ .

LEMMA 2.3. – Given  $X \in \mathcal{X}^1(M^d)$  and T > 0, there is  $\beta_T > 0$  such that for any regular point x,  $\mathcal{P}_{x,\phi_T(x)}$  is well defined on  $\mathcal{N}_x(\beta_T |X(x)|)$ .

*Proof.* – After taking an orthonormal basis  $\{e_1, e_2, \ldots, e_d\}$  of  $T_x M$ , we get a coordinate system:

$$\operatorname{Exp}_{x}: \mathbb{R}^{d} \to M^{d},$$

such that

$$\operatorname{Exp}_{x}(z) = \operatorname{exp}_{x}(\sum_{i=1}^{d} z_{i}e_{i}).$$

In the coordinate, the flow generated by the vector field *X* satisfies the following differential equation:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \hat{X}(z),$$

where

$$\hat{X}(z) = D \operatorname{Exp}_{x}^{-1} \circ X(\operatorname{Exp}_{x}(z)).$$

Note that  $\hat{X}$  depends on  $x \in M^d$ .

Assume that  $\beta_*$  is small enough such that for any  $x \in M^d$ ,  $|z| \leq \beta_*$ , and any two unit vectors v and w,

 $0.999 < |D \operatorname{Exp}_{x}(z)v| < 1.001, |\angle (D \operatorname{Exp}_{x}(z)v, D \operatorname{Exp}_{x}(z)w) - \angle (v, w)| < 0.001.$ 

We will prove the lemma in the local coordinate. Note that  $\text{Exp}_x(0) = x$  and  $|X(x)| = |\hat{X}(0)|$ .

Denote by

$$K = \sup_{x \in M^d, |z| \le \beta_*} \{ |\hat{X}(z)|, \|D\hat{X}(z)\| \}.$$

Since  $DExp_x$  and  $DExp_x^{-1}$  are uniformly bounded with respect to x, we have

$$K < \infty$$
.

Assume  $\beta_0 < \min\{1, \beta_*\}/(1000K)$ . For any regular point x, take  $e_1 = X(x)/|X(x)|$ . Then  $\hat{X}(0) = (|X(x)|, 0, ..., 0)$ . For  $|z| \le \beta_0 |X(x)|$ , according to the mean-value theorem in integral,

$$|\hat{X}(z)| = |\hat{X}(0) + \int_0^1 D\hat{X}(tz)zdt| \ge |X(x)| - K\beta_0|X(x)| \ge 0.999|X(x)| > 0.$$

This implies that  $N_x(\beta_0|X(x)|) \cap \text{Sing}(X) = \emptyset$ .

Denote by  $\hat{\phi}_t(z) = (\hat{\phi}^1, \dots, \hat{\phi}^d)$  the solution of  $\hat{X}(z)$  such that  $\hat{\phi}_0(z) = z$ . Then for  $0 \le t \le \beta_0$ , if  $|z| \le \frac{\beta_0}{3} |X(x)|$  and  $|\phi_s(z)| \le \beta_0 |X(x)|$  for  $s \in [0, t]$ , we have

$$|\hat{\phi}_t(z)| = |z + \int_0^t \hat{X}(\hat{\phi}_t(z))dt| \le (\frac{\beta_0}{3} + 1.001t)|X(x)|$$

Let t be the time such that  $|\hat{\phi}_s(z)| \leq \beta_0 |X(x)|$  for  $s \in [0, t)$  and  $|\hat{\phi}_t(z)| = \beta_0 |X(x)|$ . From the above estimation, we have that

$$t \ge \frac{2}{3}\beta_0/1.001 > \frac{1}{2}\beta_0.$$

By reducing  $\beta_0$  if necessary, for  $|z| \leq \beta_0 |X(x)|$ ,

. . . . .

$$\sup_{t \in (-\beta_0, \beta_0)} \frac{|X(z)|}{|\hat{X}(\hat{\phi}_t(z))|} < \frac{1}{1000K}, \quad \sup_{t \in (-\beta_0, \beta_0)} \angle (\hat{X}(z), \ \hat{X}(\hat{\phi}_t(z))) < \frac{1}{1000K}.$$

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Since  $e_1 = X(x)/|X(x)|$ ,  $\mathcal{N}_x = \{z : z_1 = 0\}$ , and  $\hat{X}(0) = (|X(x)|, 0, ..., 0)$ , we have that for  $|z| \le \beta_0 |X(x)|$ ,  $\hat{X}_1(z) \in [0.999|X(x)|, 1.001|X(x)|]$ .

CLAIM. – For  $0 < r < \beta_0$ , for any z = (0, y),  $y \in \mathbb{R}^{d-1}$ ,  $|y| \leq r|X(x)|/2$ , for any  $t \in [\beta_0/3, 2\beta_0/3]$ , there exists a unique  $\tau = \tau(t, y) \in (0, \beta_0]$  such that  $\hat{\phi}_{\tau}(0, y) \in \hat{N}_{\phi_t(x)}(r)$ , where

$$\hat{N}_x(r) = \operatorname{Exp}^{-1}(N_x(r)).$$

**Proof of Claim.** – Since  $|z| < 1000\beta_0 K$ ,  $t \in [0, \beta_0]$  and  $\beta_0$  is small,  $\operatorname{Exp}_x$  is almost an isometry, and  $N_{\phi_t(x)}$  is the graph of a map  $f = f_t : \mathbb{R}^{d-1} \to \mathbb{R}$  with  $|\frac{\partial f}{\partial y}| < 0.001$ . Let z = (0, y) with  $|y| \le \beta_0 |X(x)|$ . Then

$$\operatorname{proj}_{1}\hat{\phi}_{\beta_{0}}(z) = \int_{0}^{\beta_{0}} \hat{X}_{1}(\hat{\phi}_{t}(z)) dt \ge 0.999 |X(x)|\beta_{0}.$$

This means that z and  $\hat{\phi}_{\beta_0}(z)$  are on the different sides of the graph of  $f_t$  for  $t \in [\beta_0/3, 2\beta_0/3]$ , from which one can define  $\tau(t, y)$ . Notice that f is  $C^1$  and X is  $C^1$ , we have that  $\tau(t, y)$  is  $C^1$  w.r.t. t and y.

For any  $T \ge \beta_0$ , let  $n = [3T/\beta_0]$  and a partition

$$0 = t_0 < t_1 < \dots < t_{n-1} < t_n = T,$$

such that  $t_i = i\beta_0/3$ , i = 0, 1, ..., n - 1. Then we have  $t_n - t_{n-1} \in [\beta_0/3, 2\beta_0/3]$ . Then we can define  $\beta_T$  inductively.

LEMMA 2.4. – Let  $X \in \mathcal{X}^1(M^d)$  and T > 0. By reducing  $\beta_T > 0$  as in Lemma 2.3 if necessary, for any  $x \in M^d \setminus \text{Sing}(X)$ , for the sectional Poincaré map

$$\mathscr{P}_{x,\phi_T(x)}: \mathscr{N}_x(\beta_T|X(x)|) \to \mathscr{N}_{\phi_T(x)}(\beta_*),$$

 $D \mathcal{P}_{x,\phi_T(x)}(y)$  is uniformly continuous in the following sense: for any  $\epsilon > 0$  there exists  $\delta \in (0, \beta_T]$  such that for any  $x \in M^d \setminus \operatorname{Sing}(X)$  and  $y, y' \in \mathcal{N}_x(\beta_T |X(x)|)$ , if  $|y - y'| \leq \delta |X(x)|$ , then

$$|D \mathcal{P}_{x,\phi_T(x)}(y) - D \mathcal{P}_{x,\phi_T(x)}(y')| < \epsilon.$$

(Note that  $D \mathcal{P}_{x,\phi_T(x)}(0) = \psi_T|_{\mathcal{N}_x}$ .) And hence there exists  $K_T > 0$  (independent of x) such that

$$|D \mathcal{P}_{x,\phi_T(x)}| \leq K_T.$$

*Proof.* – We will still use the notations and terminologies as in the proof of Lemma 2.3. We first assume that  $T < \beta_0$ . Notice that we are in a local Euclidean coordinate,  $N_z = {}_{\mathcal{O}} \mathcal{N}_z$  for each regular point z.

In the local coordinate, assume that the vector field  $\hat{X}$  has the following form  $\hat{X}(z, y) = (f(z, y), g(z, y))$ , where  $z \in \mathbb{R}^1$ ,  $y \in \mathbb{R}^{d-1}$ ,  $f : \mathbb{R}^d \to \mathbb{R}^1$  and  $g : \mathbb{R}^d \to \mathbb{R}^{d-1}$  are continuously  $C^1$  maps such that

- $\hat{X}(0,0) = (f(0,0), 0)$ , where f(0,0) > 0.
- for any (z, y) in the local coordinate, one has  $f(z, y) \in (0.999f(0, 0), 1.001f(0, 0))$  and |g(z, y)| < f(0, 0)/1000.

The flow of  $\hat{X}$  satisfies the following differential equations:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = f(z, y),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(z, y).$$

Assume that the solution of these differential equations is  $(\widehat{\varphi}_t(z, y), \widehat{\psi}_t(z, y))$ , where  $\widehat{\varphi}_t(z, y) : \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^1$  and  $\widehat{\psi}_t(z, y) : \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^{d-1}$ .

Now we consider the expression of the sectional Poincaré map  $\mathscr{P}_T : \mathscr{N}_x(\beta_0|X(x)|) \to \mathscr{N}_{\phi_T(x)}(\beta_*)$  in this local coordinate. The local coordinate of x is (0, 0). Thus  $\mathscr{N}_{(0,0)}(\beta_*) \subset \{0\} \times \mathbb{R}^{d-1}$ . Now we consider  $(\widehat{\varphi}_T(0, 0), \widehat{\psi}_T(0, 0))$ , whose normal manifold is contained in a graph of an affine map  $h : \mathbb{R}^{d-1} \to \mathbb{R}$  such that |Dh| < 0.001.

For  $y \in \mathbb{R}^{d-1}$ ,

$$\mathcal{P}(0, y) = \mathcal{P}(y) = (\widehat{\varphi}_{\tau}(0, y), \widehat{\psi}_{\tau}(0, y))^T$$

where the time function  $\tau$  :  $\mathbb{R}^{d-1} \to \mathbb{R}^1$  satisfies

$$\widehat{\varphi}_{\tau(y)}(0, y) = h(\widehat{\psi}_{\tau(y)}(0, y)).$$

By differentiating y in the above equality, one has

$$\frac{\partial\widehat{\varphi}}{\partial t}\Big|_{t=\tau(y)}\frac{\partial\tau}{\partial y}+\frac{\partial\widehat{\varphi}}{\partial y}=\frac{\partial h}{\partial y}(\frac{\partial\widehat{\psi}}{\partial t}\frac{\partial\tau}{\partial y}+\frac{\partial\widehat{\psi}}{\partial y}).$$

Notice that in the above equality,  $\partial \tau / \partial y$ ,  $\partial \hat{\varphi} / \partial y$  and  $\partial h / \partial y$  are row vectors with d - 1 elements,  $\partial \hat{\psi} / \partial t$  is a column vector with (d - 1) elements,  $\partial \hat{\psi} / \partial y$  is a  $(d - 1) \times (d - 1)$  matrix.

By solving the above equality, one has

$$\frac{\partial \tau}{\partial y} = \frac{\frac{\partial h}{\partial y} \frac{\partial \psi}{\partial y} - \frac{\partial \hat{\varphi}}{\partial y}}{\frac{\partial \hat{\varphi}}{\partial t} - \frac{\partial h}{\partial y} \frac{\partial \hat{\psi}}{\partial t}}$$

Thus,

$$\frac{\partial \mathscr{P}}{\partial y} = \begin{pmatrix} \frac{\partial \widehat{\varphi}}{\partial t} \frac{\partial \tau}{\partial y} + \frac{\partial \widehat{\varphi}}{\partial y} \\ \frac{\partial \widehat{\psi}}{\partial t} \frac{\partial \tau}{\partial y} + \frac{\partial \widehat{\psi}}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \widehat{\varphi}}{\partial t} \\ \frac{\partial \widehat{\psi}}{\partial t} \end{pmatrix} \frac{\frac{\partial h}{\partial y} \frac{\partial \psi}{\partial y} - \frac{\partial \widehat{\varphi}}{\partial y}}{\frac{\partial \widehat{\varphi}}{\partial t} - \frac{\partial h}{\partial y} \frac{\partial \widehat{\psi}}{\partial t}} + \begin{pmatrix} \frac{\partial \widehat{\varphi}}{\partial y} \\ \frac{\partial \widehat{\psi}}{\partial y} \end{pmatrix}.$$

By the expression of the differential equations, one has

$$\frac{\partial \mathcal{P}}{\partial y} = \widehat{X}(\mathcal{P}(y)) \frac{\frac{\partial h}{\partial y} \frac{\partial \psi}{\partial y} - \frac{\partial \widehat{\varphi}}{\partial y}}{\frac{\partial \widehat{\varphi}}{\partial t} - \frac{\partial h}{\partial y} \frac{\partial \widehat{\psi}}{\partial t}} + \begin{pmatrix} \frac{\partial \widehat{\varphi}}{\partial y} \\ \frac{\partial \widehat{\psi}}{\partial y} \end{pmatrix}.$$

In another form,

$$\frac{\partial \mathcal{P}}{\partial y} = \frac{\widehat{X} \circ \mathcal{P}(y)}{\widehat{f} \circ \mathcal{P}(y) - \frac{\partial h}{\partial y}\widehat{g} \circ \mathcal{P}(y)} \left(\frac{\partial h}{\partial y}\frac{\partial \widehat{\psi}}{\partial y} - \frac{\partial \widehat{\varphi}}{\partial y}\right) + \left(\frac{\frac{\partial \widehat{\varphi}}{\partial y}(0, y)}{\frac{\partial \widehat{\psi}}{\partial y}(0, y)}\right)$$

Since  $0.999 < |f|/|\widehat{X}| < 1.001$  and |g|/|X| < 0.001, there is a uniform constant  $\widehat{K} > 0$  such that

$$\left\|\frac{\partial \mathcal{P}}{\partial y}\right\| \leq \widehat{K}.$$

For any  $\varepsilon > 0$ , since the tangent flow  $\Phi_t$  is uniformly continuous, there is  $\delta > 0$  such that for any  $y, y' \in N_x(\beta_T | X(x) |)$ , if  $|y - y'| \le \delta |X(x)|$ , one has

$$\|\frac{\partial\widehat{\varphi}}{\partial y}(0,y) - \frac{\partial\widehat{\varphi}}{\partial y}(0,y')\| < \varepsilon/4, \quad \|\frac{\partial\widehat{\psi}}{\partial y}(0,y) - \frac{\partial\widehat{\psi}}{\partial y}(0,y')\| < \varepsilon/4.$$

Let 
$$\alpha(y) = \widehat{X} \circ \mathcal{P}(y)$$
 and  $\beta(y) = \widehat{f} \circ \mathcal{P}(y) - \frac{\partial h}{\partial y}(\widehat{g} \circ \mathcal{P}(y))$ , then we have  

$$\frac{\alpha(y)}{\beta(y)} - \frac{\alpha(y')}{\beta(y')} = \frac{\alpha(y) - \alpha(y')}{\beta(y)} + \frac{\beta(y') - \beta(y)}{\beta(y)\beta(y')}\alpha(y').$$

We have the following estimation by the mean value theorem:

$$\|\alpha(y) - \alpha(y')\| \le \|D\widehat{X}\| \|\mathscr{P}(y) - \mathscr{P}(y')\| \le \widehat{K} \|D\widehat{X}\| \|y - y'\|$$

By reducing  $\delta$  if necessary, for any  $y, y' \in N_x(\beta_0|X(x)|)$ , if  $|y - y'| \le \delta|X(x)|$ , one has

$$\left\|\frac{\alpha(y) - \alpha(y')}{\beta(y)}\right\| \le 1.001 \frac{\widehat{K} \|D\widehat{X}\|\delta|X(x)|}{|X(x)|} \le 2\delta\widehat{K}$$

We just need to choose  $\delta < \varepsilon/8\hat{K}$ 

$$\left\|\frac{\beta(y) - \beta(y')}{\beta(y)\beta(y')}\alpha(y')\right\| \le \frac{\|\alpha(y')\|}{|\beta(y')|} \frac{\widehat{K}\|D\widehat{f} + \widehat{K}D\widehat{g}\||y - y'|}{|\widehat{X}|} < \varepsilon/4,$$

if we reduce  $\delta$  again.

Let

$$F(y) = \frac{\widehat{X} \circ \mathcal{P}(y)}{\widehat{f} \circ \mathcal{P} - \partial h / \partial y} \widehat{g} \circ \mathcal{P}, \quad G(y) = \frac{\partial h}{\partial y} \frac{\partial \widehat{\psi}}{\partial y} - \frac{\partial \widehat{\varphi}}{\partial y}.$$

We know that F(y) is uniformly continuous and G(y) is uniformly continuous. Thus F(y)G(y) is uniformly continuous.

Combining all above estimations, we can know that  $\mathcal{P}_{x,\phi_T(x)}$  is uniformly continuous. For any  $T \ge \beta_0$ , let  $n = [3T/\beta_0]$  and a partition

$$0 = t_0 < t_1 < \dots < t_{n-1} < t_n = T,$$

such that  $t_i = i\beta_0/3$ , i = 0, 1, ..., n - 1. Then we have  $t_n - t_{n-1} \in [\beta_0/3, 2\beta_0/3]$ . Then by using the prolongation, we know the result is true.

Sometimes one needs to consider the *scaled sectional Poincaré map*  $\mathcal{P}^*$  which is defined in the following way:

$$\mathcal{P}_{x,\phi_T(x)}^*(y) = \frac{\mathcal{P}_{x,\phi_T(x)}(|X(x)|y)}{|X(\phi_T(x))|}$$

for each  $y \in \mathcal{N}_x(\beta_T)$ . Thus  $\mathcal{P}^*_{x,\phi_T(x)}$  is a map from  $\mathcal{N}_x(\beta_T)$  to  $\mathcal{N}_{\phi_T(x)}$ .

LEMMA 2.5. – Given  $X \in \mathcal{X}^1(M^d)$  and T > 0, there are constants  $\beta_T > 0$  and  $K_T > 0$ such that for any  $t \in (0, T)$  and any regular point  $x \in M^d$ ,

- 1.  $\mathcal{P}^*_{x,\phi_t(x)}$  can be defined on  $\mathcal{N}_x(\beta_T)$ .
- 2.  $D \mathscr{P}^*_{x,\phi_t(x)}$  is uniformly continuous: for any  $\varepsilon > 0$ , there is  $\delta > 0$ , such that for any  $y, z \in \mathscr{N}_x(\beta_T)$ , if  $d(y, z) < \delta$ , one has  $\|D \mathscr{P}^*_{x,\phi_t(x)}(y) D \mathscr{P}^*_{x,\phi_t(x)}(z)\| < \varepsilon$ .
- 3.  $D \mathcal{P}_{x,\phi_T(x)}^*(y)|_{y=0} = \psi_T^*(x).$
- 4.  $\|D \mathcal{P}^*_{x,\phi_t(x)}(y)\| \leq K_T \text{ for any } y \in \mathcal{N}_x(\beta_T).$

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Proof. - Item 1 is true because of Lemma 2.3.

For any  $y, z \in \mathcal{N}_x(\beta_T)$ , one has

$$D \mathcal{P}_{x,\phi_T(x)}^*(y) - D \mathcal{P}_{x,\phi_T(x)}^*(z) = \frac{|X(x)|}{|X(\phi_T(x))|} (D \mathcal{P}_{x,\phi_T(x)}(y|X(x)|) - D \mathcal{P}_{x,\phi_T(x)}(z|X(x)|)).$$

Since  $|X(x)|/|X(\phi_T(x))|$  is uniformly bounded, item 2 follows from Lemma 2.4.

For item 3, we have

$$D_{y} \mathcal{P}_{x,\phi_{T}(x)}^{*}(y) = D_{y} \left( \frac{\mathcal{P}_{x,\phi_{T}(x)}(y|X(x)|)}{|X(\phi_{T}(x))|} \right) = D_{y} \mathcal{P}_{x,\phi_{T}(x)}(y|X(x)|) \frac{|X(x)|}{|X(\phi_{t}(x))|}.$$

Thus, when y = 0, one has

$$D_{y} \mathcal{P}_{x,\phi_{T}(x)}^{*}(0) = D_{y} \mathcal{P}_{x,\phi_{T}(x)} \frac{|X(x)|}{|X(\phi_{T}(x))|} = \frac{\psi_{T}(x)}{\|\Phi_{T}\|_{< X(x)>}\|} = \psi_{T}^{*}(x),$$

where the second equality in the formula follows from Lemma 2.2.

Item 4 holds because  $\psi_T^*$  is uniformly bounded and item 2.

# 2.3. Franks' Lemma and dominated splittings

As in the diffeomorphism case, one needs Franks' lemma [16] to get some information on the derivative along periodic orbits. We state a version of Franks' lemma for flows which is taken from [11, Theorem A.1]. Liao also had a version by using his standard differential equations [24, Proposition 3.4].

LEMMA 2.6. – Given  $X \in \mathcal{X}^1(M^d)$  and a  $C^1$  neighborhood  $\mathcal{U} \subset \mathcal{X}^1(M)$  of X, there is a neighborhood  $\mathcal{V} \subset \mathcal{U}$  of X and  $\varepsilon > 0$  such that for any  $Y \in \mathcal{V}$ , for any periodic orbit  $\operatorname{Orb}(x)$  of Y with period  $T \ge 1$ , any neighborhood U of  $\operatorname{Orb}(x)$  and any partition of [0, T]:

$$0 = t_0 < t_1 < \dots < t_l = T, \quad 1 \le t_{i+1} - t_i \le 2, \ i = 0, 1, \dots, l-1,$$

and any linear isomorphisms  $L_i$ :  $\mathcal{N}_{\phi_{l_i}^Y(x)} \to \mathcal{N}_{\phi_{l_{i+1}}^Y(x)}, i = 0, 1, \dots, l-1$ with  $||L_i - \psi_{t_{i+1}-t_i}^Y|_{\mathcal{N}_{\phi_{l_i}(x)}}|| \leq \varepsilon$ , there exists  $Z \in \mathcal{U}$  such that  $\psi_{t_{i+1}-t_i}^Z|_{\mathcal{N}_{\phi_{l_i}(x)}} = L_i$ and Z = Y on  $(M^d \setminus U) \cup \operatorname{Orb}(x)$ .

REMARK. – For simplicity, the time length of the partition is restricted to [1, 2]. We remark that this is not a serious restriction. Sometimes, the system may contain periodic orbits with period less than 1. But in our consideration, usually singularities are all hyperbolic. So there is a lower bound for the periods of periodic orbits. And after scaling, we may assume that the lower bound is 1.

In this paper, we will use two estimations obtained by Franks' lemma: dominated splittings for vector fields away from homoclinic tangencies and uniform estimation along a sequence of periodic orbits restricted on the stable bundle.

2.3.0.1. Dichotomy for the stable bundle: – For any hyperbolic periodic orbit  $\gamma$ ,  $\mathcal{N}_{\gamma}$  admits a natural splitting  $\mathcal{N}_{\gamma} = \mathcal{N}^s \oplus \mathcal{N}^u$  such that  $\mathcal{N}^s$  is contracting and  $\mathcal{N}^u$  is expanding w.r.t. the linear Poincaré flow  $\psi_t$ .

By using the methods of periodic linear systems as in [7, 11, 30, 52], one can have the following dichotomy result for a hyperbolic periodic orbit with large period. An abstract version for diffeomorphisms can be found in [30, Lemma II.3 and Lemma II.5].

LEMMA 2.7. – Given  $X \in \mathcal{X}^1(M^d)$ , for any  $C^1$  neighborhood  $\mathcal{U}$  of X, there are  $C_{\mathcal{U}} > 0$ ,  $\eta_{\mathcal{U}} > 0$  and  $\iota_{\mathcal{U}} > 0$  and a neighborhood  $\mathcal{V} \subset \mathcal{U}$  of X such that for any hyperbolic periodic orbit  $\gamma$  of index i of  $Y \in \mathcal{V}$  with  $\tau(\gamma) > \iota_{\mathcal{U}}$ , then

- either, there is  $Z \in \mathcal{U}$  such that  $\gamma$  is a hyperbolic periodic orbit of Z of index i 1;
- or, for any  $x \in \gamma$ , for any time partition

$$0 = t_0 < t_1 < \cdots < t_n = \tau(\gamma),$$

verifying  $t_{i+1} - t_i \ge \iota_{\mathcal{U}}$  for  $0 \le i \le n-1$ , one has  $\prod_{i=0}^{n-1} \|\psi_{t_{i+1}-t_i}\|_{\mathcal{N}^s(\phi_{t_i}(x))}\| \le C_{\mathcal{U}} \exp\{-\eta_{\mathcal{U}}\tau(\gamma)\}.$ 

DEFINITION 2.8. – Let C > 0,  $\eta > 0$  and T > 0. For a hyperbolic periodic orbit  $\gamma$ , for a  $\psi_t$ -invariant bundle  $E \subset \mathcal{N}_{\gamma}$ , we say that  $\gamma$  is called  $(C, \eta, T, E)$ -contracting at the period (w.r.t.  $\psi_t$ ) if there is  $m \in \mathbb{N}$ , and for any  $x \in \gamma$ , there is some time partition

$$0 = t_0 < t_1 < \cdots < t_n = m\tau(\gamma),$$

with  $t_{i+1} - t_i \leq T$  for  $0 \leq i \leq n - 1$ , such that

$$\prod_{i=0}^{n-1} \|\psi_{t_{i+1}-t_i}\|_{E(\phi_{t_i}(x))}\| \leq C \exp\{-\eta m\tau(\gamma)\}.$$

For an  $\psi_t$ -invariant subbundle  $F \subset \mathcal{N}_{\gamma}$ , we say that  $\gamma$  is called  $(C, \eta, T, F)$ -expanding at the period if it is  $(C, \eta, T, F)$ -contracting at the period for -X.

**REMARK.** – Since for a periodic orbit  $\gamma$ , for any  $x \in \gamma$ , one has  $\Phi_{\tau(\gamma)}X(x) = X(\phi_{\tau(\gamma)}(x)) = X(x)$ , one can give the above definition by using  $\psi_t^*$ .

COROLLARY 2.9. – Given  $X \in \mathcal{X}^1(M^d)$ , assume that  $\Lambda$  is a compact invariant set and not reduced to a critical element. If there is a sequence of vector fields  $\{X_n\}$  such that

- $-\lim_{n\to\infty}X_n=X,$
- each  $X_n$  has a sink  $\gamma_n$  such that  $\lim_{n\to\infty} \gamma_n = \Lambda$ ,

then one has the following dichotomy:

- either, there is a sequence of vector fields  $\{Y_n\}$  such that  $\lim_{n\to\infty} Y_n = X$ , and  $\gamma_n$  is a hyperbolic periodic orbit of  $Y_n$  of index  $d 2 = \dim_{\mathbb{C}} \mathcal{N} 1$ .
- or, there are C > 0,  $\eta > 0$  and T > 0 such that for n large enough,  $\gamma_n$  is a  $(C, \eta, T, \mathcal{N})$ -contracting at the period w.r.t.  $\psi_t^{Y_n}$ .

*Proof.* – Note that  $\lim_{n\to\infty} \tau(\gamma_n) = \infty$ . If the "either" case is not true, then by Lemma 2.7, one can get constants  $\eta_{\mathcal{U}}$  and  $\iota_{\mathcal{U}}$ . Then the "or" case is true by taking  $C = C_{\mathcal{U}}, \eta = \eta_{\mathcal{U}}$  and  $T = 2\iota_{\mathcal{U}}$ .

2.3.0.2. Vector fields away from homoclinic tangencies:- For any invariant set  $\Lambda$  (maybe not compact) without singularities, if there are  $\iota > 0$  and an invariant continuous splitting<sup>(5)</sup>  $\mathscr{N}_{\Lambda} = \mathscr{N}^{cs} \oplus \mathscr{N}^{cu}$  w.r.t.  $\psi_t$  satisfying  $\|\psi_t\|_{\mathscr{N}^{cs}(x)} \|\|\psi_{-\iota}\|_{\mathscr{N}^{cu}(\phi_t(x))}\| \le 1/2$  for any  $x \in \Lambda$ , then we say that  $\Lambda$  admits a  $\iota$ -dominated splitting w.r.t.  $\psi_t$ . If dim  $\mathscr{N}^{cs}(x)$  is independent of x, then it is called the *index* of this dominated splitting. Note that for any linear flow defined on some linear bundle, one can define the notion of dominated splitting for that linear flow. Recall that

$$\mathcal{HT} = \{X \in \mathcal{X}^1(M^d) : X \text{ has a homoclinic tangency}\}$$

By the similar arguments as diffeomorphisms and by using Franks' lemma for flows, from [52, 53], we have

LEMMA 2.10. – For any  $X \in \mathcal{X}^1(M^d) \setminus \overline{\mathcal{HT}}$ , there is a  $C^1$  neighborhood  $\mathcal{U}$  and constants C > 0,  $\lambda > 0$ ,  $\delta > 0$  and  $\iota > 0$  such that for any periodic orbit  $\gamma$  of  $Y \in \mathcal{U}$  with period  $\pi(\gamma) > \iota$ .

- $\Phi_{\pi(\gamma)}$  has at most two exponents in  $(-\delta, \delta)$ , and  $\Phi_{\pi(\gamma)}$  has at least one zero exponent, and this exponent corresponds to the flow direction.
- There is an invariant splitting  $\mathscr{N}_{\gamma} = G^s \oplus G^c \oplus G^u$  with respect to the linear Poincaré flow  $\psi_t^Y$ , where  $G^s$  is the invariant space corresponding to the exponents less than  $-\delta$ ,  $G^c$  is the invariant space corresponding to the exponents in  $(-\delta, \delta)$  and the dimension of  $G^c$  is zero or one,  $G^u$  is the invariant space corresponding to the exponents larger than  $\delta$ ; moreover for any  $x \in \gamma$ , for any time partition

$$0 = t_0 < t_1 < \cdots < t_n = \tau(\gamma),$$

*verifying*  $t_{i+1} - t_i \ge \iota$  *for*  $0 \le i \le n - 1$ *, one has* 

$$\prod_{i=0}^{n-1} \|\psi_{t_{i+1}-t_i}\|_{G^s(\phi_{t_i}(x))}\| \le C \exp\{-\lambda \tau(\gamma)\},$$
$$\prod_{i=0}^{n-1} \|\psi_{t_i-t_{i+1}}\|_{G^u(\phi_{t_{i+1}}(x))}\| \le C \exp\{-\lambda \tau(\gamma)\}.$$

- If  $\gamma$  is hyperbolic, and  $\mathcal{N}_{\gamma} = \mathcal{N}^s \oplus \mathcal{N}^u$  is the hyperbolic splitting with respect to  $\psi_t^Y$ , then for any  $x \in \gamma$  and  $T > \iota$ , one has

$$\left\|\psi_{T}^{Y}\right\|_{\mathcal{N}^{s}(x)}\left\|\left\|\psi_{-T}^{Y}\right\|_{\mathcal{N}^{u}(\phi_{T}^{Y}(x))}\right\| \leq \frac{1}{2}.$$

COROLLARY 2.11. – Let  $X \in \mathcal{X}^1(M^d) \setminus \overline{\mathcal{HT}}$ . Assume that

- there is a sequence of vector fields  $\{X_n\}$  such that  $\lim_{n\to\infty} X_n = X$ ,
- each  $X_n$  has a hyperbolic periodic orbit  $\gamma_n$  of index *i* such that  $\Lambda = \lim_{n \to \infty} \gamma_n$  in the Hausdorff topology.

Then

-  $\mathcal{N}_{\Lambda \setminus \operatorname{Sing}(X)}$  admits a dominated splitting of index *i* with respect to the linear Poincaré flow  $\psi_t$ ,

<sup>&</sup>lt;sup>(5)</sup> Notice that the continuity may be only held on some non-compact set.

