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DURFEE'S CONJECTURE ON THE SIGNATURE OF SMOOTHINGS OF SURFACE SINGULARITIES

BY János KOLLÁR AND András NÉMETHI
WITH AN APPENDIX BY Tommaso DE FERNEX

ABSTRACT. – In 1978 Durfee conjectured various inequalities between the signature σ and the geometric genus p_g of a normal surface singularity. Since then a few counter examples have been found and positive results established in some special cases.

We prove a 'strong' Durfee-type inequality for any smoothing of a Gorenstein singularity, provided that the intersection form of the resolution is unimodular. We also prove the conjectured 'weak' inequality for all hypersurface singularities and for sufficiently large multiplicity strict complete intersections. The proofs establish general inequalities valid for any numerically Gorenstein normal surface singularity.

RÉSUMÉ. – En 1978 Durfee a conjecturé plusieurs inégalités entre la signature σ et le genre géométrique p_g d'une singularité normale de surface. Depuis, quelques contre-exemples ont été trouvés et des résultats positifs établis dans des cas particuliers.

Nous montrons ici une inégalité 'forte' de type Durfee pour toute lissification d'une singularité de Gorenstein, sous la condition que la forme d'intersection de la résolution est unimodulaire. Nous prouvons aussi l'inégalité 'faible' pour toute singularité d'hypersurface et pour les intersections complètes strictes de multiplicité suffisamment grande. Les preuves établissent des inégalités générales valables pour toute singularité normale et numériquement Gorenstein de surface.

1. Introduction

Durfee's conjectures. – Let $(X, 0)$ be a complex analytic normal surface singularity and $\tilde{X} \to X$ a resolution. The *geometric genus* p_g is defined as $h^1(\mathcal{O}_{\tilde{X}})$. For any one-parameter smoothing with generic (Milnor) fiber F, the rank of the second homology $H_2(F, \mathbb{Z})$ is the *Milnor number* of the smoothing μ. Furthermore, $H_2(F, \mathbb{Z})$ has a natural intersection form with Sylvester invariants (μ_+, μ_0, μ_-). Then $\mu = \mu_+ + \mu_0 + \mu_-$ and $\sigma := \mu_+ - \mu_-$ is called the *signature* of the smoothing. The Milnor number and the signature usually depend on the choice of the smoothing; but if $(X, 0)$ is Gorenstein, they depend only on $(X, 0)$ satisfying explicit formulas. For more details see the monographs [2, 1, 17, 20] or [16, 18, 35]. Formulas for various classes of singularities can be found in [8, 9, 10, 11, 14, 15, 12, 22].

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These local invariants should be viewed as analogs of the most important global invariants: Todd genus, Euler number and signature.

Durfee proved that $2p_g = \mu_0 + \mu_+$ [5]. Furthermore, μ_0 equals the first Betti number $b_1(L_X)$ of the link L_X of $(X,0)$.

Examples show that for a surface singularity μ_- is usually large compared to the other Sylvester invariants. Equivalently, p_g is substantially smaller than μ and σ tends to be rather negative. These observations led to the formulation of Durfee's Conjectures [5].

Strong inequality. – If $(X,0)$ is an isolated complete intersection surface singularity (ICIS) then $6p_g \le \mu$.

Weak inequality. – If $(X,0)$ is a normal surface singularity, then for any smoothing $4p_g \le \mu + \mu_0$. Equivalently, $\sigma \le 0$.

Semicontinuity of σ. – If $\{(X_t,0)\}_{t\in(\mathbb{C},0)}$ is a flat family of isolated surface singularities then $\sigma(X_{t=0}) \le \sigma(X_{t\neq 0})$.

Other invariants are provided by the combinatorics of a resolution $\pi : \tilde{X} \to X$. Let s denote the number of irreducible π-exceptional curves and K the canonical class of $\tilde{X}$. Then $K^2 + s$ is independent of the resolution and, for smoothable Gorenstein singularities,

$$\mu = 12p_g + K^2 + s - \mu_0 \quad \text{and} \quad -\sigma = 8p_g + K^2 + s; \tag{1}$$

see [5, 16, 32, 35]. Therefore, an inequality of type $\mu + \mu_0 \ge C \cdot p_g$ (for some constant C) transforms into

$$(12 - C)p_g + K^2 + s \ge 0, \quad \text{or} \quad -\sigma \ge (C-4)p_g. \tag{2}$$

In particular, one can ask for these inequalities (2) even in the non-Gorenstein case.

The resolution defines the *maximal (ideal) cycle* $Z_{\max}$, which is the divisorial part of the ideal sheaf $\pi^{-1}\mathfrak{m}_{X,0} \cdot \mathcal{O}_{\tilde{X}}$ (well defined even if this ideal sheaf is not principal).

Other invariants of $(X,0)$ are the *multiplicity,* denoted by ν, and the *embedding dimension,* denoted by e.

