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SYMBOLIC DYNAMICS FOR NON-UNIFORMLY HYPERBOLIC SURFACE MAPS WITH DISCONTINUITIES

BY YURI LIMA AND CARLOS MATHEUS

ABSTRACT. – This work constructs symbolic dynamics for non-uniformly hyperbolic surface maps with a set of discontinuities \mathcal{D} . We allow the derivative of points nearby \mathcal{D} to be unbounded, of the order of a negative power of the distance to \mathcal{D} . Under natural geometrical assumptions on the underlying space M , we code a set of non-uniformly hyperbolic orbits that do not converge exponentially fast to \mathcal{D} . The results apply to non-uniformly hyperbolic planar billiards, e.g., Bunimovich billiards.

RÉSUMÉ. – Nous construisons une dynamique symbolique pour les applications non uniformément hyperboliques d’une surface ayant un ensemble de discontinuités \mathcal{D} . La dérivée de l’application peut ne pas être bornée, de l’ordre d’une puissance négative de la distance à \mathcal{D} . Sous certaines conditions géométriques naturelles sur l’espace des phases M , nous codifions un ensemble d’orbites non uniformément hyperboliques qui ne s’approchent pas exponentiellement vite de \mathcal{D} . Notre résultat s’applique aux billards planaires non uniformément hyperboliques tels que les billards de Bunimovich.

1. Introduction

Given a compact domain $T \subset \mathbb{R}^2$ with piecewise smooth boundary, consider the straight line motion of a particle inside T , with specular reflections in ∂T . Let $f : M \rightarrow M$ be the *billiard map*, where $M = \partial T \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ with the convention that $(r, \theta) \in M$ represents $r =$ collision position at ∂T and $\theta =$ angle of collision. The map f has a natural invariant Liouville measure $d\mu = \cos \theta dr d\theta$. Sinai proved that dispersing billiards are uniformly hyperbolic systems with discontinuities [24], hence the Liouville measure is ergodic.

For a while uniform hyperbolicity was the only mechanism to generate chaotic billiards, until Bunimovich constructed examples of ergodic nowhere dispersing billiards [10, 11, 7]. These billiards, known as *Bunimovich billiards*, are non-uniformly hyperbolic: μ -almost every point has one positive Lyapunov exponent and one negative Lyapunov exponent, see [14, Chapter 8]. In this paper we construct symbolic models for non-uniformly hyperbolic billiard maps such as Bunimovich billiards. Assume that the billiard table T satisfies the conditions of [17, Part V], and let h be the Kolmogorov-Sinai entropy of μ .

THEOREM 1.1. – *If μ is ergodic and $h > 0$ then there exists a topological Markov shift (Σ, σ) and a Hölder continuous map $\pi : \Sigma \rightarrow M$ s.t.:*

- (1) $\pi \circ \sigma = f \circ \pi$.
- (2) π is surjective and finite-to-one on a set of full μ -measure.

Other examples of non-uniformly hyperbolic billiard maps are [25, 9]. See Section 1.3 for the definition of topological Markov shifts.

COROLLARY 1.2. – *Under the above assumptions, $\exists C > 0$ and $p \geq 1$ s.t. f has at least Ce^{hnp} periodic points of period np for all $n \geq 1$.*

Corollary 1.2 is consequence of Theorem 1.1 and the work of Gurevič [15, 16], as in [21, Thm. 1.1]. It is related to an estimate of Chernov [13]. The integer p is the period of (Σ, σ) , hence $p = 1$ iff (Σ, σ) is topologically mixing. Since μ is mixing, we expect that the symbolic coding of Theorem 1.1 can be improved to give a topologically mixing (Σ, σ) . Theorem 1.1 is consequence of the main result of this paper, Theorem 1.3, and of an argument of Katok and Strelcyn [17, Section I.3]. The statement of Theorem 1.3 is technical, so we first introduce some notation.

Let M be a smooth Riemannian surface with finite diameter, possibly with boundary. We assume that the diameter of M is smaller than one⁽¹⁾. Let $\mathcal{D}^+, \mathcal{D}^-$ be closed subsets of M . Fix $f : M \setminus \mathcal{D}^+ \rightarrow M$ a diffeomorphism onto its image, s.t. f has an inverse $f^{-1} : M \setminus \mathcal{D}^- \rightarrow M$ that is a diffeomorphism onto its image.

Set of discontinuities \mathcal{D} . – The set of discontinuities of f is $\mathcal{D} := \mathcal{D}^+ \cup \mathcal{D}^-$.

If $x \notin \bigcup_{n \in \mathbb{Z}} f^n(\mathcal{D})$ then $f^n(x)$ is well-defined for all $n \in \mathbb{Z}$, and for every $y = f^n(x)$ there is a neighborhood $U \ni y$ s.t. $f|_U, f^{-1}|_U$ are diffeomorphisms onto their images. We require some regularity conditions on M, f . The first four assumptions are on the geometry of M . Given $x \in M \setminus \mathcal{D}$, let $\text{inj}(x)$ denote the injectivity radius of M at x , and let \exp_x be the exponential map at x , wherever it can be defined. Given $r > 0$, let $B_x[r] \subset T_x M$ be the ball with center 0 and radius r . The Riemannian metric on M induces a Riemannian metric on TM , called the Sasaki metric, see e.g., [12, §2]. Denote the Sasaki metric by $d_{\text{Sas}}(\cdot, \cdot)$. Similarly, we denote the Sasaki metric on $TB_x[r]$ by the same notation, and the context will be clear in which space we are. For nearby small vectors, the Sasaki metric is almost a product metric in the following sense. Given a geodesic γ joining y to x , let $P_\gamma : T_y M \rightarrow T_x M$ be the parallel transport along γ . If $v \in T_x M, w \in T_y M$ then $d_{\text{Sas}}(v, w) \asymp d(x, y) + \|v - P_\gamma w\|$ as $d_{\text{Sas}}(v, w) \rightarrow 0$, see e.g., [12, Appendix A]. The rate of convergence depends on the curvature tensor of the metric on M . Here are the first two assumptions on M .

Regularity of \exp_x . – $\exists a > 1$ s.t. for all $x \in M \setminus \mathcal{D}$ there is $d(x, \mathcal{D})^a < \mathfrak{r}(x) < 1$ s.t. for $D_x := B(x, 2\mathfrak{r}(x))$ the following holds:

- (A1) If $y \in D_x$ then $\text{inj}(y) \geq 2\mathfrak{r}(x)$, $\exp_y^{-1} : D_x \rightarrow T_y M$ is a diffeomorphism onto its image, and $\frac{1}{2}(d(x, y) + \|v - P_{y,x}w\|) \leq d_{\text{Sas}}(v, w) \leq 2(d(x, y) + \|v - P_{y,x}w\|)$ for all $y \in D_x$ and $v \in T_x M, w \in T_y M$ s.t. $\|v\|, \|w\| \leq 2\mathfrak{r}(x)$, where $P_{y,x} := P_\gamma$ for the radial geodesic γ joining y to x .

⁽¹⁾ Just multiply the metric by a sufficiently small constant.

(A2) If $y_1, y_2 \in D_x$ then $d(\exp_{y_1} v_1, \exp_{y_2} v_2) \leq 2d_{\text{Sas}}(v_1, v_2)$ for $\|v_1\|, \|v_2\| \leq 2\tau(x)$, and $d_{\text{Sas}}(\exp_{y_1}^{-1} z_1, \exp_{y_2}^{-1} z_2) \leq 2[d(y_1, y_2) + d(z_1, z_2)]$ for $z_1, z_2 \in D_x$ whenever the expression makes sense. In particular $\|d(\exp_x)_v\| \leq 2$ for $\|v\| \leq 2\tau(x)$, and $\|d(\exp_x^{-1})_y\| \leq 2$ for $y \in D_x$.

The next two assumptions are on the regularity of $d\exp_x$. For $x, x' \in M \setminus \mathcal{D}$, let $\mathcal{L}_{x,x'} := \{A : T_x M \rightarrow T_{x'} M : A \text{ is linear}\}$ and $\mathcal{L}_x := \mathcal{L}_{x,x}$. Then the parallel transport $P_{y,x}$ considered in (A1) is in $\mathcal{L}_{y,x}$. Given $y \in D_x, z \in D_{x'}$ and $A \in \mathcal{L}_{y,z}$, let $\widetilde{A} \in \mathcal{L}_{x,x'}$, $\widetilde{A} := P_{z,x'} \circ A \circ P_{x,y}$. By definition, \widetilde{A} depends on x, x' but different base points define a map that differs from \widetilde{A} by pre and post composition with isometries. In particular, $\|\widetilde{A}\|$ does not depend on the choice of x, x' . Similarly, if $A_i \in \mathcal{L}_{y_i, z_i}$ then $\|\widetilde{A}_1 - \widetilde{A}_2\|$ does not depend on the choice of x, x' . Define the map $\tau = \tau_x : D_x \times D_x \rightarrow \mathcal{L}_x$ by $\tau(y, z) = d(\exp_y^{-1})_z$, where we use the identification $T_v(T_y M) \cong T_y M$ for all $v \in T_y M$.

Regularity of $d\exp_x$. – The following holds:

- (A3) If $y_1, y_2 \in D_x$ then $\|d(\widetilde{\exp_{y_1}})_{v_1} - d(\widetilde{\exp_{y_2}})_{v_2}\| \leq d(x, \mathcal{D})^{-a} d_{\text{Sas}}(v_1, v_2)$ for all $\|v_1\|, \|v_2\| \leq 2\tau(x)$, and $\|\tau(y_1, z_1) - \tau(y_2, z_2)\| \leq d(x, \mathcal{D})^{-a} [d(y_1, y_2) + d(z_1, z_2)]$ for all $z_1, z_2 \in D_x$.
- (A4) If $y_1, y_2 \in D_x$ then the map $\tau(y_1, \cdot) - \tau(y_2, \cdot) : D_x \rightarrow \mathcal{L}_x$ has Lipschitz constant $\leq d(x, \mathcal{D})^{-a} d(y_1, y_2)$.

Conditions (A1)–(A2) guarantee that the exponential maps and their inverses are well-defined and have uniformly bounded Lipschitz constants in balls of radii $d(x, \mathcal{D})^a$. Condition (A3) controls the Lipschitz constants of the derivatives of these maps, and condition (A4) controls the Lipschitz constants of their second derivatives. Here are some cases when (A1)–(A4) are satisfied, in increasing order of generality:

- The curvature tensor R of M is globally bounded, e.g., when M is the phase space of a billiard map.
- $R, \nabla R, \nabla^2 R, \nabla^3 R$ grow at most polynomially fast with respect to the distance to \mathcal{D} , e.g., when M is a moduli space of curves equipped with the Weil-Petersson metric [12].

Now we discuss the assumptions on f .

Regularity of f . – There are constants $0 < \beta < 1 < b$ s.t. for all $x \in M \setminus \mathcal{D}$:

- (A5) If $y \in D_x$ then $\|df_y^{\pm 1}\| \leq d(x, \mathcal{D})^{-b}$.
- (A6) If $y_1, y_2 \in D_x$ and $f(y_1), f(y_2) \in D_{x'}$ then $\|\widetilde{df_{y_1}} - \widetilde{df_{y_2}}\| \leq \mathfrak{K}d(y_1, y_2)^\beta$, and if $y_1, y_2 \in D_x$ and $f^{-1}(y_1), f^{-1}(y_2) \in D_{x''}$ then $\|\widetilde{df_{y_1}^{-1}} - \widetilde{df_{y_2}^{-1}}\| \leq \mathfrak{K}d(y_1, y_2)^\beta$.

Although technical, conditions (A5)–(A6) hold in most cases of interest, e.g., if $\|df^{\pm 1}\|, \|d^2 f^{\pm 1}\|$ grow at most polynomially fast with respect to the distance to \mathcal{D} . We finally define the measures we code. Fix $\chi > 0$.

χ -hyperbolic measure. – An f -invariant probability measure on M is called χ -hyperbolic if μ -a.e. $x \in M$ has one Lyapunov exponent $> \chi$ and another $< -\chi$.

f-adapted measure. – An *f*-invariant measure on *M* is called *f*-adapted if

$$\int_M \log d(x, \mathcal{D}) d\mu(x) > -\infty.$$

A fortiori $\mu(\mathcal{D}) = 0$.

THEOREM 1.3. – Let *M*, *f* satisfy conditions (A1)–(A6). For all $\chi > 0$, there exists a topological Markov shift (Σ, σ) and a Hölder continuous map $\pi : \Sigma \rightarrow M$ s.t.:

- (1) $\pi \circ \sigma = f \circ \pi$.
- (2) $\pi[\Sigma^\#]$ has full μ -measure for every *f*-adapted χ -hyperbolic measure μ .
- (3) For all $x \in \pi[\Sigma^\#]$, $\#\{\underline{v} \in \Sigma^\# : \pi(\underline{v}) = x\} < \infty$.

Above, $\Sigma^\#$ is the recurrent set of Σ , see Section 1.3. Every σ -invariant measure $\widehat{\mu}$ is carried by $\Sigma^\#$, hence its projection $\mu = \widehat{\mu} \circ \pi^{-1}$ has the same entropy as $\widehat{\mu}$ (this follows from the Abramov-Rokhlin formula [1]). In particular, the topological entropy of (Σ, σ) is at most that of (M, f) . On the other direction, every *f*-adapted χ -hyperbolic measure μ has a lift $\widehat{\mu}$ with the same entropy. If we know that χ -hyperbolic measures are *f*-adapted then the topological entropies of (Σ, σ) and (M, f) coincide, and their measures of maximal entropy are related. In this case, Corollary 1.2 has a potentially stronger statement: for every $\varepsilon > 0$, $\exists C > 0$ and $p \geq 1$ s.t. *f* has at least $Ce^{(H-\varepsilon)np}$ periodic points of period np for all $n \geq 1$, where *H* is the topological entropy of Σ . At the moment, we are not aware of general results assuring that χ -hyperbolic measures are *f*-adapted, except when the measure is Liouville [17, Section I.3].

We now discuss the applicability of Theorem 1.3. Let us restrict ourselves to billiard tables with finitely many boundary components, otherwise many degeneracies can occur (see e.g., [17, Part V]). Assumptions (A1)–(A6) are satisfied if all boundary components are C^3 . The precise conditions that guarantee non-uniform hyperbolicity are unknown, so we mention two classes of billiard tables *T* whose billiard maps are non-uniformly hyperbolic:

- Sinai billiard: every component of ∂T is dispersing. In this case, the billiard map exhibits uniform hyperbolicity.
- Bunimovich billiard: ∂T is the union of finitely many segments and arcs of circles s.t. each of these arcs belongs to a disk contained in *T*. When this happens, non-uniform hyperbolicity is ensured via a focusing-defocusing mechanism, see [14, Chapter 8]. See Figure 1 for some examples.

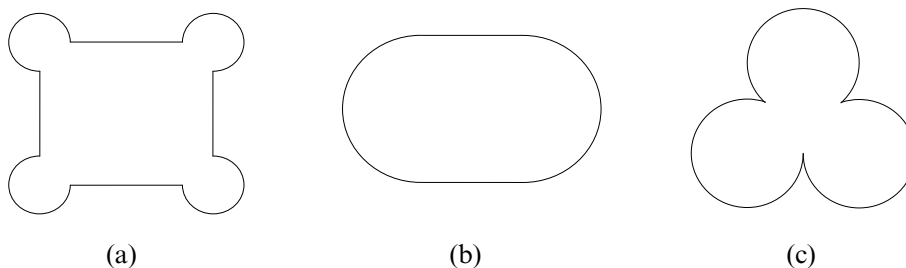


FIGURE 1. Examples of Bunimovich billiards: (a) pool table with pockets, (b) stadium, (c) flower.

1.1. Related literature

The construction of Markov partitions and symbolic dynamics for uniformly hyperbolic diffeomorphisms and flows in compact manifolds laid its foundation during the late sixties and early seventies through the works of Adler & Weiss [2, 3], Sinaï [22, 23], Bowen [4, 5], and Ratner [19, 20]. Below we discuss other contexts.

Billiards. – These are the main examples of maps with discontinuities. Katok and Strelcyn constructed invariant manifolds for non-uniformly hyperbolic billiard maps which include Bunimovich billiards [17]. Bunimovich, Chernov and Sinaï constructed countable Markov partitions for two-dimensional dispersing billiard maps [8]. All these results are for Liouville measures. Up to our knowledge, our result is the first symbolic coding of uniformly and non-uniformly hyperbolic billiard maps for general measures.

Tower extensions of billiard maps. – Young constructed tower extensions for certain two-dimensional dispersing billiard maps [26]. Contrary to our case, Young's tower extensions provide codings which are usually infinite-to-one, hence it is unclear that χ -hyperbolic measures can be lifted to the symbolic space without increasing its entropy. Nevertheless, such tower extensions guarantee exponential decay of correlations for certain two-dimensional dispersing billiard maps.

Non-uniformly hyperbolic three-dimensional flows. – The first author and Sarig constructed symbolic models for non-uniformly hyperbolic three-dimensional flows with positive speed [18]. The idea is to take a Poincaré section and analyze the Poincaré return map f . The Poincaré map f has discontinuities, but its derivative is uniformly bounded inside the set of continuities. Hence the methods of [21] apply more easily.

Weil-Petersson flow. – Moduli spaces of curves possess natural negatively curved incomplete Kähler metrics, called *Weil-Petersson metrics*. The geodesic flow of one such metric is called the *Weil-Petersson flow*, and it preserves a canonical Liouville measure. The properties of the Weil-Petersson metric are intimately related to the hyperbolic geometry of surfaces, and this partly explains the recent interest in the dynamics of the Weil-Petersson flow. Burns, Masur and Wilkinson proved that the Liouville measure is hyperbolic [12]. For that, they combined results of Wolpert and McMullen to show that the Weil-Petersson metric explodes at most polynomially fast while approaching the boundary of the Deligne-Mumford compactification of the moduli space of curves, hence the Weil-Petersson flow satisfies the assumptions of Katok and Strelcyn [17]. The construction of symbolic dynamics for the Weil-Petersson flow is still open.

As pointed out by Sarig [21, pp. 346], our main result (Theorem 1.3) can be regarded as a step towards the construction of Markov partitions capturing measures of maximal entropy for surface maps with discontinuities with positive topological entropy, such as Bunimovich billiards. Motivated by this, we ask the following question.

Question. – Let f be a billiard map with topological entropy $H > 0$. Does f have a measure of maximal entropy? If it does, is it f -adapted? Is it Bernoulli?

A positive answer to this question would imply that $\exists C > 0$ s.t. f has at least Ce^{Hn} periodic points of period n , for all $n \geq 1$.

In [12, pp. 858] it was suggested that one of the assumptions (in their notation, the compactness of \bar{N}) can be relaxed to the assumption that N has finite diameter. The main reason not to claim this is that they use [17], whose framework assumes \bar{N} to be compact. We only assume finite diameter, hence our work is a step towards the relaxation of the assumptions of [17] to the context mentioned in [12].

1.2. Methodology

The proof of Theorem 1.3 is based on [21] and [18], and it follows the steps below:

- (1) If μ is f -adapted and χ -hyperbolic, then μ -a.e. $x \in M$ has a Pesin chart $\Psi_x : [-Q_\varepsilon(x), Q_\varepsilon(x)]^2 \rightarrow M$ s.t. $\lim_{n \rightarrow \infty} \frac{1}{n} \log Q_\varepsilon(f^n(x)) = 0$.
- (2) Define ε -double charts $\Psi_x^{p^s, p^u}$, the two-sided versions of Pesin charts that control separately the local forward and local backward hyperbolicity at x .
- (3) Construct a countable collection \mathcal{A} of ε -double charts that are dense in the space of all ε -double charts. The notion of denseness is defined in terms of finitely many parameters of x .
- (4) Define the transition between ε -double charts s.t. p^s, p^u are as maximal as possible. This is important to establish the inverse theorem (Theorem 6.1).
- (5) Apply a Bowen-Sinaï refinement (following [6]). The resulting partition defines a topological Markov shift (Σ, σ) and a map $\pi : \Sigma \rightarrow M$ satisfying Theorem 1.3.

Contrary to [21, 18], we do not require M to be compact (not even to have bounded curvature) neither f to have uniformly bounded $C^{1+\beta}$ norm. As a consequence, we have to control the parameters appearing in the construction more carefully. In the methodology of proof above, this is reflected in steps (1), (3), (4). Steps (2) and (5) work almost verbatim as in [21].

1.3. Preliminaries

Let $\mathcal{G} = (V, E)$ be an oriented graph, where $V =$ vertex set and $E =$ edge set. We denote edges by $v \rightarrow w$, and we assume that V is countable.

Topological Markov shift (TMS). – A *topological Markov shift* (TMS) is a pair (Σ, σ) where

$$\Sigma := \{\mathbb{Z}\text{-indexed paths on } \mathcal{G}\} = \{\underline{v} = \{v_n\}_{n \in \mathbb{Z}} \in V^{\mathbb{Z}} : v_n \rightarrow v_{n+1}, \forall n \in \mathbb{Z}\}$$

and $\sigma : \Sigma \rightarrow \Sigma$ is the left shift, $[\sigma(\underline{v})]_n = v_{n+1}$. The *recurrent set* of Σ is

$$\Sigma^\# := \left\{ \underline{v} \in \Sigma : \exists v, w \in V \text{ s.t. } \begin{array}{l} v_n = v \text{ for infinitely many } n > 0 \\ v_n = w \text{ for infinitely many } n < 0 \end{array} \right\}.$$

We endow Σ with the distance $d(\underline{v}, \underline{w}) := \exp[-\min\{|n| \in \mathbb{Z} : v_n \neq w_n\}]$.

Write $a = e^{\pm\varepsilon}b$ when $e^{-\varepsilon} \leq \frac{a}{b} \leq e^\varepsilon$, and $a = \pm b$ when $-|b| \leq a \leq |b|$. Given an open set $U \subset \mathbb{R}^n$ and $h : U \rightarrow \mathbb{R}^m$, let $\|h\|_0 := \sup_{x \in U} \|h(x)\|$ denote the C^0 norm of h . For $0 < \beta < 1$, let $\text{Hol}_\beta(h) := \sup \frac{\|h(x) - h(y)\|}{\|x - y\|^\beta}$ where the supremum ranges over distinct elements $x, y \in U$. If h is differentiable, let $\|h\|_1 := \|h\|_0 + \|dh\|_0$ denote its C^1 norm, and $\|h\|_{1+\beta} := \|h\|_{C^1} + \text{Hol}_\beta(dh)$ its $C^{1+\beta}$ norm. Given $x \in M$, remember that $B_x[r] \subset T_x M$ is the ball with center $0 \in T_x M$ and radius r . Also define $R[r] := [-r, r]^2 \subset \mathbb{R}^2$.

