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SMALL GAPS IN THE SPECTRUM OF THE RECTANGULAR BILLIARD

BY VALENTIN BLOMER, JEAN BOURGAIN, MAKSYM RADZIWIŁŁ AND ZEÉV RUDNICK

ABSTRACT. – We study the size of the minimal gap between the first N eigenvalues of the Laplacian on a rectangular billiard having irrational squared aspect ratio α, in comparison to the corresponding quantity for a Poissonian sequence. If α is a quadratic irrationality of certain type, such as the square root of a rational number, we show that the minimal gap is roughly of size $1/N$, which is essentially consistent with Poisson statistics. We also give related results for a set of α's of full measure. However, on a fine scale we show that Poisson statistics is violated for all α. The proofs use a variety of ideas of an arithmetical nature, involving Diophantine approximation, the theory of continued fractions, and results in analytic number theory.

RÉSUMÉ. – On étudie l'écart minimal dans les N premières valeurs propres du Laplacien d'un billard rectangulaire dont le rapport des côtés est égal à $1/\sqrt{\alpha}$. On compare nos résultats avec l'écart minimal des points provenant d'une suite aléatoire poissonienne. Pour α un irrationnel quadratique d'un certain type, par exemple la racine d'un nombre rationnel, nous démontrons que l'écart minimal est approximativement de taille $1/N$. Cela est en accord avec les statistiques poissoniennes. Nous démontrons aussi un phénomène semblable pour presque tout α au sens de la mesure de Lebesgue. Cependant, à une échelle fine, de taille $1/N$, nous démontrons que l'écart minimal entre les valeurs propres et celui d'une suite poissonienne ont un comportement différent. Les démonstrations utilisent plusieurs résultats d'origine arithmétique, tels que l'approximation diophantienne, la théorie des fractions continues, et des résultats provenant de la théorie analytique des nombres.

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1. Introduction

The local statistics of the energy levels of several integrable systems are believed to follow Poisson statistics [2]. In this note we examine a variant of these statistics, the size of the minimal gap between levels, for the energy levels of a particularly simple system, a rectangular billiard. If the rectangle has width $\pi/\sqrt{\alpha}$ and height π, with aspect ratio $\sqrt{\alpha}$, then the energy levels, meaning the eigenvalues of the Dirichlet Laplacian, consist of the numbers $\alpha m^2 + n^2$ with integers $m, n \geq 1$.

The case of rational α is special: The eigenvalues lie in a lattice, in particular the nonzero gaps are bounded away from zero, and there are arbitrarily large multiplicities. We exclude this case from our discussion. If α is irrational, we get a simple spectrum $0 < \lambda_1 < \lambda_2 < \cdots$, with growth (Weyl's law)

$$\#\{j : \lambda_j \leq X\} = \#\{(m,n) : m,n \geq 1,\ \alpha m^2 + n^2 \leq X\} \sim \frac{\pi}{4\sqrt{\alpha}} X$$

as $X \to \infty$. In this setting, the pair correlation function has been shown to be Poissonian [8] for Diophantine α, see also [21] for a related problem.

We wish to study the size of the minimal gap function of the spectrum, defined as

$$\delta_{\min}^{(\alpha)}(N) = \min(\lambda_{i+1} - \lambda_i : 1 \leq i < N).$$

To set expectations, it is worth comparing with the size of the analogous quantity for some random sequences, when measured on the scale of the mean spacing between the levels in the sequence, which in our case is constant (equal to $4\sqrt{\alpha}/\pi$). For a Poissonian sequence of N uncorrelated levels with unit mean spacing, the smallest gap is almost surely of size $\approx 1/N$ [15]. In comparison, the smallest gap between the eigenphases of a random $N \times N$ unitary matrix is, on the scale of the mean spacing, almost surely of size $\approx N^{-1/3}$ [22, 1], in particular much larger than the Poisson case. The same behavior persists for the eigenvalues of random $N \times N$ Hermitian matrices (the Gaussian Unitary Ensemble) [22, 1]. For the Gaussian Orthogonal Ensemble of random symmetric matrices, is expected (though as of now not proved) that the minimal gap is of size $N^{-1/2}$. We note that the local statistics of the eigenvalues of the Laplacian for generic chaotic systems, such as non-arithmetic surfaces of negative curvature, are expected to follow the Gaussian Orthogonal Ensemble [3], while the local statistics of the zeros of the Riemann zeta function are expected to follow the Gaussian Unitary Ensemble [16, 20].

1.1. Order of growth of $\delta_{\min}^{(\alpha)}(N)$

Returning to our rectangular billiard, it is not hard to obtain lower bounds for $\delta_{\min}^{(\alpha)}(N)$, see §2.1. In the case of quadratic irrationalities, the gap function cannot shrink faster than $1/N$: for each quadratic irrationality α, there is some $c(\alpha) > 0$ so that

$$\delta_{\min}^{(\alpha)}(N) \geq \frac{c(\alpha)}{N}. \tag{1.1}$$

More generally, both for *algebraic* irrationalities and for almost every α (in the measure theoretic sense) the same argument shows

$$\delta_{\min}^{(\alpha)}(N) \gg 1/N^{1+\varepsilon} \tag{1.2}$$

for any $\varepsilon > 0$, see Proposition 2.1 below. Both (1.1) and (1.2) depend on general results in diophantine approximation.

In (1.2) and elsewhere in the paper, we use Vinogradov's notation $f(N) \ll g(N)$ to mean that there are $c > 0$ and $N_0 \geq 1$ so that $|f(N)| \leq c|g(N)|$ for all $N > N_0$; and the notation $f(N) \asymp g(N)$ to mean both $f(N) \ll g(N)$ and $g(N) \ll f(N)$. Implied constants may always depend on α and ε where applicable.

Much more work needs to be done to obtain good upper bounds for $\delta_{\min}^{(\alpha)}(N)$, i.e., to explicitly construct small gaps.

We show in Proposition 2.2 below that for any irrational α, we have

$$\delta_{\min}^{(\alpha)}(N) \ll N^{-1/2} \tag{1.3}$$

for all N. By the same argument, we can also display α where $\delta_{\min}^{(\alpha)}(N) \ll N^{-A}$ for any $A > 0$ by taking α to be suitable Liouville numbers. However these form a measure zero set and are atypical.

For certain quadratic irrationalities we show that the minimal gap can be almost as small as $1/N$:

THEOREM 1.1. – *If the squared aspect ratio is a quadratic irrationality of the form $\alpha = \sqrt{r}$, with r rational, then*

$$\delta_{\min}^{(\alpha)}(N) \ll \frac{1}{N^{1-\varepsilon}}$$

for every $\varepsilon > 0$ and all N.

