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REAL MILNOR FIBRES AND PUISEUX SERIES

BY GOULWEN FICHOU AND MASAHIRO SHIOTA

ABSTRACT. – Given a real polynomial function and a point in its zero locus, we defined a set consisting of algebraic real Puiseux series naturally attached to these data. We prove that this set determines the topology and the geometry of the real Milnor fibre of the function at this point. To achieve this goal, we balance between the tameness properties of this set of Puiseux series, considered as a real algebraic object over the field of algebraic Puiseux series, and its behavior as an infinite dimensional object over the real numbers.

RÉSUMÉ. – On associe à un zéro d'une fonction polynomiale réelle, un ensemble de séries de Puiseux algébriques. Cet ensemble détermine à la fois la topologie et la géométrie des fibres de Milnor de la fonction en ce point. Pour démontrer cela, on combine les propriétés de modération de cet ensemble de séries de Puiseux, considéré comme un objet de la géométrie réelle sur le corps des séries de Puiseux algébriques, avec son comportement en tant qu'espace de dimension infini sur le corps des nombres réels.

Let $\mathbb{R}$ denote the field of real numbers, and $\mathbb{N}$ denote the non-negative integers. Denote by $\tilde{\mathbb{R}}$ the field of continuous semialgebraic curve germs $\gamma : (0,\varepsilon) \to \mathbb{R}$ at 0, which we identify with the field $\tilde{\mathbb{R}} = \mathbb{R}_{alg}((t^{\mathbb{Q}}))$ of algebraic Puiseux series over $\mathbb{R}$ (cf. [1]). Recall that the subring of algebraic Puiseux series of the form $\sum_{i\in\mathbb{N}} a_i t^{i/p}$ is carried to the subring of continuous semialgebraic curve germs $[0,\infty) \to \mathbb{R}$ at 0.

Let f be a polynomial function on $\mathbb{R}^n$, with $n \in \mathbb{N}^*$, and denote by $\tilde{f} : \tilde{\mathbb{R}}^n \to \tilde{\mathbb{R}}$ the extension of f defined by $\tilde{f}(\gamma(t)) = f \circ \gamma(t)$ for $\gamma \in \tilde{\mathbb{R}}$ an algebraic Puiseux series. Let $x_0 \in \mathbb{R}^n$ be a vanishing point for f. The object of study of the present paper is the subset $\mathcal{F}_{f,x_0} \subset \tilde{\mathbb{R}}^n$ of continuous semialgebraic curve germs $\gamma : (\mathbb{R},0) \to (\mathbb{R}^n, x_0)$ such that $f \circ \gamma(t) = t$, namely

$$\mathcal{F}_{f,x_0} = \{\gamma \in \tilde{\mathbb{R}}^n : \ \gamma(0) = x_0, \ \ f \circ \gamma(t) = t\}.$$

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We aim to relate the topology and the geometry of $\mathcal{F}_{f,x_0}$ to the topology and the geometry of the Milnor fibre associated with the real polynomial function f at x_0, which can be described as the semialgebraic set

$$F_{r,a}(f) = \{x \in \mathbb{R}^n : \ |x - x_0| < r, \ \ f(x) = a\}$$

for $0 < a \ll r \ll 1$.

The local study of the singular points of complex hypersurfaces has a rich story initiated by J. Milnor in its fundamental book [15], where he established the so-called Milnor fibration Theorem. More recently, the development of Motivic Integration [6] has brought a new enlightenment on the subject, via the motivic zeta function introduced by J. Denef and F. Loeser, together with the analytic Milnor fibre defined by J. Nicaise and J. Sebag [16]. Even more recently, E. Hrushovski and F. Loeser have established a direct connection between the analytic Milnor fibre and the topological Milnor fibre, together with the motivic Milnor fibre [10], passing through the integration into valued fields developed by E. Hrushovski and D. Kazhdan [9].

In the real context, the action of the monodromy operator on the Milnor fibre disappears, which gives rise to the notion of positive and negative Milnor fibres, as studied for example by C. McCrory and A. Parusiński [13]. However the motivic counterpart, initiated by G. Comte and G. Fichou [2] and Y. Yin [22], does not provide yet a full understanding of the global feature. Our motivation to consider the set $\mathcal{F}_{f,x_0}$ is to study a naive real version of the analytic Milnor fibre. Looking at points, we obtain a set of real algebraic Puiseux series, which can almost (or more precisely weakly, see below) be considered as a classical semialgebraic object in real algebraic geometry, replacing the field of real numbers by the real closed field of algebraic Puiseux series over $\mathbb{R}$. Note that, if the definition of $\mathcal{F}_{f,x_0}$ makes sense for any continuous semialgebraic function f, however $\mathcal{F}_{f,x_0}$ is not necessarily a semialgebraic subset of $\tilde{\mathbb{R}}^n$ due to the condition that the arcs considered have their origin in $x_0 \in \mathbb{R}^n$. This condition can be described by a valuation condition, saying that we consider those arcs with strictly positive valuation after the translation by $\gamma \mapsto \gamma - x_0$. Such sets are sometimes called T-convex, or weakly o-minimal [12].

Our aim in this paper is to show that the topology and the geometry of $\mathcal{F}_{f,x_0}$, which is definitively a natural and intrinsic object in real geometry, determine the topology and the geometry of the (positive) real Milnor fibre of f at x_0, which can be considered as a semialgebraic set, but only well-defined up to the choice of (sufficiently small) constants. Note that a similar result holds for the negative Milnor fibre, a negative sign in front of t being necessary in the definition of $\mathcal{F}_{f,x_0}$ in that case. We prove also that some natural homology groups of $\mathcal{F}_{f,x_0}$ coincide with the classical homology groups of the (positive) real Milnor fibre. The main achievement of the paper states that $\mathcal{F}_{f,x_0}$ completely determines the (positive) real Milnor fibre up to semialgebraic homeomorphism (cf Theorem 4.1.(1)), and determines it up to Nash diffeomorphism in dimensions different from 5 and 6 (cf Theorem 4.1.(2)).

As a real closed field, $\tilde{\mathbb{R}}$ is naturally equipped with the ordered topology which coincides with the t-adic topology. But we may also regard $\tilde{\mathbb{R}}$ as a $\mathbb{R}$-vector subspace of $\mathbb{R}^{\mathbb{Q}} = \prod_{r\in\mathbb{Q}} \mathbb{R}\, t^r$, and consider $\mathbb{R}^{\mathbb{Q}}$ as a topological space with the product topology. Then $\tilde{\mathbb{R}}$ may also be equipped with the induced topology.

In the paper, we balance between the tameness properties of $\mathcal{F}_{f,x_0}$ as a subset of $\tilde{\mathbb{R}}^n$ close to be semialgebraic (up to the valuation condition), and its behavior as a topological set of infinite dimension in $\mathbb{R}^{\mathbb{Q}}$. This situation leads to the study in the first part of the paper of the notion of weak continuity, and its relationship with the continuity of semialgebraic maps defined over $\tilde{\mathbb{R}}$. We discuss in part 2 the associated homology theories, preparing the material for the comparison of homologies given in the third part as Theorem 3.6. In the fourth part, we focus on the semialgebraic characterization of the real Milnor fibre, using notion from piecewise linear topology [17]. The last part is dedicated to the Nash characterization, using the theory of topological microbundles of J. Milnor [14].

Note that all the results in the paper work verbatim for a Nash function in place of a polynomial function f, contrary to the case of a real analytic function for the reason that a globally subanalytic triviality theorem, analog to the Nash triviality theorem in [4], is not yet available.

We denote by $\mathbb{R}$ the field of real numbers, and by $\tilde{\mathbb{R}}$ the field of real Puiseux series in one variable that are algebraic over $\mathbb{R}[X]$. The field $\tilde{\mathbb{R}}$ is a real closed field, a non zero element $\gamma \in \tilde{\mathbb{R}}$ is given (uniquely if we impose that q is the smallest common denominator of the exponents) by

$$\gamma(t) = \sum_{i \geq p} a_i t^{i/q}, \quad a_i \in \mathbb{R},\ a_p \neq 0,\ (p,q) \in \mathbb{Z} \times \mathbb{N}^*,$$

and γ is positive if $a_p > 0$.

To distinguish an interval (a, b) in $\mathbb{R}$ and in $\tilde{\mathbb{R}}$, we write (a, b) in $\tilde{\mathbb{R}}$ as $(a, b)_{\tilde{\mathbb{R}}}$. A semialgebraic set is a semialgebraic set over $\mathbb{R}$ and an $\tilde{\mathbb{R}}$-semialgebraic set is a semialgebraic over $\tilde{\mathbb{R}}$. For a semialgebraic set $X \subset \mathbb{R}^n$, denote by $\tilde{X}$ the set of germs at $0 \in \mathbb{R}$ of continuous semialgebraic functions from $(0; \infty)$ to X. For $x \in X$, we denote $\tilde{x} \in \tilde{X}$ the germ of the constant function equal to x. Let X and Y be semialgebraic sets and $h : X \to Y$ be a continuous semialgebraic map. Let $\tilde{h} : \tilde{X} \to \tilde{Y}$ be defined by $\tilde{h}(\gamma(t)) = h \circ \gamma(t)$ for $\gamma \in \tilde{\mathbb{R}}$. We know from [8] that $\tilde{h}$ is continuous for the t-adic topology, but not necessarily for the product topology.

We define $\mathfrak{m}_+ \subset \tilde{\mathbb{R}}$ to be the set of infinitely small positive elements in $\tilde{\mathbb{R}}$, namely:

$$\mathfrak{m}_+ = \{\gamma \in \tilde{\mathbb{R}} :\ \ 0 < \gamma < \tilde{x} \text{ for all } x \in (0, +\infty)\}.$$

1. Weak continuity

A *Nash manifold* is a semialgebraic C^∞-submanifold of some $\mathbb{R}^n$ and a *Nash map* between Nash manifolds is a C^∞-map with semialgebraic graph. The Milnor fibre $F_f(r, a)$ as considered in the introduction is a Nash manifold with boundary, however the set $\mathcal{F}_{f,x_0}$ is not so, even by changing $\mathbb{R}$ with the field $\tilde{\mathbb{R}}$ of algebraic Puiseux series over $\mathbb{R}$. We will regard $\mathcal{F}_f$ as a local $\tilde{\mathbb{R}}$-Nash manifold, see Definition 1.1 below. Properties of semialgebraic sets, Nash manifolds and Nash maps are explained in [1] and [18]. We will recall some of them for the convenience of the reader.

For any ordered field R, we define a topology on R by considering as open sets the open intervals, and we called it the R-topology. We denote by $(a, b)_R$ the open interval defined by a and b in R, and called it an R-interval in order to emphasize again the dependence on R (if R is not the field of real numbers). In the same way as in the real number case, we define R-polynomial functions on R^n, R-algebraic sets in R^n, R-semialgebraic sets in R^n and R-continuous maps from R^n to R^m.

If R is a real closed field, we call a map $\phi : R \to R$ of *class* R-C^1 if, for every $x_0 \in R$, the difference quotient

$$\frac{\phi(x + x_0) - \phi(x_0)}{x}$$

converges in R as x tends to 0 in R. In the same way, we define an R-C^k map $R^n \to R^m$, an R-Nash manifold and an R-Nash map. Note that an R-Nash manifold M admits a finite system of R-Nash coordinate neighborhoods of the form $R^n \to M$.

Consider the case $R = \tilde{\mathbb{R}}$. The set $\mathfrak{m}_+ \subset \tilde{\mathbb{R}}$ of infinitely small positive elements is open but is not $\tilde{\mathbb{R}}$-semialgebraic. This lack of semialgebraicity leads us to enlarge the sets under consideration as follows.

Definition 1.1. – A *local* $\tilde{\mathbb{R}}$*-Nash manifold* is a subset of an $\tilde{\mathbb{R}}$-Nash manifold $\mathscr{M}$ of the form $\phi^{-1}(\mathfrak{m}_+)$ for some positive $\tilde{\mathbb{R}}$-Nash function ϕ on $\mathscr{M}$ (note that an $\tilde{\mathbb{R}}$-Nash manifold is in particular a local $\tilde{\mathbb{R}}$-Nash manifold). A *local* $\tilde{\mathbb{R}}$*-semialgebraic* $\tilde{\mathbb{R}}$*-continuous map* $\mathscr{M}_1 \to \mathscr{M}_2$ between local $\tilde{\mathbb{R}}$-Nash manifolds is the restriction to $\mathscr{M}_1$ of an $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-continuous map between the ambient $\tilde{\mathbb{R}}$-Nash manifolds.

Example 1.2. – Let f be a polynomial function on $\mathbb{R}^n$. Then the set

$$\mathscr{A}_f = \{\gamma \in \tilde{\mathbb{R}}^n : \tilde{f}(\gamma) = t\}$$

is a non-singular $\tilde{\mathbb{R}}$-algebraic set and hence a $\tilde{\mathbb{R}}$-Nash manifold because the critical value set of f is finite, and therefore there exist no non-real critical value of $\tilde{f}$. Moreover the subset

$$\{\gamma \in \mathscr{A}_f : \gamma(0) = 0\}$$

is equal to $\phi^{-1}(\mathfrak{m}_+)$, where $\phi : \mathscr{A}_f \to \tilde{\mathbb{R}}$ is defined by $\phi(\gamma) = |\gamma|$, and thus it is a local $\tilde{\mathbb{R}}$-Nash manifold. Such a set is sometimes called T-convex or definable in a weakly o-minimal structure [12].

1.1. Topology on $\tilde{\mathbb{R}}$

The $\tilde{\mathbb{R}}$-topology on $\tilde{\mathbb{R}}$ has many good properties when we treat $\tilde{\mathbb{R}}$ as an abstract real closed field (cf. [1]). However, this is not the case in this paper where we see the elements of $\tilde{\mathbb{R}}^n$ as germs of curves in $\mathbb{R}^n$. For instance, the topology on $\mathbb{R}$ induced from the $\tilde{\mathbb{R}}$-topology on $\tilde{\mathbb{R}}$ is discrete. We need to introduce another topology on $\tilde{\mathbb{R}}$, which we call *the product topology*.

Describe an element $\gamma \in \tilde{\mathbb{R}}$ as

$$\gamma(t) = \sum_{i \geq p} a_i t^{i/q}, \quad a_i \in \mathbb{R},\ (p, q) \in \mathbb{Z} \times \mathbb{N}^*$$

and regard $\tilde{\mathbb{R}}$ as a $\mathbb{R}$-vector subspace of $\Pi_{r \in \mathbb{Q}} \mathbb{R}\, t^r$ by the correspondence

$$\gamma \to (..., a_i t^{i/q}, ...) \in \cdots \times \mathbb{R}\, t^{i/q} \times \cdots$$

(which does not depend on the description of γ). Give the product topology to $\Pi_{r \in \mathbb{Q}} \mathbb{R}\, t^r$ and the induced topology on $\tilde{\mathbb{R}}$. For an $\tilde{\mathbb{R}}$-Nash manifold $\mathscr{M}$ included in $\tilde{\mathbb{R}}^n$, we give to $\mathscr{M}$ the induced topology.

However there are not enough many continuous maps in the product topology for our purpose. Indeed, an $\tilde{\mathbb{R}}$-Nash map is not necessary continuous in this topology. For example, the map $\tilde{\mathbb{R}} \ni \gamma \to \gamma^2 \in \tilde{\mathbb{R}}$ is not continuous. Actually, for $\gamma_l = lt^{1/l} + t^{1-1/l}$, with $l \in \mathbb{N}^*$, we have $\gamma_l \to 0$ because when we fix a finite number of exponents, the corresponding coefficients are equal to zero for l big enough, whereas $\gamma_l^2 \to \infty$ as $l \to \infty$ since the coefficient of t is equal to $2l$.

We need to introduce a weaker notion of continuity. Actually, we introduce two kinds of weaker continuity, which happen to be equivalent to each other in the semialgebraic context (cf. Proposition 1.6). Our main result, Theorem 4.1.(2), holds for continuity in this topology.

For $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$, let $\tilde{\mathbb{R}}_{p,q}$ denote the subset of $\tilde{\mathbb{R}}$ consisting of all Puiseux series that admit a description of the form $\sum_{i \geq p} a_i t^{i/q}$, with $a_i \in \mathbb{R}$.

Definition 1.3. – We call a map $\phi : \tilde{\mathbb{R}}^n \to \tilde{\mathbb{R}}^m$ *weakly continuous* if, for any $(p,q) \in \mathbb{Z} \times \mathbb{N}^*$, its restriction

$$\phi_{|\tilde{\mathbb{R}}^n_{p,q}} : \tilde{\mathbb{R}}^n_{p,q} \to \tilde{\mathbb{R}}^m$$

to $\tilde{\mathbb{R}}^n_{p,q}$ is continuous with respect to the induced topology on $\tilde{\mathbb{R}}^n_{p,q}$ and the product topology on $\tilde{\mathbb{R}}^m$.

We call ϕ *finitely continuous* if $\phi_{|\mathcal{D}} : \mathcal{D} \to \tilde{\mathbb{R}}^m$ is continuous for any finite dimensional $\mathbb{R}$-linear subspace $\mathcal{D}$ of $\tilde{\mathbb{R}}^n$.

Note that this definition is somehow natural from a singularity theory point of view, thinking for example that a sequence of C^∞-maps converges if the sequence of the maps itself, together with the sequences of sufficiently many derivatives of the maps, converge.

Example 1.4. – A $\tilde{\mathbb{R}}$-polynomial map is a weakly and finitely continuous map. Indeed, for $P \in \tilde{\mathbb{R}}[X_1, \ldots, X_n]$, providing that γ belongs to $\tilde{\mathbb{R}}^n_{p,q}$, the t^k-coefficient of $P(\gamma)$ depends only on a finite number of coefficients of γ. This proves the weak continuity, and the finite continuity can be proved in the same way.

In the remaining of this section, we discuss the relationships between $\tilde{\mathbb{R}}$-continuity, weakly continuity and finitely continuity for $\tilde{\mathbb{R}}$-semialgebraic maps.

Lemma 1.5. – *Let $\phi : \tilde{\mathbb{R}}^n \longrightarrow \tilde{\mathbb{R}}$ be a $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-continuous function. Then for any $(p,q) \in \mathbb{Z} \times \mathbb{N}^*$, there exists $(r,s) \in \mathbb{Z} \times \mathbb{N}^*$ such that*

$$\phi(\tilde{\mathbb{R}}^n_{p,q}) \subset \tilde{\mathbb{R}}_{r,s} .$$

Proof. – Note first that there exists a non zero polynomial $P \in \tilde{\mathbb{R}}[X_1, \ldots, X_n, X]$ such that $P(\gamma, \phi(\gamma)) = 0$ for any $\gamma \in \tilde{\mathbb{R}}^n$ since ϕ is $\tilde{\mathbb{R}}$-semialgebraic. Let $\tilde{\mathbb{C}}$ denote the algebraic closure of $\tilde{\mathbb{R}}$, that is the complex algebraic Puiseux series field. For a fixed $\gamma \in \tilde{\mathbb{R}}^n$, the number of $\tilde{\mathbb{C}}$-solutions of $P(\gamma, X) = 0$ is finite, and we can choose $l \in \mathbb{N}$ so that this number is less than or equal to l for any choice of $\gamma \in \tilde{\mathbb{R}}^n$, thanks to the semialgebraicity of ϕ. First we prove that

$$\phi(\tilde{\mathbb{R}}^n_{p,q}) \subset \bigcup_{j=1}^{l} \bigcup_{r \in \mathbb{Z}} \tilde{\mathbb{R}}_{r,j} .$$

Otherwise there exist $k > l$, $p_0 \in \mathbb{Z}$ and $\gamma \in \tilde{\mathbb{R}}^n_{p,q}$ such that the equation $P(\gamma, X) = 0$ admits a solution δ of the form

$$\delta \in \tilde{\mathbb{R}}_{p_0,k} \setminus \bigcup_{j=1}^{k-1} \tilde{\mathbb{R}}_{p_0,j} .$$

But in that case the number of $\tilde{\mathbb{C}}$-roots of $P(\gamma, X)$ is strictly greater than l, a contradiction.