KNOWN RESULTS 3. – A counterexample to the *weak inequality* was given by Wahl [35, p. 240]; it is a minimally elliptic normal surface singularity (not ICIS) with $\nu = 12$, $\mu = 3$, $\mu_0 = 0$, $p_g = 1$ and $\sigma = 1$. If one combines the results from [35, 2.2(d)] with [21] or [31], examples with arbitrary large positive σ can be constructed.

Nevertheless, both the strong and the weak inequalities hold in most examples and the intrinsic structure responsible for the positivity/negativity of the signature of a given germ has not been understood.

A counterexample to the *semicontinuity* of the signature was found in [13]: the semicontinuity already fails for some degenerations of hypersurfaces with non-degenerate Newton principal part. This excludes degeneration arguments in possible proofs of the inequalities.

The articles [14, 15] show that the *strong inequality* also fails for some non-hypersurface ICIS, and without other restrictions the best that we can expect is the weak inequality.

For hypersurfaces we have the following 'positive' results:

$8p_g < \mu$ for $(X,0)$ of multiplicity 2, Tomari [33],

$6p_g \leq \mu - 2$ for $(X, 0)$ of multiplicity 3, Ashikaga [3],
$6p_g \leq \mu - \nu + 1$ for quasi-homogeneous singularities, Xu-Yau [36],
$6p_g \leq \mu$ for suspension singularities $\{g(x, y) + z^k = 0\}$, Némethi [24, 26],
$6p_g \leq \mu$ for absolutely isolated singularities, Melle-Hernández [19].
For a short proof of $\sigma \leq 0$ in the suspension case see [25].

In this note we estimate the expression $8p_g + K^2 + s$ using properties of the dual graph of the minimal resolution. For smoothable Gorenstein singularities we obtain the following.

THEOREM 4. – *Let $(X, 0)$ be a normal Gorenstein surface singularity with embedding dimension e and geometric genus p_g. Let σ denote the signature of a smoothing. Then*

1. *If the resolution intersection form is unimodular then $-\sigma \geq 2^{4-e}(p_g + 1)$.*
2. *If $(X, 0)$ is a hypersurface singularity then $-\sigma \geq p_g + s_{\min}$, where $s_{\min}$ is the number of irreducible exceptional curves in the minimal resolution.*

The intersection form is unimodular if and only if the integral homology of the link is torsion-free [23]. Part (1) is a generalization of the following result, valid for a special family of ICIS's with unimodular lattice, namely for splice type singularities of Neumann-Wahl [30]. The *Casson Invariant Conjecture,* proved in [29, 28], states that the Casson invariant of the link is minus one-eighth the signature. As the Casson invariant is additive under splicing, and each splice component is a Brieskorn complete intersection with positive Casson invariant, the negativity of the signature follows.

We prove several inequalities that hold without the Gorenstein assumption. In fact, the strategy is to prove general inequalities using the combinatorial properties of the resolution lattice. In order to simplify the technicalities we will assume that the lattice is numerically Gorenstein. Then we apply these primary inequalities in different analytic situations.

At each step we 'lose something'. Analyzing these steps should lead to better estimates in many cases. Our aim is not to over-exploit these technicalities, but to show conceptually the general principles behind the inequalities.

It seems that $-\sigma \geq 0$ for all 'sufficiently complicated' complete intersections, but we can prove this only for *strict complete intersection* singularities, where a local ring $(\mathcal{O}_{X,0}, \mathfrak{m}_{X,0})$ is called a strict complete intersection iff the corresponding graded ring $\mathrm{Gr}_{\mathfrak{m}_{X,0}}(\mathcal{O}_{X,0})$ is a complete intersection; see [4].

PROPOSITION 5. – *Fix e and consider the set of strict ICIS of embedding dimension e. Then $-\sigma$ tends to infinity whenever the multiplicity ν tends to infinity.*

EXAMPLE 6. – [14, 15] Assume that $(X, 0)$ is a homogeneous ICIS of codimension $r = e - 2$ and multidegree $(d, \ldots, d)$. If $r = 1$ then $6p_g = \mu + 1 - \nu$. For any r the inequality $4p_g \leq \mu + 1 - \nu$ is valid. Moreover, if $r \geq 2$ is fixed, then $\frac{\mu}{p_g}$ asymptotically tends to $C_{2,r} := \frac{4(r+1)}{r+1/3}$, although $C_{2,r} \cdot p_g \leq \mu + 1$ does not hold in general. (The constant 4 is the best bound valid for any d and r.) For precise formulae see [loc.cit.].

Finally we wish to emphasize that the 'strong inequality' $6p_g \leq \mu$, conjecturally valid for all hypersurface singularities, still remains open.

The structure of the article. – In the introduction in (1) and (2) we recall signature formulae for smoothable Gorenstein singularities. The minimal (analytic) Euler characteristic of a resolution is introduced and discussed in Section 2; the key Proposition 8 gives a graphical inequality relating this object with the geometric genus and embedding dimension. Its proof uses a commutative algebra result from the Appendix. When the graph is unimodular, a theorem of Elkies combined with Proposition 8 gives a strong inequality (Section 3). The non-unimodular case is treated in Section 4.