The diameter of M is less than one, hence we can assume that $a = b$: just change a, b to $\max\{a, b\}$. For symmetry and simplification purposes, we will sometimes use (A3)–(A5) in the weaker forms below. Define $\rho(x) := d(\{f^{-1}(x), x, f(x)\}, \mathcal{D})$, then (A3)–(A5) imply that for all $x \in M \setminus \mathcal{D}$:

- (A3)' If $y_1, y_2 \in D_x$ then $\|d(\widetilde{\exp_{y_1}})_{v_1} - d(\widetilde{\exp_{y_2}})_{v_2}\| \leq \rho(x)^{-a} d_{\text{Sas}}(v_1, v_2)$ for all $\|v_1\|, \|v_2\| \leq 2\tau(x)$, and $\|\tau(y_1, z_1) - \tau(y_2, z_2)\| \leq d(x, \mathcal{D})^{-a} [d(y_1, y_2) + d(z_1, z_2)]$ for all $z_1, z_2 \in D_x$.
- (A4)' If $y_1, y_2 \in D_x$ then the map $\tau(y_1, \cdot) - \tau(y_2, \cdot) : D_x \rightarrow \mathcal{L}_x$ has Lipschitz constant $\leq \rho(x)^{-a} d(y_1, y_2)$.
- (A5)' If $y \in D_x$ then $\|df_y^{\pm 1}\| \leq \rho(x)^{-a}$.

Here is a consequence of (A5) and the inverse theorem, written in symmetric form:

$$(A7) \quad \|df_x^{\pm 1}\| \geq m(df_x^{\pm 1}) \geq \rho(x)^a.$$

Above, $m(A) := \|A^{-1}\|^{-1}$. For the ease of reference, we collect (A1)–(A7) in Appendix A in the format we will use in the text.

We note that μ is f -adapted iff $\int \log \rho(x) d\mu > -\infty$. If μ is f -adapted then by μ -invariance the functions $-\log d(f^{-1}(x), \mathcal{D})$, $-\log d(x, \mathcal{D})$, $-\log d(f(x), \mathcal{D})$ are in $L^1(\mu)$, hence is also their maximum $-\log \rho(x)$. The reverse implication is proved similarly.

2. Linear Pesin theory

In this section we construct changes of coordinates that make df a hyperbolic matrix. Since we are dealing with the action of the derivative only, the closeness of x to \mathcal{D} is irrelevant.

Fix $\chi > 0$, and let NUH_χ be the set of $x \in M \setminus \bigcup_{n \in \mathbb{Z}} f^n(\mathcal{D})$ for which there are vectors $\{e_{f^n(x)}^s\}_{n \in \mathbb{Z}}, \{e_{f^n(x)}^u\}_{n \in \mathbb{Z}}$ s.t. for every $y = f^n(x)$, $n \in \mathbb{Z}$, it holds:

- (1) $e_y^{s/u} \in T_y M$, $\|e_y^{s/u}\| = 1$.
- (2) $\text{span}(df_y^m e_y^{s/u}) = \text{span}(e_{f^m(y)}^{s/u})$ for all $m \in \mathbb{Z}$.
- (3) $\lim_{m \rightarrow \pm\infty} \frac{1}{m} \log \|df_y^m e_y^s\| < -\chi$ and $\lim_{m \rightarrow \pm\infty} \frac{1}{m} \log \|df_y^m e_y^u\| > \chi$.
- (4) $\lim_{m \rightarrow \pm\infty} \frac{1}{m} \log |\sin \alpha(f^m(y))| = 0$, where $\alpha(f^m(y)) = \angle(e_{f^m(y)}^s, e_{f^m(y)}^u)$.

2.1. Oseledets-Pesin reduction

We represent df_x as a hyperbolic matrix.

Parameters $s(x), u(x)$. – For $x \in \text{NUH}_\chi$, define

$$s(x) := \sqrt{2} \left(\sum_{n \geq 0} e^{2n\chi} \|df_x^n e_x^s\|^2 \right)^{1/2} \quad \text{and} \quad u(x) := \sqrt{2} \left(\sum_{n \geq 0} e^{2n\chi} \|df_x^{-n} e_x^u\|^2 \right)^{1/2}.$$

These numbers are well-defined because $x \in \text{NUH}_\chi$, and $s(x), u(x) \geq \sqrt{2}$.

Let $e_1 = (1, 0), e_2 = (0, 1)$ be the canonical basis of \mathbb{R}^2 .

Linear map $C_\chi(x)$. – For $x \in \text{NUH}_\chi$, let $C_\chi(x) : \mathbb{R}^2 \rightarrow T_x M$ be the linear map s.t.

$$C_\chi(x) : e_1 \mapsto \frac{e^s}{s(x)}, \quad C_\chi(x) : e_2 \mapsto \frac{e^u}{u(x)}.$$

Given a linear transformation, let $\|\cdot\|$ denote its sup norm and $\|\cdot\|_{\text{Frob}}$ its Frobenius norm⁽²⁾. The Frobenius norm is equivalent to the usual sup norm, with $\|\cdot\| \leq \|\cdot\|_{\text{Frob}} \leq \sqrt{2}\|\cdot\|$.

LEMMA 2.1. – For all $x \in \text{NUH}_\chi$, the following holds:

- (1) $\|C_\chi(x)\| \leq \|C_\chi(x)\|_{\text{Frob}} \leq 1$ and $\|C_\chi(x)^{-1}\|_{\text{Frob}} = \frac{\sqrt{s(x)^2 + u(x)^2}}{|\sin \alpha(x)|}$.
- (2) $C_\chi(f(x))^{-1} \circ df_x \circ C_\chi(x)$ is a diagonal matrix with diagonal entries $A, B \in \mathbb{R}$ s.t. $|A| < e^{-\chi}$ and $|B| > e^\chi$.

Proof. – (a) In the basis $\{e_1, e_2\}$ of \mathbb{R}^2 and the basis $\{e_x^s, (e_x^s)^\perp\}$ of $T_x M$, $C_\chi(x)$ takes the form $\begin{bmatrix} \frac{1}{s(x)} & \frac{\cos \alpha(x)}{u(x)} \\ 0 & \frac{\sin \alpha(x)}{u(x)} \end{bmatrix}$, hence $\|C_\chi(x)\|_{\text{Frob}}^2 = \frac{1}{s(x)^2} + \frac{1}{u(x)^2} \leq 1$. The inverse of $C_\chi(x)$ is $\begin{bmatrix} s(x) & -\frac{s(x) \cos \alpha(x)}{\sin \alpha(x)} \\ 0 & \frac{u(x)}{\sin \alpha(x)} \end{bmatrix}$, therefore $\|C_\chi(x)^{-1}\|_{\text{Frob}} = \frac{\sqrt{s(x)^2 + u(x)^2}}{|\sin \alpha(x)|}$.

(b) It is clear that e_1, e_2 are eigenvectors of $C_\chi(f(x))^{-1} \circ df_x \circ C_\chi(x)$. We calculate the eigenvalue of e_1 (the calculation of the eigenvalue of e_2 is similar). Since $df_x e_x^s = \pm \|df_x e_x^s\| e_{f(x)}^s$, $[df_x \circ C_\chi(x)](e_1) = \pm df_x \left[\frac{e_x^s}{s(x)} \right] = \pm \frac{\|df_x e_x^s\|}{s(x)} e_{f(x)}^s$, hence $[C_\chi(f(x))^{-1} \circ df_x \circ C_\chi(x)](e_1) = \pm \|df_x e_x^s\| \frac{s(f(x))}{s(x)} e_1$. Thus $A := \pm \|df_x e_x^s\| \frac{s(f(x))}{s(x)}$ is the eigenvalue of e_1 . Note that

$$s(f(x))^2 = \frac{2}{e^{2\chi} \|df_x e_x^s\|^2} \sum_{n \geq 1} e^{2n\chi} \|df_x^n e_x^s\|^2 = \frac{s(x)^2 - 2}{e^{2\chi} \|df_x e_x^s\|^2} < \frac{s(x)^2}{e^{2\chi} \|df_x e_x^s\|^2},$$

therefore $|A| < e^{-\chi}$. □

2.2. The set NUH_χ^*

We need to control the exponential rate decay of the distance of trajectories to the set of discontinuities \mathcal{D} .

Regular set. – We define the regular set of f by

$$\text{Reg} := \left\{ x \in M \setminus \mathcal{D} : \lim_{n \rightarrow \pm\infty} \frac{1}{|n|} \log \rho(f^n(x)) = 0 \right\}.$$

(2) The Frobenius norm of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\|A\|_{\text{Frob}} = \sqrt{a^2 + b^2 + c^2 + d^2}$.

The set NUH_χ^* . – It is the set of $x \in \text{NUH}_\chi$ with the following properties:

- (1) $x \in \text{Reg}$.
- (2) There exist sequences $n_k, m_k \rightarrow \infty$ s.t. $C_\chi(f^{n_k}(x)), C_\chi(f^{-m_k}(x)) \rightarrow C_\chi(x)$.
- (3) $\lim_{n \rightarrow \pm\infty} \frac{1}{|n|} \log \|C_\chi(f^n(x))\| = 0$.
- (4) $\lim_{n \rightarrow \pm\infty} \frac{1}{|n|} \log \|C_\chi(f^n(x))^{-1}\| = 0$.

The next lemma shows that relevant measures are carried by NUH_χ^* .

LEMMA 2.2. – If μ is f -adapted and χ -hyperbolic, then $\mu[\text{NUH}_\chi^*] = 1$.

Proof. – By (A5) and the f -adaptedness of μ , $\int \log^+ \|df^{\pm 1}\| d\mu < \infty$ hence the Oseledets theorem applies to the cocycle df^n and measure μ . Since μ is χ -hyperbolic, $\mu[\text{NUH}_\chi] = 1$. By f -adaptedness and the Birkhoff ergodic theorem⁽³⁾, $\mu(\text{Reg}) = 1$. By the Poincaré recurrence theorem, (2) holds μ -a.e. It remains to check (3)–(4).

For $x \in \text{NUH}_\chi$, let $D_\chi(x) := C_\chi(f(x))^{-1} \circ df_x \circ C_\chi(x)$. This defines a cocycle $D_\chi^{(n)}$ on NUH_χ . We first show that we can apply the Oseledets theorem for $D_\chi^{(n)}$ and μ . By Lemma 2.1 and its proof, $D_\chi(x) = \begin{bmatrix} A(x) & 0 \\ 0 & B(x) \end{bmatrix}$ where $A(x)^2 = e^{-2\chi} \frac{s(x)^2 - 2}{s(x)^2}$ and $B(x)^2 = e^{2\chi} \frac{u(f(x))^2}{u(f(x))^2 - 2}$. We have $\|D_\chi(x)\| = |B(x)|$ and $\|D_\chi(x)^{-1}\| = |A(x)|^{-1}$, therefore we wish to show that

$$\int \log |A(x)| d\mu(x) > -\infty \text{ and } \int \log |B(x)| d\mu(x) < \infty.$$

We prove the first inequality (the second inequality is proved similarly). By (A6), $s(x)^2 \geq 2(1 + e^{2\chi} \|df_x e_x^s\|^2) \geq 2(1 + e^{2\chi} \rho(x)^{2a})$ hence

$$A(x)^2 = e^{-2\chi} \frac{s(x)^2 - 2}{s(x)^2} = e^{-2\chi} \left(1 - \frac{2}{s(x)^2}\right) \geq \frac{\rho(x)^{2a}}{1 + e^{2\chi} \rho(x)^{2a}} \geq \frac{\rho(x)^{2a}}{1 + e^{2\chi}}.$$

Therefore

$$\int \log |A(x)| d\mu(x) \geq a \int \log \rho(x) d\mu(x) - \frac{1}{2} \log(1 + e^{2\chi}) > -\infty.$$

By a similar reasoning, $\int \log |B(x)| d\mu(x) < \infty$. Therefore we can apply the Oseledets theorem for $D_\chi^{(n)}$ and μ : there is an f -invariant set $X \subset \text{NUH}_\chi$ with $\mu(X) = 1$ s.t. every $x \in X$ satisfies (2) and $\lim_{n \rightarrow \infty} \frac{1}{n} \log \|D_\chi^{(n)}(x)\|$ exists. We claim that (3)–(4) hold in X .

We first show that the Lyapunov exponents of $D_\chi^{(n)}$ and df^n coincide in X . Fix $x \in X$, and take $n_k \rightarrow \infty$ s.t. $C_\chi(f^{n_k}(x)) \rightarrow C_\chi(x)$. Since $\|D_\chi^{(n)}(x)\| \leq \|C_\chi(f^n(x))^{-1}\| \|df_x^n\| \|C_\chi(x)\| \leq \|C_\chi(f^n(x))^{-1}\| \|df_x^n\|$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \log \|D_\chi^{(n)}(x)\| &= \limsup_{k \rightarrow \infty} \frac{1}{n_k} \log \|D_\chi^{(n_k)}(x)\| \\ &\leq \limsup_{k \rightarrow \infty} \frac{1}{n_k} \log \|C_\chi(f^{n_k}(x))^{-1}\| + \limsup_{k \rightarrow \infty} \frac{1}{n_k} \log \|df_x^{n_k}\| = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|df_x^n\|. \end{aligned}$$

⁽³⁾ Here we are using that if $\varphi : M \rightarrow \mathbb{R}$ satisfies $\int |\varphi| d\mu < \infty$ then $\liminf_{n \rightarrow \pm\infty} \frac{1}{n} \varphi(f^n(x)) = 0$ μ -a.e. Indeed, by the Birkhoff theorem $\tilde{\varphi}(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x))$ exists μ -a.e., hence $\lim_{n \rightarrow \infty} \frac{1}{n} \varphi(f^n(x)) = \lim_{n \rightarrow \infty} \left[\frac{1}{n} \sum_{i=0}^n \varphi(f^i(x)) - \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \right] = 0$ μ -a.e. The same argument works for $n \rightarrow -\infty$.

Similarly, $\|df_x^n\| \leq \|C_\chi(f^n(x))\| \|D_\chi^{(n)}(x)\| \|C_\chi(x)^{-1}\| \leq \|D_\chi^{(n)}(x)\| \|C_\chi(x)^{-1}\|$, thus

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \log \|df_x^n\| &= \limsup_{k \rightarrow \infty} \frac{1}{n_k} \log \|df_x^{n_k}\| \leq \limsup_{k \rightarrow \infty} \frac{1}{n_k} \log \|D_\chi^{(n_k)}(x)\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \|D_\chi^{(n)}(x)\|. \end{aligned}$$

Hence $\lim_{n \rightarrow \infty} \frac{1}{n} \log \|D_\chi^{(n)}(x)\| = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|df_x^n\|$. Applying the same argument along the sequence $m_k \rightarrow \infty$ for which $C_\chi(f^{-m_k}(x)) \rightarrow C_\chi(x)$, we obtain

$$(2.1) \quad \lim_{n \rightarrow \pm\infty} \frac{1}{|n|} \log \|D_\chi^{(n)}(x)\| = \lim_{n \rightarrow \pm\infty} \frac{1}{|n|} \log \|df_x^n\|.$$

Since $\|C_\chi(\cdot)\| \leq 1$, $\limsup_{n \rightarrow \pm\infty} \frac{1}{|n|} \log \|C_\chi(f^n(x))\| \leq 0$. Reversely, the inequality $\|df_x^n\| \leq \|C_\chi(f^n(x))\| \|D_\chi^{(n)}(x)\| \|C_\chi(x)^{-1}\|$ implies

$$\liminf_{n \rightarrow \pm\infty} \frac{1}{|n|} \log \|C_\chi(f^n(x))\| \geq \lim_{n \rightarrow \pm\infty} \frac{1}{|n|} \log \|df_x^n\| - \lim_{n \rightarrow \pm\infty} \frac{1}{|n|} \log \|D_\chi^{(n)}(x)\| = 0.$$

This proves (3). A similar argument to the proof of (3) does *not* give (4). For that, we introduce some normalizing matrices. Let $\lambda_1(x), \lambda_2(x)$ be the Lyapunov exponents of df_x^n at x .

By (2.1), $D_\chi^{(n)}$ has the same Lyapunov exponents at x . Taking $\Lambda_\chi(x) := \begin{bmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{bmatrix}$, we

have $\lim_{n \rightarrow \pm\infty} \frac{1}{|n|} \log \|(D_\chi^{(n)}(x)\Lambda_\chi(x)^{-n})^{\pm 1}\| = 0$.

Similarly, we can define $\Lambda(x) : T_x M \rightarrow T_x M$ by $\Lambda(x)e_x^s = \lambda_1(x)e_x^s$ and $\Lambda(x)e_x^u = \lambda_2(x)e_x^u$ and observe that $\lim_{n \rightarrow \pm\infty} \frac{1}{|n|} \log \|(df_x^n \Lambda(x)^{-n})^{\pm 1}\| = 0$. Since $\Lambda_\chi(x) = C_\chi(x)^{-1} \Lambda(x) C_\chi(x)$, it follows that

$$\begin{aligned} C_\chi(f^n(x))^{-1} &= D_\chi^{(n)}(x) C_\chi(x)^{-1} (df_x^n)^{-1} \\ &= [D_\chi^{(n)}(x) \Lambda_\chi(x)^{-n}] [\Lambda_\chi(x)^n C_\chi(x)^{-1} \Lambda(x)^{-n}] [df_x^n \Lambda(x)^{-n}]^{-1} \\ &= [D_\chi^{(n)}(x) \Lambda_\chi(x)^{-n}] C_\chi(x)^{-1} [df_x^n \Lambda(x)^{-n}]^{-1} \end{aligned}$$

and hence

$$\begin{aligned} \limsup_{n \rightarrow \pm\infty} \frac{1}{|n|} \log \|C_\chi(f^n(x))^{-1}\| \\ \leq \lim_{n \rightarrow \pm\infty} \frac{1}{|n|} \log \|D_\chi^{(n)}(x) \Lambda_\chi(x)^{-n}\| + \lim_{n \rightarrow \pm\infty} \frac{1}{|n|} \log \|(df_x^n \Lambda(x)^{-n})^{-1}\| = 0. \end{aligned}$$

Since $\liminf_{n \rightarrow \pm\infty} \frac{1}{|n|} \log \|C_\chi(f^n(x))^{-1}\| \geq 0$, property (4) holds. Hence X satisfies (2)–(4) and $\mu[X] = 1$. Therefore $X \cap \text{Reg} \subset \text{NUH}_\chi^*$ has full μ -measure. \square

3. Non-linear Pesin theory

We now define charts that make f itself look like a hyperbolic matrix.

Pesin chart Ψ_x . – For $x \in \text{NUH}_\chi$, let $\Psi_x : R[\tau(x)] \rightarrow M$, $\Psi_x := \exp_x \circ C_\chi(x)$. Ψ_x is called the *Pesin chart at x*.

Given $x \in M \setminus \mathcal{D}$, let $\iota_x : T_x M \rightarrow \mathbb{R}^2$ be an isometry. If $y \in D_x$ and $A : \mathbb{R}^2 \rightarrow T_y M$ is a linear map, we can define $\widetilde{A} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\widetilde{A} := \iota_x \circ P_{y,x} \circ A$. Again, \widetilde{A} depends on x but $\|\widetilde{A}\|$ does not.

LEMMA 3.1. – *The Pesin chart Ψ_x is a diffeomorphism onto its image. Moreover:*

- (1) Ψ_x is 2-Lipschitz and Ψ_x^{-1} is $2\|C_\chi(x)^{-1}\|$ -Lipschitz.
- (2) $\|d(\widetilde{\Psi_x})_{v_1} - d(\widetilde{\Psi_x})_{v_2}\| \leq d(x, \mathcal{D})^{-a} \|v_1 - v_2\|$ for all $v_1, v_2 \in R[\tau(x)]$.

Proof. – Since $C_\chi(x)$ is a contraction, $C_\chi(x)R[\tau(x)] \subset B_x[2\tau(x)]$ and so Ψ_x is well-defined with inverse $C_\chi(x)^{-1} \circ \exp_x^{-1}$. It is a diffeomorphism because $C_\chi(x)$ and \exp_x are.

- (1) By (A2), Ψ_x is 2-Lipschitz and Ψ_x^{-1} is $2\|C_\chi(x)^{-1}\|$ -Lipschitz.
- (2) Since $C_\chi(x)v_i \in B_x[2\tau(x)]$, (A3) implies that

$$\begin{aligned} \|d(\widetilde{\Psi_x})_{v_1} - d(\widetilde{\Psi_x})_{v_2}\| &= \|d(\widetilde{\exp_x})_{C_\chi(x)v_1} \circ C_\chi(x) - d(\widetilde{\exp_x})_{C_\chi(x)v_2} \circ C_\chi(x)\| \\ &\leq d(x, \mathcal{D})^{-a} \|C_\chi(x)v_1 - C_\chi(x)v_2\| \leq d(x, \mathcal{D})^{-a} \|v_1 - v_2\|. \quad \square \end{aligned}$$

Given $\varepsilon > 0$, let $I_\varepsilon := \{e^{-\frac{1}{3}\varepsilon n} : n \geq 0\}$.

Parameter $Q_\varepsilon(x)$. – For $x \in \text{NUH}_\chi$, let $Q_\varepsilon(x) := \max\{q \in I_\varepsilon : q \leq \widetilde{Q}_\varepsilon(x)\}$, where

$$\widetilde{Q}_\varepsilon(x) = \varepsilon^{3/\beta} \min \left\{ \|C_\chi(x)^{-1}\|_{\text{Frob}}^{-24/\beta}, \|C_\chi(f(x))^{-1}\|_{\text{Frob}}^{-12/\beta} \rho(x)^{72a/\beta} \right\}.$$

The term $\varepsilon^{3/\beta}$ will allow to absorb multiplicative constants. The choice of $Q_\varepsilon(x)$ guarantees that the composition $\Psi_{f(x)}^{-1} \circ f \circ \Psi_x$ is well-defined in $R[10Q_\varepsilon(x)]$ and it is close to a linear hyperbolic map (Theorem 3.3), and it allows to compare nearby Pesin charts (Proposition 3.4). We have the following bounds:

$$\begin{aligned} Q_\varepsilon(x) &\leq \varepsilon^{3/\beta}, & \|C_\chi(x)^{-1}\| Q_\varepsilon(x)^{\beta/24} &\leq \varepsilon^{1/8}, \\ \|C_\chi(f(x))^{-1}\| Q_\varepsilon(x)^{\beta/12} &\leq \varepsilon^{1/4}, & \rho(x)^{-a} Q_\varepsilon(x)^{\beta/72} &< \varepsilon^{1/24}. \end{aligned}$$

LEMMA 3.2 (Temperedness lemma). – *If $x \in \text{NUH}_\chi^*$, then*

$$\lim_{n \rightarrow \pm\infty} \frac{1}{|n|} \log Q_\varepsilon(f^n(x)) = 0.$$

Proof. – Clearly $\limsup_{n \rightarrow \pm\infty} \frac{1}{|n|} \log Q_\varepsilon(f^n(x)) \leq 0$. Reversely, $x \in \text{Reg}$ implies that $\lim_{n \rightarrow \pm\infty} \frac{1}{|n|} \log \rho(f^n(x)) = 0$. By property (4) in the definition of NUH_χ^* , $\lim_{n \rightarrow \pm\infty} \frac{1}{|n|} \log \|C_\chi(f^n(x))^{-1}\| = 0$ hence $\liminf_{n \rightarrow \pm\infty} \frac{1}{|n|} \log Q_\varepsilon(f^n(x)) \geq 0$. \square

3.1. The map f in Pesin charts

THEOREM 3.3. – *The following holds for all $\varepsilon > 0$ small enough: If $x \in \text{NUH}_x$ then $f_x := \Psi_{f(x)}^{-1} \circ f \circ \Psi_x$ is well-defined on $R[10Q_\varepsilon(x)]$ and satisfies:*

- (1) $d(f_x)_0 = C_\chi(f(x))^{-1} \circ df_x \circ C_\chi(x)$.
- (2) $f_x(v_1, v_2) = (Av_1 + h_1(v_1, v_2), Bv_2 + h_2(v_1, v_2))$ for $(v_1, v_2) \in R[10Q_\varepsilon(x)]$ where:
 - (a) $|A| < e^{-\lambda}$ and $|B| > e^\lambda$, cf. Lemma 2.1.
 - (b) $h_1(0, 0) = h_2(0, 0) = 0$ and $\nabla h_1(0, 0) = \nabla h_2(0, 0) = 0$.
 - (c) $\|h_1\|_{1+\beta/2} < \varepsilon$ and $\|h_2\|_{1+\beta/2} < \varepsilon$.
- (3) $\|df_x\|_0 < \frac{2(1+e^{2\lambda})}{\rho(x)^a}$.