Note moreover that, if s denotes the lowest common multiple of $1, \dots, l$, then

$$\bigcup_{j=1}^{l} \bigcup_{r\in\mathbb{Z}} \tilde{\mathbb{R}}_{r,j} \subset \bigcup_{r\in\mathbb{Z}} \tilde{\mathbb{R}}_{r,s} .$$

Finally, note that $\tilde{\mathbb{R}}_{p,q}$ is included in $[-t^{(p-1)/q}, t^{(p-1)/q}]$ so $\tilde{\mathbb{R}}_{p,q}$ is bounded, and thus the image of $\tilde{\mathbb{R}}^n_{p,q}$ by the $\tilde{\mathbb{R}}$-continuous function ϕ is bounded in $\tilde{\mathbb{R}}$. This boundedness of $\phi(\tilde{\mathbb{R}}^n_{p,q})$ enables to conclude that there exists $r \in \mathbb{Z}$ such that $\phi(\tilde{\mathbb{R}}^n_{p,q}) \subset \tilde{\mathbb{R}}_{r,s}$. □

Proposition 1.6. – *Let $\phi : \tilde{\mathbb{R}}^n \longrightarrow \tilde{\mathbb{R}}^m$ be a $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-continuous map. Then ϕ is linearly continuous if and only if ϕ is weakly continuous.*

Proof. – It is sufficient to prove the case $m = 1$.

Assume ϕ is weakly continuous. Any finite set of points in $\tilde{\mathbb{R}}^n$ is included in some $\tilde{\mathbb{R}}^n_{p,q}$, and the $\mathbb{R}$-linear space generated by those vectors is also included in $\tilde{\mathbb{R}}^n_{p,q}$, since $\tilde{\mathbb{R}}^n_{p,q}$ is also a $\mathbb{R}$-linear space. As a consequence ϕ is finitely continuous.

Assume now that ϕ is finitely continuous, and take $(p, q) \in \mathbb{Z}\times\mathbb{N}^*$. Note that, as a $\tilde{\mathbb{R}}$-continuous $\tilde{\mathbb{R}}$-semialgebraic map on the bounded set $\tilde{\mathbb{R}}^n_{p,q}$, the function ϕ is uniformly $\tilde{\mathbb{R}}$-continuous, namely

$$\forall k \in \mathbb{N}, \exists l \in \mathbb{N}, \forall \gamma \in \tilde{\mathbb{R}}^n_{p,q}, \ \ \phi(\gamma + [-t^l, t^l]) \subset \phi(\gamma) + [-t^k, t^k].$$

Moreover, there exists $(r, s) \in \mathbb{Z}\times\mathbb{N}^*$ such that

$$\phi(\tilde{\mathbb{R}}^n_{p,q}) \subset \tilde{\mathbb{R}}_{r,s}$$

by Lemma 1.5. In particular, for any $j \geq r$, the $t^{j/s}$-coefficient of the image $\phi(\gamma)$ in $\tilde{\mathbb{R}}$ of a series $\gamma \in \tilde{\mathbb{R}}^n_{p,q}$ is decided by a finite number of coefficients of γ. More precisely, if we denote by $\sum_{i\geq p} a_i t^{i/q}$, with $a_i \in \mathbb{R}^n$, the elements of $\tilde{\mathbb{R}}^n_{p,q}$ and by $\sum_{i\geq r} b_i t^{i/s}$, with $b_i \in \mathbb{R}$, the elements of $\tilde{\mathbb{R}}_{r,s}$, we can consider ϕ as a map

$$(a_1, a_2, \dots) \mapsto (b_1, b_2, \dots).$$

Then, for any $j \geq r$ there exists $i_j \in \mathbb{N}$ such that the map ϕ induces a map

$$(a_1, a_2, \dots, a_{i_j}) \mapsto b_j$$

which is continuous by finite continuity of ϕ. Since we consider only a finite number of such j in the product topology, we conclude that ϕ is continuous on $\tilde{\mathbb{R}}^n_{p,q}$, and so ϕ is weakly continuous. □

Lemma 1.7. – *Let $\phi : \tilde{\mathbb{R}}^n \longrightarrow \tilde{\mathbb{R}}$ be a $\tilde{\mathbb{R}}$-semialgebraic function. If ϕ is weakly continuous, then ϕ is $\tilde{\mathbb{R}}$-continuous.*

Proof. – Using the curve selection lemma (Theorem 2.5.5 in [1]), it is sufficient to treat the case $n = 1$. The function ϕ is piecewise $\tilde{\mathbb{R}}$-continuous as a $\tilde{\mathbb{R}}$-semialgebraic function. Assume that $\phi(0) = 0$ and that ϕ is not $\tilde{\mathbb{R}}$-continuous at 0. Let assume that ϕ admits a finite limit $\beta \in \tilde{\mathbb{R}}^*$ when approaching 0 from above (note that the proof is similar if the limit is infinite), namely

$$\forall \varepsilon \in \tilde{\mathbb{R}}^*_+, \exists \eta_\varepsilon \in \tilde{\mathbb{R}}^*_+, \forall \gamma \in (0, \eta_\varepsilon], \ |\phi(\gamma) - \beta| < \varepsilon.$$

Assume $\beta(t) = bt^{p/q} + \cdots$ with $b \in \mathbb{R}^*$, and choose $\varepsilon = t^k$ with $k > p/q$. Then for $l \in \mathbb{N}$ big enough, the series t^l belongs to $(0, \eta_{t^k}]$, so that $\phi(t^l) = bt^{p/q} + \cdots$. But the series t^l also belongs to $\tilde{\mathbb{R}}_{0,1}$, so by weak continuity of ϕ the $t^{p/q}$-coefficient of $\phi(t^l)$ should converge to 0 as l goes to infinity, and this gives a contradiction with the expression of $\phi(t^l)$. □

PROPOSITION 1.8. – *The composition of weakly continuous $\tilde{\mathbb{R}}$-semialgebraic maps is weakly continuous.*

Proof. – By Lemma 1.7 such maps are $\tilde{\mathbb{R}}$-continuous, so we can use Lemma 1.5 to conclude. □

1.2. Weak continuity in dimension one

In this section, we discuss some properties of weak continuity specific to the one dimensional case. The results exposed here will be useful for the study of the general case in next section.

Let us begin with an illustrative example.

EXAMPLE 1.9. – Let $\phi : [0, +\infty)_{\tilde{\mathbb{R}}} \longrightarrow \tilde{\mathbb{R}}$ denote the function defined by $\phi(\gamma) = \gamma^{p/q}$ with $p, q \in \mathbb{N}^*$. If ϕ is weakly continuous, then $p/q \in \mathbb{N}$.

Actually, suppose on the contrary that $p/q \notin \mathbb{N}$. Denote $k = [p/q]$, where $[\cdot]$ stands for the floor function. Note that ϕ is of class C^∞ on $\tilde{\mathbb{R}}^*_+$, so that for $x \in \mathbb{R}^*_+$ there exists $\delta \in [0, 1]_{\tilde{\mathbb{R}}}$ such that

$$\phi(\tilde{x} + t) = \phi(x) + \phi'(x)t + \cdots + \frac{\phi^{(k+1)}(x)t^{k+1}}{(k+1)!} + \frac{\phi^{(k+2)}(\tilde{x} + \delta(t)t)t^{k+2}}{(k+2)!}$$

by the mean value theorem. Due to the particular form of ϕ, note that $\phi(x), \cdots, \phi^{(k)}(x)$ tends to 0 in $\mathbb{R}$ as x tends to 0 in $\mathbb{R}$, whereas $\phi^{(k+1)}(x)$ tends to infinity. Note moreover that $\phi^{(k+2)}(\tilde{x}+\delta(t)t)$ is continuous at $t = 0$, so that the t^{k+1}-coefficient of $\phi^{(k+2)}(\tilde{x}+\delta(t)t)t^{k+2}$ is zero, and thus the t^{k+1}-coefficient of the right hand side of the equality above tends to infinity as x goes to 0 in $\mathbb{R}$. This fact leads to a contradiction because on the left hand side, since $\tilde{x} + t \in \tilde{\mathbb{R}}_{0,1}$ and ϕ is continuous for the product topology on $\tilde{\mathbb{R}}_{0,1}$, the t^{k+1}-coefficient of $\phi(\tilde{x} + t)$ should converge in $\mathbb{R}$ as x goes to 0 in $\mathbb{R}$.

Next two results give some relations between $\tilde{\mathbb{R}}$-Nash functions and weakly continuous ones.

LEMMA 1.10. – *Let $\phi : [0, 1]_{\tilde{\mathbb{R}}} \longrightarrow \tilde{\mathbb{R}}$ be a $\tilde{\mathbb{R}}$-Nash function. Then there exists $\varepsilon \in (0, 1]_{\tilde{\mathbb{R}}}$ such that ϕ is weakly continuous on $[0, \varepsilon]_{\tilde{\mathbb{R}}}$.*

Proof. – Let assume that there exist $\varepsilon \in (0,1]_{\tilde{\mathbb{R}}}$ and $\beta \in (0,\infty)_{\tilde{\mathbb{R}}}$ such that

$$\forall \gamma \in [0,\varepsilon]_{\tilde{\mathbb{R}}}, \forall k \in \mathbb{N}, \ \ |\phi^{(k)}(\gamma)| \leq \beta^k.$$

Then we can prove that ϕ is linearly continuous on $[0,\varepsilon]_{\tilde{\mathbb{R}}}$, and so weakly continuous by Proposition 1.6. Actually, for $\gamma_1,\dots,\gamma_l \in [0,\varepsilon]_{\tilde{\mathbb{R}}}$, we are going to prove that ϕ is continuous on the $\mathbb{R}$-linear space $\mathscr{D}$ generated by $\gamma_1,\dots,\gamma_l$. Note that $\mathscr{D}$ is included in $[-\varepsilon,\varepsilon]_{\tilde{\mathbb{R}}}$, shrinking ε if necessary. Choose $\delta \in \mathscr{D}$. Since the mean value theorem holds for ϕ, for any $k \in \mathbb{N}$ and any $\gamma \in \mathscr{D}$, there exists $\theta \in [0,1]_{\tilde{\mathbb{R}}}$ such that

$$\phi(\delta+\gamma)-\phi(\delta) = \phi'(\delta)\gamma + \cdots + \frac{\phi^{(k)}(\delta)}{k!}\gamma^k + \frac{\phi^{(k+1)}(\delta+\theta\gamma)}{(k+1)!}\gamma^{k+1}.$$

Fix $r \in \mathbb{Q}$. We want to prove that the t^r-coefficient of $\phi(\delta+\gamma)-\phi(\delta)$ goes to zero in $\mathbb{R}$ as γ goes to zero in $\mathscr{D}$. Let us write $\gamma = \sum_{j=1}^{l} b_j\gamma_j$, with $b_1,\dots,b_l \in \mathbb{R}$, and $\gamma_j = \sum_{i\in\mathbb{N}} c_{j,i}t^{i/q}$, with $c_{j,i} \in \mathbb{R}$ and $q \in \mathbb{N}^*$ a common denominator for $\gamma_1,\dots,\gamma_l$ (note that the index i runs in $\mathbb{N}$ because $\gamma_1,\dots,\gamma_l \in [0,\varepsilon]_{\tilde{\mathbb{R}}}$).

Shrinking ε if necessary, let assume that $\varepsilon\beta < t$. Now $\delta+\theta\gamma \in [0,\varepsilon]_{\tilde{\mathbb{R}}}$, so that

$$|\phi^{(k+1)}(\delta+\theta\gamma)\gamma^{k+1}| \leq \beta^{k+1}|\gamma^{k+1}| \leq \beta^{k+1}\varepsilon^{k+1} < t^{k+1}.$$

Then, choosing k such that for $k+1 > r$, the t^r-coefficient of $\phi^{(k+1)}(\delta+\theta\gamma)\gamma^{k+1}$ is equal to zero. Therefore the t^r-coefficient of $\phi(\delta+\gamma)-\phi(\delta)$ is a polynomial in $b_1,\dots,b_l$, which proves the linear continuity of ϕ.

To achieve the proof, it remains to justify the existence of ε and β such that the inequality upstairs is valid. There exists a non zero polynomial $P \in \tilde{\mathbb{R}}[x,y]$ such that $P(\gamma,\phi(\gamma)) = 0$ since ϕ is $\tilde{\mathbb{R}}$-semialgebraic. Set $P(x,y) = P_0(x)y^d + \cdots + P_d(x)$ and $P_0(x) = x^e Q_0(x)$, with $e \in \mathbb{N}$ and $Q_0 \in \tilde{\mathbb{R}}[x]$ verifying $Q_0(0) \neq 0$. For $s \in \mathbb{N}$, multiply the equality $P(\gamma,\phi(\gamma)) = 0$ by γ^{sd-e} so that we obtain

$$Q_0(\gamma)\psi^d(\gamma) + \gamma^{s-e}P_1(\gamma)\psi^{d-1}(\gamma) + \cdots + \gamma^{sd-e}P_d(\gamma) = 0,$$

where $\psi(\gamma) = \gamma^s\phi(\gamma)$. Fix $s > e$ and set $Q_i(\gamma) = \gamma^{si-e}P_i(\gamma)$ for $i \in \{1,\dots,d\}$. Then the polynomial $Q(x,y) = Q_0(x)y^d + \cdots + Q_d(x)$ satisfies $Q(\gamma,\psi(\gamma)) = 0$ with $Q_0(0) \neq 0$ and $Q_i(0) = 0$ for $i \in \{1,\dots,d\}$. Note that it is sufficient to prove the result for ψ instead of ϕ. By deriving $Q(\gamma,\psi(\gamma))$, we find

$$0 = \psi'(dQ_0\psi^{d-1} + \cdots + Q_{d-1}) + Q_0'\psi^d + \cdots + Q_d'.$$

If ψ is not constant on a neighborhood of zero, there exists $\alpha \in \tilde{\mathbb{R}}_+^*$ such that

$$|dQ_0\psi^{d-1} + \cdots + Q_{d-1}| \geq \alpha$$

on $[0,\varepsilon]_{\tilde{\mathbb{R}}}$ for $\varepsilon \in \tilde{\mathbb{R}}_+^*$ sufficiently small. Then

$$|\psi'| \leq \frac{1}{\alpha}|Q_0'\psi^d + \cdots + Q_d'|$$

on $[0,\varepsilon]_{\tilde{\mathbb{R}}}$, and define β to be the maximum on $[0,\varepsilon]_{\tilde{\mathbb{R}}}$ of the function in the right hand side of the inequality. Repeating the derivation of $Q(\gamma,\psi(\gamma))$, we see that

$$|\psi''| \leq \frac{1}{\alpha}|\psi'\big((d(d-1)Q_0\psi^{d-2} + \cdots + 2Q_{d-2}) + (dQ_0' + \cdots + Q_1')\big) + Q_0''\psi^d + \cdots + Q_d''|$$

so that $|\psi''| \leq \beta^2$ on $[0,\varepsilon]_{\tilde{\mathbb{R}}}$, by enlarging β if necessary. We conclude that we can find a common β such that $\psi^{(k)} \leq \beta^k$ on $[0,\varepsilon]_{\tilde{\mathbb{R}}}$ for all $k \in \mathbb{N}$ because the k-derivatives of Q_i, for $i \in \{0,\dots,d\}$, vanish for k big enough since the Q_i are polynomials. □

Now we show that a weakly continuous $\tilde{\mathbb{R}}$-semialgebraic function is not only $\tilde{\mathbb{R}}$-continuous as in Lemma 1.7, but moreover $\tilde{\mathbb{R}}$-Nash. The basic idea for the proof comes from Example 1.9.

Proposition 1.11. – *Let $\phi : \tilde{\mathbb{R}} \longrightarrow \tilde{\mathbb{R}}$ be a $\tilde{\mathbb{R}}$-semialgebraic function. If ϕ is weakly continuous, then ϕ is $\tilde{\mathbb{R}}$-Nash.*

Proof. – As a semialgebraic function, the function ϕ is $\tilde{\mathbb{R}}$-Nash on the open intervals of a finite partition of $\tilde{\mathbb{R}}$. The function ϕ is also $\tilde{\mathbb{R}}$-continuous by Lemma 1.7. As a first step, we are going to prove that if ϕ is $\tilde{\mathbb{R}}$-Nash on $(0,\alpha]_{\tilde{\mathbb{R}}}$, then it is $\tilde{\mathbb{R}}$-Nash on $[0,\alpha]_{\tilde{\mathbb{R}}}$. In a second step, we prove the Nash property at 0. Assume $\phi(0) = 0$.

Step 1. – First of all, there exist $q \in \mathbb{N}^*$ and $\varepsilon \in \tilde{\mathbb{R}}_+^*$ so that the function $\psi : \gamma \mapsto \phi(\gamma^q)$ is Nash on $[0,\varepsilon]_{\tilde{\mathbb{R}}}$ since ϕ is $\tilde{\mathbb{R}}$-semialgebraic (Proposition 8.1.12 in [1]). Choose q as small as possible. If $q = 1$ we are done, so assume $q > 1$. We have $\psi(0) = 0$, and denote by $p \in \mathbb{N}$ the order of ψ at 0. Let assume that $\psi^{(p)}(0) = 1$, multiplying ϕ (and ψ) by a positive constant if necessary. We can assume moreover that q does not divide p (otherwise subtract to ψ the corresponding Taylor polynomial).

Finally, we suppose that there exists a real number $x \in \mathbb{R}_+^*$ such that $\tilde{x}$ belongs to $(0,\varepsilon]_{\tilde{\mathbb{R}}}$, since otherwise the multiplication by ε gives a $\tilde{\mathbb{R}}$-Nash diffeomorphism $\xi : [0,1]_{\tilde{\mathbb{R}}} \longrightarrow [0,\varepsilon]_{\tilde{\mathbb{R}}}$ which is moreover weakly continuous, so that $\phi \circ \xi$ is weakly continuous on $[0,1]_{\tilde{\mathbb{R}}}$.

Denote $k = [p/q]$. By the mean value theorem, there exists $\delta \in [0,1]_{\tilde{\mathbb{R}}}$ such that

$$\phi(\tilde{x}+t) = \phi(\tilde{x}) + \phi'(\tilde{x})t + \cdots + \frac{\phi^{(k+1)}(\tilde{x})t^{k+1}}{(k+1)!} + \frac{\phi^{(k+2)}(\tilde{x}+\delta(t)t)t^{k+2}}{(k+2)!}.$$

Consider the t^{k+1}-coefficient of this equality.