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2. Minimal Euler characteristic of a resolution

Let $(X,0)$ be a normal surface singularity with *minimal* resolution $\tilde{X} \to X$. We write $L = H_2(\tilde{X}, \mathbb{Z})$, $(\cdot,\cdot)$ denotes the intersection form on L, and L' is the dual lattice $\mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$ with natural inclusions $L \subset L' \subset L \otimes \mathbb{Q}$.

Let $Z_K \in L'$ be the anticanonical cycle, that is, $(Z_K, E_i) = -(K, E_i)$ for every exceptional curve E_i. By the minimality of the resolution $(Z_K, l) \leq 0$ for any effective rational cycle l and $Z_K \geq 0$. A singularity is called *numerically Gorenstein* if $Z_K \in L$.

Set $\chi(l') = -(l', l' - Z_K)/2$ for any $l' \in L \otimes \mathbb{Q}$. By Riemann-Roch and the adjunction formula, $\chi(l) = \chi(\mathcal{O}_l)$ for any non-zero effective cycle $l \in L$. We set

$$\min \chi := \min_{l \in L} \chi(l).$$

It is a topological invariant of $(X,0)$, strongly related to arithmetical properties of the lattice L. It takes some effort to compute in explicit examples. In the literature $1 - \min \chi = p_a$ is called the *arithmetic genus* of $(X,0)$ [34].

(The expression $\min \chi$ is also the normalization term of the Seiberg-Witten invariant of the link expressed in terms of the lattice cohomology [27]. The comparison of $\min \chi$ with the d-invariant of the link provided by the Heegaard-Floer theory and the involved topological inequalities lead the authors to the ideas of the present note.)

If $(X,0)$ is a rational singularity (that is, $p_g = 0$) then $\min \chi(l)$ is realized by the empty cycle $l = 0$. (Under the condition that the lattice is numerically Gorenstein, rational singularities are exactly the Du Val singularities with $Z_K = 0$.) Since the realization of $\min \chi(l)$ by $l = 0$ mess up our formulas, we exclude the rational case in the sequel.

The quantity $\min \chi$ satisfies two obvious inequalities. Since $h^0(\mathcal{O}_l) - h^1(\mathcal{O}_l) \geq 1 - p_g$ for any non-zero effective cycle l, and $\min \chi$ is realized (in the non-rational) case by a non-zero effective cycle (see Lemma 7 below), we get $\min \chi \geq 1 - p_g$. Also, since the real quadratic function $\chi(x) = -(x, x - Z_K)/2$ has its minimum at $Z_K/2$, and $\chi(Z_K/2) = K^2/8$, we get that $\min \chi \geq K^2/8$.

We wish to understand how sharp these inequalities are. The first inequality $\min \chi \geq 1 - p_g$ will be improved to $\min \chi \geq -Cp_g$ for a certain constant $0 < C < 1$. This will be applied in the form $p_g + \chi(l) \geq (1 - C)p_g$ for any l.

On the other hand, we wish to bound the difference $\min\chi - K^2/8$ from above. The strategy is the following. Assume that for some rational cycle ξ one has $Z_K - \xi = 2l \in 2L$. Then $\chi(l) = (K^2 - \xi^2)/8$, hence $\chi(l) - K^2/8$ is minimal exactly when $-\xi^2/8$ is minimal among the rational cycles ξ satisfying $Z_K - \xi \in 2L$. Thus, if there exists a cycle ξ with $\xi^2 + s \geq 0$ and $Z_K - \xi \in 2L$, then $(K^2 + s)/8 \geq \min\chi$. This combined with the first inequality gives $p_g + (K^2 + s)/8 \geq (1 - C)p_g$.

LEMMA 7. – *The minimum* $\min\chi$ *is achieved by an effective cycle. If* $(X, 0)$ *is numerically Gorenstein, then* $\min\chi$ *is achieved by a cycle* $l \in L$ *satisfying* $Z_K/2 \leq l \leq Z_K$.

Proof. – Assume that $\chi(l) = \min\chi$ and write $l = a - b$, where $a, b \in L$ are effective and have no common components. Then $\chi(a + b) - \chi(a - b) = (b, Z_K - 2a) \leq 0$, thus $\chi(a + b) \leq \chi(a - b)$. This proves the first part. Similarly, write $l = Z_K - a + b$. Then $\chi(Z_K - a + b) - \chi(Z_K - a - b) = (b, 2a - Z_K) \geq 0$. These two inequalities applied repeatedly show that the minimum is achieved for some $l \in L$ with $0 \leq l \leq Z_K$.

Take such a cycle and write it as $l = Z_K/2 + a - b$, $a, b \in \frac{1}{2}L$, effective and without common components. Then $\chi(Z_K/2 + a + b) - \chi(l) = -2(a, b) \leq 0$. □

If $(X, 0)$ is a Du Val singularity, then $Z_K = 0$ and $\min\chi(l)$ is realized by the empty cycle $l = 0$. If $(X, 0)$ is numerically Gorenstein but not Du Val then the support of Z_K, and hence the support of $l \geq Z_K/2$, is the whole exceptional set of the resolution.