The norms above are taken in $R[10Q_\varepsilon(x)]$. A similar statement holds for $f_x^{-1} := \Psi_x^{-1} \circ f^{-1} \circ \Psi_{f(x)}$.

Proof. – The first step is to show that $f_x : R[10Q_\varepsilon(x)] \rightarrow \mathbb{R}^2$ is well-defined. Using that $C_\chi(x)$ is a contraction, $C_\chi(x)R[10Q_\varepsilon(x)] \subset B_x[20Q_\varepsilon(x)]$. Since $C_\chi(f(x))^{-1}$ is globally defined, it is enough to show that

$$(f \circ \exp_x)(B_x[20Q_\varepsilon(x)]) \subset \exp_{f(x)}(B_{f(x)}[2\tau(f(x))]).$$

For small $\varepsilon > 0$ we have:

- $20Q_\varepsilon(x) < 2\tau(x) \Rightarrow \exp_x$ is well-defined on $B_x[20Q_\varepsilon(x)]$. By (A2), \exp_x maps $B_x[20Q_\varepsilon(x)]$ diffeomorphically into $B(x, 40Q_\varepsilon(x))$.
- $40Q_\varepsilon(x) < 2\tau(x) \Rightarrow B(x, 40Q_\varepsilon(x)) \subset B(x, 2\tau(x))$. By (A5), f maps $B(x, 40Q_\varepsilon(x))$ diffeomorphically into $B(f(x), 40\rho(x)^{-a}Q_\varepsilon(x))$.
- $40\rho(x)^{-a}Q_\varepsilon(x) < \frac{\tau(f(x))}{2} \Rightarrow B(f(x), 40\rho(x)^{-a}Q_\varepsilon(x)) \subset B\left(f(x), \frac{\tau(f(x))}{2}\right)$. By (A2), $\exp_{f(x)}^{-1}$ maps $B\left(f(x), \frac{\tau(f(x))}{2}\right)$ diffeomorphically into $B_{f(x)}[\tau(f(x))]$.

Therefore $f_x : R[10Q_\varepsilon(x)] \rightarrow \mathbb{R}^2$ is a diffeomorphism onto its image.

We check (1)–(2). Property (1) is clear since $d(\Psi_x)_0 = C_\chi(x)$ and $d(\Psi_{f(x)})_0 = C_\chi(f(x))$.

By Lemma 2.1, $d(f_x)_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ with $|A| < e^{-\lambda}$ and $|B| > e^\lambda$. Define $h_1, h_2 : R[10Q_\varepsilon(x)] \rightarrow \mathbb{R}$ by $f_x(v_1, v_2) = (Av_1 + h_1(v_1, v_2), Bv_2 + h_2(v_1, v_2))$. Then (a)–(b) are automatically satisfied. It remains to prove (c).

Claim. – $\|d(f_x)_{w_1} - d(f_x)_{w_2}\| \leq \frac{\varepsilon}{3}\|w_1 - w_2\|^{\beta/2}$ for all $w_1, w_2 \in R[10Q_\varepsilon(x)]$.

Before proving the claim, let us show how to conclude (c). Let $h = (h_1, h_2)$. If $\varepsilon > 0$ is small enough then $R[10Q_\varepsilon(x)] \subset B_x[1]$. Applying the claim with $w_2 = 0$, we get $\|dh_w\| \leq \frac{\varepsilon}{3}\|w\|^{\beta/2} < \frac{\varepsilon}{3}$. By the mean value inequality, $\|h(w)\| \leq \frac{\varepsilon}{3}\|w\| < \frac{\varepsilon}{3}$, hence $\|h\|_{1+\beta/2} < \varepsilon$.

Proof of the claim. – For $i = 1, 2$, define

$$A_i = d(\exp_{f(x)}^{-1})_{(f \circ \exp_x)(w_i)}, \quad B_i = d\widehat{f}_{\exp_x(w_i)}, \quad C_i = d(\widehat{\exp_x})_{w_i}.$$

We first estimate $\|A_1 B_1 C_1 - A_2 B_2 C_2\|$.

– By (A2), $\|A_i\| \leq 2$. By (A2), (A3), (A5):

$$\begin{aligned} \|A_1 - A_2\| &\leq d(f(x), \mathcal{D})^{-a} d((f \circ \exp_x)(w_1), (f \circ \exp_x)(w_2)) \\ &\leq 2d(x, \mathcal{D})^{-a} d(f(x), \mathcal{D})^{-a} \|w_1 - w_2\| \leq 2\rho(x)^{-2a} \|w_1 - w_2\|. \end{aligned}$$

– By (A5), $\|B_i\| \leq \rho(x)^{-a}$. By (A2) and (A6):

$$\|B_1 - B_2\| \leq \mathfrak{K}d(\exp_x(w_1), \exp_x(w_2))^\beta \leq 2\mathfrak{K}\|w_1 - w_2\|^\beta.$$

– By (A2), $\|C_i\| \leq 2$. By (A3):

$$\|C_1 - C_2\| \leq d(x, \mathcal{D})^{-a} \|w_1 - w_2\| \leq \rho(x)^{-a} \|w_1 - w_2\|.$$

By a crude approximation, we get $\|A_1 B_1 C_1 - A_2 B_2 C_2\| \leq 24\mathfrak{K}\rho(x)^{-3a} \|w_1 - w_2\|^\beta$. Now we estimate $\|d(f_x)_{w_1} - d(f_x)_{w_2}\|$:

$$\begin{aligned} \|d(f_x)_{w_1} - d(f_x)_{w_2}\| &\leq \|C_\chi(f(x))^{-1}\| \|A_1 B_1 C_1 - A_2 B_2 C_2\| \|C_\chi(x)\| \\ &\leq 24\mathfrak{K}\rho(x)^{-3a} \|C_\chi(f(x))^{-1}\| \|w_1 - w_2\|^\beta. \end{aligned}$$

Since $\|w_1 - w_2\| < 40Q_\varepsilon(x)$, if $\varepsilon > 0$ is small enough then

$$\begin{aligned} 24\mathfrak{K}\rho(x)^{-3a} \|C_\chi(f(x))^{-1}\| \|w_1 - w_2\|^{\beta/2} \\ \leq 200\mathfrak{K}\rho(x)^{-3a} \varepsilon^{3/2} \|C_\chi(f(x))^{-1}\|^{-5} \rho(x)^{36a} \leq 200\mathfrak{K}\varepsilon^{3/2} < \varepsilon. \end{aligned}$$

This completes the proof of the claim. \square

(3) In the proof of Lemma 2.2 we showed that $\|d(f_x)_0\| = |B(x)| \leq \frac{\sqrt{1+e^{2x}}}{\rho(x)^a} < \frac{1+e^{2x}}{\rho(x)^a}$. By part (2) above, if $w \in R[10Q_\varepsilon(x)]$ then $\|d(f_x)_w\| \leq \varepsilon\|w\|^{\beta/2} + \frac{1+e^{2x}}{\rho(x)^a} < \frac{2(1+e^{2x})}{\rho(x)^a}$, since $\varepsilon\|w\|^{\beta/2} < 1 < \frac{1+e^{2x}}{\rho(x)^a}$ for small $\varepsilon > 0$. \square

3.2. The overlap condition

We now want to change coordinates from Ψ_x to Ψ_y when x, y are ‘‘sufficiently close’’. Even when x and y are very close, the behavior of $C_\chi(x)$ and $C_\chi(y)$ might differ, so we need to compare them. We will eventually consider Pesin charts with different domains.

Pesin chart Ψ_x^η . – It is the restriction of Ψ_x to $R[\eta]$, where $0 < \eta \leq Q_\varepsilon(x)$.

ε -overlap. – Two Pesin charts $\Psi_{x_1}^{\eta_1}, \Psi_{x_2}^{\eta_2}$ are said to ε -overlap if $\frac{\eta_1}{\eta_2} = e^{\pm\varepsilon}$ and if there is $x \in M$ s.t. $x_1, x_2 \in D_x$ and $d(x_1, x_2) + \|\widetilde{C_\chi(x_1)} - \widetilde{C_\chi(x_2)}\| < (\eta_1 \eta_2)^4$.

We write $\Psi_{x_1}^{\eta_1} \stackrel{\varepsilon}{\approx} \Psi_{x_2}^{\eta_2}$. We claim that if $\varepsilon > 0$ is small enough, then $\Psi_{x_1}^{\eta_1} \stackrel{\varepsilon}{\approx} \Psi_{x_2}^{\eta_2}$ implies that $\Psi_{x_i}(R[10Q_\varepsilon(x_i)]) \subset D_{x_1} \cap D_{x_2}$ (and hence we can apply (A1)–(A3) without mentioning x). We prove the inclusion for $i = 1$. Start noting that, since $d(x_1, x_2) < \varepsilon d(x_2, \mathcal{D})$, $d(x_1, \mathcal{D}) = d(x_2, \mathcal{D}) \pm d(x_1, x_2) = (1 \pm \varepsilon)d(x_2, \mathcal{D})$. By Lemma 3.1(1), $\Psi_{x_1}(R[10Q_\varepsilon(x_1)]) \subset B(x_1, 40Q_\varepsilon(x_1))$. This ball is contained in D_{x_1} since $40Q_\varepsilon(x_1) \ll 40\varepsilon^{3/\beta} \rho(x_1)^a < \tau(x_1)$. We have

$$\Psi_{x_1}(R[10Q_\varepsilon(x_1)]) \subset B(x_1, 40Q_\varepsilon(x_1)) \subset B(x_2, 40Q_\varepsilon(x_1) + d(x_1, x_2)).$$

Since $40Q_\varepsilon(x_1) + d(x_1, x_2) \leq 40\varepsilon^{3/\beta} (1+\varepsilon)^a d(x_2, \mathcal{D})^a + d(x_2, \mathcal{D})^a < 2\tau(x_2)$ for small $\varepsilon > 0$, it follows that $\Psi_{x_1}(R[10Q_\varepsilon(x_1)]) \subset D_{x_2}$. The next proposition shows that ε -overlap is strong enough to guarantee that the Pesin charts are close.

PROPOSITION 3.4. – *The following holds for $\varepsilon > 0$ small enough. If $\Psi_{x_1}^{\eta_1} \stackrel{\varepsilon}{\approx} \Psi_{x_2}^{\eta_2}$ then:*

- (1) CONTROL OF s, u : $\frac{s(x_1)}{s(x_2)} = e^{\pm(\eta_1\eta_2)^3}$ and $\frac{u(x_1)}{u(x_2)} = e^{\pm(\eta_1\eta_2)^3}$.
- (2) CONTROL OF α : $\frac{|\sin \alpha(x_1)|}{|\sin \alpha(x_2)|} = e^{\pm(\eta_1\eta_2)^3}$.
- (3) OVERLAP: $\Psi_{x_i}(R[e^{-2\varepsilon}\eta_i]) \subset \Psi_{x_j}(R[\eta_j])$ for $i, j = 1, 2$.
- (4) CHANGE OF COORDINATES: For $i, j = 1, 2$, the map $\Psi_{x_i}^{-1} \circ \Psi_{x_j}$ is well-defined in $R[d(x_j, \mathcal{D})^a]$, and $\|\Psi_{x_i}^{-1} \circ \Psi_{x_j} - \text{Id}\|_{1+\beta/2} < \varepsilon(\eta_1\eta_2)^2$ where the norm is taken in $R[d(x_j, \mathcal{D})^{2a}]$.

Proof. – Assume $x_1, x_2 \in D_x$, and let $C_i = \widetilde{C_\chi}(x_i)$.

By assumption, $d(x_1, x_2) + \|C_1 - C_2\| < (\eta_1\eta_2)^4$. Note that $\Psi_{x_i} = \exp_{x_i} \circ P_{x, x_i} \circ C_i$.

- (1) We prove the estimate for s (the calculation for u is similar). Since $\varepsilon > 0$ is small, it is enough to prove that $\left| \frac{s(x_1)}{s(x_2)} - 1 \right| < \varepsilon^{3/\beta} (\eta_1\eta_2)^3$. We have $s(x_i)^{-1} = \|C_\chi(x_i)e_1\| = \|C_i e_1\|$, hence $|s(x_1)^{-1} - s(x_2)^{-1}| = \| \|C_1 e_1\| - \|C_2 e_1\| \| \leq \|C_1 - C_2\| < (\eta_1\eta_2)^4$. Also $s(x_1) = \|C_\chi(x_1)e_1\|^{-1} \leq \|C_\chi(x_1)^{-1}\| < \frac{\varepsilon^{3/\beta}}{\mathcal{Q}_\varepsilon(x_1)} < \frac{\varepsilon^{3/\beta}}{\eta_1\eta_2}$, therefore

$$\left| \frac{s(x_1)}{s(x_2)} - 1 \right| = s(x_1) |s(x_1)^{-1} - s(x_2)^{-1}| < \varepsilon^{3/\beta} (\eta_1\eta_2)^3.$$

- (2) We use the general inequality for an invertible linear transformation L :

$$(3.1) \quad \frac{1}{\|L\| \|L^{-1}\|} \leq \frac{|\sin \angle(Lv, Lw)|}{|\sin \angle(v, w)|} \leq \|L\| \|L^{-1}\|.$$

Apply this to $L = C_1 C_2^{-1}$, $v = C_2 e_1$, $w = C_2 e_2$ to get that

$$\frac{1}{\|C_1 C_2^{-1}\| \|C_2 C_1^{-1}\|} \leq \frac{\sin \alpha(x_1)}{\sin \alpha(x_2)} \leq \|C_1 C_2^{-1}\| \|C_2 C_1^{-1}\|.$$

We have $\|C_1 C_2^{-1} - \text{Id}\| \leq \|C_1 - C_2\| \|C_2^{-1}\| < \varepsilon^{3/\beta} (\eta_1\eta_2)^3$, and by symmetry $\|C_2 C_1^{-1} - \text{Id}\| < \varepsilon^{3/\beta} (\eta_1\eta_2)^3$, therefore $\|C_1 C_2^{-1}\| \|C_2 C_1^{-1}\| < [1 + \varepsilon^{3/\beta} (\eta_1\eta_2)^3]^2 < e^{2\varepsilon^{3/\beta} (\eta_1\eta_2)^3} < e^{(\eta_1\eta_2)^3}$. The left hand side estimate is proved similarly.

- (3) We prove that $\Psi_{x_1}(R[e^{-2\varepsilon}\eta_1]) \subset \Psi_{x_2}(R[\eta_2])$. If $v \in R[e^{-2\varepsilon}\eta_1]$ then $\|C_\chi(x_1)v\| \leq \sqrt{2}e^{-2\varepsilon}\eta_1 < 2\tau(x)$, hence by (A1):

$$d_{\text{Sas}}(C_\chi(x_1)v, C_\chi(x_2)v) \leq 2(d(x_1, x_2) + \|C_1 v - C_2 v\|) \leq 2(\eta_1\eta_2)^4.$$

By (A2), $d(\Psi_{x_1}(v), \Psi_{x_2}(v)) \leq 4(\eta_1\eta_2)^4 \Rightarrow \Psi_{x_1}(v) \in B(\Psi_{x_2}(v), 4(\eta_1\eta_2)^4)$. By Lemma 3.1(1), $B(\Psi_{x_2}(v), 4(\eta_1\eta_2)^4) \subset \Psi_{x_2}(B)$ where $B \subset \mathbb{R}^2$ is the ball with center v and radius $8\|C_2^{-1}\|(\eta_1\eta_2)^4$, hence it is enough that $B \subset R[\eta_2]$. If $w \in B$ then $\|w\|_\infty \leq \|v\|_\infty + 8\|C_2^{-1}\|(\eta_1\eta_2)^4 \leq (e^{-\varepsilon} + 8\varepsilon^{3/\beta})\eta_2 < \eta_2$ for $\varepsilon > 0$ small enough.

- (4) The proof that $\Psi_{x_2}^{-1} \circ \Psi_{x_1}$ is well-defined in $R[d(x_1, \mathcal{D})^a]$ is similar to the proof of (3). The only difference is in the last estimate: if $\varepsilon > 0$ is small enough then for $w \in B$ it holds

$$\begin{aligned} \|w\| &\leq \|v\| + 8\|C_2^{-1}\|(\eta_1\eta_2)^4 \leq \sqrt{2}d(x_1, \mathcal{D})^a + 8(\eta_1\eta_2)^3 \\ &\leq [\sqrt{2}(1 + \varepsilon)^a + 8\varepsilon^{3/\beta}]d(x_2, \mathcal{D})^a < 2\tau(x_2). \end{aligned}$$

Now:

$$\begin{aligned}\Psi_{x_2}^{-1} \circ \Psi_{x_1} - \text{Id} &= C_2^{-1} \circ \exp_{x_2}^{-1} \circ \exp_{x_1} \circ C_1 - \text{Id} \\ &= [C_2^{-1} \circ P_{x_2,x}] \circ [\exp_{x_2}^{-1} \circ \exp_{x_1} - P_{x_1,x_2}] \circ [P_{x,x_1} \circ C_1] + C_2^{-1}(C_1 - C_2) \\ &= [C_2^{-1} \circ P_{x_2,x}] \circ [\exp_{x_2}^{-1} - P_{x_1,x_2} \circ \exp_{x_1}^{-1}] \circ \Psi_{x_1} + C_2^{-1}(C_1 - C_2).\end{aligned}$$

We calculate the $C^{1+\beta/2}$ norm of $[\exp_{x_2}^{-1} - P_{x_1,x_2} \circ \exp_{x_1}^{-1}] \circ \Psi_{x_1}$ in the domain $R[d(x_1, \mathcal{D})^{2a}]$. By Lemma 3.1(1), $\|d\Psi_{x_1}\|_0 \leq 2$ and

$$\text{Hol}_{\beta/2}(d\Psi_{x_1}) \leq d(x_1, \mathcal{D})^{-a} 4d(x_1, \mathcal{D})^{2a(1-\beta/2)} = 4d(x_1, \mathcal{D})^{a(1-\beta)} < 4.$$

Call $\Theta := \exp_{x_2}^{-1} - P_{x_1,x_2} \circ \exp_{x_1}^{-1}$. For $\varepsilon > 0$ small enough, inside D_{x_1} we have:

- By (A2), $\|\Theta(v)\| \leq d_{\text{Sas}}(\exp_{x_2}^{-1}(v), \exp_{x_1}^{-1}(v)) \leq 2d(x_1, x_2) \leq 2\varepsilon^{6/\beta}(\eta_1\eta_2)^3$ thus $\|\Theta \circ \Psi_{x_1}\|_0 < \varepsilon^{2/\beta}(\eta_1\eta_2)^3$.
- By (A3), $\|d\Theta_v\| = \|\tau(x_2, v) - \tau(x_1, v)\| \leq d(x_1, \mathcal{D})^{-a}d(x_1, x_2) < \varepsilon^{3/\beta}(\eta_1\eta_2)^3$. Hence $\|d\Theta\|_0 < \varepsilon^{3/\beta}(\eta_1\eta_2)^3$ and $\|d(\Theta \circ \Psi_{x_1})\|_0 \leq 2\varepsilon^{3/\beta}(\eta_1\eta_2)^3 < \varepsilon^{2/\beta}(\eta_1\eta_2)^3$.
- By (A4),

$$\begin{aligned}\|\widetilde{d\Theta}_v - \widetilde{d\Theta}_w\| &= \|[\tau(x_2, v) - \tau(x_1, v)] - [\tau(x_2, w) - \tau(x_1, w)]\| \\ &\leq d(x_1, \mathcal{D})^{-a}d(x_1, x_2)\|v - w\|\end{aligned}$$

hence $\text{Lip}(d\Theta) \leq d(x_1, \mathcal{D})^{-a}d(x_1, x_2)$.

- Using that

$$\text{Hol}_{\beta/2}(d(\Theta_1 \circ \Theta_2)) \leq \|d\Theta_1\|_0 \text{Hol}_{\beta/2}(d\Theta_2) + \text{Lip}(d\Theta_1)\|d\Theta_2\|_0^2 4d(x_1, \mathcal{D})^{2a(1-\beta/2)}$$

for Θ_2 with domain $R[d(x_1, \mathcal{D})^{2a}]$, we get that

$$\begin{aligned}\text{Hol}_{\beta/2}[d(\Theta \circ \Psi_{x_1})] &\leq \|d\Theta\|_0 \text{Hol}_{\beta/2}(d\Psi_{x_1}) + \text{Lip}(d\Theta)\|d\Psi_{x_1}\|_0^2 4d(x_1, \mathcal{D})^{2a(1-\beta/2)} \\ &< 4\varepsilon^{3/\beta}(\eta_1\eta_2)^3 + d(x_1, \mathcal{D})^{-a}d(x_1, x_2)16d(x_1, \mathcal{D})^{2a(1-\beta/2)} \\ &< 4\varepsilon^{3/\beta}(\eta_1\eta_2)^3 + 16\varepsilon^{6/\beta}(\eta_1\eta_2)^3 < \varepsilon^{2/\beta}(\eta_1\eta_2)^3.\end{aligned}$$

This implies that $\|\Theta \circ \Psi_{x_1}\|_{1+\beta/2} < 3\varepsilon^{2/\beta}(\eta_1\eta_2)^3$, hence

$$\|C_2^{-1} \circ P_{x_2,x} \circ \Theta \circ \Psi_{x_1}\|_{1+\beta/2} \leq \|C_2^{-1}\| 3\varepsilon^{2/\beta}(\eta_1\eta_2)^3 \leq 3\varepsilon^{2/\beta}(\eta_1\eta_2)^2.$$

Thus $\|\Psi_{x_2}^{-1} \circ \Psi_{x_1} - \text{Id}\|_{1+\beta/2} \leq 3\varepsilon^{2/\beta}(\eta_1\eta_2)^2 + \|C_2^{-1}\|(\eta_1\eta_2)^4 < 3\varepsilon^{2/\beta}(\eta_1\eta_2)^2 + \varepsilon^{3/\beta}(\eta_1\eta_2)^3 < 4\varepsilon^{2/\beta}(\eta_1\eta_2)^2 < \varepsilon(\eta_1\eta_2)^2$. \square

3.3. The map $f_{x,y}$

Let $x, y \in \text{NUH}_\chi$, and assume that $\Psi_{f(x)}^\eta \stackrel{\varepsilon}{\approx} \Psi_y^{\eta'}$. We want to change $\Psi_{f(x)}$ by Ψ_y in f_x and obtain a result similar to Theorem 3.3.