- On the left hand side, this coefficient converges at x goes to zero in $\mathbb{R}$ by linear continuity of ϕ (and the same is true for that of $\phi(x)$).
- When $x \in \mathbb{R}_+^*$ is fixed, the function $t \mapsto \phi^{(k+2)}(\tilde{x}+\delta(t)t)t^{k+2}$ is continuous at 0, so that the t^{k+1}-coefficient of $\phi^{(k+2)}(\tilde{x}+\gamma(t)t)t^{k+2}$ is zero.
- To estimate the remaining terms, we write ψ as $\psi(\gamma) = \gamma^p + \zeta(\gamma)$, with ζ a $\tilde{\mathbb{R}}$-Nash function with $\zeta^{(j)}(0) = 0$ for $j \in \{0,\dots,p\}$. Then we use the classical formula expressing the n-th derivative of a composite function, so that

$$\phi^{(j)}(\tilde{x}) = (\psi(\tilde{x}^{1/q}))^{(j)} = \frac{p}{q}\cdot(\frac{p}{q}-1)\cdots(\frac{p}{q}-j+1)\tilde{x}^{\frac{p}{q}-j} + (\zeta(\tilde{x}^{1/q}))^{(j)},$$

  where $(\zeta(\tilde{x}^{1/q}))^{(j)}$ is equal to

$$\sum_{k_1+2k_2+\cdots+jk_j=j} \frac{j!}{k_1!\cdots k_j!}\zeta^{(k_1+\cdots+k_j)}(\tilde{x})\prod_{i=1}^{j}(\frac{1/q\cdots(1/q-i+1)\cdot\tilde{x}^{1/q-i}}{i!})^{k_i}. \tag{1}$$

  Note moreover that, by the mean value theorem again, there exist $\delta_j \in [0,1]_{\tilde{\mathbb{R}}}$ such that

$$\zeta^{(j)}(\tilde{x}^{1/q}) = \frac{1}{(p+1-j)!}\psi^{(p+1)}(\delta_j\tilde{x}^{1/q})\tilde{x}^{(p+1-j)/q}.$$

As a consequence, each term in the sum (1) is, up to a non zero rational constant, of the form

$$\zeta^{(k_1+\cdots+k_j)}(\tilde{x})\tilde{x}^{(k_1+\cdots+k_j)/q-j} = \frac{1}{(p+1-(k_1+\cdots+k_j))!}\zeta^{(p+1)}(\delta_{k_1+\cdots+k_j}\tilde{x}^{1/q})x^{p/q+1/q-j}.$$

Coming back to $\phi^{(j)}(\tilde{x})$, with $j \in \{1,\dots,k+1\}$, we obtain, replacing the (non zero) rational constants by a $*$ for more clarity, that

$$\begin{aligned}\phi^{(j)}(\tilde{x}) &= *\,\tilde{x}^{\frac{p}{q}-j} + \sum_{k_1+2k_2+\cdots+jk_j=j} *\,\zeta^{(p+1)}(\delta_{k_1+\cdots+k_j}\tilde{x}^{1/q})\tilde{x}^{p/q+1/q-j}\\ &= \tilde{x}^{\frac{p}{q}-j}\big(* + \sum_{k_1+2k_2+\cdots+jk_j=j} *\,\zeta^{(p+1)}(\delta_{k_1+\cdots+k_j}\tilde{x}^{1/q})\tilde{x}^{1/q}\big).\end{aligned}$$

Then the term inside the parenthesis tends to a non zero constant as x goes to 0 in $\mathbb{R}$, so that the t^{k+1}-coefficient of $\phi^{(j)}(\tilde{x})t^j$ converges to 0 as far as $j \in \{1,\dots,k\}$, whereas it diverges for $j = k+1$.

As a consequence $q = 1$ and ϕ is $\tilde{\mathbb{R}}$-Nash on $[0,\alpha]_{\tilde{\mathbb{R}}}$.

Step 2. – To achieve the proof, we need to see that if ϕ is of class $\tilde{\mathbb{R}}$-Nash on $[-\varepsilon,0]_{\tilde{\mathbb{R}}}$ and on $[0,\varepsilon]_{\tilde{\mathbb{R}}}$, then ϕ is of class $\tilde{\mathbb{R}}$-Nash on $[-\varepsilon_1,\varepsilon_1]_{\tilde{\mathbb{R}}}$ for some $\varepsilon_1 \in \tilde{\mathbb{R}}^*_+$. As ϕ is $\tilde{\mathbb{R}}$-Nash on $[-\varepsilon,0]_{\tilde{\mathbb{R}}}$, there exist $\varepsilon_1 \in \tilde{\mathbb{R}}^*_+$ and a $\tilde{\mathbb{R}}$-Nash function ϕ_- defined on $[-\varepsilon,\varepsilon_1]_{\tilde{\mathbb{R}}}$ which coincides with ϕ on $[-\varepsilon,0]_{\tilde{\mathbb{R}}}$. Shrinking ε_1 if necessary, there exists a $\tilde{\mathbb{R}}$-Nash function ϕ_+ defined on $[-\varepsilon_1,\varepsilon]_{\tilde{\mathbb{R}}}$ which coincides with ϕ on $[0,\varepsilon]_{\tilde{\mathbb{R}}}$. Define a function ψ on $[\varepsilon_1,\varepsilon_1]_{\tilde{\mathbb{R}}}$ to be zero on $[-\varepsilon_1,0]_{\tilde{\mathbb{R}}}$ and to be $\phi_+ - \phi_-$ on $[0,\varepsilon_1]_{\tilde{\mathbb{R}}}$. Then ψ is $\tilde{\mathbb{R}}$-semialgebraic and weakly continuous. If ψ is not identically zero, we can assume (as before) that $\psi(\gamma) = \gamma^k$ on $[0,\tilde{a}]_{\tilde{\mathbb{R}}}$ for some $k \in \mathbb{N}^*$ and $a \in \mathbb{R}^*_+$. Then

$$\psi(x+t) = \begin{cases} 0 & \text{for } x \in [-a,\,0),\\ x^k + ktx^{k-1} + \cdots + t^k & \text{for } x \in [0,\,a).\end{cases}$$

In particular, the t^k-coefficient of $\psi(x+t)$ is equal to 1 for $x \in [0,a)$ and to 0 for $x \in [a,0)$, in contradiction with the linear continuity of ψ in restriction to the $\mathbb{R}$-linear space generated by 1 and t. As a consequence ψ is constant equal to zero, and therefore ϕ is of class $\tilde{\mathbb{R}}$-Nash. □

1.3. Finite dimensional case

For the definition of manifold with corners, we refer to [7].

PROPOSITION 1.12. – *Let $g : M_1 \longrightarrow M_2$ be a Nash map between Nash manifolds possibly with corners. Then for any closed and bounded $\tilde{\mathbb{R}}$-semialgebraic subset $\mathcal{X}$ of $\tilde{M}_1$, the restriction $\tilde{g}|_{\mathcal{X}}$ is weakly continuous.*

Proof. – We can assume that $M_2 = \mathbb{R}$ without loss of generality. Assume $M_1 \subset \mathbb{R}^m$.

For simplicity, we begin by reducing the proof to the case where M_1 is a n-dimensional compact Nash manifold with corners of $\mathbb{R}^n$. First, it suffices to prove that $\tilde{g}$ is continuous in restriction to $\tilde{U}$ for any compact semialgebraic subset $U \subset M_1$. In particular we may assume that M_1 is the graph of a Nash map h from a n-dimensional compact Nash manifold with corners $M_3 \subset \mathbb{R}^n$ to $\mathbb{R}^{m-n}$. Since the projection map from $\tilde{M}_1$ to $\tilde{M}_3$ is clearly weakly continuous, it is sufficient (using Proposition 1.8) to prove that the map $\tilde{g} \circ (\mathrm{id} \times h)$ from $\tilde{M}_3$ to $\tilde{M}_2$ is weakly continuous.

So we assume that g is a Nash map from a n-dimensional compact Nash manifold with corners $M_1 \subset \mathbb{R}^n$ to $\mathbb{R}$. We are going to prove that $\tilde{g}$ is finitely continuous, and conclude using Proposition 1.6. Note that $\tilde{M}_1 \subset \bigcup_{q\in\mathbb{N}^*} \tilde{\mathbb{R}}^n_{0,q}$ since M_1 is compact, so that an element $\gamma \in \tilde{M}_1$ has a well-defined endpoint $\gamma(0) \in M_1$.

Let $\mathcal{D}$ be a $\mathbb{R}$-linear subspace of $\bigcup_{q\in\mathbb{N}^*} \tilde{\mathbb{R}}^n_{0,q}$ generated over $\mathbb{R}$ by $\gamma_1, \ldots, \gamma_l$, and chose $\delta \in \mathcal{D}$. Set $\delta(0) = x_0 \in M_1$. Since g is Nash over the real numbers, describe g ([1] Proposition 8.1.8) around x_0 as a series

$$g(x) = \sum_{I\in\mathbb{N}^n} a_I (x - x_0)^I, \ \ a_I \in \mathbb{R},$$

and similarly, for $j \in \{1, \ldots, l\}$, denote

$$\gamma_j = \sum_{i\in\mathbb{N}} c_{j,i} t^{i/q}, \ \ c_{j,i} \in \mathbb{R}^n$$

where $q \in \mathbb{N}^*$ is a common denominator for $\gamma_1, \ldots, \gamma_l$. Then, for $b_1, \ldots, b_l \in \mathbb{R}$, we have

$$\tilde{g}(\sum_{j=1}^{l} b_j \gamma_j)(t) = g(\sum_{j=1}^{l} b_j \gamma_j(t)) = \sum_{I\in\mathbb{N}^n} a_I (\sum_{j=1}^{l} \sum_{i\in\mathbb{N}} b_j c_{j,i} t^{i/q} - x_0)^I$$

with $t \in [0, r]$, for $r \in \mathbb{R}^*_+$ sufficiently small so that the series converge. As a consequence, for any $k \in \mathbb{Q}$, there exists $i_0 \in \mathbb{N}$ such that the t^k-coefficient of $\tilde{g}(\sum_{j=1}^{l} b_j \gamma_j)(t)$ is equal to the t^k-coefficient of

$$\sum_{I\in\mathbb{N}^n} a_I (\sum_{j=1}^{l} \sum_{i=0}^{i_0} b_j c_{j,i} t^{i/q} - x_0)^I,$$

for t sufficiently small and $b_1, \ldots, b_l$ such that $\sum_{j=1}^{l} b_j \gamma_j(0)$ is closed enough to x_0. The latter function is Nash, therefore the t^k-coefficient of $\tilde{g}(\sum_{j=1}^{l} b_j \gamma_j)(t)$ is continuous in $b_1, \ldots, b_l$ and $\tilde{g}$ is continuous in restriction to $\mathcal{D}$. □

REMARK 1.13. – 1. If M_1 is not compact, the map $\tilde{g}$ is not necessarily weakly continuous. Consider for example the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = (1 + x^2)^{-1}$. Then $\tilde{g}(ct^{-1}) = t^2/c^2 - t^4/c^4 + \cdots$ for $c \in \mathbb{R}^*$, so that $\tilde{g}(ct^{-1})$ does not tend to $\tilde{g}(0) = 1$ as c goes to 0 in $\mathbb{R}$.

2. A general $\tilde{\mathbb{R}}$-Nash map defined on a closed and bounded $\tilde{\mathbb{R}}$-Nash manifold is not necessarily weakly continuous (in Proposition 1.12 the map was already defined over $\mathbb{R}$). Consider for example the map ϕ defined by $\phi(\gamma) = \tilde{g}(\gamma/t)$ on $[0, 1]_{\tilde{\mathbb{R}}}$, where g is defined upstairs.

Next result is a weaker analog of Proposition 1.11 in higher dimensions.

PROPOSITION 1.14. – *Let $\phi : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ be a local $\tilde{\mathbb{R}}$-semialgebraic map between local $\tilde{\mathbb{R}}$-Nash manifolds possibly with corners. If ϕ is weakly continuous, then it is of class $\tilde{\mathbb{R}}$-C^1.*

Proof. – As the problem is local, we assume that ϕ is a $\tilde{\mathbb{R}}$-semialgebraic weakly continuous function on $\tilde{\mathbb{R}}^n$. The partial derivatives of ϕ exist by Proposition 1.11, therefore it suffices to prove that they are $\tilde{\mathbb{R}}$-continuous. Consider the partial derivative $\frac{\partial\phi}{\partial\gamma_1}$ and let assume that $\frac{\partial\phi}{\partial\gamma_1}$ is not $\tilde{\mathbb{R}}$-continuous at $0 \in \tilde{\mathbb{R}}^n$. In that case, by the curve selection lemma there exists a $\tilde{\mathbb{R}}$-continuous $\tilde{\mathbb{R}}$-semialgebraic curve $\eta : [0,1]_{\tilde{\mathbb{R}}} \longrightarrow \tilde{\mathbb{R}}^n$ satisfying $\eta(0) = 0$, whose composition $\frac{\partial\phi}{\partial\gamma_1} \circ \eta$ is $\tilde{\mathbb{R}}$-continuous on $(0,1]_{\tilde{\mathbb{R}}}$ but the limit $\theta \in \tilde{\mathbb{R}}^n$ of $\frac{\partial\phi}{\partial\gamma_1} \circ \eta(\gamma)$ as γ tends to 0 in $[0,1]_{\tilde{\mathbb{R}}}$ is not equal to $\frac{\partial\phi}{\partial\gamma_1}(0)$.

We are going to obtain a contradiction with the weak continuity of ϕ by particularizing this limit to a relevant set of series. Note that there exist $q \in \mathbb{N}$ and $\varepsilon \in \tilde{\mathbb{R}}^*_+$ such that the curve ξ defined by $\xi(\gamma) = \eta(\varepsilon\gamma^q)$ is a weakly continuous $\tilde{\mathbb{R}}$-Nash map on $[0,1]_{\tilde{\mathbb{R}}}$ by Proposition 8.1.12 in [1] and Lemma 1.10.

As θ is different from $\frac{\partial\phi}{\partial\gamma_1}(0)$, there exist $r \in \mathbb{Q}^*_+$ and $c \in \mathbb{R}^*_+$ such that

$$ct^r < |\theta - \frac{\partial\phi}{\partial\gamma_1}(0)| < 2ct^r.$$

As a consequence, for $s \in \mathbb{Q}$ big enough, the inequalities

$$\frac{c}{2}t^r < |\frac{\partial\phi}{\partial\gamma_1}(\xi(at^s)) - \frac{\partial\phi}{\partial\gamma_1}(0)| < 3ct^r \tag{1}$$

hold for any $a \in \mathbb{R}^*_+$. We are going to replace the derivative of ϕ by a difference quotient in order to make use of the weak continuity of ϕ. More precisely, for any $\gamma \in [0,1]_{\tilde{\mathbb{R}}}$, there exist $\beta \in \tilde{\mathbb{R}}^*_+$ such that

$$\left| \frac{\phi\big(\xi_1(\gamma) + \beta, \xi_2(\gamma), \dots, \xi_n(\gamma)\big) - \phi(\xi(\gamma))}{\beta} - \frac{\partial\phi}{\partial\gamma_1}(\xi(\gamma)) \right| < \frac{c}{8}t^r \tag{2}$$

by the mean value theorem, and moreover the set of all such (γ, β) is a $\tilde{\mathbb{R}}$-semialgebraic subset of $[0,1]_{\tilde{\mathbb{R}}} \times \tilde{\mathbb{R}}^*_+$. In particular there exists a strictly increasing $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-continuous function ζ on $[0,1]_{\tilde{\mathbb{R}}}$ such that $\zeta(0) = 0$ and (2) holds for any $(\gamma,\beta) \in (0,1]_{\tilde{\mathbb{R}}} \times \tilde{\mathbb{R}}^*_+$ satisfying $\beta \le \zeta(\gamma)$. Restricting ζ to $\{at^s : a \in \mathbb{R}^*_+\}$, we see that there exists $\beta_0 \in \tilde{\mathbb{R}}^*_+$ such that $\beta_0 < \zeta(at^s)$ for any $a \in \mathbb{R}^*_+$. As a consequence (2) holds for $\beta = \beta_0$ and any $\gamma \in \{at^s : a \in \mathbb{R}_+\}$. Combining (1) and (2) provides

$$\frac{c}{4}t^r < |\frac{\phi\big(\xi_1(at^s) + \beta_0, \xi_2(at^s), \dots, \xi_n(at^s)\big) - \phi(\xi(at^s))}{\beta_0} - \frac{\phi(\beta_0, 0, \dots, 0) - \phi(0)}{\beta_0}| < 4ct^r \tag{3}$$

for any $a \in \mathbb{R}^*_+$. But (3) is in contradiction with the fact that, by weak continuity of ϕ and ξ, the t^r-coefficient of

$$\frac{\phi\big(\xi_1(at^s) + \beta_0, \xi_2(at^s), \dots, \xi_n(at^s)\big) - \phi(\xi(at^s))}{\beta_0}$$

converges to the t^r-coefficient of

$$\frac{\phi(\beta_0, 0, \dots, 0) - \phi(0)}{\beta_0}$$

as a tends to 0 in $\mathbb{R}$. □

REMARK 1.15. – In Proposition 1.14, we do not know whether the partial derivatives of ϕ are weakly continuous. This is the reason why we cannot prove that ϕ is of class $\tilde{\mathbb{R}}$-Nash as in the one dimensional case.

We end this section with a technical result that will be useful in the proof of Lemma 2.10. Let $\Upsilon : \tilde{\mathbb{R}} \longrightarrow \mathbb{R}$ denote the projection onto the constant term, namely if $\gamma = \sum_{i \geq p} a_i t^{i/q} \in \tilde{\mathbb{R}}$, then $\Upsilon(\gamma) = a_0$.

LEMMA 1.16. – *Let ϕ be an $\tilde{\mathbb{R}}$-semialgebraic weakly continuous function on* $[0,\,1]^n_{\tilde{\mathbb{R}}}$ *such that* $\operatorname{Im}\phi \subset [-\tilde{a}, \tilde{a}]_{\tilde{\mathbb{R}}}$, *with $a \in \mathbb{R}^*_+$. Then:*

1. *the equality*
$$\Upsilon \circ \phi = \Upsilon \circ \phi \circ (\Upsilon \times \cdots \times \Upsilon)$$
holds on $[0,\,1]^n_{\tilde{\mathbb{R}}}$,
2. *restricting to the real numbers, the function $\Upsilon \circ \phi|_{[0,\,1]^n}$ is semialgebraic and continuous.*

Proof. – We prove the equality announced in (1) by reduction to absurdity. Let $\gamma_0 \in [0,\,1]^n_{\tilde{\mathbb{R}}}$ such that $\Upsilon \circ \phi(\gamma_0) \neq \Upsilon \circ \phi \circ (\Upsilon \times \cdots \times \Upsilon)(\gamma_0)$. We can assume than $n = 1$ by restricting ϕ to the line over $\tilde{\mathbb{R}}$ passing through γ_0 and $(\Upsilon \times \cdots \times \Upsilon)(\gamma_0)$, regarding that line as $\tilde{\mathbb{R}}$ via the $\tilde{\mathbb{R}}$-linear $\tilde{\mathbb{R}}$-homeomorphism θ from $\tilde{\mathbb{R}}$ to that line defined by
$$\gamma \mapsto (1-\gamma)\gamma_0 + \gamma(\Upsilon \times \cdots \times \Upsilon)(\gamma_0).$$
Note that $\Upsilon \circ \theta^{-1}(\gamma_0) = 0$.