PROPOSITION 8. – *Set* $\varepsilon = 1$ *if* $(X, 0)$ *is Gorenstein, and* $\varepsilon = 0$ *otherwise. Then for any numerically Gorenstein, non-Du Val surface singularity* $p_g + \min\chi \geq 2^{\varepsilon - e}(p_g + 1)$.

Proof. – Fix $l \in L$ such that $Z_K/2 \leq l \leq Z_K$ and $\min\chi = \chi(l)$. In the non-Du Val case $Z_K > 0$, hence $l > 0$ too and

$$p_g + \chi(l) = p_g - h^1(\mathcal{O}_l) + h^0(\mathcal{O}_l) \geq h^0(\mathcal{O}_l).$$

Note that for any effective $m \in L$ we have

$$h^0(\mathcal{O}_m) \geq \dim\big(H^0(\mathcal{O}_{\tilde{X}})/H^0(\mathcal{O}_{\tilde{X}}(-m))\big).$$

The inequality is usually strict but if $m = Z_K$ then the $h^1(\mathcal{O}_{\tilde{X}}(-Z_K)) = 0$ vanishing implies

$$h^0(\mathcal{O}_{Z_K}) = \dim\big(H^0(\mathcal{O}_{\tilde{X}})/H^0(\mathcal{O}_{\tilde{X}}(-Z_K))\big) = p_g. \tag{9}$$

Note that $H^0(\mathcal{O}_{\tilde{X}})$ equals the local ring R of $(X, 0)$ and each $H^0(\mathcal{O}_{\tilde{X}}(-m))$ can be identified with an ideal sheaf $I(m) \subset R$. This correspondence is sub-multiplicative, that is, $I(m_1) \cdot I(m_2) \subset I(m_1 + m_2)$. Thus, for every m, Lemma 26 from Appendix shows that

$$\dim\big(H^0(\mathcal{O}_{\tilde{X}})/H^0(\mathcal{O}_{\tilde{X}}(-m))\big) \geq 2^{-e}\big(1 + \dim\big(H^0(\mathcal{O}_{\tilde{X}})/H^0(\mathcal{O}_{\tilde{X}}(-2m))\big)\big).$$

Putting these together gives that

$$\begin{aligned} p_g + \chi(l) &\geq \dim\big(H^0(\mathcal{O}_{\tilde{X}})/H^0(\mathcal{O}_{\tilde{X}}(-l))\big) \\ &\geq \tfrac{1}{2^e}\big(1 + \dim\big(H^0(\mathcal{O}_{\tilde{X}})/H^0(\mathcal{O}_{\tilde{X}}(-2l))\big)\big) \\ &\geq \tfrac{1}{2^e}\big(1 + \dim\big(H^0(\mathcal{O}_{\tilde{X}})/H^0(\mathcal{O}_{\tilde{X}}(-Z_K))\big)\big) \\ &= \tfrac{1}{2^e}(p_g + 1). \end{aligned}$$

In the Gorenstein case this can be improved as follows. Let $0 \le m \le Z_K$ be a cycle and set $\bar{m} = Z_K - m$. The Gorenstein duality gives

$$h^1(\mathcal{O}_m) = h^0(\mathcal{O}_m(-\bar{m})) = \dim\big(H^0(\mathcal{O}_{\tilde{X}}(-\bar{m}))/H^0(\mathcal{O}_{\tilde{X}}(-Z_K))\big)$$
$$= p_g - \dim\big(H^0(\mathcal{O}_{\tilde{X}})/H^0(\mathcal{O}_{\tilde{X}}(-\bar{m}))\big),$$

hence, using in the 3rd line Lemma 26 from Appendix again, we get that

$$\begin{aligned} p_g + \chi(m) &= p_g - h^1(\mathcal{O}_m) + h^0(\mathcal{O}_m) \\ &\ge \dim\big(H^0(\mathcal{O}_{\tilde{X}})/H^0(\mathcal{O}_{\tilde{X}}(-\bar{m}))\big) + \dim\big(H^0(\mathcal{O}_{\tilde{X}})/H^0(\mathcal{O}_{\tilde{X}}(-m))\big) \\ &\ge \tfrac{1}{2^{e-1}}\left(1 + \dim\big(H^0(\mathcal{O}_{\tilde{X}})/H^0(\mathcal{O}_{\tilde{X}}(-Z_K))\big)\right) \\ &= \tfrac{1}{2^{e-1}}\left(p_g + 1\right). \end{aligned} \tag{10}$$

For $m = l$ this gives the claimed inequality. □

EXAMPLE 11. – Assume that $Z_K \in 2L$. Then $\min\chi = K^2/8$, and, by Proposition 8, $p_g + (K^2+s)/8 = p_g + \min\chi + s/8 \ge 2^{\varepsilon-e}(p_g+1) + s/8$. Hence, if additionally $(X,0)$ is smoothable Gorenstein (i.e., $\varepsilon = 1$), then one has $-\sigma \ge 2^{4-e}(p_g+1) + s$.