We can also deal with other quadratic irrationalities, such as the golden mean. We refer to Section 6 for more general results. In particular, we show in this section that there exist quadratic irrationalities α such that the stronger result

$$\delta_{\min}^{(\alpha)}(N) \ll 1/N \tag{1.4}$$

holds for all N. An explicit example is the square of the golden mean $\alpha = (3 + \sqrt{5})/2$.

Moving away from quadratic irrationalities, where our results are deterministic, we turn to generic in measure α.

THEOREM 1.2. – *For almost all $\alpha > 0$ (in the sense of Lebesgue measure) we have*

$$\delta_{\min}^{(\alpha)}(N) \ll \frac{1}{N^{1-\varepsilon}} \tag{1.5}$$

for any $\varepsilon > 0$ and all N.

We summarize the preceding results by stating that the order of growth of $\delta_{\min}^{(\alpha)}(N) \approx 1/N$ is consistent with Poisson statistics for certain special and also generic in measure α. However, as we now explain, finer details of Poisson statistics are always violated.

1.2. Deviations from Poisson statistics

Given a sequence of points, let $\delta_{\min,k}(N)$ be the k-th smallest gap ($k \geq 1$) among the first N points in the sequence, so that in particular $\delta_{\min,1}(N) = \delta_{\min}(N)$. For a Poisson sequence with unit mean spacing (by which we mean N points picked independently and uniformly in $[0, N]$), Devroye [6] showed that for any fixed $k \geq 1$ and any sequence $\{u_n\}$ of positive numbers such that u_n/n^2 is decreasing we have

$$\text{Prob}\Big(N\delta_{\min,k}(N) \leq u_N \text{ infinitely often}\Big) = \begin{cases} 1, & \sum_n u_n^k/n = \infty, \\ 0, & \sum_n u_n^k/n < \infty. \end{cases} \tag{1.6}$$

Choosing for instance $u_n = 1/\log n$ for $k = 1$, one has

$$\delta_{\min}(N) \leq \frac{1}{N \log N} \qquad \text{infinitely often} \tag{1.7}$$

almost surely, while choosing $u_n = 1/(\log n)^{2/3}$ for $k = 2$ one has

$$\delta_{\min,2}(N) \geq \frac{1}{N(\log N)^{2/3}} \qquad \text{for all sufficiently large } N \tag{1.8}$$

almost surely. Similarly, it is shown in [6, Theorem 4.2] that

$$\delta_{\min}(N) \geq \frac{\log\log N}{N} \qquad \text{infinitely often} \tag{1.9}$$

almost surely, but by [6, Theorem 4.1] we have

$$\text{Prob}\left(\delta_{\min}(N) \geq \frac{(\log\log N)^2}{N} \text{ infinitely often}\right) = 0. \tag{1.10}$$

For our sequence $\{\alpha m^2 + n^2\}$, we infer from (1.1) that in the case of quadratic irrationalities (1.7) is violated. The following result shows that (1.10) is violated for *almost all* α:

THEOREM 1.3. – *For almost all $\alpha > 0$ (in the sense of Lebesgue measure) we have*

$$\delta_{\min}^{(\alpha)}(N) \gg \frac{(\log N)^c}{N} \qquad \textit{infinitely often} \tag{1.11}$$

where $c = 1 - \frac{\log(e\log 2)}{\log 2} = 0.086\ldots$.

In fact, for *all* α we show

THEOREM 1.4. – *For any $\alpha > 0$, at least one of the conditions* (1.7) *or* (1.8) *is violated.*

It is also of interest to study the distribution of the *largest* gap. One does expect arbitrarily large gaps, and it is a challenging problem to prove this for Diophantine α.

1.3. About the proofs

The proofs draw from a variety of methods. We show in Section 3 (see Lemma 3.1) that the size of $\delta_{\min}^{(\alpha)}(N)$ depends on the existence of good rational approximants p/q to α, where both p and q are *evenly divisible*, by which we mean integers n having a divisor $d \mid n$ roughly of size square root:

$$\min(d, n/d) \gg n^{1/2-\varepsilon}$$

for any $\varepsilon > 0$. We will see that the concept of evenly divisible numbers comes up naturally in the context of finding small gaps, although we have not seen it in other number theoretical applications.

To find such approximants for certain quadratic irrationalities, for instance $\alpha = \sqrt{D}$ as in Theorem 1.1, for integer $D > 1$ not a perfect square, we use the theory of Pell's equation to show that there are many approximants p_n/q_n for which both of the sequences $\{p_n\}$ and $\{q_n\}$ satisfy a "strong divisibility" condition of the form

$$\gcd(a_m, a_n) = a_{\gcd(m,n)}, \quad m, n \text{ odd}.$$

This condition can be used to produce "good" divisors.

Theorem 1.2 uses a second moment approach to obtain a result valid for almost all α. The corresponding counting problem that produces evenly divisible approximants is analyzed by exponential sums, and becomes naturally a problem in 4 variables, so that the second moment produces an eighth moment of the Riemann zeta-function. In absence of the Lindelöf hypothesis, we introduce artificially a bilinear structure, separating the 4 variables into 4 short ones and 4 long ones; we obtain an unconditional saving on the short variables using strong bounds for the Riemann zeta function $\zeta(s)$ near the line $\mathrm{Re}(s) = 1$ based on Vinogradov's method, and handle the contribution of the long variables using a mean-value theorem. The general scheme of this method has already found further applications in connection with the Oppenheim conjecture for ternary quadratic forms [4, 5].

To prove the lower bound in Theorem 1.3, we invoke Ford's quantitative version [9] of the result first proved by Erdős [7] that a multiplication table of side length X contains $o(X^2)$ different entries, which gives restrictions on the arithmetic properties of approximants.

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2. Some general results

2.1. Lower bounds

An irrational α is *badly approximable* if for all integers (p, q) with $q \geq 1$ we have

$$|q\alpha - p| \gg \frac{1}{q}. \tag{2.1}$$

It is (strongly) Diophantine if we have the weaker inequality

$$|q\alpha - p| \gg \frac{1}{q^{1+\varepsilon}} \quad \text{for all} \quad \varepsilon > 0. \tag{2.2}$$

We recall [13] that α being badly approximable is equivalent to having bounded partial quotients in the continued fraction expansion of α. Thus quadratic irrationalities are badly approximable. The set of badly approximable reals has measure zero. However the set of (strongly) Diophantine numbers has full measure. Roth's theorem says that all algebraic irrationalities are (strongly) Diophantine. For a full measure set of α, one in fact has a stronger lower bound [13]: For every $\varepsilon > 0$, we have

$$|q\alpha - p| \gg \frac{1}{q(\log q)^{1+\varepsilon}}, \tag{2.3}$$

for all $q \geq 2$.