The maps $f_{x,y}$ and $f_{x,y}^{-1}$. – If $\Psi_{f(x)}^\eta \stackrel{\varepsilon}{\approx} \Psi_y^{\eta'}$, define the map $f_{x,y} := \Psi_y^{-1} \circ f \circ \Psi_x$. If $\Psi_x^\eta \stackrel{\varepsilon}{\approx} \Psi_{f^{-1}(y)}^{\eta'}$, define $f_{x,y}^{-1} := \Psi_x^{-1} \circ f^{-1} \circ \Psi_y$.

Any meaningful estimate of the regularity of $f_{x,y}$ in the $C^{1+\beta/2}$ norm cannot be better than that of Theorem 3.3. In order to keep estimates of size ε , we consider the $C^{1+\beta/3}$ norm.

THEOREM 3.5. – *The following holds for all $\varepsilon > 0$ small enough: If $x, y \in \text{NUH}_X$ and $\Psi_{f(x)}^\eta \stackrel{\varepsilon}{\approx} \Psi_y^{\eta'}$, then $f_{x,y}$ is well-defined in $R[10Q_\varepsilon(x)]$ and can be written as $f_{x,y}(v_1, v_2) = (Av_1 + h_1(v_1, v_2), Bv_2 + h_2(v_1, v_2))$ where:*

- (a) $|A| < e^{-\lambda}$, $|B| > e^\lambda$, cf. Lemma 2.1.
- (b) $\|h_i(0)\| < \varepsilon\eta$, $\|\nabla h_i(0)\| < \varepsilon\eta^{\beta/3}$, and $\text{Hol}_{\beta/3}(\nabla h_i) < \varepsilon$ where the norm is taken in $R[10Q_\varepsilon(x)]$.

If $\Psi_x^\eta \stackrel{\varepsilon}{\approx} \Psi_{f^{-1}(y)}^{\eta'}$ then a similar statement holds for $f_{x,y}^{-1}$.

Proof. – We write $f_{x,y} = (\Psi_y^{-1} \circ \Psi_{f(x)}) \circ f_x =: g \circ f_x$ and see it as a small perturbation of f_x . By Theorem 3.3(2–3),

$$f_x(0) = 0, \quad \|d(f_x)\|_0 < \frac{2(1+e^{2\lambda})}{\rho(x)^a}, \quad \|d(f_x)_v - d(f_x)_w\| \leq \varepsilon\|v - w\|^{\beta/2}$$

for $v, w \in R[10Q_\varepsilon(x)]$, where the C^0 norm is taken in $R[10Q_\varepsilon(x)]$, and by Proposition 3.4(4) we have

$$\|g - \text{Id}\| < \varepsilon(\eta\eta')^2, \quad \|d(g - \text{Id})\|_0 < \varepsilon(\eta\eta')^2, \quad \|dg_v - dg_w\| \leq \varepsilon(\eta\eta')^2\|v - w\|^{\beta/2}$$

for $v, w \in R[d(f(x), \mathcal{D})^{2a}]$, where the C^0 norm is taken in this same domain.

We first prove that $f_{x,y}$ is well-defined in $R[10Q_\varepsilon(x)]$. We have

$$f_x(R[10Q_\varepsilon(x)]) \subset B(0, 40(1 + e^{2\lambda})\rho(x)^{-a}Q_\varepsilon(x)) \subset R[d(f(x), \mathcal{D})^{2a}]$$

since $40(1 + e^{2\lambda})\rho(x)^{-a}Q_\varepsilon(x) < 40(1 + e^{2\lambda})\varepsilon^{3/\beta}d(f(x), \mathcal{D})^{2a} < d(f(x), \mathcal{D})^{2a}$ for $\varepsilon > 0$ small enough. By Proposition 3.4(4), $f_{x,y}$ is well-defined.

Now we prove (b). Let $h := (h_1, h_2) = g \circ f_x - d(f_x)_0$. Then $\|h(0)\| = \|g(0)\| < \varepsilon(\eta\eta')^2 < \varepsilon\eta$ and for $\varepsilon > 0$ small enough:

$$\begin{aligned} \|\nabla h(0)\| &\leq \|dg_0 \circ d(f_x)_0 - d(f_x)_0\| \leq \|d(g - \text{Id})_0\| \|d(f_x)_0\| \\ &< \varepsilon(\eta\eta')^2 2(1 + e^{2\lambda})\rho(x)^{-a} < \varepsilon\eta\eta' 2\varepsilon^{3/\beta}(1 + e^{2\lambda}) < \varepsilon\eta^{\beta/3}. \end{aligned}$$

Finally, since $f_x(R[10Q_\varepsilon(x)]) \subset R[d(f(x), \mathcal{D})^{2a}]$, if $\varepsilon > 0$ is small enough then for all $v, w \in R[10Q_\varepsilon(x)]$ it holds:

$$\begin{aligned} \|dh_v - dh_w\| &= \|dg_{f_x(v)} \circ d(f_x)_v - dg_{f_x(w)} \circ d(f_x)_w\| \\ &\leq \|dg_{f_x(v)} - dg_{f_x(w)}\| \|d(f_x)_v\| + \|dg_{f_x(w)}\| \|d(f_x)_v - d(f_x)_w\| \\ &\leq \varepsilon(\eta\eta')^2 \|f_x(v) - f_x(w)\|^{\beta/2} \|d(f_x)\|_0 + \varepsilon \|dg\|_0 \|v - w\|^{\beta/2} \\ &\leq (\varepsilon(\eta\eta')^2 \|d(f_x)\|_0^{1+\beta/2} + 40\varepsilon \|dg\|_0 Q_\varepsilon(x)^{\beta/6}) \|v - w\|^{\beta/3} \\ &\leq (4\eta^2(1 + e^{2\lambda})^2 \rho(x)^{-2a} + 80Q_\varepsilon(x)^{\beta/6}) \varepsilon \|v - w\|^{\beta/3} \\ &\leq (4\varepsilon^{6/\beta}(1 + e^{2\lambda})^2 + 80\varepsilon^{1/2}) \varepsilon \|v - w\|^{\beta/3} < \varepsilon \|v - w\|^{\beta/3}. \quad \square \end{aligned}$$

4. Double charts and the graph transform method

We now define ε -double charts. For $\varepsilon > 0$ small, define $\delta_\varepsilon := e^{-\varepsilon n} \in I_\varepsilon$ where n is the unique positive integer s.t. $e^{-\varepsilon n} < \varepsilon \leq e^{-\varepsilon(n-1)}$. In particular, $\delta_\varepsilon < \varepsilon$.

ε -double chart. – An ε -double chart is a pair of Pesin charts $\Psi_x^{p^s, p^u} = (\Psi_x^{p^s}, \Psi_x^{p^u})$ where $p^s, p^u \in I_\varepsilon$ with $0 < p^s, p^u \leq \delta_\varepsilon Q_\varepsilon(x)$.

The parameters p^s/p^u control the local forward/backward hyperbolicity at x . They are a way of separating the future and past dynamics. This will be better explained below, when we introduce the parameters $q_\varepsilon, q_\varepsilon^s, q_\varepsilon^u$.

Edge $v \xrightarrow{\varepsilon} w$. – Given ε -double charts $v = \Psi_x^{p^s, p^u}$ and $w = \Psi_y^{q^s, q^u}$, we draw an edge from v to w if the following conditions are satisfied:

- (GPO1) $\Psi_{f(x)}^{q^s \wedge q^u} \stackrel{\varepsilon}{\approx} \Psi_y^{q^s \wedge q^u}$ and $\Psi_{f^{-1}(y)}^{p^s \wedge p^u} \stackrel{\varepsilon}{\approx} \Psi_x^{p^s \wedge p^u}$.
 (GPO2) $p^s = \min\{e^\varepsilon q^s, \delta_\varepsilon Q_\varepsilon(x)\}$ and $q^u = \min\{e^\varepsilon p^u, \delta_\varepsilon Q_\varepsilon(y)\}$.

(GPO1) allows to pass from an ε -double chart at x to an ε -double chart at y and vice-versa. (GPO2) is a greedy recursion that implies that the local hyperbolicity parameters are the largest as possible. It implies that $\frac{p^s \wedge p^u}{q^s \wedge q^u} = e^{\pm\varepsilon}$. (GPO2) will be crucial in the proof of the inverse theorem (Theorem 6.1).

ε -generalized pseudo-orbit (ε -gpo). – An ε -generalized pseudo-orbit (ε -gpo) is a sequence $\underline{v} = \{v_n\}_{n \in \mathbb{Z}}$ of ε -double charts s.t. $v_n \xrightarrow{\varepsilon} v_{n+1}$ for all $n \in \mathbb{Z}$.

4.1. The parameters $q_\varepsilon(x), q_\varepsilon^s(x), q_\varepsilon^u(x)$

A transition between Pesin charts only makes sense if their sizes η, η' satisfy $\frac{\eta}{\eta'} = e^{\pm\varepsilon}$ (see Theorem 3.5). Since the ratio $\frac{Q_\varepsilon(f(x))}{Q_\varepsilon(x)}$ might be different from $e^{\pm\varepsilon}$, we introduce the parameter $q_\varepsilon(x)$ below.

Parameter $q_\varepsilon(x)$. – For $x \in \text{NUH}_\chi^*$, let $q_\varepsilon(x) := \delta_\varepsilon \min\{e^{\varepsilon|n|} Q_\varepsilon(f^n(x)) : n \in \mathbb{Z}\}$.

The above minimum is the greedy way of defining values in I_ε smaller than $\varepsilon Q_\varepsilon$ with the required regularity property.

LEMMA 4.1. – For all $x \in \text{NUH}_\chi^*$, $0 < q_\varepsilon(x) < \varepsilon Q_\varepsilon(x)$ and $\frac{q_\varepsilon(f(x))}{q_\varepsilon(x)} = e^{\pm\varepsilon}$.

Proof. – By Lemma 3.2, $\inf\{e^{\varepsilon|n|} Q_\varepsilon(f^n(x)) : n \in \mathbb{Z}\} > 0$. Since zero is the only accumulation point of I_ε , $q_\varepsilon(x)$ is well-defined and positive. It is clear that $q_\varepsilon(x) \leq \delta_\varepsilon Q_\varepsilon(x) < \varepsilon Q_\varepsilon(x)$. Since

$$\min\{e^{\varepsilon|n|} Q_\varepsilon(f^{n+1}(x)) : n \in \mathbb{Z}\} \leq e^\varepsilon \min\{e^{\varepsilon|n+1|} Q_\varepsilon(f^{n+1}(x)) : n \in \mathbb{Z}\},$$

we have $q_\varepsilon(f(x)) \leq e^\varepsilon q_\varepsilon(x)$. Reversely,

$$e^{-\varepsilon} \min\{e^{\varepsilon|n+1|} Q_\varepsilon(f^{n+1}(x)) : n \in \mathbb{Z}\} \leq \min\{e^{\varepsilon|n|} Q_\varepsilon(f^{n+1}(x)) : n \in \mathbb{Z}\},$$

therefore $e^{-\varepsilon} q_\varepsilon(x) \leq q_\varepsilon(f(x))$. □

We want to separate the dependence of $q_\varepsilon(x)$ on the future from its dependence on the past, hence we define the one-sided versions of $q_\varepsilon(x)$.

Parameters $q_\varepsilon^s(x), q_\varepsilon^u(x)$. – For $x \in \text{NUH}_\chi^*$, define

$$\begin{aligned} q_\varepsilon^s(x) &:= \delta_\varepsilon \min\{e^{\varepsilon|n|} Q_\varepsilon(f^n(x)) : n \geq 0\} \\ q_\varepsilon^u(x) &:= \delta_\varepsilon \min\{e^{\varepsilon|n|} Q_\varepsilon(f^n(x)) : n \leq 0\}. \end{aligned}$$

LEMMA 4.2. – For all $x \in \text{NUH}_\chi^*$, the following holds:

- (1) GOOD DEFINITION: $0 < q_\varepsilon^s(x), q_\varepsilon^u(x) < \varepsilon Q_\varepsilon(x)$ and $q_\varepsilon^s(x) \wedge q_\varepsilon^u(x) = q_\varepsilon(x)$.
- (2) GREEDY ALGORITHM: For all $n \in \mathbb{Z}$ it holds

$$\begin{aligned} q_\varepsilon^s(f^n(x)) &= \min\{e^\varepsilon q_\varepsilon^s(f^{n+1}(x)), \delta_\varepsilon Q_\varepsilon(f^n(x))\} \\ q_\varepsilon^u(f^n(x)) &= \min\{e^\varepsilon q_\varepsilon^u(f^{n-1}(x)), \delta_\varepsilon Q_\varepsilon(f^n(x))\}. \end{aligned}$$

Proof. – As in the proof of Lemma 4.1, $q_\varepsilon^s(x)$ and $q_\varepsilon^u(x)$ are well-defined and positive, and by definition $q_\varepsilon^s(x) \wedge q_\varepsilon^u(x) = q_\varepsilon(x)$. This proves (1). We prove the first equality in (2): for a fixed $n \in \mathbb{Z}$ we have

$$\begin{aligned} q_\varepsilon^s(f^n(x)) &= \delta_\varepsilon \min\{e^{\varepsilon|m|} Q_\varepsilon(f^m(f^n(x))) : m \geq 0\} \\ &= \min\{\delta_\varepsilon \min\{e^{\varepsilon|m|} Q_\varepsilon(f^{m+n}(x)) : m \geq 1\}, \delta_\varepsilon Q_\varepsilon(f^n(x))\} \\ &= \min\{e^\varepsilon \delta_\varepsilon \min\{e^{\varepsilon|m|} Q_\varepsilon(f^m(f^{n+1}(x))) : m \geq 0\}, \delta_\varepsilon Q_\varepsilon(f^n(x))\} \\ &= \min\{e^\varepsilon q_\varepsilon^s(f^{n+1}(x)), \delta_\varepsilon Q_\varepsilon(f^n(x))\}. \end{aligned}$$

The second equality is proved similarly. □

The set $\text{NUH}_\chi^\#$. – It is the set of $x \in \text{NUH}_\chi^*$ s.t.

$$\limsup_{n \rightarrow +\infty} q_\varepsilon(f^n(x)) > 0 \text{ and } \limsup_{n \rightarrow -\infty} q_\varepsilon(f^n(x)) > 0.$$

4.2. The graph transform method

Let $v = \Psi_x^{p^s, p^u}$ be an ε -double chart.

Admissible manifolds. – We define an s -admissible manifold at v as a set of the form $\Psi_x\{(t, F(t)) : |t| \leq p^s\}$ where $F : [-p^s, p^s] \rightarrow \mathbb{R}$ is a $C^{1+\beta/3}$ function s.t.:

- (AM1) $|F(0)| \leq 10^{-3}(p^s \wedge p^u)$.
- (AM2) $|F'(0)| \leq \frac{1}{2}(p^s \wedge p^u)^{\beta/3}$.
- (AM3) $\|F'\|_0 + \text{Hol}_{\beta/3}(F') \leq \frac{1}{2}$ where the norms are taken in $[-p^s, p^s]$.

Similarly, a u -admissible manifold at v is a set of the form $\Psi_x\{(G(t), t) : |t| \leq p^u\}$ where $G : [-p^u, p^u] \rightarrow \mathbb{R}$ is a $C^{1+\beta/3}$ function satisfying (AM1)–(AM3), where the norms are taken in $[-p^u, p^u]$.

The functions F, G are called the *representing functions*. We let $\mathcal{M}^s(v)$ (resp. $\mathcal{M}^u(v)$) denote the set of all s -admissible (resp. u -admissible) manifolds at v .

LEMMA 4.3. – The following holds for $\varepsilon > 0$ small enough. If $v = \Psi_x^{p^s, p^u}$ is an ε -double chart, then for every $V^s \in \mathcal{M}^s(v)$ and $V^u \in \mathcal{M}^u(v)$ it holds:

- (1) V^s and V^u intersect at a single point $P = \Psi_x(w)$, and $\|w\|_\infty < 10^{-2}(p^s \wedge p^u)$.
- (2) $\frac{\sin \angle(V^s, V^u)}{\sin \alpha(x)} = e^{\pm(p^s \wedge p^u)^{\beta/4}}$ and $|\cos \angle(V^s, V^u) - \cos \alpha(x)| < 2(p^s \wedge p^u)^{\beta/4}$, where $\angle(V^s, V^u)$ = angle of intersection of the tangents to V^s and V^u at P .

When M is compact and f is a $C^{1+\beta}$ diffeomorphism, this is [21, Prop. 4.11]. The same proof works almost verbatim, see the appendix for the necessary adaptations.

Let $v = \Psi_x^{p^s, p^u}$, $w = \Psi_y^{q^s, q^u}$ be ε -double charts with $v \xrightarrow{\varepsilon} w$. We now define the *graph transforms*: these are two maps that work in different directions of the edge $v \xrightarrow{\varepsilon} w$, one of them sends u -admissible manifolds at v to u -admissible manifolds at w , the other sends s -admissible manifolds at w to s -admissible manifolds at v .

Graph transforms $\mathcal{F}_{v,w}^s$ and $\mathcal{F}_{v,w}^u$. – The graph transform $\mathcal{F}_{v,w}^s : \mathcal{M}^s(w) \rightarrow \mathcal{M}^s(v)$ is the map that sends an s -admissible manifold at w with representing function $F : [-q^s, q^s] \rightarrow \mathbb{R}$ to the unique s -admissible manifold at v with representing function $G : [-p^s, p^s] \rightarrow \mathbb{R}$ s.t. $\{(t, G(t)) : |t| \leq p^s\} \subset f_{x,y}^{-1}\{(t, F(t)) : |t| \leq q^s\}$. Similarly, the graph transform $\mathcal{F}_{v,w}^u : \mathcal{M}^u(v) \rightarrow \mathcal{M}^u(w)$ is the map sending a u -admissible manifold at v with representing function $F : [-p^u, p^u] \rightarrow \mathbb{R}$ to the unique u -admissible manifold at w with representing function $G : [-q^u, q^u] \rightarrow \mathbb{R}$ s.t. $\{(G(t), t) : |t| \leq q^u\} \subset f_{x,y}\{(F(t), t) : |t| \leq p^u\}$.

In other words, the representing functions of s, u -admissible manifolds change by the application of $f_{x,y}^{-1}$, $f_{x,y}$ respectively. For $V_1, V_2 \in \mathcal{M}^s(v)$ with representing functions F_1, F_2 and for $i \geq 0$, define $d_{C^i}(V_1, V_2) := \|F_1 - F_2\|_i$ where the norm is taken in $[-p^s, p^s]$. A similar definition holds in $\mathcal{M}^u(v)$.

PROPOSITION 4.4. – *The following holds for $\varepsilon > 0$ small enough. If $v \xrightarrow{\varepsilon} w$ then $\mathcal{F}_{v,w}^s$ and $\mathcal{F}_{v,w}^u$ are well-defined. Furthermore, if $V_1, V_2 \in \mathcal{M}^u(v)$ then:*

- (1) $d_{C^0}(\mathcal{F}_{v,w}^u(V_1), \mathcal{F}_{v,w}^u(V_2)) \leq e^{-\chi/2} d_{C^0}(V_1, V_2)$.
- (2) $d_{C^1}(\mathcal{F}_{v,w}^u(V_1), \mathcal{F}_{v,w}^u(V_2)) \leq e^{-\chi/2} (d_{C^1}(V_1, V_2) + d_{C^0}(V_1, V_2)^{\beta/3})$.
- (3) $f(V_i)$ intersects every element of $\mathcal{M}^s(w)$ at exactly one point.

An analogous statement holds for $\mathcal{F}_{v,w}^s$.

When M is compact and f is a $C^{1+\beta}$ diffeomorphism, this is [21, Prop. 4.12 and 4.14]. The proof in our case requires some minor changes, see Appendix B.

4.3. Stable and unstable manifolds of ε -gpo's

Call a sequence $\underline{v}^+ = \{v_n\}_{n \geq 0}$ a *positive ε -gpo* if $v_n \xrightarrow{\varepsilon} v_{n+1}$ for all $n \geq 0$. Similarly, a *negative ε -gpo* is a sequence $\underline{v}^- = \{v_n\}_{n \leq 0}$ s.t. $v_{n-1} \xrightarrow{\varepsilon} v_n$ for all $n \leq 0$.

Stable/unstable manifold of positive/negative ε -gpo. – The *stable manifold* of a positive ε -gpo $\underline{v}^+ = \{v_n\}_{n \geq 0}$ is

$$V^s[\underline{v}^+] := \lim_{n \rightarrow \infty} (\mathcal{F}_{v_0, v_1}^s \circ \cdots \circ \mathcal{F}_{v_{n-2}, v_{n-1}}^s \circ \mathcal{F}_{v_{n-1}, v_n}^s)(V_n)$$

for some (any) choice of $(V_n)_{n \geq 0}$ with $V_n \in \mathcal{M}^s(v_n)$. The *unstable manifold* of a negative ε -gpo $\underline{v}^- = \{v_n\}_{n \leq 0}$ is

$$V^u[\underline{v}^-] := \lim_{n \rightarrow -\infty} (\mathcal{F}_{v_{-1}, v_0}^u \circ \cdots \circ \mathcal{F}_{v_{n+1}, v_{n+2}}^u \circ \mathcal{F}_{v_n, v_{n+1}}^u)(V_n)$$

for some (any) choice of $(V_n)_{n \leq 0}$ with $V_n \in \mathcal{M}^u(v_n)$.

For an ε -gpo $\underline{v} = \{v_n\}_{n \in \mathbb{Z}}$, let $V^s[\underline{v}] := V^s[\{v_n\}_{n \geq 0}]$ and $V^u[\underline{v}] := V^u[\{v_n\}_{n \leq 0}]$.

PROPOSITION 4.5. – *The following holds for all $\varepsilon > 0$ small enough.*

- (1) **ADMISSIBILITY:** $V^s[\underline{v}^+], V^u[\underline{v}^-]$ are well-defined admissible manifolds at v_0 .

(2) INVARIANCE:

$$f(V^s[\{v_n\}_{n \geq 0}]) \subset V^s[\{v_n\}_{n \geq 1}] \text{ and } f^{-1}(V^u[\{v_n\}_{n \leq 0}]) \subset V^u[\{v_n\}_{n \leq -1}].$$

(3) SHADOWING: If $\underline{v}^+ = \{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n \geq 0}$ then

$$V^s[\underline{v}^+] = \{x \in \Psi_{x_0}(R[p_0^s]) : f^n(x) \in \Psi_{x_n}(R[10Q_\varepsilon(x_n)]), \forall n \geq 0\}.$$

An analogous statement holds for $V^u[\underline{v}^-]$.

(4) HYPERBOLICITY: If $x, y \in V^s[\underline{v}^+]$ then $d(f^n(x), f^n(y)) \xrightarrow{n \rightarrow \infty} 0$, if $x, y \in V^u[\underline{v}^-]$ then $d(f^n(x), f^n(y)) \xrightarrow{n \rightarrow -\infty} 0$, and the rates are exponential.

(5) HÖLDER PROPERTY: The map $\underline{v}^+ \mapsto V^s[\underline{v}^+]$ is Hölder continuous, i.e., there exists $K > 0$ and $\theta < 1$ s.t. for all $N \geq 0$, if $\underline{v}^+, \underline{w}^+$ are positive ε -gpo's with $v_n = w_n$ for $n = 0, \dots, N$ then $d_{C^1}(V^s[\underline{v}^+], V^s[\underline{w}^+]) \leq K\theta^N$. The same holds for the map $\underline{v}^- \mapsto V^u[\underline{v}^-]$.