REMARK 12. – The property $Z_K \in 2L$ has a conceptual meaning as well. The (almost) complex structure on $\tilde{X}$ gives a spin^c structure $\sigma_{\tilde{X}}$ on $\tilde{X}$. On the other hand, by the adjunction formula, L is an even lattice if and only if $Z_K \in 2L'$. In this case $\tilde{X}$ has a unique spin structure, say $\varepsilon_{\tilde{X}}$. The point is that, in general, $\sigma_{\tilde{X}} \ne \varepsilon_{\tilde{X}}$, and their restrictions $r(\sigma_{\tilde{X}})$ and $r(\varepsilon_{\tilde{X}})$ to the link can be different as well, even if $r(\sigma_{\tilde{X}})$ is spin. One has the following facts: $r(\sigma_{\tilde{X}})$ is spin if and only if $Z_K \in L$; and $r(\sigma_{\tilde{X}}) = r(\varepsilon_{\tilde{X}})$ if and only if $Z_K \in 2L$.

3. Inequalities in the unimodular case.

Assume that the intersection form of L is unimodular, that is $L = L'$. Note that this holds iff the first integral homology of the link of $(X,0)$ is torsion free since this torsion group is isomorphic to L'/L by [23].

THEOREM 13. – *Let $(X,0)$ be a normal surface singularity of embedding dimension e. Let $\tilde{X} \to X$ be the minimal resolution with canonical class K and s irreducible exceptional curves. Assume that the resolution intersection form is unimodular. Then*

1. $(K^2+s)/8 \ge \min\chi$ *and*
2. $p_g + (K^2+s)/8 \ge 2^{\varepsilon-e}\,(p_g+1)$, *equivalently,* $(K^2+s)/8 \ge -(1-2^{\varepsilon-e})\,p_g + 2^{\varepsilon-e}$, *where ε is as in Proposition 8.*

Proof. – By a result of Elkies [6, Theorem, p. 4], there is a $\xi \in L$ such that $\xi^2 + s \ge 0$ and $(m, m-\xi)$ is even for every $m \in L$. (That is, ξ is a *characteristic element* of 'small' norm.) If E is an irreducible exceptional curve then $(E, E-Z_K) = 2g(E) - 2$ is even, thus $(m, m-Z_K)$ is even for every $m \in L$. Therefore $(m, Z_K - \xi)$ is even for every $m \in L$ and $l := \frac{1}{2}(Z_K - \xi) \in L$. (We used unimodularity here and it is also needed in [6].)

Then $(K^2+s)/8 = \chi(l) + (\xi^2+s)/8 \ge \chi(l)$ and we can apply Proposition 8. □

If, in addition, $(X,0)$ is smoothable and Gorenstein, then $\varepsilon = 1$ thus (2) of Theorem 13 and the second formula of (1) from the introduction give that

$$-\sigma = 8p_g + K^2 + s \geq 2^{4-e}(p_g + 1). \tag{14}$$

This completes the proof of part (1) of Theorem 4. □

The above theorem shows that the torsionfreeness of the first homology of the link has more substantial effect on the negativity of the signature than the embedded properties, like being a hypersurface or an ICIS.

EXAMPLE 15. – Assume that $(X,0)$ is a hypersurface singularity with $L = L'$. Then $-\sigma \geq 2p_g + 2$, or equivalently, $\mu + \mu_0 \geq 6p_g + 2$. In particular, if the link of a hypersurface singularity is an integral homology sphere (hence $\mu_0 = 0$ too), then it satisfies the strong Durfee inequality $6p_g \leq \mu - 2$ with the optimal asymptotic constant 6.

4. The non-unimodular case

In the previous section we used the strong result of Elkies, valid for unimodular definite lattices. This statement has no analog in the non-unimodular case. Therefore, it is somewhat surprising that combinatorial manipulation on the lattice can produce a similar (though weaker) inequality, at the price of introducing a negative contribution given by the multiplicity. This is what we present next.

Let $\{\ \}$ and $\lfloor\ \rfloor$ denote the rational/integral part of a cycle. In this section we assume that $(X,0)$ is numerically Gorenstein but not Du Val. Set $x := 2\{Z_K/2\} \in L$ and $\bar{x} := E - x$, where E is the reduced exceptional curve. Then $m := (Z_K - x)/2 = \lfloor Z_K/2 \rfloor \in L$. We write Σ for $8p_g + K^2 + s$. (Thus, in the smoothable Gorenstein case, $\sigma = -\Sigma$.)