PROPOSITION 2.1. – *Let $\alpha > 0$.*

i) *Suppose $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is badly approximable. Then for all N we have*

$$\delta_{\min}^{(\alpha)}(N) \gg \frac{1}{N}.$$

ii) *If $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is (strongly) Diophantine, then for all $\varepsilon > 0$ and all N we have*

$$\delta_{\min}^{(\alpha)}(N) \gg \frac{1}{N^{1+\varepsilon}}.$$

iii) *For Lebesgue almost all α, for all $\varepsilon > 0$ and all N we have*

$$\delta_{\min}^{(\alpha)}(N) \gg \frac{1}{N(\log N)^{1+\varepsilon}}.$$

Proof. – Indeed if α is badly approximable then for any two distinct eigenvalues $\lambda := \alpha m^2 + n^2$ and $\lambda' := \alpha m'^2 + n'^2$ with $\max(\lambda, \lambda') \leq N$ we obtain

$$|\lambda - \lambda'| = |(m^2 - m'^2)\alpha - (n'^2 - n^2)| \gg \frac{1}{|m^2 - m'^2|} \gg \frac{1}{\max(\lambda, \lambda')} \geq \frac{1}{N}$$

using (2.1). The same argument with (2.2) and (2.3) in place of (2.1) proves ii) and iii). □

2.2. A general upper bound

PROPOSITION 2.2. – *For any irrational $\alpha > 0$, we have*

$$\delta_{\min}^{(\alpha)}(N) \ll N^{-1/2},$$

for all N.

Proof. – Let $Q \geq 1$ be sufficiently large. By Dirichlet's approximation theorem there exist integers $a \in \mathbb{Z}$, $1 \leq q \leq Q$ such that $0 < |a - q\alpha| \leq 1/Q$, and since $\alpha > 0$ we must have $a \geq 1$. With $m = 2q + 1, m' = 2q - 1, n = 2a - 1, n' = 2a + 1$ we have $1 \leq m, m', n, n' \ll Q$ and

$$|\alpha m^2 + n^2 - (\alpha m'^2 - n'^2)| = 8|\alpha q - a| \leq \frac{8}{Q}.$$

Choosing Q to be of exact order $N^{1/2}$ gives the desired bound. □

3. The general strategy

From now on, we deal with getting a bound of the form $\delta_{\min}^{(\alpha)}(N) \ll N^{-1+\varepsilon}$. We will frequently use the relation $\lambda_i \asymp i$ for $i \geq 1$.

We recall the notion of "evenly divisible", introduced in Section 1.3.

DEFINITION 1. – We call an integer n is evenly divisible if there is a divisor $d \mid n$, such that $\min(d, n/d) \gg n^{1/2-\varepsilon}$ for all $\varepsilon > 0$. We call n strongly evenly divisible if there is a divisor $d \mid n$, such that $\min(d, n/d) \gg n^{1/2}$.

So primes are not evenly divisible, but perfect squares are, even strongly so. Suppose we have found a good rational approximation

$$|\alpha q - p| \ll \frac{1}{q} \tag{3.1}$$

with p, q both evenly divisible, say $d \mid q$, $q^{1/2-\varepsilon} \ll d \leq \sqrt{q}$, and $e \mid p$, $p^{1/2-\varepsilon} \ll e \leq \sqrt{p}$ (note that $p \asymp q$ since p/q is an approximation to α). It is useful to observe that we may assume without loss of generality that neither p nor q is a perfect square. Indeed, at least one of the pairs (p, q), $(2p, 2q)$, $(3p, 3q)$ contains two non-squares, and so we can simply replace (p, q) with $(2p, 2q)$ or $(3p, 3q)$ in (3.1) if necessary.

Now find $m > m' \geq 1, n' > n \geq 1$ solving

$$m - m' = 2d, \quad m + m' = 2\frac{q}{d}, \quad n - n' = 2e, \quad n + n' = 2\frac{p}{e},$$

namely

$$m = \frac{q}{d} + d, \quad m' = \frac{q}{d} - d, \quad n = \frac{p}{e} - e, \quad n' = \frac{p}{e} + e.$$

Notice that all variables are non-zero by our assumption that neither p nor q is a perfect square. Clearly

$$m^2 - m'^2 = 4q, \quad n'^2 - n^2 = 4p$$

and moreover by our assumptions on the size of d and e, we have

$$q^{1/2} \ll m, n' \quad \text{and} \quad m, m', n, n' \ll q^{1/2+\varepsilon}.$$

Hence the corresponding eigenvalues

$$\lambda := \alpha m^2 + n^2, \quad \lambda' := \alpha m'^2 + n'^2$$

satisfy (maybe with a different value of ε)

$$q \ll \lambda, \lambda' \ll q^{1+\varepsilon}$$

and give a gap in the spectrum of size at most

$$|\lambda - \lambda'| = |\alpha(m^2 - m'^2) - (n'^2 - n^2)| = 4|\alpha q - p| \ll \frac{1}{q} \ll \frac{1}{\max(\lambda, \lambda')^{1-\varepsilon}},$$

where we used (3.1) in the penultimate step. We conclude

$$\delta_{\min}^{(\alpha)}(N) \ll \frac{1}{N^{1-\varepsilon}}$$

for $N \asymp \max(\lambda, \lambda')$. This argument shows the following:

LEMMA 3.1. – *If $\alpha > 0$ has infinitely many good rational approximations p_n/q_n with $q_1 < q_2 < \cdots$ as in* (3.1) *with both p and q evenly divisible (resp. strongly evenly divisible), then $\delta^{(\alpha)}_{\min}(N) \ll N^{-1+\varepsilon}$ for all $\varepsilon > 0$ (resp. $\delta^{(\alpha)}_{\min}(N) \ll N^{-1}$) infinitely often.*

If in addition $q_n \geq cq_{n+1}$, for some constant $c > 0$ (possibly depending on α, but not on n), then these inequalities hold for all N.