When M is compact and f is a $C^{1+\beta}$ diffeomorphism, this is [21, Prop. 4.15]. The same proof works in our case: it uses the hyperbolicity of $f_{x,y}$ (Theorem 3.5), and the contracting properties of the graph transforms (Proposition 4.4). Proposition 4.5 ensures that every ε -gpo is associated to a unique point.

Shadowing. – We say that an ε -gpo $\{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n \in \mathbb{Z}}$ shadows a point $x \in M$ when $f^n(x) \in \Psi_{x_n}(R[p_n^s \wedge p_n^u])$ for all $n \in \mathbb{Z}$.

LEMMA 4.6. – Every ε -gpo shadows a unique point.

Proof. – The proof is identical to the proof of [18, Theorem 4.2], so we just mention the main steps. Let $\underline{v} = \{v_n\}_{n \in \mathbb{Z}} = \{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n \in \mathbb{Z}}$ be an ε -gpo.

- By Proposition 4.5(3), any point shadowed by \underline{v} must lie in $V^s[\{v_n\}_{n \geq 0}] \cap V^u[\{v_n\}_{n \leq 0}]$.
- The definition of shadowing is equivalent to the following weaker definition: \underline{v} shadows $x \in M$ when $f^n(x) \in \Psi_{x_n}(R[10Q_\varepsilon(x_n)])$ for all $n \in \mathbb{Z}$. The definitions are equivalent because of the hyperbolicity of the maps $f_{x_n, x_{n+1}}^\pm$ (see Thm. 3.5).
- The weaker definition and Proposition 4.5(2) imply that \underline{v} shadows $V^s[\{v_n\}_{n \geq 0}] \cap V^u[\{v_n\}_{n \leq 0}]$. \square

5. Coarse graining

We now pass to a countable set of ε -double charts that define a topological Markov shift that shadows all relevant orbits.

THEOREM 5.1. – For all $\varepsilon > 0$ sufficiently small, there exists a countable family \mathcal{A} of ε -double charts with the following properties:

- (1) DISCRETENESS: For all $t > 0$, the set $\{\Psi_x^{p^s, p^u} \in \mathcal{A} : p^s, p^u > t\}$ is finite.
- (2) SUFFICIENCY: If $x \in \text{NUH}_\chi^*$ then there is a sequence $\underline{v} \in \mathcal{A}^{\mathbb{Z}}$ that shadows x .
- (3) RELEVANCE: For all $v \in \mathcal{A}$ there is an ε -gpo $\underline{v} \in \mathcal{A}^{\mathbb{Z}}$ with $v_0 = v$ that shadows a point in NUH_χ^* .

Parts (1) and (3) will be crucial to prove the inverse theorem (Theorem 6.1). Part (2) says that the ε -gpo's in \mathcal{A} shadow a.e. point with respect to every f -adapted χ -hyperbolic measure, see Lemma 2.2.

REMARK 5.2. – In part (2) we only assume that $x \in \text{NUH}_\chi^*$, while [18, 21] require the stronger assumption $x \in \text{NUH}_\chi^\#$. The reason of the improvement is that here $q_\varepsilon(x)$ is defined as a minimum instead of a sum, and hence Lemma 4.2(1) holds.

Proof. – When M is compact and f is a diffeomorphism, the above statement is consequence of Propositions 3.5, 4.5 and Lemmas 4.6, 4.7 of [21]. When M is compact (with boundary) and f is a local diffeomorphism with bounded derivatives, this is Proposition 4.3 of [18]. We follow the same strategy, adapted to our context.

For $t > 0$, let $M_t = \{x \in M : d(x, \mathcal{D}) \geq t\}$. Since M has finite diameter (remember we are even assuming it is smaller than one), each M_t is precompact⁽⁴⁾. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Fix a countable open cover $\mathcal{P} = \{D_i\}_{i \in \mathbb{N}_0}$ of $M \setminus \mathcal{D}$ s.t.:

- $D_i := D_{z_i} = B(z_i, 2\tau(z_i))$ for some $z_i \in M$.
- For every $t > 0$, $\{D \in \mathcal{P} : D \cap M_t \neq \emptyset\}$ is finite.

Let $X := M^3 \times \text{GL}(2, \mathbb{R})^3 \times (0, 1]$. For $x \in \text{NUH}_\chi^*$, let $\Gamma(x) = (\underline{x}, \underline{C}, \underline{Q}) \in X$ with

$$\underline{x} = (f^{-1}(x), x, f(x)), \quad \underline{C} = (C_\chi(f^{-1}(x)), C_\chi(x), C_\chi(f(x))), \quad \underline{Q} = Q_\varepsilon(x).$$

Let $Y = \{\Gamma(x) : x \in \text{NUH}_\chi^*\}$. We want to construct a countable dense subset of Y . Since the maps $x \mapsto C_\chi(x)$, $Q_\varepsilon(x)$ are usually just measurable, we apply a precompactness argument. For each triple of vectors $\underline{k} = (k_{-1}, k_0, k_1)$, $\underline{\ell} = (\ell_{-1}, \ell_0, \ell_1)$, $\underline{a} = (a_{-1}, a_0, a_1) \in \mathbb{N}_0^3$ and $m \in \mathbb{N}_0$, define

$$Y_{\underline{k}, \underline{\ell}, \underline{a}, m} := \left\{ \Gamma(x) \in Y : \begin{array}{l} e^{-k_i-1} \leq d(f^i(x), \mathcal{D}) < e^{-k_i}, \quad -1 \leq i \leq 1 \\ e^{\ell_i} \leq \|C_\chi(f^i(x))^{-1}\| < e^{\ell_i+1}, \quad -1 \leq i \leq 1 \\ f^i(x) \in D_{a_i}, \quad -1 \leq i \leq 1 \\ e^{-m-1} \leq Q_\varepsilon(x) < e^{-m} \end{array} \right\}.$$

Claim 1. – $Y = \bigcup_{\substack{\underline{k}, \underline{\ell}, \underline{a} \in \mathbb{N}_0^3 \\ m \in \mathbb{N}_0}} Y_{\underline{k}, \underline{\ell}, \underline{a}, m}$, and each $Y_{\underline{k}, \underline{\ell}, \underline{a}, m}$ is precompact in X .

Proof of claim 1. – The first statement is clear. We focus on the second. Fix $\underline{k}, \underline{\ell}, \underline{a} \in \mathbb{N}_0^3$, $m \in \mathbb{N}_0$. Take $\Gamma(x) \in Y_{\underline{k}, \underline{\ell}, \underline{a}, m}$. Then

$$\underline{x} \in M_{e^{-k_{-1}-1}} \times M_{e^{-k_0-1}} \times M_{e^{-k_1-1}},$$

a precompact subset of M^3 . For $|i| \leq 1$, $C_\chi(f^i(x))$ is an element of $\text{GL}(2, \mathbb{R})$ with norm ≤ 1 and inverse norm $\leq e^{\ell_i+1}$, hence it belongs to a compact subset of $\text{GL}(2, \mathbb{R})$. This guarantees that \underline{C} belongs to a compact subset of $\text{GL}(2, \mathbb{R})^3$. Also, $\underline{Q} \in [e^{-m-1}, 1]$, a compact subinterval of $(0, 1]$. Since the product of precompact sets is precompact, the claim is proved. \square

Let $j \geq 0$. By claim 1, there exists a finite set $Y_{\underline{k}, \underline{\ell}, \underline{a}, m}(j) \subset Y_{\underline{k}, \underline{\ell}, \underline{a}, m}$ s.t. for every $\Gamma(x) \in Y_{\underline{k}, \underline{\ell}, \underline{a}, m}$ there exists $\Gamma(y) \in Y_{\underline{k}, \underline{\ell}, \underline{a}, m}(j)$ s.t.:

- (a) $d(f^i(x), f^i(y)) + \|C_\chi(\widetilde{f^i(x)}) - C_\chi(\widetilde{f^i(y)})\| < e^{-8(j+2)}$ for $-1 \leq i \leq 1$.
- (b) $\frac{Q_\varepsilon(x)}{Q_\varepsilon(y)} = e^{\pm \varepsilon/3}$.

⁽⁴⁾ M_t might not be compact, since M might have boundaries.

The alphabet \mathcal{A} . – Let \mathcal{A} be the countable family of $\Psi_x^{p^s, p^u}$ s.t.:

(CG1) $\Gamma(x) \in Y_{\underline{k}, \underline{\ell}, \underline{a}, m}(j)$ for some $(\underline{k}, \underline{\ell}, \underline{a}, m, j) \in \mathbb{N}_0^3 \times \mathbb{N}_0^3 \times \mathbb{N}_0^3 \times \mathbb{N}_0 \times \mathbb{N}_0$.

(CG2) $0 < p^s, p^u \leq \delta_\varepsilon Q_\varepsilon(x)$ and $p^s, p^u \in I_\varepsilon$.

(CG3) $e^{-j-2} \leq p^s \wedge p^u \leq e^{-j+2}$.

Proof of discreteness. – We will use the following fact, whose proof is in the appendix:

$$(5.1) \quad \|C_\chi(f^{-1}(x))^{-1}\| \leq 2\rho(x)^{-2a}(1 + e^\chi \rho(x)^{-a})\|C_\chi(x)^{-1}\|.$$

Fix $t > 0$, and let $\Psi_x^{p^s, p^u} \in \mathcal{A}$ with $p^s, p^u > t$. Note that $\rho(x) > \rho(x)^{2a} > Q_\varepsilon(x) > p^s, p^u > t$. If $\Gamma(x) \in Y_{\underline{k}, \underline{\ell}, \underline{a}, m}(j)$ then:

– Finiteness of \underline{k} : for $|i| \leq 1$, $e^{-k_i} > d(f^i(x), \mathcal{D}) \geq \rho(x) > t$, hence $k_i < |\log t|$.

– Finiteness of $\underline{\ell}$: for $i = 0, 1$, $e^{\ell_i} \leq \|C_\chi(f^i(x))^{-1}\| < Q_\varepsilon(x)^{-1} < t^{-1}$, hence $\ell_i < |\log t|$.

By inequality (5.1) above,

$$e^{\ell_{-1}} \leq \|C_\chi(f^{-1}(x))^{-1}\| < 2t^{-1}(1 + e^\chi t^{-1})t^{-1} < 4e^\chi t^{-3},$$

hence $\ell_{-1} < \log 4 + \chi + 3|\log t| =: T_t$, which is bigger than $|\log t|$.

– Finiteness of \underline{a} : $f^i(x) \in D_{a_i} \cap M_t$, hence D_{a_i} belongs to the set $\{D \in \mathcal{D} : D \cap M_t \neq \emptyset\}$.

– Finiteness of m : $e^{-m} > Q_\varepsilon(x) > t$, hence $m < |\log t|$.

– Finiteness of j : $t < p^s \wedge p^u \leq e^{-j+2}$, hence $j \leq |\log t| + 2$.

– Finiteness of (p^s, p^u) : $t < p^s, p^u$, hence $\#\{(p^s, p^u) : p^s, p^u > t\} \leq \#(I_\varepsilon \cap (t, 1])^2$ is finite.

The first five items above give that, for $\underline{a} \in \mathbb{N}_0^3$ and $t > 0$,

$$\#\left\{\Gamma(x) : \begin{array}{l} \Psi_x^{p^s, p^u} \in \mathcal{A} \text{ s.t. } p^s, p^u > t \\ \text{and } f^i(x) \in D_{a_i}, |i| \leq 1 \end{array}\right\} \leq \sum_{j=0}^{\lceil |\log t| \rceil + 2} \sum_{m=0}^{\lceil |\log t| \rceil} \sum_{\substack{-1 \leq i \leq 1 \\ k_i, \ell_i = 0}}^{T_t} \#Y_{\underline{k}, \underline{\ell}, \underline{a}, m}(j)$$

is the finite sum of finite terms, hence finite. Together with the last item above, we conclude that

$$\begin{aligned} \#\left\{\Psi_x^{p^s, p^u} \in \mathcal{A} : p^s, p^u > t\right\} &\leq \sum_{j=0}^{\lceil |\log t| \rceil + 2} \sum_{m=0}^{\lceil |\log t| \rceil} \sum_{\substack{-1 \leq i \leq 1 \\ k_i, \ell_i = 0}}^{T_t} \#Y_{\underline{k}, \underline{\ell}, \underline{a}, m}(j) \\ &\quad \times (\#\{D \in \mathcal{D} : D \cap M_t \neq \emptyset\})^3 \times (\#(I_\varepsilon \cap (t, 1]))^2 \end{aligned}$$

is finite. This proves the discreteness of \mathcal{A} . \square

Proof of sufficiency. – Let $x \in \text{NUH}_\chi^*$. Take $(k_i)_{i \in \mathbb{Z}}, (\ell_i)_{i \in \mathbb{Z}}, (m_i)_{i \in \mathbb{Z}}, (a_i)_{i \in \mathbb{Z}}, (j_i)_{i \in \mathbb{Z}}$ s.t.:

$$d(f^i(x), \mathcal{D}) \in [e^{-k_i-1}, e^{-k_i}), \|C_\chi(f^i(x))^{-1}\| \in [e^{\ell_i}, e^{\ell_i+1}),$$

$$Q_\varepsilon(f^i(x)) \in [e^{-m_i-1}, e^{-m_i}), f^i(x) \in D_{a_i}, q_\varepsilon(f^i(x)) \in [e^{-j_i-1}, e^{-j_i+1}).$$

For $n \in \mathbb{Z}$, define

$$\underline{k}^{(n)} = (k_{n-1}, k_n, k_{n+1}), \underline{\ell}^{(n)} = (\ell_{n-1}, \ell_n, \ell_{n+1}), \underline{a}^{(n)} = (a_{n-1}, a_n, a_{n+1}).$$

Then $\Gamma(f^n(x)) \in Y_{\underline{k}^{(n)}, \underline{\ell}^{(n)}, \underline{a}^{(n)}, m_n} \cdot$ Take $\Gamma(x_n) \in Y_{\underline{k}^{(n)}, \underline{\ell}^{(n)}, \underline{a}^{(n)}, m_n}(j_n)$ s.t.:

(a_n) $d(f^i(f^n(x)), f^i(x_n)) + \|C_\chi(\widetilde{f^i(f^n(x))}) - C_\chi(\widetilde{f^i(x_n)})\| < e^{-8(j_n+2)}$ for $|i| \leq 1$.

(b_n) $\frac{Q_\varepsilon(f^n(x))}{Q_\varepsilon(x_n)} = e^{\pm \varepsilon/3}$.

Define $p_n^s = \delta_\varepsilon \min\{e^{\varepsilon|k|} Q_\varepsilon(x_{n+k}) : k \geq 0\}$ and $p_n^u = \delta_\varepsilon \min\{e^{\varepsilon|k|} Q_\varepsilon(x_{n+k}) : k \leq 0\}$. We claim that $\{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n \in \mathbb{Z}}$ is an ε -gpo in $\mathcal{A}^{\mathbb{Z}}$ that shadows x .

Claim 2. – $\Psi_{x_n}^{p_n^s, p_n^u} \in \mathcal{A}$ for all $n \in \mathbb{Z}$.

(CG1) By definition, $\Gamma(x_n) \in Y_{\underline{k}^{(n)}, \underline{\ell}^{(n)}, \underline{a}^{(n)}, m_n}(j_n)$.

(CG2) By (b_n) , $\inf\{e^{\varepsilon|k|} Q_\varepsilon(x_{n+k}) : k \geq 0\} = e^{\pm\varepsilon/3} \inf\{e^{\varepsilon|k|} Q_\varepsilon(f^{n+k}(x)) : k \geq 0\}$ is positive. Since the only accumulation point of I_ε is zero, it follows that p_n^s, p_n^u are well-defined and positive. The other conditions are clear from the definition.

(CG3) Again by (b_n) , we have

$$\min\{e^{\varepsilon|k|} Q_\varepsilon(x_{n+k}) : k \geq 0\} = e^{\pm\varepsilon/3} \min\{e^{\varepsilon|k|} Q_\varepsilon(f^{n+k}(x)) : k \geq 0\}$$

hence $\frac{p_n^s}{q_\varepsilon^s(f^n(x))} = e^{\pm\varepsilon/3}$, and analogously $\frac{p_n^u}{q_\varepsilon^u(f^n(x))} = e^{\pm\varepsilon/3}$. By Lemma 4.2(1), $p_n^s \wedge p_n^u = e^{\pm\varepsilon/3} q_\varepsilon(f^n(x)) \in [e^{-jn-2}, e^{-jn+2}]$.

Claim 3. – $\Psi_{x_n}^{p_n^s, p_n^u} \xrightarrow{\varepsilon} \Psi_{x_{n+1}}^{p_{n+1}^s, p_{n+1}^u}$ for all $n \in \mathbb{Z}$.

(GPO1) We have $f(x_n), x_{n+1} \in D_{a_{n+1}}$, and by (a_n) with $i = 1$ and (a_{n+1}) with $i = 0$, we have

$$\begin{aligned} & d(f(x_n), x_{n+1}) + \|C_\chi(\widetilde{f(x_n)}) - C_\chi(\widetilde{x_{n+1}})\| \\ & \leq d(f^{n+1}(x), f(x_n)) + \|C_\chi(\widetilde{f^{n+1}(x)}) - C_\chi(\widetilde{f(x_n)})\| \\ & \quad + d(f^{n+1}(x), x_{n+1}) + \|C_\chi(\widetilde{f^{n+1}(x)}) - C_\chi(\widetilde{x_{n+1}})\| \\ & < e^{-8(j_n+2)} + e^{-8(j_{n+1}+2)} \leq e^{-8} (q_\varepsilon(f^n(x)))^8 + q_\varepsilon(f^{n+1}(x))^8 \\ & \stackrel{!}{\leq} e^{-8} (1 + e^{8\varepsilon}) q_\varepsilon(f^{n+1}(x))^8 \leq e^{-8+8\varepsilon/3} (1 + e^{8\varepsilon}) (p_{n+1}^s \wedge p_{n+1}^u)^8 \\ & \stackrel{!!}{<} (p_{n+1}^s \wedge p_{n+1}^u)^8, \end{aligned}$$

where in $\stackrel{!}{\leq}$ we used Lemma 4.1 and in $\stackrel{!!}{<}$ we used that $e^{-8+8\varepsilon/3} (1 + e^{8\varepsilon}) < 1$ when $\varepsilon > 0$ is sufficiently small. This proves that $\Psi_{f(x_n)}^{p_{n+1}^s \wedge p_{n+1}^u} \stackrel{\varepsilon}{\approx} \Psi_{x_{n+1}}^{p_{n+1}^s \wedge p_{n+1}^u}$. Similarly, we prove that

$$\Psi_{f^{-1}(x_{n+1})}^{p_n^s \wedge p_n^u} \stackrel{\varepsilon}{\approx} \Psi_{x_n}^{p_n^s \wedge p_n^u}.$$

(GPO2) The very definitions of p_n^s, p_n^u guarantee that $p_n^s = \min\{e^\varepsilon p_{n+1}^s, \delta_\varepsilon Q_\varepsilon(x_n)\}$ and $p_{n+1}^u = \min\{e^\varepsilon p_n^u, \delta_\varepsilon Q_\varepsilon(x_{n+1})\}$.

Claim 4. – $\{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n \in \mathbb{Z}}$ shadows x .

By (a_n) with $i = 0$, we have $\Psi_{f^n(x)}^{p_n^s \wedge p_n^u} \stackrel{\varepsilon}{\approx} \Psi_{x_n}^{p_n^s \wedge p_n^u}$, hence by Proposition 3.4(3) we have $f^n(x) = \Psi_{f^n(x)}(0) \in \Psi_{x_n}(R[p_n^s \wedge p_n^u])$, thus $\{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n \in \mathbb{Z}}$ shadows x .

This concludes the proof of sufficiency. \square

Proof of relevance. – The alphabet \mathcal{A} might not a priori satisfy the relevance condition, but we can easily reduce it to a sub-alphabet \mathcal{A}' satisfying (1)–(3). Call $v \in \mathcal{A}$ relevant if there is $\underline{v} \in \mathcal{A}^{\mathbb{Z}}$ with $v_0 = v$ s.t. \underline{v} shadows a point in NUH_χ^* . Since NUH_χ^* is f -invariant, every v_i is relevant. Then $\mathcal{A}' = \{v \in \mathcal{A} : v \text{ is relevant}\}$ is discrete because $\mathcal{A}' \subset \mathcal{A}$, it is sufficient and relevant by definition. \square

Let Σ be the TMS associated to the graph with vertex set \mathcal{A} given by Theorem 5.1 and edges $v \xrightarrow{\varepsilon} w$. An element of Σ is an ε -gpo, hence we define $\pi : \Sigma \rightarrow M$ by

$$\{\pi[\{v_n\}_{n \in \mathbb{Z}}]\} := V^s[\{v_n\}_{n \geq 0}] \cap V^u[\{v_n\}_{n \leq 0}].$$

Here are the main properties of the triple (Σ, σ, π) .

PROPOSITION 5.3. – *The following holds for all $\varepsilon > 0$ small enough.*

- (1) *Each $v \in \mathcal{A}$ has finite ingoing and outgoing degree, hence Σ is locally compact.*
- (2) *$\pi : \Sigma \rightarrow M$ is Hölder continuous.*
- (3) *$\pi \circ \sigma = f \circ \pi$.*
- (4) *$\pi[\Sigma] \supset \text{NUH}_\chi^*$.*

Part (1) follows from (GPO2), part (2) follows from Proposition 4.4, part (3) is obvious, and part (4) follows from Theorem 5.1(2). It is important noting that (Σ, σ, π) does *not* satisfy Theorem 1.3, since π might be (and usually is) infinite-to-one. We use π to induce a locally finite cover of NUH_χ^* , which will then be refined to a cover of NUH_χ^* by pairwise disjoint sets that will lead to the proof of Theorem 1.3.

6. The inverse problem

Our goal is to analyze when π loses injectivity. More specifically, given that $\pi(\underline{v}) = \pi(\underline{w})$ we want to compare v_n and w_n , and show that they are uniquely defined “up to bounded error”. We do this under the additional assumption that $\underline{v}, \underline{w} \in \Sigma^\#$. Remind that $\Sigma^\#$ is the *recurrent set* of Σ :

$$\Sigma^\# := \left\{ \underline{v} \in \Sigma : \exists v, w \in V \text{ s.t. } \begin{array}{l} v_n = v \text{ for infinitely many } n > 0 \\ v_n = w \text{ for infinitely many } n < 0 \end{array} \right\}.$$

The main result is the following.