Since $8\chi(m) = K^2 - x^2$, by Proposition 8

$$\Sigma = 8\big(p_g + \chi(m)\big) + x^2 + s \geq 2^{\varepsilon+3-e}(p_g + 1) + x^2 + s. \tag{16}$$

Similarly,

$$\Sigma = 8(p_g + \chi(m + E)) + (E + \bar{x})^2 + s \geq 2^{\varepsilon+3-e}(p_g + 1) + (E + \bar{x})^2 + s. \tag{17}$$

Since $x = E - \bar{x}$, adding the equations (16) and (17) gives that

$$\Sigma \geq 2^{\varepsilon+3-e}(p_g + 1) + E^2 + \bar{x}^2 + s. \tag{18}$$

For each cycle $y = x,\ \bar{x}$ and E write the relation $y^2 = -2\chi(y) + (y, Z_K)$ and add the equations (16) and (18). We get that

$$\Sigma \geq 2^{\varepsilon+3-e}(p_g + 1) + s - \chi(x) - \chi(\bar{x}) - \chi(E) + (E, Z_K). \tag{19}$$

Since $x,\ \bar{x},\ E$ are reduced, $\chi(x) + \chi(\bar{x}) + \chi(E) \leq s + 1 - b_1(L_X)$ (since $b_1(L_X) = b_1(E) \geq h^1(\mathcal{O}_E)$). Hence (19) can be rewritten as

PROPOSITION 20. – *$\Sigma \geq 2^{\varepsilon+3-e}(p_g + 1) - 1 + b_1(L_X) + (E, Z_K)$ where (E, Z_K) also equals $E^2 + 2\chi(E)$. Furthermore, $-1 + b_1(L_X) + (E, Z_K) = E^2 + \chi(\Gamma)$ where $\chi(\Gamma)$ is the Euler characteristic of the topological realization of the resolution graph Γ.* □

Although the term (E, Z_K) is negative, in many cases (e.g., hypersurfaces, ICIS) it is much smaller than p_g. We do not have a good general estimate, but the following argument gives a bound that implies the negativity of the signature in several cases.

In order to simplify the notation let us denote the constant $2^{\varepsilon+3-e} - 1 + b_1(L_X)$ by A. Let $Z = Z_{\max} \in L$ be the maximal cycle. Hence $Z \geq E$, which implies that $(E, Z_K) \geq (Z, Z_K)$. For any $t \geq e - \varepsilon - 3$ write $(2^{t+1}Z, Z_K)$ as $(2^{t+1}Z)^2 + 2\chi(2^{t+1}Z)$, hence we obtain that

$$\Sigma \geq \big(\tfrac{1}{2^{e-\varepsilon-3}} - \tfrac{1}{2^t}\big)p_g + \tfrac{1}{2^t}\big(p_g + \chi(2^{t+1}Z)\big) + 2^{t+1}Z^2 + A. \tag{21}$$

Then using $Z^2 \geq -\nu$ (cf. [34]) and Proposition 8 we get the following.

LEMMA 22. – *For $t \geq e - \varepsilon - 3$ one has*

$$\Sigma \geq \big(\tfrac{1}{2^{e-\varepsilon-3}} - \tfrac{1}{2^t} + \tfrac{1}{2^{t+e-\varepsilon}}\big)p_g - 2^{t+1}\nu + A + \tfrac{1}{2^{t+e-\varepsilon}}.$$

With different choices of t the coefficient of p_g can be arranged to be as close to $1/2^{e-\varepsilon-3}$ as we wish, but the price is a more negative coefficient for ν. This expression shows that for an arbitrary normal surface singularity we should expect an inequality of the form

$$\Sigma \geq C_1 p_g - C_2\nu + C_3 \quad \text{for some constants} \quad C_1, C_2 > 0 \text{ and } C_3 > -1$$

that depend on the embedding dimension e. If ν dominates p_g—as in the example of Wahl—then Σ can be negative. However, if p_g dominates the multiplicity, then Σ becomes positive, as in the next examples.

The case of strict complete intersections. – A strict ICIS $(X, 0)$ is a normally flat deformation of its tangent cone, which, by definition, is a homogeneous complete intersection singularity [4] (though this cone might have non-isolated singularities). In the next argument we concentrate on the right hand side of the inequality, valid for $(X, 0)$,

$$\Sigma \geq \big(\tfrac{1}{2^{e-\varepsilon-3}} - \tfrac{1}{2^t} + \tfrac{1}{2^{t+e-\varepsilon}}\big)p_g - 2^{t+1}\nu + 2^{\varepsilon+3-e} - 1 + \tfrac{1}{2^{t+e-\varepsilon}}, \tag{23}$$

obtained from Lemma 22 by $b_1(L_X) \geq 0$. (Now $\varepsilon = 1$ and e and t are fixed.)

We wish to show that the right hand side tends to infinity whenever ν tends to infinity.

We may assume that the lowest degree parts of the equations of the strict ICIS $(X, 0)$ are the equations of the tangent cone. Then we proceed in two steps. First, we deform lowest degree parts into generic homogeneous equations; in this way we achieve that the tangent cone of the new ICIS is isolated. Then the multiplicity of the general fiber is the same, and the geometric genus is less that or equal to the original. Then, we degenerate the new singularity to its tangent cone (via a positive weight deformation of the isolated cone), replacing the new equations by their lowest degree parts. Under this second step the multiplicity and the geometric genus are both constant. In particular, the right hand side of the inequality (23) will not increase during this procedure.