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- $\mathcal{N}_{\Lambda \setminus \operatorname{Sing}(X)}$  admits a dominated splitting of index *i* with respect to the scaled linear Poincaré flow  $\psi_t^*$ ,
- if one considers  $\widetilde{\Lambda}$ , then  $\widetilde{\mathcal{N}}_{\widetilde{\Lambda}}$  admits a dominated splitting of index *i* with respect to the flow  $\widetilde{\psi}_t$ .

*Proof.* – Since  $X \in \mathcal{X}^1(M^d) \setminus \overline{\mathcal{HT}}, X_n \to X$  and  $\gamma_n$  is a hyperbolic periodic orbit of  $X_n$  of index *i*,

$$\mathscr{N}_{\gamma_n} = \mathscr{N}^s(\gamma_n) \oplus \mathscr{N}^u(\gamma_n)$$

is an *i*-dominated splitting of index *i* w.r.t.  $\psi_t^{X_n}$  for some uniform  $\iota > 0$ .

For each  $x \in \Lambda \setminus \text{Sing}(X)$ , by taking a subsequence if necessary, one can assume that there is  $x_n \in \gamma_n$  such that  $\lim_{n\to\infty} x_n = x$ . After taking another subsequence, one can assume that  $\mathcal{N}^{cs}(x) = \lim_{n\to\infty} \mathcal{N}^{s}(x_n)$  and  $\mathcal{N}^{cu}(x) = \lim_{n\to\infty} \mathcal{N}^{u}(x_n)$ .

Thus  $\mathcal{ON}_{\Lambda \setminus Sing(X)} = \mathcal{ON}^{cs} \oplus \mathcal{ON}^{cu}$  is an *i*-dominated splitting of index *i*. One can see [23] for more details.

Since

$$\psi_t^*(x) = \frac{\psi_t(x)}{\|\Phi_t\|_{< X(x)>}}\|,$$

any dominated splitting of  $\psi_t$  is also a dominated splitting of  $\psi_t^*$ .

The dominated splitting of the linear Poincaré flow can be extended to the closure of its representation in the sphere bundle. See [23] for more details.  $\Box$ 

LEMMA 2.12. – For every  $X \in \mathcal{X}^1(M^d) \setminus \overline{\mathcal{HT}}$ , there are  $\iota > 0$ , C > 0,  $\eta > 0$  and a  $C^1$  neighborhood  $\mathcal{U}$  of X such that for any  $Y \in \mathcal{U}$ , if  $\gamma$  is a periodic sink of Y with period  $\tau(\gamma) > \iota$ , then

- either,  $\mathcal{N}_{\gamma}$  admits an  $\iota$ -dominated splitting of index d-2 with respect to  $\psi_t^Y$ ,
- or  $\gamma$  is  $(C, \eta, 2\iota, \mathcal{N})$ -contracting at the period w.r.t.  $\psi_t^Y$ .

*Proof.* – Let *C* and *i* be as in Lemma 2.10. If the conclusion is not true, there exist  $\eta_n \to 0$ ,  $X_n \to X$  and a periodic sink  $\gamma_n$  of  $X_n$  with  $\tau(\gamma_n) > \iota$ , neither item 1 nor item 2 is satisfied. Then according to Franks' Lemma, after a small perturbation of  $X_n$  of size  $\eta_n$ , we get a  $Y_n$  such that  $\gamma_n$  is a periodic orbit of index d-2. Since  $Y_n \to X$ , for *n* large enough,  $\psi_t^{Y_n}$  has an  $\iota$ -dominated splitting over  $\gamma_n$  of index d-2, and then we get a dominated splitting for the extended linear Poincaré flow over the limit. But the limits of  $X_n$  and  $Y_n$  are the same since  $X_n|_{\gamma_n} = Y_n|_{\gamma_n}$ . By the continuity of dominated splitting of the extended linear Poincaré flow, we know that for *n* large enough,  $X_n$  has also an  $\iota$ -dominated splitting over  $\gamma_n$  of index d-2, which gives a contradiction.

### 2.4. Mixed dominated splittings: from linear Poincaré flow to tangent flow

For two linear normed spaces E and F, and a linear operator  $A : E \to F$ , the mini-norm m(A) is defined by

$$m(A) = \inf_{v \in E, \ |v|=1} |Av|.$$

We use L(E, F) to denote the space of bounded linear maps from E to F.

The following lemma concerns how we can get the dominated splitting of the tangent flow from the dominated splitting of the linear Poincaré flow. This kind of ideas is also used in [23, Lemma 5.5, Lemma 5.6]. Here we give a general version.

LEMMA 2.13. – Let  $\widehat{\Lambda} \subset SM^d$  be a compact invariant set of  $\Phi_t^I$ . Suppose –  $\widetilde{\mathcal{N}}_{\widehat{\Lambda}} = \Delta^{cs} \oplus \Delta^{cu}$  is a dominated splitting w.r.t.  $\widetilde{\psi}_t$ . – There are C > 0 and  $\lambda > 0$  such that for any  $u \in \widehat{\Lambda}$ , for any t > 0, one has  $\frac{\|\widetilde{\psi}_t\|_{\Delta^{cs}(u)}\|}{\|\Phi_t(u)\|} \leq Ce^{-\lambda t}.$ 

Then the projection 
$$\pi(\widehat{\Lambda})$$
 admits a dominated splitting  $T_{\pi(\widehat{\Lambda})}M^d = E \oplus F$  w.r.t the tangent flow  $\Phi_t$ , where dim  $E = \dim \Delta^{cs}$ .

*Proof.* – For each point  $u \in \widetilde{\Lambda} \subset SM^d$ , the direct-sum splitting  $T_{\pi(u)}M^d = \Delta^{cs} \oplus \langle u \rangle \oplus \Delta^{cu}$  is continuous w.r.t. u. With respect to this decomposition, the tangent flow  $\Phi_T$  has the following form:

$$\begin{pmatrix} \widetilde{\psi}_T|_{\Delta^{cs}(u)} & 0 & 0\\ B(u) & \Phi_T|_{} & C(u)\\ 0 & 0 & \widetilde{\psi}_T|_{\Delta^{cu}(u)} \end{pmatrix}$$

By the definitions of  $\chi_t$  and  $\widetilde{\psi}_t$  (see Subsection 2.1), one has that  $F(u) = \langle u \rangle \oplus \Delta^{cu}(u)$  is an invariant sub-bundle of  $\Phi_t$ . Let's find another invariant sub-bundle of  $\Phi_t$ .

CLAIM. – There is  $C_1 > 0$  and  $\lambda_1 > 0$  such that for any  $u \in \widehat{\Lambda}$  and any  $t \ge 0$ , one has

$$\frac{\|\overline{\psi}_t\|_{\Delta^{cs}(u)}\|}{m(\Phi_t)_{F(u)}} \le C_1 \mathrm{e}^{-\lambda_1 t}.$$

*Proof of the claim.* – By enlarging T if necessary, one can assume that for any  $u \in \widehat{\Lambda} \subset SM^d$ , one has

$$\frac{\|\psi_{T}\|_{\Delta^{cs}(u)}\|}{\|\Phi_{T}\|_{\leq u>}\|} \leq \frac{1}{2}, \quad \frac{\|\psi_{T}\|_{\Delta^{cs}(u)}\|}{m(\widetilde{\psi}_{T})_{\Delta^{cu}(u)}} \leq \frac{1}{2}.$$

Since  $\Phi_T$  is bounded, by the continuity of the splitting, there is K > 0 such that  $||\Phi_T|| \le K$ and  $m(\Phi_T) \ge 1/K$ . Denote by

$$D(u) = \begin{pmatrix} \Phi_T |_{} & C(u) \\ 0 & \widetilde{\psi}_T |_{\Delta^{cu}(u)} \end{pmatrix}.$$

For any  $n \in \mathbb{N}$ , one has

$$\Phi_{-nT}|_{F(\Phi_{nT}^{I}(u))} = \prod_{i=0}^{n-1} D^{-1}(\Phi_{iT}^{I}(\Phi_{nT}^{I}(u))) = D^{-n}(\Phi_{nT}^{I}(u)).$$

Since

$$D^{n}(u) = \begin{pmatrix} \Phi_{nT}|_{} \sum_{i=0}^{n-1} \Phi_{(n-1-i)T}|_{<\Phi^{I}_{(i+1)T}(u)>} C(\Phi^{I}_{iT}(u))\widetilde{\psi}_{iT}|_{\Delta^{cu}(u)} \\ 0 \qquad \qquad \widetilde{\psi}_{nT}|_{\Delta^{cu}(u)} \end{pmatrix},$$

we have

$$D^{-n}((\Phi_{nT}^{I}(u))) = \begin{pmatrix} \Phi_{-nT}|_{<\Phi_{nT}^{I}(u)>} \sum_{i=0}^{n-1} \Phi_{(-1-i)T}|_{<\Phi_{(i+1)T}^{I}(u)>} C(\Phi_{iT}^{I}(u))\widetilde{\psi}_{(i-n)T}|_{\Delta^{cu}(\Phi_{nT}^{I}(u))} \\ 0 \qquad \widetilde{\psi}_{-nT}|_{\Delta^{cu}(\Phi_{nT}^{I}(u))} \end{pmatrix}.$$

This implies

$$\begin{split} \|D^{-n}|_{\Phi_{nT}^{I}(u)}\| &\leq \|\Phi_{-nT}|_{\langle\Phi_{nT}^{I}(u)\rangle}\| + \|\widetilde{\psi}_{-nT}|_{\Delta^{cu}(\Phi_{nT}^{I}(u))}\| \\ &+ \|\sum_{i=0}^{n-1} \Phi_{(-1-i)T}|_{\langle\Phi_{(i+1)T}^{I}(u)\rangle} C(\Phi_{iT}^{I}(u))\widetilde{\psi}_{(i-n)T}|_{\Delta^{cu}(\Phi_{nT}^{I}(u))}\| \\ &\leq 2 \cdot 2^{-n} \frac{1}{\|\widetilde{\psi}_{nT}|_{\Delta^{cs}(u)}\|} + K \sum_{i=0}^{n-1} \|\Phi_{(-1-i)T}|_{\langle\Phi_{(i+1)T}^{I}(u)\rangle}\| \|\widetilde{\psi}_{(i-n)T}|_{\Delta^{cu}(\Phi_{nT}^{I}(u))}\| \\ &\leq \frac{2^{-n+1}}{\|\widetilde{\psi}_{nT}|_{\Delta^{cs}(u)}\|} + K \sum_{i=0}^{n-1} \frac{2^{-(i+1)}}{\|\widetilde{\psi}_{(i+1)T}|_{\Delta^{cs}(u)}\|} \frac{2^{i-n}}{\|\widetilde{\psi}_{(n-i)T}|_{\Delta^{cs}(\Phi_{iT}^{I}(u))}\|} \\ &\leq \frac{2^{-n+1}}{\|\widetilde{\psi}_{nT}|_{\Delta^{cs}(u)}\|} + K^{2} \sum_{i=0}^{n-1} \frac{2^{-n-1}}{\|\widetilde{\psi}_{iT}|_{\Delta^{cs}(u)}\| \|\widetilde{\psi}_{(n-i)T}|_{\Delta^{cs}(\Phi_{iT}^{I}(u))}\|} \\ &\leq \frac{2^{-n+1}}{\|\widetilde{\psi}_{nT}|_{\Delta^{cs}(u)}\|} + K^{2} \frac{n2^{-n-1}}{\|\widetilde{\psi}_{nT}|_{\Delta^{cs}(u)}\|}. \end{split}$$

Thus, when *n* large enough, one has

$$\frac{\|\widetilde{\psi}_{nT}|_{\Delta^{cs}(u)}\|}{m(\Phi_{nT}|_{F(u)})} \leq \frac{1}{2}.$$

This inequality implies the claim.

Now we will start to find a  $\Phi_t$ -invariant bundle E(u) and T' > 0 such that for any  $u \in \widehat{\Lambda}$ ,

$$\frac{\|\Phi_{T'}\|_{E(u)}\|}{m(\Phi_{T'}\|_{F(u)})} \leq \frac{1}{2}.$$

By the claim above, there is  $T_0 > 0$  such that

$$\frac{\|\widetilde{\psi}_{T_0}|_{\Delta^{cs}(u)}\|}{m(\Phi_{T_0}|_{F(u)})} \leq \frac{1}{2}.$$

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Let  $L(\widehat{\Lambda}) = \prod_{u \in \widehat{\Lambda}} L(\Delta^{cs}(u), F(u))$ . For each  $\Pi \in L(\widehat{\Lambda})$ , one can define the norm  $\|\Pi\| = \sup_{u \in \widehat{\Lambda}} \|\Pi(u)\|$ . Under this norm, one knows that  $L(\widehat{\Lambda})$  is a Banach space. For any  $\Pi \in L(\widehat{\Lambda})$ , one has

$$\begin{pmatrix} \widetilde{\psi}_{T_0|_{\Delta^{cs}(u)}} & 0\\ B(u) & \Phi_{T_0|_{F(u)}} \end{pmatrix} \begin{pmatrix} \Delta^{cs}(u)\\ \Pi(u)\Delta^{cs}(u) \end{pmatrix} = \begin{pmatrix} \widetilde{\psi}_{T_0|_{\Delta^{cs}(u)}}\Delta^{cs}(u)\\ B(u)\Delta^{cs}(u) + \Phi_{T_0|_{F(u)}}\Pi(u)\Delta^{cs}(u) \end{pmatrix}.$$

Thus, if we want to find an invariant bundle w.r.t.  $\Phi_t$ , we need to require that

$$B(u)\Delta^{cs}(u) + \Phi_{T_0|_{F(u)}}\Pi(u)\Delta^{cs}(u) = \Pi(\Phi^I_{T_0}(u))\widetilde{\psi}_{T_0|_{\Delta^{cs}(u)}}\Delta^{cs}(u).$$

In the spirit of the above equality, one can define a map  $\mathcal{F} : L(\widehat{\Lambda}) \to L(\widehat{\Lambda})$  by the following form:

$$\mathcal{F}\Pi(u) = (\Phi_{-T_0}|_{F(\Phi_{T_0}^I u)})(\Pi(\Phi_{T_0}^I(u))\widetilde{\psi}_{T_0}|_{\Delta^{cs}(u)} - B(u)).$$

Given  $\Pi_1, \Pi_2 \in L(\widehat{\Lambda})$ , one has

$$\mathcal{F}\Pi_1 - \mathcal{F}\Pi_2 = (\Phi_{-T_0}|_{F(\Phi_{T_0}^I u)})(\Pi_1 - \Pi_2)(\widetilde{\psi}_{T_0}|_{\Delta^{cs}(u)}).$$

Thus,

$$\|\mathscr{F}\Pi_{1} - \mathscr{F}\Pi_{2}\| \le \|\Phi_{-T_{0}}|_{F(\Phi_{T_{0}}^{I}u)}\|\|\Pi_{1} - \Pi_{2}\|\|\widetilde{\psi}_{T_{0}}|_{\Delta^{cs}(u)}\| \le \frac{1}{2}\|\Pi_{1} - \Pi_{2}\|.$$

So,  $\mathcal{F}$  is a contracting map. By the contraction mapping principle,  $\mathcal{F}$  has a unique fixed point  $\Pi \in L(\widehat{\Lambda})$ , i.e.,

$$B(u)\Delta^{cs}(u) + \Phi_{T_0|_{F(u)}}\Pi(u)\Delta^{cs}(u) = \Pi(\Phi^{I}_{T_0}(u))\widetilde{\psi}_{T_0|_{\Delta^{cs}(u)}}\Delta^{cs}(u)$$

As a corollary,  $E = (id, \Pi) \Delta^{cs}$  is an invariant bundle of  $\Phi_{T_0}$ .

Since E, F and  $\Delta^{cs}$  are continuous bundles w.r.t.  $u \in \widehat{\Lambda}$ , there is L > 0, which depends on the angles between each two bundles, such that for any non-zero vector  $v_E \in E$ ,  $v^{cs} \in \Delta^{cs}$ , if  $v_E = (id, \Pi)v^{cs}$ , then

$$v_E| \le \frac{1}{L} |v^{cs}|.$$

Thus, for each n, one has

$$\|\Phi_{nT_0}\|_{E(u)}\| \leq \frac{1}{L} \|\widetilde{\psi}_{nT_0}\|_{\Delta^{cs}(u)}\| \leq \frac{1}{L} \frac{1}{2^n} m(\Phi_{nT_0}\|_{F(u)}).$$

From these, we get a  $\Phi_t$ -invariant splitting  $E \oplus F$  over  $\Lambda$ , which satisfies the condition of the dominated splitting.

#### 2.5. The existence of invariant manifolds

Assume that  $\Lambda$  is a compact invariant set and  $\mathscr{N}_{\Lambda \setminus \operatorname{Sing}(X)}$  admits a dominated splitting with respect to the linear Poincaré flow. If  $\Lambda \cap \operatorname{Sing}(X) = \emptyset$ , then  $\Lambda$  will have a plaque family ([20, Theorem 5.5]) as in the case of diffeomorphisms. If  $\Lambda \cap \operatorname{Sing}(X) \neq \emptyset$ , then  $\Lambda$  won't have uniform size of plaque family: the plaque family is defined on a non-compact set and the size is scaled by the norm of the vector field.

The scaled Poincaré sectional map  $\mathcal{P}^*$  can be defined in a uniform neighborhood of the zero section of  $\mathcal{N}$ . Moreover, we have uniform estimations on  $D \mathcal{P}^*_{x,\phi_T(x)}(y)$  by Lemma 2.5. For getting plaque families of dominated splittings, one needs the following abstract lemma.

2.5.0.1. Notation. For a linear normed space A and r,  $A(r) = \{v \in A, |v| \le r\}$ .

LEMMA 2.14. – For any  $d \in \mathbb{N}$ , L > 0, r > 0, and  $\alpha > 0$ , there is  $\gamma_0 > 0$ ,  $\varepsilon_0 > 0$  such that: for any  $\gamma \leq \gamma_0$ , there exists  $\delta > 0$ , if a sequence of  $C^1$  diffeomorphisms

$$f_i: \mathbb{R}^d(r) \to \mathbb{R}^d, \ i \in \mathbb{Z}$$

satisfies the following properties:

1.  $f_i(0) = 0$ ,

- 2.  $\sup_{i \in \mathbb{Z}} \max\{|Df_i(0)|, |Df_i^{-1}(0)|\} \le L$
- 3. There is a sequence of invariant decompositions  $\mathbb{R}^d = E_i \oplus F_i$  with the following properties:
  - $Df_i(0)(E_i) = E_{i+1}, Df_i(0)(F_i) = F_{i+1},$
  - $\angle (E_i, F_i) > \alpha$ ,  $\|Df_i(0)\|_{E_i} \| = 1$

• 
$$\overline{m(Df_i(0)|_{F_i})} \geq \overline{2}$$
.

4. Lip $(f_i - Df_i(0)) < \varepsilon_0$ .

Then there are two sequences of embedding maps  $\phi_i^{cs} \in \operatorname{Emb}(E_i(\gamma), F_i)$  and  $\phi_i^{cu} \in$  $\operatorname{Emb}(F_i(\gamma), E_i)$  such that

- $\phi_i^{cs/cu}(0) = 0, \ D\phi_i^{cs/cu}(0) = 0,$
- $f_i W_i^{cs/cu}(\delta) \subset W_{i+1}^{cs/cu}(\gamma)$ , where  $W_i^{cs}(\gamma)$  is the graph of  $\phi_i^{cs}$  restricted to  $E_i(\gamma)$  and  $W_i^{cu}(\gamma)$  is the graph of  $\phi_i^{cu}$  restricted to  $F_i(\gamma)$ .

Moreover, the invariant manifolds are continuous with respect to the sequence of  $f = (f_i)$ . Precisely, for two sequences  $f = (f_i), g = (g_i)$ , define their metric as

$$|f - g|_{C^1} = \sum_{i=-\infty}^{\infty} \frac{|f_i - g_i|_{C^1}}{2^{|i|}}.$$

Then both  $W_i^{cs}(\gamma, f)$  and  $W_i^{cu}(\gamma, f)$  are continuous with respect to f, i.e., for every  $i \in \mathbb{Z}$ , if  $f^{(n)} \to f$ ,  $x_n \in W_i^{cs}(\gamma, f^{(n)})$ ,  $x_n \to x$ , then  $x \in W_i^{cs}(\gamma, f)$ , and  $T_{x_n}W_i^{cs}(\gamma, f^{(n)}) \to$  $T_x W_i^{cs}(\gamma, f).$ 

The proof of Lemma 2.14 needs to adapt the argument of [20, Theorem 5.5]. We omit the proof here.

For diffeomorphisms, we know that plaque family of compact invariant set with dominated splittings exists. For vector fields, if a compact invariant singular set has a dominated splitting w.r.t. the linear Poincaré flow, we also have some similar results, but the form is changed: one should modify the size of the manifolds. Recall that  $P_{x,\phi_t(x)}$  is the sectional Poincaré map between  $N_x$  and  $N_{\phi_t(x)}$ .

LEMMA 2.15. – Let  $\Lambda$  be a compact invariant set of  $X \in \mathcal{X}^1(M^d)$ . Assume that  $\Lambda \setminus \operatorname{Sing}(X)$  admits a dominated splitting  $\mathscr{N}_{\Lambda \setminus \operatorname{Sing}(X)} = \Delta^{cs} \oplus \Delta^{cu}$  of index i with respect to the linear Poincaré flow  $\psi_t$ . For any T > 0, there exists  $\xi_0 > 0$  and two families of continuous  $C^1$  maps  $\eta^{cs}(x) : \Delta^{cs}(x)(\xi_0) \to \mathcal{N}(x)$  and  $\eta^{cu}(x) : \Delta^{cu}(x)(\xi_0) \to \mathcal{N}(x)$ ,  $x \in \Lambda \setminus \operatorname{Sing}(X)$ , verifying the following properties:

1.  $\eta^{cs}(x)(0_x) = 0_x$  and  $\eta^{cu}(x)(0_x) = 0_x$ , where  $0_x$  is the origin of  $T_x M$ .

2.  $\eta^{cs}(x)$  and  $\eta^{cu}(x)$  are  $C^1$  embeddings.

3.  $D\eta^{cs}(x)$  and  $D\eta^{cu}(x)$  are continuous: for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that if  $d(x, x') < \delta$  and  $d(y, y') < \delta$ ,  $d(z, z') < \delta$  then

$$|D\eta^{cs}(x)(y) - D\eta^{cs}(x')(y')| < \varepsilon, \quad |D\eta^{cu}(x)(z) - D\eta^{cu}(x')(z')| < \varepsilon,$$

where  $y, y' \in \Delta^{cs}(x)(\xi_0), z, z' \in \Delta^{cu}(x)(\xi_0)$ .

4. For any  $\xi \in (0, \xi_0]$ , we define two sub-manifolds by

$$W_{\xi|X(x)|}^{cs}(x) = \exp_{x}(|X(x)|\eta^{cs}(x)(\Delta^{cs}(x)(\xi)))$$
  
and  $W_{\xi|X(x)|}^{cu}(x) = \exp_{x}(|X(x)|\eta^{cu}(x)(\Delta^{cu}(x)(\xi))),$ 

then one has

- $T_x W^{cs}_{\xi|X(x)|}(x) = \Delta^{cs}(x)$  and  $T_x W^{cu}_{\xi|X(x)|}(x) = \Delta^{cu}(x)$ , for any  $\varepsilon > 0$ , there is  $\delta > 0$  such that for any regular point  $x \in \Lambda$ , one has  $P_{x,\phi_T(x)}(W^{cs}_{\delta|X(x)|}(x)) \subset W^{cs}_{\varepsilon|X(\phi_T(x))|}(\phi_T(x))$  and  $P_{x,\phi_T(x)}(W^{cu}_{\delta|X(x)|}(x)) \subset$  $W^{cu}_{\varepsilon|X(\phi_T(x))|}(\phi_T(x)).$

*Proof.* – We will mainly use Lemma 2.14 to prove this lemma. For each point x,  $\mathcal{N}_x$  is isomorphic to  $\mathbb{R}^{d-1}$ . Since  $\Lambda \setminus \text{Sing}(X)$  admits a dominated splitting of index *i* w.r.t. the linear Poincaré flow, there is T > 0 such that

$$\frac{\|\psi_T^*|_{\Delta^{cs}(x)}\|}{m(\psi_T^*|_{\Delta^{cu}(x)})} \le \frac{1}{2}, \quad \forall x \in \Lambda \setminus \operatorname{Sing}(X).$$

For each  $i \in \mathbb{Z}$ , one takes  $f_i = \mathcal{P}^*_{\phi_{iT}(x),\phi_{(i+1)T}(x)}$  on  $\mathcal{N}_{\phi_{iT}(x)}$ . By Lemma 2.5, all assumptions of Lemma 2.14 are satisfied. Then by Lemma 2.14, we get the existence of plaque family:  $\eta^{cs/cu}(x) = \phi_0^{cs/cu}$ .

 $W^{cs}(x)$  and  $W^{cu}(x)$  are called *central stable plaques* and *central unstable plaques* respectively.

COROLLARY 2.16. – Let  $\Lambda$  be a compact invariant set of  $X \in \mathcal{X}^1(M^d)$ . Assume that  $\Lambda \setminus \operatorname{Sing}(X)$  admits a dominated splitting  $\mathcal{N}_{\Lambda \setminus \operatorname{Sing}(X)} = \Delta^{cs} \oplus \Delta^{cu}$  of index *i* with respect to the linear Poincaré flow  $\psi_t$ . Then for any  $T > 0, \xi > 0, \varepsilon > 0$ , there is  $\delta > 0$  such that for any  $x, y \in \Lambda$ , if  $d(x, y) < \delta$ ,  $d(x, \operatorname{Sing}(X)) > \varepsilon$ ,  $d(y, \operatorname{Sing}(X)) > \varepsilon$ , then  $W_{\xi}^{cs}(x) \cap \phi_{[-T,T]}(W_{\xi}^{cu}(y)) \neq \emptyset.$ 

The proof of this corollary is based on the uniform continuity of plaque families when points are far away from singularities.

#### 2.6. Estimations on the size of stable/unstable manifolds

**DEFINITION** 2.17. – Let  $\Lambda$  be an invariant set and  $E \subset \mathcal{N}_{\Lambda \setminus Sing(X)}$  an invariant subbundle of the linear Poincaré flow  $\psi_t$ . For C > 0,  $\eta > 0$  and T > 0,  $x \in \Lambda \setminus \text{Sing}(X)$  is called  $(C, \eta, T, E) - \psi_t^*$ -contracting if there exists a partition:  $0 = t_0 < t_1 < \cdots < t_n < \cdots$  verifying  $t_{n+1} - t_n \leq T$  for any  $n \in \mathbb{N}$  and  $t_n \to \infty$  as  $n \to \infty$ , and for any  $n \in \mathbb{N}$ ,

$$\prod_{i=0}^{n-1} \|\psi_{t_{i+1}-t_i}^*|_{E(\phi_{t_i}(x))}\| \leq C e^{-\eta t_n}.$$

 $x \in \Lambda \setminus \text{Sing}(X)$  is called  $(C, \eta, T, E) \cdot \psi_t^*$ -expanding if it's  $(C, \eta, T, E) \cdot \psi_t^*$ -contracting for -X.

An increasing homeomorphism  $\theta : \mathbb{R} \to \mathbb{R}$  is called a *reparametrization* if  $\theta(0) = 0$ . For any orbit Orb(x), one defines

$$W^{s}(\operatorname{Orb}(x)) = \{y \in M^{d}, \exists \text{ a reparametrization } \theta \text{ s.t., } \lim_{t \to \infty} d(\phi_{\theta(t)}(y), \phi_{t}(x)) = 0\},\$$

 $W^{u}(\operatorname{Orb}(x)) = \{y \in M^{d}, \exists \text{ a reparametrization } \theta \text{ s.t.}, \lim_{t \to -\infty} d(\phi_{\theta(t)}(y), \phi_{t}(x)) = 0\}.$ 

In above definitions, we need to use the reparametrization  $\theta$  because the stable/unstable set of the sectional Poincaré maps along an orbit is in the stable/unstable set of the flow  $\phi_t$  after a reparametrization.

LEMMA 2.18. – Let  $\Lambda$  be a compact invariant set of  $X \in \mathcal{X}^1(M^d)$ . Assume that  $\Lambda \setminus \operatorname{Sing}(X)$  admits a dominated splitting  $\mathcal{N}_{\Lambda \setminus \operatorname{Sing}(X)} = \Delta^{cs} \oplus \Delta^{cu}$  of index i with respect to the linear Poincaré flow  $\psi_t$ . For C > 0,  $\eta > 0$  and T > 0, there is  $\delta > 0$  such that

- For any regular point  $x \in \Lambda$ , if x is  $(C, \eta, T, \Delta^{cs}) \cdot \psi_t^*$ -contracting, then  $W_{\delta|X(x)|}^{cs} \subset W^s(\operatorname{Orb}(x))$ ;
- For any regular point  $x \in \Lambda$ , if x is  $(C, \eta, T, \Delta^{cu}) \psi_t^*$ -expanding, then  $W^{cu}_{\delta|X(x)|} \subset W^u(\operatorname{Orb}(x))$ .

*Proof.* – We need to prove that there is  $\delta > 0$  such that

$$\lim_{t \to \infty} \operatorname{diam} \left( \mathcal{P}_{x,\phi_t(x)}^*(\eta^{cs}(x)(\Delta^{cs}(\delta))) \right) = 0.$$

By the uniform continuity of  $D \mathscr{P}_{x,\phi_t(x)}^*$  in Lemma 2.5, we have a uniform linearized neighborhood of  $0_z$  in  $\mathscr{N}_z$  for each regular point z. Then the proof parallels to [45, Corollary 3.3].

COROLLARY 2.19. – Under the assumption of Lemma 2.18, for any compact set  $\Lambda_0 \subset \Lambda \setminus \text{Sing}(X)$ , there is  $\varepsilon > 0$  such that for any  $x, y \in \Lambda_0$ , if

- $d(x, y) < \varepsilon$ ;
- x is  $(C, \eta, T, \Delta^{cs})$ - $\psi_t^*$ -contracting and y is  $(C, \eta, T, \Delta^{cu})$ - $\psi_t^*$ -expanding;

then  $W^{s}(\operatorname{Orb}(x)) \cap W^{u}(\operatorname{Orb}(y)) \neq \emptyset$ .

Similar to the proof of Lemma 2.18, we have

LEMMA 2.20. – Let  $X \in \mathcal{X}^1(M^d)$ . For any C > 0,  $\eta > 0$  and T > 0, there is  $\delta = \delta(X, C, \eta, T) > 0$  such that if a regular point  $x \in M^d$  is  $(C, \eta, T, \mathcal{N})-\psi^*$ -contracting, then

$$\lim_{t \to +\infty} \operatorname{diam} \left( P_{x,\phi_t(x)}(N_x(\delta|X(x)|)) \right) = 0.$$

In other words,  $N_{x,\delta|X(x)|} \subset W^s(\operatorname{Orb}(x))$ .

Notice that not only x may be close to a singularity, but also  $\omega(x)$  may contain singularities.

THEOREM 2.21 ([28, Theorem 4.1, Proposition 6.2]). – Given  $X \in \mathcal{X}^1(M^d)$  and a hyperbolic singularity  $\sigma$  of X, for any C > 0,  $\eta > 0$  and T > 1, there exists a neighborhood U of  $\sigma$ such that there is no  $(C, \eta, T, \mathcal{N})$ - $\psi^*$ -contracting periodic point in U.

