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Bjørn Ian DUNDAS & Matthew MORROW

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

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Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : smf@smf.univ-mrs.fr

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FINITE GENERATION AND CONTINUITY OF TOPOLOGICAL HOCHSCHILD AND CYCLIC HOMOLOGY

BY BJØRN IAN DUNDAS AND MATTHEW MORROW

ABSTRACT. – The goal of this paper is to establish fundamental properties of the Hochschild, topological Hochschild, and topological cyclic homologies of commutative, Noetherian rings, which are assumed only to be F-finite in the majority of our results. This mild hypothesis is satisfied in all cases of interest in finite and mixed characteristic algebraic geometry. We prove firstly that the topological Hochschild homology groups, and the homotopy groups of the fixed point spectra TR^r, are finitely generated modules (after p-completion in the mixed characteristic setting). We use this to establish the continuity of these homology theories for any given ideal. A consequence of such continuity results is the pro Hochschild-Kostant-Rosenberg theorem for topological Hochschild and cyclic homology. Finally, we show more generally that the aforementioned finite generation and continuity properties remain true for any proper scheme over such a ring.

RÉSUMÉ. – Le but de cet article est d'établir des propriétés fondamentales des homologies de Hochschild, de Hochschild topologique et cyclique topologique d'anneaux commutatifs et noethériens, qu'on ne suppose être que F-finis pour la majorité de nos résultats. Cette hypothèse faible est satisfaite en tous cas d'intérêts en géométrie algébrique en caractéristique finie et mixte. Nous démontrons d'abord que les groupes d'homologie de Hochschild topologique, ainsi que les groupes d'homotopie du spectre des points fixés TR^r, sont des modules de type fini (après la p-complétion dans le cadre de caractéristique mixte). En l'utilisant, nous établissons la continuité de ces homologies pour n'importe quel idéal. Une conséquence de ces résultats de continuité est le théorème de Hochschild-Kostant-Rosenberg pro pour les homologies de Hochschild topologique et cyclique topologique. Finalement, nous démontrons que ces résultats de génération finie et ces propriétés de continuité sont toujours valables pour les schémas propres et lisses sur un tel anneau.

1. Introduction

The aim of this paper is to prove fundamental finite generation, continuity, and pro Hochschild-Kostant-Rosenberg theorems for the Hochschild, topological Hochschild, and topological cyclic homologies of commutative, Noetherian rings. As far as we are aware, these are the first general results on the finite generation and continuity of topological

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Hochschild and cyclic homology, despite the obvious foundational importance of such problems in the subject.

The fundamental hypothesis for the majority of our theorems is the classical notion of F-finiteness:

Definition 1.1. – A $\mathbb{Z}_{(p)}$-algebra (always commutative) is said to be *F-finite* if and only if the $\mathbb{F}_p$-algebra A/pA is finitely generated over its subring of p-th powers.

This is a mild condition: it is satisfied as soon as A/pA is obtained from a perfect field by iterating the following constructions any finite number of times: passing to finitely generated algebras, localising, completing, or Henselising; see Lemma 3.8.

To state our main finite generation result, we first remark that the Hochschild homology $HH_n(A)$ of a ring A is always understood in the derived sense (see Section 2.2). Secondly, $THH_n(A)$ denotes the topological Hochschild homology groups of a ring A, while $TR^r_n(A;p)$ denotes the homotopy groups of the fixed point spectrum $TR^r(A;p)$ for the action of the cyclic group $C_{p^{r-1}}$ on the topological Hochschild homology spectrum $THH(A)$. It is known that $TR^r_n(A;p)$ is naturally a module over the p-typical Witt ring $W_r(A)$. Note that $W_1(A) = A$ and $TR^1(A,p) = THH(A)$. The obvious notation will be used for the p-completed, or finite coefficient, versions of these theories, and for their extensions to quasi-separated, quasi-compact schemes following [9].