For later purposes we record the following variation. If we replace (3.1) with the weaker condition

$$\left|\alpha - \frac{p}{q}\right| \ll \frac{1}{T} \tag{3.2}$$

for some $T \leq q^2$, we obtain the following:

LEMMA 3.2. – *If $\alpha > 0$ has infinitely many good rational approximations p_n/q_n with $q_1 < q_2 < \cdots$ as in* (3.2) *with both p and q evenly divisible and $q_n \geq cq_{n+1}$ for some constant $c > 0$ and all $n \geq 1$, then*

$$\delta^{(\alpha)}_{\min}(N) \ll N^{1+\varepsilon}T^{-1}$$

for all N and all $\varepsilon > 0$.

4. Interlude: Strong divisibility sequences and Chebyshev polynomials

A sequence of integers $\{a_n\}$ is a *divisibility sequence* if $m \mid n$ implies that $a_m \mid a_n$. It is a *strong* divisibility sequence if

$$\gcd(a_m, a_n) = a_{\gcd(m,n)}.$$

A classical example is the sequence of Fibonacci numbers (see [14, Section 1.2.8]), and it is known that second order recurrence sequences with constant coefficients of the form

$$a_{n+1} = ba_n + da_{n-1}, \quad (b, d) = 1, \quad a_0 = 0, \quad a_1 = 1 \tag{4.1}$$

satisfy this property, see e.g., [11, Proposition 2.2].

One can generate families of such sequences with Chebyshev polynomials. We recall that the Chebyshev polynomials of the first and second kind T_n and U_n are defined as (see e.g., [19])

$$T_n(x) = \frac{1}{2}\left((x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n\right)$$

and

$$U_n(x) = \frac{(x + \sqrt{x^2-1})^{n+1} - (x - \sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}.$$

They satisfy the second order recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \tag{4.2}$$

and they are solutions of a polynomial Pell equation

$$T_n(x)^2 - (x^2-1)U_{n-1}(x)^2 = 1. \tag{4.3}$$

Also useful is the formula

$$T_{n+m}(x) = 2T_n(x)T_m(x) - T_{n-m}(x), \quad n \geq m \geq 0, \tag{4.4}$$

which can be easily verified from the definition of T_n. One checks by induction using (4.2) that

$$U_n(x/2),\ 2T_n(x/2) \in \mathbb{Z} \quad \text{for } x \in \mathbb{Z}. \tag{4.5}$$

Any half-integral specialization of shifted Chebyshev polynomials of the second kind forms a strong divisibility sequence:

$$(U_{n-1}(x/2), U_{m-1}(x/2)) = U_{(n,m)-1}(x/2) \tag{4.6}$$

for all $n, m, x \in \mathbb{N}$. This follows, for instance, from noting that the sequence $a_n = U_{n-1}(x/2)$ satisfies (4.1) with $d = -1$, $b = x$, see also [17].

A little less known is a slightly weaker corresponding statement for Chebyshev polynomials of the first kind: we have

$$(2T_n(x/2), 2T_m(x/2)) = 2T_{(n,m)}(x/2) \tag{4.7}$$

for all $x \in \mathbb{N}$ and all *odd* positive integers n, m. A variation of this is proved in [17, Theorem 2], but for convenience we give a proof of this fact:

Let $x \in \mathbb{Z}$, and let $a_n = 2T_n(x/2)$. We write $y = \frac{1}{2}(x + \sqrt{x^2-4})$, so that $2T_n(x/2) = y^n + y^{-n}$. Clearly y is a quadratic algebraic integer of norm 1, since it is the root of a monic integral quadratic polynomial. Let m be odd. Then clearly

$$2T_{nm}(x/2) = 2T_n(x/2) \sum_{j=0}^{m-1} (-1)^j y^{n(2j-(m-1))},$$

and by basic Galois theory, the second factor is rational and an algebraic integer, hence integral. This shows $a_n \mid a_{nm}$ for every odd m. Next suppose that n, m are both odd. We know already $a_{(n,m)} \mid (a_n, a_m)$, and we want to show equality here. Write

$$2(n,m) = rn - sm$$

with *odd* positive integers r, s. Then

$$(a_n, a_m) \mid (a_{rn}, a_{sm}) = (a_{sm+2(n,m)}, a_{sm}).$$

Applying (4.4) recursively with $(sm - 2j(n,m), (n,m))$, $j = 0, 1, \ldots$, in place of (n,m) we see that

$$(a_{sm+2(n,m)}, a_{sm}) \mid (a_{sm}, a_{sm-2(n,m)}) \mid \cdots \mid (a_{3(n,m)}, a_{(n,m)}) = a_{(n,m)},$$

as desired.

5. Rational approximants of $\sqrt{D}$

In this section we prove Theorem 1.1. Let $\alpha = \sqrt{r} \notin \mathbb{Q}$, $r \in \mathbb{Q}_{>0}$, be given. By Lemma 3.1 it suffices to find a sequence p_n/q_n, $q_1 < q_2 < \cdots$, of approximations $|\alpha - p_n/q_n| \ll 1/q_n^2$ such that p_n and q_n are simultaneously evenly divisible, and $q_n \gg q_{n+1}$. To simplify things, we observe that we can restrict r to be an integer divisible by 4, say $r = 4D$ with $D \in \mathbb{N}$ not a perfect square, since fixed rational factors can be distributed among the p_n and q_n without changing the notion of evenly divisible, nor the quality of the approximation, nor the inequality $q_n \gg q_{n+1}$.

By the theory of Pell's equation there exists a non-trivial solution $(x, y) \in \mathbb{N} \times \mathbb{N}$ to the diophantine equation

$$x^2 - Dy^2 = 1.$$

Consider the sequences

$$x_n := T_n(x), \quad y_n := yU_{n-1}(x).$$

By (4.3), these are also (obviously pairwise different) solutions of the Pell equation, since

$$x_n^2 - Dy_n^2 = T_n(x)^2 - Dy^2U_{n-1}(x)^2 = 1.$$

Therefore

$$\left|\sqrt{4D} - \frac{2x_n}{y_n}\right| \le \frac{2}{\sqrt{D}\,y_n^2} \ll \frac{1}{y_n^2}.$$

It is clear from the definition of the Chebyshev polynomials that

$$\log x_n,\, \log y_n = n \log(x + \sqrt{x^2 - 1}) + O(1) \tag{5.1}$$

for $n \to \infty$.