The proof of Theorem 2.21 is not short and it is contained in [28]. Now we give some idea of Theorem 2.21: if Theorem 2.21 is not true, then there is a sequence of  $(C, \eta, T, \mathcal{N})-\psi^*$ -contracting periodic points  $\{p_n\}$  such that  $\lim_{n\to\infty} p_n = \sigma$ . This holds only if  $\sigma$  is a saddle. By the property of  $\{p_n\}$ , we have

- $T_{\sigma}M^d$  admits a dominated splitting  $T_{\sigma}M^d = E^{cs} \oplus E^{uu}$  w.r.t. the tangent flow, where dim  $E^{uu} = 1$  and  $E^{uu}$  is strong unstable,
- there is  $\delta > 0$ ,  $N_{p_n}(\delta |X(p_n)|) \subset W^s(p_n)$ .

Then by a careful estimation (which is not obvious), one has for *n* large enough,  $W^{uu}(\sigma) \cap W^s(p_n) \neq \emptyset$ , where  $W^{uu}(\sigma)$  is the strong unstable manifold corresponding to  $E^{uu}$ . But  $W^{uu}(\sigma) \setminus \{\sigma\}$  contains only two orbits, and  $p_n$  are distinct periodic orbits. This gives us a contradiction.

An available proof can be found in [57].

#### 2.7. Pliss Lemma

We use the following lemma of Pliss type to get the points which can have uniform estimations to infinity.

LEMMA 2.22. – Given  $X \in \mathcal{X}^1(M^d)$ , C > 0, T > 0 and  $\eta > 0$ , for any  $\eta' \in (0, \eta)$ , there is  $N = N(C, T, \eta, \eta') > 0$ , such that if  $\gamma$  is a periodic orbit with period  $\tau(\gamma) > N$ ,  $E \subset \mathcal{N}_{\gamma}$  is an invariant bundle w.r.t.  $\psi_t$ , and if  $\gamma$  is  $(C, \eta, T, E)$ -contracting at the period w.r.t.  $\psi_t$ , then there is  $x \in \gamma$  such that x is  $(1, \eta', T, E)$ - $\psi_t^*$ -contracting.

*Proof.* – Since  $\gamma$  is  $(C, \eta, T, E)$ -contracting at the period w.r.t.  $\psi_t$ , there is  $m \in \mathbb{N}$  and a time partition

$$0 = t_0 < t_1 < \cdots < t_n = m\tau(\gamma),$$

with  $t_{i+1} - t_i \leq T$  for  $0 \leq i \leq n - 1$ , such that

$$\prod_{i=0}^{n-1} \|\psi_{t_{i+1}-t_i}\|_{E(\phi_{t_i}(x))}\| \le C \exp\{-\eta m\tau(\gamma)\}.$$

Since  $\Phi_{\tau(\gamma)}(X(x)) = X(\phi_{\tau(\gamma)}(x)) = X(x)$ , the above estimation is also true for  $\psi_t^*$ .

$$\prod_{i=0}^{n-1} \|\psi_{t_i+1}^*|_{E(\phi_{t_i}(x))}\| \le C \exp\{-\eta m\tau(\gamma)\}.$$

When  $\eta' < \eta$ , if  $\tau(\gamma)$  is large enough, one can cancel the constant C. Following [15, Lemma 2.14], one can get a  $(1, \eta', T, E) - \psi_t^*$ -contracting point x.

LEMMA 2.23. – Assume that every critical element of  $X \in \mathfrak{X}^1(M^d)$  is hyperbolic. For any C > 0, T > 0 and  $\eta > 0$ , X can only have finitely many  $(C, \eta, T, \mathcal{N})$ -contracting periodic orbits.

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*Proof.* – If the conclusion is not true, then X has infinitely many distinct periodic orbits  $\{\gamma_n\}$  such that each  $\gamma_n$  is  $(C, \eta, T, \mathcal{N})$ -contracting at the period w.r.t.  $\psi_t$ . We claim that  $\lim_{n\to\infty} \tau(\gamma_n) = \infty$ . Indeed, by taking a subsequence if necessary, assume that  $\lim_{n\to\infty} \gamma_n = \Lambda$ . If  $\tau(\gamma_n)$  are uniformly bounded, then every point in  $\Lambda$  is a critical point. Since we assume that every critical element of X is hyperbolic, every (critical) orbit in  $\Lambda$  is hyperbolic. It is easy to show that  $\Lambda$  consists of finitely many orbits. Since  $\Lambda$  is connected, it reduces to a single orbit. Since a hyperbolic critical orbit is isolated, there is no other periodic orbits contained in a neighborhood of  $\Lambda$ .

By Lemma 2.22, for each *n* large enough, there is  $x_n \in \gamma_n$  such that  $x_n$  is a  $(1, \eta/2, T, \mathcal{N})-\psi_t^*$ -contracting point. Thus, there is  $\delta = \delta(X, \eta, T) > 0$  such that  $N_{x_n}(\delta|X(x_n)|)$  is contained in the stable manifold of  $x_n$ . If  $x_n$  accumulates on singularities, then one can get a contradiction by Theorem 2.21. If  $x_n$  dose not accumulate on singularities, then the basin of  $\gamma_n$  covers an open set with uniform size. This also gives a contradiction because the volume of  $M^d$  is finite and  $\{\gamma_n\}$  are distinct periodic orbits.

# 2.8. Liao's shadowing lemma for $\psi_t^*$ and Liao's sifting lemma

DEFINITION 2.24. – Let  $\eta > 0$  and T > 0. For any  $x \in M^d \setminus \text{Sing}(X)$  and  $T_0 > T$ , the orbit arc  $\phi_{[0,T_0]}(x)$  is called  $(\eta, T) \cdot \psi_t^*$ -quasi hyperbolic with respect to a direct sum splitting  $\mathcal{N}_x = E(x) \oplus F(x)$  if there is a partition

$$0 = t_0 < t_1 < \dots < t_l = T, \quad t_{i+1} - t_i \le T,$$

such that for  $k = 0, 1, \ldots, l - 1$ , we have

$$\begin{split} & \prod_{i=0}^{k-1} \|\psi_{t_i+1-t_i}^*\|_{\psi_{t_i}(E(x))}\| \le e^{-\eta t_k}, \\ & \prod_{i=k}^{l-1} m(\psi_{t_i+1-t_i}^*\|_{\psi_{t_i}(F(x))}) \ge e^{\eta(t_l-t_k)}, \\ & \frac{\|\psi_{t_k+1-t_k}^*\|_{\psi_{t_k}(E(x))}\|}{m(\psi_{t_k+1-t_k}^*\|_{\psi_{t_i}(F(x))})} \le e^{-\eta(t_{k+1}-t_k)}. \end{split}$$

REMARK. – The third inequality is usually satisfied in an invariant set with a  $T^*$ -dominated splitting in the normal bundle with respect to the linear Poincaré flow.

Note that this definition is similar to the usual quasi hyperbolic orbit arc for linear Poincaré flow, while the only difference is that now we use the scaled linear Poincaré flow  $\psi_t^*$ instead of linear Poincaré flow  $\psi_t$ . Let  $d_T$  be the metric in  $TM^d$  induced by the Riemannian metric. For  $x, y \in M^d$  and two linear subspaces E(x) and F(y), one defines

$$d(E(x), F(y)) = \max\{\sup_{u \in E(x), |u|=1} \inf_{v \in F(y), |v|=1} \{d_T(u, v)\}, \sup_{v \in F(y), |v|=1} \inf_{u \in E(x), |u|=1} \{d_T(u, v)\}\}.$$

The following shadowing lemma for singular flows was given by Liao [27]. We restate it by using modern languages.

THEOREM 2.25 ([27]). – Let  $X \in \mathcal{X}^1(M^d)$ ,  $\Lambda \subset M^d \setminus \operatorname{Sing}(X)$  be a compact set, and  $\eta > 0, T > 1$ . For any  $\varepsilon > 0$  there exists  $\delta > 0$ , such that for any  $(\eta, T) - \psi_t^*$ -quasi hyperbolic orbit arc  $\phi_{[0,T]}(x)$  with respect to some direct sum splitting  $\mathcal{N}_x = E(x) \oplus F(x)$  satisfying x,  $\phi_T(x) \in \Lambda$  and  $\tilde{d}(E(x), \psi_T(E(x))) \leq \delta$  and  $\tilde{d}(F(x), \psi_T(F(x))) \leq \delta$ , there exists a point  $p \in M^d$  and a  $C^1$  strictly increasing function  $\theta : [0, T] \to \mathbb{R}$  such that

- $\theta(0) = 0$  and  $1 \varepsilon < \theta'(t) < 1 + \varepsilon$ ,
- *p* is periodic:  $\phi_{\theta(T)}(p) = p$ ,
- $d(\phi_t(x), \phi_{\theta(t)}(p)) \le \varepsilon |X(\phi_t(x))|, \quad t \in [0, T],$
- there is a direct-sum splitting  $\mathcal{N}_p = E(p) \oplus F(p)$  such that  $\psi^*_{\theta(T)}(E(p)) = E(p)$ ,  $\psi^*_{\theta(T)}(F(p)) = F(p)$ , and for any  $t \in [0, T]$ ,

$$d(\psi_t^*(E(x)), \psi_{\theta(t)}^*(E(p))) \le \varepsilon,$$
  
$$\tilde{d}(\psi_t^*(F(x)), \psi_{\theta(t)}^*(F(p))) \le \varepsilon.$$

REMARK. – If we consider an invariant set with a dominated splitting in the normal bundle w.r.t. the linear Poincaré flow, we can replace  $\tilde{d}(E(x), \psi_T(E(x))) \leq \delta$  by  $d(x, \phi_T(x)) < \delta$ .

Note that in this version of shadowing lemma, we only need that the head and tail of orbit arc are far from singularities, while other part of the orbit arc can approximate singularities. This enables us to deal with some problems where regular orbits approximate singularities.

We also need Liao's sifting lemma [25, 26], whose aim is to find quasi-hyperbolic orbit segments. One can see [54] for a proof.

LEMMA 2.26. – Let  $\phi_t : \Lambda \to \Lambda$  be a continuous flow on a compact metric space  $\Lambda$  and  $f : \Lambda \to \mathbb{R}$  a continuous function. Let  $\eta_2 > \eta_1 > 0$  and T > 0. Assume that

- there is  $b \in \Lambda$  such that for any  $n \in \mathbb{N}$ ,

$$\sum_{i=0}^{n-1} f(\phi_{iT}(b)) \ge 0;$$

- for any  $c \in \Lambda$  verifying for any  $n \in \mathbb{N}$ ,

$$\sum_{i=0}^{n-1} f(\phi_{iT}(c)) \ge -n\eta_1,$$

there is  $g \in \omega(c)$  such that for any  $n \in \mathbb{N}$ ,

$$\sum_{i=0}^{n-1} f(\phi_{iT}(g)) \le -n\eta_2.$$

Then for any  $\eta_3$ ,  $\eta_4$  verifying  $\eta_2 > \eta_3 > \eta_4 > \eta_1$ , for any  $k \in \mathbb{N}$ , there is y in the positive orbit of x and integers  $0 = n_0 < n_1 < \cdots < n_k$  such that for each integer  $i \in [0, k - 1]$ , for any integer  $m \in [1, n_{i+1} - n_i]$  one has

$$\sum_{j=0}^{m-1} f(\phi_{jT}(\phi_{n_jT}(y))) \leq -m\eta_4,$$

$$\sum_{j=m-1}^{n_{i+1}-n_i-1} f(\phi_{jT}(\phi_{(n_i+m-1)T})) \ge -(n_{i+1}-n_i-m+1)\eta_3.$$

We need the following folklore lemma to prove hyperbolicity for compact sets. Its proof mainly uses some compact arguments. Hence we omit the proof.

LEMMA 2.27. – Let  $\phi_t : \Lambda \to \Lambda$  be a continuous flow on a compact metric space  $\Lambda$  and  $f : \Lambda \to \mathbb{R}$  a continuous function. Given T > 0, if for any  $x \in \Lambda$ , there is  $n_x \in \mathbb{N}$  such that

$$\sum_{i=0}^{n_x - 1} f(\phi_{iT}(x)) < 0$$

then there are  $C \ge 0$  and  $\lambda < 0$  such that for any  $x \in \Lambda$  and any  $n \in \mathbb{N}$ , one has

$$\sum_{i=0}^{n-1} f(\phi_{iT}(x)) \le C + n\lambda.$$

#### 3. Chain recurrence and genericity

#### 3.1. Conley theory

A chain recurrent class is called *non-trivial* if it is not reduced to a critical element; otherwise, it is called trivial. For each hyperbolic critical element p of X, since Orb(p) has a well-defined continuation  $Orb(p_Y)$  for Y close to X, C(p) also has a well-defined continuation  $C(p_Y, Y)$ .

A compact invariant set  $\Lambda$  of X (if it has a continuation) is called *lower semi-continuous* if for any sequence of vector fields  $\{X_n\}$  verifying  $\lim_{n\to\infty} X_n = X$ , one has  $\lim \inf_{n\to\infty} \Lambda_{X_n} \supset \Lambda$ . A compact invariant set  $\Lambda$  of X (if it has a continuation) is called *upper semi-continuous* if for any sequence of vector fields  $\{X_n\}$  verifying  $\lim_{n\to\infty} X_n = X$ , one has  $\limsup_{n\to\infty} \Lambda_{X_n} \subset \Lambda$ . It is well known that the closure of hyperbolic periodic orbits is lower semi-continuous and the chain recurrent set is upper semi-continuous. There is a classical result saying that lower semi-continuous sets and upper semi-continuous sets are continuous for generic vector fields.

LEMMA 3.1. – For a hyperbolic critical element p, C(p) is upper semi-continuous. As a corollary, if  $p_1$  and  $p_2$  are two critical elements of X with the property  $C(p_1) \cap C(p_2) = \emptyset$ , then there is a neighborhood  $\mathcal{U}$  of X such that for any  $Y \in \mathcal{U}$ , one has  $C(p_{1,Y}) \cap C(p_{2,Y}) = \emptyset$ .

*Proof.* – The fact that C(p) is upper semi-continuous because of the continuity of the flows with respect to the vector fields.

By [12], if we have two chain recurrent classes  $C(p_1)$  and  $C(p_2)$  satisfying  $C(p_1) \cap C(p_2) = \emptyset$ , then there is an open set U such that

- $\phi_t(\overline{U}) \subset U$  for t > 0;
- $C(p_1) \subset U$  and  $C(p_2) \subset Int(M \setminus \overline{U})$  or  $C(p_2) \subset U$  and  $C(p_1) \subset Int(M \setminus \overline{U})$ .

Then by the continuity of vector fields, there is a  $C^1$  neighborhood  $\mathcal{U}$  of X such that for any  $Y \in \mathcal{U}$ , one has  $C(p_{1,Y}) \subset U$  and  $C(p_{2,Y}) \subset \operatorname{Int}(M \setminus \overline{U})$  or  $C(p_{2,Y}) \subset U$  and  $C(p_{1,Y}) \subset \operatorname{Int}(M \setminus \overline{U})$ . As a corollary, one has  $C(p_{1,Y}) \cap C(p_{2,Y}) = \emptyset$ .

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For each point  $x \in M^d$ , one can define the strong stable manifold  $W^{ss}(x)$  and the strong unstable manifold  $W^{uu}(x)$  as

$$W^{ss}(x) \stackrel{\Delta}{=} \{ y \in M^d : \lim_{t \to +\infty} d(\phi_t(x), \phi_t(y)) = 0 \},\$$
$$W^{uu}(x) \stackrel{\Delta}{=} \{ y \in M^d : \lim_{t \to -\infty} d(\phi_t(x), \phi_t(y)) = 0 \}.$$

But for flows, this definition is not enough in many cases. By the difference with diffeomorphisms, sometimes we need to reparametrize the time variable. This leads us to give the definition of  $W^{s}(Orb(x))$  and  $W^{u}(Orb(x))$  as in Section 2.

By the definitions, one has that  $W^{s}(Orb(x))$  and  $W^{u}(Orb(x))$  are invariant sets. The proof of the following lemma is forklore.

LEMMA 3.2. – For any hyperbolic critical point p, one has

- 1. for any critical element p, one has  $W^{s}(\operatorname{Orb}(p)) = \bigcup_{t \ge 0} W^{ss}(\phi_{t}(p))$  and  $W^{u}(\operatorname{Orb}(p)) = \bigcup_{t \ge 0} W^{uu}(\phi_{t}(p)).$
- 2. If C(p) is non-trivial, then  $C(p) \cap W^s(\operatorname{Orb}(p)) \setminus \operatorname{Orb}(p) \neq \emptyset$  and  $C(p) \cap W^u(\operatorname{Orb}(p)) \setminus \operatorname{Orb}(p) \neq \emptyset$ .

*Proof.* – For a critical element p,  $W^{ss}(p)$  coincides with the stable manifold of p for the time-one map  $\phi_1$ . p is a hyperbolic point of  $\phi_1$  when p is a singularity and  $\operatorname{Orb}(p)$  is a normally hyperbolic circle when p is periodic. In any case,  $\bigcup_{t\geq 0} W^{ss}(\phi_t(p))$  is the set of point whose  $\omega$ -limit is the orbit of p. Item 1 follows from this fact.

One can find a proof of Item 2 in [10, Lemma 2.7].

For a compact invariant set  $\Lambda$ , one says that  $\Lambda$  is *Lyapunov stable* for X if for any neighborhood U of  $\Lambda$ , there is a neighborhood V of  $\Lambda$  such that  $\phi_t(V) \subset U$  for any  $t \geq 0$ .

LEMMA 3.3. – If  $\Lambda$  is Lyapunov stable, then  $W^u(\operatorname{Orb}(x)) \subset \Lambda$  for each  $x \in \Lambda$ .

*Proof.* – Given any  $y \in W^u(\text{Orb}(x))$ , for any neighborhood U of Λ, since Λ is Lyapunov stable, there is a neighborhood V of Λ such that  $\phi_t(V) \subset U$  for any  $t \ge 0$ . For any  $y \in W^u(\text{Orb}(x))$ , there is an increasing homeomorphism  $\theta : \mathbb{R} \to \mathbb{R}$  such that  $d(\phi_t(x), \phi_{\theta(t)}(y)) \to 0$  as  $t \to -\infty$ . Thus there is  $t_V > 0$  such that  $\phi_{\theta(-t_V)} \in V$ . By the Lyapunov stability one has  $y = \phi_{-\theta(-t_V)}(\phi_{\theta(-t_V)}(y)) \in U$ . By the arbitrary property of U, one has  $y \in \Lambda$ .

By the definition, if  $\Lambda$  is not Lyapunov stable, then there is a neighborhood  $U_0$  of  $\Lambda$  such that there is a sequence of neighborhoods  $\{V_n\}$  of  $\Lambda$  such that

-  $V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots$ , and  $\bigcap_{n \ge 0} V_n = \Lambda$ . -  $\phi_{t_n}(V_n) \subset U_0$  for some  $t_n > 0$ .

Since  $\Lambda$  is an invariant set, one has that  $\lim_{n\to\infty} t_n = \infty$ . Hence, if  $\Lambda$  is not Lyapunov stable, then there are  $\{t_n\} \subset \mathbb{R}$  and a sequence of points  $\{x_n\}$  such that

- $-\lim_{n\to\infty} x_n \in \Lambda.$
- $-\lim_{n\to\infty}t_n=+\infty.$
- $\lim_{n\to\infty} \phi_{t_n}(x_n)$  exists and  $\lim_{n\to\infty} \phi_{t_n}(x_n) \notin \Lambda$ .

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For  $x \in M^d$ , one can define the *chain unstable set*  $W^{ch,u}(x)$  of x and the *chain stable set*  $W^{ch,s}(x)$  of x in the following way:

$$W^{ch,u}(x) = \{ y \in M^d : \forall \varepsilon > 0, \exists an \varepsilon - pseudo orbit \{ x_i \}_{i=0}^n \text{ s.t. } x_0 = x, x_n = y \}, \\ W^{ch,s}(x) = \{ y \in M^d : \forall \varepsilon > 0, \exists an \varepsilon - pseudo orbit \{ x_i \}_{i=0}^n \text{ s.t. } x_0 = y, x_n = x \}.$$

Using the notation of chain (un)stable set, we have that y is *chain attainable* from x iff  $y \in W^{ch,u}(x)$ , and x, y are *chain bi-attainable* iff  $y \in W^{ch,u}(x) \cap W^{ch,s}(x)$ .

# 3.2. The $C^1$ connecting lemmas and the ergodic closing lemma for flows

Arnaud, Wen and Xia [3, 55] gave the following extension of Hayashi's  $C^1$  connecting lemma [19]. We will use it in Section 6.

LEMMA 3.4. – For any vector field  $X \in \mathcal{X}^1(M^d)$ , for any point  $z \notin Per(X) \cup Sing(X)$ , for any  $\varepsilon > 0$ , there are L > 0 and two neighborhoods  $\widetilde{W_z} \subset W_z$  of z such that

- one can choose  $W_z$  and  $\widetilde{W_z}$  to be arbitrarily small neighborhoods of z,
- for any p and q in  $M^d$ , if the positive orbit of p and the negative orbit of q enter  $\widetilde{W_z}$ , but the orbit segments  $\{\phi_t(p): 0 \le t \le L\}$  and  $\{\phi_t(q): -L \le t \le 0\}$  don't intersect  $W_z$ ,

then there is a vector field  $Y \in C^1$ -close to X such that

- q is in the positive orbit of p with respect to the flow  $\phi_t^Y$  generated by Y,
- Y(x) = X(x) for any  $x \in M^d \setminus W_{L,z}$ , where  $W_{L,z} = \bigcup_{0 \le t \le L} \phi_t^X(W_z)$ .

Mañé's ergodic closing lemma [30] is also useful in this paper. First we state a flow version of Mañé's ergodic closing lemma taken from [51].

DEFINITION 3.5. – Let  $X \in \mathcal{X}^1(M^d)$ . A regular point  $x \in M^d$  is called strongly closable if for any  $C^1$  neighborhood  $\mathcal{U}$  of X, there is L > 0, and any  $\varepsilon > 0$ , there are  $Y \in \mathcal{U}$  and a periodic point  $y \in M^d$  of Y with period  $\tau(y)$  such that

$$- X(z) = Y(z) \text{ for any } z \in M^d \setminus \left( \bigcup_{t \in [0,L]} \phi_t(B(x,\varepsilon)) \right),$$

 $- d(\phi_t^X(x), \phi_t^Y(y)) < \varepsilon \text{ for each } t \in [0, \tau(y)].$ 

Denote by  $\Sigma(X)$  the set of strongly closable points of X.

The ergodic closing lemma [30, 51] states:

LEMMA 3.6. – We have  $\mu(\Sigma(X) \cup \text{Sing}(X)) = 1$  for every T > 0 and every  $\phi_T^X$ -invariant probability Borel measure  $\mu$ .

One needs the following corollary which asserts that one can get a periodic orbit with some additional properties after a small perturbation which preserves a compact invariant subset in a transitive set. For the applications in this paper, one takes the compact invariant subset as the union of finitely homoclinic orbits of singularities. See Section 6.

COROLLARY 3.7. – Let  $f : M^d \to \mathbb{R}$  be a continuous function. Let  $\mu$  be an ergodic measure of a flow  $\phi_t$  generated by  $X \in \mathcal{X}^1(M^d)$ , which is not supported on a singularity. Assume that  $\Lambda$  is a compact invariant set such that  $\mu(\Lambda) = 0$ . Then for any  $\varepsilon > 0$ , for any  $C^1$  neighborhood  $\mathcal{V}$  of X, and any neighborhood U of  $\operatorname{supp}(\mu)$ , there exists  $Y \in \mathcal{V}$  and a periodic orbit  $\gamma \subset U$  of Y such that

- $\Lambda$  is a compact invariant set of Y.
- $|\int f d\delta_{\gamma} \int f d\mu| < \varepsilon$ , where  $\delta_{\gamma}$  is the uniform distribution measure on  $\gamma$ , i.e., for any continuous function  $g: M^d \to \mathbb{R}$ ,

$$\int g(x) \mathrm{d}\delta_{\gamma}(x) = \frac{1}{\tau(\gamma)} \int_0^{\tau(\gamma)} g(\phi_t(p)) \mathrm{d}t,$$

where  $p \in \gamma$ .

*Proof.* – Without loss of generality, one can assume that  $\mu$  is not supported on a periodic orbit. Since  $\mu$  is not supported on a singularity, one has  $\mu(\Sigma(X)) = 1$ . Also choose  $x \in \Sigma$  with  $x \notin \Lambda$  satisfying

$$\lim_{T\to\infty}\delta_{x,T}=\mu$$

where  $\delta_{x,T}$  is the uniform distribution measure supported on  $\phi_{[0,T]}(x)$ , i.e., for any continuous function  $g: M^d \to \mathbb{R}$ ,

$$\int g(y) \mathrm{d}\delta_{x,T}(y) = \frac{1}{T} \int_0^T g(\phi_t(x)) \mathrm{d}t.$$

For any  $\varepsilon > 0$ , there is  $T_0 > 0$  such that for any  $T > T_0$ , one has

$$\left|\int f\,\mathrm{d}\delta_{x,T}-\int f\,\mathrm{d}\mu\right|<\varepsilon/2$$

Take  $\delta > 0$  small enough such that for any  $d(y, z) < \delta$ , one has  $|f(y) - f(z)| < \varepsilon/2$ . By reducing  $\delta$  if necessary, one can assume that  $B(x, \delta) \cap \Lambda = \emptyset$ . Since  $\Lambda$  is invariant, one has that  $\phi_t(B(x, \delta)) \cap \Lambda = \emptyset$  for any  $t \in \mathbb{R}$ . By Lemma 3.6, there is  $Y \in \mathcal{U}$  which has a periodic point p with period  $\tau(p)$  such that

- $\operatorname{Orb}(p, Y) \subset U;$
- $\Lambda$  is a compact invariant set of *Y*;
- $d(\phi_t^X(x), \phi_t^Y(p)) < \delta \text{ for each } t \in [0, \tau(p)].$

x is not periodic because we assume that  $\mu$  is not supported on a periodic orbit. Thus, one can assume that  $\tau(p) > T_0$ . Let  $\gamma = \operatorname{Orb}(p, Y)$ . Then

$$\begin{split} \left| \int f \, \mathrm{d}\delta_{\gamma} - \int f \, \mathrm{d}\mu \right| &\leq \left| \int f \, \mathrm{d}\delta_{\gamma} - \int f \, \mathrm{d}_{\delta_{X,\tau(\gamma)}} \right| + \left| \int f \, \mathrm{d}_{\delta_{X,\tau(\gamma)}} - \int f \, \mathrm{d}\mu \right| \\ &\leq \frac{1}{\tau(\gamma)} \int_{0}^{\tau(\gamma)} \left| f(\phi_{t}^{Y}(p)) - f(\phi_{t}^{X}(x)) \right| \, \mathrm{d}t + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{split}$$

A vector field  $X \in \mathcal{X}^r(M^d)$  is called *Kupka-Smale* if every critical element of X is hyperbolic, and the stable manifold of any critical element intersects the unstable manifold of any critical element transversely. A classical generic result is: Kupka-Smale vector fields form a residual set in  $\mathcal{X}^r(M^d)$ . We need the following weak terminology:

DEFINITION 3.8. – A vector field  $X \in \mathcal{X}^r(M^d)$  is called weak Kupka-Smale if every critical element of X is hyperbolic.

Since Kupka-Smale is  $C^r$  generic in  $\mathcal{X}^r(M^d)$ , we have that weak Kupka-Smale is also a  $C^r$  generic property in  $\mathcal{X}^r(M^d)$ .

We will state a connecting lemma for pseudo orbits, which was proved in [6]. Bonatti and Crovisier studied the connecting lemma for pseudo-orbits for weak Kupka-Smale diffeomorphisms. The assumption of weak Kupka-Smale is used since

- by using  $\lambda$ -lemma, for every non-periodic point x which is not in the stable/unstable manifold of a periodic point, the positive/negative iteration of x will be in a *topological* tower.

Now we give the definition of a topological tower for a vector field. For any L > 0, denote by

$$\operatorname{Crit}_L(X) = \{x : \exists l \in [0, L], \text{ s.t. } \phi_l(x) = x, \operatorname{Orb}(x) \text{ is hyperbolic} \}.$$

Given a vector field X, for  $\delta > 0$  and L > 0, a sequence of cross sections  $\{\Sigma_i\}_{i=1}^N$  is called a topological tower, if

- $\{\widetilde{\Sigma}_i\}_{i=1}^N$  are mutually disjoint, where  $\widetilde{\Sigma}_i = \phi_{[0,L]}\Sigma_i$ , if x is not contained in  $\bigcup_{p \in \operatorname{Crit}_L(X)} W^s_{\delta}(\operatorname{Orb}(p))$ , then the forward orbit of x will
- intersect  $\bigcup_{i=1}^{N} \Sigma_i$ , if x is not contained in  $\bigcup_{p \in \operatorname{Crit}_L(X)} W^u_{\delta}(\operatorname{Orb}(p))$ , then the backward orbit of x will intersect  $\bigcup_{i=1}^{N} \Sigma_i$ .

A flow version of [6, Théorème 3.1] states that such a topological tower exists for small  $\delta > 0$ and large L > 0 when X is weak Kupka-Smale. Moreover, one can require that

- the diameter of each  $\Sigma_i$  is as small as we want;
- each  $\Sigma_i$  is almost orthogonal to the vector field X.

When L is large enough, the perturbation for connecting orbit as in [6, Théorme 1.2] will be realized in  $\bigcup_{i=1}^{N} \widetilde{\Sigma}_i$ .

For flows,  $\lambda$ -lemma is true for both hyperbolic singularities and hyperbolic periodic orbits. The orbit structure is clear near hyperbolic critical elements. So the connecting lemma for pseudo-orbits is true for vector fields.

LEMMA 3.9 ([6, Theorem 1.2]). – Let  $X \in \mathcal{X}^1(M^d)$  be a weak Kupka-Smale vector field. Given any finite set  $\hat{F}$  of periodic orbits of X,, for any  $C^1$  neighborhood  $\mathcal{U}$  of X, and for any  $x, y \in M^d$ , if y is chain attainable from x, then there are  $Y \in \mathcal{U}$ , some neighborhood U of  $\hat{F}$ and t > 0 such that  $\phi_t^Y(x) = y$  and Y(z) = X(z) for any  $z \in U$ .

Moreover, if X is  $C^r$ , we can require that Y is also  $C^r$ .

The statement here is a little bit different from [6], but it is essentially contained there. See [6, Remarque 1.1].

REMARK. – When we prove Theorem C, we first consider some generic vector field. And then we consider some new vector field in a small neighborhood of the original one. Thus we need Lemma 3.9 to do some perturbations when the new one is weak Kupka-Smale. See Lemma 3.15 and Section 6 for more details.

Sometimes, one needs to perturb a vector field to be a weak Kupka-Smale vector field while preserving some non-transverse connection. This was used by Palis [38]. One can see R. Xi's master thesis [56] for a proof.

THEOREM 3.10 ([38, 56]). – For any vector field  $X \in \mathcal{X}^r(M^d)$ , any  $n \in \mathbb{N}$  and any hyperbolic critical elements  $\{P_1, Q_1, \ldots, P_n, Q_n\}$ , if  $\operatorname{Orb}(x_i) \subset W^s(P_i) \cap W^u(Q_i)$  is a non-transverse orbit for  $1 \leq i \leq n$ , then for any  $C^r$  neighborhood  $\mathcal{U}$  of X there exists  $Y \in \mathcal{U}$ ,

- $\operatorname{Orb}(x_i) \subset W^s(P_i) \cap W^u(Q_i)$  is still a non-transverse orbit of Y,
- any other critical element of Y is hyperbolic, i.e., Y is weak Kupka-Smale.

#### 3.3. Generic results

Recall that  $\mathcal{R} \subset \mathcal{X}^1(M^d)$  is *residual* if it contains a dense  $G_\delta$  subsets of  $\mathcal{X}^1(M^d)$ . A property of vector fields is called  $C^1$  generic if it holds for vector fields in a residual set in  $\mathcal{X}^1(M^d)$ . Sometimes we use the terminology "for  $C^1$  generic X" which is equivalent to say that "there is a residual set  $\mathcal{R} \subset \mathcal{X}^1(M^d)$  and  $X \in \mathcal{R}$ ". Since  $\mathcal{X}^1(M^d)$  is a Banach space, every countable intersection of open dense subsets of  $\mathcal{X}^1(M^d)$  is dense. Usually we can get a dense open property via a generic way.