Our main finite generation result is the following, where $\hat{A_p} = \varprojlim_s A/p^sA$ denotes the p-completion of a ring A:

Theorem 1.2 (see Corol. 4.8). – *Let A be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra, and $n \geq 0$, $r \geq 1$. Then $HH_n(A;\mathbb{Z}_p)$ and $THH_n(A;\mathbb{Z}_p)$ are finitely generated $\hat{A_p}$-modules, and $TR^r_n(A;p,\mathbb{Z}_p)$ is a finitely generated $W_r(\hat{A_p})$-module.*

The key step towards proving Theorem 1.2 is the following finite generation result for the André-Quillen homology of $\mathbb{F}_p$-algebras:

Theorem 1.3 (see Thm. 4.6). – *Let A be a Noetherian, F-finite $\mathbb{F}_p$-algebra. Then the André-Quillen homology groups $D^i_n(A/\mathbb{F}_p)$ are finitely generated for all $n, i \geq 0$.*

Next we turn to "degree-wise continuity" for the homology theories HH, THH, and TR^r, by which we mean the following: given an ideal $I \subseteq A$, we examine when the natural map of pro A-modules

$$\{HH_n(A) \otimes_A A/I^s\}_s \longrightarrow \{HH_n(A/I^s)\}_s$$

is an isomorphism, and similarly for THH and TR^r. This question was first raised by L. Hesselholt in 2001 [4], who later proved with T. Geisser the THH isomorphism in the special case that $A = R[X_1,\dots,X_d]$ and $I = \langle X_1,\dots,X_s\rangle$ for any ring R [12, §1].

In Section 5.1 we prove the following:

THEOREM 1.4 (see Thm. 5.3). – *Let A be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra, and $I \subseteq A$ an ideal. Then, for all $n \geq 0$ and $v, r \geq 1$, the canonical maps*

$$\{HH_n(A;\mathbb{Z}/p^v) \otimes_A A/I^s\}_s \longrightarrow \{HH_n(A/I^s;\mathbb{Z}/p^v)\}_s$$
$$\{TR^r_n(A;p,\mathbb{Z}/p^v) \otimes_{W_r(A)} W_r(A/I^s)\}_s \longrightarrow \{TR^r_n(A/I^s;p,\mathbb{Z}/p^v)\}_s$$

are isomorphisms of pro A-modules and pro $W_r(A)$-modules respectively.

Applying Theorem 1.4 simultaneously to A and its completion $\widehat{A} = \varprojlim_s A/I^s$ with respect to the ideal I, we obtain Corollary 5.4, stating that both the maps

$$HH_n(A;\mathbb{Z}/p^v) \otimes_A \widehat{A} \longrightarrow HH_n(\widehat{A};\mathbb{Z}/p^v) \longrightarrow \varprojlim_s HH_n(A/I^s;\mathbb{Z}/p^v)$$

are isomorphisms, and similarly for THH and TR^r.

Of a more topological nature than such degree-wise continuity statements are spectral continuity, namely the question of whether the canonical map of spectra

$$THH(A) \longrightarrow \operatorname{holim}_s THH(A/I^s)$$

is a weak equivalence, at least after p-completion. The analogous continuity question for K-theory was studied for discrete valuation rings by A. Suslin [36], I. Panin [28], and the first author [7], and for power series rings $A = R[[X_1,\ldots,X_d]]$ over a regular, F-finite $\mathbb{F}_p$-algebra R by Geisser and Hesselholt [12], with $I = \langle X_1,\ldots,X_d\rangle$. Geisser and Hesselholt proved the continuity of K-theory in such cases by establishing it first for THH and TR^r.

We use the previous degree-wise continuity results to prove the following:

THEOREM 1.5 (see Thm. 5.5). – *Let A be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra, and $I \subseteq A$ an ideal; assume that A is I-adically complete. Then, for all $r \geq 1$, the canonical map of spectra*

$$TR^r(A;p) \longrightarrow \operatorname{holim}_s TR^r(A/I^s;p)$$

is a weak equivalence after p-completion. Similarly for THH, TR, TC^r, and TC.

There are two important special cases in which the results so far can be analysed further: when p is nilpotent, and when p generates I. Firstly, if p is nilpotent in A, for example if A is a Noetherian, F-finite, $\mathbb{F}_p$-algebra, then Theorem 1.2 – Theorem 1.5 are true integrally, without p-completing or working with finite coefficients; see Corollaries 4.9 and 5.6 for precise statements. Secondly, if $I = pA$, then Theorems 1.4 and 1.5 simplify significantly; see Corollary 5.8 for the precise statement and Remark 5.9 for related work.

We present our pro Hochschild-Kostant-Rosenberg (HKR) theorems in Section 5.2. Given a geometrically regular (e.g., smooth) morphism $k \to A$ of Noetherian rings, the classical HKR theorem, combined with Néron-Popescu desingularisation, states that the antisymmetrisation map $\Omega^n_{A/k} \to HH^k_n(A)$ is an isomorphism of A-modules for all $n \geq 0$. In Theorem 5.11 we establish its pro analogue: if $I \subseteq A$ is an ideal, then the canonical map of pro A-modules

$$\{\Omega^n_{(A/I^s)/k}\}_s \longrightarrow \{HH^k_n(A/I^s)\}_s$$

is an isomorphism for all $n \geq 0$.