As a consequence, we suppose $n = 1$ and $\Upsilon(\gamma_0) = 0$ (so that $\gamma_0 \in \mathfrak{m}_+$), and without loss of generality let assume moreover $\phi(0) = 0$, so that $\Upsilon \circ \phi(\gamma_0) \neq 0$. Assume $\Upsilon \circ \phi(\gamma_0) < 0$ for example. The function ϕ is piecewise monotone because ϕ is a $\tilde{\mathbb{R}}$-semialgebraic function, so we may suppose that ϕ is increasing on $[\gamma_0, 1]_{\tilde{\mathbb{R}}} \cap \mathfrak{m}_+$. Note that $\Upsilon \circ \phi$ is also increasing on $[\gamma_0, 1]_{\tilde{\mathbb{R}}} \cap \mathfrak{m}_+$.

Let $\gamma_1 \in (\gamma_0, 1]_{\tilde{\mathbb{R}}} \cap \mathfrak{m}_+$ be such that $\gamma_0 < x\gamma_1$ for any $x \in \mathbb{R}^*_+$. Denote by $\mathcal{L}_1$ the $\mathbb{R}$-line passing through 0 and γ_1, and by $\mathcal{L}_1^+$ the positive half line $\mathcal{L}_1^+ = \{x\gamma_1,\ \ x \in \mathbb{R}^*_+\}$. Then the function $\Upsilon \circ \phi_{|\mathcal{L}_1}$ is continuous by weak continuity of ϕ, it is increasing on $\mathcal{L}_1^+$ and $\Upsilon \circ \phi_{|\mathcal{L}_1}(0) = 0$. Therefore $\Upsilon \circ \phi(\gamma) \geq 0$ for any $\gamma \in \mathcal{L}_1^+$.

Denote by $\mathcal{L}_2$ the $\mathbb{R}$-line passing through γ_0 and γ_1 and by $[\gamma_0, \gamma_1]_{\mathbb{R}}$ the segment between γ_0 and γ_1 in $\mathcal{L}_2$. For any $\gamma \in (\gamma_0, \gamma_1]_{\mathbb{R}}$, note that we have $\gamma_0 < x\gamma$ for any $x \in \mathbb{R}^*_+$. Indeed, if $\gamma = a\gamma_0 + (1-a)\gamma_1$ with $a \in [0, 1)$, then for $x \in \mathbb{R}^*_+$ we have the following alternative:

- either $ax \geq 1$, so that $x\gamma = ax\gamma_0 + (1-a)x\gamma_1 > ax\gamma_0 \geq \gamma_0$,
- or $0 \leq ax < 1$, and then $\frac{x(1-a)}{1-ax} > 0$ and thus $\gamma_0 < \frac{x(1-a)}{1-ax}\gamma_1$. It follows that $\gamma_0 < x(a\gamma_0 + (1-a)\gamma_1) = x\gamma$ as required.

So we have proved that $\gamma_0 < x\gamma$ for any $x \in \mathbb{R}^*_+$, therefore repeating the same argument as before show that $\Upsilon \circ \phi(\gamma) \geq 0$. However $\Upsilon \circ \phi$ should be continuous on $[\gamma_0, \gamma_1]_{\mathbb{R}}$ by weak continuity of ϕ, which is in contradiction with $\Upsilon \circ \phi(\gamma_0) < 0$.

Concerning (2), it suffices to prove that $\Upsilon \circ \phi$ is a semialgebraic function on $[0, 1]^n \subset \mathbb{R}^n$, since the continuity of $\Upsilon \circ \phi_{|[0,1]^n}$ follows from the weak continuity of ϕ. As ϕ is $\tilde{\mathbb{R}}$-semialgebraic, let $P \in \tilde{\mathbb{R}}[x_1, \ldots, x_{n+1}]$ be a non zero polynomial such that $P(\gamma, \phi(\gamma)) = 0$ for any $\gamma \in [0,\,1]^n_{\tilde{\mathbb{R}}}$. Multiplying P by the relevant power of t, we may suppose that all the coefficients of P are bounded, and moreover some of them are not in $\mathfrak{m}_+$. Denote by $Q \in \mathbb{R}[x_1, \dots, x_{n+1}]$ the polynomial obtained from P by replacing the coefficients of P with their value under Υ. Then Q is non zero and $Q(x, \Upsilon \circ \phi(x)) = 0$ for any $x \in [0,1]^n$, so that $\Upsilon \circ \phi$ is semialgebraic on $[0,1]^n$. □

2. Comparison of homologies

In this section we compare different homology theories on closed and bounded ($\tilde{\mathbb{R}}$-)Nash manifolds, the usual singular homology, the $\tilde{\mathbb{R}}$-semialgebraic singular homology (cf. [5] for the first definition of semialgebraic homology) where we consider $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-continuous chains, and another sort of singular homology where we consider $\tilde{\mathbb{R}}$-semialgebraic weakly continuous chains (cf. Definition 2.3).

2.1. Algebraic topology over Puiseux series

Let $\triangle^n$ denote the n-simplex spanned by $0, (1, 0, \dots,), \dots, (0, \dots, 0, 1)$ in $\mathbb{R}^n$. For a topological space X, let $S_n(X)$ denote the set of singular n-simplexes from $\triangle^n$ to X. We denote by $H_*(X)$ the singular homology groups of X with coefficient in $\mathbb{Z}$.

We begin this section with the elementary remark that the singular homology groups $H_*(X)$ of a topological space X can be defined using $\tilde{\mathbb{R}}$-simplexes as well as by usual $\mathbb{R}$-simplexes. Actually, replacing $\triangle^n$ with its extension $\tilde{\triangle}^n$ to $\tilde{\mathbb{R}}^n$, consider the set $S'_n(X)$ of singular n-$\tilde{\mathbb{R}}$-simplexes from $\tilde{\triangle}^n$ to X and denote the corresponding homology groups by $H'_*(X)$.

Lemma 2.1. – *The singular homology groups $H_*(X)$ of a topological space X are isomorphic to the homology groups $H'_*(X)$ defined using $\tilde{\mathbb{R}}$-simplexes.*

Proof. – Note that for $\gamma \in \tilde{\triangle}^n$, the order of γ is positive or equal to zero so that $\gamma(0)$ is well-defined. For $u \in S_n(X)$, define $\alpha(u) \in S'_n(X)$ by $\alpha(u)(\gamma) = u(\gamma(0))$ for $\gamma \in \tilde{\triangle}^n$, so that we obtain a map $\alpha : S_n(X) \longrightarrow S'_n(X)$. Similarly, for $\sigma \in S'_n(X)$, define $h(\sigma) \in S_n(X)$ by $h(\sigma)(x) = \sigma(\tilde{x})$ for $x \in \triangle^n$, and this gives a map $h : S'_n(X) \longrightarrow S_n(X)$. Then α and h define an homotopy equivalence. Indeed, $h \circ \alpha$ is the identity map of $S_n(X)$. Moreover the $\tilde{\mathbb{R}}$-simplex $\alpha \circ h(\sigma)$, for $\sigma \in S'_n(X)$, is given by $\alpha \circ h(\sigma)(\gamma) = \sigma(\widetilde{\gamma(0)})$ for any $\gamma \in \tilde{\triangle}^n$. As a consequence, the map

$$\begin{aligned} \tilde{\triangle}^n \times [0,1] &\longrightarrow X \\ (\gamma, s) &\mapsto \sigma\big((1-s)\gamma + s\widetilde{\gamma(0)}\big) \end{aligned}$$

defines an homotopy between $\sigma \in S'_n(X)$ and $\alpha \circ h(\sigma)$. □

As we have already noticed, we are interested in $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-continuous maps and $\tilde{\mathbb{R}}$-semialgebraic weakly continuous maps rather than simply $\tilde{\mathbb{R}}$-semialgebraic continuous maps. We define below two kinds of homology groups corresponding to these notions of continuity. We will prove some isomorphisms between the corresponding homologies, in the spirit of Lemma 2.1 but with more involved proofs.

Definition 2.2. – Let $\mathcal{X}$ and $\mathcal{Y}$ be $\tilde{\mathbb{R}}$-semialgebraic sets. An $\tilde{\mathbb{R}}$-*semialgebraic singular* n-*simplex* of $\mathcal{X}$ is an $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-continuous map from $\tilde{\Delta}^n$ to $\mathcal{X}$. We denote by $\tilde{S}_n(\mathcal{X})$ the set of $\tilde{\mathbb{R}}$-semialgebraic singular n-simplex of $\mathcal{X}$, by $\tilde{H}_*(\mathcal{X})$ the associated homology groups, and we call them the $\tilde{\mathbb{R}}$-*semialgebraic singular homology groups*.

An $\tilde{\mathbb{R}}$-*semialgebraic* $\tilde{\mathbb{R}}$-*homotopy* $\theta_\lambda : \mathcal{X} \to \mathcal{Y}$, with $\lambda \in [0, 1]_{\tilde{\mathbb{R}}}$, means an $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-continuous map $(\gamma, \lambda) \mapsto \theta_\lambda(\gamma)$ from $\mathcal{X}\times[0, 1]_{\tilde{\mathbb{R}}}$ to $\mathcal{Y}$. If the maps θ_λ are embeddings for any $\lambda \in [0, 1]_{\tilde{\mathbb{R}}}$, the $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-homotopy is called an $\tilde{\mathbb{R}}$-*semialgebraic* $\tilde{\mathbb{R}}$-*isotopy*.

The $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-continuous maps between $\tilde{\mathbb{R}}$-semialgebraic sets and the homology groups $\tilde{H}_*$ satisfy the Eilenberg-Steenrod axioms of homology groups. In the case where $\mathcal{X}$ is a local $\tilde{\mathbb{R}}$-Nash manifold, we define $\tilde{H}_*(\mathcal{X})$ similarly.

Note that the family of local $\tilde{\mathbb{R}}$-Nash manifolds possibly with corners and the family of $\tilde{\mathbb{R}}$-semialgebraic weakly continuous maps between them form a category as shown already.

Definition 2.3. – Let $\mathcal{M}$ be a local $\tilde{\mathbb{R}}$-Nash manifold possibly with corners. An $\tilde{\mathbb{R}}$-*semialgebraic weak singular* n-*simplex* of $\mathcal{M}$ is a $\tilde{\mathbb{R}}$-semialgebraic weakly continuous map from $\tilde{\Delta}^n$ to $\mathcal{M}$. We denote by $\tilde{H}^w_*(\mathcal{M})$ the associated homology groups, and we call them the $\tilde{\mathbb{R}}$-semialgebraic weak singular homology groups of $\mathcal{M}$.

For $\mathcal{N}$ another local $\tilde{\mathbb{R}}$-Nash manifold possibly with corners, we define similarly a *local* $\tilde{\mathbb{R}}$-*semialgebraic weak homotopy* $\theta_\lambda : \mathcal{M} \to \mathcal{N}$, with $\lambda \in [0, 1]_{\tilde{\mathbb{R}}}$. The $\tilde{\mathbb{R}}$-semialgebraic weakly continuous maps between local $\tilde{\mathbb{R}}$-Nash manifolds possibly with corners and the homology groups $\tilde{H}^w_*$ satisfy the Eilenberg-Steenrod axioms of homology groups.

Lemma 2.4. – *Let $\mathcal{M}$ be a local $\tilde{\mathbb{R}}$-Nash manifold possibly with corners. There exists a functorial morphism between the covariant functors $\mathcal{M} \to \tilde{H}^w_*(\mathcal{M})$ and $\mathcal{M} \to \tilde{H}_*(\mathcal{M})$.*

Proof. – An $\tilde{\mathbb{R}}$-semialgebraic weak singular n-simplex is in particular an $\tilde{\mathbb{R}}$-semialgebraic singular n-simplex by Lemma 1.7. As a consequence, we have a natural map from $\tilde{H}^w_*(\mathcal{M})$ to $\tilde{H}_*(\mathcal{M})$. □

The goal of this section is to prepare the material to prove Theorem 3.6. We begin with comparing $\tilde{H}_*(\tilde{X})$ with $H_*(X)$ for a compact semialgebraic set X.

Lemma 2.5. –
1. *A closed and bounded $\tilde{\mathbb{R}}$-semialgebraic set is $\tilde{\mathbb{R}}$-semialgebraically $\tilde{\mathbb{R}}$-homeomorphic to the $\tilde{\mathbb{R}}$-extension of some compact semialgebraic set X.*
2. *For such a set $\tilde{X}$, there exists a natural isomorphism $\tilde{H}_*(\tilde{X}) \to H_*(X)$.*

The proof of (2) follows from usual arguments in algebraic topology combined with the simplicial homotopy theorem (Lemma 3.1 in [20]). However, we write down the proof in full details because we will use similar arguments under more involved situations in Lemma 2.9 and Lemma 2.10.

Let X be a compact semialgebraic set, and let K be a simplicial decomposition of X. Denote by $\tilde{K}$ the extension of the simplexes of K, namely $\tilde{K} = \{\tilde{\sigma} : \sigma \in K\}$. Then $\tilde{K}$ is an $\tilde{R}$-simplicial complex whose simplicial homology $H_*(\tilde{K})$ is isomorphic to the simplicial homology $H_*(K)$ of K (and which is also isomorphic to the singular homology groups of X). We are going to relate $H_*(K)$ to $\tilde{H}_*(\tilde{X})$ passing through homology groups related to X and K.

Let denote by $\tilde{S}_n^L(\tilde{X})$ the set of $\tilde{\mathbb{R}}$-linear (relatively to $\tilde{K}$) maps from $\tilde{\Delta}^n$ to $\tilde{X}$, and by $\tilde{H}_n^L(\tilde{X})$ the associated homology groups. Denote similarly by $\tilde{S}_n^{PL}(\tilde{X})$ the set of $\tilde{\mathbb{R}}$-piecewise linear maps from $\tilde{\Delta}^n$ to $\tilde{X}$, and by $\tilde{H}_n^{PL}(\tilde{X})$ the associated homology groups. Then we have natural maps

$$H_n(X) \longrightarrow H_n(K) \longrightarrow H_n(\tilde{K}) \longrightarrow \tilde{H}_n^L(\tilde{X}) \longrightarrow \tilde{H}_n^{PL}(\tilde{X}) \longrightarrow \tilde{H}_n(\tilde{X}),$$

and the first fourth ones are isomorphisms by usual arguments in algebraic topology. The goal of the proof of the second part of Lemma 2.5 is to see that the fifth one is also an isomorphism.

Before entering into the details of the proof, we begin by recalling the statement of the simplicial homotopy theorem for the convenience of the reader.

Lemma 2.6 (Lemma 3.1 in [20]). – *Let $\mathcal{X}$ and $\mathcal{Y}$ be closed and bounded $\tilde{\mathbb{R}}$-polyhedra, and let $\phi : \mathcal{X} \longrightarrow \mathcal{Y}$ be a $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-continuous map which is $\tilde{\mathbb{R}}$-piecewise linear in restriction to a closed and bounded $\tilde{\mathbb{R}}$-polyhedron $\mathcal{X}_0 \subset \mathcal{X}$. Then ϕ is $\tilde{\mathbb{R}}$-homotopic to a $\tilde{\mathbb{R}}$-piecewise linear map, and the homotopy can be chosen to be fixed on $\mathcal{X}_0$.*

Proof of Lemma 2.5. – For the proof of (1), note that for any real closed field R, a closed and bounded R-semialgebraic set is R-homeomorphic to a closed and bounded R-polyhedron ([17], Theorem 2.2). Moreover, a closed and bounded R-polyhedron is the underlying polyhedron of some finite R-simplicial complex ([17], Theorem 2.11). Finally, a finite R-simplicial complex is defined by a finite set and finite relations between the elements of the set. As a consequence, a closed and bounded $\tilde{\mathbb{R}}$-semialgebraic set is $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-homeomorphic to $\tilde{X}$ for a compact polyhedron X.

For the proof of (2), consider a compact semialgebraic set X and let K be a simplicial decomposition of X. We are going to prove that the natural map

$$\tilde{H}_n^{PL}(\tilde{X}) \longrightarrow \tilde{H}_n(\tilde{X})$$

is an isomorphism.

Let us focus first on injectivity. Let ξ be a $\tilde{\mathbb{R}}$-piecewise linear n-chain on $\tilde{X}$, and assume that ξ is the boundary of a $\tilde{\mathbb{R}}$-semialgebraic singular $n+1$-chains, namely there exist $l \in \mathbb{N}$, $\sigma_i \in \tilde{S}_{n+1}(\tilde{X})$ and $m_i \in \mathbb{Z}$ for $i \in \{1, \dots, l\}$, such that

$$\xi = \sum_{i=1}^{l} m_i \partial \sigma_i$$

as a $\tilde{\mathbb{R}}$-semialgebraic singular n-chain. We are going to deform the simplexes $\sigma_1, \dots, \sigma_l$ into $\tilde{\mathbb{R}}$-piecewise linear $n+1$-simplexes using Lemma 2.6. Let Σ denote the disjoint union of l copies of $\tilde{\Delta}^{n+1}$, and define

$$\sigma : \Sigma \longrightarrow \tilde{X}$$

to be the $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-continuous map whose restriction to the i-th copy of $\tilde{\Delta}^{n+1}$ coincides with σ_i, for $i \in \{1, \dots, l\}$. Let $\Sigma_0 \subset \Sigma$ denote the union of those faces of the i-th copy of $\tilde{\Delta}^{n+1}$ where the $\tilde{\mathbb{R}}$-semialgebraic map $\partial \sigma_i$ is already $\tilde{\mathbb{R}}$-piecewise linear, for any $i \in \{1, \dots, l\}$. Identify two n-dimensional faces $\tilde{\Delta}_{i_1}$ and $\tilde{\Delta}_{i_2}$ of the copies of $\tilde{\Delta}^{n+1}$ in Σ through an $\tilde{\mathbb{R}}$-linear isomorphism $\psi : \tilde{\Delta}_{i_1} \longrightarrow \tilde{\Delta}_{i_2}$ as soon as $\partial \sigma_{i_1} = \partial \sigma_{i_2} \circ \psi$ on $\tilde{\Delta}_{i_1}$. We denote by Σ' the resulting closed and bounded $\tilde{\mathbb{R}}$-polyhedron and by $\pi : \Sigma \longrightarrow \Sigma'$ the associated projection. By this construction, we can define a $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-continuous map $\sigma' : \Sigma' \longrightarrow \tilde{X}$ by $\sigma' \circ \pi = \sigma$, and let us set $\Sigma'_0 = \pi(\Sigma_0)$. By Lemma 2.6, there exists a $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-continuous map $\theta : \Sigma' \times [0,1]_{\tilde{\mathbb{R}}} \longrightarrow \tilde{X}$ such that

- $\theta_{|\Sigma' \times \{0\}}$ coincides with σ',
- the restriction to Σ'_0 remains fixed, namely $\theta_{|\Sigma'_0 \times [0,1]_{\tilde{\mathbb{R}}}} = \sigma'_{|\Sigma'_0}$,
- and finally $\theta_{|\Sigma' \times \{1\}}$ is $\tilde{\mathbb{R}}$-piecewise linear.