One knows that lower semi-continuous maps and upper semi-continuous maps defined on a complete metric space are continuous on a residual set.

LEMMA 3.11. – For  $C^1$  generic  $X \in \mathcal{X}^1(M^d)$ , for every critical element p, C(p, X) is continuous at X. This means, if  $\{X_n\}$  is a sequence of vector fields and  $\lim_{n\to\infty} X_n = X$  in the  $C^1$  topology, then  $\lim_{n\to\infty} C(p_{X_n}, X_n) = C(p, X)$  in the Hausdorff topology.

*Proof.* – The proof of this lemma just uses the upper semi-continuity of chain recurrent class.  $\Box$ 

LEMMA 3.12. – For  $C^1$  generic  $X \in \mathcal{X}^1(M^d)$ , if  $p_1$  and  $p_2$  are two different critical elements such that  $C(p_1) = C(p_2)$ , then there is a  $C^1$  neighborhood  $\mathcal{U}$  of X such that for any  $Y \in \mathcal{U}$ , one has  $C(p_{1,Y}, Y) = C(p_{2,Y}, Y)$ .

*Proof.* – Let  $\mathcal{C}$  be the metric space of all compact subsets of  $M^d$ , endowed with the Hausdorff metric.  $\mathcal{C}$  is a compact metric space. Let  $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n, \ldots$ , be a countable basis of  $\mathcal{C}$ . Let  $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n, \ldots$ , be the list of finite unions of elements of the countable basis. For each n and m, we define the sets  $\mathcal{H}_{n,m}$  and  $\mathcal{N}_{n,m}$  of vector fields as following:

- $-X \in \mathcal{H}_{n,m}$  iff there is a neighborhood  $\mathcal{U}$  of X such that for any  $Y \in \mathcal{U}$ , for any hyperbolic critical element  $p_n \in \mathcal{O}_n$  and any hyperbolic critical element  $p_m \in \mathcal{O}_m$ , one has  $C(p_n) = C(p_m)$  for Y,
- X ∈  $\mathcal{N}_{n,m}$  iff there is a hyperbolic critical element  $p_n \in \mathcal{O}_n$  and a hyperbolic critical element  $p_m \in \mathcal{O}_m$  such that  $C(p_n) \cap C(p_m) = \emptyset$ . Since the chain recurrent class of any hyperbolic critical element is upper semi-continuous, one knows that  $\mathcal{N}_{n,m}$  is open.

From the definitions,  $\mathcal{H}_{n,m} \cup \mathcal{N}_{n,m}$  is open and dense in  $\mathcal{X}^1(M^d)$ . Let

$$\mathcal{R} = \bigcap_{n,m\in\mathbb{N}} (\mathcal{H}_{n,m} \cup \mathcal{N}_{n,m}).$$

 $\mathcal{R}$  is a residual subset. We will verify that every  $X \in \mathcal{R}$  satisfies the conclusion of the lemma. Let  $X \in \mathcal{R}$ . Thus, for each *n* and *m* one has  $X \in \mathcal{H}_{n,m} \cup \mathcal{N}_{n,m}$ . For any two hyperbolic critical elements  $p_1$  and  $p_2$ , there are  $l \in \mathbb{N}$  and  $k \in \mathbb{N}$  and a  $C^1$  neighborhood  $\mathcal{U}$  of X such that

- for any  $Y \in \mathcal{U}$ ,  $p_{1,Y}$  and  $p_{2,Y}$  are the maximal compact invariant sets in  $\mathcal{O}_l$  and  $\mathcal{O}_k$  respectively.

If  $C(p_1) = C(p_2)$  for X, then  $X \notin \mathcal{N}_{k,l}$ . This implies that  $X \in \mathcal{H}_{k,l}$ . Let  $\mathcal{U}_0 = \mathcal{U} \cap \mathcal{H}_{k,l}$ . For each  $Y \in \mathcal{U}_0$ ,

- since  $Y \in \mathcal{H}_{k,l}$ , there is a critical element  $p'_1 \in \mathcal{O}_l$  and a critical element  $p'_2 \in \mathcal{O}_k$  of Y, one has  $C(p'_1) = C(p'_2)$ ,
- since  $Y \in \mathcal{U}$ , the unique critical element in  $\mathcal{O}_l$  is  $p_{1,Y}$  and the unique critical element in  $\mathcal{O}_k$  is  $p_{2,Y}$ . As a corollary,  $p_{1,Y} = p'_1$  and  $p_{2,Y} = p'_2$ .

Thus, one has  $C(p_{1,Y}) = C(p_{2,Y})$  for any  $Y \in \mathcal{U}_0$ .

LEMMA 3.13. – For  $C^1$  generic  $X \in \mathcal{X}^1(M^d)$ , and any hyperbolic critical element p of X, if  $\overline{W^u(p)} \subset C(p)$ , then there is a  $C^1$  neighborhood  $\mathcal{U}$  of X such that  $\overline{W^u(p_Y, Y)} \subset C(p_Y, Y)$ .

*Proof.* – Let C be the metric space of all compact subsets of  $M^d$ , endowed with the Hausdorff metric. C is a compact metric space. Let  $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n, \ldots$ , be a countable basis of C. Let  $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n, \ldots$ , be the list of finite unions of elements of the countable basis. For each n, one can define sets  $\mathcal{H}_n$  and  $\mathcal{N}_n$  as following:

- $-X \in \mathcal{H}_n$  iff there is a  $C^1$  neighborhood  $\mathcal{U}$  of X such that for any  $Y \in \mathcal{U}$ , every hyperbolic critical element  $p_Y \in \mathcal{O}_n$  of Y satisfies  $\overline{W^u(p_Y, Y)} \subset C(p_Y, Y)$ . By definition,  $\mathcal{H}_n$  is open,
- $X \in \mathcal{N}_n$  iff X has a hyperbolic critical element  $p \in \mathcal{O}_n$  such that  $\overline{W^u(p, X)} C(p, X)$ .  $\overline{W^u(p, X)}$  varies lower semi-continuously with respect to X and C(p, X) varies upper semi-continuously with respect to X. So if  $\overline{W^u(p, X)} C(p, X)$ , there is a neighborhood  $\mathcal{U}$  of X such that  $\overline{W^u(p_Y, Y)} C(p_Y, Y)$  for any  $Y \in \mathcal{U}$ . This implies that  $\mathcal{N}_n$  is an open set in  $\mathcal{X}^1(M^d)$ .

It is clear that  $\mathcal{H}_n \cup \mathcal{N}_n$  is open and dense in  $\mathcal{X}^1(M^d)$ . Let

$$\mathcal{R} = \bigcap_{n \in \mathbb{N}} (\mathcal{H}_n \cup \mathcal{N}_n).$$

 $\mathcal{R}$  is a residual subset of  $\mathcal{X}^1(M^d)$ . Take  $X \in \mathcal{R}$ . If p is a hyperbolic critical element of X, then there are n and a neighborhood  $\mathcal{U}$  of X such that for each  $Y \in \mathcal{U}$ ,  $p_Y$  is the unique hyperbolic critical element in  $\mathcal{O}_n$ . Since  $\overline{W^u(p)} \subset C(p)$ , one has  $X \notin \mathcal{N}_n$ . As a corollary,  $X \in \mathcal{H}_n$ . Let  $\mathcal{U}_0 = \mathcal{U} \cap \mathcal{H}_n$ . For any  $Y \in \mathcal{U}_0$ ,

- since  $Y \in \mathcal{H}_n$ , every hyperbolic critical element  $q \in \mathcal{O}_n$  of Y, one has  $\overline{W^u(q,Y)} \subset C(q,Y)$ ,
- since  $Y \in \mathcal{U}$ , the unique critical element in  $\mathcal{O}_n$  is  $p_Y$ .
Thus,  $\overline{W^u(p_Y, Y)} \subset C(p_Y, Y)$  for any  $Y \in \mathcal{U}_0$ .

LEMMA 3.14. – For  $C^1$  generic  $X \in \mathcal{X}^1(M^d)$ , let p be a hyperbolic critical element of X. If  $W^u(p) \subset C(p)$ , then C(p) is Lyapunov stable.

Lemma 3.14 is folklore. The proof is based on the connecting lemma for pseudo orbits [6].

The following lemma claims that for  $C^1$  generic vector fields, Lyapunov stability is a robust property under perturbations.

LEMMA 3.15. – For  $C^1$  generic  $X \in \mathcal{X}^1(M^d)$ , let p be a hyperbolic critical element of X. If C(p, X) is Lyapunov stable, then there is a  $C^1$  neighborhood  $\mathcal{U}$  of X such that for any weak Kupka-Smale vector field  $Y \in \mathcal{U}$ ,  $C(p_Y, Y)$  is also Lyapunov stable.

*Proof.* – Let  $\mathscr{R} \subset \mathscr{U}^1(M^d)$  be the residual subset as in Lemma 3.13:  $X \in \mathscr{R}$  iff for any hyperbolic critical element p of X, if  $\overline{W^u(p, X)} \subset C(p, X)$ , then there is a  $C^1$  neighborhood  $\mathscr{U} = \mathscr{U}_{X,p}$  such that  $\overline{W^u(p_Y, Y)} \subset C(p_Y, Y)$  for any  $Y \in \mathscr{U}$ . We will prove that  $C(p_Y, Y)$  is Lyapunov stable for each weak Kupka-Smale  $Y \in \mathscr{U}$ . If not, there is a weak Kupka-Smale vector field  $X_0 \in \mathscr{U}$  such that  $C(p_{X_0}, X_0)$  is not Lyapunov stable. Thus we have

 $- \overline{W^u(p_{X_0}, X_0)} \subset C(p_{X_0}, X_0),$ 

- there is  $y \notin C(p_{X_0}, X_0)$  such that  $y \in W^{ch,u}(C(p_{X_0}, X_0))$ .

Then, there is a  $C^1$  neighborhood  $\mathcal{U}_0 \subset \mathcal{U}$  of  $X_0$  such that for any  $Y \in \mathcal{U}_0$ ,  $y \notin C(p_Y, Y)$  by the upper semi-continuity of chain recurrent classes. Choose  $z \in W^u(p_{X_0}, X_0) \setminus \{p_{X_0}\}$ . Since  $z \in C(p_{X_0}, X_0)$ , y is chain attainable from z.

Take a small neighborhood U of  $p_{X_0}$ . One can assume that the negative orbit of z is contained in U. By Lemma 3.9, there is a vector field  $Y \in \mathcal{U}_0$  such that

- $Y(x) = X_0(x)$  for any  $x \in U$ ,
- y is in the positive orbit of z with respect to  $\phi_t^Y$ .

As a corollary, y is in the unstable manifold of  $p_Y$  with respect to Y. This fact gives a contradiction.

The following lemma asserts that for a generic vector field, if the perturbed system has a periodic orbit which is  $(C, \eta, d, \mathcal{N})$ -contracting at the period, then the original generic system already have by relaxing the constants.

LEMMA 3.16. – There is a dense  $G_{\delta}$  set  $\mathcal{G} \subset \mathcal{X}^{1}(M^{d})$  such that for any  $X \in \mathcal{G}$ , given two open sets  $U, V \subset M^{d}$  with  $\overline{U} \subset V$ , and given three number  $K \in \mathbb{N}$ ,  $\eta > 0$  and T > 0, if for any  $C^{1}$  neighborhood  $\mathcal{U}$  of X such that if there is some  $Y \in \mathcal{U}$  has a hyperbolic periodic orbit  $\gamma_{Y}$  which is  $(K, \eta, T, \mathcal{N})$ -contracting at the period w.r.t.  $\psi_{t}^{Y}$  satisfying  $\gamma_{Y} \cap U \neq \emptyset$ , then X has a periodic orbit  $\gamma$  which is  $(K, \eta/2, 2T, \mathcal{N})$ -contracting at the period w.r.t.  $\psi_{t}$  satisfying  $\gamma \cap V \neq \emptyset$ .

*Proof.* – The proof of this lemma is still an application of fundamental tricks for generic properties. Thus we just give a sketch. Let  $O_1, O_2, \ldots, O_n, \cdots$  be a topological basis of  $M^d$ . Let  $\{\eta_m\}$  and  $\{d_\ell\}$  be the sets of positive rational numbers. For each  $n \in \mathbb{N}, K \in \mathbb{N}, m \in \mathbb{N}$  and  $\ell \in \mathbb{N}$ , one defines:

 $-X \in \mathcal{H}_{n,K,m,\ell}$  iff X has a hyperbolic periodic orbit  $\gamma$  such that

•  $\gamma \cap O_n \neq \emptyset$ .

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• There is  $x \in \gamma$  and a time partition  $0 = t_0 < t_1 < \cdots < t_q = \alpha \pi(\gamma)$  for some positive integer  $\alpha$  satisfying  $t_{i+1} - t_i \leq d_\ell$  for  $0 \leq i \leq q - 1$  such that

$$\prod_{i=0}^{q-1} \|\psi_{t_{i+1}-t_i}\|_{\mathcal{N}_{\phi_{t_i}}(x)} \| < K \mathrm{e}^{-\eta_m \alpha \pi(\gamma)}.$$

- $-X \in \mathcal{N}_{n,K,m,\ell}$  iff there is a neighborhood  $\mathcal{U}$  of X such that for any  $Y \in \mathcal{U}$ , one has
  - either Y has no hyperbolic periodic orbit intersecting  $O_n$ ;
  - Or, *Y* has a hyperbolic periodic orbit  $\gamma$  such that  $\gamma \cap O_n \neq \emptyset$  and for any  $x \in \gamma$  and any time partition  $0 = t_0 < t_1 < \cdots < t_q = \alpha \pi(\gamma)$  for any positive integer  $\alpha$  satisfying  $t_{i+1} t_i \leq d_\ell$  for  $0 \leq i \leq q 1$ , one has

$$\prod_{i=0}^{q-1} \|\psi_{t_{i+1}-t_{i}}^{Y}|_{\mathcal{O}_{\phi_{t_{i}}^{Y}(x)}}\| \geq K e^{-\eta_{m}\alpha\pi(\gamma)}.$$

By the definitions,  $\mathscr{H}_{n,K,m,\ell} \cup \mathscr{N}_{n,K,m,\ell}$  is a dense open set in  $\mathscr{X}^1(M^d)$ . Thus,

$$\mathcal{G} = \bigcap_{n,K,m,\ell \in \mathbb{N}} (\mathcal{H}_{n,K,m,\ell} \cup \mathcal{N}_{n,K,m,\ell})$$

is a residual subset of  $\mathcal{X}^1(M^d)$ . Now we check that every  $X \in \mathcal{C}$  satisfies the properties of the lemma.

For any  $\eta > 0$ , T > 0, one can take a rational number  $\eta_{m_0} \in (\eta/2, \eta)$  and  $T_{\ell_0} \in (T, 2T)$ . If there is a sequence of vector fields  $\{X_n\}$  such that

- $\lim_{n\to\infty} X_n = X$ ,
- each  $X_n$  has a hyperbolic periodic orbit  $\gamma_n$  which is  $(K, \eta, T, \mathcal{N})$ -contracting at the period such that  $\gamma_n \cap U \neq \emptyset$ .

There is  $x \in \overline{U}$  such that x is an accumulating point of  $\gamma_n$ . Thus, there is  $n_0$  such that  $x \in O_{n_0} \subset V$ . Since  $X \in \mathcal{G} \subset \mathcal{H}_{n_0,K,m_0,\ell_0} \cup \mathcal{N}_{n_0,K,m_0,\ell_0}$ , one has  $X \in \mathcal{H}_{n_0,K,m_0,\ell_0}$  by the definitions. Thus X itself has a periodic orbit in  $O_{n_0}$  which is  $(K, \eta_{m_0}, T_{\ell_0}, \mathcal{N})$ -contracting at the period w.r.t.  $\psi_t$ . It's  $(K, \eta/2, 2T, \mathcal{N})$ -contracting at the period w.r.t.  $\psi_t$  in V.

LEMMA 3.17. – There is a dense  $G_{\delta}$  set  $\mathcal{C} \subset \mathcal{X}^1(M^d)$  such that for any  $X \in \mathcal{C}$  and  $x \in M^d$ , for any  $K \in \mathbb{N}$ ,  $\eta > 0$  and d > 0, one has

- either, x is contained in a periodic sink which is  $(K, \eta/2, 2d, \mathcal{N})$ -contracting at the period w.r.t.  $\psi_t$ ;
- or, there is a  $C^1$  neighborhood  $\mathcal{U}$  of X and a neighborhood U of x such that for any  $Y \in \mathcal{U}$ , Y has no periodic sink which is  $(K, \eta, d, \mathcal{N})$ -contracting at the period w.r.t.  $\psi_t^Y$ , and intersects U.

*Proof.* – Let  $\mathcal{G}$  be as in Lemma 3.16. If the conclusion of this lemma is not true, one has that

- x is not contained in a periodic sink which is  $(K, \eta/2, 2d, \mathcal{N})$ -contracting at the period w.r.t.  $\psi_t$ ,
- for any  $C^1$  neighborhood  $\mathcal{U}$  of X and any neighborhood U of x, some  $Y \in \mathcal{U}$  has a periodic sink, which is  $(K, \eta, d, \mathcal{N})$ -contracting at the period w.r.t.  $\psi_t^Y$ , and intersects U.

Thus, by Lemma 3.16, for any neighborhood U of x, X itself has a periodic sink intersects U, which is  $(K, \eta/2, 2d, \mathcal{O})$ -contracting at the period w.r.t.  $\psi_t$ . In other words, there is a sequence of periodic points  $x_n$  such that

- $\lim_{n\to\infty} x_n = x$ ,
- $x_n$  is contained in a period sink  $\gamma_n$ , which is  $(K, \eta, d, \mathcal{N})$ -contracting at the period for each *n*. Moreover,  $\{\gamma_n\}$  are distinct.

We assert that  $\tau(\gamma_n) \to \infty$  as  $n \to \infty$ . Otherwise, by taking a limit, x would be in a periodic sink which is  $(K, \eta, d, \mathcal{N})$ -contracting at the period. Thus one can get a contradiction by Lemma 2.23.

COROLLARY 3.18. – Assume that dim  $M^3 = 3$ . There is a residual set  $\mathcal{C} \subset \mathcal{X}^1(M^3) \setminus \overline{\mathcal{H}}\overline{\mathcal{J}}$ such that for any  $X \in \mathcal{C}$ , there exists  $\iota > 0$  such that for any  $\sigma \in \operatorname{Sing}(X)$ , there is a  $C^1$  neighborhood  $\mathcal{U}$  of X and a neighborhood U of  $\sigma$  such that for any periodic orbit  $\gamma$  of Y, if  $\gamma \cap U \neq \emptyset$ , then  $\mathcal{N}_{\gamma}$  admits an  $\iota$ -dominated splitting of index 1 w.r.t. the linear Poincaré flow  $\psi_t$ .

*Proof.* – If  $ind(\gamma) = 1$ , then it is done by Lemma 2.10. Thus, one can assume that  $\gamma$  is a sink or source. Without loss of generality, assume that it is a sink. By Lemma 2.12, if  $\gamma$  dose not admits an  $\iota$ -dominated splitting for some  $\iota$ , then there are C > 0,  $\eta > 0$  and T > 0 such that  $\gamma$  is  $(C, \eta, T, \mathcal{N})$ -contracting at the period w.r.t.  $\psi_t$ . Since  $\sigma$  is a singularity, not a periodic point, by Lemma 3.17, one can get a contradiction.

We would like to list some other generic results we need in this paper.

LEMMA 3.19. – There is a dense  $G_{\delta}$  set  $\mathcal{C} \subset \mathcal{X}^1(M^d)$  such that for each  $X \in \mathcal{C}$ , one has

- 1. For any non-trivial chain recurrent class  $C(\sigma)$ , where  $\sigma$  is a hyperbolic singularity of index d 1, then every separatrix of  $W^u(\sigma)$  is dense in  $C(\sigma)$ . In particular,  $C(\sigma)$  is transitive and Lyapunov stable.
- 2. Let  $i \in [0, \dim M 1]$ . If there is a sequence of vector fields  $\{X_n\}$  such that
  - $\lim_{n\to\infty} X_n = X$ ,
  - each  $X_n$  has a hyperbolic periodic orbits  $\gamma_{X_n}$  of index *i* such that  $\lim_{n\to\infty} \gamma_{X_n} = \Lambda$ .

Then there is a sequence of hyperbolic periodic orbits  $\gamma_n$  of index *i* of *X* such that  $\lim_{n\to\infty} \gamma_n = \Lambda$ .

3. There exists a neighborhood  $\mathcal{U}$  of X such that for any  $Y \in \mathcal{U}$ , Y has only finitely many singularities. Moreover, for every singularity  $\sigma$  of Y, the eigenvalues  $\lambda_1, \lambda_2, \ldots, \lambda_d$  of  $DY(\sigma)$  satisfy:

$$\operatorname{Re}(\lambda_i) + \operatorname{Re}(\lambda_i) \neq 0$$
,

for any  $1 \leq i, j \leq d$ .

- 4. For any hyperbolic periodic orbit P of X, C(P) = H(P), where H(P) is the homoclinic class of P.
- 5. Every non-trivial chain transitive set is the limit of a sequence of periodic orbits in the Hausdorff topology.
- 6. X is Kupka-Smale.
- 7. Given a critical element p of X, if for any neighborhood  $\mathcal{U}$  of X, there is  $Y \in \mathcal{U}$  such that  $C(p_Y)$  is singular hyperbolic, then C(p) is singular hyperbolic itself.

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REMARK. – Item 1 is a corollary of the connecting lemma for pseudo-orbits [6]. There is no explicit version like this. [32, Section 4] gave some ideas about the proof of Item 1 without using the terminology of chain recurrence. Item 2 is fundamental (see [53] for instance). Item 3 is fundamental. It is true because generic X can only have finitely many singularities. Moreover, the eigenvalues of the singularities have some continuous property. Item 4 is also a result in [6]. Item 5 is the main result in [13]. Item 6 is the classical Kupka-Smale theorem [21, 22, 46]. Singular hyperbolicity is an open property. Thus, by using a standard generic argument, we know that Item 7 is true.

Let  $\mathcal{G}_0$  be a dense  $G_\delta$  set of  $\mathcal{X}^1(M^d)$  such that if  $X \in \mathcal{G}_0$  then X satisfies all generic properties mentioned in this subsection.

### 4. Reduction of the main theorems

# 4.1. Prove Theorem A from Theorem B

Proof of Theorem A. - Now we give the proof of Theorem A by assuming the result of Theorem B. Suppose on the contrary that  $\mathfrak{X}^1(M^3) \setminus \overline{\mathcal{MS} \cup \mathcal{HS}}$  is not empty. Choose a  $C^1$  generic  $X \in \mathcal{X}^1(M^3) \setminus \overline{\mathcal{MS} \cup \mathcal{HS}}$ . Since every homoclinic tangency of a hyperbolic periodic orbit can be perturbed to be a transverse homoclinic intersection by an arbitrarily small perturbation, we have that X is far away from ones with a homoclinic tangency. Thus, by Theorem B, every non-trivial chain recurrent class is a homoclinic class. Since we assume that Theorem A is not true, one has that every chain recurrent class is trivial: it is reduced to be a critical element. If there are finitely many chain recurrent classes, then we have that X is Morse-Smale. Thus, one can assume that X has infinitely many chain recurrent classes, and each chain recurrent class is a hyperbolic critical element. It is known that a  $C^{1}$  generic vector field can only have finitely many singularities since it is Kupka-Smale. Thus X has countably many distinct hyperbolic periodic orbits  $\{\gamma_n\}$ . By taking a subsequence if necessary, we can assume that  $\{\gamma_n\}$  converges in the Hausdorff topology. Let  $\Lambda$  be the limit. Then,  $\Lambda$  is chain transitive, which implies that  $\Lambda$  is contained in a chain recurrent class. Because we know that every chain recurrent class of X is a hyperbolic critical element,  $\Lambda$  is a hyperbolic critical element. This cannot happen because hyperbolic critical elements are locally maximal. 

### 4.2. The reduction of the proofs of Theorem B and Theorem C

To prove Theorem B, we need to focus on non-trivial singular chain recurrent classes without periodic orbits.

We need the following theorem to guarantee the existence of the dominated splitting for the tangent flow. The proof will be completed later.

THEOREM 4.1. – For a  $C^1$  generic  $X \in \mathcal{X}^1(M^3) \setminus \overline{\mathcal{HT}}$ , for the non-trivial chain recurrent class  $C(\sigma)$  of some singularity  $\sigma$ , if  $C(\sigma)$  is not a homoclinic class, then  $C(\sigma)$  admits a dominated splitting  $T_{C(\sigma)}M^3 = E \oplus F$  w.r.t. the tangent flow.

In fact, we have the following theorem from [10, Theorem B] where it is proved that one bundle is uniform hyperbolic:  $^{(6)}$ 

THEOREM 4.2. – For a  $C^1$  generic  $X \in \mathcal{X}^1(M^3)$ , and a non-trivial chain recurrent class  $C(\sigma)$  of some singularity  $\sigma$ , if  $C(\sigma)$  admits a dominated splitting  $T_{C(\sigma)}M^3 = E \oplus F$  w.r.t. the tangent flow, then  $C(\sigma)$  admits a partially hyperbolic splitting; more precisely,  $\operatorname{ind}(\sigma) = 2$  iff dim E = 1 and E is contracting.

In general, we give the definition of singular hyperbolic sets as the following: <sup>(7)</sup>

DEFINITION 4.3. – Let  $\Lambda$  be a compact invariant set of  $X \in \mathcal{X}^1(M^3)$ ,  $E \subset T_{\Lambda}M^3$  be a two dimensional invariant sub-bundle, we say that E is area-contracting, if there are C > 0,  $\lambda > 0$  such that for any  $x \in \Lambda$  and any t > 0,  $|\det \Phi_t|_{E(x)}| \leq C e^{-\lambda t}$ ; we say that E is area-expanding if it is area-contracting for -X.

A compact invariant set  $\Lambda$  of  $X \in \mathcal{X}^1(M^3)$  is called singular hyperbolic, if

- either,  $\Lambda$  admits a partially hyperbolic splitting  $T_{\Lambda}M^3 = E^{ss} \oplus E^{cu}$ , where dim  $E^{ss} = 1$ ,  $E^{ss}$  is contracting and  $E^{cu}$  is area-expanding;
- or,  $\Lambda$  admits a partially hyperbolic splitting  $T_{\Lambda}M^3 = E^{cs} \oplus E^{uu}$ , where dim  $E^{uu} = 1$ ,  $E^{uu}$  is expanding and  $E^{cs}$  is area-contracting.

Note that in the above definition, we don't require  $\Lambda$  contains a singularity or not. So every non-trivial hyperbolic set which is disjoint from the singular set is singular hyperbolic.

We have the next theorem which is mainly proven in Section 6:

THEOREM 4.4. – For a  $C^1$  generic  $X \in \mathcal{X}^1(M^3)$  and a non-trivial chain recurrent class  $C(\sigma)$  of some singularity  $\sigma$ , if  $C(\sigma)$  admits a partially hyperbolic splitting  $T_{C(\sigma)}M^3 = E^s \oplus F$  w.r.t. the tangent flow, where dim  $E^s = 1$ , and if  $C(\sigma)$  contains no periodic orbits, then  $C(\sigma)$  is singular hyperbolic.

Morales and Pacifico [32] proved the following results:

THEOREM 4.5. – For a  $C^1$  generic  $X \in \mathcal{X}^1(M^3)$  and a singularity  $\sigma$  of X, if  $C(\sigma)$  is singular hyperbolic and Lyapunov stable, then  $C(\sigma)$  is an attractor. As a corollary,  $C(\sigma)$  contains periodic orbits.

We also needs the following two results from [10]. The version of the first one is the Proposition 3.1 of [10]. It essentially follows the elegant works of Pujals-Sambarino [45] and Arroyo-Rodriguez Hertz [4].

THEOREM 4.6. – For a  $C^1$  generic  $X \in \mathcal{X}^1(M^3)$ , if  $\Lambda$  is a non-singular chain transitive set and admits a dominated splitting  $\mathcal{N}_{\Lambda} = \Delta^s \oplus \Delta^u$  with respect to  $\psi_t$ , then  $\Lambda$  is hyperbolic.

<sup>&</sup>lt;sup>(6)</sup> The statement is a little bit stronger than [10]. But the proof is contained there.

<sup>&</sup>lt;sup>(7)</sup> We postpone to give the definition because we want the introduction to be easier to read.

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THEOREM 4.7. – For a  $C^1$  generic  $X \in \mathcal{X}^1(M^3)$ , if the chain recurrent class  $C(\sigma)$  of a singularity  $\sigma$  contains a periodic orbit and admits a dominated splitting  $T_{C(\sigma)}M = E \oplus F$  with respect to  $\Phi_t$ , then  $C(\sigma)$  is singular hyperbolic. Consequently,  $C(\sigma)$  is an attractor or a repeller depending if the index of  $\sigma$  equals to 2 or 1.

Proof of Theorem B. – If Theorem B is not true, then there is a  $C^1$  generic  $X \in \mathcal{X}^1(M^3) \setminus \overline{\mathcal{HT}}$  and a non-trivial chain recurrent class  $\mathcal{C}$  of X such that  $\mathcal{C}$  is not a homoclinic class. Now we have two cases:

4.2.0.1. *C* contains no singularity. – Since  $\mathcal{C}$  is chain transitive, there is a sequence of periodic orbits  $\{\gamma_n\}$  such that  $\gamma_n \to \mathcal{C}$  as  $n \to \infty$  in the Hausdorff metric. Since *C* is not reduced to a periodic orbit, one can assume that  $\{\gamma_n\}$  are distinct periodic orbits and  $\tau(\gamma_n) \to \infty$ . By Corollary 2.9, if we cannot perturb  $\gamma_n$  to be a hyperbolic periodic orbit of index 1 for *n* large enough, then there are constants C > 0, T > 0,  $\eta > 0$  such that  $\gamma_n$  are  $(C, \eta, T, \mathcal{N})$ -contracting at the period w.r.t.  $\psi_t$  for *X* or -X. Then by Lemma 2.23, one can get a contradiction.

Thus, one can assume that the index of every  $\gamma_n$  is 1. From this, we have that C admits a dominated splitting  $\mathcal{N}_C = \Delta^{cs} \oplus \Delta^{cu}$  of index 1 w.r.t.  $\psi_t$ . By Theorem 4.6, we have that C is hyperbolic. This fact shows that C is a homoclinic class, which gives a contradiction.

4.2.0.2. *C* contains a singularity  $\sigma$ . – Since *C* is not a homoclinic class, by Theorem 4.1, *C* admits a dominated splitting  $T_C M^3 = E \oplus F$  w.r.t. the tangent flow  $\Phi_t$ . Moreover, by Theorem 4.2, it is a partially hyperbolic splitting. If *C* contains a singularity and contains no periodic orbits, by Theorem 4.4, *C* is singular hyperbolic. By the theorem of Morales-Pacifico (Theorem 4.5), *C* is a homoclinic class, which gives a contradiction.

Proof of Theorem C. – Given a  $C^1$  generic  $X \in \mathcal{X}(M^3)$ , assume that  $C(\sigma)$  admits a dominated splitting w.r.t. the tangent flow. If C contains a periodic orbit, Theorem 4.7 implies C is a singular hyperbolic attractor or repeller. To prove that C contains a periodic orbit, suppose on the contrary that C contains no periodic orbits. Then by Theorem 4.2, the dominated splitting is a partially hyperbolic splitting. And then by Theorem 4.4, C is singular hyperbolic and hence Theorem 4.5 implies C is a homoclinic class. This contradiction proves that C contains a periodic orbit.