Now given $0 < \varepsilon < 1/2$, we can find distinct odd primes $2 < \ell_1 < \cdots < \ell_J$ coprime to y so that

$$\frac{1}{2} - \varepsilon < 1 - \prod_{j=1}^{J} \left(1 - \frac{1}{\ell_j}\right) < \frac{1}{2}. \tag{5.2}$$

This is because $\{1/\ell : \ell \text{ prime}\}$ is a zero sequence whose sum is divergent (this is a form of the Riemann rearrangement theorem). For instance, take $\ell_1 = 3, \ell_2 = 5, \ell_3 = 17, \ell_4 = 257$ with

$$1 - \prod_{j=1}^{4} \left(1 - \frac{1}{\ell_j}\right) \approx 0.499992.$$

From now on we consider indices n of the form

$$n := \ell_1 \cdot \ell_2 \cdot \cdots \cdot \ell_J \cdot P, \tag{5.3}$$

where P is any odd large positive integer coprime to $\ell_1 \cdot \cdots \cdot \ell_J$ (note that such n's are odd). Put $p_n = 2x_n$, $q_n = y_n$. By (4.6) and (4.7), $q_{n/\ell_j} \mid q_n$ and $p_{n/\ell_j} \mid p_n$ for each j. Therefore setting

$$Q := \operatorname{lcm}(q_{n/\ell_1}, \ldots, q_{n/\ell_J}), \quad P := \operatorname{lcm}(p_{n/\ell_1}, \ldots, p_{n/\ell_J}),$$

we get divisors $Q \mid q_n$ and $P \mid p_n$.

$$\left(\frac{1}{2} - \varepsilon\right) \log q_n + O(1) \le \log Q \le \left(\frac{1}{2} + \varepsilon\right) \log q_n + O(1). \tag{5.4}$$

To this end we recall the inclusion-exclusion formula for the least common multiple

$$\begin{aligned} \operatorname{lcm}(a_1, \ldots, a_r) &= \prod_{\substack{S \subseteq \{1, \ldots J\} \\ |S| \ge 1}} \gcd(\{a_j \mid j \in S\})^{(-1)^{|S|-1}}, \\ &= \prod_{1 \le j \le J} a_j \prod_{1 \le i < j \le J} \gcd(a_i, a_j)^{-1} \prod_{1 \le i < j < k \le J} \gcd(a_i, a_j, a_k) \cdots, \end{aligned}$$

so that by (4.6) we obtain

$$\begin{aligned}\log Q &= \sum_{1\le j\le J} \log q_{n/\ell_j} - \sum_{1\le i<j\le J} \log q_{\gcd(n/\ell_i, n/\ell_j)} + \cdots \\ &= \sum_{1\le j\le J} \log q_{n/\ell_j} - \sum_{1\le i<j\le J} \log q_{n/(\ell_i\ell_j)} + \cdots,\end{aligned}$$

where in the second step we used that $\ell_1, \dots, \ell_J$ are pairwise coprime divisors of n and hence

$$\gcd(\{n/\ell_j \mid j \in S\}) = \frac{n}{\prod_{j\in S} \ell_j}.$$

By (5.1) this equals

$$\begin{aligned}&\log\Big(x+\sqrt{x^2-1}\Big)\Big(\sum_{1\le j\le J}\frac{n}{\ell_j} - \sum_{1\le i<j\le J}\frac{n}{\ell_i\ell_j} + \cdots\Big) + O(1) \\ &\qquad = \log\Big(x+\sqrt{x^2-1}\Big)\, n\left(1 - \prod_{j=1}^{J}\Big(1-\frac{1}{\ell_j}\Big)\right) + O(1) \\ &\qquad = \log q_n\left(1 - \prod_{j=1}^{J}\Big(1-\frac{1}{\ell_j}\Big)\right) + O(1),\end{aligned}$$

which gives (5.4).

Likewise, $\log P \sim \frac{1}{2}\log p_n$, so that p_n is evenly divisible.

Finally we observe that the admissible indices n as in (5.3) is an integer sequence with bounded gaps (e.g., by $(\ell_1 \cdot \cdots \cdot \ell_J)^2$), and it follows directly from the definition of $q_n = yU_{n-1}(x)$ that $q_n \gg q_{n+1}$. This completes the proof of Theorem 1.1. □

6. Some other quadratic irrationalities

We can leverage our results about irrationalities of the form $\sqrt{D}$ to obtain the same result on $\delta_{\min}^{(\alpha)}(N)$ for other quadratic irrationalities.

THEOREM 6.1. – *For all positive real quadratic irrationalities of the form*

$$\alpha = \alpha(x;a,b,\varepsilon,r) = r\cdot\left(\frac{x+\sqrt{x^2+4\varepsilon}}{2}\right)^a \cdot \Big(\sqrt{x^2+4\varepsilon}\Big)^b \tag{6.1}$$

with

$$a\in\mathbb{Z},\quad b=0,1,\quad x\in\mathbb{Z}\backslash\{0\},\quad \varepsilon=\pm1,\quad r\in\mathbb{Q}^\times,$$

we have $\delta_{\min}^{(\alpha)}(N) \ll N^{-1+\varepsilon}$ *infinitely often.*

In order to have $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ we need $(a,b)\neq 0$, and in addition $x\notin\{0,\pm1,\pm2\}$ if $\varepsilon=-1$. We can also assume $x>0$, since $\alpha(-x,a,b,\varepsilon,r) = \alpha(x,-a,b,\varepsilon,(-1)^{\varepsilon a}r)$.

Note that we can display any irrationality of the form $\sqrt{D}$, with integral $D>1$ not a perfect square, as such α: Indeed, let $z^2 - Dw^2 = 1$ be a nontrivial solution to the corresponding Pell equation. Choosing

$$r = 1/(2w),\quad a=0,\quad b=1,\quad x=2z,\quad \varepsilon=-1$$

gives $\sqrt{D} = \alpha(2z, 0, 1, -1, 1/(2w))$. In particular, Theorem 1.1 is a special case of Theorem 6.1. The golden ratio $(1+\sqrt{5})/2$ is obtained by taking $r = 1, a = 1, b = 0, x = 1$, $\varepsilon = 1$. There are many other examples, but we do not know how to cover all quadratic irrationalities.