As a consequence, denoting by $\sigma''_i : \tilde{\Delta}^{n+1} \longrightarrow \tilde{X}$ the restriction of $\theta_{|\Sigma' \times \{1\}}$ to the i-th copy of $\tilde{\Delta}^{n+1}$, we obtain $\tilde{\mathbb{R}}$-piecewise linear $n+1$-simplexes $\sigma''_i \in \tilde{S}^{PL}_{n+1}(\tilde{X})$, for $i \in \{1, \dots, l\}$. Moreover the equality

$$\sum_{i=1}^{l} m_i \partial \sigma_i = \sum_{i=1}^{l} m_i \partial \sigma''_i$$

holds by construction of Σ', so that ξ is the boundary of a $\tilde{\mathbb{R}}$-piecewise linear chain.

Concerning the surjectivity, consider a $\tilde{\mathbb{R}}$-semialgebraic n-cycle ξ on $\tilde{X}$, namely $\xi = \sum_{i=1}^{l} m_i \sigma_i$ with $\sigma_i \in \tilde{S}_n(\tilde{X})$ and $\partial \xi = 0$. We look for a $\tilde{\mathbb{R}}$-piecewise linear n-chain μ on $\tilde{X}$ and a $\tilde{\mathbb{R}}$-semialgebraic $n+1$-chain ν on $\tilde{X}$ such that $\xi = \mu - \partial \nu$. Define the polyhedrons Σ and Σ', the projection $\pi : \Sigma \longrightarrow \Sigma'$ together with the maps $\sigma : \Sigma \longrightarrow \tilde{X}$ and $\sigma' : \Sigma' \longrightarrow \tilde{X}$ similarly as above in the proof of the injectivity.

We use Lemma 2.6 which gives the existence of an $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-continuous map $\theta' : \Sigma' \times [0,1]_{\tilde{\mathbb{R}}} \longrightarrow \tilde{X}$ such that $\theta'_{|\sigma' \times \{0\}}$ is equal to σ' and $\theta'_{|\sigma' \times \{1\}}$ is $\tilde{\mathbb{R}}$-piecewise linear. Set $\theta = \theta' \circ (\pi \times \mathrm{id})$. Then

$$\theta : \Sigma \times [0,1]_{\tilde{\mathbb{R}}} \longrightarrow \tilde{X}$$

is a $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-continuous map such that $\theta_{|\sigma \times \{0\}}$ is equal to σ and $\theta_{|\sigma \times \{1\}}$ is $\tilde{\mathbb{R}}$-piecewise linear. Restricting to the i-th copy of $\tilde{\Delta}^n$ in Σ, we obtain moreover $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-continuous maps

$$\theta_i : \tilde{\Delta}^n \times [0,1]_{\tilde{\mathbb{R}}} \longrightarrow \tilde{X}$$

for $i \in \{1, \dots, l\}$. These maps satisfy

$$\sum_{i=1}^{l} m_i \partial (\theta_{i\,|\tilde{\Delta}^n \times \{\lambda\}}) = 0 \tag{1}$$

for any $\lambda \in [0,1]_{\tilde{\mathbb{R}}}$. We set

$$\mu = \sum_{i=1}^{l} m_i \theta_{i\,|\tilde{\Delta}^n \times \{1\}}.$$

Then μ is a $\tilde{\mathbb{R}}$-piecewise linear n-chain on $\tilde{X}$ and $\partial \mu = 0$. Note moreover that

$$\xi - \mu = \sum_{i=1}^{l} m_i (\theta_{i\,|\tilde{\Delta}^n \times \{0\}} - \theta_{i\,|\tilde{\Delta}^n \times \{1\}}). \tag{2}$$

Denote by $\mathscr{K}$ the natural simplicial decomposition of Σ, and by $\mathscr{K}^0$ the 0-skeleton of $\mathscr{K}$. In order to define ν, consider the canonical $\tilde{\mathbb{R}}$-simplicial decomposition $\mathscr{L}$ of $\tilde{\Delta}^n \times [0,\, 1]_{\tilde{\mathbb{R}}}$, namely each $\tilde{\mathbb{R}}$-simplex in $\mathscr{L}$ of dimension $n+1$ is spanned by

$$(\tilde{v}_1, 0), \dots, (\tilde{v}_k, 0), (\tilde{v}_k, 1), \dots, (\tilde{v}_{n+1}, 1)$$

(and the vertices have this order) where

$$v_1 = 0,\ v_2 = (1, 0, \ldots, 0), \ldots, v_{n+1} = (0, \ldots, 0, 1).$$

We denote by $\mathcal{L}_i$ the $\tilde{\mathbb{R}}$-simplicial decomposition of the i-th copy of $\tilde{\Delta}^n \times [0,\ 1]_{\tilde{\mathbb{R}}}$ in $\Sigma \times [0,\ 1]_{\tilde{\mathbb{R}}}$. Then the induced $\tilde{\mathbb{R}}$-simplicial decomposition $\mathcal{L}_\Sigma = \bigcup_{i=1}^{l} \mathcal{L}_i$ of $\Sigma \times [0,\ 1]_{\tilde{\mathbb{R}}}$, satisfies

$$\mathcal{L}_\Sigma|_{\Sigma\times\{0\}} = \mathcal{K} \times \{0\},\ \mathcal{L}_\Sigma|_{\Sigma\times\{1\}} = \mathcal{K} \times \{1\},\ \mathcal{L}_\Sigma^0 = \mathcal{K}^0 \times \{0, 1\}.$$

Note that the vertices of each $\tilde{\mathbb{R}}$-simplex in $\mathcal{L}_\Sigma$ have an order. For each $i \in \{1, \ldots, l\}$ and $\omega \in \mathcal{L}_i$ of dimension $n+1$, consider ω as the standard $n+1$-simplex $\tilde{\Delta}_{n+1}$ through the unique $\tilde{\mathbb{R}}$-linear isomorphism from $\tilde{\Delta}_{n+1}$ to ω which preserves the orders of the vertices, and set $\theta_{i,\omega} = \theta_i|_\omega$. Set

$$\nu = \sum_{i=1}^{l} \sum_{\omega\in\mathcal{L}, \dim\omega=n+1} m_i \theta_{i,\omega}.$$

Then the boundary of ν is equal to $\xi - \mu$ by (1) and (2). □

We are going to use the method of the proof of Lemma 2.5 in order to prove the following.

PROPOSITION 2.7. – *Let M be a compact Nash manifold possibly with corners. Then there is a natural isomorphism $\tilde{H}_*^w(\tilde{M}) \to H_*(M)$.*

To prove Proposition 2.7, we introduce, as an intermediate tool, Nash homology groups.

DEFINITION 2.8. – Let M be a Nash manifold possibly with corners. Denote by $S_n^N(M)$ the set of Nash n-simplexes, namely Nash maps from Δ^n to M, and denote by $H_*^N(M)$ the corresponding homology groups. Similarly, denote by $\tilde{S}_n^N(\tilde{M})$ the set of $\tilde{\mathbb{R}}$-extensions $\tilde{u} : \tilde{\Delta}^n \longrightarrow \tilde{M}$ of Nash n-simplexes $u \in S_n^N(M)$, and denote the corresponding homology groups by $\tilde{H}_*^N(\tilde{M})$.

Proof of Proposition 2.7. – There exist natural maps

$$\tilde{H}_*^w(\tilde{M}) \longleftarrow \tilde{H}_*^N(\tilde{M}) \longleftarrow H_*^N(M) \longrightarrow H_*(M),$$

the middle one being clearly an isomorphism. We prove in Lemma 2.9 and Lemma 2.10 below that the two other ones are also isomorphisms. □

LEMMA 2.9. – *Let M be a compact Nash manifold possibly with corners. Then the natural map from $H_*^N(M)$ to $H_*(M)$ is an isomorphism.*

Proof. – The proof follows the same lines as the proof of Lemma 2.5. The only difference is that we need a counterpart for Lemma 2.1 in [20].

If M has corners, let N be a compact Nash manifold with corners which contains M in its interior and such that there exists a Nash isotopy $H_s : M \longrightarrow N$, with $s \in [0, 1]$, such that $H_0 = \mathrm{id}$ and $\mathrm{Im}\, H_1 = N$ (such an isotopy exists by the proof of Theorem VI.2.1 in [18]). In case M has no corners, set $N = M$. Let Σ be the union of $l \in \mathbb{N}$ copies of the standard n-simplex Δ^n, and consider as in the proof of Lemma 2.5 a union Σ_0 of some faces of the copies of Δ^n in Σ, together with a quotient space Σ' obtained by identifying some proper faces of the copies of Δ^n via linear isomorphisms, and denote the quotient map by $\pi : \Sigma \longrightarrow \Sigma'$.

Let $g' : \Sigma' \longrightarrow M$ be a continuous semialgebraic map such that the restriction of the continuous semialgebraic map $g = g' \circ \pi$ to any face of Σ_0 is a Nash map. What we are going to prove is the following statement.

($*$) *There exists an homotopy $h'_s : \Sigma' \longrightarrow M_1$, for $s \in [0,1]$, such that*

- *the induced homotopy $h_s = h'_s \circ \pi$ satisfies $h_0 = g$,*
- *in restriction to Σ_0, the maps h_s coincide with g,*
- *h_1 is a Nash map.*

In order to prove this statement, consider first the case where $l = 1$ so that $\Sigma = \triangle^n$, and assume moreover that $g_{|\partial\triangle^n}$ is of class Nash. Thus we can suppose $\Sigma_0 = \partial\Sigma$ and $\Sigma' = \Sigma$. Assume N is included in $\mathbb{R}^m$ and let $v : U \longrightarrow N$ be a Nash tubular neighborhood of N in $\mathbb{R}^m$. In order to solve the problem for such g, it suffices to find a strong Nash approximation $g_1 : \triangle^n \longrightarrow \mathbb{R}^m$ of g in the C^0-topology such that $g_1 = g$ on $\partial\triangle^n$ and $\operatorname{Im} g_1 \subset U$ because in that case the maps defined by

$$h_s(x) = v\big((1-s)g(x) + sg_1(x)\big), \quad s \in [0,1]$$

give a relevant homotopy. To construct such an approximation, it suffices to consider the case $m = 1$. Extend $g_{|\partial\triangle^n}$ to a Nash function g_2 on $\triangle^n$ by Proposition 0.7 in [3], so that, replacing g with $g - g_2$ if necessary, we can suppose that g vanishes on the boundary of $\triangle^n$. Replacing again g with $g \circ w$, where $w : \triangle^n \longrightarrow \triangle^n$ is a continuous semialgebraic map closed to the identity such that the inverse image $w^{-1}(\partial\triangle^n)$ of the boundary of $\triangle^n$ is a neighborhood of $\partial\triangle^n$ in $\triangle^n$, we can even suppose that g vanishes on a neighborhood of the boundary of $\triangle^n$. Now we construct the approximation as follows. Let l_j be linear functions on $\mathbb{R}^n$ whose zero sets are the linear spaces spanned by the faces of $\triangle^n$ of dimension $n-1$. Define a continuous semialgebraic function q on $\triangle^n$ by

$$q = \begin{cases} 0 & \text{on } \partial\triangle^n \\ \frac{g}{\prod_j l_j} & \text{on } \operatorname{Int}\triangle^n. \end{cases}$$

Let p be a Nash approximation (for example a polynomial approximation) of q. Then $p\prod_j l_j$ is a relevant Nash approximation of g.

The proof of the general statement *(*)* follows from this particular case by induction on the number l of copies of $\triangle^n$ in Σ and on the dimension n. More precisely, if $l > 1$, let Σ_1 be one copy of $\triangle^n$ in Σ and let Σ_2 be the union of the other copies. By the induction hypothesis we obtain a homotopy $h_s : \pi(\Sigma_2) \longrightarrow M_1$, and this homotopy induces maps from the union of those faces in Σ_1 that are identified with faces in Σ_2 via π, namely maps from $\Sigma_1 \cap \pi^{-1}(\pi(\Sigma_1) \cap \pi(\Sigma_2))$. We can extend this maps to give an homotopy on $\pi(\Sigma_1)$, fixed on $\pi(\Sigma_2 \cap \Sigma_0)$. Using the induction hypothesis for the case $l = 1$, and extending if necessary N to a bigger Nash manifold with corners, we can ask moreover that the homotopy ends with a Nash map. Therefore it suffices to treat the case $l = 1$. Furthermore, we can assume that the restriction of g to $\partial\Sigma$ is Nash by the induction hypothesis on the dimension, so that the particular case treated upstairs enables to achieve the proof. □

LEMMA 2.10. – *Let M be a compact Nash manifold possibly with corners. The natural map $\tilde{H}^N_*(\tilde{M}) \to \tilde{H}^w_*(\tilde{M})$ is an isomorphism.*

Proof. – We proceed as in the proof of Lemma 2.9 (note that the orthogonal projection of a tubular $\tilde{\mathbb{R}}$-neighborhood of $\tilde{M}$ in $\tilde{\mathbb{R}}^m$ is weakly continuous because it is induced from a tubular neighborhood of M in its ambient Euclidean space $\mathbb{R}^m$). In particular, it suffices to find a substitute for the approximation argument. More precisely, we are going to prove that for any $a \in \mathbb{R}_+^*$, an $\tilde{\mathbb{R}}$-semialgebraic weakly continuous function ϕ on $\tilde{\Delta}^n$, whose restriction to $\partial\tilde{\Delta}^n$ is the $\tilde{\mathbb{R}}$-extension of some Nash function h on $\partial\Delta^n$, is $\tilde{a}$-approximated by the $\tilde{\mathbb{R}}$-extension $\tilde{g}$ of some Nash function g on Δ^n whose restriction to $\partial\Delta^n$ coincides with h.

Indeed, the function

$$\Upsilon \circ \phi_{|\Delta^n} : \Delta^n \longrightarrow \mathbb{R}$$

is a continuous semialgebraic function by Lemma 1.16, and its restriction to $\partial\Delta^n$ coincides with the Nash function h. Let g be a Nash $a/2$-approximation of $\Upsilon \circ \phi_{|\Delta^n}$ whose restriction to $\partial\Delta^n$ coincides with h. Then $\tilde{g}$ gives a relevant approximation of ϕ. □

3. Homology of the Milnor fibre

Let f be a polynomial function on $\mathbb{R}^n$. We may associate to f a positive and a negative Milnor fibre (as in [13] in a more topological context). The positive Milnor fibre $F_f(r,a)$ of f at $x_0 \in f^{-1}(0) \subset \mathbb{R}^n$ is the semialgebraic subset of $\mathbb{R}^n$ defined by

$$F_f(r,a) = \{x \in \mathbb{R}^n : \ |x - x_0| \leq r, \ \ f(x) = a\}$$

where $a > 0 \in \mathbb{R}$ and $r > 0 \in \mathbb{R}$ are sufficiently small so that f is a locally trivial fibration on a neighborhood of x_0 over $(0,a]$, with fibre $F_f(r,a)$. We associate to f and x_0 a set of Puiseux series $\mathcal{F}_f$ by

$$\mathcal{F}_{f,x_0} = \{\gamma \in \tilde{\mathbb{R}}^n : \ \gamma(0) = x_0, \ \ f \circ \gamma(t) = t\}.$$

Note that $\mathcal{F}_{f,x_0}$ is a local $\tilde{\mathbb{R}}$-Nash manifold (as defined in Definition 1.1), so that we can consider the $\tilde{\mathbb{R}}$-semialgebraic singular homology groups and the $\tilde{\mathbb{R}}$-semialgebraic weak singular homology groups of $\mathcal{F}_{f,x_0}$. In this part, we are going to compare these homology groups of $\mathcal{F}_{f,x_0}$ with the (classical) homology groups of $F_f(r,a)$. To this aim, we study the Nash triviality of the family of Milnor fibres $F_f(r,a)$, with $a > 0 \in \mathbb{R}$ and $r > 0 \in \mathbb{R}$, together with its analog after extension into the field of algebraic real Puiseux series, namely the semialgebraic subsets $\tilde{F}_{\tilde{f}}(\rho,\alpha) \subset \tilde{\mathbb{R}}^n$ defined by

$$\tilde{F}_{\tilde{f}}(\rho,\alpha) = \{\gamma \in \tilde{\mathbb{R}}^n : \ |\gamma - \tilde{x}_0| \leq \rho, \ \ \tilde{f}(\gamma) = \alpha\},$$

with $\alpha > 0 \in \tilde{\mathbb{R}}$ and $\rho > 0 \in \tilde{\mathbb{R}}$.

Note that $F_f(r,a)$ is a Nash manifold with boundary and that $\tilde{F}_{\tilde{f}}(\rho,\alpha)$ is an $\tilde{\mathbb{R}}$-Nash manifold with boundary.

In the following, we fix $x_0 = 0 \in \mathbb{R}^n$ and denote $\mathcal{F}_{f,x_0} = \mathcal{F}_f$ for simplicity of notation. In that situation, remark that $\mathcal{F}_f \subset \{\pm\gamma : \gamma \in \mathfrak{m}_+\}^n$. Note moreover that, specializing α to the particular value t, the extension $\tilde{F}_{\tilde{f}}(\rho,t)$ is equal to

$$\tilde{F}_{\tilde{f}}(\rho,t) = \{\gamma \in \tilde{\mathbb{R}}^n : \ |\gamma| \leq \rho, \ \ \tilde{f}(\gamma) = t\},$$

and we can recover $\mathcal{F}_f$ as a union of some of these extended Milnor fibres.

LEMMA 3.1. – *For any $\rho_0 \in \mathfrak{m}_+ \cup \{0\}$, the set $\mathscr{F}_f$ is equal to the union of the $\tilde{\mathbb{R}}$-semialgebraic sets $\tilde{F}_{\tilde{f}}(\rho, t)$ over $\rho \in \mathfrak{m}_+$ with $\rho > \rho_0$, namely*

$$\mathscr{F}_f = \bigcup_{\rho \in \mathfrak{m}_+,\ \rho > \rho_0} \tilde{F}_{\tilde{f}}(\rho, t).$$

Proof. – Note first that $\tilde{F}_{\tilde{f}}(\rho, t) \subset \mathscr{F}_f$ for $\rho \in \mathfrak{m}_+$ since for $\gamma \in \tilde{\mathbb{R}}$, if $|\gamma| \leq |\rho|$ then the limit of $\gamma(t)$ when t goes to zero exists and satisfies $|\gamma(0)| \leq |\rho(0)| = 0$. Conversely, it suffices to notice that any $\gamma \in \mathfrak{m}_+$ is less than some element in $\mathfrak{m}_+$. □

3.1. Nash triviality

Let M_1 and M_2 be Nash manifolds possibly with corners, and let M_3 be a semialgebraic subset of M_2. A Nash map $g : M_1 \to M_2$ is called *Nash trivial* over M_3 if there exists a Nash diffeomorphism $h : g^{-1}(x) \times M_3 \to g^{-1}(M_3)$ for some $x \in M_3$, such that $g \circ h$ coincides with the projection $g^{-1}(x) \times M_3 \to M_3$.

We define similarly the $\tilde{\mathbb{R}}$-*Nash triviality* of a $\tilde{\mathbb{R}}$-Nash map.