# 4.3. Proof of Theorem 4.1

The proof of Theorem 4.1 can be divided into the following two propositions:

**PROPOSITION 4.8.** – There is a dense  $G_{\delta}$  set  $\mathcal{C} \subset \mathcal{X}^1(M^3) \setminus \overline{\mathcal{H}\mathcal{T}}$  such that for any  $X \in \mathcal{C}$ , if  $\sigma$  is a hyperbolic saddle of X and  $C(\sigma)$  is Lyapunov stable, then every singularity  $\rho \in C(\sigma)$  is Lorenz-like, i.e., the eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  of  $DX(\rho)$  satisfy:

$$\lambda_1 < \lambda_2 < 0 < -\lambda_2 < \lambda_3.$$

Moreover, there is a  $C^1$  neighborhood  $\mathcal{U}_X$  of X such that for any  $Y \in \mathcal{U}_X$  and any  $\rho \in C(\sigma) \cap \operatorname{Sing}(X)$ , one has  $\rho_Y \in C(\sigma_Y, Y)$  and  $W^{ss}(\rho_Y) \cap C(\sigma_Y, Y) = \{\rho_Y\}$ .

Proposition 4.8 will be proven in Subsection 5.1.

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**PROPOSITION 4.9.** – Let  $\Lambda$  be a compact invariant set of  $X \in \mathcal{X}^1(M^d)$  verifying the following properties:

- $\Lambda \setminus \text{Sing}(X)$  admits an index *i* dominated splitting  $\mathcal{N}_{\Lambda \setminus \text{Sing}(X)} = \Delta^{cs} \oplus \Delta^{cu}$  in the normal bundle w.r.t. the linear Poincaré flow  $\psi_t$ ,
- every singularity  $\sigma \in \Lambda$  is hyperbolic and  $\operatorname{ind}(\sigma) > i$ . Moreover,  $T_{\sigma}M^d$  admits a partially hyperbolic splitting  $T_{\sigma}M^d = E^{ss} \oplus E^{cu}$  with respect to the tangent flow, where dim  $E^{ss} = i$  and for the corresponding strong stable manifolds  $W^{ss}(\sigma)$ , one has  $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$ ,
- for every  $x \in \Lambda$ , one has  $\omega(x) \cap \text{Sing}(X) \neq \emptyset$ .

Then one has

- either  $\Lambda$  admits a partially hyperbolic splitting  $T_{\Lambda}M^d = E^{ss} \oplus F$  with respect to the tangent flow  $\Phi_t$ , where dim  $E^{ss} = i$ ,
- or,  $\Lambda$  intersects a homoclinic class.

Proposition 4.9 will be proven in Subsection 5.2.

Now one can give a proof of Theorem 4.1 by Proposition 4.8 and Proposition 4.9.

Proof of Theorem 4.1. – Since dim  $M^3 = 3$ , without loss generality, one can assume that  $\operatorname{ind}(\sigma) = 2$ . Otherwise, one considers -X. By Lemma 3.19,  $C(\sigma)$  is Lyapunov stable. By Proposition 4.8, every singularity  $\rho$  in  $C(\sigma)$  is Lorenz-like and  $W^{ss}(\rho) \cap C(\sigma) = \{\rho\}$ . Since  $X \in \mathcal{X}^1(M^3) \setminus \overline{\mathcal{HT}}$ , one has

- the normal bundle of  $C(\sigma) \setminus \text{Sing}(X)$  admits a dominated splitting with respect to the linear Poincaré flow (See more details from Corollary 3.18),
- since X is C<sup>1</sup> generic and C(σ) is not a homoclinic class, C(σ) contains no periodic orbit. As a corollary, for every regular point x ∈ C(σ), ω(x) contains a singularity. Otherwise, if ω(x) contains no singularity, then by Theorem 4.6, ω(x) is hyperbolic. Then one can get a periodic orbit in C(σ) by the shadowing lemma, which is a contradiction.

By Proposition 4.9, either  $C(\sigma)$  admits a partially hyperbolic splitting, or  $C(\sigma) \cap H(\gamma) \neq \emptyset$ for some hyperbolic saddle  $\gamma$ . But the fact that  $C(\sigma) \cap H(\gamma) \neq \emptyset$  gives a contradiction.  $\Box$ 

# 4.4. Comments on Theorem 4.4

The proof of Theorem 4.4 will use more notations and definitions. We will give the proof in Section 6.

# 5. Partial hyperbolicity: the proof of Theorem 4.1

### 5.1. Lorenz-like singularities

The goal of this subsection is to prove Proposition 4.8.

LEMMA 5.1. – Let  $\Lambda$  be a non-trivial chain transitive set of  $X \in \mathcal{X}^1(M^d)$ . Assume

- every singularity in  $\Lambda$  is hyperbolic,
- $\mathcal{N}_{\Lambda \setminus \text{Sing}(X)}$  admits a dominated splitting of index *i* w.r.t the linear Poincaré flow  $\psi_t$ .

Then for every hyperbolic singularity  $\sigma \in \Lambda$  with  $ind(\sigma) > i$ ,  $T_{\sigma}M^d$  admits a dominated splitting  $T_{\sigma}M^d = E^{ss} \oplus E^{cu}$  with respect to the tangent flow  $\Phi_t$ , where dim  $E^{ss} = i$ .

*Proof.* – For a hyperbolic singularity  $\sigma \in \Lambda$ , since  $\Lambda$  is a non-trivial chain transitive set, one has  $W^s(\sigma) \cap \Lambda \setminus \{\sigma\} \neq \emptyset$  and  $W^u(\sigma) \cap \Lambda \setminus \{\sigma\} \neq \emptyset$ . One can see [10, Lemma 2.6] for more details about the proof. Recall the definition of  $\widetilde{\Lambda}$ : the lift of  $\Lambda$  in the sphere bundle as in Subsection 2.1. One has that there is  $v \in E^u(\sigma) \cap \widetilde{\Lambda}$ .

Since  $\mathscr{N}_{\Lambda \setminus Sing(X)}$  admits a dominated splitting of index *i* with respect to the linear Poincaré flow,  $\widetilde{\mathscr{N}}_{\Lambda}$  admits a dominated splitting of index *i* with respect to the extended linear Poincaré flow  $\widetilde{\psi}_t$  by Corollary 2.11. We will consider the negative limit set  $\alpha(v)$  with respect to the flow  $\Phi_t^I$ .

By changing the Riemannian metric in a small neighborhood of  $\sigma$ , without loss of generality, one can assume  $E^{s}(\sigma) \perp E^{u}(\sigma)$ . Thus,

- $\tilde{\mathcal{N}}_{\alpha(v)}$  admits a dominated splitting  $\Delta^{cs} \oplus \Delta^{cu}$  of index *i* w.r.t  $\tilde{\psi}_t$  since  $\mathcal{N}_{\Lambda \setminus \text{Sing}(X)}$  admits a dominated splitting of index *i* with respect to  $\psi_t$ ,
- the hyperbolic splitting on  $T_{\sigma}M^d$  implies that:  $\widetilde{\mathcal{N}}_{\alpha(v)}$  admits a dominated splitting  $E^s \oplus F$  of index dim  $E^s$  for some F w.r.t.  $\widetilde{\psi_t}$ , since  $\sigma$  is hyperbolic and  $E^s(\sigma) \perp E^u(\sigma)$ .

Thus by the properties of dominated splittings, one has  $\Delta^{cs}(\alpha(v)) \subset E^s(\sigma)$ . Thus, there are C > 0 and  $\lambda > 0$  such that for any  $u \in E^u \cap S_\sigma M^d$ , there is a splitting  $\Delta^{cs} \oplus \Delta^{cu} = \widetilde{\mathcal{N}}_{(\sigma,u)}$  with the following property:

- By the natural hyperbolic splitting on the singularity  $\sigma$ , we have

$$\frac{\|\psi_t|_{\Delta^{cs}(u)}\|}{\|\Phi_t|_{}\|} \le C e^{-\lambda t}, \quad \forall t \ge 0.$$

- By extending the dominated splitting to the sphere bundle, we have

$$\frac{\|\psi_t|_{\Delta^{cs}(u)}\|}{m(\widetilde{\psi}_t|_{\Delta^{cu}(u)})} \le C \,\mathrm{e}^{-\lambda t}, \quad \forall t \ge 0.$$

Then by Lemma 2.13, we know that  $\sigma$  admits a dominated splitting  $T_{\sigma}M^d = E \oplus F$  w.r.t. the tangent flow  $\Phi_t$ , where dim E = i. Since ind $\sigma > i$ , one can get the splitting as required.

LEMMA 5.2. – There is a residual subset  $\mathcal{G} \subset \mathcal{K}^1(M^3) \setminus \overline{\mathcal{H}} \mathcal{T}$  such that for any hyperbolic singularity  $\sigma$  of index 2 of  $X \in \mathcal{G}$ , if  $C(\sigma)$  is non-trivial, then  $T_{\sigma}M^3$  admits a dominated splitting  $T_{\sigma}M^3 = E^{ss} \oplus F$  w.r.t. the tangent flow  $\Phi_t$ , where dim  $E^{ss} = 1$ ,  $E^{ss}$  is contracting, and  $W^{ss}(\sigma) \cap C(\sigma) = \{\sigma\}$ .

Similarly, if  $ind(\sigma) = 1$  and  $C(\sigma)$  is non-trivial, then  $W^{uu}(\sigma) \cap C(\sigma) = \{\sigma\}$ .

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*Proof.* – Assume that X satisfies all generic properties in  $\S3$ . We focus on the case of ind( $\sigma$ ) = 2. By Lemma 3.19, there is a sequence of periodic orbit  $\gamma_n$  such that  $\lim_{n\to\infty} \gamma_n = C(\sigma)$  in the Hausdorff topology. By Corollary 3.18, there is  $\iota > 0$  such that each  $\gamma_n$  admits an *i*-dominated splitting in  $\mathcal{N}_{\gamma_n}$  w.r.t. the linear Poincaré flow. Then by Corollary 2.11,  $\mathcal{N}_{C(\sigma) \setminus Sing(X)}$  admits a dominated splitting of index 1 with respect to the linear Poincaré flow  $\psi_t$ . As a corollary of Lemma 5.1,  $T_{\sigma}M$  has dominated splitting  $T_{\sigma}M = E^{ss} \oplus F$  w.r.t. the tangent flow  $\Phi_t$ . Thus what we need to prove now is  $W^{ss}(\sigma) \cap C(\sigma) = \{\sigma\}$ . We will prove this by absurd. If this is not true, there is  $x_0 \in W^{ss}(\sigma) \cap C(\sigma) \setminus \{\sigma\}$ . One also notice that  $C(\sigma)$  is Lyapunov stable since  $ind(\sigma) = 2$  by Lemma 3.19. Hence  $W^u(\sigma) \subset C(\sigma)$  and  $W^u(\sigma)$  is dense in  $C(\sigma)$ . By changing the Riemannian metric in a small neighborhood, we may assume that  $E^{ss}(\sigma)$ ,  $E^{cs}(\sigma)$  and  $E^{u}(\sigma)$  are mutually orthogonal, where  $E^{cs}(\sigma) = E^{s}(\sigma) \cap F$ .

Thus, by using the  $C^1$  connecting lemma (Lemma 3.4), there is an arbitrarily small perturbation Y of X such that

- Y has a strong connection with respect to  $\sigma$ : there is  $y \in M^3$  such that  $y \in W^{ss}(\sigma) \cap$  $W^u(\sigma) \setminus \{\sigma\},\$
- Y(x) = X(x) if x is in a small neighborhood of  $\sigma$ .

By an extra small perturbation, one can assume that Y has the following extra properties:

- Y is linear in a small neighborhood of  $\sigma$  in some local chart;
- $E^{ss}(\sigma, Y)$ ,  $E^{cs}(\sigma, Y)$  and  $E^{u}(\sigma, Y)$  are still mutually orthogonal.

Let P be the plane spanned by  $E^{ss}$  and  $E^{u}$  in the local chart. One has that P is locally invariant: there is a neighborhood  $O_1$  of  $\sigma$  such that for any  $x \in P \cap O_1$ , if  $\phi_{[0,t]}^Y(x) \in O_1$ , then  $\phi_t^Y(x) \in P$ . Now for Y, by an extra perturbation, there is a sequence of vector fields  $Y_n$ and a smaller neighborhood  $O_2$  of  $\sigma$  such that

- $\lim_{n\to\infty} Y_n = Y;$
- for each  $n, Y_n = Y$  in  $O_2$ ;
- $Y_n$  has a periodic orbit  $\gamma_n$  such that  $\gamma_n \cap O_2 \subset P$  and  $\lim_{n \to \infty} \gamma_n = \operatorname{Orb}(y, Y) \cup \{\sigma_Y\}$ .
- By Corollary 3.18, one has
- there are  $\iota = \iota(X)$  such that  $\mathscr{N}_{\gamma_n}$  has an  $\iota$ -dominated splitting  $\Delta^{cs} \oplus \Delta^{cu}$  of index 1 with respect to the linear Poincaré flow  $\psi_t^{Y_n}$ .

Thus,  $\mathcal{N}_{Orb}(y,Y)$  admits an *i*-dominated splitting w.r.t.  $\psi_t^Y$ . Let  $\Gamma_Y = Orb(y,Y) \cup \sigma_Y$ . It is a compact invariant set. Over the lift  $\widetilde{\Gamma_Y}$  (see Subsection 2.1), there exists a dominated splitting  $\widetilde{\mathcal{N}}_{\widetilde{\Gamma}_{V}} = \Delta^{cs} \oplus \Delta^{cu}$  w.r.t.  $\widetilde{\psi}_{t}$  such that

- for v<sup>u</sup> ∈ E<sup>u</sup>(σ) ∩ Γ̃<sub>Y</sub>, one has Δ<sup>cs</sup>(v<sup>u</sup>) = E<sup>ss</sup> and Δ<sup>cu</sup>(v<sup>u</sup>) = E<sup>cs</sup>,
  For v<sup>ss</sup> ∈ E<sup>ss</sup>(σ) ∩ Γ̃<sub>Y</sub>, one has Δ<sup>cs</sup>(v<sup>ss</sup>) = E<sup>cs</sup> and Δ<sup>cu</sup>(v<sup>ss</sup>) = E<sup>u</sup>.

By the continuity of the dominated splitting of  $\widetilde{\psi}_t$  over  $\widetilde{\mathcal{W}}_{\Gamma_v}$ , one can choose  $t_1 > 0$  and  $t_2 > 0$  large enough such that

- φ<sub>t1</sub>(y) ∈ W<sup>ss</sup><sub>loc</sub>(σ<sub>Y</sub>) and φ<sub>-t2</sub>(y) ∈ W<sup>u</sup><sub>loc</sub>(σ<sub>Y</sub>),
  Δ<sup>cs</sup>(φ<sub>-t2</sub>(y)) is close to E<sup>ss</sup>(σ<sub>Y</sub>) and Δ<sup>cs</sup>(φ<sub>t1</sub>(y)) is close to E<sup>cs</sup>(σ<sub>Y</sub>).

Then we take a point  $p_n \in \gamma_n$  and  $t_n > 0$  such that

•  $p_n$  is close to  $\phi_{-t_2}^Y(y)$  and  $\phi_{-t_n}^Y(p_n)$  is close to  $\phi_{t_1}^Y(y)$ ,



FIGURE 1. Local dynamics near a strong connection

- $\phi_{[-t_n,0]}^Y(p_n)$  is contained in the plane P spanned by  $E^{ss}$  and  $E^u$  in the local chart,
- $\phi_t^Y(p_n) = \phi_t^{Y_n}(p_n)$  for any  $t \in [-t_n, 0]$  by the construction of  $Y_n$ .

Thus we have that  $\Delta^{cs}(p_n)$  is close to  $E^{ss}(\sigma_Y)$  and  $\Delta^{cs}(\phi_{-t_n}^Y(p_n))$  is close to  $E^{cs}(\sigma_Y)$ . In other words, in the local linearized chart, if we extend  $E^{ss}$ ,  $E^{cs}$  and  $E^u$  to every point in a small neighborhood of  $\sigma_Y$ , we have  $E^{ss}$  is contained in a cone (with a prescribed small size) of  $\Delta^{cs}(p_n)$ . By the invariance of *cs*-cone field by backward iterations, we have  $\psi_{-t_n}(E^{ss}(p_n))$  is also contained in the cone of  $\Delta^{cs}(\phi_{-t_n}(p_n))$  with a prescribed small size.

Recall that we assume the dynamics is linear in a small neighborhood of  $\sigma_Y$  and P is spanned by  $E^{ss} \oplus E^u$ , we have  $\psi^Y_{-t_n}(E^{ss}(p_n))$  is also contained in P; in other words, it is almost orthogonal to  $E^{cs}(\sigma_Y)$ . This shows that  $\Delta^{cs}(\phi_{-t_n}(p_n))$  is almost orthogonal to  $E^{cs}(\sigma_Y)$ . We get a contradiction since we have  $\Delta^{cs}(\phi^Y_{-t_n}(p_n))$  is close to  $E^{cs}(\sigma_Y)$ .

One can also see the analysis in the proof of Lemma 4.3 in [23, page 255-256].  $\Box$ 

COROLLARY 5.3. – There is a residual subset  $\mathcal{G} \subset \mathcal{K}^1(M^3) \setminus \overline{\mathcal{HT}}$  such that if a chain recurrent class contains singularities, then all the singularities in the chain recurrent class have the same index.

*Proof.* – Let  $\mathcal{G} = \mathcal{G}_0 \setminus \overline{\mathcal{HT}}$ , where  $\mathcal{G}_0$  is as in the end of Subsection 3.3. We will prove this corollary by absurd. If it's not true, then there is  $X \in \mathcal{G}$  and a chain recurrent class C of X such that C contains singularities of different indices. Thus, one can assume that C contains two singularities  $\sigma_1$  and  $\sigma_2$  satisfying  $\operatorname{ind}(\sigma_1) = 1$  and  $\operatorname{ind}(\sigma_2) = 2$ . By Lemma 5.2,

 $T_{\sigma_1}M^3 = E^{cs} \oplus E^{uu}$  is a dominated splitting w.r.t.  $\Phi_t$ , where dim  $E^{uu} = 1$ , and for the corresponding strong unstable manifold  $W^{uu}(\sigma_1) \cap C = \{\sigma_1\}$ . Since  $\sigma_2$  is codimension 1, by Item 1 of Lemma 3.19,  $W^u(\sigma_2) \subset C$ . This fact implies that C is Lyapunov stable by Lemma 3.14. As a corollary, one has  $W^{uu}(\sigma_1) \subset C$ . This gives us a contradiction.

COROLLARY 5.4. – There is a residual subset  $\mathcal{G} \subset \mathcal{X}^1(M^3) \setminus \overline{\mathcal{H}T}$  such that for any hyperbolic singularity  $\sigma$  of index 2 of  $X \in \mathcal{G}$ , there is a  $C^1$  neighborhood  $\mathcal{U}$  of X such that for any  $Y \in \mathcal{U}$  and for any singularity  $\rho \in C(\sigma)$ , one has

- $\operatorname{ind}(\rho) = 2$  and  $\rho_Y \in C(\sigma_Y)$ ,
- $T_{\rho}M^3 = E^{ss} \oplus E^{cu}$  is a dominated splitting w.r.t.  $\Phi_t$ , where dim  $E^{ss} = 1$ ,
- for the corresponding stable manifolds of  $E^{ss}$ , one has  $W^{ss}(\rho_Y, Y) \cap C(\sigma_Y) = \{\rho_Y\}$ .

*Proof.* – This is true just because X is  $C^1$  generic and the continuous property of the local strong stable manifolds.

Furthermore, we have

THEOREM 5.5. – For a generic  $X \in \mathcal{X}^1(M^3) \setminus \overline{\mathcal{HT}}$ , and a hyperbolic singularity  $\sigma$  of index 2, if  $C(\sigma)$  is non trivial, then  $\sigma$  is Lorenz-like for X, i.e., the eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  of  $DX(\sigma)$  are all real and satisfy

$$\lambda_1 < \lambda_2 < 0 < -\lambda_2 < \lambda_3.$$

*Proof.* – First by Lemma 5.2, for the three eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  of  $DX(\sigma)$ , they are all real and

$$\lambda_1 < \lambda_2 < 0 < \lambda_3.$$

What's left is to prove that  $\lambda_2 + \lambda_3 > 0$ . The three corresponding eigenspaces are denoted by  $E^{ss}(\sigma)$ ,  $E^{cs}(\sigma)$  and  $E^u(\sigma)$ . By changing the Riemannian metric in a small neighborhood of  $\sigma$ , we can assume that they are mutually orthogonal. We will prove this by absurd, i.e., assume that  $\lambda_2 + \lambda_3 \leq 0$ . Since X is  $C^1$  generic, by Lemma 3.19 one has  $\lambda_2 + \lambda_3 < 0$ . Moreover,

- $W^{u}(\sigma) \subset C(\sigma)$  and  $W^{u}(\sigma)$  is dense in  $C(\sigma)$  by Lemma 3.19,
- $W^{s}(\sigma) \cap C(\sigma) \setminus \{\sigma\} \neq \emptyset$  since  $C(\sigma)$  is non-trivial.

By using Lemma 3.4 (the  $C^1$  connecting lemma), for any  $C^1$  neighborhood  $\mathcal{U}$  of X, there is  $Y \in \mathcal{U}$  such that

• *Y* has a homoclinic orbit  $\Gamma$  associated to  $\sigma_Y$ ,

• for the three eigenvalues  $\lambda_1^Y < \lambda_2^Y < 0 < \lambda_3^Y$  of  $DY(\sigma_Y)$ , one still has  $\lambda_2^Y + \lambda_3^Y < 0$ ,

•  $W^{ss}(\sigma_Y) \cap C(\sigma_Y) = \{\sigma_Y\}$  by Corollary 5.4.

By simple perturbations, there is a sequence of vector fields  $Y_n$  such that

- $\lim_{n\to\infty} Y_n = Y$ ,
- each  $Y_n$  has a hyperbolic periodic orbit  $\gamma_n$  such that  $\lim_{n\to\infty} \gamma_n = \Gamma \cup \sigma$ .

Under our assumption,  $\{\gamma_n\}$  can be sinks or saddles. Let C > 0,  $\eta > 0$ ,  $\iota > 0$  be as in Lemma 2.12. By Corollary 3.18  $_{\mathcal{O}}\mathcal{N}_{\gamma_n}$  admits an  $\iota$ -dominated splitting of index 1 with respect to  $\psi_{\tau_n}^{Y_n}$  for *n* large enough.

As a corollary,  $\mathscr{N}_{\Gamma}$  admits an *i*-dominated splitting w.r.t.  $\psi_t^Y$ . Thus,  $\widetilde{\mathscr{N}}_{\Gamma \cup \sigma}$  admits an *i*-dominated splitting  $\widetilde{\mathscr{N}}_{\Gamma \cup \sigma} = \Delta^{cs} \oplus \Delta^{cu}$  w.r.t.  $\widetilde{\psi}_t^Y$ .

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CLAIM. – For every  $v \in T_{\sigma}M^3 \cap (\widetilde{\Gamma \cup \sigma})$ , one has  $\Delta^{cs}(v) = E^{ss}(\sigma)$ .

*Proof.* – For each  $v \in T_{\sigma}M^3$ , if  $v \in \widetilde{\Gamma \cup \sigma}$ , then  $v \in E^{cs}(\sigma)$  or  $v \in E^u(\sigma)$ . Since  $E^{ss}(\sigma)$ ,  $E^{cs}(\sigma)$  and  $E^u(\sigma)$  are mutually orthogonal, one has

- if  $v \in E^{cs}(\sigma)$ , since  $\widetilde{\mathcal{O}}_v = E^{ss}(\sigma) \oplus E^u(\sigma)$  is a dominated splitting w.r.t.  $\widetilde{\psi}_t$ , one has  $\Delta^{cs}(v) = E^{ss}(\sigma)$ ;
- if  $v \in E^u(\sigma)$ , since  $\widetilde{\mathcal{O}}_v = E^{ss}(\sigma) \oplus E^{cs}(\sigma)$  is a dominated splitting w.r.t.  $\widetilde{\psi}_t$ , one has  $\Delta^{cs}(v) = E^{ss}(\sigma)$ .

Since the unique ergodic measure is supported on  $\{\sigma\}$  for  $\phi_t|_{\Gamma \cup \sigma}$ , one has that there are C > 0 and  $\lambda > 0$  such that for any t > 0 and any  $(x, v) \in \widetilde{\Gamma \cup \sigma}$ ,

$$\frac{\left\|\psi_{t}\right\|_{\Delta^{cs}(x,v)}}{\left\|\Phi_{t}\right\|_{}} \leq Ce^{-\lambda t}.$$

By Lemma 2.13,  $\Gamma \cup \sigma$  admits a dominated splitting  $T_{\Gamma \cup \sigma} M^3 = E \oplus F$  w.r.t. the tangent flow  $\Phi_t$ , where dim E = 1. Thus  $E(\sigma) = E^{ss}(\sigma)$  by the uniqueness of dominated splittings. Since the unique ergodic measure is supported on  $\{\sigma\}$ , one has that E is uniformly contracting. Thus  $\gamma_n$  is also  $(C, \eta, d, \mathcal{N})$ -contracting at the period w.r.t.  $\psi_t^{Y_n}$  for some C > 0, d > 0 and  $\eta \in (0, -(\lambda_2 + \lambda_3))$  which depends on X since  $\gamma_n$  stays in a small neighborhood of the singularity for most time. This also gives a contradiction by Lemma 3.17.

Now we are ready to conclude Proposition 4.8.

Proof of Proposition 4.8. – Since  $\sigma$  is a hyperbolic saddle, we have  $\operatorname{ind}(\sigma) = 1$  or  $\operatorname{ind}(\sigma) = 2$ . If  $\operatorname{ind}(\sigma) = 1$ , by Lemma 5.2, we have  $W^{uu}(\sigma) \cap C(\sigma) = \{\sigma\}$ . The Lyapunov stability of  $C(\sigma)$  implies  $C(\sigma) \supset W^u(\sigma) \supset W^{uu}(\sigma)$ . Thus we get a contradiction.

So we have  $ind(\sigma) = 2$ . By Corollary 5.3, every singularity  $\rho$  contained in  $C(\sigma)$  is of index 2. By Theorem 5.5,  $\rho$  is Lorenz-like, i.e., the eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  of  $DX(\rho)$  satisfy:

$$\lambda_1 < \lambda_2 < 0 < -\lambda_2 < \lambda_3.$$

By Lemma 5.2, for any  $\rho \in C(\sigma)$ , we have  $W^{ss}(\rho) \cap C(\sigma) = \{\rho\}$ .

The above properties also hold in a small neighborhood of X. Now we give the proof. Since there are only finitely many singularities, by Lemma 3.12, for any  $\rho \in C(\sigma)$ , its continuation  $\rho_Y$  is also contained in  $C(\sigma_Y)$  for Y close to X. The inequality of eigenvalues is a robust property. For the strong stable manifold, we assume by contradiction there is a sequence of vector fields  $\{X_n\}$  such that  $\lim_{n\to\infty} X_n = X$  and one separatrix of  $W^{ss}(\rho_{X_n})$  is contained in  $C(\sigma_{X_n})$ . Since there is some uniform  $\varepsilon > 0$  such that  $W^{ss}_{\varepsilon}(\rho_{X_n}) \to W^{ss}_{\varepsilon}(\rho)$  for  $n \to \infty$ , there is one separatrix of  $W^{ss}_{\varepsilon}(\rho)$  is contained in  $C(\sigma)$ . Thus we get a contradiction.

# 5.2. Proof of Proposition 4.9

*Proof of Proposition 4.9.* – For proving this proposition, one assumes that the second case of the conclusion cannot occur.

By changing the Riemannian metric in a small neighborhood of singularities, one can assume that  $E^{ss}(\sigma) \perp E^{cu}(\sigma)$  for any singularity  $\sigma \in \Lambda$ . There is  $T^* > 0$  such that

• for any  $\sigma \in \Lambda$  and any unit vector  $v \in E^{cu}(\sigma)$ , one has

$$\frac{\|\Phi_{T^*}|_{E^{ss}}\|}{|\Phi_{T^*}(v)|} \le \frac{1}{4},$$

•  $\mathcal{N}_{\Lambda \setminus \text{Sing}(X)} = \Delta^{cs} \oplus \Delta^{cu}$  is a  $T^*$ -dominated splitting w.r.t.  $\psi_t$ .

We consider  $\widetilde{\Lambda}$ , the lift of  $\Lambda$  in the sphere bundle as in Subsection 2.1. By considering the dynamics of  $\chi_t$ , one has  $\widetilde{\mathcal{O}}_{\widetilde{\Lambda}} = \widetilde{\Delta^{cs}} \oplus \widetilde{\Delta^{cu}}$  is a  $T^*$ -dominated splitting of index iw.r.t.  $\widetilde{\psi}_t = \operatorname{proj}_2(\chi_t)$  verifying the following property:  $\Delta^{cs}(x) = \widetilde{\Delta^{cs}}(X(x)/|X(x)|)$  and  $\Delta^{cu}(x) = \widetilde{\Delta^{cu}}(X(x)/|X(x)|)$  for any regular point  $x \in \Lambda$ .

Since  $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$ , one has if  $v \in \widetilde{\Lambda} \cap T_{\sigma}M^d$ , then  $v \in E^{cu}(\sigma)$ . On  $\widetilde{\Lambda}$ , one defines the function  $\widetilde{\xi}$  by

$$\widetilde{\xi}: \widetilde{\Lambda} \to \mathbb{R}, \\ v \mapsto \log \|\widetilde{\psi}_{T^*}|_{\widetilde{\Delta^{cs}}(v)}\| - \log \|\Phi_{T^*}(v)\|.$$

Since  $\widetilde{\Delta}^{cs}$  is a continuous bundle,  $\widetilde{\xi}$  is a continuous function. On  $\Lambda \setminus \text{Sing}(X)$ , one can define the function  $\xi$  by

$$\xi: \Lambda \setminus \operatorname{Sing}(X) \to \mathbb{R},$$
$$x \mapsto \log \|\psi_{T^*}|_{\Delta^{cs}(x)}\| - \log \|\Phi_{T^*}|_{\langle X(x) \rangle}\|.$$

By the definitions, for every regular point  $x \in \Lambda$ ,  $\xi(x) = \tilde{\xi}(X(x)/|X(x)|)$ .  $\tilde{\xi}$  is defined on a *compact* set and  $\xi$  is defined on a *non-compact* set.

CLAIM. – There is C > 1 and  $0 < \lambda < 1$  such that for any  $v \in \widetilde{\Lambda}$  and  $n \in \mathbb{N}$ , one has

$$\frac{\|\widetilde{\psi}_{nT^*}|_{\widetilde{\Delta^{cs}}(v)}\|}{|\Phi_{nT^*}(v)|} \leq C\lambda^n.$$

*Proof of the Claim.* – The claim is equivalent to the following statement: There are C > 1 and  $0 < \lambda < 1$  such that for any  $v \in \widetilde{\Lambda}$  and  $n \in \mathbb{N}$ ,

$$\sum_{i=0}^{n-1} \widetilde{\xi}(\Phi_{iT^*}^I(v)) \le \log C + n \log \lambda.$$

If the claim is not true, by Lemma 2.27, for any  $n \in \mathbb{N}$ , there is  $v_n \in \widetilde{\Lambda}$  such that for any integer  $\ell \in [1, n]$ , one has

$$\sum_{j=0}^{\ell-1}\widetilde{\xi}(\Phi_{iT^*}^I(v_n)) \ge 0.$$

Let  $b \in \widetilde{\Lambda}$  be an accumulation point of  $\{v_n\}$ . Then for any  $n \in \mathbb{N}$ ,

$$\sum_{i=0}^{n-1} \tilde{\xi}(\Phi_{iT^*}^I(b)) \ge 0.$$

Since we assume that  $E^{ss}(\sigma) \perp E^{cu}(\sigma)$ , one has for any  $v \in T_{\sigma}M^{d} \cap \widetilde{\Lambda}$ ,  $\widetilde{\Delta^{cs}}(v) = E^{ss}(\sigma)$ and  $v \in E^{cu}$ . As a corollary, one has for any  $n \in \mathbb{N}$ ,

$$\sum_{i=0}^{n-1} \tilde{\xi}(\Phi_{iT^*}^I(v)) \le -n\log 4.$$

For every point  $x \in \Lambda$ , by our assumptions,  $\omega(x)$  contains a hyperbolic singularity. Thus for its lift  $\tilde{\Lambda}$ , for any point  $c \in \tilde{\Lambda}$ , there is a singularity  $\sigma \in \Lambda$  and a unit vector  $v \in E^{cu}(\sigma)$ such that  $v \in \omega(c)$ . Thus for the function  $\tilde{\xi}$  and the flow  $\phi_t^I$ , the conditions of Lemma 2.26 (Liao's sifting lemma) are satisfied.