THEOREM 6.1 (Inverse theorem). – *The following holds for $\varepsilon > 0$ small enough. If $\{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n \in \mathbb{Z}}, \{\Psi_{y_n}^{q_n^s, q_n^u}\}_{n \in \mathbb{Z}} \in \Sigma^\#$ satisfy $\pi[\{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n \in \mathbb{Z}}] = \pi[\{\Psi_{y_n}^{q_n^s, q_n^u}\}_{n \in \mathbb{Z}}]$ then:*

- (1) $d(x_n, y_n) < 25^{-1} \max\{p_n^s \wedge p_n^u, q_n^s \wedge q_n^u\}$.
- (2) $\frac{\sin \alpha(x_n)}{\sin \alpha(y_n)} = e^{\pm \sqrt{\varepsilon}}$ and $|\cos \alpha(x_n) - \cos \alpha(y_n)| < \sqrt{\varepsilon}$.
- (3) $\frac{s(x_n)}{s(y_n)} = e^{\pm 4\sqrt{\varepsilon}}$ and $\frac{u(x_n)}{u(y_n)} = e^{\pm 4\sqrt{\varepsilon}}$.
- (4) $\frac{Q_\varepsilon(x_n)}{Q_\varepsilon(y_n)} = e^{\pm \sqrt[3]{\varepsilon}}$.
- (5) $\frac{p_n^s}{q_n^s} = e^{\pm \sqrt[3]{\varepsilon}}$ and $\frac{p_n^u}{q_n^u} = e^{\pm \sqrt[3]{\varepsilon}}$.
- (6) $(\Psi_{y_n}^{-1} \circ \Psi_{x_n})(v) = (-1)^{\sigma_n} v + \delta_n + \Delta_n(v)$ for $v \in R[10Q_\varepsilon(x_n)]$, where $\sigma_n \in \{0, 1\}$, δ_n is a vector with $\|\delta_n\| < 10^{-1}(q_n^s \wedge q_n^u)$ and Δ_n is a vector field s.t. $\Delta_n(0) = 0$ and $\|d\Delta_n\|_0 < \sqrt[3]{\varepsilon}$ on $R[10Q_\varepsilon(x_n)]$.

The difference from Theorem 6.1 to [21, Thm 5.2] is that the estimate on our part (6) holds only in the smaller rectangle $R[10Q_\varepsilon(x_n)]$. Part (1) is proved as in [21, Prop. 5.3]. Here is one of its consequences. We have $d(x_n, y_n) < 25^{-1}(p_n^s \wedge p_n^u + q_n^s \wedge q_n^u) < \varepsilon[d(x_n, \mathcal{D})^a + d(y_n, \mathcal{D})^a]$, hence

$$d(x_n, \mathcal{D}) = d(y_n, \mathcal{D}) \pm d(x_n, y_n) = d(y_n, \mathcal{D}) \pm \varepsilon[d(x_n, \mathcal{D})^a + d(y_n, \mathcal{D})^a].$$

These estimates have two consequences. The first is that

$$(6.1) \quad \frac{1 - \varepsilon}{1 + \varepsilon} \leq \frac{d(x_n, \mathcal{D})}{d(y_n, \mathcal{D})} \leq \frac{1 + \varepsilon}{1 - \varepsilon}$$

and so, for $\varepsilon > 0$ is sufficiently small, it holds $\frac{1}{2} \leq \frac{d(x_n, \mathcal{D})^a}{d(y_n, \mathcal{D})^a} \leq 2$. The second consequence is that $x_n \in D_{y_n}$ and $y_n \in D_{x_n}$, since

$$d(x_n, y_n) < \varepsilon[d(x_n, \mathcal{D})^a + d(y_n, \mathcal{D})^a] < 3\varepsilon \min\{d(x_n, \mathcal{D})^a, d(y_n, \mathcal{D})^a\} < \min\{\tau(x_n), \tau(y_n)\}.$$

Therefore we can take parallel transport with respect to either x_n or y_n .

The proofs of parts (2)–(6) use, as in [21], some auxiliary facts about admissible manifolds. Let $\underline{v}^+ = \{v_n\}_{n \geq 0}$ be a positive ε -gpo with $v_n = \Psi_{x_n}^{p_n^s, p_n^u}$. By Proposition 4.5, $V^s[\underline{v}^+]$ has the following property: $f^n(V^s[\underline{v}^+]) \subset V^s[\{v_k\}_{k \geq n}] \subset \Psi_{x_n}(R[10Q_\varepsilon(x_n)])$. This motivates the definition of *staying in windows* as in [21]: given an ε -double chart, say that $V^s \in \mathcal{M}^s(v)$ stays in windows if there exists a positive ε -gpo \underline{v}^+ with $v_0 = v$ and s -admissible manifolds $W_n^s \in \mathcal{M}^s(v_n)$ s.t. $f^n(V^s) \subset W_n^s$ for all $n \geq 0$. In particular, every $V^s[\underline{v}^+]$ stays in windows, and the reverse statement is also true. An analogous definition holds for u -admissible manifolds. Given $V^s \in \mathcal{M}^s[v]$ and $x \in V^s$, let $e_x^s \in T_x M$ denote the positively oriented vector tangent to V^s at x .

PROPOSITION 6.2. – *The following holds for all $\varepsilon > 0$ small enough.*

- (1) *If $V^s \in \mathcal{M}^s[\Psi_x^{p^s, p^u}]$ stays in windows then for all $y, z \in V^s$ and $n \geq 0$:*
 - (a) $d(f^n(y), f^n(z)) < 6p^s e^{-\frac{\xi}{2}n}$.
 - (b) $\|df_y^n e_y^s\| \leq 6\|C_\chi(x)^{-1}\|e^{-\frac{\xi}{2}n}$.
 - (c) $|\log \|df_y^n e_y^s\| - \log \|df_z^n e_z^s\|| < Q_\varepsilon(x)^{\beta/4}$.
 - (2) *If $V^s \in \mathcal{M}^s[\Psi_x^{p^s, p^u}]$, $U^s \in \mathcal{M}^s[\Psi_x^{q^s, q^u}]$ stay in windows then either $V^s \subset U^s$ or $U^s \subset V^s$.*
- Analogous statements hold for u -admissible manifolds that stay in windows.*

When M is compact and f is a $C^{1+\beta}$ diffeomorphism, this is [21, Prop. 6.3 and 6.4]. The only adaptation we need to make is in part (1)(c), see Appendix B. Because of part (1)(c), if $y, z \in V^s$ then $\frac{s(y)}{s(z)} = e^{\pm Q_\varepsilon(x)^{\beta/4}}$, therefore we can define $s(V^s) := s(\Psi_x(0, F(0)))$, where F is the representing function of V^s . Note that $s(V^s)$ might be infinite, in which case $s(y)$ is infinite for all $y \in V^s$. A similar definition holds for u -admissible manifolds that stay in windows.

The proof of part (2) of Theorem 6.1 is analogous to [21, Prop. 6.5]. In the sequel we adapt the methods of [21] to prove parts (3)–(6).

6.1. Control of $s(x_n)$ and $u(x_n)$

As in [21], the hyperbolicity of f induces an improvement for s and u . Because of symmetry, we only state the result for s .

LEMMA 6.3 (Improvement lemma). – *The following holds for all $\varepsilon > 0$ small enough. Let $v \xrightarrow{\varepsilon} w$ with $v = \Psi_x^{p^s, p^u}$, $w = \Psi_y^{q^s, q^u}$, and assume $V^s \in \mathcal{M}^s[w]$ stays in windows.*

- (1) *If $s(V^s) < \infty$ then $s[\mathcal{F}_{v,w}^s(V^s)] < \infty$.*
- (2) *For $\xi \geq \sqrt{\varepsilon}$, if $s(V^s) < \infty$ and $\frac{s(V^s)}{s(y)} = e^{\pm \xi}$ then $\frac{s(\mathcal{F}_{v,w}^s(V^s))}{s(x)} = e^{\pm(\xi - Q_\varepsilon(x)^{\beta/4})}$.*

Note that the ratio improves.

Proof. – When M is compact and f is a $C^{1+\beta}$ diffeomorphism, this is [21, Lemma 7.2], and the proof of part (1) is identical. Part (2) requires some finer estimates.

Let F, G be the representing functions of V^s , $\mathcal{F}_{v,w}^s(V^s)$, and let $q := \Psi_y(0, F(0))$, $p := \Psi_x(0, G(0))$. Then $\frac{s(\mathcal{F}_{v,w}^s(V^s))}{s(x)} = \frac{s(p)}{s(x)} = \frac{s(p)}{s(f^{-1}(q))} \cdot \frac{s(f^{-1}(q))}{s(f^{-1}(y))} \cdot \frac{s(f^{-1}(y))}{s(x)}$. We have:

- $p, f^{-1}(q) \in \mathcal{F}_{v,w}^s(V^s)$, hence Proposition 6.2(1)(c) implies $\frac{s(p)}{s(f^{-1}(q))} = e^{\pm Q_\varepsilon(x)^{\beta/4}}$.
- Since $(p^s \wedge p^u)^3 (q^s \wedge q^u)^3 \ll Q_\varepsilon(x)^{\beta/4}$, Proposition 3.4(1) implies $\frac{s(f^{-1}(y))}{s(x)} = e^{\pm Q_\varepsilon(x)^{\beta/4}}$.

Thus it is enough to show that $\frac{s(f^{-1}(q))}{s(f^{-1}(y))} = e^{\pm(\xi-3Q_\varepsilon(x)^{\beta/4})}$. We show one side of the inequality (the other is similar). Note that this is the term that gives the improvement. As in [21, pp. 375], we have

$$\frac{s(f^{-1}(q))^2}{s(f^{-1}(y))^2} \leq \underbrace{\left(\frac{2+e^{2\xi+2\chi} s(y)^2 \|dfe_{f^{-1}(y)}^s\|^2}{2+e^{2\chi} s(y)^2 \|dfe_{f^{-1}(y)}^s\|^2} \right)}_{=I} \exp \left(\underbrace{2|\log \|dfe_{f^{-1}(q)}^s\| - \log \|dfe_{f^{-1}(y)}^s\||}_{=II} \right).$$

We estimate I as in [21, pp. 376]: $I \leq e^{2\xi-7Q_\varepsilon(x)^{\beta/4}}$. Therefore it suffices to show that $II \leq e^{Q_\varepsilon(x)^{\beta/4}}$. Since $\|dfe_{f^{-1}(z)}^s\| = \|df^{-1}e_z^s\|^{-1}$, $II = \exp(2|\log \|df^{-1}e_q^s\| - \log \|df^{-1}e_y^s\||)$, hence by the claim in the proof of Proposition 6.2 (Appendix B):

$$(6.2) \quad \log(II) \leq 2\mathfrak{K}\rho(y)^{-2a} [d(q, y)^\beta + \|e_q^s - P_{y,q}e_y^s\|].$$

Since $q = \Psi_y(0, G(0))$ and $y = \Psi_y(0, 0)$, Lemma 3.1(1) implies that $d(q, y) \leq 2|G(0)| \leq 500^{-1}(q^s \wedge q^u) \leq 500^{-1}e^\varepsilon(p^s \wedge p^u)$, therefore $d(q, y) < Q_\varepsilon(x), Q_\varepsilon(y)$. Hence for small $\varepsilon > 0$:

$$\begin{aligned} 2\mathfrak{K}\rho(y)^{-2a} d(q, y)^\beta &\leq 2\mathfrak{K}\rho(y)^{-2a} Q_\varepsilon(y)^{3\beta/4} Q_\varepsilon(x)^{\beta/4} \leq 2\mathfrak{K}\rho(y)^{-2a} Q_\varepsilon(y)^{\beta/36} Q_\varepsilon(x)^{\beta/4} \\ &\leq 2\mathfrak{K}\varepsilon^{1/12} Q_\varepsilon(x)^{\beta/4} < \frac{1}{2} Q_\varepsilon(x)^{\beta/4}. \end{aligned}$$

To bound the second term of (6.2), we first estimate $\sin \angle(e_q^s, P_{y,q}e_y^s)$. Since e_y^s is the unitary vector in the direction of $d(\Psi_y)_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = d(\exp_y)_0 \circ C_\chi(y) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and e_q^s is the unitary vector in the direction of $d(\Psi_y)_{(0,G(0))} \begin{bmatrix} 1 \\ G'(0) \end{bmatrix} = d(\exp_y)_{C_\chi(y) \begin{bmatrix} 0 \\ G(0) \end{bmatrix}} \circ C_\chi(y) \begin{bmatrix} 1 \\ G'(0) \end{bmatrix}$, the angles they define are the same. In other words, if

$$A = d(\exp_y)_0 \widetilde{\circ} C_\chi(y), B = d(\exp_y)_{C_\chi(y) \begin{bmatrix} 0 \\ G(0) \end{bmatrix}} \widetilde{\circ} C_\chi(y), v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ G'(0) \end{bmatrix}$$

then $\sin \angle(e_q^s, P_{y,q}e_y^s) = \sin \angle(Av_1, Bv_2)$. Using (3.1) with $L = A$, $v = v_1$, $w = A^{-1}Bv_2$, we get

$$\begin{aligned} |\sin \angle(Av_1, Bv_2)| &\leq \|A\| \|A^{-1}\| |\sin \angle(v_1, A^{-1}Bv_2)| \\ &\leq \|C_\chi(y)^{-1}\| [|\sin \angle(v_1, v_2)| + |\sin \angle(v_2, A^{-1}Bv_2)|]. \end{aligned}$$

We have $|\sin \angle(v_1, v_2)| \leq |G'(0)| \leq \frac{1}{2}(q^s \wedge q^u)^{\beta/3} \leq \frac{e^{\beta\varepsilon}}{2}(p^s \wedge p^u)^{\beta/3}$, therefore for small $\varepsilon > 0$ it holds $|\sin \angle(v_1, v_2)| \leq Q_\varepsilon(x)^{\beta/3}, Q_\varepsilon(y)^{\beta/3}$.

In particular $|\sin \angle(v_1, v_2)| \leq Q_\varepsilon(y)^{\beta/12} Q_\varepsilon(x)^{\beta/4}$. Also, by (A3):

$$\begin{aligned} \|A^{-1}B - \text{Id}\| &\leq \|A^{-1}\| \|A - B\| \leq \|C_\chi(y)^{-1}\| \left\| \widetilde{d(\text{exp}_y)_0} - \widetilde{d(\text{exp}_y)_{C_\chi(y)[\begin{smallmatrix} 0 \\ G(0) \end{smallmatrix}]} \right\| \\ &\leq \|C_\chi(y)^{-1}\| \rho(y)^{-a} |G(0)| \leq \|C_\chi(y)^{-1}\| \rho(y)^{-a} Q_\varepsilon(y)^{1-\frac{\beta}{4}} Q_\varepsilon(x)^{\beta/4} \\ &\leq Q_\varepsilon(y)^{1-\frac{11\beta}{36}} Q_\varepsilon(x)^{\beta/4} < \frac{1}{4} Q_\varepsilon(y)^{\beta/12} Q_\varepsilon(x)^{\beta/4} \ll 1. \end{aligned}$$

This implies that $v_2, A^{-1}Bv_2$ are almost unitary vectors, therefore

$$|\sin \angle(v_2, A^{-1}Bv_2)| \leq 2\|v_2 - A^{-1}Bv_2\| \leq 4\|A^{-1}B - \text{Id}\| < Q_\varepsilon(y)^{\beta/12} Q_\varepsilon(x)^{\beta/4},$$

thus $|\sin \angle(e_q^s, P_{y,q}e_y^s)| < 2\|C_\chi(y)^{-1}\| Q_\varepsilon(y)^{\beta/12} Q_\varepsilon(x)^{\beta/4}$. Since $\|e_q^s\| = \|P_{y,q}e_y^s\| = 1$ and the angle between them is small,

$$\|e_q^s - P_{y,q}e_y^s\| \leq 2|\sin \angle(e_q^s, P_{y,q}e_y^s)| < 4\|C_\chi(y)^{-1}\| Q_\varepsilon(y)^{\beta/12} Q_\varepsilon(x)^{\beta/4}.$$

The conclusion is that for small $\varepsilon > 0$:

$$\begin{aligned} 2\mathfrak{K}\rho(y)^{-2a} \|e_q^s - P_{y,q}e_y^s\| &\leq 8\mathfrak{K}\|C_\chi(y)^{-1}\| \rho(y)^{-2a} Q_\varepsilon(y)^{\beta/12} Q_\varepsilon(x)^{\beta/4} \\ &\leq 8\mathfrak{K}\|C_\chi(y)^{-1}\| Q_\varepsilon(y)^{\beta/24} \rho(y)^{-2a} Q_\varepsilon(y)^{\beta/36} Q_\varepsilon(x)^{\beta/4} \\ &\leq 8\mathfrak{K}\varepsilon^{5/24} Q_\varepsilon(x)^{\beta/4} < \frac{1}{2} Q_\varepsilon(x)^{\beta/4}. \end{aligned}$$

Hence (6.2) implies that $\Pi < e^{Q_\varepsilon(x)^{\beta/4}}$. \square

We are now ready to prove part (3) of Theorem 6.1.

PROPOSITION 6.4. – *The following holds for all $\varepsilon > 0$ small enough. If $\{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n \in \mathbb{Z}}, \{\Psi_{y_n}^{q_n^s, q_n^u}\}_{n \in \mathbb{Z}} \in \Sigma^\#$ satisfy $\pi[\{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n \in \mathbb{Z}}] = \pi[\{\Psi_{y_n}^{q_n^s, q_n^u}\}_{n \in \mathbb{Z}}]$ then for all $n \in \mathbb{Z}$:*

$$\frac{s(x_n)}{s(y_n)} = e^{\pm 4\sqrt{\varepsilon}} \text{ and } \frac{u(x_n)}{u(y_n)} = e^{\pm 4\sqrt{\varepsilon}}.$$

When M is compact and f is a $C^{1+\beta}$ diffeomorphism, this is [21, Prop. 7.3], and the proof is identical. Let $\underline{v} = \{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n \in \mathbb{Z}}$ and $\underline{w} = \{\Psi_{y_n}^{q_n^s, q_n^u}\}_{n \in \mathbb{Z}}$. We sketch the proof for the first estimate:

- If $\pi(\underline{v}) = x$ then $s(x) < \infty$: this follows from the relevance of \mathcal{A} (Thm. 5.1(3)).
- Apply Lemma 6.3 along \underline{v} and the orbit of x : if $v_n = v$ for infinitely many $n > 0$, then the ratio improves at each of these indices. The conclusion is that $\frac{s(V^s[\{v_k\}_{k \geq n}])}{s(x_n)} = e^{\pm \sqrt{\varepsilon}}$, and analogously $\frac{s(V^s[\{w_k\}_{k \geq n}])}{s(y_n)} = e^{\pm \sqrt{\varepsilon}}$.
- Since $f^n(x) \in V^s[\{v_k\}_{k \geq n}] \cap V^s[\{w_k\}_{k \geq n}]$, Proposition 6.2(1)(c) implies that $\frac{s(V^s[\{v_k\}_{k \geq n}])}{s(f^n(x))} = e^{\pm \sqrt{\varepsilon}}$ and $\frac{s(V^s[\{w_k\}_{k \geq n}])}{s(f^n(x))} = e^{\pm \sqrt{\varepsilon}}$.

$$\text{Hence } \frac{s(x_n)}{s(y_n)} = \frac{s(x_n)}{s(V^s[\{v_k\}_{k \geq n}])} \cdot \frac{s(V^s[\{v_k\}_{k \geq n}])}{s(f^n(x))} \cdot \frac{s(f^n(x))}{s(V^s[\{w_k\}_{k \geq n}])} \cdot \frac{s(V^s[\{w_k\}_{k \geq n}])}{s(y_n)} = e^{\pm 4\sqrt{\varepsilon}}.$$

6.2. Control of $Q_\varepsilon(x_n)$

Remind that $Q_\varepsilon(x) := \max\{q \in I_\varepsilon : q \leq \tilde{Q}_\varepsilon(x)\}$ where

$$\tilde{Q}_\varepsilon(x) = \varepsilon^{3/\beta} \min \left\{ \|C_\chi(x)^{-1}\|_{\text{Frob}}^{-24/\beta}, \|C_\chi(f(x))^{-1}\|_{\text{Frob}}^{-12/\beta} \rho(x)^{72a} \right\},$$

so we first control $\tilde{Q}_\varepsilon(x_n)$. By parts (2)–(3), $\frac{\|C_\chi(x_n)^{-1}\|_{\text{Frob}}}{\|C_\chi(y_n)^{-1}\|_{\text{Frob}}} = e^{\pm 5\sqrt{\varepsilon}}$.

Using that $\Psi_{f(x_n)}^{p_{n+1}^s \wedge p_{n+1}^u} \approx \Psi_{x_{n+1}}^{p_{n+1}^s \wedge p_{n+1}^u}$, Proposition 3.4(1)–(2) implies that $\frac{\|C_\chi(f(x_n))^{-1}\|_{\text{Frob}}}{\|C_\chi(x_{n+1})^{-1}\|_{\text{Frob}}} = e^{\pm\sqrt{\varepsilon}}$, and similarly $\frac{\|C_\chi(f(y_n))^{-1}\|_{\text{Frob}}}{\|C_\chi(y_{n+1})^{-1}\|_{\text{Frob}}} = e^{\pm\sqrt{\varepsilon}}$. Hence

$$\frac{\|C_\chi(f(x_n))^{-1}\|_{\text{Frob}}}{\|C_\chi(f(y_n))^{-1}\|_{\text{Frob}}} = \frac{\|C_\chi(f(x_n))^{-1}\|_{\text{Frob}}}{\|C_\chi(x_{n+1})^{-1}\|_{\text{Frob}}} \cdot \frac{\|C_\chi(x_{n+1})^{-1}\|_{\text{Frob}}}{\|C_\chi(y_{n+1})^{-1}\|_{\text{Frob}}} \cdot \frac{\|C_\chi(y_{n+1})^{-1}\|_{\text{Frob}}}{\|C_\chi(f(y_n))^{-1}\|_{\text{Frob}}} = e^{\pm 7\sqrt{\varepsilon}}.$$

We now estimate the ratio $\frac{\rho(x_n)}{\rho(y_n)}$. For that we obtain estimates similar to (6.1) for $f^{\pm 1}(x_n)$, $f^{\pm 1}(y_n)$. By symmetry, we only need to get the inequalities for $f(x_n)$, $f(y_n)$. Start by noting that $d(f(x_n), x_{n+1}) \leq (p_{n+1}^s \wedge p_{n+1}^u)^8 < \varepsilon d(x_{n+1}, \mathcal{D})$, hence $d(f(x_n), \mathcal{D}) = d(x_{n+1}, \mathcal{D}) \pm d(f(x_n), x_{n+1}) = (1 \pm \varepsilon)d(x_{n+1}, \mathcal{D})$ and thus $d(f(x_n), x_{n+1}) < 2\varepsilon d(f(x_n), \mathcal{D})$. Similarly $d(f(y_n), y_{n+1}) < 2\varepsilon d(f(y_n), \mathcal{D})$. Using part (1), $d(x_{n+1}, y_{n+1}) < \varepsilon[d(x_{n+1}, \mathcal{D}) + d(y_{n+1}, \mathcal{D})] < 2\varepsilon[d(f(x_n), \mathcal{D}) + d(f(y_n), \mathcal{D})]$, therefore

$$\begin{aligned} d(f(x_n), f(y_n)) &\leq d(f(x_n), x_{n+1}) + d(x_{n+1}, y_{n+1}) + d(y_{n+1}, f(y_n)) \\ &< 4\varepsilon[d(f(x_n), \mathcal{D}) + d(f(y_n), \mathcal{D})]. \end{aligned}$$

This implies that $d(f(x_n), \mathcal{D}) = d(f(y_n), \mathcal{D}) \pm 4\varepsilon[d(f(x_n), \mathcal{D}) + d(f(y_n), \mathcal{D})]$ and so $\frac{1-4\varepsilon}{1+4\varepsilon} \leq \frac{d(f(x_n), \mathcal{D})}{d(f(y_n), \mathcal{D})} \leq \frac{1+4\varepsilon}{1-4\varepsilon}$. The same estimate holds for f^{-1} . Together with (6.1), we get that $\frac{1-4\varepsilon}{1+4\varepsilon} \leq \frac{\rho(x_n)}{\rho(y_n)} \leq \frac{1+4\varepsilon}{1-4\varepsilon}$. If $\varepsilon > 0$ is small enough then $e^{-\sqrt{\varepsilon}} < \left(\frac{1-4\varepsilon}{1+4\varepsilon}\right)^{72a} < \left(\frac{1+4\varepsilon}{1-4\varepsilon}\right)^{72a} < e^{\sqrt{\varepsilon}}$, hence $\frac{\rho(x_n)^{72a}}{\rho(y_n)^{72a}} = e^{\pm\sqrt{\varepsilon}}$. The conclusion is that $\frac{\tilde{Q}_\varepsilon(x_n)}{\tilde{Q}_\varepsilon(y_n)} = \exp[\pm(\frac{120}{\beta}\sqrt{\varepsilon})]$, which implies that $\frac{Q_\varepsilon(x_n)}{Q_\varepsilon(y_n)} = \exp[\pm(\frac{2}{3}\varepsilon + \frac{120}{\beta}\sqrt{\varepsilon})]$. Hence if $\varepsilon > 0$ is small enough it holds $\frac{Q_\varepsilon(x_n)}{Q_\varepsilon(y_n)} = e^{\pm\sqrt[3]{\varepsilon}}$.