REMARK 3.2. – If $g : M_1 \to M_2$ is Nash trivial over M_3, then $\tilde{g} : \tilde{M}_1 \to \tilde{M}_2$ is $\tilde{\mathbb{R}}$-Nash trivial over $\tilde{M}_3$. Indeed, let $h : g^{-1}(x) \times M_3 \to g^{-1}(M_3)$ be a Nash diffeomorphism which gives the Nash trivialization of $g : M_1 \to M_2$. Then $\tilde{h} : \tilde{g}^{-1}(\tilde{x}) \times \tilde{M}_3 \to \tilde{g}^{-1}(\tilde{M}_3)$ is a $\tilde{\mathbb{R}}$-Nash $\tilde{\mathbb{R}}$-diffeomorphism of $\tilde{\mathbb{R}}$-Nash trivialization of $\tilde{g} : \tilde{M}_1 \to \tilde{M}_2$.

We use classical triviality results in the Nash category (cf. [4]) to deal with the triviality of real Milnor fibres.

LEMMA 3.3. –
1. *There exist $r_0 \in \mathbb{R}_+^*$ and a non-negative continuous semialgebraic function v defined on $[0, r_0]$, with zero set $\{0\}$, such that the projection map*

$$\pi : \bigcup_{0<\alpha<\rho \in \tilde{\mathbb{R}}} \tilde{F}_{\tilde{f}}(\rho, \alpha) \times \{(\rho, \alpha)\} \longrightarrow \tilde{\mathbb{R}}^2$$

given by $(\gamma, \rho, \alpha) \mapsto (\rho, \alpha)$ is $\tilde{\mathbb{R}}$-Nash trivial over

$$\mathscr{D} = \{(\rho, \alpha) \in \tilde{\mathbb{R}}^2 : 0 < \rho \leq \tilde{r}_0,\ 0 < \alpha \leq \tilde{v}(\rho)\}.$$

In particular, $\tilde{F}_{\tilde{f}}(\rho_1, \alpha_1)$ and $\tilde{F}_{\tilde{f}}(\rho_2, \alpha_2)$ are $\tilde{\mathbb{R}}$-Nash $\tilde{\mathbb{R}}$-diffeomorphic for (ρ_1, α_1) and (ρ_2, α_2) in $\mathscr{D}$.

2. *Moreover, the map π is weakly continuously $\tilde{\mathbb{R}}$-Nash trivial over*

$$\mathscr{D}' = \{(\rho, \alpha) \in \tilde{\mathbb{R}}^2 : \rho_0 \leq \rho \leq \tilde{r}_0,\ \alpha_0 \leq \alpha \leq \tilde{v}(\rho)\}$$

for any $0 < \alpha_0 < \rho_0 \in \mathfrak{m}_+$. In particular, $\tilde{F}_{\tilde{f}}(\rho_1, \alpha_1)$ and $\tilde{F}_{\tilde{f}}(\rho_2, \alpha_2)$ are weakly continuously $\tilde{\mathbb{R}}$-Nash $\tilde{\mathbb{R}}$-diffeomorphic for (ρ_1, α_1) and (ρ_2, α_2) in $\mathscr{D}'$.

Proof. – The proof of (1) is a consequence of Theorem 3 in [4]. To see this, note that the projection map

$$p : \bigcup_{0<a<r \in \mathbb{R}} F_f(r, a) \times \{(r, a)\} \longrightarrow \mathbb{R}^2$$

given by $(x, r, a) \mapsto (r, a)$ and its restriction to the boundaries

$$\bigcup_{0<a<r\in\mathbb{R}} \partial F_f(r, a) \times \{(r, a)\} \longrightarrow \mathbb{R}^2$$

are proper and submersive onto

$$\{(r, a) \in \mathbb{R}^2 : \ 0 < a \ll r \ll 1\}.$$

In particular, there exist $r_0 \in \mathbb{R}_+^*$ and a non-negative continuous semialgebraic function v defined on $[0, r_0]$, with zero set $\{0\}$, such that these maps are proper and submersive onto

$$D = \{(r, a) \in \mathbb{R}^2 : \ r \leq r_0, \ 0 < a \leq v(r)\}.$$

As a consequence p is Nash trivial over D by Theorem 3 in [4], and therefore π is $\tilde{R}$-Nash trivial over $\tilde{D} = \mathscr{D}$. The second statement of the lemma follows from Proposition 1.12. $\square$

Recall that $\mathscr{A}_f$ is the $\tilde{\mathbb{R}}$-algebraic set defined by $\tilde{f}$, namely

$$\mathscr{A}_f = \{\gamma \in \tilde{\mathbb{R}}^n : \tilde{f}(\gamma) = t\}.$$

Corollary 3.4. – *The map*

$$\bigcup_{\rho\in\tilde{\mathbb{R}}} \{\gamma \in \mathscr{F}_f : |\gamma| \leq \rho\} \times \{\rho\} \longrightarrow \tilde{\mathbb{R}}$$

given by $(\gamma, \rho) \mapsto \rho$, and hence the map $\gamma \mapsto |\gamma|$ from $\mathscr{F}_f$ to $\tilde{\mathbb{R}}$, are weakly continuously trivial over $\{\rho \in \mathfrak{m}_+ : \rho \geq \rho_0\}$ for some $\rho_0 \in \mathfrak{m}_+$.

Moreover, the map

$$\bigcup_{\rho\in\tilde{\mathbb{R}}} \{\gamma \in \mathscr{A}_f, \ |\gamma| \leq \rho\} \times \{\rho\} \longrightarrow \tilde{\mathbb{R}}$$

given by $(\gamma, \rho) \mapsto \rho$, and the map $\gamma \mapsto |\gamma|$ from $\mathscr{A}_f$ to $\tilde{\mathbb{R}}$, are weakly continuously trivial over $[\rho_0, \tilde{r}_0]_{\tilde{\mathbb{R}}}$ for some $r_0 \in \mathbb{R}$.

Proof. – We apply Lemma 3.3 by specifying α to the value t. This is possible because $t < \tilde{v}(\tilde{r}_0)$ since the semialgebraic function v provided by Lemma 3.3 is defined over $\mathbb{R}$, so that $\tilde{v}(\tilde{r}_0)$ belongs to $\mathbb{R}_+^* \subset \tilde{\mathbb{R}}$. It remains to recall that, by Lemma 3.1,

$$\mathscr{F}_f = \bigcup_{\rho\in\mathfrak{m}_+} \tilde{F}_{\tilde{f}}(\rho, t). \qquad \square$$

Remark 3.5. – We have deduced Corollary 3.4 from Lemma 3.3 by specifying α to the value $t \in (0, \tilde{v}(\tilde{r}_0)]_{\tilde{\mathbb{R}}}$. In the proof of Theorem 3.6 below, we will use the same result for a real value of α, so belonging to $(0, v(r_0)]$.

3.2. Comparison of homology groups

The goal of this section is to prove that the natural maps from $\tilde{H}^w_*(\mathcal{J}_f)$ to $\tilde{H}_*(\mathcal{J}_f)$ and from $\tilde{H}_*(\mathcal{J}_f)$ to $H_*(F_f(a,r))$ are isomorphisms, for $a \in \mathbb{R}^*_+$ and $r \in \mathbb{R}^*_+$ small enough.

THEOREM 3.6. – *Let f be a polynomial function on $\mathbb{R}^n$ vanishing at 0. Then there exist natural isomorphisms*

$$\tilde{H}^w_*(\mathcal{J}_f) \to \tilde{H}_*(\mathcal{J}_f) \to H_*(F_f(r,a)),$$

with $0 < a \ll r \ll 1$ small enough.

First, we reduce the problem to the closed and bounded case by the following lemma.

LEMMA 3.7. – *Let $\mathcal{M}$ be a local $\tilde{\mathbb{R}}$-Nash manifold possibly with corners, and $\mathcal{N}$ be a closed and bounded $\tilde{\mathbb{R}}$-Nash manifold possibly with corners. Let $\phi : \mathcal{N} \to \tilde{\mathbb{R}}$ be a positive $\tilde{\mathbb{R}}$-Nash function such that $\phi^{-1}(\mathfrak{m}_+) = \mathcal{M}$. Assume that the map*

$$\pi : \bigcup_{\rho \in \tilde{\mathbb{R}}} \{\gamma \in \mathcal{N} : \phi(\gamma) \leq \rho\} \times \{\rho\} \longrightarrow \tilde{\mathbb{R}}$$

defined by $(\gamma, \rho) \to \rho$ is weakly continuously $\tilde{\mathbb{R}}$-Nash trivial over $[\rho_0,\, \tilde{r}_0]_{\tilde{\mathbb{R}}}$ for some $\rho_0 \in \mathfrak{m}_+$ and $r_0 \in \mathbb{R}^*_+$, via a trivialization already defined over $\mathbb{R}$.

Then the inclusions $\mathcal{M} \to \phi^{-1}((0,\, \tilde{r}_0]_{\tilde{\mathbb{R}}})$ and $\phi^{-1}((0,\, \rho_0]_{\tilde{\mathbb{R}}}) \to \mathcal{M}$ are local $\tilde{\mathbb{R}}$-semialgebraic weak homotopy equivalences.

Proof. – We are going to construct a relevant $\tilde{\mathbb{R}}$-semialgebraic weak homotopy. Let denote by $\mathcal{N}_{\rho_0}$ and $\mathcal{N}_{\tilde{r}_0}$ respectively, the $\tilde{\mathbb{R}}$-Nash manifolds $\phi^{-1}((0,\, \rho_0]_{\tilde{\mathbb{R}}})$ and $\phi^{-1}((0,\, \tilde{r}_0]_{\tilde{\mathbb{R}}})$. By assumption, there exist a weakly continuous $\tilde{\mathbb{R}}$-Nash $\tilde{\mathbb{R}}$-diffeomorphism

$$\psi : \mathcal{N}_{\tilde{r}_0} \times [\rho_0, \tilde{r}_0]_{\tilde{\mathbb{R}}} \longrightarrow \bigcup_{\rho \in [\rho_0, \tilde{r}_0]_{\tilde{\mathbb{R}}}} \{\gamma \in \mathcal{N} : \phi(\gamma) \leq \rho\} \times \{\rho\}$$

such that $\pi \circ \psi$ is equal to the projection map

$$\mathcal{N}_{\tilde{r}_0} \times [\rho_0, \tilde{r}_0]_{\tilde{\mathbb{R}}} \longrightarrow [\rho_0, \tilde{r}_0]_{\tilde{\mathbb{R}}}.$$

We can assume moreover that ψ is the identity map on restriction to $\mathcal{N}_{\tilde{r}_0} \times \{\tilde{r}_0\}$. Let denote by ν the projection

$$\nu : \bigcup_{\rho \in \tilde{\mathbb{R}}} \{\gamma \in \mathcal{N} : \phi(\gamma) \leq \rho\} \times \{\rho\} \longrightarrow \tilde{\mathbb{R}}^n$$

defined by $(\gamma, \rho) \to \gamma$. Then we construct a $\tilde{\mathbb{R}}$-semialgebraic weak homotopy $\theta_\lambda : \mathcal{N}_{\tilde{r}_0} \longrightarrow \mathcal{N}_{\tilde{r}_0}$ by defining

$$\theta_\lambda(\gamma) = \nu \circ \psi\big(\gamma, \lambda(\rho_0 - \tilde{r}_0) + \tilde{r}_0\big)$$

for $\lambda \in [0,1]_{\tilde{\mathbb{R}}}$ and $\gamma \in \mathcal{N}_{\tilde{r}_0}$. In particular θ_0 is the identity map on $\mathcal{N}_{\tilde{r}_0}$, the image of $\mathcal{N}_{\tilde{r}_0}$ under θ_1 is equal to $\mathcal{N}_{\rho_0}$ and moreover $\theta_\lambda(\mathcal{M}) \subset \mathcal{M}$ for any $\lambda \in [0,1]_{\tilde{\mathbb{R}}}$ since the trivialization ψ comes from a trivialization defined over $\mathbb{R}$. □

The following result is an immediate consequence of Lemma 3.7 and Lemma 1.7.

COROLLARY 3.8. – *Under the assumptions of Lemma 3.7, inclusions $\mathscr{M} \to \phi^{-1}((0,\ \tilde{r}_0]_{\tilde{\mathbb{R}}})$ and $\phi^{-1}((0,\ \rho_0]_{\tilde{\mathbb{R}}}) \to \mathscr{M}$ are $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-homotopy equivalences. Therefore, the maps*

$$\tilde{H}_*(\phi^{-1}((0,\ \rho_0]_{\tilde{\mathbb{R}}})) \to \tilde{H}_*(\mathscr{M}) \to \tilde{H}_*(\phi^{-1}((0,\ \tilde{r}_0]_{\tilde{\mathbb{R}}}))$$

and

$$\tilde{H}_*^w(\phi^{-1}((0,\ \rho_0]_{\tilde{\mathbb{R}}})) \to \tilde{H}_*^w(\mathscr{M}) \to \tilde{H}_*^w(\phi^{-1}((0,\ \tilde{r}_0]_{\tilde{\mathbb{R}}}))$$

are isomorphisms.

REMARK 3.9. – The proof of Lemma 3.7 shows also that the inclusions $\mathscr{M} \to \phi^{-1}([0, \tilde{r}_0]_{\tilde{\mathbb{R}}})$ and $\phi^{-1}([0, \rho_0/2]_{\tilde{\mathbb{R}}}) \to \mathscr{M}$ are homotopy equivalences in the sense of the product topology, and the maps

$$H_*(\phi^{-1}((0, \rho_0]_{\tilde{\mathbb{R}}})) \longrightarrow H_*(\mathscr{M}) \longrightarrow H_*(\phi^{-1}((0, \tilde{r}_0]_{\tilde{\mathbb{R}}}))$$

are therefore isomorphisms.

Proof of Theorem 3.6. – The set $\mathscr{F}_f$ is the local $\tilde{\mathbb{R}}$-Nash manifold associated with the $\tilde{\mathbb{R}}$-Nash manifold $\mathscr{A}_f$ (actually an $\tilde{\mathbb{R}}$-algebraic set) and to the $\tilde{\mathbb{R}}$-Nash function $\phi : \mathscr{A}_f \longrightarrow \tilde{\mathbb{R}}$ defined by $\phi(\gamma) = |\gamma|$ for $\gamma \in \tilde{\mathbb{R}}^n$, cf. Example 1.2. Note moreover that for $\rho \in \tilde{\mathbb{R}}_+^*$, we have $\tilde{F}_{\tilde{f}}(\rho, t) = \phi^{-1}((0, \rho]_{\tilde{\mathbb{R}}})$.

Using Corollary 3.4 combined with Corollary 3.8, we obtain therefore that the morphisms

$$\tilde{H}_*(\tilde{F}_{\tilde{f}}(\rho_0, t)) \longrightarrow \tilde{H}_*(\mathscr{F}_f) \longrightarrow \tilde{H}_*(\tilde{F}_{\tilde{f}}(\tilde{r}_0, t))$$

and

$$\tilde{H}_*^w(\tilde{F}_{\tilde{f}}(\rho_0, t)) \longrightarrow \tilde{H}_*^w(\mathscr{F}_f) \longrightarrow \tilde{H}_*^w(\tilde{F}_{\tilde{f}}(\tilde{r}_0, t))$$

are isomorphisms. Let us concentrate on $\tilde{F}_{\tilde{f}}(\tilde{r}_0, t)$. As noted in Remark 3.5, for a sufficiently small value $a_0 \in \mathbb{R}_+^*$, we have similarly natural isomorphisms

$$\tilde{H}_*(\tilde{F}_{\tilde{f}}(\tilde{r}_0, t)) \longrightarrow \tilde{H}_*(\tilde{F}_{\tilde{f}}(\tilde{r}_0, \tilde{a}_0))$$

and

$$H_*^w(\tilde{F}_{\tilde{f}}(\tilde{r}_0, t)) \longrightarrow H_*^w(\tilde{F}_{\tilde{f}}(\tilde{r}_0, \tilde{a}_0)).$$

We achieve the proof using Lemma 2.5 combined with Proposition 2.7, since Lemma 2.5 provides a natural isomorphism from $\tilde{H}_*(\tilde{F}_{\tilde{f}}(\tilde{r}_0, \tilde{a}_0))$ to $H_*(F_f(r_0, a_0))$, whereas Proposition 2.7 gives a natural isomorphism between $H_*^w(\tilde{F}_{\tilde{f}}(\tilde{r}_0, \tilde{a}_0))$ and $H_*(F_f(r_0, a_0))$. □

4. A characterisation of Milnor fibres

The final goal of the paper is to give a characterization of the Milnor fibre $F_f(r, a)$ of a polynomial function f in terms of its associated set of Puiseux series $\mathscr{F}_f$. We propose a semialgebraic (together with a piecewise linear) characterization in Theorem 4.1.(1), valid in any dimension. We propose also a Nash characterization, for which we need to exclude some small dimensions for topological reasons.

Recall that if h is a semialgebraic homeomorphism between semialgebraic neighborhoods of 0 in $\mathbb{R}^n$, then $\tilde{h}$ is an $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-homeomorphism between $\tilde{\mathbb{R}}$-semialgebraic $\tilde{\mathbb{R}}$-neighborhoods of 0 in $\tilde{\mathbb{R}}^n$.

THEOREM 4.1. – *Let f and g be polynomial functions on $\mathbb{R}^n$ vanishing at 0, and let h be a semialgebraic homeomorphism between semialgebraic neighborhoods of 0 in $\mathbb{R}^n$.*

1. *If $\tilde{h}(\mathcal{J}_f) = \mathcal{J}_g$ then $F_f(r,a)$ and $F_g(r,a)$ are semialgebraically homeomorphic for $0 < a \ll r \ll 1$.*

   In particular, $F_f(r,a)$ and $F_g(r,a)$ are piecewise linear homeomorphic (in the sense that C^∞-semialgebraic triangulations of $F_f(r,a)$ and $F_g(r,a)$ are piecewise linear homeomorphic).
2. *Assume moreover that $\tilde{h}_{|\mathcal{J}_f}$ is a weakly continuous homeomorphism onto $\mathcal{J}_g$. Then $F_f(r,a)$ and $F_g(r,a)$ are Nash diffeomorphic for $0 < a \ll r \ll 1$, under the condition that n is not equal to 5 or 6. If $n \neq 5$, then* Int $F_f(r,a)$ *and* Int $F_g(r,a)$ *are analytically diffeomorphic for $0 < a \ll r \ll 1$.*

The proof of Theorem 4.1.(1) is based on piecewise linear topology [17], and is exposed in Section 4.1. The proof of Theorem 4.1(2) is more involved, and we use the theory of microbundles [11, 14] to discuss Nash structures on the Milnor fibres. Note that Theorem 4.1.(2) is false without the additional condition that $\tilde{h}_{|\mathcal{J}_f}$ is a weakly continuous homeomorphism, as illustrated by the example of Kervaire's exotic sphere presented in Section 4.2. We postpone the proof of this fact after the proof of Theorem 4.1.(1) since we need it to prove Proposition 4.6.

4.1. Proof of the piecewise linear characterization

The goal of the section is to understand to which extend the set of Puiseux series $\mathcal{J}_f$ determines the Milnor fibre $F_f(r,a)$ of a polynomial function f. Here is a first result is this direction.