For any four numbers  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$  and  $\lambda_4$  satisfying  $-\log 4/2 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < 0$ . Let

$$\widetilde{\Lambda}_{\lambda_2} = \{ v \in \widetilde{\Lambda} : \ \widetilde{\xi}(\Phi^I_{-T^*}(v)) \ge \lambda_2 \}.$$

Since there are only finite many singularities, there is  $\varepsilon_0 = \varepsilon_0(\lambda_2) > 0$ , such that for any singularity  $\sigma \in \Lambda$ , and any unit vector  $v \in T_{\sigma}M^d \cap \widetilde{\Lambda}$ ,  $d_T(v, \widetilde{\Lambda}_{\lambda_2}) \ge \varepsilon_0$ . By Lemma 2.26, for any  $k \in \mathbb{N}$ , there is *u* in the positive orbit of *b* and integers  $0 = n_0 < n_1 < \cdots < n_k$  such that for each integer  $\ell \in [0, k - 1]$ , for any integer  $m \in [1, n_{\ell+1} - n_{\ell}]$  one has

$$\sum_{j=0}^{m-1} \tilde{\xi}(\Phi_{jT^*}^I(\Phi_{n_{\ell}T^*}^I(u))) \le m\lambda_3,$$
$$\sum_{j=m-1}^{n_{\ell+1}-n_{\ell}-1} \tilde{\xi}(\Phi_{jT^*}^I(\Phi_{(n_{\ell}+m-1)T^*}^I(u))) \ge (n_{\ell+1}-n_{\ell}-m+1)\lambda_2.$$

Thus  $\Phi_{n_{\ell}T^*}^I(u) \in \widetilde{\Lambda}_{\lambda_2}$  for  $1 \le \ell \le k$ . Assume that b = X(x)/|X(x)| (i.e.,  $\pi(u) = x$ ) and  $u = \Phi_{t_0}^I(b)$ . Thus,

$$d(\phi_{n_{\ell}T^*}(\phi_{t_0}(x)), \operatorname{Sing}(X)) \geq \varepsilon_0$$

There exists  $\varepsilon > 0$  small enough such that for any regular point  $\beta \in \Lambda$ , for any point  $\theta \in B(\beta, \varepsilon | X(\beta) |)$ , for any  $T' \in [(1 - \varepsilon)T^*, (1 + \varepsilon)T^*]$ , for any two subspaces  $G(\beta) \subset {}_{\mathcal{O}}\mathcal{N}_{\beta}$  and  $G(\theta) \subset {}_{\mathcal{O}}\mathcal{N}_{\theta}$  satisfying  $\tilde{d}(G(\beta), G(\theta)) < \varepsilon$ , one has

$$\begin{split} \lambda_1 - \lambda_2 &\leq \log \|\psi_{T^*}^*|_{G(\beta)}\| - \log \|\psi_{T'}^*|_{G(\theta)}\| \leq \lambda_4 - \lambda_3, \\ &\max\{\frac{\|\psi_{T'}|_{G(\theta)}\|}{\|\psi_{T^*}|_{G(\beta)}\|}, \frac{m(\psi_{T^*}|_{G(\beta)})}{m(\psi_{T'}|_{G(\theta)})}\} \leq \frac{\sqrt{2}}{2}. \end{split}$$

For this  $\varepsilon > 0$ , there is  $\delta = \delta(\varepsilon)$  as in Theorem 2.25 (Liao's shadowing). There is  $k_{\delta} \in \mathbb{N}$  such that for any  $k_{\delta}$  points  $\{x_1, x_2, \ldots, x_{k_{\delta}}\} \subset \widetilde{\Lambda}_{\lambda_2}$ , there are  $1 \le i_1 < i_2 \le k_{\delta}$  such that  $d(x_{i_1}, x_{i_2}) < \delta$ . For this  $k_{\delta}$ , there are  $n_1 < n_2 < \cdots < n_{k_{\delta}}$  and a point  $y' \in \operatorname{Orb}^+(x)$  such that for the function  $\xi$  and  $0 \le \ell \le k_{\delta} - 1$  and  $m \in [1, n_{\ell+1} - n_{\ell}]$ , one has

$$\sum_{j=0}^{m-1} \xi(\phi_{jT^*}(\phi_{n_{\ell}T^*}(y'))) \le m\lambda_3,$$
$$\sum_{j=m-1}^{n_{\ell+1}-n_{\ell}-1} \xi(\phi_{jT^*}(\phi_{(n_{\ell}+m-1)T^*}(y'))) \ge (n_{\ell+1}-n_{\ell}-m+1)\lambda_2.$$

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Let  $y_{\ell} = \phi_{n_{\ell}T^*}(y')$ . By the dominated properties, one has for each  $y_{\ell}$ , for any integer  $m \in [1, n_{i+1} - n_i]$ , one has

$$\prod_{j=0}^{m-1} \|\psi_{T^*}^*|_{\Delta^{cs}}(\phi_{jT^*}(y_\ell))\| \le e^{m\lambda_3},$$
$$\prod_{j=m-1}^{n_{\ell+1}-n_{\ell}-1} \|\psi_{-T^*}^*|_{\Delta^{cu}(\phi_{j+1}(y_\ell))}\| \le (\frac{1}{4})^{n_{\ell+1}-n_{\ell}-m} e^{-(n_{\ell+1}-n_{\ell}-m)\lambda_2}.$$

Let  $\eta = \min\{-\lambda_3, \log 4 + \lambda_2\}$ . One has that

- $\phi_{[0,(n_{\ell+1}-n_{\ell})T^*]}(y_{\ell})$  is an  $(\eta, T^*)$ - $\psi_t^*$ -quasi hyperbolic strings.
- $d(y_{\ell}, \operatorname{Sing}(X)) \geq \varepsilon_0$ .

By the choice of  $k_{\delta}$ , there are  $y_{\alpha}$  and  $y_{\beta}$  such that  $d(y_{\alpha}, y_{\beta}) < \delta$ . Thus by Theorem 2.25, the orbit segment from  $y_{\alpha}$  to  $y_{\beta}$  can be shadowed: there is a periodic orbit  $P_{\varepsilon}$  with period  $\tau(P_{\varepsilon})$  and  $p_{\varepsilon} \in P_{\varepsilon}$  and a monotone increasing function  $\theta_{\varepsilon}(t), \theta_{\varepsilon}(0) = 0$  such that

- $d(\phi_{\theta_{\varepsilon}(t)}(p_{\varepsilon}), \phi_t(y_{\alpha})) < \varepsilon |X(\phi_t(y_{\alpha}))|$  for any  $0 \le t \le (n_{\beta} n_{\alpha})T^*$ .
- $1 \varepsilon \leq \theta_{\varepsilon}'(t) \leq 1 + \varepsilon$  and  $\theta_{\varepsilon}((n_{\beta} n_{\alpha})T^*) = \tau(P_{\varepsilon}),$
- there is a direct-sum splitting  $\mathcal{N}_{p_{\varepsilon}} = E(p_{\varepsilon}) \oplus F(p_{\varepsilon})$  such that

$$\begin{split} \psi^*_{\theta((n_{\beta}-n_{\alpha})T^*)}(E(p_{\varepsilon})) &= E(p_{\varepsilon}), \\ \psi^*_{\theta((n_{\beta}-n_{\alpha})T^*)}(F(p_{\varepsilon})) &= F(p_{\varepsilon}), \end{split}$$

and for any  $t \in [0, (n_{\beta} - n_{\alpha})T^*]$ ,

$$\begin{split} \tilde{d}(\psi_t^*(E(y_\alpha)),\psi_{\theta(t)}^*(E(p_\varepsilon))) &\leq \varepsilon, \\ \tilde{d}(\psi_t^*(F(y_\alpha)),\psi_{\theta(t)}^*(F(p_\varepsilon))) &\leq \varepsilon. \end{split}$$

Let  $\eta' = \min\{-\lambda_4, \log 2 + \lambda_1\}$  and  $T' = (1 + \varepsilon)T$ . By the choosing of  $\varepsilon$ , one has that  $P_{\varepsilon}$  is an  $(\eta', T') - \psi_t^*$ -quasi hyperbolic string. For each q and  $\varepsilon = 1/q$ , there is a periodic orbit  $P_{1/q}$  with period  $\tau(P_{1/q})$  such that

- $\limsup_{q\to\infty} P_{1/q} \subset \Lambda$ ,
- there is  $p_{1/q} \in P_{1/q}$  such that  $d(p_{1/q}, \operatorname{Sing}(X)) \geq \varepsilon_0$ ,  $p_{1/q}$  is  $(1, \eta', T', E) \psi_t^*$ -contracting and  $(1, \eta', T', F) \psi_t^*$ -expanding,
- $P_{1/q}$  admits a T\*-dominated splitting w.r.t.  $\psi_t^*$ .

Without loss of generality, one can assume that  $\{p_{1/q}\}_{q \in \mathbb{N}}$  converges. By Lemma 2.18<sup>(8)</sup>, for any two  $w, m \in \mathbb{N}$  large enough, one has

$$W^{s}(P_{1/w}) \pitchfork W^{u}(P_{1/m}) \neq \emptyset, \quad W^{u}(P_{1/w}) \pitchfork W^{s}(P_{1/m}) \neq \emptyset.$$

In other words, they are homoclinically related. Thus for w large enough, one has  $\Lambda \cap H(P_{1/w}) \neq \emptyset$ . Thus we have the second case of Proposition 4.9. This contradiction proves the claim.

<sup>&</sup>lt;sup>(8)</sup> This is the part that we need the existence of central plaques as in Section 2.

By Lemma 2.13, and the claim above, one has  $\Lambda$  admits a dominated splitting  $T_{\Lambda}M = E \oplus F$  w.r.t. the tangent flow, where dim E = i and  $X(x) \subset F(x)$  for any regular point  $x \in \Lambda$ . Assume that it is a *T*-dominated splitting. We define the function  $f : \Lambda \to \mathbb{R}$  by  $f(x) = \log \|\Phi_T\|_{F(x)}\|$ .

CLAIM. - We have

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{\ell=0}^{n-1}f(\phi_{\ell T}(x))<0,\quad\forall x\in\Lambda.$$

*Proof.* – For every point  $x \in \Lambda$ , there are two cases: either  $\omega(x) \subset \text{Sing}(X)$ , or  $\omega(x)$  contains a regular point  $a \in \Lambda$ . In the first case, since every singularity in  $\Lambda$  admits a partially hyperbolic splitting, one has that the claim is true. Now we consider the second case:  $\omega(x)$  contains a regular point a. We fix a neighborhood  $U_a$  of a such that for any  $z, y \in U_a$ , one has

$$\frac{1}{2} \le \frac{|X(z)|}{|X(y)|} \le 2.$$

Let  $f_0(z) = \log \|\Phi_T|_{E(z)}\| - \log \|\Phi_T|_{\langle X(z) \rangle}\|$  for every regular point  $z \in \Lambda$ . Since  $T_{\Lambda}M^d = E \oplus F$  is a dominated splitting and  $X(z) \subset F(z)$  for every regular point  $z \in \Lambda$ , one has  $f_0(z) \leq -\log 2$  for any regular point z. Since  $a \in \omega(x)$ , one can take  $x_0 \in \operatorname{Orb}^+(x)$  such that  $x_0 \in U_a$  and there is a sequence of times  $\{t_n\}_{n \in \mathbb{N}}$  such that  $\lim_{n \to \infty} \phi_{t_n}(x_0) = a$  and  $\lim_{n \to \infty} t_n = \infty$ .

For *n* large enough, assume that  $t_n = kT + t$ , where  $t \in [0, T]$ . Thus we have

$$\frac{1}{k} \sum_{\ell=0}^{k-1} f(\phi_{\ell T}(x_0)) \le -\log 2 + \frac{1}{k} (\log \|\Phi_T\|_{< X(\phi_{kT}(x_0))>} \| -\log \|\Phi_T\|_{X(x_0)} \|).$$

Since  $k \to \infty$  as  $n \to \infty$ , one has

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{\ell=0}^{n-1} f(\phi_{\ell T}(x_0)) < 0$$

Thus the same inequality holds for x.

By the above claim, one has that for any  $x \in \Lambda$ , there exists  $n_x \in \mathbb{N}$  such that  $\sum_{\ell=0}^{n_x-1} f(\phi_{\ell T}(x)) < 0$ . By Lemma 2.27, we have that *E* is uniformly contracting. This ends the proof Proposition 4.9.

#### 6. Perturbations in partially hyperbolic Lyapunov stable chain recurrent classes

The purpose of this section is to prove Theorem 4.4: for a  $C^1$  generic  $X \in \mathcal{X}^1(M^3)$  and a non-trivial chain recurrent class  $C(\sigma)$  of some singularity  $\sigma$ , if  $C(\sigma)$  admits a partially hyperbolic splitting  $T_{C(\sigma)}M^3 = E^s \oplus F$  w.r.t. the tangent flow, where dim  $E^s = 1$ , and if  $C(\sigma)$  contains no periodic orbits, then  $C(\sigma)$  is singular hyperbolic.

What's left is to prove that F is area-expanding. We will prove Theorem 4.4 by absurd, i.e., F is not area-expanding. It suffices to consider three-dimensional vector fields in  $\mathcal{G}_0$  ( $\mathcal{G}_0$  was defined at the end of Section 3). We assume that there is  $X \in \mathcal{G}_0$  such that X has a non-trivial chain recurrent class  $C(\sigma)$  with a partially hyperbolic splitting  $T_{C(\sigma)}M^3 = E^{ss} \oplus F$  w.r.t.

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the tangent flow, where  $E^{ss}$  is one-dimensional and contracting,  $C(\sigma)$  contains no periodic orbits and F is not area-expanding. Hereafter, we will fix this X. By the discussions before, there is a neighborhood  $\mathcal{U}_X$  of X such that:

- A.1  $C(\sigma_Y)$  is not singular hyperbolic for any  $Y \in \mathcal{U}_X$  (Lemma 3.19). Notice that  $C(\sigma_Y)$  may contain periodic orbits for non-generic Y.
- A.2 For every singularity  $\rho \in C(\sigma_Y)$  of any  $Y \in \mathcal{U}_X$ , one has that  $\rho$  is Lorenz-like and  $W^{ss}(\rho) \cap C(\sigma_Y) = \{\rho\}$  (Proposition 4.8).
- A.3 For every weak Kupka-Smale  $Y \in \mathcal{U}_X$ ,  $C(\sigma_Y)$  is Lyapunov stable (Lemma 3.15). As a corollary, for any sink  $\gamma$  of Y, we have  $\overline{\text{Basin}(\gamma)} \cap C(\sigma_Y) = \emptyset$ .
- A.4 There are  $\varepsilon_0 > 0$  and a neighborhood U of  $C(\sigma)$ , such that for any  $Y \in \mathcal{U}_X$  and any  $x \in U$ , the strong stable manifold  $W_{\varepsilon_0}^{ss}(x, Y)$  exists (e.g., see [8, page 289]).
- A.5 A singularity  $\rho_Y$  is contained in  $C(\sigma_Y)$  if and only if it is the continuation of a singularity  $\rho \in C(\sigma)$  (Lemma 3.12).

The strategy of the proof of Theorem 4.4 is to construct some "good" cross sections and some "good" return maps. It follows the steps below:

- 1. We construct some cross sections and some return map with good properties. For example, we can choose each cross section is to be *thin* and the boundary of the cross section is to be *disjoint* from  $C(\sigma)$ . Notice that this is not true for the geometric Lorenz attractor. When the cross section of a geometric Lorenz attractor is thin, then the stable boundary of the cross section will intersect the attractor. Moreover, the properties of cross sections and return maps are robust.  $C(\sigma_Y)$  is not singular hyperbolic for  $Y \in \mathcal{U}_X$ , but  $C(\sigma_Y)$  may contain periodic orbits.
- 2. By considering some special  $C^2$  weak Kupka-Smale vector field Y close to X and by considering some special sets and measures of Y, we can get *better* properties of the return map of the cross sections.
- 3. By doing some extra perturbation if necessary, one can get some sink whose basin accumulates to  $C(\sigma_Y)$  and get a contradiction to the robustness of Lyapunov stability. To get a sink by perturbation, it is always easy by Mañé's ergodic closing lemma. But we don't have the control of the basin. Here, the basin can be small by size; but it is enough to achieve the class.

We have the following three subsections to detail the three steps above.

# 6.1. Cross sections of partially hyperbolic Lyapunov stable chain recurrent classes

DEFINITION 6.1. – For  $Z \in \mathcal{X}^1(M^3)$ , S is called a cross-section of Z if

- *S* is a  $C^1$  surface which is homeomorphic to  $(-1, 1)^2$ ,
- $\measuredangle(T_xS, \langle Z(x) \rangle) > \pi/4, \forall x \in S.$

DEFINITION 6.2. – Let  $\rho$  be a Lorenz-like singularity in some partially hyperbolic Lyapunov stable chain recurrent class. A cross section S is called a singular cross-section associated to  $\rho$  if the following conditions are satisfied:

- 1. There is a homeomorphism  $h = h(x, y) : [-1, 1]^2 \to \overline{S}$  such that  $h((-1, 1)^2) = S$ .
- 2. There is some uniform  $\alpha > 0$  such that  $h(x, \cdot) = W_{loc}^{ss}(\phi_{[-\alpha,\alpha]}(h(x,0))) \cap S$ .
- 3.  $S \cap W^s_{\text{loc}}(\rho) = h(\{0\} \times (-1, 1)).$

In fact,  $(h, [-1, 1]^2)$  (or h for short) is a coordinate system of the surface S.

Denote by  $\ell = S \cap W^s_{loc}(\rho)$  and  $S \setminus \ell = S^l \cup S^r$ , i.e., the local stable manifold of the singularity (in fact  $\ell$ ) cut the S into two pieces: the left part  $S^l$  and the right part  $S^r$  (see Figure 2).



FIGURE 2. Cross-section

For every Lorenz-like singularity  $\rho$ , we will take two singular cross-sections  $S^+$ ,  $S^-$ , which are on the opposite sides of  $E^{ss}(\rho) \oplus E^u(\rho)$ . Denote by  $\ell^{\pm} = W^s_{loc}(\rho) \cap S^{\pm}_i$  for  $\pm \in \{+, -\}$ .

For every Lorenz-like singularity  $\rho$ , there is an orientation  $\eta$  defined in a neighborhood of  $\rho$ . More precisely, one can define the bundle  $E^u(\rho)$  and the unstable cone  $\mathcal{C}^u$  (of some size) in a neighborhood of  $\rho$  continuously. We say an orientated curve  $\gamma$  tangent to the cone field  $\mathcal{C}^u$  has the orientation  $\eta$ , if  $\pi^u(\gamma(t_1)) > \pi^u(\gamma(t_0))$  for any  $1 \ge t_1 > t_0 \ge 0$ , where  $\pi^u$  is the projection to  $E^u$  in a local chart. Since each separatrix of the local unstable manifold is a curve tangent to the cone field  $\mathcal{C}^u$ , we will say one separatrix of local unstable manifold is in the direction  $\eta$ , and the other one is in the direction  $-\eta$ .

For  $Y \in \mathcal{U}_X$ , put  $C(\sigma_Y) \cap \text{Sing}(Y) = \{\sigma_{1,Y}, \ldots, \sigma_{k,Y}\}$  and

$$\Sigma = \bigcup_{1 \le i \le k} (S_i^+ \cup S_i^-),$$

where  $S_i^+$  and  $S_i^-$  are two singular cross-sections associated to  $\sigma_{i,Y}$ .  $S_i^+$  and  $S_i^-$  are disjoint since they are on the opposite sides of  $E^{ss}(\sigma_{i,Y}) \oplus E^u(\sigma_{i,Y})$ . The partially hyperbolic splitting on  $C(\sigma_Y)$  induces two cone fields on  $\Sigma$ : the strong stable cone field  $\mathcal{C}^{ss}$  and the centerunstable cone field  $\mathcal{C}^{cu}$ . A curve is called an *ss curve* if it is tangent to  $\mathcal{C}^{ss}$ ; it is called a *cu-curve* if it is tangent to  $\mathcal{C}^{cu}$ . Notice that by choosing  $S_i^{\pm}$  is small and almost orthogonal to Y, and by choosing the width of  $\mathcal{C}^{cu}$  small enough, we have that the  $\mathcal{C}^{cu}$  (defined in the two-dimensional space) is contained in  $\mathcal{C}^u$  (defined in the three-dimensional space). The boundary of  $\Sigma$  is composed by *ss* curves and *cu*-curves, which are called *ss-boundaries* and *cu-boundaries*.

For each  $S_i^{\pm}$ , let  $h_i^{\pm}$ :  $[-1, 1]^2 \rightarrow \overline{S_i^{\pm}}$  be the homeomorphism in the definition of singular cross-section. For every  $p \in \Sigma$ , there exists a unique coordinate system  $h_i^{\pm}$  for some

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 $i \in \{1, 2, ..., k\}$  and  $\pm \in \{+, -\}$ . We denote  $h_i^{\pm}$  by  $h_p$ . For each  $p \in \Sigma$ , one can associate it to  $(x_p, y_p) \in [-1, 1]^2$  such that  $h_p((x_p, y_p)) = p$ . Note that for every  $p, p' \in \overline{S_i^{\pm}}, h_p = h_{p'}$ . For each  $p \in \Sigma$ , if  $\{t > 0 : \phi_t(p) \in \Sigma\} \neq \emptyset$ , we can define the first return time

to each  $p \in \mathbb{Z}$ , if  $\{t \ge 0 : \phi_t(p) \in \mathbb{Z}\} \neq \emptyset$ , we can define the first return time  $t_p = \min\{t \ge 0 : \phi_t(p) \in \Sigma\}$  and define  $R(p) = \phi_{t_p}(p)$ .  $R : \text{Dom}(R) \to \Sigma$  is called the *first return map* associated to  $\Sigma$ , where

$$Dom(R) = \{ p \in \Sigma : \exists t > 0 \text{ s.t. } \phi_t^Y(p) \in \Sigma \}.$$

Notice that one should denote them by  $\Sigma_Y$  and  $R_Y$ . When there is no confusion, we just denote them by  $\Sigma$  and R for simplicity.

**REMARK.** – In the above definition, the local stable manifolds of the singularities also have the exponentially contracting property with respect to the flow  $\phi_t$ . Thus,

$$\bigcup_{1 \le i \le k} \bigcup_{\pm \in \{+,-\}} \bigcup_{x \in (-1,1)} h_i^{\pm}(\{x\} \times (-1,1))$$

can be regarded as a stable foliation of the first return map.

Since  $S_i^{\pm}$  is transverse to the vector field, Dom(R) is open and  $t_x$  is upper semi-continuous with x.

DEFINITION 6.3. – In the notations above,  $(\Sigma, R)$  is a cross-section system of  $(C(\sigma_Y), Y)$  if the following conditions are satisfied:

- 1.  $\partial \Sigma \cap C(\sigma_Y) = \emptyset$ , where  $\partial \Sigma$  is the boundary of  $\Sigma$ .
- 2. For each  $p \in \Sigma$ , there is  $\varepsilon > 0$  such that  $W_{loc}^{ss}(\phi_{(-\varepsilon,\varepsilon)}(p)) \cap \Sigma \subset h_p(\{x_p\} \times (-1,1))$ , where  $h_p(x_p, y_p) = p$ .
- 3. For each  $1 \leq i \leq k$ , and  $x \in W^s_{loc}(\sigma_{i,Y}) \cap C(\sigma_Y) \setminus \{\sigma_{i,Y}\}$ , there exists  $t \in \mathbb{R}$  such that  $\phi_t(x) \in \ell_i^{\pm}$  and if t > 0, then  $\phi_{[0,t]}(x) \subset W^s_{loc}(\sigma_{i,Y})$  and if  $t < 0, \phi_{[t,0]}(x) \subset W^s_{loc}(\sigma_{i,Y})$ .
- 4. The ss-adapted property: there is  $\alpha_0 \in (0, 1)$  such that for any  $p = h_p((x_p, y_p)) \in Dom(R)$ ,  $q = R(p) = h_q((x_q, y_q))$ , we have that  $h_p((x_p, t))$  is in the domain of R for any  $t \in [-1, 1]$  and  $R(h_p(\{x_p\} \times [-1, 1])) \subset h_q(\{x_q\} \times (-\alpha_0, \alpha_0))$ .
- 5. For any  $x \in C(\sigma_Y) \setminus \bigcup_{1 \le i \le k} W^s_{loc}(\sigma_{i,Y})$ , the positive orbit of x will intersect  $\Sigma$ . In particular,

$$C(\sigma_Y) \cap \Sigma \setminus \bigcup_{1 \leq i \leq k} W^s_{\text{loc}}(\sigma_{i,Y}) \subset \text{Dom}(R).$$

REMARK. – In the above definition, Item 1 will help us to define all positive iterations of the return map R. The geometric Lorenz attractor has cross sections with this property. But for general singular hyperbolic attractors, we don't know this is true or not. Item 4 implies some adapted property: the iteration of each stable leaf under the first return map is totally contained in the cross section. Usually we don't have the adapted property in the *cu* direction.

Recall that X is a  $C^1$  generic three-dimensional vector field,  $C(\sigma)$  is a non-trivial chain recurrent class of X,  $C(\sigma)$  admits a partially hyperbolic splitting  $T_{C(\sigma)}M^3 = E^{ss} \oplus F$  with dim F = 2, and  $C(\sigma)$  contains no periodic orbits. Thus, we have the properties (A.1)-(A.5) mentioned before.

**PROPOSITION 6.4.** – For X,  $C(\sigma)$  admits a cross-section system  $(\Sigma, R)$ .

*Proof. The preparation: curves in the stable manifolds.* – Assume that

 $C(\sigma) \cap \operatorname{Sing}(X) = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}.$ 

For each singularity  $\sigma_i$ , one can choose a local chart  $\phi_i^s : \mathbb{R}^2 \to W^s_{\text{loc}}(\sigma_i)$  such that

- $\phi_i^s(\mathbb{R} \times \{0\}) = W_{\text{loc}}^{ss}(\sigma_i),$
- $\phi_i^s(\{0\} \times \mathbb{R})$  is an invariant central manifold.

Take two curves  $\gamma_{i,1}, \gamma_{i,2} : \mathbb{R} \to W^s(\sigma_i)$  as the images of y = x and y = -x under  $\phi_i^s$  such that  $\gamma_{i,i}(0) = \sigma_i$  for j = 1, 2.

CLAIM. – There is  $\rho_i > 0$  such that  $C(\sigma) \cap \phi_i^s([-\rho_i, \rho_i]^2) \subset \phi_i^s(\{(x, y) : |x| < |y|\})$ . As a corollary,  $\gamma_{i,1}([-\rho_i, \rho_i]) \cap C(\sigma) = \{\sigma_i\}$  and  $\gamma_{i,2}([-\rho_i, \rho_i]) \cap C(\sigma) = \{\sigma_i\}$ .

*Proof of the claim.* – If the claim is not true, there are  $x_n \in \phi_i^s(\{(x, y) : |x| \ge |y|\}) \cap C(\sigma)$  such that  $\lim_{n\to\infty} x_n = \sigma_i$  and  $x_n \ne \sigma_i$ . The negative iterations of  $x_n$  are still in  $C(\sigma)$ . Choose a small neighborhood  $B_i$  of  $\sigma_i$ . Let

$$t_n = \sup\{t : \phi_{-s}(x_n) \in B_i, \forall 0 \le s \le t\}.$$

We have that  $t_n \to \infty$  as  $x_n \to \sigma_i$ . Let *a* be an accumulation point of  $\phi_{-t_n}(x_n)$ . Then  $\phi_t(a) \in \{(x, y) : |x| \ge |y|\}$  for  $t \ge 0$ . Hence  $a \in C(\sigma) \cap W^{ss}(\sigma_i) \cap \partial B_i$ . This contradicts the fact that  $W^{ss}(\sigma_i) \cap C(\sigma) = \{\sigma_i\}$  (Property (A.2)). One can also see [23, Lemma 4.4].

The first step of the construction. – Let  $\rho = \min\{\rho_i : 1 \le i \le k\}$ . For each  $\sigma_i$ , there are two connected components  $\Theta_i^{\pm}$  of  $W_{loc}^s(\sigma_i) \setminus W_{loc}^{ss}(\sigma_i)$ . In the following, we will use  $\Theta_i^+$  to construct  $S_i^+$ , while  $S_i^-$  can be constructed similarly.

There are two points  $x_{i,1} \in \gamma_{i,1} \cap \Theta_i^+$  and  $x_{i,2} \in \gamma_{i,2} \cap \Theta_i^+$  such that

- $x_{i,1}, x_{i,2} \notin C(\sigma)$ ,
- $x_{i,1} \in W^{ss}_{loc}(x_{i,2}).$

Thus there is a cross-section  $\widetilde{S}_i^+ = h_i^+((-1,1)^2)$ , where  $h_i^+ : [-1,1]^2 \to \overline{\widetilde{S}_i^+}$  is a homeomorphism, such that

- $h_i^+((0,-1)) = x_{i,1}, h_i^+((0,1)) = x_{i,2}$  and  $h_i^+(\{0\} \times (-1,1))$  is a connected part of a strong stable manifold of  $\phi_t$ ,
- $h_i^+((-1,1) \times \{-1,1\}) \cap C(\sigma) = \emptyset$ ,
- $\widetilde{S}_i^+$  is foliated by strong stable foliation in the following sense: for each  $x \in \widetilde{S}_i^+$ , one defines  $\mathscr{F}^s(x)$  to be the connected component of  $\bigcup_{t \ge [-T,T]} \phi_t(W_{\varepsilon_0}^{ss}(x)) \cap \widetilde{S}_i^+$  (for some T > 0),  $\widetilde{S}_i^+$  can be foliated by  $\mathscr{F}^s$ . Moreover,  $h_i^+(\{z\} \times (-1,1))$  is a leaf of the strong stable foliation,
- For any arbitrarily small number  $\alpha > 0$ , one can require that the horizontal width of  $S_i^+$  (the *cu*-diameter) is less than  $\alpha$ . This implies the *cu*-boundary of  $\widetilde{S}_i^+$  is disjoint from  $C(\sigma)$ ,
- $\bigcup_{x \in (-1,1)} h_i^+(\{x\} \times (-1,1))$  is a family of  $C^1$  curves, and as a  $C^1$  family, it varies continuously with respect to x.

One can construct  $\widetilde{S}_i^-$  in  $\Theta_i^-$  similarly.

6.1.0.1. *Refine the construction:*- We take a neighborhood  $U = B_{\delta}(C(\sigma))$  with  $\delta$  small enough such that  $B_{\delta}(\sigma_i)$  and  $B_{\delta}(\sigma_j)$  are disjoint for  $i \neq j$  and  $\overline{U}$  is disjoint from  $h_i^{\pm}((-\beta_0, \beta_0) \times \{-1, 1\})$  for any *i* and any  $\pm \in \{+, -\}$ .