Therefore, in order to prove that $-\sigma = \Sigma$ is positive for large ν and fixed $e = r+2$, by (23) and the above deformation argument, it is enough to show that p_g/ν tends to infinity with ν for homogeneous complete intersections. In that case, if $d_1, \ldots, d_r$ ($d_i \geq 2$) are the degrees of the defining equations, then

$$\frac{p_g}{\nu} = \sum_i \frac{(d_i-1)(d_i-2)}{6} + \sum_{i<j} \frac{(d_i-1)(d_j-1)}{4} \tag{24}$$

and $\nu = \prod_i d_i$, cf. [14, 15].

Note that (24) does not imply the negativity of the signature for every strict ICIS, but it gives a much stronger result asymptotically. This suggests that the positivity of Σ (or, the negativity of the signature in the presence of Gorenstein smoothing) is guided by the ratio p_g/ν. This seems to be a general phenomenon, not specifically related to embedded properties.

The case of hypersurfaces. – We apply Lemma 26 from Appendix to the ideal $\mathfrak{a}_1 = \mathfrak{a}_2 = H^0(\mathcal{O}_{\tilde{X}}(-Z_K)) \subset R := H^0(\mathcal{O}_{\tilde{X}})$, where $\tilde{X} \to X$ is the minimal resolution. Since $e = 3$, and using (9), we get

$$(25) \qquad 8p_g \geq \dim\, H^0(\mathcal{O}_{\tilde{X}})/H^0(\mathcal{O}_{\tilde{X}}(-Z_K))^{\otimes 2} \geq \dim\, H^0(\mathcal{O}_{\tilde{X}})/H^0(\mathcal{O}_{\tilde{X}}(-2Z_K)).$$

Using the cohomology sequence of $0 \to \mathcal{O}_{\tilde{X}}(-2Z_K) \to \mathcal{O}_{\tilde{X}}(-Z_K) \to \mathcal{O}_{Z_K}(-Z_K) \to 0$, the vanishings $H^1(\mathcal{O}_{\tilde{X}}(-Z_K)) = H^1(\mathcal{O}_{\tilde{X}}(-2Z_K)) = 0$ and Riemann-Roch we get that $\dim\, H^0(\mathcal{O}_{\tilde{X}}(-Z_K))/H^0(\mathcal{O}_{\tilde{X}}(-2Z_K)) = -K^2_{\min}$. Hence (9) and (25) reads as

$$7p_g + K^2_{\min} \geq 0.$$

This via (1) and (2) transforms into

$$\mu + \mu_0 \geq 5p_g + s_{\min} \quad \text{and} \quad -\sigma \geq p_g + s_{\min}.$$

5. Appendix by Tommaso de Fernex: Colength of a product of ideals

Let R be a local ring with maximal ideal $\mathfrak{m}$, essentially of finite type over a field k. Let e be the embedded dimension of R. For any $\mathfrak{m}$-primary ideal $\mathfrak{a}$, denote by $\ell(R/\mathfrak{a})$ the length of $R/\mathfrak{a}$.

LEMMA 26. – *For any finite collection of* $\mathfrak{m}$*-primary ideals* $\mathfrak{a}_1, \ldots, \mathfrak{a}_d \subset R$*, we have*

$$d^{e-1}\textstyle\sum_{i=1}^{d}\ell(R/\mathfrak{a}_i) \geq \ell\big(R/(\mathfrak{a}_1 \cdots \mathfrak{a}_d)\big),$$

and the inequality is strict if $d \geq 2$ *and* $e \geq 2$.

Proof. – By Cohen's structure theorem, there is a surjection $k[[x_1, \ldots, x_e]] \to \widehat{R}$, where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R. After taking the inverse image of the ideals $\mathfrak{a}_i \widehat{R}$ to $k[[x_1, \ldots, x_e]]$ and restricting to $k[x_1, \ldots, x_e]$, we reduce to prove the lemma when $R = k[x_1, \ldots, x_e]$ and $\mathfrak{m} = (x_1, \ldots, x_e)$. If we fix a monomial order which gives a flat degeneration to monomial ideals, and denote by $\mathrm{in}(\mathfrak{a})$ the initial ideal of an ideal $\mathfrak{a} \subset R$, then $\ell(R/\mathfrak{a}) = \ell(R/\mathrm{in}(\mathfrak{a}))$ and $\prod_{i=1}^d \mathrm{in}(\mathfrak{a}_i) \subset \mathrm{in}(\prod_{i=1}^d \mathfrak{a}_i)$. We can therefore assume that each $\mathfrak{a}_i$ is monomial.