Proof. – Good rational approximants for α are obtained from the relations

$$\begin{aligned}
\alpha\Big(x, a, 0, -1, \frac{c}{d}\Big) &= \frac{cU_{n+a}(x/2)}{dU_n(x/2)} + O\Big(\frac{1}{U_n(x/2)^2}\Big),\\
\alpha\Big(x, a, 0, +1, \frac{c}{d}\Big) &= \frac{ci^{-n-a}U_{n+a}(ix/2)}{di^{-n}U_n(ix/2)} + O\Big(\frac{1}{|U_n(ix/2)|^2}\Big),\\
\alpha\Big(x, a, 1, -1, \frac{c}{d}\Big) &= \frac{c2T_{n+a}(x/2)}{dU_{n-1}(x/2)} + O\Big(\frac{1}{U_{n-1}(x/2)^2}\Big),\\
\alpha\Big(x, a, 1, +1, \frac{c}{d}\Big) &= \frac{c2i^{-n-a}T_{n+a}(ix/2)}{di^{1-n}U_{n-1}(ix/2)} + O\Big(\frac{1}{|U_{n-1}(ix/2)|^2}\Big),
\end{aligned} \tag{6.2}$$

which follow immediately from the definition of the Chebyshev polynomials. By the above remarks, this covers all α considered in Theorem 6. Notice that numerators and denominators are integers in each case. We now proceed similarly as in the previous section. We choose odd primes $2 < \ell_1 < \cdots < \ell_r$, with $\ell_i = 1 \mod 4$, so that

$$\frac{1}{2} - \varepsilon < 1 - \prod_{j=1}^{r}\Big(1 - \frac{1}{\ell_j}\Big) < \frac{1}{2}$$

and we choose another set of distinct odd primes $2 < \ell_1' < \cdots < \ell_s'$, with $\ell_j' = 3 \mod 4$, so that

$$\frac{1}{2} - \varepsilon < 1 - \prod_{j=1}^{s}\Big(1 - \frac{1}{\ell_j'}\Big) < \frac{1}{2}.$$

Put

$$L = \prod_{j=1}^{r}\ell_j, \quad L' = \prod_{j=1}^{r}\ell_j'.$$

By construction $(L, L') = 1$. We put $n + a = Lm$; moreover, in the first two cases of (6.2) we put $n + 1 = L'm'$, in the last two cases of (6.2) we put $n = L'm'$ with Lm odd and $(L, m) = (L', m') = 1$. Then by the argument of the previous section, numerators and denominators of the approximations are evenly divisible. It remains to show that we can pick infinitely many such pairs (m, m'). To this end we put

$$m' = \mu' L' + 1, \quad m = 2\mu L + 1,$$

so that $(L, m) = (L', m') = 1$ and mL is odd, and the linear diophantine equation

$$Lm - L'm' = 2L^2\mu - \mu'(L')^2 + (L - L') = b$$

has, for any $b \in \mathbb{Z}$, infinitely many pairs of solutions (μ, μ'), since $(2L^2, (L')^2) = 1$.

In certain cases we can do a little better, and we conclude this section with a proof of (1.4) for all $\alpha(x, a, 0, \pm 1, c/d)$ with a even. In this case we are dealing exclusively with Chebyshev polynomials of the second kind, for which slightly better divisibility conditions hold. In

particular, restricting the first two cases of (6.2) to odd n and assuming that a is even, the indices in numerator and denominator are odd, and it follows from (4.6) that

$$U_{(n+a-1)/2}(x/2) \mid U_{n+a}(x/2), \quad U_{(n-1)/2}(x/2) \mid U_n(x/2),$$

so that every second approximant of α has numerator and denominator that are strongly evenly divisible. □

7. Almost all α, lower bound: Proof of Theorem 1.3

Without loss of generality we will prove Theorem 1.3 for almost all $\alpha \in [1,2]$. Of course the same argument works for any other interval. For $N, X \geq 1$ and $q \in \mathbb{N}$ let

$$S_{X,N} = \{\alpha \in [1,2] \mid \delta_{\min}^{(\alpha)}(N) \leq 1/X\}$$

and

$$S_X^{(q)} = \{\alpha \in [1,2] \mid \|\alpha q\| \leq 1/X\},$$

where as usual $\|.\|$ denotes the distance to the nearest integer. Clearly

$$\mu\big(S_X^{(q)}\big) = \frac{2}{X}.$$

Then $\alpha \in S_{X,N}$ implies that there exist $m, m', n, n' \ll N^{1/2}$ such that

$$|\alpha(m^2 - m'^2) - (n'^2 - n^2)| \leq 1/X$$

and in particular there exist $u = m - m'$, $v = m + m'$ with $u, v \ll N^{1/2}$ such that

$$\alpha \in S_X^{(uv)}.$$

We conclude that

$$S_{X,N} \subseteq \bigcup_{u,v \leq CN^{1/2}} S_X^{(uv)}$$

for a suitable constant $C > 0$ (depending on α). Note that the sets $S_X^{(uv)}$ are indexed by the integers which are products $u \cdot v$ with $u, v \leq CN^{1/2}$. These are just the distinct elements in a multiplication table of side length $CN^{1/2}$. Erdős showed [7] that a multiplication table of side length X contains $o(X^2)$ different entries. We now invoke Ford's quantitative version [9, Corollary 3], which shows that the union is over $\ll N(\log N)^{-c}(\log\log N)^{-2/3}$ pairs with $c = 1 - \frac{\log(e\log 2)}{\log 2} = 0.086\ldots$. We obtain

$$\mu(S_{X,N}) \ll \frac{N}{(\log N)^c (\log\log N)^{2/3}} \cdot \frac{1}{X}.$$

Now let

$$S := \big\{\alpha \in [1,2] \mid \delta_{\min}^{(\alpha)}(N) \leq (\log N)^c/N \text{ for all sufficiently large } N\big\}.$$

Then

$$S = \liminf_{N\to\infty} S_{N/(\log N)^c, N},$$

and since

$$\mu(S_{N/(\log N)^c,N}) \ll \frac{1}{(\log\log N)^{2/3}} \to 0,$$

it is clear that $\mu(S) = 0$.

8. Almost all α: bounds for all N

In this section we prove Theorem 1.2 for almost all $\alpha \in \mathscr{J}$ (without loss of generality), where $\mathscr{J} \subseteq (0,\infty)$ is some fixed compact interval. In the following, all implied constants may depend on $\mathscr{J}$.