Denote by  $S_i^{\pm}(\beta) = h_i^{\pm}((-\beta,\beta) \times (-1,1))$  for  $\beta \in (0,\beta_0]$ , and denote by

$$\Sigma(\beta) = \bigcup_{1 \le i \le k, \pm \in \{+, -\}} S_i^{\pm}(\beta)$$

As before, we consider the first return map R with respect to  $\Sigma(\beta)$ .

CLAIM. – There is  $\alpha_0 \in (0, 1)$  such that if  $\beta$  is small enough, for any  $p \in \text{Dom}(R) \cap \Sigma(\beta)$ , q = R(p), we have  $R(h_p(\{x_p\} \times [-1, 1])) \subset h_q(x_q \times (-\alpha_0, \alpha_0))$ , where  $h_p((x_p, y_p)) = p$  and  $h_q((x_q, y_q)) = q$ .



FIGURE 3. Cross-section and return map:  $\Lambda$  does not intersect the shaded area and the image of any strong stable leaf under the return map does not intersect the shaded area.

*Proof of the claim.* – By the construction of h, for each point  $p \in \Sigma(1)$ ,  $\mathcal{F}^{s}(p)$  is uniformly close to the strong stable manifold of p by the time-one map  $\varphi_1$ . If  $\beta$  is small, the orbit of  $p \in \Sigma(\beta)$  will intersect a small neighborhood of a singularity and then return to the cross section  $\Sigma(1)$ . This implies the return time is long. Thus the stable foliation of p in the cross section will be contracted a lot.

Since X is a  $C^1$  generic vector field,  $C(\sigma)$  can be accumulated by periodic orbits. By assumptions,  $C(\sigma)$  contains no hyperbolic periodic orbit. We will use this fact ( $C(\sigma)$  contains no periodic orbits) to prove that the left and the right boundary of  $\Sigma$  have empty intersection with  $C(\sigma)$ .

CLAIM. - Given 
$$\beta \in (0, \beta_0]$$
, for every  $i = 1, 2, ..., k$  and  $\pm \in \{+, -\}$ ,  
 $J_i^{\pm} = \{z \in (-\beta, \beta) : h_i^{\pm}(\{z\} \times (-1, 1)) \cap C(\sigma) = \emptyset\}$ 

*is open and dense in*  $(-\beta, \beta)$ *.* 

Proof of the claim. – We only have to prove that: Given  $a, b \in (-\beta, \beta), a < b$ , there exists  $z \in (a, b)$  such that  $h_i^{\pm}(\{z\} \times (-1, 1)) \cap C(\sigma) = \emptyset$ . Otherwise, since  $C(\sigma)$  can be accumulated by periodic orbits by Lemma 3.19, there is a periodic point  $p \in S_i^{\pm}$  close to  $h_i^{\pm}(\{(a + b)/2\} \times (-1, 1))$ . Thus  $W_{\varepsilon_0}^{ss}(p) \cap C(\sigma) \neq \emptyset$ . This contradicts the fact that  $C(\sigma) \cap \operatorname{Per}(X) = \emptyset$ .

For  $\beta > 0$  small enough,

$$G = \bigcap_{1 \le i \le k, \pm \in \{+, -\}} (0, \beta) \cap J_i^{\pm} \cap (-J_i^{\pm})$$

is open and dense in  $(0, \beta)$ .

So, take  $\beta' \in G$  and let  $\Sigma = \Sigma(\beta')$ . For this cross section  $\Sigma(\beta')$ , we define the return map R. Notice the return time is longer. After scaling, we may assume coordinate mappings  $h_i^{\pm}$  are defined on  $(-1, 1)^2$ . Then  $(\Sigma, R)$  satisfies item 1)-4) in the definition of cross-section system.

6.1.0.2. The domain of R:- For any  $x \in C(\sigma)$ , if  $\omega(x)$  contains no singularity, then  $\omega(x)$  is a (non-singular) hyperbolic set by Theorem 4.6. By using the shadowing lemma,  $\omega(x)$  is shadowed by periodic orbits in the same homoclinic class. Thus,  $C(\sigma)$  contains periodic orbits. This will contradict our assumption that  $C(\sigma)$  contains no periodic orbits. Now for every  $x \in C(\sigma)$ , if x is not in the local stable manifold of some singularity, then the positive iterations of x will be close to stable manifold of some singularity in  $C(\sigma)$ . Some of its iterations will close to the intersection  $\Sigma$  and local stable manifolds of singularities. This finishes the proof of the existence of the cross-section system.

By summarizing the construction as in the above proof, we first find some "large" crosssection  $\Sigma(1)$ , then we just take some smaller part  $\Sigma(\beta')$  which is modified by the strong stable foliation. Since the local strong stable manifolds are continuous with respect to the vector fields, for any  $Y \ C^1$  close to X, the intersection of the strong stable manifolds  $W_{\text{loc}}^{ss}(z, Y)$ of  $z \in \Sigma(\beta')$  and  $\Sigma(1)$  is close to  $\Sigma(\beta')$ . The cross-section system has some continuous property.

**PROPOSITION 6.5.** – By reducing  $\mathcal{U}_X$  if necessary,  $C(\sigma_Y)$  admits a cross-section system  $(\Sigma_Y, R_Y)$  for  $Y \in \mathcal{U}_X$ . Moreover, one can require that  $\Sigma_Y$  is close to  $\Sigma_X$ : it is just obtained by modifying the boundary of  $\Sigma_X$  slightly.

*Proof.* – As explained above, we take

$$\Sigma_Y = (\bigcup_{z \in \Sigma_X} \phi_{[-\varepsilon(z),\varepsilon(z)]}(W^{ss}_{\text{loc}}(z,Y))) \cap \Sigma(1),$$

where  $(\Sigma_X, R_X)$  is constructed as in Proposition 6.4. Moreover, one can define the first return map  $R_Y$  by using this cross section.

By the continuity of the local strong stable manifolds w.r.t. the vector fields, we have that  $\Sigma_Y$  is close to  $\Sigma = \Sigma_X$  when Y is close to X. Since  $C(\sigma_Y)$  is continuous w.r.t. Y, we have that  $C(\sigma_Y) \cap \partial \Sigma_Y = \emptyset$ . By the definition of  $\Sigma_Y$ , Item 2 of the cross section system is satisfied. Item 3 is true because  $C(\sigma_Y) \cap W^s_{loc}(\sigma_Y)$  is continuous w.r.t. Y. Item 4 is true if the return time is long, which can be guaranteed if the cross section is thin. We have Item 5 because

- the return time is uniformly continuous when the point is not close to the local stable manifolds of the singularities,
- when the point is close to the local stable manifolds of the singularities, the return will follow the unstable manifolds of the singularities, which are stably contained in the new cross-section.

By using the Lyapunov stability for the chain recurrent class, we have the following lemma:

LEMMA 6.6. – For any weak Kupka-Smale  $Y \in \mathcal{U}_X$ , there is a neighborhood  $U_s$  of  $\ell_i^{\pm}$  such that for any cu-curve  $\gamma \subset U_s$ , we have

- either, there is  $N \in \mathbb{N}$  such that  $R_Y^n(\gamma)$  is a connected cu-curve for any  $n \leq N$ , and  $R_Y^N(\gamma)$  intersects  $\ell_i^{\pm}$  for some  $i \in \{1, ..., k\}$  and  $\pm \in \{+, -\}$ ;
- or,  $R_Y^n(\gamma)$  is a connected cu-curve that is contained in  $\Sigma_Y$  for any  $n \in \mathbb{N}$ .

*Proof.* – Since  $Y \in \mathcal{U}_X$  is weak Kupka-Smale, we have that  $C(\sigma_Y)$  is Lyapunov stable. Since the boundary of  $\Sigma_Y$  is disjoint from  $C(\sigma_Y)$ , there is a neighborhood U of  $C(\sigma)$  such that the closure of U is disjoint from the boundary of  $\Sigma_Y$ . By reducing U if necessary, for any point  $x \in U$ , either it is contained in the local stable manifold of some singularity, or its forward iteration will intersect  $\Sigma_Y$ .

By the Lyapunov stability, there is a neighborhood V of  $C(\sigma_Y)$  such that  $\phi_t(V) \subset U$ . By choosing a small neighborhood  $U_s$  of  $\ell_i^{\pm}$ , we know that the iteration of  $U_s$  will be close to  $\sigma_{i,Y}$ . Hence the iteration of  $U_s$  will be contained in V.

Now for a *cu*-curve  $\gamma \subset U_s$ , we know that all its positive iteration will be contained in U. Thus for any  $N \in \mathbb{N}$ , if  $R_Y^N(\gamma)$  doesn't intersect some  $\ell_i^{\pm}$ , then it is a *cu*-curve and in the domain of  $R_Y$ . Thus one can define  $R_Y^{N+1}(\gamma)$  by Lyapunov stability. The conclusion follows from an inductive argument.

We can give more details about the structure of the return map R for points close to  $\ell_i^{\pm}$ . For each singularity  $\sigma_i$ , we can fix two points  $z_l$  and  $z_r$  in the left and right separatrix of the local unstable manifold of  $\sigma_{i,Y}$ . At these two points, we put two cross sections  $\Sigma_{i,l}^u$  and  $\Sigma_{i,r}^u$  at  $z_l$  and  $z_r$ , respectively. Then, we have

- for points close to  $\ell_i^{\pm}$  but not in  $\ell_i^{\pm}$ , the flow induces a map  $R_1$  from  $\Sigma$  to  $\Sigma_{i,l}^u \cup \Sigma_{i,r}^u$ ,
- if the diameters of  $\Sigma_{i,l}^u$  and  $\Sigma_{i,r}^u$  are small enough, then the flow induces a map  $R_2$  from  $\Sigma_{i,l}^u \cup \Sigma_{i,r}^u$  to  $\Sigma$ . The times used to define  $R_2$  is uniformly bounded,
- finally,  $R = R_2 \circ R_1$  for points in a small neighborhood of  $\ell_i^+ \cup \ell_i^-$ .

From the proof of Proposition 6.4 and Proposition 6.5, in fact we have the following additional information:

LEMMA 6.7. – For any  $Y \in \mathcal{U}_X$ , for any sequence of points  $\{x_n\}$  in  $C(\sigma_Y)$  such that  $x_n \notin W^u_{loc}(\sigma_Y)$  for any  $n \in \mathbb{N}$  and  $x_n \to z \in W^u_{loc}(\sigma_Y) \setminus \{\sigma_Y\}$ , denoting  $t_n = \sup\{t < 0 | \phi_t^Y(x_n) \in \Sigma\}$ , any limit point of  $\phi_{t_n}^Y(x_n)$  is contained in  $\ell_i^+ \cup \ell_i^-$ .



FIGURE 4. The composition of the return map

*Proof.* – We first show that  $t_n$  is well-defined for n large enough. Take a small enough closed neighborhood U of  $\sigma_Y$  so that if  $\phi_t^Y(x) \subset U$  for t > 0, then  $x \in W_{loc}^s(\sigma_Y)$ . Let  $s_n = \inf\{t < 0 \mid \phi_{[t,0]}^Y(x_n) \subset U\}$ . Since  $x_n \to z \in W_{loc}^u(\sigma_Y)$ , we have that  $s_n \to -\infty$ . Let y be a limit point of  $\phi_{s_n}^Y(x_n)$ . Then  $\phi_t^Y(y) \in U$  for any t > 0. Hence  $y \in W_{loc}^s(\sigma_Y) \cap \partial U \cap C(\sigma_Y)$ . According to the construction of  $\Sigma$  in Proposition 6.4, we have that for some  $\tau < 0$  such that  $\phi_\tau^Y(y) \in \ell_i^+ \cup \ell_i^-$ . Hence  $t_n$  is well-defined for n large enough.

By taking a subsequence, we may assume that  $\lim_{n\to\infty} \phi_{t_n}^Y(x_n) = y$ .

Since  $x_n \to z \in W^u_{\text{loc}}(\sigma_Y), t_n \to -\infty$ . Hence  $y \in W^s_{\text{loc}}(\sigma) \cap \Sigma \cap C(\sigma_Y)$ . According to the construction of  $\Sigma$  in Proposition 6.4, we have that  $y \in \ell_i^+ \cup \ell_i^-$ .

We have the following uniform continuity for the return map *R*:

LEMMA 6.8. – For any  $\varepsilon > 0$ , there is  $\delta > 0$  such that for any cu-curve  $\gamma \subset \Sigma$ , if the length of  $\gamma$  is less than  $\delta$ , and  $\gamma$  does not intersect the local stable manifold of singularities, then the length of  $R(\gamma)$  is less than  $\varepsilon$ .

*Proof.* – We first prove that for any  $i \in \{1, 2, ..., k\}$ , for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that for any *cu*-curve  $\gamma \subset S_i^{+,r}$ , if the length of  $\gamma$  is less than  $\delta$ , then the length of  $R_1(\gamma)$  is less than  $\varepsilon$ . Suppose on the contrary, for some  $\varepsilon > 0$ , there is a sequence of *cu*-curves  $\gamma_n \subset S_i^{+,r}$  such that the length of  $\gamma_n$  tend to zero, but the length of  $R_1(\gamma_n) \subset \Sigma_{i,r}^u$  equals  $\varepsilon$ . By taking a subsequence, we may assume that  $R_1(\gamma_n)$  tends to  $\gamma'$ . Then both two ends of  $\gamma_n$  tend to some point on  $l_i^+$  and at least one end of  $\gamma'$  is not  $z_r$ , where  $z_r$  is the point chosen in the right separatrix of the local unstable manifold. This contradiction finishes the proof.

Since  $R_2$  is a sectional map of bounded times, the conclusion holds for R.

Near the stable manifolds of singularities, we have the expansion for the return map:

LEMMA 6.9. – For any K > 0, there is  $\delta > 0$  such that for any cu-curve  $\gamma \subset \Sigma$  in the  $\delta$ -neighborhood of  $\bigcup \ell_i^{\pm}$ , then  $\|DR|_{T_{\times \gamma}}\| > K$  for any  $x \in \gamma$ .

*Proof.* – We prove this lemma for *cu*-curves close to  $\ell_i^{\pm}$ . For the singularity  $\sigma_i$ , we choose two points  $z_l$  and  $z_r$  in the local unstable manifold of  $\sigma_i$  such that they are in the different separatices. There is a time T > 0 such that  $\phi_{[0,T]}(z_l)$  and  $\phi_{[0,T]}(z_r)$  intersect  $\Sigma$ . Thus there is a constant C such that for any *cu*-curves in  $\Sigma^u$ , we have the derivative of  $DR_2$  along to this curve is larger than C. Since  $\sigma_i$  is Lorenz-like, there exists a partially hyperbolic splitting  $T_{\sigma_i}M = E^{ss} \oplus E^{cu}$  such that  $E^{cu}$  is area-expanding ( $\sigma_i$  is Lorenz-like). Assume that  $R_1(x) = \phi_{\tau(x)}(x)$ . Then  $\tau(x) \to \infty$  as  $x \to \ell_i^{\pm}$ . So, if the *cu*-curve  $\gamma$  is arbitrarily close to  $\ell_i^{\pm}$ , we have that  $|DR_1|_{T_x\gamma}|$  can be arbitrarily large. Since  $R = R_2 \circ R_1$  and  $R_2$  is defined by using bounded times, we get the conclusion.

## 6.2. Vector fields close to X

In this subsection, we will consider the properties of some vector fields close to X and to realize the Step 2 of the strategy.

**DEFINITION 6.10.** – A nonempty compact invariant set  $\Lambda$  is called an  $\mathcal{N}$ -set, if F is not area-expanding on  $\Lambda$ , and F is area-expanding on every proper compact invariant set of  $\Lambda$ .

As in the case of minimally non-hyperbolic set, by Zorn's lemma, for any compact invariant set  $\Lambda$ , if F is not area-expanding on  $\Lambda$ , then  $\Lambda$  contains an  $\mathcal{N}$ -set.

LEMMA 6.11. –  $C(\sigma_Y)$  contains a transitive  $\mathcal{N}$ -set  $\Lambda_Y$  for any  $Y \in \mathcal{U}_X$ .

*Proof.* – By Properties (A.1)-(A.4),  $C(\sigma_Y, Y)$  is not singular hyperbolic. Hence by Zorn's lemma, there exists an  $c\mathcal{N}$ -set  $\Lambda_Y \subset C(\sigma_Y)$ .

We will prove that  $\Lambda_Y$  is transitive. Notice that  $C(\sigma_Y, Y)$  admits a partially hyperbolic splitting  $T_{C(\sigma_Y,Y)}M^3 = E^{ss} \oplus F$ , where dim  $E^{ss} = 1$ . If  $\Lambda_Y$  is not transitive, for every  $x \in \Lambda_Y$ ,  $\alpha(x)$  is a proper subset of  $\Lambda_Y$ . As a consequence, F is area-expanding on  $\alpha(x)$  for every  $x \in \Lambda_Y$ . This implies  $\limsup_{t\to\infty} \log |\text{Det}\Phi_{-t}^Y|_{F(x)}| < 0$ . Since every  $x \in \Lambda_Y$  has this property, by a compact argument (e.g., see Lemma 2.27), one can prove that F is areaexpanding on  $\Lambda_Y$ . This contradicts the assumption that  $\Lambda_Y$  is an cN-set.

COROLLARY 6.12. – Under the assumption of Lemma 6.11, for every  $\mathcal{N}$ -set  $\Lambda_Y$ , there is an ergodic measure  $\mu_Y$  of  $\phi_t^Y$  such that the support of  $\mu_Y$  is  $\Lambda_Y$ , and for any t > 0, one has

$$\int \log |\mathrm{Det}\Phi_t|_{F(x)} |\mathrm{d}\mu_Y \leq 0.$$

*Proof.* – By Lemma 2.27, there is a point  $x \in \Lambda_Y$  such that  $\log |\text{Det}\Phi_t|_{F(x)}| \le 0$  for any  $t \ge 0$ . By using a standard method, we can have an invariant measure  $\nu$  such that for any t > 0, one has

$$\int \log |\mathrm{Det}\Phi_t|_{F(x)} |\mathrm{d}\nu \leq 0.$$

By using the ergodic decomposition theorem, there is an ergodic component  $\mu_Y$  of  $\nu$  such that

$$\int \log |\mathrm{Det}\Phi_t|_{F(x)} |\mathrm{d}\mu_Y| \le 0.$$

 $\operatorname{supp}(\mu_Y) = \Lambda_Y$ : otherwise,  $\operatorname{supp}(\mu_Y)$  is a proper compact invariant set of  $\Lambda_Y$ ; hence F is area-expanding on  $\operatorname{supp}(\mu_Y)$ , which implies that the inequality above is false.

We know some structures about minimally non-hyperbolic *non-singular* set for  $C^2$  vector fields when the set admits a dominated splitting w.r.t. the linear Poincaré flow, which is the main theorem in [4].

THEOREM 6.13. – Let  $Z \in \mathcal{X}^2(M^3)$ . For a compact invariant transitive set  $\Lambda$  of Z, if

- $\Lambda \cap \operatorname{Sing}(Z) = \emptyset$ ,
- the linear Poincaré flow  $\psi_t^Z$  admits a dominated splitting on  $\mathcal{N}_{\Lambda}$ ,
- every periodic point in  $\Lambda$  is hyperbolic, but  $\Lambda$  is not hyperbolic.

Then  $\Lambda$  is a normally hyperbolic 2-dimensional torus with respect to  $\Phi_t^Z$ , and  $\phi_t^Z|_{\Lambda}$  is equivalent to an irrational flow.

COROLLARY 6.14. – Under the assumption of Lemma 6.11, every  $\mathcal{N}$ -set  $\Lambda_Y$  contains a singularity for every  $C^2$  weak Kupka-Smale vector field  $Y \in \mathcal{U}_X$ . Moreover,  $\Lambda_Y$  is not reduced to a singularity.

*Proof.* – Notice that  $C(\sigma_Y)$  cannot contain a normally hyperbolic torus without a singularity. If not, one assumes that  $C(\sigma_Y)$  contains a normally hyperbolic torus without a singularity. *Y* is a three-dimensional vector field, the torus is normally contracting or normally expanding. Without loss of generality, we assume that the torus is normally contracting. As a corollary,  $\sigma_Y$  is not chain attainable from any point in the torus. But  $C(\sigma_Y)$  is chain recurrent, every two points in  $C(\sigma)$  are chain bi-attainable. This is a contradiction.

Consequently,  $\Lambda_Y$  is not a normally hyperbolic torus without a singularity. If  $\Lambda_Y$  contains no singularity, then by Theorem 6.13,  $\Lambda_Y$  is hyperbolic since Y is weak Kupka-Smale. This contradicts the definition of  $\Lambda_Y$ . Hence  $\Lambda_Y$  contains a singularity. Since every singularity is Lorenz-like, so F is area-expanding on every singularity. Thus we have that  $\Lambda_Y$  is not reduced to a singularity.

REMARK. – This is another place that we need to use the assumption of "weak Kupka-Smale" besides the usage of the connecting lemma for pseudo orbits.

We choose another neighborhood  $\mathcal{V}_X$  of X such that the closure of  $\mathcal{V}_X$  is contained in the interior of  $\mathcal{U}_X$ . For each  $Y \in \mathcal{V}_X$ , one defines n(Y) to be the number of homoclinic orbits of singularities contained in  $C(\sigma_Y)$ . Since there are only k singularities in  $C(\sigma)$ ,  $n(Y) \leq 2k$ . Let

 $n = \max\{n(Y) : Y \in \mathcal{D}_X, Y \text{ is } C^2 \text{ and is weak Kupka-Smale}\}.$ 

Let  $\mathcal{M}_n \subset \mathcal{V}_X$  be the set of  $C^2$  weak Kupka-Smale vector fields Y with n(Y) = n.

LEMMA 6.15. – For  $Y \in \mathcal{M}_n$  and the transitive  $\mathcal{N}$ -set  $\Lambda_Y \subset C(\sigma_Y)$  as in Lemma 6.11, then any singularity  $\sigma_{i,Y} \in \Lambda_Y$  has a homoclinic orbit  $\Gamma_i^{\pm}$ .

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*Proof.* – Suppose on the contrary,  $\Lambda_Y$  contains a singularity  $\sigma_{i,Y}$ , but  $C(\sigma_Y)$  does not contain a homoclinic orbit of  $\sigma_{i,Y}$ . Since  $\Lambda_Y$  is transitive and  $\Lambda$  is not reduced to a singularity by Corollary 6.14,  $\Lambda_Y \cap W^s(\sigma_{i,Y}) \setminus \{\sigma_{i,Y}\} \neq \emptyset$  and  $\Lambda_Y \cap W^u(\sigma_{i,Y}) \setminus \{\sigma_{i,Y}\} \neq \emptyset$ . Take  $x^s \in \Lambda_Y \cap W^s(\sigma_{i,Y}) \setminus \{\sigma_{i,Y}\}$  and  $x^u \in \Lambda_Y \cap W^u(\sigma_{i,Y}) \setminus \{\sigma_{i,Y}\}$ . By the assumptions,  $x^s$  and  $x^u$  are not in a homoclinic orbit.

6.2.0.1. Construction of perturbation boxes. For  $Y \in \mathcal{M}_n$ , one can choose  $\varepsilon > 0$ , such that  $B(Y,\varepsilon) \subset \mathcal{V}_X$ . By Lemma 3.4, one can choose  $L^s = L^s(x^s) > 0$  and  $L^u = L^u(x^u)$  (related to  $\varepsilon$ ) and neighborhoods  $\widetilde{W}_{x^s} \subset W_{x^s}$  of  $x^s$ ,  $\widetilde{W}_{x^u} \subset W_{x^u}$  of  $x^u$  as in Lemma 3.4 such that by taking  $L = \max\{L^s, L^u\},\$ 

- $W_{L_x} x^s \cap W_{L_x} u = \emptyset$
- $\Lambda_Y \setminus (W_{L,x^s} \cup W_{L,x^u}) \neq \emptyset.$
- $W_{L,x^s} \cup W_{L,x^u}$  is disjoint from any other homoclinic orbits of singularities in  $C(\sigma_Y)$ .
- $W_{L,x^s} \cup W_{L,x^u}$  is disjoint from  $\sigma_{i,Y}$ .

6.2.0.2. Choosing the orbits. Since  $\Lambda_Y$  is transitive, there is  $z \in \Lambda_Y \setminus (W_{L,x^s} \cup W_{L,x^u})$  such that  $\alpha(z) = \omega(z) = \Lambda$ . Choose  $t_1, t_2 > 0$  such that  $\phi_{-t_1}^Y(z) \in \widetilde{W}_{x^s}$  and  $\phi_{t_2}^Y(z) \in \widetilde{W}_{x^u}$ . Choose  $t_s > 0$  and  $t_u > 0$  such that  $\phi_{t_s}^Y(x^s) \notin W_{L,x^s} \cup W_{L,x^u}$  and  $\phi_{-t_u}^Y(x^u) \notin W_{L,x^s} \cup W_{L,x^u}$ .

6.2.0.3. Connecting the orbit from  $\phi_{t_2}^Y(z)$  to  $\phi_{t_s}^Y(x^s)$ . Since the negative orbit of  $\phi_{t_s}^Y(x^s)$ and the positive orbit of  $\phi_{-t_2}^Y(z)$  both enter  $\widetilde{W}_{x^s}$ , by using Lemma 3.4, there is  $Y_1$  which is  $\varepsilon$ -close to Y such that

- there is T<sub>1</sub> > 0 such that φ<sup>Y<sub>1</sub></sup><sub>-T<sub>1</sub></sub>(φ<sup>Y</sup><sub>ts</sub>(x<sup>s</sup>)) = φ<sup>Y</sup><sub>-t<sub>2</sub></sub>(z),
  Y<sub>1</sub>(x) = Y(x) for any x ∈ M<sup>3</sup> \ W<sub>L,x<sup>s</sup></sub>.

As a corollary, we have

- any homoclinic orbit of Y is still a homoclinic orbit of  $Y_1$ ,
- $\sigma_{i,Y}$  is still a singularity of  $Y_1$ ,  $\phi_{t_s}^Y(x^s)$  is still in the stable manifold of  $\sigma_{i,Y}$  and  $\phi_{-t_u}^Y(x^u)$  is still in the unstable manifold of  $\sigma_{i,Y}$  with respect to  $Y_1$ .

6.2.0.4. Connecting the orbit from  $\phi_{-t_u}^Y(x^u)$  to  $\phi_{t_s}^Y(x^s)$ . Since  $\phi_{-T_1}^{Y_1}(\phi_{t_s}^Y(x^s)) = \phi_{-t_2}^Y(z)$  is contained in  $\widetilde{W}_{x^{u}}$ , by using Lemma 3.4 again, there is  $Y_{2}$  which is  $\varepsilon$ -close to  $Y_{1}$  such that

- there is  $T_2 > 0$  such that  $\phi_{-T_2}^{Y_1}(\phi_{t_s}^Y(x^s)) = \phi_{-t_u}^Y(x^u)$ ,
- $Y_2(x) = Y_1(x)$  for any  $x \in M^3 \setminus W_{L,x^u}$ .

As a corollary, we have

- $Y(x) = Y_2(x)$  for any  $x \in M^3 \setminus (W_{L,x^s} \cup W_{L,x^u})$ ,.
- $\phi_{t_s}^Y(x^s)$  is in a homoclinic orbit of  $\sigma_{i,Y}$ .

The meaning of  $Y \in \mathcal{M}_n$  is that Y has n homoclinic orbits of singularities in  $C(\sigma_Y)$ . By Property (A.5), any singularity  $C(\sigma_Y)$  has its continuation in  $C(\sigma_{Y_2})$ . Thus, if  $\Gamma_Y$  is a homoclinic orbit of  $\rho_Y \in C(\sigma_Y)$ , by the choice of  $W_{L,X^s} \cup W_{L,X^u}$  (which is disjoint from  $\Gamma_Y \cup \rho_Y$ ), we have that  $\rho_{Y_2} = \rho_Y$  is still contained in  $C(\sigma_{Y_2})$  and  $\Gamma_Y$  is still a homoclinic orbit of  $\rho_{Y_2} = \rho_Y$ . Since  $\Gamma_Y \cup \rho_{Y_2}$  is a chain transitive set, it is contained in  $C(\sigma_Y) =$  $C(\rho_{Y_2})$ . This implies that any homoclinic orbit of Y is still a homoclinic orbit of  $Y_2$ . Now by perturbation we have one more homoclinic orbit of  $\sigma_{Y_2}$ . Since  $Y_2 \in \mathcal{V}_X$  and we have that  $Y_2$  has n + 1 homoclinic orbits of singularities in  $C(\sigma_Y)$ , which contradicts the maximality of n.

By Lemma 6.15, one knows that  $\sigma_{i,Y} \in \Lambda_Y$  contains a homoclinic orbit. However, we don't know whether this homoclinic orbit is contained in  $\Lambda_Y$  in advance.

However, with some additional assumption, we have

LEMMA 6.16. – For  $Y \in \mathcal{M}_n$ , if  $\sigma_{i,Y} \in \Lambda_Y$  has only one homoclinic orbit, then

- this homoclinic orbit is contained in  $\Lambda_Y$ ;
- if moreover, the other separatrix of  $W^u(\sigma_{i,Y})$  is contained in  $\Lambda_Y$ , then  $\Lambda_Y$  intersects  $\ell_i^+ \cup \ell_i^-$  at only one point.

*Proof.* – If the first item is not true, we know that  $\Lambda_Y$  will intersect other orbit in the stable manifold and the unstable manifold of  $\sigma_{i,Y}$ . Similar to the proof of Lemma 6.15, by using the connecting lemma, we get one more homoclinic orbit. This contradicts the maximality of the number of homoclinic orbits.

The other item can be proven similarly. Notice that since the homoclinic orbit is contained in  $\Lambda_Y$ , the intersection point of this homoclinic orbit and  $\ell_i^+ \cup \ell_i^-$  is contained in  $\Lambda_Y$ . Now we assume that the other separatrix  $\Gamma$  of  $W^u(\sigma_{i,Y})$  is contained in  $\Lambda_Y$ . We argue by contradiction. If  $\Lambda_Y$  intersects  $\ell_i^+ \cup \ell_i^-$  at least two points, i.e., there is a point  $z \in \ell_i^+ \cup \ell_i^$ such that z is not contained in the homoclinic orbit. One chooses a point  $w \in \Gamma \setminus \{\sigma_{i,Y}\}$ . Since  $\Lambda$  is transitive, there is a sequence of points  $\{w_n\} \subset \Lambda_Y$  and  $\{t_n\}$  such that  $\lim_{n\to\infty} w_n = w$ and  $\lim_{n\to\infty} \phi_{t_n}^Y(w_n) = z$ . Then one can choose perturbation tubes around w and z such that they are disjoint from all homoclinic orbits of singularities as in Lemma 6.15. By applying the connecting lemma, after a perturbation, one have that the other separatrix is also a homoclinic orbit; meanwhile, all homoclinic orbits of Y are preserved. Thus, one gets one more homoclinic orbit. This contradicts the maximality of n.

#### 6.3. Infinitesimal cu adapted returns via homoclinic orbits of singularities

We will define the notion of *infinitesimal cu adapted returns* (infinitesimal adapted for short). We will prove (Proposition 6.20) that every  $C^2$  weak Kupka-Smale vector field in  $\mathcal{O}_X$  has no infinitesimal adapted returns for the cross section in Proposition 6.5. On the other hand, for any  $Y \in \mathcal{M}_n$ , we will prove (Proposition 6.25) that there is Z arbitrarily close to Y such that Z has an infinitesimal adapted return. A contradiction is got.

The idea of defining the infinitesimal adapted property is to find some local stable manifold of the singularity in the cross section (some  $\ell_i^{\pm}$ ) such that any small *cu*-curve which starts from  $\ell_i^{\pm}$  can be iterated infinitely many times by the return map  $R_Y$ .