Let $\mathfrak{a} = \prod_{i=1}^d \mathfrak{a}_i$. For $\mathbf{u} = (u_1, \ldots, u_e) \in \mathbb{Z}^e_{\geq 0}$, we denote $\mathbf{x}^{\mathbf{u}} = \prod_{j=1}^e x_j^{u_j}$. Let

$$Q_i = \bigcup_{\mathbf{x}^{\mathbf{u}} \in \mathfrak{a}_i} \left(\mathbf{u} + \mathbb{R}^e_{\geq 0}\right) \quad \text{and} \quad Q = \bigcup_{\mathbf{x}^{\mathbf{u}} \in \mathfrak{a}} \left(\mathbf{u} + \mathbb{R}^e_{\geq 0}\right).$$

Notice that $\ell(R/\mathfrak{a}_i) = \mathrm{Vol}\left(\mathbb{R}^e_{\geq 0} \setminus Q_i\right)$ and $\ell(R/\mathfrak{a}) = \mathrm{Vol}\left(\mathbb{R}^e_{\geq 0} \setminus Q\right)$, where the volumes are computed with respect to the Euclidean metric. We consider the radial sum

$$Q' = \bigstar_{i=1}^{d} Q_i := \bigcup_W \sum_{i=1}^{d} (Q_i \cap W)$$

introduced in [7]: the union runs over all rays $W \subset \mathbb{R}^e_{\geq 0}$, and the sum appearing in the right-hand side is the usual sum of subsets of a vector space.

For every $\mathbf{v} \in Q'$, we can find $\mathbf{v}_i \in Q_i$ such that $\mathbf{v} = \sum_{i=1}^d \mathbf{v}_i$. For each i, we have $\mathbf{v}_i \in \left(\mathbf{u}_i + \mathbb{R}^e_{\geq 0}\right)$ for some $\mathbf{u}_i \in \mathbb{Z}^e_{\geq 0}$ such that $\mathbf{x}^{\mathbf{u}_i} \in \mathfrak{a}_i$. Then, setting $\mathbf{u} = \sum_{i=1}^d \mathbf{u}_i$, we have $\mathbf{x}^{\mathbf{u}} \in \mathfrak{a}$ and $\mathbf{v} \in \left(\mathbf{u} + \mathbb{R}^e_{\geq 0}\right)$, and therefore $\mathbf{v} \in Q$. This means that $Q' \subset Q$, and hence

$$\mathrm{Vol}\left(\mathbb{R}^e_{\geq 0} \setminus Q'\right) \geq \mathrm{Vol}\left(\mathbb{R}^e_{\geq 0} \setminus Q\right). \tag{27}$$

Then, to prove the inequality stated in the lemma, it suffices to show that

$$d^{e-1}\left(\sum_{i=1}^{d} \mathrm{Vol}\left(\mathbb{R}^e_{\geq 0} \setminus Q_i\right)\right) \geq \mathrm{Vol}\left(\mathbb{R}^e_{\geq 0} \setminus Q'\right). \tag{28}$$

To this end, we fix spherical coordinates $(\theta, \rho) \in S \times \mathbb{R}_{\geq 0}$ where S is the intersection of the unit sphere with $\mathbb{R}^e_{\geq 0}$. For any $\theta \in S$, we define $r_i(\theta) = \inf\{\rho \mid (\theta, \rho) \in Q_i\}$ and $r(\theta) = \inf\{\rho \mid (\theta, \rho) \in Q'\}$. By the definition of Q', we have $r(\theta) = \sum_{i=1}^d r_i(\theta)$. We have

$$\mathrm{Vol}(\mathbb{R}^e_{\geq 0} \setminus Q_i) = \int_S \int_0^{r_i(\theta)} \rho^{e-1}\, d\rho\, \omega(\theta) = \int_S \frac{r_i(\theta)^e}{e}\, \omega(\theta)$$

and

$$\mathrm{Vol}(\mathbb{R}^e_{\geq 0} \setminus Q') = \int_S \int_0^{r(\theta)} \rho^{e-1}\, d\rho\, \omega(\theta) = \int_S \frac{r(\theta)^e}{e}\, \omega(\theta)$$

for some volume form ω on S. Then the desired inequality follows from

$$d^{e-1} \sum_{i=1}^{d} r_i(\theta)^e \geq r(\theta)^e, \tag{29}$$

which follows from Hölder's inequality.

To conclude, we show that the inequality is strict if $d \geq 2$ and $e \geq 2$. First observe that (28) is a strict inequality unless (29) is an equality for almost all $\theta \in S$, which can only happen if $\mathfrak{a}_i = \mathfrak{a}_1$ for every i. Suppose this is the case, so that $\mathfrak{a} = \mathfrak{a}_1^d$. Notice that in this case Q' is a polyhedron. Let a, b be the smallest integers such that $x_1^a \in \mathfrak{a}_1$ and $x_1^{a'} x_2^b \in \mathfrak{a}_1$ for some $a' < a$. Then $x_1^{(d-1)a+a'} x_2^b \in \mathfrak{a}$, and hence the vector $\mathbf{v} = ((d-1)a + a', b, 0, \dots, 0)$ belongs to Q. Note, on the contrary, that $\mathbf{v}$ is not in Q'. Hence $Q' \quad Q$, and since these sets are polyhedra, it follows that (27) is a strict inequality. Therefore the inequality stated in the lemma, which follows as a combination of (27) and (28), is strict. □

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