LEMMA 4.2. – *Let f and g be continuous semialgebraic function germs at 0 in $\mathbb{R}^n$. If $\mathcal{J}_f = \mathcal{J}_g$ then $F_f(r,a) = F_g(r,a)$ for $0 < a \ll r \ll 1$.*

Proof. – It suffices to prove that $g = f$ on $\{g > 0\} \cup \{f > 0\}$. Assuming it is not the case, there exists a continuous semialgebraic curve $\gamma : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}^n, 0)$ along which f is not equal to g and either $f \circ \gamma(s)$ or $g \circ \gamma(s)$ is strictly positive for $s > 0$ small enough. Assume for example that $g \circ \gamma(s)$ is strictly positive. We can suppose $g \circ \gamma(s) = s$ by changing the parameter s, so that $\gamma \in \mathcal{J}_g$. By assumption we obtain therefore $\gamma \in \mathcal{J}_f$, which contradicts that f is not equal to g along γ. □

Next result is the key argument in the proof of Theorem 4.1.(1).

PROPOSITION 4.3. – *Let f be a continuous semialgebraic function on a compact semialgebraic subset X of $\mathbb{R}^n$, with $0 \in X$ and $f(0) = 0$. Let d_i be a non-negative continuous semialgebraic function on X whose zero set is reduced to $\{0\}$, for $i \in \{1,2\}$. Set*

$$N_i(a,r) = \{x \in X : d_i(x) \leq r,\ f(x) = a\}$$

for real numbers $a, r \in \mathbb{R}$, with $i \in \{1,2\}$. Then $N_1(a,r)$ and $N_2(a,r)$ are semialgebraically homeomorphic for $0 < a \ll r \ll 1$.

We postpone the proof of Proposition 4.3 in order to show how we use it in the proof of Theorem 4.1.(1).

Proof of Theorem 4.1.(1). – The continuous semialgebraic functions f and $g \circ h$ share the same set of Puiseux series $\mathcal{F}_f = \mathcal{F}_{g\circ h}$ since the set $\tilde{h}^{-1}(\mathcal{F}_g)$, which coincides with $\mathcal{F}_f$ by assumption, is equal to $\mathcal{F}_{g\circ h}$. Therefore the sets $F_f(r,a)$ and $F_{g\circ h}(r,a)$ are equal for $0 < a \ll r \ll 1$ by Lemma 4.2.

Define a distance function d on a compact neighborhood X of 0 in $\mathbb{R}^n$ by $d(x) = |h^{-1}(x)|$ for $x \in X$. Then $F_g(r,a)$ is semialgebraically homeomorphic to

$$N(a,r) = \{x \in X : d(x) \le r,\ g(x) = a\}$$

by Proposition 4.3, for $0 < a \ll r \ll 1$. This last set is carried to

$$\{y \in h^{-1}(X) : |y| \le r,\ g \circ h(y) = a\}$$

by the semialgebraic homeomorphism h^{-1}, and thus it is equal to $F_{g\circ h}(r,a)$ for $0 < a \ll r \ll 1$. As a consequence $F_f(r,a) = F_{g\circ h}(r,a)$ is semialgebraically homeomorphic to $F_g(r,a)$. ☐

The proof of Proposition 4.3 is classical piecewise linear topology in the case all the data are piecewise linear. The most delicate part is to come back to this situation. We begin with a lemma in the piecewise linear case.

Lemma 4.4. – *Let f be a piecewise linear function on a compact polyhedron X of $\mathbb{R}^n$, with $0 \in X$ and $f(0) = 0$. Let d_i be a non-negative piecewise linear function on X whose zero set is reduced to $\{0\}$, for $i \in \{1,2\}$. Then $N_1(a,r)$ and $N_2(a,r)$ are piecewise linear homeomorphic for $0 < a \ll r \ll 1$.*

Proof. – Let K be a simplicial complex such that X is the underlying polyhedron of K and f together with d_1, d_2 are simplicial on K, i.e., linear on each simplex in K.

Let K' denote the barycentric subdivision of K, and choose ε so small that the set

$$\{x \in X : 0 < d_i(x) \le \varepsilon\},$$

for $i \in \{1,2\}$, does not contain any vertex in K'', the double barycentric subdivision of K. Then by the uniqueness of regular neighborhoods (cf. Theorem 3.8 in [17]), there exist simplicial isomorphisms α_i from K'' to some simplicial subdivisions K_i of K such that

$$\alpha_i(|\operatorname{st}(0,K'')|) = \{x \in X : \phi_i(x) \le \varepsilon\},$$

for $i \in \{1,2\}$, where the notation $\operatorname{st}(0,K'')$ denotes the star of K'' at 0 (i.e., the simplicial complex obtained by taking all simplexes adjacent to 0) and $|\operatorname{st}(0,K'')|$ denotes its underlying polyhedron. Explicitly, we define α_i to be the identity map on $\{0\} \cup (X - |\operatorname{st}(0,K'')|)$ and $\alpha_i(v)$ to be equal to $\phi_i(\varepsilon) \cap l_v$ for each vertex v in the link $\operatorname{lk}(0,K'')$, and we extend linearly α_i to each simplex in K'' (where the notation l_v denotes the segment with ends 0 and v). In particular, note that $\alpha_1^{-1}(f^{-1}(0))$ is equal to $\alpha_2^{-1}(f^{-1}(0))$ by linearity of f on the simplexes.

Hence, by replacing f with $f \circ \alpha_i$ and X with $|\operatorname{st}(0,K'')|$, we have reduced the problem to prove that, for f_1 and f_2 simplicial functions on K such that $f_1^{-1}(0) = f_2^{-1}(0)$, the sets $f_1^{-1}(a)$ and $f_2^{-1}(a)$ are piecewise linear homeomorphic for small values of $a > 0$. This statement also follows from the uniqueness of regular neighborhoods. ☐

In order to prove Proposition 4.3, it is natural to hope to triangulate the maps $(f, d_i) : X \to \mathbb{R}^2$, for $i \in \{1, 2\}$. However a global triangulation of a continuous semialgebraic map from a compact set to $\mathbb{R}^2$ is impossible in general. We overcome this difficulty using a weak triangulation (given in Lemma 7 in [21]) which will be sufficient to treat our local situation.

Before entering into the details of the proof, we recall the notion of cell complex (from [17]) that will be useful for the proof. A cell means a compact convex polyhedron in $\mathbb{R}^n$. A cell is piecewise linear homeomorphic to a simplex. A cell complex means a family of cells such that the boundary of each cell is the union of some cells and the union of the interior of the cells is a locally finite disjoint union. Note also that, given a cell complex, the interior of the elements form a cell complex in the usual sense in topology. A cellular map $h : L_1 \to L_2$ between cell complexes means a piecewise linear map $h : |L_1| \to |L_2|$ which linearly carries each element of L_1 to some element of L_2, where $|L_1|$ denotes the underlying polyhedron of L_1. The Alexander trick ([17], p. 37) is the statement which states that a piecewise linear homeomorphism between the boundaries of two cells can be extended to a piecewise linear homeomorphism between the cells. We can see some examples of application of the Alexander trick in [17] and [20] for instance.

Proof of Proposition 4.3. – By the triangulation theorem of semialgebraic functions (Theorem 3.2 in [20]), we can suppose that X is the underlying polyhedron of a simplicial complex K and that f is simplicial on K, with $f^{-1}(0)$ a union of simplexes. Using Lemma 4.4, we are reduced to prove that we can suppose that d_1 and d_2 are piecewise linear.

By Lemma 7 in [21], there exist a semialgebraic homeomorphism h_i of X such that $h_i(\sigma) = \sigma$ for each $\sigma \in K$, and a neighborhood U of 0 in $f^{-1}(0)$ together with a compact polyhedral neighborhood V of $U - \{0\}$ in X such that $(f, d_i) \circ h_i$ is piecewise linear on V for $i \in \{1, 2\}$.

Note that:

1. V is not necessarily a neighborhood of 0,
2. $d_i \circ h_i$ is piecewise linear only on V, but not necessarily on $f^{-1}([0, a])$,
3. $f \circ h_i$ is not necessarily piecewise linear on $f^{-1}([0, a])$ globally.

The point (2) is not annoying for the reduction to the case where d_1 and d_2 are piecewise linear because we are interested only in the set

$$\{x \in X : d_i(x) \leq \varepsilon,\ f(x) = a\},$$

that is, we need the condition that d_i is piecewise linear only on a neighborhood of $\{x \in X : d_i(x) = \varepsilon,\ f(x) = a\}$ in X. However, the main difficulty consists in point (3). In the sequel, we modify the semialgebraic homeomorphism h_i so that f becomes equal to $f \circ h_i$ on $f^{-1}([0, a])$.

We compare first the functions f and $f \circ h_i$ on V. By subdividing K, we can assume that V is the underlying polyhedron of some subcomplex of K, and that the function $f \circ h_i$ is simplicial on $K|_V$ for $i \in \{1, 2\}$, although we may lose the property that $h_i(\sigma) = \sigma$ for $\sigma \in K$. Set

$$K|_{V \cap f^{-1}(a)} = \{\sigma \cap V \cap f^{-1}(a) : \sigma \in K\},$$

for $a \in \mathbb{R}$. Then $K|_{V\cap f^{-1}(a)}$ is a cell complex (not necessarily simplicial). Then, for $0 < a \ll 1 \in \mathbb{R}$, there is a unique cellular isomorphism

$$k_{i,a} : K|_{V\cap f^{-1}(a)} \to K|_{V\cap(f\circ\alpha_i)^{-1}(a)}$$

such that for each $\sigma \in K|_{V\cap f^{-1}(a)}$, the cells σ and $k_{i,a}(\sigma)$ are included in some simplex in K of dimension equal to $\dim\sigma + 1$. Hence, for some $a > 0$, there is exists a piecewise linear homeomorphism

$$k_i : V \cap f^{-1}([0, a]) \to V \cap (f \circ h_i)^{-1}([0, a])$$

such that $f = f \circ h_i \circ k_i$ on $V \cap f^{-1}([0, a])$ and $k_i(\sigma) = \sigma$ for $\sigma \in K|_V$. Moreover, we can extend h_i to a piecewise linear homeomorphism h_i of X so that $h_i(\sigma) = \sigma$ for $\sigma \in K$ by the Alexander trick. Note that $d_i \circ h_i \circ k_i$ continues to be piecewise linear on $V \cap f^{-1}([0, a])$.

So replacing h_i with $h_i \circ k_i$, we have obtained that $f = f \circ h_i$ on $V \cap f^{-1}([0, a])$. Finally, we are going to modify h_i outside of $V \cap f^{-1}([0, a])$ so that $f = f \circ h_i$ on $f^{-1}([0, a])$. Set

$$L = K|_{f^{-1}([0,a])} = \{\sigma \cap f^{-1}([0, a]) : \sigma \in K\}$$

and consider the restriction $h_i' = h_{i|V\cap|L|}$. Then L is a cellular decomposition of $f^{-1}([0, a])$, the set $V \cap |L|$ is the underlying polyhedron of some subcomplex L' of L such that $h_i'(\sigma) = \sigma$ for $\sigma \in L'$, and $f = f \circ h_i'$ on $|L'|$. Hence by the Alexander trick again, we can extend h_i' to a semialgebraic homeomorphism h_i'' of $|L|$ so that $f = f \circ h_i''$ on $|L|$. □

REMARK 4.5. –

1. As a consequence of the proof above, we can refine Theorem 4.1.(1) as follows. Choose $0 < a_1 \ll r_1 \ll 1$ and $0 < a_2 \ll r_2 \ll 1$ so that

   $$h(F_f(r_1, a_1)) \subset F_g(r_2, a_2).$$

   Then

   $$h_{|F_f(r_1,a_1)} : F_f(r_1, a_1) \to F_g(r_2, a_2)$$

   is semialgebraically isotopic to a semialgebraic homeomorphism onto $F_g(r_2, a_2)$. We will use this refined version in the proof of Theorem 4.1.(2).
2. In the same spirit of the proof of Theorem 4.1.(1), we can prove that if f and g are polynomial functions on $\mathbb{R}^n$ vanishing at 0, if h is a semialgebraic continuous map from a semialgebraic neighborhood of 0 in $f^{-1}(0)$ to $\mathbb{R}^n$, and if the restriction of $\tilde{h}$ to

   $$\{\gamma \in \tilde{f}^{-1}(0) : \gamma(0) = 0\}$$

   is a bijection onto

   $$\{\gamma \in \tilde{g}^{-1}(0) : \gamma(0) = 0\},$$

   then the sets $\{x \in f^{-1}(0) : |x| \le r\}$ and $\{x \in g^{-1}(0) : |x| \le r\}$ are semialgebraically homeomorphic, for $r > 0$ small enough.

4.2. Kervaire's exotic sphere

Regard $\mathbb{C}$ as $\mathbb{R}^2$, and define polynomials f and g on $\mathbb{C}^6 \times \mathbb{R}$ by

$$f(z,x) = |z_1|^2 + x^2$$

and

$$g(z,x) = |z_1^2 + \cdots + z_5^2 + z_6^3|^2 + x^2,$$

for $(z,x) = (z_1,\dots,z_6,x) \in \mathbb{C}^6 \times \mathbb{R}$. Let S denote the sphere in $\mathbb{C}^6 \times \mathbb{R}$ with center 0 and with radius 1. The set $g^{-1}(0)\cap S$ is the Kervaire's exotic sphere (cf. [15] p. 72), i.e., $g^{-1}(0)\cap S$ is a topological sphere of dimension 9 (Theorem 8.5 and 9.1 in [15]) which is not diffeomorphic to the standard 9-sphere. Note that $f^{-1}(0) \cap S$ is a standard sphere, also of dimension 9.

PROPOSITION 4.6. – 1. *There exists a semialgebraic homeomorphism h of $\mathbb{C}^6 \times \mathbb{R}$ such that $h(0) = 0$ and $\tilde{h}(\mathcal{J}_f) = \mathcal{J}_g$.*

2. *$F_f(r,a)$ and $F_g(r,a)$ are not Nash diffeomorphic for $0 < a \ll r \ll 1$.*

Proof. – 1. We begin with constructing a semialgebraic homeomorphism H of S such that

$$H(f^{-1}(0) \cap S) = g^{-1}(0) \cap S$$

and $f = g \circ H$ on a neighborhood of $f^{-1}(0) \cap S$ in S.

Regard $g^{-1}(0) \cap S$ and $f^{-1}(0) \cap S$ as piecewise linear manifolds by semialgebraic C^1 triangulations (cf. Proposition I.3.13 and Remark I.3.22 in [19] for semialgebraic C^1 triangulations). Then these manifolds are piecewise linear homeomorphic to a standard piecewise linear sphere, since a topological sphere of dimension greater than or equal to 5 admits a unique piecewise linear manifold structure [11]. Regard moreover the pairs $(S, g^{-1}(0) \cap S)$ and $(S, f^{-1}(0) \cap S)$ as piecewise linear manifold pairs (by semialgebraic C^1 triangulations). Then they are unknotted piecewise linear sphere pairs by the Zeeman's unknotting theorem since

$$\dim S - \dim f^{-1}(0) \cap S = \dim S - \dim g^{-1}(0) \cap S = 3$$

(see Theorem 7.1 in [17]). Hence both of them are standard piecewise linear sphere pairs, and therefore there exists a semialgebraic homeomorphism H such that $H(f^{-1}(0) \cap S) = g^{-1}(0) \cap S$.

Moreover, we can modify H so that the additional condition that $f = g \circ H$ on a neighborhood of $f^{-1}(0) \cap S$ in S holds, as we did in the proof of Proposition 4.3.

Let $y = (z,x) \in S$. Define two semialgebraic maps

$$l_y : [0,+\infty) \to \mathbb{C}^6 \times \mathbb{R}, \quad s \mapsto sy$$

and

$$c_y : [0,+\infty) \to \mathbb{C}^6 \times \mathbb{R}, \quad s \mapsto (s^{1/2}z_1,\dots,s^{1/2}z_5, s^{1/3}z_6, sx)$$

so that the image L_y of l_y is the half line passing through $y \in S$, and the image C_y of c_y is a semialgebraic curve with origin $0 \in \mathbb{C}^6 \times \mathbb{R}$. Note moreover that

$$f \circ l_y(s) = s^2 f(y) \qquad \text{and} \qquad g \circ c_y(s) = s^2 g(y)$$

for $s \in [0,+\infty)$, and that the functions $f \circ l_y$ and $g \circ c_y$ are strictly increasing and converging to $+\infty$ if $f(y) \neq 0$, respectively $g(y) \neq 0$. In particular, for $y \in S$

there exists an homeomorphism H_y from L_y onto $C_{H(y)}$ such that $H_y \circ l_y = c_{H(y)}$ if $f(y) = 0$, and $g \circ H_y = f$ on L_y otherwise.

We use these homeomorphisms to define the map

$$h : C^6 \times \mathbb{R} \to C^6 \times \mathbb{R}$$

so that $h_{|L_y}$ coincides with $H_y : L_y \to C_{H(y)}$, for any $y \in S$. Then h preserves the origin, and h is well-defined outside the origin since $C^6 \times \mathbb{R} \setminus \{0\}$ is the disjoint union of the half lines $L_y \setminus \{0\}$, for $y \in S$. Moreover h is a bijection because $C^6 \times \mathbb{R} \setminus \{0\}$ is the disjoint union of the curves $C_y \setminus \{0\}$, for $y \in S$, and it satisfies $f = g \circ h$ by construction. Finally h is clearly a semialgebraic map continuous at $S \setminus f^{-1}(0)$, and the continuity of h at $S \cap f^{-1}(0)$ follows from the condition that $f = g \circ H$ on a neighborhood of $f^{-1}(0) \cap S$ in S. As a consequence h is a semialgebraic homeomorphism such that $\tilde{h}(\mathscr{F}) = \mathscr{F}_g$, as required.

2. The set $g^{-1}(0) \cap S$ is the Kervaire's exotic sphere (cf. [15] p. 72), whereas $f^{-1}(0) \cap S$ is a standard sphere. As a consequence the sets $F_f(r, a)$ and $F_g(r, a)$, whose boundary is diffeomorphic to $f^{-1}(0) \cap S$ and $g^{-1}(0) \cap S$ respectively, cannot be diffeomorphic. □

5. Microbundles and the Nash characterization

We begin with recalling some notions about microbundles, as introduced by J. Milnor in [14]. Then we use the notion of concordance of microbundles to deduce the required isomorphisms, using [11].

5.1. Microbundles

A *microbundle* of rank $n \in \mathbb{N}$ is a diagram

$$B \xrightarrow{i} E \xrightarrow{j} B$$

where B and E are topological spaces, such that the composition $j \circ i$ is the identity map, and for each $b \in B$ there exist an open neighborhood U of b, an open neighborhood V of $i(b)$ and a homeomorphism $h : V \to U \times \mathbb{R}^n$, with $i(U) \subset V$ and $j(V) \subset U$, such that the map $h \circ i_{|_U} : U \to U \times \mathbb{R}^n$ coincides with the map $x \to (x, 0)$ and the map $p_1 \circ h : V \to U$ coincides with $j_{|_V}$, where p_1 denotes the projection $U \times \mathbb{R}^n \to U$ onto the first coordinate.

A *smooth microbundle* is a microbundle such that B and E are C^∞-manifolds possibly with boundary, such that i and j are of class C^∞ and h is a diffeomorphism. A *smooth structure* on a topological manifold possibly with boundary is a C^∞-equivalence class of atlases on it. A smooth structure on a microbundle is the data of smooth structures on B and E such that the microbundle becomes a smooth microbundle.