Recall the cross section system  $(\Sigma_Y, R_Y)$  for  $Y \in \mathcal{V}_X \subset \mathcal{U}_X$  as in Proposition 6.5. For a *cu*-curve  $\gamma : [0, 1] \to S_i^{\pm}$ , we define its representation  $\tilde{\gamma} : [0, 1] \to [-1, 1]^2$  by

$$\widetilde{\gamma} = (h_i^{\pm})^{-1} \circ \gamma$$

Recall that one can define an orientation  $\eta$  in a neighborhood of a Lorenz-like singularity, i.e., one can define the orientation for any curve tangent to the unstable cone field  $\mathcal{C}^{\mu}$  for the ambient manifold  $M^3$ . From the construction of the cross section, by choosing the cross

section carefully and the center-unstable cone field (with smaller width) of the cross section, one can assume that the center-unstable cone field in the cross section is contained in  $\mathcal{C}^{u}$ .



FIGURE 5. The local orientations in a neighborhood of a Lorenz-like singularity

DEFINITION 6.17 (Orientation). – For a cu-curve  $\gamma : [0, 1] \rightarrow S_i^{\pm}$  (and its representation  $\widetilde{\gamma}$ ), if the first coordinate of  $\widetilde{\gamma}(t_1)$  is strictly larger than the first coordinate of  $\widetilde{\gamma}(t_0)$  for any  $1 \ge t_1 > t_0 \ge 0$ , then we say that the orientation of  $\gamma$  is  $\eta$ ; otherwise, the orientation of  $\gamma$  is  $-\eta$ . By defining the equivalent class, each point  $x \in \Sigma$  has two orientations:  $\eta$  and  $-\eta$ . We use  $\eta_x$ and  $-\eta_x$  to emphasize the orientation at x.

If  $x, y \in \Sigma$  can be connected by a cu-curve  $\gamma$  such that  $\gamma(0) = x$  and  $\gamma(1) = y$ , then we use [x, y] to denote the orientation of  $\gamma$ .

Furthermore, for a stable manifold  $W^{s}(z)$  (of the return map), denote by  $[x, W^{s}(z)]$  the orientation of cu-curves which start at x and end at  $W^{s}(z)$ .

Notice that this orientation coincides with the orientation in a neighborhood of the Lorenz-like singularity and we have defined the direction of a separatrix of the unstable manifold of the Lorenz-like singularity.

For a point  $x \in \Sigma_Y$  and a direction  $\zeta_x \in \{\eta_x, -\eta_x\}$ , we say a sequence of points  $\{x_n\} \subset \Sigma_Y$ accumulates x in the direction  $\zeta_x$  if  $[x, W^s(x_n)] = \zeta_x$  for n large enough.

By local dynamics of Lorenz-like singularities, we have the following lemma:

LEMMA 6.18. – For  $Y \in \mathcal{O}_X$ , there is a sequence of points  $\{x_n\}$  which accumulates a point in  $\ell_i^+$  (or  $\ell_i^-$ ) in the direction  $\zeta \in \{\eta, -\eta\}$  if and only if there is a sequence of times  $\{t_n\}$  such that  $\{\phi_{t_n}(x_n)\}$  accumulates the separatrix of the unstable manifold of  $\sigma_{i,Y}$  in the direction  $\zeta$ .

*Proof.* – For a small neighborhood U of  $\sigma_{i,Y}$ , we take  $t_n = \sup\{t : \phi_{[0,t]}(x_n) \subset U\}$ . By the hyperbolicity of  $\sigma_{i,Y}$ , after taking a subsequence if necessary, we have  $\lim_{n\to\infty} \phi_{t_n}(x_n)$  is a point contained in the local unstable manifold of  $\sigma_{i,Y}$ . The accumulation point will be contained in the separatrix in the direction  $\zeta$ : the local orbit cannot cross  $W_{loc}^s(\sigma_{i,Y})$ . The other part comes from Lemma 6.7.

Recall the definition of the orientations at each point in a neighborhood of a Lorenz-like singularity. Every point  $x \in \Sigma$  has two orientations  $\{\eta_x, -\eta_x\}$  which are given by the oriented *cu*-curves starting at *x*. Now we can define the iteration on the orientations. For any point  $x \in \Sigma \setminus \bigcup_{i,\pm} \ell_i^{\pm}$ , the map *DR* induces a map on the orientations naturally. However, one can also define the iteration for any points in  $\Sigma$ . For any point  $x \in \Sigma$ , take a small *cu*-curve  $\gamma$ started at *x*, whose orientation is  $\zeta_x$ , the orientation of  $R(\gamma)$  is defined to be  $\Theta(\eta_x)$ . Notice that for a point  $x \notin \ell_i^{\pm}$ , we have  $\Theta(-\zeta_x) = -\Theta(\zeta_x)$ ; but we do not have this property for points in  $\ell_i^{\pm}$ .



FIGURE 6. The possibilities of the iterations of the orientations

 $\Theta$  may have periodic orbits. { $(\zeta_{x_1}, \zeta_{x_2}, \dots, \zeta_{x_N})$ } is a periodic orbit of  $\Theta$  if  $\Theta(\zeta_{x_j}) = \zeta_{x_{j+1}}$  for  $1 \le j \le N$ , where by convention  $\zeta_{x_{N+1}} = \zeta_{x_1}$ .

For a periodic orbit  $\{(\zeta_{x_1}, \zeta_{x_2}, \dots, \zeta_{x_N})\}$  of  $\Theta$ , if  $x_1 \in \ell_i^{\pm}$  and if  $x_j \notin \bigcup_{1 \le m \le k, \pm \in \{+, -\}} \ell_m^{\pm}$  for any  $1 < j \le N$ , then the vector field Y has a homoclinic orbit containing  $x_1$ . But  $\Theta$  may not have a periodic orbit when the vector field has a homoclinic orbit.

Recall that we are considering the chain recurrent class  $C(\sigma)$  of the vector field X.  $\sigma_Y$  is the continuation of  $\sigma$  for Y in some neighborhood of X.

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DEFINITION 6.19 (The infinitesimal adapted return). – For  $Y \in \mathcal{V}_X$ , we say that the cross-section system  $(\Sigma_Y, R_Y)$  has an infinitesimal adapted return (or infinitesimal cu adapted return) if  $\Theta$  has a periodic orbit  $\{(\zeta_{x_1}, \zeta_{x_2}, \dots, \zeta_{x_N})\}$  such that

- 1.  $x_1$  is contained in the stable manifold  $\ell_i^{\pm}$  for some  $i \in \{1, \ldots, k\}$  and  $\pm \in \{+, -\}$ ,
- 2. for  $j \neq 1$ , if  $x_j$  is in the cross section  $S_{i_j} \setminus \ell_{i_j}^{\pm}$  and if  $[x_j, \ell_{i_j}^{\pm}]$  is  $\zeta_{x_j}$ , then there is a sink contained in the region bounded by  $W^s(x_j)$  and  $\ell_{i_j}^{\pm}$ .



FIGURE 7. An infinitesimal adapted return

**PROPOSITION 6.20.** – Any weak Kupka-Smale  $Y \in \mathcal{V}_X$  has no infinitesimal adapted return.

*Proof.* – We will prove this by contradiction. Assume that there is a weak Kupka-Smale  $Y \in \mathcal{D}_X$  having an infinitesimal adapted return as in Definition 6.19 (the periodic orbit of  $\Theta$  is  $\{\zeta_{x_1}, \zeta_{x_2}, \ldots, \zeta_{x_N}\}$ ). Without loss of generality, we take  $x_1 \in \ell_i^+$  and  $\zeta_{x_1} = \eta_{x_1}$ . In the following, we want to represent the dynamics in the standard rectangle.

For the standard rectangle  $[-1, 1]^2$ , the two canonical projections  $p_c$  and  $p_s$  are given by

$$p_c((x, y)) = x, \quad p_s((x, y)) = y.$$

Now we consider the map  $R(x, y) = p_c \circ (h_i^+)^{-1} \circ R^N \circ h_i^+(x, y)$  if  $h_i^+(x, y)$  is in the domain of  $R^N$ . Note that  $(h_i^+)^{-1} \circ R^N \circ h_i^+$  is the representation of  $R^N$  in the standard rectangle. Since  $h_i^+(x, \cdot)$  is a stable leaf of R, R(x, y) is constant with respect to y. This allows us to define the map  $R^{cu}$  by  $R^{cu}(x) = p_c \circ (h_i^+)^{-1} \circ R^N \circ h_i^+(x, 0)$  if  $h_i^+(x, 0)$  is in the domain of  $R^N$ .

By Lemma 6.8,  $R^{cu}$  is uniformly continuous in a small neighborhood of 0. Thus,  $R^{cu}$  can be extended to be a continuous function at 0, which is still denoted by  $R^{cu}$ . We have 0 is a fixed point of  $R^{cu}$ .

Moreover, by Lemma 6.9, there is  $\beta_Y > 0$  such that for any  $\beta \in (0, \beta_Y]$ , there is  $N(\beta) \in \mathbb{N}$  such that  $(R^{cu})^{N(\beta)}(0, \beta) \supset (0, \beta_Y]$ .

From this fact, we have that the fixed point 0 of  $R^{cu}$  is "topologically expanding": there is  $\beta_Y$  such that for any  $x \in (0, \beta_Y)$ , we have  $(R^{cu})^{-1}(x)$  is defined,  $(R^{cu})^{-1}(x) < x$  and  $(R^{cu})^{-n}(x) \to 0$  as  $n \to \infty$ .

Consider the set I which is the connected component of  $\{x : R^{cu}(x) > x\}$  that contains  $\beta_Y$ . By the definition of I, we have for any  $z \in I$ ,  $(R^{cu})^{-n}(z) \to 0$  as  $n \to \infty$ . Notice that 0 is contained in the closure of I.

By Lemma 6.6, for any *cu*-curve  $\gamma$  satisfying  $p_c(h_i^+)^{-1}(\gamma) = (0, \beta_{\gamma}), \beta_{\gamma} < \beta_Y$ , we have that either there is  $N(\gamma) \in \mathbb{N}$  such that  $R_Y^{N(\gamma)}(\gamma)$  intersects some  $\ell_i^{\pm}$ , or  $R_Y^n(\gamma)$  is defined for any *n*. Denote by  $\gamma_s \subset \gamma$  such that  $p_c((h_i^+)^{-1}(\gamma_s)) = (0, s)$  for any  $s \in (0, \beta_{\gamma})$ . we have:

- Either, there is  $y \in (0, \beta_{\gamma})$  such that one point of  $\{R(h_i^+(y, 0)), \ldots, R^{N-1}((h_i^+(y, 0))\}\)$  is contained in the stable manifold of singularity. By Item 2 of Definition 6.19, there is  $1 < L \leq N$  such that there is a sink in the region bounded by  $\ell_{iL}^{\pm}$  and  $W^s(x_L)$  and the orientation  $[x_L, \ell_{iL}^{\pm}]$  is  $\zeta_{x_L}$ . Thus there is  $z \in (0, y)$  such that  $R^L(h_i^+((z, \cdot)))$  is in the stable manifold of the sink. By the choice of  $\beta_Y$ , for any s > 0, there is N(s) such that  $(R^{cu})^{N(s)}((0, s)) \supset (0, \beta_Y]$ . Thus, there is  $N_0(s) \leq N(s)$  such that  $(R^{cu})^{N(s)}((0, s)) \supset (0, y) \supset \{z\}$ . This implies that  $R^{N_0(s)N+L}(\gamma_s)$  cuts the stable manifold of the sink, for all s. Thus  $x_1$  can be accumulated by the basin of the sink. This contradicts the Property (A.3).
- Or,  $(R^{cu})^n(\beta_\gamma) < (R^{cu})^{n+1}(\beta_\gamma)$  for any  $n \in \mathbb{N}$ . This implies the limit point y of  $\{(R^{cu})^n(\beta_\gamma)\}$  is a fixed point of the map  $R^{cu}$ . Moreover, y is not topologically expanded. By the ss adapted property, we have  $R^N(h_i^+(\{y\} \times [-1, 1]))$  is contained in the interior of  $h_i^+(\{y\} \times [-1, 1])$ . Since Y is weak Kupka-Smale, we have  $h_i^+(y, 0)$  is a stable manifold of a sink. This also implies that  $x_1$  can be accumulated by the basin of the sink. It contradicts the Property (A.3) again.

To get a contradiction, we will manage to prove that for any  $Y \in \mathcal{M}_n$ , there is a perturbation of Y such that the perturbation is weak Kupka-Smale, and has an infinitesimal adapted return.

DEFINITION 6.21. – A point  $p \in M$  is called a typical point of a probability ergodic measure  $\mu$  of a vector field Y, if the following conditions are satisfied:

- 1. *p* is strongly closable;
- 2.  $\omega(p) = \operatorname{supp}(\mu)$ ;
- 3. for every continuous function  $f : M \to \mathbb{R}$ ,

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T f(\phi_t^Y(p)) dt = \int f(x) d\mu(x).$$

According to Ergodic Closing Lemma and Birkhoff Ergodic Theorem, the set of typical points of  $\mu$  has  $\mu$ -full measure.

Recall  $\Lambda_Y$  and  $\mu_Y$  as in Lemma 6.11 and Corollary 6.12. We have the following definition:

DEFINITION 6.22. – We say a point  $x \in \Lambda_Y$  is accumulated by  $(\Lambda_Y, \mu_Y)$  if there is a sequence of  $\mu_Y$ -typical points  $\{x_n\}$  such that  $\lim_{n\to\infty} x_n = x$ .

For a point  $x \in \Sigma_Y$  and a direction  $\zeta_x \in \{\eta_x, -\eta_x\}$ , we say that it is accumulated by  $(\Lambda_Y, \mu_Y)$ in the direction  $\zeta_x \in \{\eta_x, -\eta_x\}$  if there is a sequence of  $\mu_Y$ -typical points  $\{x_n\} \subset \Sigma_Y$  such that  $\lim_{n\to\infty} x_n = x$  and  $[x, W^s(x_n)] = \zeta_x$ .

**PROPOSITION 6.23.** – For any  $C^2$  weak Kupka-Smale  $Y \in \mathcal{V}_X$ , and  $\sigma_{i,Y} \in \Lambda_Y \cap \text{Sing}(Y)$ , denoting by  $\mu_Y$  the ergodic measure as in Corollary 6.12, assume that

- 1. There is a periodic point  $\{\zeta_{x_1}, \zeta_{x_2}, \dots, \zeta_{x_n}\}$  of  $\Theta$  such that  $x_1$  is contained in  $\ell_i^{\pm}$ , where  $\zeta_{x_i} \in \{\eta_{x_j}, -\eta_{x_j}\}$  for  $1 \le j \le n$ .
- 2.  $(\Lambda_Y, \mu_Y)$  accumulates  $x_1$  in the direction  $\zeta_{x_1}$ .

Then there is a weak Kupka-Smale vector field Z arbitrarily close to Y such that  $(\Sigma_Z, R_Z)$  has an infinitesimal adapted return.

*Proof.* – Assume that  $x_j$  is contained in the cross section  $S_{i_j,Y}^{\pm}$ . We need to find a weak Kupka-Smale vector field Z arbitrarily  $C^1$ -close to Y such that

- $\{\zeta_{x_1}, \zeta_{x_2}, \dots, \zeta_{x_n}\}$  is a periodic orbit of  $\Theta_Z$  such that  $x_1$  is contained in  $\ell_i^{\pm}$ , where  $\zeta_{x_j} \in \{\eta_{x_j}, -\eta_{x_j}\}$  for  $1 \le j \le n$ .
- there is a sink  $\gamma, \gamma \cap S_{i_i, \gamma}^{\pm} = \{p_j\}$  such that  $[x_j, W^s(p_j)]$  is  $\zeta_{x_j}$ .

Given T > 0, define  $f_T(x) = \log |\text{Det}\Phi_T^Y|_{F(x)}|$  for any  $x \in C(\sigma_Y)$ . One knows that  $f_T$  is a continuous function on  $C(\sigma)$ . Since F can be extended continuously in a small neighborhood of  $C(\sigma)$ ,  $f_T$  can be also extended continuously. Denote by  $\widehat{F}$  and  $\widehat{f}_T$  the extension of F and  $f_T$  respectively. Note that we don't require that  $\widehat{F}$  is invariant. By the property of  $\mu_Y$ , one has  $\int f_T d\mu_Y \leq 0$ 

Since  $\mu_Y$  is ergodic and  $\operatorname{supp}(\mu_Y) = \Lambda_Y$ , one has that the set of homoclinic orbits of singularities has zero measure w.r.t.  $\mu_Y$ . By the assumption, there is a typical point x of  $\mu_Y$  that is close to  $x_1$  and in the direction  $\zeta_{x_1}$ . Thus, the forward orbit of x will be very close to  $x_j$  in the direction  $\zeta_{x_j}$ . Since x is typical, we have that x is a strongly closable point. Now by Corollary 3.7, for any  $\varepsilon > 0$ , there is Z  $\varepsilon$ -close to Y such that

- Z(w) is the same as Y(w) when w in an arbitrary small neighborhood of prescribed piece of orbit of x. Thus, Y and Z coincide on a neighborhood of the closure of the orbits of the points  $x_1, x_2, \ldots, x_n$ .
- Z has a periodic orbit  $\gamma$  such that

$$\int \widehat{f}_T \mathrm{d}\delta_{\gamma} - \int \widehat{f}_T \mathrm{d}\mu_Y \bigg| < \varepsilon.$$

•  $\gamma$  contains a point  $y_i$  such that the orientation from  $x_i$  to  $y_i$  is  $\zeta_{x_i}$ .

As a corollary of the first item, we have that  $\{\zeta_{x_1}, \zeta_{x_2}, \dots, \zeta_{x_n}\}$  is a periodic orbit of  $\Theta_Z$ such that  $x_1$  is still contained in  $\ell_i^{\pm}$ . Moreover, the orbit  $\gamma$  intersects an arbitrarily small neighborhood of  $x_j$  in the direction  $\zeta_{x_j}$  for any  $1 \le j \le n$ . In other words, for any  $1 \le j \le n$ , for any neighborhood  $U_j \subset \Sigma$  of  $x_j$ , by choosing the pertubation Z close to Y, there is  $c_j \in \gamma \cup U_j$  such that the orientation  $[x_j, W^s(c_j)]$  is  $\zeta_{x_j}$ .

Since  $\gamma$  is close to  $\Lambda_Y$ , one knows that  $\gamma$  admits a partially hyperbolic splitting  $T_{\gamma}M^3 = E^{ss,Z} \oplus F^Z$ . By the property of dominated splittings, for any  $x \in \gamma$ , one has that, for the

distance between two bundles,  $\widetilde{d}(\widehat{F}(x), F^Z(x)) = o(1)$ , where  $o(1) \to 0$  as  $\varepsilon \to 0$ . Thus, one has that  $|\int \log |\text{Det}\Phi_T^Z|_{Fcu,Z} |d\delta_\gamma - \int f_T d\mu_Y| = o(1)$ . As a consequence,

$$\int \log |\mathrm{Det}\Phi_T^Z|_{E^{cu,Z}} |\mathrm{d}\delta_{\gamma} \leq \mathrm{o}(1)$$

Thus, by using the Franks Lemma (Lemma 2.6), by an extra small perturbation in an arbitrarily small neighborhood of  $\gamma$ , one gets that  $\gamma$  is a periodic sink. By the construction of  $\gamma$ ,  $\gamma$  contains a point  $p_j$  that is very close to  $y_j$ . Finally, by Theorem 3.10, one can assume that Z is weak Kupka-Smale. Thus Z has an infinitesimal adapted return.

**PROPOSITION 6.24.** – For any  $Y \in \mathcal{M}_n$ , if  $\sigma_{i,Y} \in \Lambda_Y$  has two homoclinic orbits, then for any neighborhood  $\mathcal{U}$  of Y, there is  $Z \in \mathcal{U}$  such that  $(\Sigma_Z, R_Z)$  has an infinitesimal adapted return.

*Proof.* – Recall that by Lemma 6.11,  $\Lambda_Y$  is transitive; by Corollary 6.14,  $\Lambda_Y$  is not reduced to a single singularity. Thus, at least one of the separatrice of  $\sigma_{i,Y}$  is contained in  $\Lambda_Y$  for  $\sigma_{i,Y} \in \Lambda_Y$ .

Without loss of generality, we assume that the separatrix of the local unstable manifold in the direction  $\eta$  and the stable manifold  $\ell_i^+$  form a homoclinic orbit. Moreover, this homoclinic orbit can be accumulated by  $(\Lambda_Y, \mu_Y)$ . Assume that  $x \in \ell_i^+$  is contained in this homoclinic orbit and this homoclinic orbit intersect the cross section  $\Sigma$  at N times.

If  $\{\eta_x, \ldots, \Theta^N(\eta_x)\}$  is periodic and  $(\Lambda_Y, \mu_Y)$  accumulates x in the direction of  $\eta_x$ , then by Proposition 6.23, we get the conclusion. Thus, one can assume this will not happen.

If  $(\Lambda_Y, \mu_Y)$  accumulates x in the direction  $\eta_x$ , then  $\Theta^N(\eta_x) = -\eta_x$  by the previous discussion. This implies that  $(\Lambda_Y, \mu_Y)$  also accumulates x in the direction  $-\eta_x$ . In any case  $(\Lambda_Y, \mu_Y)$  accumulates x in the direction  $-\eta_x$ . In particular, by Lemma 6.18, the separatrix of  $W^u(\sigma_{i,Y})$  in the direction  $-\eta_x$  is contained in  $\Lambda_Y$ .

From our assumption, the separatrix of  $W^u(\sigma_{i,Y})$  in the direction  $-\eta_x$  is a homoclinic orbit of  $\sigma_{i,Y}$ . Since we know that it is contained in  $\Lambda_Y$ , the argument above also applies to this orbit. It intersects  $\Sigma$  in L points and intersects  $\ell_i^+ \cup \ell_i^-$  in a (unique) point y. The argument above also applies and shows that y is accumulated by  $\Lambda$  in the direction  $\eta_y$ .

Now we can assume the following possibilities:

- 1. Both are orientation reversing:  $\Theta^N(\eta_x) = -\eta_x$  and  $(\Lambda_Y, \mu_Y)$  accumulates x in the directions  $-\eta_x$ ;  $\Theta^L(-\eta_y) = \eta_y$  and  $(\Lambda_Y, \mu_Y)$  accumulates y in the direction  $\eta_y$
- 2. Both are orientation preserving:  $\Theta^N(\eta_x) = \eta_x$  and  $(\Lambda_Y, \mu_Y)$  accumulates x in the direction  $-\eta_x$ ;  $\Theta^L(-\eta_y) = -\eta_y$  and  $(\Lambda_Y, \mu_Y)$  accumulates y in the direction  $\eta_y$ .
- 3. Orientation preserving at x and orientation reversing at  $y: \Theta^N(\eta_x) = \eta_x$  and  $(\Lambda_Y, \mu_Y)$  accumulates x in the direction  $-\eta_x; \Theta^L(-\eta_y) = \eta_y$  and  $(\Lambda_Y, \mu_Y)$  accumulates y in the direction  $\eta_y$ .
- 4. Orientation reversing at x and orientation preserving at y:  $\Theta^N(\eta_x) = -\eta_x$  and  $(\Lambda_Y, \mu_Y)$  accumulates x in the direction  $-\eta_x$ ;  $\Theta^L(-\eta_y) = -\eta_y$  and  $(\Lambda_Y, \mu_Y)$  accumulates y in the direction  $\eta_y$ .

In Case 1, notice that

$$\{\eta_x, \dots, \Theta^N(\eta_x) = -\eta_x, \Theta(-\eta_x), \dots, \Theta^L(-\eta_x) = \Theta^{N+L}(\eta_x) = -\eta_y, \\ \Theta(-\eta_y), \dots, \Theta^L(-\eta_y) = \eta_y, \dots, \Theta^N(\eta_y) = \Theta^{2(N+L)}(\eta_x) \}$$

is periodic by  $\Theta$ .

From the previous discussion,  $-\eta_x$  is periodic for  $\Theta$ . Since x is accumulated by  $(\Lambda_Y, \mu_Y)$  in the direction  $-\eta_x$ , one concludes by applying Proposition 6.23.

The other cases are proven similarly. For the completeness of the proof, we give the periodic orbit of  $\Theta$  and accumulation direction.

– Case 2: the periodic orbit of  $\Theta$  is:

$$\{-\eta_x,\ldots,\Theta^L(-\eta_x)=\eta_y,\ldots,\Theta^L(\eta_y)=\Theta^{N+L}(-\eta_x)\}$$

is periodic by  $\Theta$ ; x can be accumulated by  $(\Lambda_Y, \mu_Y)$  in the direction  $-\eta_x$ .

– Case 3: the periodic orbit of  $\Theta$  is:

$$\{-\eta_x,\ldots,\Theta^L(-\eta_x)=-\eta_y,\ldots,\Theta^L(-\eta_y)=\eta_y,\ldots,\Theta^N(\eta_y)=-\eta_x;$$

x can be accumulated by  $(\Lambda_Y, \mu_Y)$  in the direction  $-\eta_x$ .

– Case 4: the periodic orbit of  $\Theta$  is:

$$\{\eta_y,\ldots,\Theta^N(\eta_y)=\eta_x,\ldots,\Theta^N(\eta_x)=-\eta_x,\ldots,\Theta^L(-\eta_x)=\eta_y;$$

y can be accumulated by  $(\Lambda_Y, \mu_Y)$  in the direction  $\eta_y$ .

**PROPOSITION 6.25.** – For any  $Y \in \mathcal{M}_n$  and any neighborhood  $\mathcal{U}$  of Y, there is a weak Kupka-Smale  $Z \in \mathcal{U}$  such that  $(\Sigma_Z, R_Z)$  has an infinitesimal adapted return.

*Proof.* – We will prove this by absurd, i.e., we assume that there is a neighborhood  $\mathcal{U}_Y \subset \mathcal{U}_X$  of Y such that  $(\Sigma_Z, R_Z)$  has no infinitesimal adapted return for any  $Z \in \mathcal{U}_Y$ . Now we consider the set  $\Lambda_Y$ . By Lemma 6.15, every singularity  $\sigma_{i,Y}$  has one homoclinic orbit. By Proposition 6.24,  $\sigma_{i,Y}$  has only one homoclinic orbit. By Lemma 6.16, this homoclinic orbit is contained in  $\Lambda_Y$ .

Without loss of generality, we assume that the separatrix of  $W^u(\sigma_{i,Y})$  in the direction  $\eta$  is a homoclinic orbit  $\Gamma_i$  of  $\sigma_{i,Y}$ ,  $\Gamma_i \cap \ell_i^+ = \{x\}$  and  $\Gamma_i$  intersects  $\Sigma_Y$  in N-points.

There are two cases: the periodic case  $\Theta^N(\eta_x) = \eta_x$  or the non-periodic case  $\Theta^N(\eta_x) = -\eta_x$ .

Now we show the non-periodic case is impossible. Assume by contradiction that we have  $\Theta^N(\eta_x) = -\eta_x$ . In this case, we always have that  $(\Lambda_Y, \mu_Y)$  accumulates x in the directions  $-\eta_x$ , and a priori may or may not accumulates x in the direction  $\eta_x$ . Thus the separatrix of  $W^u(\sigma_{i,Y})$  in the direction  $-\eta$  is contained in  $\Lambda_Y$  by Lemma 6.18. By Lemma 6.16,  $\Lambda_Y$  intersects  $\ell_i^+ \cup \ell_i^-$  at only one point and the homoclinic orbit is contained in  $\Lambda_Y$ . Choose a point z in the local unstable manifold  $W^u(\sigma_{i,Y})$  in the direction  $\eta$ . The point z is contained in the homoclinic orbit. Thus, there is a sequence of  $\mu_Y$  typical points  $\{z_n\}$  such that  $\lim_{n\to\infty} z_n = z$ . Choose a minimal  $\tau_n$  such that  $\phi_{-\tau_n}(z_n)$  is contained in  $\Sigma_Y$ . The limit point of  $\{x_n = \phi_{-\tau_n}(z_n)\}$  is in  $\Lambda$  and is contained in  $\ell_i^+ \cup \ell_i^-$ . Since they only intersect at x, the limit point is x. So there is a sequence of  $\mu_Y$  typical points  $\{x_n\}$  which accumulates x in the direction  $\eta_x$  by Lemma 6.18. There is T > 0 such that  $\{\phi_{-T}(x_n)\}$  accumulates the local unstable manifold of  $\sigma_{i,Y}$  in the direction  $\eta$ . Thus, there is a sequence of times  $\{s_n\}$  such that



FIGURE 8. The position of the points

- $\phi_{-T-s_n}(x_n)$  accumulates  $\ell_i^+ \cup \ell_i^-$  in the direction  $\eta_x$ .
- $\phi_{-s-T}(x_n)$  is contained in the local neighborhood of  $\sigma_{i,Y}$  for any  $s \in [0, s_n]$ .

By the choice of  $s_n$ , we have  $R^N(\phi_{-s_n-T}(x_n)) = x_n$ . By Lemma 6.16,  $\{\phi_{-s_n-T}(x_n)\}$  accumulates x. By Lemma 6.18, it accumulates x in the direction  $\eta$  (See Figure 8). But this implies that  $\Theta^N(\eta_x) = \eta_x$ . We also get a contradiction.

Thus we are in the periodic case  $\Theta^N(\eta_x) = \eta_x$ .

By Proposition 6.23,  $(\Lambda_Y, \mu_Y)$  can only accumulate x in the direction  $-\eta_x$ . By Lemma 6.11 and its proof, there is a point  $x^* \in \Lambda_Y$  such that  $\alpha(x^*) = \Lambda_Y$ . Thus,  $\Lambda_Y$  is not reduced to the closure of a homoclinic orbit. Thus, x is accumulated by points outside of  $\ell_i^+$ . This means that there is a sequence of typical points  $\{x_n\}$  of  $\mu_Y$  such that  $\lim_{n\to\infty} x_n = x$  and the orientation  $[x_n, \ell_i^+]$  coincides with  $\eta_x$ . By Lemma 6.18, the separatrix of  $W^u(\sigma_{i,Y})$  in the direction  $-\eta$  is also contained in  $\Lambda_Y$ .

By the choice of  $\{x_n\}$ , there is a sequence of times  $\{t_n\}$  that tends to  $\infty$  such that the limit  $\lim_{n\to\infty} \phi_{t_n}(x_n) \in W^u_{loc}(\sigma_{i,Y})$ . Since  $x_n$  is  $\mu_Y$ -typical, we have  $\omega(x_n) = \Lambda_Y$ . As a corollary, there is another sequence of times  $\{s_n\}$  that tends to  $\infty$  such that  $\lim_{n\to\infty} \phi_{t_n+s_n}(x_n)$  exists and is contained in the local stable manifold of  $\sigma_{i,Y}$ . Without loss of generality, by Lemma 6.7, the accumulation point is contained in  $\ell_i^+ \cup \ell_i^-$ . By Lemma 6.16,  $\Lambda_Y$  intersects  $\ell_i^+ \cup \ell_i^-$  at only one point. Since there is no other choice, this point is x. This implies there is T > 0 such that  $\{\phi_{s_n+t_n-T}(x_n)\}$  accumulates the local separatrix unstable manifold in the direction  $\eta_x$ . As a corollary, there is a time sequence  $\{\tau_n\}$  such that  $\{\phi_{s_n+t_n-T-\tau_n}(x_n)\}$  accumulates  $\ell_i^+ \cup \ell_i^-$  in the direction  $\eta_x$  by Lemma 6.18. Thus, the sequence accumulate  $\ell_i^+$  in the direction  $\eta_x$ .
Proposition 6.25 and Proposition 6.20 give a contradiction together. The proof is complete.

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Shaobo GAN LMAM, School of Mathematical Sciences Peking University Beijing, 100871, P. R. China E-mail: gansb@pku.edu.cn

Dawei YANG School of Mathematical Sciences Soochow University Suzhou, 215006, P.R. China E-mail: yangdw1981@gmail.com, yangdw@suda.edu.cn