For $\alpha \in \mathscr{J}$, real $M \geq 1$ and $M^3 \leq T \leq M^4$, let

$$\mathscr{S}(M,T,\alpha) := \#\Big\{n_1,n_2,n_3,n_4 \asymp M : \Big|\frac{n_1n_2}{n_3n_4} - \alpha\Big| \ll \frac{1}{T}\Big\}. \tag{8.1}$$

We are interested in a lower bound for this quantity for almost all α and T as large as possible in terms of M. We will prove the following

PROPOSITION 8.1. – *For any $\eta > 0$ sufficiently small, we have $S(M, M^{4-\eta}, \alpha) \geq 1$ for all sufficiently large $M \geq M_0(\alpha)$, and all $\alpha \in \mathscr{J} \setminus \mathscr{T}_M$, where $\mathscr{T}_M$ is an exceptional set of measure $\mu(\mathscr{T}_M) \ll M^{-\rho}$ with $\rho > 0$ depending only on $\eta > 0$.*

Taking Proposition 8.1 for granted, we specialize $M = 2^\nu$, $\nu \in \mathbb{N}$, so that

$$\sum_{M=2^\nu} \mu(\mathscr{T}_M) < \infty.$$

By the Borel-Cantelli lemma we conclude $S(M, M^{4-\eta}, \alpha) \geq 1$ for almost all α, all sufficiently large $M = 2^\nu \geq M_0(\alpha)$ and η as in Proposition 8.1. It follows from Lemma 3.2 that

$$\delta^{(\alpha)}_{\min}(N) \ll N^{1-\frac{4-\eta}{2}+\varepsilon} = N^{-1+\eta/2+\varepsilon} \tag{8.2}$$

for *all* sufficiently large integers $N \geq N_0(\alpha)$. Since we allow the implied constant to depend on α, (8.2) holds in fact for all N, and the bound (1.5) in Theorem 1.2 follows.

The remainder of this section is devoted to the proof of Proposition 8.1. To prepare for the upcoming Fourier analysis, let w_1, w_2 be two non-negative smooth functions bounded by 1. We assume that w_1 takes the value 1 on some sufficiently large interval $[a_1, b_1]$ with constants $0 < a_1 < b_1$ depending on $\mathscr{J}$ and the value 0 outside $[\frac{1}{2}a_1, 2b_1]$, and that w_2 takes the value 1 on $[-1,1]$ and the value 0 outside $[-2,2]$. Note that the Fourier transform $\widehat{w}_2(y) := \int_{-\infty}^{\infty} w_2(x) e^{-2\pi i x y} dx \geq 0$ is rapidly decreasing.

Fix some small $\eta > 0$, and let as usual $\varepsilon > 0$ denote an arbitrarily small constant, not necessarily the same at each occurrence. The first key observation is that by the standard divisor bound we have

$$S(M,T,\alpha) \gg M^{-\varepsilon}\widetilde{S}(M,T,\alpha),$$

where

$$\widetilde{S}(M,T,\alpha) := \#\Big\{n_i \asymp M^{\eta/4}, m_i \asymp M^{1-\eta/4} : \Big|\frac{n_1m_1n_2m_2}{n_3m_3n_4m_4} - \alpha\Big| \ll \frac{1}{T}\Big\}.$$

Denoting $\beta = \log\alpha$, we see that $\widetilde{S}(M,T,\alpha)$ is bounded from below by

$$\sum_{\substack{n_1,n_2,n_3,n_4\\ m_1,m_2,m_3,m_4}} w_2\Big(T\Big(\log\frac{n_1m_1n_2m_2}{n_3m_3n_4m_4}-\beta\Big)\Big)\prod_{j=1}^{4} w_1\Big(\frac{n_j}{M^{\eta/4}}\Big)w_1\Big(\frac{m_j}{M^{1-\eta/4}}\Big)$$
$$= \frac{1}{T}\int_{-\infty}^{\infty} \widehat{w_2}(y/T)e^{-2\pi iy\beta}\Big|\sum_n w_1\Big(\frac{n}{M^{\eta/4}}\Big)n^{2\pi iy}\Big|^4\Big|\sum_m w_1\Big(\frac{m}{M^{1-\eta/4}}\Big)m^{2\pi iy}\Big|^4 dy$$
$$=: I_1(\beta)+I_2(\beta)$$

say, where $I_1(\beta)$ is the integral restricted to $|y|\le M^{\varepsilon}$ for some very small fixed $\varepsilon>0$, and $I_2(\beta)$ is the rest.

Let

$$\check{w}_1(s) := \int_0^{\infty} w_1(x)x^s\frac{dx}{x}$$

denote the Mellin transform of w_1. Then

$$\Sigma(N,y) := \sum_n w_1\Big(\frac{n}{N}\Big)n^{2\pi iy} = \int_{(2)} \check{w}_1(s)N^s\zeta(s-2\pi iy)\frac{ds}{2\pi i}. \tag{8.3}$$

To analyze $I_1(\beta)$ we shift the contour in (8.3) to $\operatorname{Re} s = 0$, say, and using the rapid decay of $\check{w}_1$ along vertical lines, we see that for $|y|\le M^{\varepsilon}$ and $N = M^c$ ($c=\eta/4$ or $1-\eta/4$) we have

$$\Sigma(N,y) = \check{w}_1(1+2\pi iy)N^{1+2\pi iy} + O(N^{\varepsilon}). \tag{8.4}$$

From (8.4) we conclude (using also Taylor's theorem in the second step)

$$I_1(\beta) = \frac{M^4}{T}\int_{-M^{\varepsilon}}^{M^{\varepsilon}} \widehat{w}_2\Big(\frac{y}{T}\Big)e^{-2\pi iy\beta}|\check{w}_1(1+2\pi iy)|^8dy + O\left(\frac{M^{4-\frac{\eta}{4}+\varepsilon}}{T}\right)$$
$$= c(\beta)\frac{M^4}{T} + O\left(\frac{M^{4-\frac{\eta}{4}+\varepsilon}}{T}+\frac{M^4}{T^2}\right),$$

where $c(\beta) = \widehat{w}_2(0)\int_{-\infty}^{\infty} e^{-2\pi iy\beta}|\check{w}_1(1+2\pi iy)|^8dy$. If we define $v(t) := w_1(e^t)e^t$, again a non-negative compactly supported function, then $\check{w}_1(1+2\pi iy) = \widehat{v}(-y)$, so that

$$c(\beta) = \widehat{w}_2(0)\int_{\mathbb{R}^7} v(t_1)v(t_2)\cdots v(t_7)v(-\beta+t_1+t_2+t_3+t_4-t_5-t_6-t_7)dt_1\cdots dt_7.$$

If the support of w_1 is sufficiently large, then $c(\beta)$ is bounded away from 0, uniformly for all $e^{\beta}\in\mathcal{J}$, so that

$$I_1(\beta) \gg \frac{M^4}{T}$$

uniformly in β.