6.3. Control of p_n^s and p_n^u

As in [21, Prop. 8.3], (GPO2) implies the lemma below.

LEMMA 6.5. – *If $\underline{v} = \{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n \in \mathbb{Z}} \in \Sigma^\#$ then $p_n^s = \delta_\varepsilon Q_\varepsilon(x_n)$ for infinitely many $n > 0$ and $p_n^u = \delta_\varepsilon Q_\varepsilon(x_n)$ for infinitely many $n < 0$.*

We now prove the first half of part (5) (the other half is analogous). By symmetry, it is enough to prove that $p_n^s \geq e^{-\sqrt[3]{\varepsilon}} q_n^s$ for all $n \in \mathbb{Z}$.

- If $p_n^s = \delta_\varepsilon Q_\varepsilon(x_n)$ then part (4) gives $p_n^s = \delta_\varepsilon Q_\varepsilon(x_n) \geq e^{-\sqrt[3]{\varepsilon}} \delta_\varepsilon Q_\varepsilon(y_n) \geq e^{-\sqrt[3]{\varepsilon}} q_n^s$.
- If $p_n^s \geq e^{-\sqrt[3]{\varepsilon}} q_n^s$ then (GPO2) and part (4) give:

$$p_{n-1}^s = \min\{e^\varepsilon p_n^s, \delta_\varepsilon Q_\varepsilon(x_{n-1})\} \geq e^{-\sqrt[3]{\varepsilon}} \min\{e^\varepsilon q_n^s, \delta_\varepsilon Q_\varepsilon(y_{n-1})\} = e^{-\sqrt[3]{\varepsilon}} q_{n-1}^s.$$

By Lemma 6.5, it follows that $p_n^s \geq e^{-\sqrt[3]{\varepsilon}} q_n^s$ for all $n \in \mathbb{Z}$.

6.4. Control of $\Psi_{y_n}^{-1} \circ \Psi_{x_n}$

For $z_n = x_n, y_n$, the calculations in the proof of Lemma 2.1 give that

$$\widetilde{C_\chi(z_n)} = R_{z_n} \begin{bmatrix} \frac{1}{s(z_n)} & \frac{\cos \alpha(z_n)}{u(z_n)} \\ 0 & \frac{\sin \alpha(z_n)}{u(z_n)} \end{bmatrix}$$

where R_{z_n} is the rotation that takes e_1 to $\iota_{z_n} e_{z_n}^s$.

LEMMA 6.6. – *Under the conditions of Theorem 6.1, for all $n \in \mathbb{Z}$ it holds*

$$R_{y_n}^{-1} R_{x_n} = (-1)^{\sigma_n} \text{Id} + \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix}$$

where $\sigma_n \in \{0, 1\}$ and $|\varepsilon_{jk}| < (p_n^s \wedge p_n^u)^{\beta/5} + (q_n^s \wedge q_n^u)^{\beta/5} < \sqrt{\varepsilon}$.

When M is compact and f is a $C^{1+\beta}$ diffeomorphism, this is [21, Prop. 6.7]. See Appendix B for the proof in our context.

Now we establish part (6). It is enough to prove the case $n = 0$. Write $\Psi_{x_0}^{p_0^s, p_0^u} = \Psi_x^{p^s, p^u}$, $\Psi_{y_0}^{q_0^s, q_0^u} = \Psi_y^{q^s, q^u}$, $p = p^s \wedge p^u$, $q = q^s \wedge q^u$, $\sigma = \sigma_0$. Write $\widetilde{C_\chi(x)} = R_x C_x$, $\widetilde{C_\chi(y)} = R_y C_y$. As in [21, §9], Lemma 6.6 gives $\|C_y^{-1} C_x - (-1)^\sigma \text{Id}\| < 14\sqrt{\varepsilon}$ and hence for small $\varepsilon > 0$:

$$\begin{aligned} \|\widetilde{C_\chi(x)} - \widetilde{C_\chi(y)}\| &\leq \|R_x C_x - (-1)^\sigma R_x C_y\| + \|R_x C_y - (-1)^\sigma R_y C_y\| \\ &\leq \|C_y^{-1}\| \|C_y^{-1} C_x - (-1)^\sigma \text{Id}\| + \|R_y^{-1} R_x - (-1)^\sigma \text{Id}\| \\ &< 16\sqrt{\varepsilon} \|C_y^{-1}\| < \|C_y^{-1}\|. \end{aligned}$$

We use this to show that $\Psi_y^{-1} \circ \Psi_x$ is well-defined in $R[10Q_\varepsilon(x)]$. The argument is very similar to the proof of Proposition 3.4(3). For $v \in R[10Q_\varepsilon(x)]$, (A2) and part (4) imply that for small $\varepsilon > 0$:

$$\begin{aligned} d(\Psi_x(v), \Psi_y(v)) &\leq 2d_{\text{Sas}}(C_\chi(x)v, C_\chi(y)v) \leq 4(d(x, y) + \|\widetilde{C_\chi(x)} - \widetilde{C_\chi(y)}\| \|v\|) \\ &< 4(q + \|C_y^{-1}\| \|v\|) < 100\|C_y^{-1}\| Q_\varepsilon(y). \end{aligned}$$

hence $\Psi_x(v) \in B(\Psi_y(v), 100\|C_y^{-1}\| Q_\varepsilon(y)) \subset \Psi_y[B]$ where $B \subset \mathbb{R}^2$ is the ball with center v and radius $200\|C_y^{-1}\|^2 Q_\varepsilon(y)$. If $\varepsilon > 0$ is small then for $w \in B$ we have

$$\begin{aligned} \|w\| &\leq \|v\| + 200\|C_y^{-1}\|^2 Q_\varepsilon(y) < 20Q_\varepsilon(y) + 200\varepsilon^{1/4} Q_\varepsilon(y)^{1-\beta/12} \\ &< 20\varepsilon^{3/\beta} d(y, \mathcal{D})^a + 200\varepsilon^{1/4} d(y, \mathcal{D})^a < d(y, \mathcal{D})^a < 2\tau(y), \end{aligned}$$

therefore $\Psi_y^{-1} \circ \Psi_x$ is well-defined in $R[10Q_\varepsilon(x)]$.

It remains to estimate $\Psi_y^{-1} \circ \Psi_x - (-1)^\sigma \text{Id}$. Write $\Psi_y^{-1} \circ \Psi_x = (-1)^\sigma \text{Id} + \delta + \Delta$, where $\delta \in \mathbb{R}^2$ is a constant vector and $\Delta : R[10Q_\varepsilon(x)] \rightarrow \mathbb{R}^2$. Let $v \in R[10Q_\varepsilon(x)]$. Proceeding as in [21, pp. 382] and applying (A4) we get for small $\varepsilon > 0$ that:

$$\begin{aligned} \|d(\Delta)_v\| &\leq 2\|C_y^{-1}\| d(y, \mathcal{D})^{-a} d(x, y) + 14\sqrt{\varepsilon} < 2\|C_y^{-1}\| d(y, \mathcal{D})^{-a} Q_\varepsilon(y) + 14\sqrt{\varepsilon} \\ &< 2\sqrt{\varepsilon} \|C_y^{-1}\| Q_\varepsilon(y)^{\beta/24} d(y, \mathcal{D})^{-a} Q_\varepsilon(y)^{\beta/72} + 14\sqrt{\varepsilon} < 16\sqrt{\varepsilon} < \sqrt[3]{\varepsilon}. \end{aligned}$$

The estimate of $\|\delta\|$ is identical to [21, pp. 383]. This completes the proof of part (6), and hence of Theorem 6.1.

7. Symbolic dynamics

7.1. A countable Markov partition

Let (Σ, σ) be the TMS constructed in Theorem 5.1, and let $\pi : \Sigma \rightarrow M$ as defined in the end of Section 5. In the sequel we use Theorem 6.1 to construct a cover of $\text{NUH}_\chi^\#$ that is locally finite and satisfies a (symbolic) Markov property.

The Markov cover \mathcal{Z} . – Let $\mathcal{Z} := \{Z(\underline{v}) : \underline{v} \in \mathcal{A}\}$, where

$$Z(\underline{v}) := \{\pi(\underline{v}) : \underline{v} \in \Sigma^\# \text{ and } v_0 = v\}.$$

In other words, \mathcal{Z} is the family defined by the natural partition of $\Sigma^\#$ into cylinder at the zeroth position. Admissible manifolds allow us to define *invariant fibers* inside each $Z \in \mathcal{Z}$. Let $Z = Z(\underline{v})$.

s/u-fibers in \mathcal{Z} . – Given $x \in Z$, let $W^s(x, Z) := V^s[\{v_n\}_{n \geq 0}] \cap Z$ be the *s-fiber* of x in Z for some (any) $\underline{v} = \{v_n\}_{n \in \mathbb{Z}} \in \Sigma^\#$ s.t. $\pi(\underline{v}) = x$ and $v_0 = v$. Similarly, let $W^u(x, Z) := V^u[\{v_n\}_{n \leq 0}] \cap Z$ be the *u-fiber* of x in Z .

By Proposition 6.2(2), the definitions above do not depend on the choice of \underline{v} , and any two *s-fibers* (*u-fibers*) either coincide or are disjoint. We also define $V^s(x, Z) := V^s[\{v_n\}_{n \geq 0}]$ and $V^u(x, Z) := V^u[\{v_n\}_{n \leq 0}]$. Below we collect the main properties of \mathcal{Z} .

PROPOSITION 7.1. – *The following are true.*

- (1) **COVERING PROPERTY:** \mathcal{Z} is a cover of $\text{NUH}_\chi^\#$.
- (2) **LOCAL FINITENESS:** For every $Z \in \mathcal{Z}$, $\#\{Z' \in \mathcal{Z} : Z \cap Z' \neq \emptyset\} < \infty$.
- (3) **PRODUCT STRUCTURE:** For every $Z \in \mathcal{Z}$ and every $x, y \in Z$, the intersection $W^s(x, Z) \cap W^u(y, Z)$ consists of a single point of Z .
- (4) **SYMBOLIC MARKOV PROPERTY:** If $x = \pi(\underline{v})$ with $\underline{v} \in \Sigma^\#$, then

$$f(W^s(x, Z(v_0))) \subset W^s(f(x), Z(v_1)) \text{ and } f^{-1}(W^u(f(x), Z(v_1))) \subset W^u(x, Z(v_0)).$$

Part (1) follows from Theorem 5.1(2), part (2) follows from Theorem 6.1(5), part (3) follows from Lemma 4.3(1), and part (4) is proved as in [21, Prop. 10.9]. For $x, y \in Z$, let $[x, y]_Z :=$ intersection point of $W^s(x, Z)$ and $W^u(y, Z)$, and call it the *Smale bracket* of x, y in Z .

LEMMA 7.2. – *The following holds for all $\varepsilon > 0$ small enough.*

- (1) **COMPATIBILITY:** If $x, y \in Z(v_0)$ and $f(x), f(y) \in Z(v_1)$ with $v_0 \xrightarrow{\varepsilon} v_1$ then $f([x, y]_{Z(v_0)}) = [f(x), f(y)]_{Z(v_1)}$.
- (2) **OVERLAPPING CHARTS PROPERTIES:** If $Z = Z(\Psi_x^{p^s, p^u})$, $Z' = Z(\Psi_y^{q^s, q^u}) \in \mathcal{Z}$ with $Z \cap Z' \neq \emptyset$ then:
 - (a) $Z \subset \Psi_y(R[q^s \wedge q^u])$.
 - (b) If $x \in Z \cap Z'$ then $W^{s/u}(x, Z) \subset V^{s/u}(x, Z')$.
 - (c) If $x \in Z, y \in Z'$ then $V^s(x, Z)$ and $V^u(y, Z')$ intersect at a unique point.

When M is compact and f is a diffeomorphism, part (1) is [21, Lemma 10.7] and part (2) is [21, Lemmas 10.8 and 10.10]. The same proofs work in our case, since all calculations are made in the rectangle $R[10Q_\varepsilon(x)]$, and in this domain we have Theorem 6.1(6).

Now we apply a refinement method to destroy non-trivial intersections in \mathcal{Z} . The result is a cover of $\text{NUH}_\chi^\#$ by pairwise disjoint sets with the (geometrical) Markov property. This idea, originally developed by Sinaï and Bowen for finite covers [22, 23, 6], works equally well for countable covers with the local finiteness property [21]. Write $\mathcal{Z} = \{Z_1, Z_2, \dots\}$.

The Markov partition \mathcal{R} . – For every $Z_i, Z_j \in \mathcal{Z}$, define a partition of Z_i by:

$$T_{ij}^{su} = \{x \in Z_i : W^s(x, Z_i) \cap Z_j \neq \emptyset, W^u(x, Z_i) \cap Z_j \neq \emptyset\}$$

$$T_{ij}^{s\emptyset} = \{x \in Z_i : W^s(x, Z_i) \cap Z_j \neq \emptyset, W^u(x, Z_i) \cap Z_j = \emptyset\}$$

$$T_{ij}^{\emptyset u} = \{x \in Z_i : W^s(x, Z_i) \cap Z_j = \emptyset, W^u(x, Z_i) \cap Z_j \neq \emptyset\}$$

$$T_{ij}^{\emptyset\emptyset} = \{x \in Z_i : W^s(x, Z_i) \cap Z_j = \emptyset, W^u(x, Z_i) \cap Z_j = \emptyset\}.$$

Let $\mathcal{T} := \{T_{ij}^{\alpha\beta} : i, j \geq 1, \alpha \in \{s, \emptyset\}, \beta \in \{u, \emptyset\}\}$, and let \mathcal{R} be the partition generated by \mathcal{T} .

Since $T_{ii}^{su} = Z_i$, \mathcal{R} is a cover of $\text{NUH}_\chi^\#$ by pairwise disjoint sets. Clearly, \mathcal{R} is a refinement of \mathcal{Z} . Theorem 6.1 implies two local finiteness properties for \mathcal{R} :

- For every $Z \in \mathcal{Z}$, $\#\{R \in \mathcal{R} : R \subset Z\} < \infty$.
- For every $R \in \mathcal{R}$, $\#\{Z \in \mathcal{Z} : Z \supset R\} < \infty$.

Now we show that \mathcal{R} is a Markov partition in the sense of Sinaï [23].

s/u-fibers in \mathcal{R} . – Given $x \in R \in \mathcal{R}$, we define the *s-fiber* and *u-fiber* of x by:

$$W^s(x, R) := \bigcap_{\substack{T_{ij}^{\alpha\beta} \in \mathcal{T} \\ T_{ij}^{\alpha\beta} \supset R}} W^s(x, Z_i) \cap T_{ij}^{\alpha\beta} \quad \text{and} \quad W^u(x, R) := \bigcap_{\substack{T_{ij}^{\alpha\beta} \in \mathcal{T} \\ T_{ij}^{\alpha\beta} \supset R}} W^u(x, Z_i) \cap T_{ij}^{\alpha\beta}.$$

Any two *s*-fibers (*u*-fibers) either coincide or are disjoint.

PROPOSITION 7.3. – *The following are true.*

- (1) **PRODUCT STRUCTURE:** For every $R \in \mathcal{R}$ and every $x, y \in R$, the intersection $W^s(x, R) \cap W^u(y, R)$ consists of a single point of R . Denote it by $[x, y]$.
- (2) **HYPERBOLICITY:** If $z, w \in W^s(x, R)$ then $d(f^n(z), f^n(w)) \xrightarrow[n \rightarrow \infty]{} 0$, and if $z, w \in W^u(x, R)$ then $d(f^n(z), f^n(w)) \xrightarrow[n \rightarrow -\infty]{} 0$. The rates are exponential.
- (3) **GEOMETRICAL MARKOV PROPERTY:** Let $R_0, R_1 \in \mathcal{R}$. If $x \in R_0$ and $f(x) \in R_1$ then

$$f(W^s(x, R_0)) \subset W^s(f(x), R_1) \quad \text{and} \quad f^{-1}(W^u(f(x), R_1)) \subset W^u(x, R_0).$$

When M is compact and f is a diffeomorphism, this is [21, Prop. 11.5 and 11.7] and the same proof works in our case.

7.2. A finite-to-one Markov extension

We construct a new symbolic coding of f . Let $\widehat{\mathcal{G}} = (\widehat{V}, \widehat{E})$ be the oriented graph with vertex set $\widehat{V} = \mathcal{R}$ and edge set $\widehat{E} = \{R \rightarrow S : R, S \in \mathcal{R} \text{ s.t. } f(R) \cap S \neq \emptyset\}$, and let $(\widehat{\Sigma}, \widehat{\sigma})$ be the TMS induced by $\widehat{\mathcal{G}}$. The ingoing and outgoing degree of every vertex in $\widehat{\Sigma}$ is finite.

For $\ell \in \mathbb{Z}$ and a path $R_m \rightarrow \dots \rightarrow R_n$ on $\widehat{\mathcal{G}}$ define ${}_{\ell}[R_m, \dots, R_n] := f^{-\ell}(R_m) \cap \dots \cap f^{-\ell-(n-m)}(R_n)$, the set of points whose itinerary from ℓ to $\ell + (n - m)$ visits the rectangles R_m, \dots, R_n . The crucial property that gives the new coding is that ${}_{\ell}[R_m, \dots, R_n] \neq \emptyset$. This follows by induction, using the Markov property of \mathcal{R} (Proposition 7.3(3)).

The map π defines similar sets: for $\ell \in \mathbb{Z}$ and a path $v_m \xrightarrow{\varepsilon} \dots \xrightarrow{\varepsilon} v_n$ on Σ let

$$Z_{\ell}[v_m, \dots, v_n] := \{\pi(\underline{w}) : \underline{w} \in \Sigma^{\#} \text{ and } w_{\ell} = v_m, \dots, w_{\ell+(n-m)} = v_n\}.$$

There is a relation between Σ and $\widehat{\Sigma}$ in terms of these sets: if $\{R_n\}_{n \in \mathbb{Z}} \in \widehat{\Sigma}$ then there exists $\{v_n\}_{n \in \mathbb{Z}} \in \Sigma$ s.t. ${}_{-n}[R_{-n}, \dots, R_n] \subset Z_{-n}[v_{-n}, \dots, v_n]$ for all $n \geq 0$ (in particular $R_n \subset Z(v_n)$ for all $n \in \mathbb{Z}$). This fact is proved as in [21, Lemma 12.2]. By Proposition 7.3(2), $\bigcap_{n \geq 0} {}_{-n}[R_{-n}, \dots, R_n]$ is the intersection of a descending chain of nonempty closed sets with diameters converging to zero.

The map $\widehat{\pi} : \widehat{\Sigma} \rightarrow M$. – Given $\underline{R} = \{R_n\}_{n \in \mathbb{Z}} \in \widehat{\Sigma}$, $\widehat{\pi}(\underline{R})$ is defined by the identity

$$\{\widehat{\pi}(\underline{R})\} := \bigcap_{n \geq 0} \overline{{}_{-n}[R_{-n}, \dots, R_n]}.$$

The triple $(\widehat{\Sigma}, \widehat{\sigma}, \widehat{\pi})$ is the one that satisfies Theorem 1.3.

THEOREM 7.4. – *The following holds for all $\varepsilon > 0$ small enough.*

- (1) $\widehat{\pi} : \widehat{\Sigma} \rightarrow M$ is Hölder continuous.
- (2) $\widehat{\pi} \circ \widehat{\sigma} = f \circ \widehat{\pi}$.
- (3) $\widehat{\pi}[\widehat{\Sigma}^{\#}] \supset \text{NUH}_{\chi}^{\#}$, hence $\pi[\widehat{\Sigma}^{\#}]$ carries all f -adapted χ -hyperbolic measures.
- (4) Every point of $\widehat{\pi}[\widehat{\Sigma}^{\#}]$ has finitely many pre-images in $\widehat{\Sigma}^{\#}$.

When M is compact and f is a diffeomorphism, parts (1)–(3) are [21, Thm. 12.5] and part (4) is [18, Thm. 5.6(5)]. The same proofs work in our case, and the bound on the number of pre-images is exactly the same: there is a function $N : \mathcal{R} \rightarrow \mathbb{N}$ s.t. if $x = \widehat{\pi}(\underline{R})$ with $R_n = R$ for infinitely many $n > 0$ and $R_n = S$ for infinitely many $n < 0$ then $\#\{\underline{S} \in \widehat{\Sigma}^{\#} : \widehat{\pi}(\underline{S}) = x\} \leq N(R)N(S)$.