Let $\mathfrak{b}_k$ denote a microbundle $B \xrightarrow{i_k} E_k \xrightarrow{j_k} B$, for $k \in \{1, 2\}$. We say that $\mathfrak{b}_1$ is *isomorphic* to $\mathfrak{b}_2$ if there exist open neighborhoods U_1 of $i_1(B)$ in E_1 and U_2 of $i_2(B)$ in E_2 together with a homeomorphism $h : U_1 \to U_2$ such that the following diagram is commutative.

$$\begin{tikzcd} B \arrow[r, "i_1"] \arrow[d, "\mathrm{id}"'] & U_1 \arrow[r, "j_1|_{U_1}"] \arrow[d, "h"'] & B \arrow[d, "\mathrm{id}"'] \\ B \arrow[r, "i_2"] & U_2 \arrow[r, "j_2|_{U_2}"] & B. \end{tikzcd}$$

If h is, moreover, an inclusion, we say $\mathfrak{b}_1$ is micro-identical to $\mathfrak{b}_2$.

An important example is the *tangent microbundle* $\mathfrak{t}M$ of a topological manifold M possibly with boundary, defined by

$$M \overset{D}{\to} M \times M \overset{p_1}{\to} M$$

where D is the diagonal map (and p_1 still denotes the projection onto the first factor). If M is a C^∞-manifold possibly with boundary, we regard the tangent vector bundle $p : TM \to M$, denoted by $\mathfrak{T}M$, as a microbundle $M \to TM \to M$ by defining the maps $M \to TM$ and $TM \to M$ to be respectively the zero cross-section and the projection. Then $\mathfrak{T}M$ is isomorphic to $\mathfrak{t}M$ (Theorem 2.2 in [14]). We fix such an isomorphism for each M.

Let $(\mathfrak{t}M) \times [0, 1]$ denote the microbundle

$$M \times [0, 1] \xrightarrow{D\times\mathrm{id}} M \times M \times [0, 1] \xrightarrow{p_1\times\mathrm{id}} M \times [0, 1].$$

We define similarly a microbundle bundle $\mathfrak{b} \times [0, 1]$ for any microbundle $\mathfrak{b}$.

In order to introduce a microbundle map, let us recall what a vector bundle map is. Note that any vector bundle over a topological space is a microbundle. Let $\mathfrak{b}_1$ and $\mathfrak{b}_2$ be vector bundles $E_1 \xrightarrow{j_1} B_1$ and $E_2 \xrightarrow{j_2} B_2$, respectively, and let $g : B_1 \to B_2$ be a continuous map. A *vector bundle map* $\mathfrak{b}_1 \to \mathfrak{b}_2$ covering g is the following commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "G"] \arrow[d, "j_1"'] & E_2 \arrow[d, "j_2"'] \\ B_1 \arrow[r, "g"] & B_2 \end{tikzcd}$$

such that G is a continuous map and for each $x \in B_1$, $G|_{j_1{}^{-1}(x)}$ is a linear morphism onto $j_2^{-1}(g(x))$. In a similar way, we define a microbundle map as follows. Let $\mathfrak{b}_k$ be a microbundle $B_k \xrightarrow{i_k} E_k \xrightarrow{j_k} B_k$, for $k \in \{1, 2\}$, and let $g : B_1 \to B_2$ be a continuous map. A *microbundle map* $\mathfrak{b}_1 \to \mathfrak{b}_2$ covering g is the following commutative diagram

$$\begin{tikzcd} B_1 \arrow[r, "i_1"] \arrow[d, "g"'] & U_1 \arrow[r, "j_1|_{U_1}"] \arrow[d, "G"'] & B_1 \arrow[d, "g"'] \\ B_2 \arrow[r, "i_2"] & E_2 \arrow[r, "j_2"] & B_2 \end{tikzcd}$$

such that $G|_{U_1 \cap j_1^{-1}(b_1)}$ is an open embedding into $j_2^{-1}(g(b_1))$ for each $b_1 \in B_1$, where U_1 is an open neighborhood of $i_1(B_1)$ in E_1 and G is a continuous map. For a homeomorphism $g : M_1 \to M_2$ between topological manifolds possibly with boundary, let $\boldsymbol{g}\colon \mathfrak{t}M_1 \to \mathfrak{t}M_2$

denote the microbundle map covering g defined by $U_1 = M_1 \times M_1$ and $G(x, y) = (g(x), g(y))$.

Given a microbundle $\mathfrak{b}$: $B \xrightarrow{i} E \xrightarrow{j} B$, a topological space A and a continuous map $g : A \to B$, we define the *induced microbundle* $g^*\mathfrak{b}$ by the diagram

$$A \to \{(a, e) \in A \times E : g(a) = j(e)\} \to A$$

where the maps are defined respectively by $a \mapsto (a, i \circ g(a))$ and $(a, e) \mapsto a$.

5.2. Concordance

A *concordance* between smooth structures $\mathfrak{b}_0$ and $\mathfrak{b}_1$ on a microbundle $\mathfrak{b}$ is a smooth structure $\mathfrak{A}$ on $\mathfrak{b} \times [0, 1]$ such that the restriction $\mathfrak{A}|_{B\times\{k\}}$ is micro-identical to $\mathfrak{b}_k$, for $k \in \{0, 1\}$. A *stable smooth structure* on $\mathfrak{b}$ means a smooth structure on a microbundle $B \to E \times \mathbb{R}^l \to B$ for some $l \in \mathbb{N}$, where the maps are defined by $b \mapsto (i(b), 0)$ and $(e, x) \mapsto e$. We naturally define a *stable concordance* between stable smooth structures.

A *concordance* between two smooth structures on a topological manifold possibly with boundary M is a smooth structure $\mathfrak{A}$ on $M\times[0, 1]$ such that the restriction $\mathfrak{A}|_{M\times\{0\}}$ coincides with the first smooth structure, and $\mathfrak{A}|_{M\times\{1\}}$ coincides with the second.

The homotopy theorem for microbundles (cf. §3 in [14]) is a useful tool to produce isomorphisms between microbundles. We recall its statement since we will use it in the sequel. Let A and B be topological spaces, $\mathfrak{b}$: $B \xrightarrow{i} E \xrightarrow{j} B$ be a microbundle and f and g be continuous maps from A to B. Assume A is paracompact and f and g are homotopic. Then the homotopy theorem for microbundles states that $f^*\mathfrak{b}$ and $g^*\mathfrak{b}$ are isomorphic.

The following proposition gives a criteria for two smooth structures on a topological manifold to be diffeomorphic.

Proposition 5.1. – *Let M be a topological manifold possibly with boundary of dimension different from 4 and 5 if $\partial M \neq \emptyset$, or of dimension different from 4 if $\partial M = \emptyset$. Let M_0 and M_1 be C^∞-manifolds possibly with boundary which are homeomorphic to M via $h_k : M \to M_k$, for $k \in \{0, 1\}$. Assume that there exist a vector bundle $\mathfrak{b} : V \to M \times [0, 1]$ and a microbundle isomorphism $H : \mathfrak{b} \to (\mathfrak{t}M) \times [0, 1]$ such that*

$$\mathbf{h}_k \circ (H|_{M\times\{k\}}) : \mathfrak{b}|_{M\times\{k\}} \to \mathfrak{T}M_k$$

is a vector bundle map covering h_k, for $k \in \{0, 1\}$. Then M_0 and M_1 are C^∞-diffeomorphic.

Proof. – Consider the case $\partial M = \emptyset$. The induced vector bundles $h_0^*(\mathfrak{T}M_0)$ and $h_1^*(\mathfrak{T}M_1)$ over M give two smooth structures on $\mathfrak{t}M$. By our assumption, they are concordant.

These smooth structures can also be defined as follows. Assume M be embedded in $\mathbb{R}^n$, and let $r : N \to M$ be a retraction of an open neighborhood of M in $\mathbb{R}^n$. Let $r_k : N \to M$, for $k \in \{0, 1\}$, be continuous maps homotopic to $r : N \to M$ so that $h_k \circ r_k$ are smooth. Then $h_k^*(\mathfrak{T}M_k)$ is micro-identical to $(h_k \circ r_k)^*(\mathfrak{T}M_k)|_M$, for $k \in \{0, 1\}$. Apply the pull-back rule (see p. 166 in [11]). The structures $(h_k \circ r_k)^*(\mathfrak{T}M_k)|_M$ are the smooth structures on $\mathfrak{t}M$ endowed from the smooth structures on M given by $h_k : M \to M_k$, and the smooth structures $(h_k \circ r_k)^*(\mathfrak{T}M_k)|_M$ are also the image of the smooth structures on M under the pull-back rule. Then Theorem 4.3 in Essay IV of [11] states that the two smooth structures on M given by M_0 and M_1 are concordant. As a consequence, M_0 and M_1 are diffeomorphic by the Concordance Implies Isotopy Theorem 4.1 in Essay I, ibid. Note moreover that we can choose the diffeomorphism to be isotopic to $h_1 \circ h_0^{-1}$.

If $\partial M \neq \emptyset$, we can repeat the same arguments as above for ∂M_k, with $k \in \{0,1\}$. Then we see that $\big(h_0^*(\mathfrak{T}M_0)\big)|_{\partial M}$ and $\big(h_1^*(\mathfrak{T}M_1)\big)|_{\partial M}$ are stably concordant, and we can apply Theorems 4.3 and 4.1 of [11] as above. Hence the boundaries ∂M_0 and ∂M_1 are diffeomorphic, and moreover, modifying h_k, we can give a smooth structure to ∂M so that $h_k|_{\partial M} : \partial M \to \partial M_k$ is a diffeomorphism, for $k \in \{0,1\}$. By collaring, we can extend smooth structures on ∂M to a neighborhood U of ∂M in M where h_k is a C^∞-embedding. Then we can, once more, apply the above arguments to $\operatorname{Int} M_k$, for $k \in \{0,1\}$, because the relative versions of Theorems 4.3 and 4.1 in [11] hold. Hence there exists a C^∞-diffeomorphism $F : \operatorname{Int} M_0 \to \operatorname{Int} M_1$ such that $F = h_1 \circ h_0^{-1}$ on $h_0(U') \cap \operatorname{Int} M_0$ for a closed neighborhood U' of ∂M in M included in U. Thus we see that M_0 and M_1 are diffeomorphic in the same way. □

COROLLARY 5.2. – *Let M_0 and M_1 be C^∞-manifolds possibly with boundary, of dimension different from 4 or 5 if $\partial M_k \neq \emptyset$ or of dimension different from 4 if $\partial M_k = \emptyset$, with $k \in \{0,1\}$. Assume that there exists an isotopy $g_s : M_0 \to M_1$, with $0 \le s \le 1$, such that g_0 is a homeomorphism onto M_1 and g_1 is a C^1-embedding. Then M_0 and M_1 are C^∞-diffeomorphic.*

Proof. – Set $M = M_0$, let $h_0 : M \to M_0$ denote the identity map, and let h_1 denote the map $g_0 : M \to M_1$. Let $\mathfrak{b}$ be the vector bundle $(\mathfrak{T}M) \times [0,\ 1]$, i.e.,

$$(TM) \times [0,\ 1] \to M \times [0,\ 1].$$

We want to define a microbundle map $H : \mathfrak{b} \to (\mathfrak{t}M) \times [0,\ 1]$ so that the conditions in Proposition 5.1 are satisfied. Consider the microbundle homotopy $\boldsymbol{g}_s : \mathfrak{t}M_0 \to \mathfrak{t}M_1$ covering $g_s : M_0 \to M_1$, with $0 \le s \le 1$. Then g_0 is bijective, but $\boldsymbol{g}_0$ is not necessarily a vector bundle map. Moreover g_1 is not necessarily bijective, however $\boldsymbol{g}_1$ is a vector bundle map. Our goal is to modify $\boldsymbol{g}_s$ so that g_s is bijective for any $s \in [0,\ 1]$ and $\boldsymbol{g}_1$ remains a vector bundle map.

By the homotopy theorem for microbundles (or more precisely its proof in §6 of [14]), we have a microbundle homotopy $G_s : \mathfrak{t}M_0 \to \mathfrak{t}M_1$, with $0 \le s \le 1$, covering g_0 such that $G_0 = \boldsymbol{g}_0$ and $G_1 : \mathfrak{T}M_0 \to \mathfrak{T}M_1$ is a vector bundle map, where we identify $\mathfrak{t}M_k$ with $\mathfrak{T}M_k$ by Theorem 2.2 in [14], for $k \in \{0,1\}$. Set

$$H(x,s) = (\boldsymbol{g}_0^{-1} \circ G_s(x), s) \quad \text{for } (x,s) \in (TM) \times [0,\ 1].$$

Then H is a microbundle map from $\mathfrak{b}$ to $(\mathfrak{t}M) \times [0,\ 1]$ such that $\boldsymbol{h}_0 \circ (H|_{M\times\{0\}})$ is the identity map. Moreover

$$\mathbf{h}_1 \circ (H|_{M\times\{1\}}) = \mathbf{h}_1 \circ \mathbf{g}_0^{-1} \circ G_1 = G_1$$

and hence $\boldsymbol{h}_0 \circ (H|_{M\times\{0\}})$ and $\boldsymbol{h}_1 \circ (\pi|_{M\times\{1\}})$ are vector bundle maps. Therefore $\mathfrak{b}$ and H satisfy the conditions in Proposition 5.1, and therefore M_0 and M_1 are C^∞-diffeomorphic. □

We are now equipped to handle the proof of Theorem 4.1.(2). We will prove this by applying Corollary 5.2. To this aim, we will make use of Proposition 1.14 in order to convert the weak continuity assumption on $\tilde{h}$ into a $\tilde{\mathbb{R}}$-C^1 regularity first, and finally into a C^1 regularity over $\mathbb{R}$.

Proof of Theorem 4.1.(2). – First, note that there exist a sufficiently small $r_0 \in \mathbb{R}_+^*$ and a semialgebraic neighborhood U of $(0, r_0] \times \{0\}$ in $(0, +\infty)^2$ such that the projection map

$$p : \bigcup_{0<a<r\in\mathbb{R}} F_f(r,a) \times \{(r,a)\} \to \mathbb{R}^2$$

given by $(x,r,a) \mapsto (r,a)$ is Nash trivial over U, as in the proof of Lemma 3.3. Hence there exists $l \in \mathbb{N}$ such that the projection map

$$\bigcup_{r\in(0,r_0]} F_f(r,r^l) \times \{r\} \to (0,\ r_0]$$

given by $(x,r) \mapsto r$ is Nash trivial. As a consequence, there exist a Nash diffeomorphism

$$H_f : F_f(r_0, r_0^l) \times (0,\ r_0] \to \bigcup_{r\in(0,\,r_0]} F_f(r,r^l)$$

such that $H_f(F_f(r_0,r_0^l) \times \{r\}) = F_f(r,r^l)$ and $H_f(\cdot, r_0) = \mathrm{id}$. In the same way, there exist a Nash diffeomorphism

$$H_g : F_g(r_0^{l_1}, r_0^l) \times (0,\ r_0] \to \bigcup_{r\in(0,\,r_0]} F_g(r^{l_1}, r^l)$$

such that $H_g(F_g(r_0^{l_1}, r_0^l) \times \{r\}) = F_g(r^{l_1}, r^l)$ and $H_g(\cdot, r_0) = \mathrm{id}$, for some small $l_1 \in \mathbb{Q}_+^*$, by enlarging l and shrinking r_0 if necessary. Here we can choose $l_1 \in \mathbb{Q}_+^*$ so that

$$h(\{x \in \mathbb{R}^n : |x| \le r\}) \subset \{x \in \mathbb{R}^n : |x| \le r^{l_1}\}$$

for any $r \in [0,\ r_0]$ by semialgebraicity of h.

Define a semialgebraic isotopy

$$g_s : F_f(r_0, r_0^l) \to F_g(r_0^{l_1}, r_0^l)$$

with $0 \le s \le 1/2$ (rather than $0 \le s \le 1$) using Remark 4.5.(1), so that g_0 is a homeomorphism onto $F_g(r_0^{l_1}, r_0^l)$ and $g_{1/2}(x) = h(x)$ for $x \in F_f(r_0, r_0^l)$. Note that

$$h \circ H_f(x, r_0) = H_g(g_{1/2}(x), r_0)$$

for $x \in F_f(r_0, r_0^l)$. Next, extend g_s to $g_s : F_f(r_0, r_0^l) \to F_g(r_0^{l_1}, r_0^l)$ with the values of s in $[0, 1)$, using the continuous semialgebraic embeddings

$$h|_{F_f((2-2s)r_0,(2-2s)^l r_0^l)} : F_f((2-2s)r_0, (2-2s)^l r_0^l) \to F_g((2-2s)^{l_1} r_0^{l_1}, (2-2s)^l r_0^l)$$

where $s \in [1/2,\ 1)$, so that

$$h \circ H_f(x, (2-2s)r_0) = H_g(g_s(x), (2-2s)r_0)$$

for $s \in [1/2,\ 1)$. Then g_s is similar to an isotopy, except that the parameter s moves only in $[0, 1)$.

In order to achieve the proof, we want to apply Corollary 5.2 to g_s, with $0 \le s \le s_0$, for some $s_0 \in (0,\ 1)$ close to 1. To this aim, it suffices to see that g_s is a C^1-embedding for $s \in (0,\ 1)$ sufficiently close to 1.

Extend the semialgebraic maps g_s to $\tilde{\mathbb{R}}$-semialgebraic maps

$$\tilde{g}_\lambda : \tilde{F}_{\tilde{f}}(\tilde{r}_0, \tilde{r}_0^l) \to \tilde{F}_{\tilde{g}}(\tilde{r}_0^{l_1}, \tilde{r}_0^l)$$

where $\lambda \in [0,\ 1)_{\tilde{\mathbb{R}}}$.

Using the assumption that $\tilde{h}|_{\mathcal{I}_f}$ is a weakly continuous homeomorphism, together with Proposition 1.14, we obtain that $\tilde{h}$ is of class $\tilde{\mathbb{R}}$-C^1. Then the restriction of $\tilde{h}$ to $\tilde{F}_{\tilde{f}}(\rho, \rho^l)$ is an $\tilde{\mathbb{R}}$-C^1-embedding into $\tilde{F}_{\tilde{g}}(\rho^{l_1}, \rho^l)$ for $\rho = t^{1/l}$, since $\tilde{F}_{\tilde{f}}(t^{1/l}, t)$ is included in $\mathcal{I}_f$. As a consequence $\tilde{g}_\lambda$ is an $\tilde{\mathbb{R}}$-C^1-embedding for $\lambda_0 \in \tilde{\mathbb{R}}_+^*$ such that $(2 - 2\lambda_0)\tilde{r}_0 = t^{1/l}$.

Remark that λ_0 belongs to $[1/2, 1)_{\tilde{\mathbb{R}}}$, so that the subset of $[1/2, 1)_{\tilde{\mathbb{R}}}$ defined by

$$\{\lambda \in [1/2, 1)_{\tilde{\mathbb{R}}} : \ \tilde{g}_\lambda \text{ is a } \tilde{\mathbb{R}}\text{-}C^1\text{-embedding}\}$$

is not empty. Moreover this set is the $\tilde{\mathbb{R}}$-extension of the semialgebraic subset of $[1/2, 1)$ defined by

$$\{s \in [1/2, 1) : \ g_s \text{ is a } C^1\text{-embedding}\},$$

because the former is described by the $\tilde{\mathbb{R}}$-extensions of the polynomials which describe the latter. Therefore g_s is a C^1-embedding for s close to 1. □

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