It remains to show that for almost all β the contribution $I_2(\beta)$ of the large frequencies $|y|>M^{\varepsilon}$ is of lower order of magnitude. Let

$$\mathcal{J} := \Big(\int_{\log\mathcal{J}} |I_2(\beta)|^2d\beta\Big)^{1/2}.$$

Suppose we can show

$$\mathcal{J} \ll M^{4-\rho}T^{-1} \tag{8.5}$$

for $T = M^{4-\eta}$ and $\rho > 0$ possibly depending on η. Then we conclude $\mathscr{J}_2(\beta) \ll M^{4-\rho/2}T^{-1}$ for all β except for a small set $\mathscr{T}_M$ of measure $\ll M^{-\rho}$, so that for all $\alpha \in \mathscr{T} \setminus \mathscr{T}_M$,

$$S(M, M^{4-\eta}, \alpha) \gg M^{-\varepsilon} \cdot \widetilde{S}(M, M^{4-\eta}, \alpha) \gg M^{-\varepsilon} \cdot \Big(\frac{M^4}{T} + O\Big(\frac{M^{4-\rho}}{T}\Big)\Big) \gg \frac{M^{4-\varepsilon}}{T} \geq 1$$

and the proof of Proposition 8.1 is complete.

To bound $\mathscr{J}$, we note that $I_2(\beta) = \widehat{F}(\beta)$ is the Fourier transform of

$$F(y) := \mathbf{1}(|y| > M^\varepsilon)\, \frac{1}{T} \widehat{w}_2\Big(\frac{y}{T}\Big) \big|\Sigma(M^{\frac{\eta}{4}}, y)\Sigma(M^{1-\frac{\eta}{4}}, y)\big|^4.$$

Therefore

$$\mathscr{J}^2 \leq \int_{-\infty}^{\infty} |I_2(\beta)|^2 d\beta = \int_{-\infty}^{\infty} |F(y)|^2 dy$$

by Plancherel, that is

$$\mathscr{J}^2 \ll \frac{1}{T^2} \int_{|y|>M^\varepsilon} \Big|\widehat{w_2}\Big(\frac{y}{T}\Big)\Big|^2 \Big|\Sigma(M^{\frac{\eta}{4}}, y)\Big|^8 \Big|\Sigma(M^{1-\frac{\eta}{4}}, y)\Big|^8 dy. \tag{8.6}$$

Since $|y| \geq M^\varepsilon$, we may bound $\Sigma(M^{\frac{\eta}{4}}, y)$ by shifting the contour in (8.3) to $\operatorname{Re} s = 1-\eta^4$; now the pole at $s = 1 + 2\pi i y$ is negligible due to the rapid decay of $\check{w}_1$, and hence we obtain the upper bound

$$\Sigma(M^{\frac{\eta}{4}}, y) \ll M^{(1-\eta^4)\eta/4} \int_{-\infty}^{\infty} \frac{|\zeta(1-\eta^4-2\pi i y + it)|}{1+|t|^{10}} dt.$$

The crucial input is now a bound of the type

$$|\zeta(\sigma + it)| \ll |t|^{A(1-\sigma)^{3/2}+\varepsilon}, \quad 1/2 \leq \sigma \leq 1,\ |t| \geq 2$$

where both A and the implied constant are absolute. A first result of this type was proved by Richert [18] with $A = 100$; recently Heath-Brown [10] obtained $A = 1/2$. We conclude that

$$\Big|\Sigma(M^{\frac{\eta}{4}}, y)\Big| \ll M^{(1-\eta^4)\eta/4} \cdot |y|^{A\eta^6+\varepsilon},$$

so that (8.6) is bounded by

$$\begin{aligned} &\ll M^{2(1-\eta^4)\eta} \cdot \frac{1}{T^2} \int_{\mathbb{R}} \Big|\widehat{w_2}\Big(\frac{y}{T}\Big)\Big|^2 |y|^{8A\eta^6+\varepsilon} \Big|\Sigma(M^{1-\frac{\eta}{4}}, y)\Big|^8 dy \\ &\ll M^{2(1-\eta^4)\eta} \cdot \frac{1}{T^2} \cdot T^{8A\eta^6+\varepsilon} \cdot \int_{|y|\leq T^{1+\varepsilon}} \Big|\sum_{n \ll M^{4-\eta}} a(n) n^{2\pi i y}\Big|^2 dy, \end{aligned}$$

where $a(n) = \sum_{n_1\cdot\dots\cdot n_4 = n} w_1\Big(\frac{n_1}{M^{1-\frac{\eta}{4}}}\Big) \cdot \dots \cdot w_1\Big(\frac{n_4}{M^{1-\frac{\eta}{4}}}\Big) \ll n^\varepsilon$. Using the standard mean value theorem [12, Theorem 9.1]

$$\int_0^X \Big|\sum_{n\leq N} a_n n^{it}\Big|^2 dt \ll (X+N) \sum_{n\leq N} |a_n|^2,$$

we obtain for $T = M^{4-\eta}$ that

$$\mathscr{J}^2 \ll M^{2(1-\eta^4)\eta+\varepsilon} \cdot \frac{1}{T^2} \cdot T^{8A\eta^6} \cdot \Big(T + M^{4-\eta}\Big) \cdot M^{4-\eta} \ll \frac{M^{8-2\eta^5+32A\eta^6+\varepsilon}}{T^2} \ll \frac{M^{8-\eta^5}}{T^2}$$

for all sufficiently small $\eta > 0$. This shows (8.5) with $\rho = \frac{1}{2}\eta^5$ and completes the proof of Proposition 8.1. Moreover Heath-Brown's result [10] allows us to pick $A = \frac{1}{2}$ in which case any $0 < \eta < 1/16$ is admissible. □

9. Proof of Theorem 1.4

One simple reason why the sequence of eigenvalues is non-generic as far as the behavior of minimal gaps is concerned, is that it is closed under multiplication by perfect squares, hence one small gap propagates. Indeed, let $\alpha > 0$ be arbitrary, and suppose (1.7) holds, that is $\delta_{\min}^{(\alpha)}(N) \leq \frac{1}{N \log N}$ infinitely often. Let $\lambda', \lambda' \ll N$ be two eigenvalues with

$$0 < \lambda - \lambda' \leq \frac{1}{N \log N}.$$

Then obviously $\tilde{\lambda} := 4\lambda$, $\tilde{\lambda}' := 4\lambda'$ are eigenvalues with

$$0 < \tilde{\lambda} - \tilde{\lambda}' \leq \frac{4}{N \log N}, \quad \text{so that} \quad \delta_{\min,2}^{(\alpha)}(cN) \leq \frac{4}{N \log N} \qquad \text{infinitely often}$$

for some suitable constant c, violating (1.8). □

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