Appendix A: Underlying assumptions

Remember the definition of $\widetilde{A} \in \mathcal{L}_{x, x'}$ for $A \in \mathcal{L}_{y, z}$ and $y \in D_x, z \in D_{x'}$. Remember also the definition of $\tau = \tau_x : D_x \times D_x \rightarrow \mathcal{L}_x$ by $\tau(y, z) = d(\widetilde{\exp_y^{-1}})_z$. Throughout the text, we assume that there are constants $\mathfrak{R}, a > 1$ s.t. for all $x \in M \setminus \mathcal{D}$ there is $d(x, \mathcal{D})^a < \tau(x) < 1$ s.t. for $D_x := B(x, 2\tau(x))$ it holds:

- (A1) If $y \in D_x$ then $\text{inj}(y) \geq 2\tau(x)$, $\exp_y^{-1} : D_x \rightarrow T_y M$ is a diffeomorphism onto its image, and $\frac{1}{2}(d(x, y) + \|v - P_{y,x}w\|) \leq d_{\text{Sas}}(v, w) \leq 2(d(x, y) + \|v - P_{y,x}w\|)$ for all $y \in D_x$ and $v \in T_x M$, $w \in T_y M$ s.t. $\|v\|, \|w\| \leq 2\tau(x)$, where $P_{y,x} := P_\gamma$ for the radial geodesic γ joining y to x .
- (A2) If $y_1, y_2 \in D_x$ then $d(\exp_{y_1} v_1, \exp_{y_2} v_2) \leq 2d_{\text{Sas}}(v_1, v_2)$ for $\|v_1\|, \|v_2\| \leq 2\tau(x)$, and $d_{\text{Sas}}(\exp_{y_1}^{-1} z_1, \exp_{y_2}^{-1} z_2) \leq 2[d(y_1, y_2) + d(z_1, z_2)]$ for $z_1, z_2 \in D_x$ where the expression makes sense. In particular $\|d(\exp_x)_v\| \leq 2$ for $\|v\| \leq 2\tau(x)$, and $\|d(\exp_x^{-1})_y\| \leq 2$ for $y \in D_x$.
- (A3) If $y_1, y_2 \in D_x$ then
- $$\|d(\widetilde{\exp_{y_1}})_{v_1} - d(\widetilde{\exp_{y_2}})_{v_2}\| \leq d(x, \mathcal{D})^{-a} d_{\text{Sas}}(v_1, v_2) \leq \rho(x)^{-a} d_{\text{Sas}}(v_1, v_2)$$
- for all $\|v_1\|, \|v_2\| \leq 2\tau(x)$ and
- $$\begin{aligned} \|\tau(y_1, z_1) - \tau(y_2, z_2)\| &\leq d(x, \mathcal{D})^{-a} [d(y_1, y_2) + d(z_1, z_2)] \\ &\leq \rho(x)^{-a} [d(y_1, y_2) + d(z_1, z_2)] \end{aligned}$$
- for all $z_1, z_2 \in D_x$.
- (A4) If $y_1, y_2 \in D_x$ then the map $\tau(y_1, \cdot) - \tau(y_2, \cdot) : D_x \rightarrow \mathcal{L}_x$ has Lipschitz constant $\leq d(x, \mathcal{D})^{-a} d(y_1, y_2) \leq \rho(x)^{-a} d(y_1, y_2)$.
- (A5) If $y \in D_x$ then $\|df_y^{\pm 1}\| \leq d(x, \mathcal{D})^{-a} \leq \rho(x)^{-a}$.
- (A6) If $y_1, y_2 \in D_x$ and $f(y_1), f(y_2) \in D_{x'}$ then $\|\widetilde{df_{y_1}} - \widetilde{df_{y_2}}\| \leq \mathfrak{K}d(y_1, y_2)^\beta$, and if $y_1, y_2 \in D_x$ and $f^{-1}(y_1), f^{-1}(y_2) \in D_{x''}$ then $\|\widetilde{df_{y_1}^{-1}} - \widetilde{df_{y_2}^{-1}}\| \leq \mathfrak{K}d(y_1, y_2)^\beta$.
- (A7) $\|df_x^{\pm 1}\| \geq m(df_x^{\pm 1}) \geq \rho(x)^a$.

Appendix B: Standard proofs and adaptations of [21]

In this appendix we prove some statements claimed throughout the text, most of them consisting of adaptations of proofs in [21]. The main issue is the lack of higher regularity of the exponential map. The results of [21] are technical but extremely well-written, so rewriting them to our context would probably increase the technicalities and decrease the clarity. Hence we decided to write this appendix as a tutorial: we follow the proofs of [21] as much as possible, mentioning the necessary changes. The main changes are in the geometrical estimates on M : some Lipschitz constants of [21] are substituted by terms of the form $d(x, \mathcal{D})^{-a}$. We then show that our definition of $Q_\varepsilon(x)$ is strong enough to cancel out these terms. Since the proofs of [21] have freedom in the choice of exponents, we obtain the same final results and therefore (almost always) the same statements of [21].

Proof of Lemma 4.3. – Part (1) is proved exactly as in [21, Prop. 4.11(1)–(2)]. We concentrate on part (2). Let $\eta = p^s \wedge p^u$. The estimate of $\frac{\sin \angle(V^s, V^u)}{\sin \alpha(x)}$ in [21] is divided into the analysis of four factors. The estimate of the first two factors is identical; the difference is in the estimates of the remaining two factors.

By (A3), if $x \in M \setminus \mathcal{D}$ and $\|v\| \leq 2\tau(x)$ then $|\det[d(\exp_x)_v] - 1| \leq 4d(x, \mathcal{D})^{-a} \|v\|$, i.e., we substitute K_1 in [21, pp. 407] by $4d(x, \mathcal{D})^{-a}$.

With this notation, $K_1 \eta < 4d(x, \mathcal{D})^{-a} Q_\varepsilon(x)^{\beta/72} \eta^{1-\beta/72} < 4\varepsilon^{1/24} \eta^{1-\beta/72} < \eta^{2\beta/3}$ for $\varepsilon > 0$ small, then the third factor is $e^{\pm 2\eta^{2\beta/3}}$. To estimate the fourth factor, note that

again by (A3) if $x \in M \setminus \mathcal{D}$ and $\|v\| \leq 2\tau(x)$ then $\|d(\widetilde{\exp_x})_v - \text{Id}\| \leq d(x, \mathcal{D})^{-a} \|v\|$, i.e., we substitute K_2 in [21, pp. 407] by $d(x, \mathcal{D})^{-a}$. Noting as above that $3K_2\eta < \eta^{2\beta/3}$, we get that the fourth factor is $e^{\pm \frac{1}{3}\eta^{\beta/4}}$ as in [21, pp. 408].

The estimates of $|\cos \angle(V^s, V^u) - \cos \alpha(x)|$ work as in [21] after using again that $K_2\eta < \eta^{2\beta/3}$, in which case $K_3 = 24$. \square

Proof of Proposition 4.4. – We follow the proofs of [21, Prop. 4.12 and 4.14], with the modifications below.

– Pages 411–412: in claim 3, it is enough to have $|G'(0)| < \frac{1}{2}(q^s \wedge q^u)^{\beta/3}$. Proceed as in [21] to get that

$$|G'(0)| < e^{-\chi+\varepsilon} \left[|A| |F'(0)| + \frac{2}{3}\varepsilon^{\beta/3} (p^s \wedge p^u)^{\beta/3} + 6\varepsilon (p^s \wedge p^u)^{\beta/3} \right]$$

and then note that for $\varepsilon > 0$ small enough this is at most

$$\begin{aligned} & e^{-\chi+\varepsilon} \left[\frac{1}{2}e^{-\chi} + \frac{2}{3}\varepsilon^{\beta/3} + 6\varepsilon \right] (p^s \wedge p^s)^{\beta/3} \\ & \leq e^{-\chi+\varepsilon+\varepsilon\beta/3} \left[\frac{1}{2}e^{-\chi} + \frac{2}{3}\varepsilon^{\beta/3} + 6\varepsilon \right] (q^s \wedge q^s)^{\beta/3} < \frac{1}{2}(q^s \wedge q^u)^{\beta/3}. \end{aligned}$$

– Page 412: in claim 4, it is enough to have $\|G'\|_0 + \text{Hol}_{\beta/3}(G') < \frac{1}{2}$. Proceed as in [21] to get that $\|G'\|_0 + \text{Hol}_{\beta/3}(G') < e^{-\chi+3\varepsilon} \left[\frac{1}{2}e^{-\chi} + \frac{3}{2}\varepsilon \right]$. This is $< \frac{1}{2}$ when $\varepsilon > 0$ is small.

– Pages 414–415: in the proof of part 2, proceed as in [21] to get that

$$\|G_1 - G_2\|_0 \leq (|A| + 3\varepsilon^2)(1 + \varepsilon^2 + 3\varepsilon^3)\|F_1 - F_2\|_0$$

and note that $(|A| + 3\varepsilon^2)(1 + \varepsilon^2 + 3\varepsilon^3) < (e^{-\chi} + 3\varepsilon^2)(1 + \varepsilon^2 + 3\varepsilon^3) < e^{-\chi/2}$ when $\varepsilon > 0$ is small enough. \square

Proof of inequality (5.1). – We will use assumption (A5) as stated in Section 1:

(A5) $\|df_x\| < d(x, \mathcal{D})^{-a}$ and $\|df_x^{-1}\| < d(x, \mathcal{D})^{-a}$ for all $x \in M \setminus \mathcal{D}$.

We have:

$$\begin{aligned} s(f^{-1}(x))^2 &= 2 \sum_{n \geq 0} e^{2n\chi} \|df^n e_{f^{-1}(x)}^s\|^2 = 2 + 2e^{2\chi} \|df e_{f^{-1}(x)}^s\|^2 \sum_{n \geq 0} e^{2n\chi} \|df^n e_x^s\|^2 \\ &= 2 + e^{2\chi} \|df e_{f^{-1}(x)}^s\|^2 s(x)^2 \leq (1 + e^{2\chi} \|df e_{f^{-1}(x)}^s\|^2) s(x)^2. \end{aligned}$$

By (A3), $\frac{s(f^{-1}(x))^2}{s(x)^2} \leq 1 + e^{2\chi} d(f^{-1}(x), \mathcal{D})^{-2a} \leq 1 + e^{2\chi} \rho(x)^{-2a}$. We also have that

$$\begin{aligned} u(f^{-1}(x))^2 &= 2 \sum_{n \geq 0} e^{2n\chi} \|df^{-n} e_{f^{-1}(x)}^u\|^2 = 2 \|df^{-1} e_x^u\|^{-2} \sum_{n \geq 0} e^{2n\chi} \|df^{-(n+1)} e_x^u\|^2 \\ &= 2e^{-2\chi} \|df^{-1} e_x^u\|^{-2} \sum_{n \geq 1} e^{2n\chi} \|df^{-n} e_x^u\|^2 = e^{-2\chi} \|df^{-1} e_x^u\|^{-2} (u(x)^2 - 2) \\ &< \|df^{-1} e_x^u\|^{-2} u(x)^2, \end{aligned}$$

hence by (A5) we get that $\frac{u(f^{-1}(x))^2}{u(x)^2} \leq \rho(x)^{-2a} < 1 + e^{2\chi} \rho(x)^{-2a}$. Finally, applying (3.1) for $L = df_x^{-1}$, $v = e_x^s$, $w = e_x^u$ and using (A5), we have

$$\frac{\sin \alpha(x)}{\sin \alpha(f^{-1}(x))} = \frac{\sin \angle(e_x^s, e_x^u)}{\sin \angle(df_x^{-1} e_x^s, df_x^{-1} e_x^u)} \leq \|df_x^{-1}\| \|df_{f^{-1}(x)}\| < \rho(x)^{-2a}.$$

Since $\|\cdot\| \leq \|\cdot\|_{\text{Frob}} \leq \sqrt{2}\|\cdot\|$, the above inequalities and Lemma 2.1 give that

$$\begin{aligned} \|C_\chi(f^{-1}(x))^{-1}\| &\leq \|C_\chi(f^{-1}(x))^{-1}\|_{\text{Frob}} \leq \rho(x)^{-2a} \sqrt{1 + e^{2\chi}\rho(x)^{-2a}} \|C_\chi(x)^{-1}\|_{\text{Frob}} \\ &\leq 2\rho(x)^{-2a} (1 + e^\chi \rho(x)^{-a}) \|C_\chi(x)^{-1}\|. \quad \square \end{aligned}$$

Proof of Proposition 6.2. – The proof of part (2) is identical to the proof of [21, Prop. 6.4], and the proof of part (1)(a)–(b) is identical to the proof of [21, Prop. 6.3(1)–(2)]. To prove part (1)(c), we make some modifications in the proof of [21, Prop. 6.3(3)]. We start with the claim below.

Claim. – If $y, z \in D_x$ and $v \in T_y M, w \in T_z M$ with $\|v\| = \|w\| = 1$ then

$$\begin{aligned} \|\|df_y^{\pm 1}(v)\| - \|df_z^{\pm 1}(w)\|\| &\leq \mathfrak{K}\rho(x)^{-a} [d(y, z)^\beta + \|v - P_{z,y}w\|] \quad \text{and} \\ \left| \frac{\|df_y^{\pm 1}(v)\|}{\|df_z^{\pm 1}(w)\|} - 1 \right| &\leq \mathfrak{K}\rho(x)^{-2a} [d(y, z)^\beta + \|v - P_{z,y}w\|]. \end{aligned}$$

In particular $|\log \|df_y^{\pm 1}(v)\| - \log \|df_z^{\pm 1}(w)\|| \leq \mathfrak{K}\rho(x)^{-2a} [d(y, z)^\beta + \|v - P_{z,y}w\|]$.

Proof of the claim. – The inequalities are consequences of (A5)–(A7). Since these assumptions are symmetric on f and f^{-1} , we only prove the claim for f . Note that:

$$\begin{aligned} \|\|df_y(v)\| - \|df_z(w)\|\| &\leq \|\widetilde{df}_y(P_{y,x}v) - \widetilde{df}_z(P_{z,x}w)\| \\ &\leq \|\widetilde{df}_y - \widetilde{df}_z\| + \|\widetilde{df}_z\| \|v - P_{z,y}w\| \\ &\leq \mathfrak{K}d(y, z)^\beta + \rho(x)^{-a} \|v - P_{z,y}w\| \\ &\leq \mathfrak{K}\rho(x)^{-a} [d(y, z)^\beta + \|v - P_{z,y}w\|]. \end{aligned}$$

The second inequality follows from the first one and from (A7).

Let us now prove part (1)(c). Write $V^s = V^s[\{\Psi_{x_n}^{p_n^s, p_n^s}\}_{n \geq 0}]$. By the claim,

$$\begin{aligned} |\log \|df^n e_y^s\| - \log \|df^n e_z^s\|| &\leq \sum_{k=0}^{n-1} |\log \|df e_{f^k(y)}^s\| - \log \|df e_{f^k(z)}^s\|| \\ &\leq \sum_{k=0}^{n-1} \mathfrak{K}\rho(x_k)^{-2a} [d(f^k(y), f^k(z))^\beta + \|e_{f^k(y)}^s - P_{f^k(z), f^k(y)} e_{f^k(z)}^s\|]. \end{aligned}$$

By part (1)(a) and the definition of $Q_\varepsilon(x_k)$,

$$\begin{aligned} \rho(x_k)^{-2a} d(f^k(y), f^k(z))^\beta &< \varepsilon^{1/12} Q_\varepsilon(x_k)^{-\beta/36} 6e^{-\frac{\beta\chi}{2}k} (p_0^s)^\beta \\ &< 6\varepsilon^{1/12} (p_k^s)^{-\beta/36} e^{-\frac{\beta\chi}{2}k} (p_0^s)^\beta. \end{aligned}$$

By (GPO2) we have $p_0^s \leq e^{\varepsilon k} p_k^s$, then for small $\varepsilon > 0$ the last expression above is

$$\leq 6\varepsilon^{1/12} (p_0^s)^{-\beta/36} e^{-\frac{\beta\chi}{2}k + \frac{\beta\varepsilon}{36}k} (p_0^s)^\beta < 6\varepsilon^{1/12} e^{-\frac{\beta\chi}{3}k} (p_0^s)^{\beta/4}$$

and thus

$$\sum_{k=0}^{n-1} \mathfrak{K}\rho(x_k)^{-2a} d(f^k(y), f^k(z))^\beta \leq \frac{6\mathfrak{K}\varepsilon^{1/12}}{1 - e^{-\frac{\beta\chi}{3}}} (p_0^s)^{\beta/4} < \frac{1}{2} (p_0^s)^{\beta/4}.$$

We now estimate the second sum. Call $N_k := \|e_{f^k(y)}^s - P_{f^k(z), f^k(y)} e_{f^k(z)}^s\|$. Write $f^k(y) = \Psi_{x_k}(y_k) = \Psi_{x_k}(y_k, F_k(y_k))$ and $f^k(z) = \Psi_{x_k}(z_k) = \Psi_{x_k}(z_k, F_k(z_k))$, where F_k is the

representing function of $V^s[\{\Psi_{x_n}^{p_n^s, p_n^u}\}_{n \geq k}]$. In part (1), it is proved that $\|y_k - z_k\| \leq 3p_0^s e^{-\frac{\beta}{2}k}$. As in [21, pp. 418–419],

$$N_k \leq 2\|C_\chi(x_k)^{-1}\|\|y_k - z_k\|^{\beta/3} + 4\|C_\chi(x_k)^{-1}\|\left\|d(\exp_{x_k})\widetilde{C_\chi(x_k)_{y_k}} \circ C_\chi(x_k) - d(\exp_{x_k})\widetilde{C_\chi(x_k)_{z_k}} \circ C_\chi(x_k)\right\|$$

which, by (A3), is $\leq 2\|C_\chi(x_k)^{-1}\|\|y_k - z_k\|^{\beta/3} + 4\|C_\chi(x_k)^{-1}\|\rho(x_k)^{-a}\|y_k - z_k\|$. For $\varepsilon > 0$ small enough

$$\begin{aligned} 4\rho(x_k)^{-a}\|y_k - z_k\|^{\beta/72} &\leq 12\rho(x_k)^{-a}(p_0^s)^{\beta/72}e^{-\frac{\beta\chi}{144}k} \\ &\leq 12\rho(x_k)^{-a}(p_k^s)^{\beta/72}e^{-\frac{\beta\chi}{144}k + \frac{\beta\varepsilon}{72}k} \leq 12\varepsilon^{1/24}e^{-\frac{\beta\chi}{144}k + \frac{\beta\varepsilon}{72}k} < 1, \end{aligned}$$

thus $N_k \leq 3\|C_\chi(x_k)^{-1}\|\|y_k - z_k\|^{\beta/3}$. Hence for small $\varepsilon > 0$

$$\begin{aligned} \rho(x_k)^{-2a}N_k &\leq 3\|C_\chi(x_k)^{-1}\|\rho(x_k)^{-2a}\|y_k - z_k\|^{\beta/3} \\ &\leq 9\|C_\chi(x_k)^{-1}\|\rho(x_k)^{-2a}(p_0^s)^{\beta/3}e^{-\frac{\beta\chi}{6}k} \\ &\leq 9\|C_\chi(x_k)^{-1}\|\rho(x_k)^{-2a}(p_0^s)^{\beta/12}e^{-\frac{\beta\chi}{6}k}(p_0^s)^{\beta/4} \\ &\leq 9\|C_\chi(x_k)^{-1}\|\rho(x_k)^{-2a}(p_k^s)^{\beta/12}e^{-\frac{\beta\chi}{6}k + \frac{\beta\varepsilon}{12}k}(p_0^s)^{\beta/4} \\ &\leq 9\|C_\chi(x_k)^{-1}\|(p_k^s)^{\beta/24}\rho(x_k)^{-2a}(p_k^s)^{\beta/36}e^{-\frac{\beta\chi}{6}k + \frac{\beta\varepsilon}{12}k}(p_0^s)^{\beta/4} \\ &\leq 9\varepsilon^{5/24}e^{-\frac{\beta\chi}{7}k}(p_0^s)^{\beta/4} \end{aligned}$$

and therefore

$$\sum_{k=0}^{n-1} \mathfrak{K}\rho(x_k)^{-2a}\|e_{f^k(y)}^s - P_{f^k(z), f^k(y)}e_{f^k(z)}^s\| \leq \frac{9\mathfrak{K}\varepsilon^{5/24}}{1 - e^{-\beta\chi/7}}(p_0^s)^{\beta/4} < \frac{1}{2}(p_0^s)^{\beta/4}.$$

The conclusion is that $|\log\|df^n e_y^s\| - \log\|df^n e_z^s\|| < (p_0^s)^{\beta/4} < Q_\varepsilon(x)^{\beta/4}$. \square

Proof of Lemma 6.6. – It is enough to prove the case $n = 0$. Write $\Psi_{x_0}^{p_0^s, p_0^u} = \Psi_x^{p^s, p^u}$, $\Psi_{y_0}^{q_0^s, q_0^u} = \Psi_y^{q^s, q^u}$, $p = p^s \wedge p^u$, $q = q^s \wedge q^u$. Write $\widetilde{C_\chi(x)} = R_x C_x$, $\widetilde{C_\chi(y)} = R_y C_y$. Since $R_y^{-1}R_x$ is a rotation matrix, it is enough to estimate its angle. As in [21, pp. 372], $\exists \lambda \neq 0$ s.t. $C_x \underline{a} = \lambda[d(\exp_x)\widetilde{C_\chi(x)_\xi}]^{-1}[d(\exp_y)\widetilde{C_\chi(y)_\eta}]C_y \underline{b}$ where:

$$\begin{aligned} - \underline{\xi} &\in R[10^{-2}p], \underline{a} = \begin{bmatrix} 1 \\ a \end{bmatrix} \text{ and } |a| < p^{\beta/3}. \\ - \underline{\eta} &\in R[10^{-2}q], \underline{b} = \begin{bmatrix} 1 \\ b \end{bmatrix} \text{ and } |b| < q^{\beta/3}. \end{aligned}$$

The proof is based on three claims. Write $\vec{v} \propto \vec{w}$ if $\vec{v} = t\vec{w}$ for some $t \neq 0$.

$$\textit{Claim 1.} - C_x \underline{a} \propto R_x \begin{bmatrix} 1 \pm p^{\beta/4} \\ \pm p^{\beta/4} \end{bmatrix} \text{ and } C_y \underline{b} \propto R_y \begin{bmatrix} 1 \pm q^{\beta/4} \\ \pm q^{\beta/4} \end{bmatrix}.$$

The proof is the same as in [21, pp. 372].

Claim 2. – If $x, y \in D_z$ and $\|v\|, \|w\| \leq \tau(z)$ then

$$\| [d(\widetilde{\exp_x})_v]^{-1} [d(\widetilde{\exp_y})_w] - \text{Id} \| < 2d(z, \mathcal{D})^{-a} d_{\text{Sas}}(v, w).$$

The proof is a direct consequence of (A2)–(A3).

In particular, if we write $E := [d(\widetilde{\exp_x})_{C_X(x)\underline{\zeta}}]^{-1} [d(\widetilde{\exp_y})_{C_X(y)\underline{\eta}}] - \text{Id}$ then

$$\begin{aligned} \|E\| &< 2d(y, \mathcal{D})^{-a} d_{\text{Sas}}(C_X(x)\underline{\zeta}, C_X(y)\underline{\eta}) \leq 4d(y, \mathcal{D})^{-a} [d(x, y) + \|\underline{\zeta} - \underline{\eta}\|] \\ &< 4d(y, \mathcal{D})^{-a} (p + q) < 8d(x, \mathcal{D})^{-a} p + 8d(y, \mathcal{D})^{-a} q \ll p^{\beta/3} + q^{\beta/3} \end{aligned}$$

since $d(x, y) < 25^{-1}(p + q)$ and $\|\underline{\zeta}\| + \|\underline{\eta}\| < 10^{-2}(p + q)$.

Claim 3. – $R_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \varepsilon_1 \propto R_y \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \varepsilon_2$ where $\|\varepsilon_1\|, \|\varepsilon_2\| < 3(p^{\beta/4} + q^{\beta/4}) \leq 6\varepsilon^{3/4}$.

To see this, note that since $C_x \underline{a} \propto (E + I)C_y \underline{b}$, claim 1 gives that

$$R_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \underbrace{R_x \begin{bmatrix} \pm p^{\beta/4} \\ \pm p^{\beta/4} \end{bmatrix}}_{=\varepsilon_1} \propto R_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \underbrace{R_y \begin{bmatrix} \pm q^{\beta/4} \\ \pm q^{\beta/4} \end{bmatrix} + EC_y \underline{b}}_{=\varepsilon_2}$$

and that $\|\varepsilon_1\| \leq 2p^{\beta/4}$ and $\|\varepsilon_2\| \leq 2q^{\beta/4} + 2(p^{\beta/3} + q^{\beta/3}) < 3(p^{\beta/4} + q^{\beta/4})$. The remainder of the proof is identical to [21, pp. 373]. \square

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