In the special case of finite type algebras over fields, this was proved by G. Cortiñas, C. Haesemeyer, and C. Weibel [5, Thm. 3.2]. The stronger form presented here has recently been required in the study of infinitesimal deformations of algebraic cycles [2, 26].

The analogue of the HKR theorem for THH and TR^r was established by Hesselholt [15, Thm. B]: If A is a regular $\mathbb{F}_p$-algebra, then there is a natural isomorphism of $W_r(A)$-modules $\bigoplus_{i=0}^n W_r\Omega^i_A \otimes_{W_r(\mathbb{F}_p)} TR^r_{n-i}(\mathbb{F}_p;p) \xrightarrow{\simeq} TR^r_n(A;p)$, where $W_r\Omega^\bullet_A$ denotes the de Rham-Witt complex of S. Bloch, P. Deligne, and L. Illusie. In the limit over r, Hesselholt moreover showed that the contribution from the left side vanishes except in top degree $i = n$, giving an isomorphism of pro abelian groups $\{W_r\Omega^n_A\}_r \cong \{TR^r_n(A;p)\}_r$ which deserves to be called the HKR theorem for the pro spectrum $\{TR^r\}_r$.

We prove the following pro versions of these HKR theorems:

Theorem 1.6 (see Thm. 5.14 & Corol. 5.15). – *Let A be a regular, F-finite $\mathbb{F}_p$-algebra, and $I \subseteq A$ an ideal. Then, for all $n \geq 0$ and $r \geq 1$, the canonical map of pro $W_r(A)$-modules*

$$\bigoplus_{i=0}^n \{W_r\Omega^i_{A/I^s} \otimes_{W_r(\mathbb{F}_p)} TR^r_{n-i}(\mathbb{F}_p;p)\}_s \longrightarrow \{TR^r_n(A/I^s;p)\}_s,$$

and the canonical map of pro pro abelian groups

$$\big\{\{W_r\Omega^n_{A/I^s}\}_s\big\}_r \longrightarrow \big\{\{TR^r_n(A/I^s;p)\}_s\big\}_r$$

are isomorphisms.

Finally, in Section 6, the earlier finite generation and continuity results are extended to proper schemes over Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebras. These are obtained by combining the results in the affine case with Zariski descent and Grothendieck's formal functions theorem for coherent cohomology. Our main finite generation result is the following:

Theorem 1.7 (see Corol. 6.4). – *Let A be a Noetherian, F-finite, finite Krull-dimensional $\mathbb{Z}_{(p)}$-algebra, X a proper scheme over A, and $n \geq 0$, $r \geq 1$. Then $HH_n(X;\mathbb{Z}_p)$ and $THH_n(X;\mathbb{Z}_p)$ are finitely generated $\widehat{A_p}$-modules, and $TR^r_n(X;p,\mathbb{Z}_p)$ is a finitely generated $W_r(\widehat{A_p})$-module.*

Given an ideal $I \subseteq A$ and a proper scheme X over A, we next consider the natural map of pro A-modules $\{HH_n(X) \otimes_A A/I^s\}_s \longrightarrow \{HH_n(X_s)\}_s$, where $X_s := X \times_A A/I^s$, and similarly for THH and TR^r. In this situation we establish the exact analogues of Theorem 1.4 and its Corollary 5.4 mentioned above; see Theorem 6.6 and Corollary 6.7 for precise statements. As in the affine case, we use these degree-wise continuity statements to deduce continuity of topological cyclic homology for proper schemes over our usual base rings:

Theorem 1.8 (see Thm. 6.8). – *Let A be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra, $I \subseteq A$ an ideal, and X be a proper scheme over A; assume A is I-adically complete. Then, for all $r \geq 1$, the canonical map of spectra*

$$TR^r(X;p) \longrightarrow \operatorname*{holim}_s TR^r(X_s;p)$$

is a weak equivalence after p-completion. Similarly for THH, TR, TC^r, and TC.

As in the affine case, it is particularly interesting to consider the cases that p is nilpotent or is the generator of I; see Corollaries 6.9 and 6.10.

Notation, etc.

With the strict exception of Proposition 5.1 (which holds also in the non-commutative case), all rings from Section 3 onwards are commutative. Modules are understood to be symmetric bimodules whenever a bimodule structure is required for Hochschild homology.

Given a positive integer n, the n-torsion of an abelian group M is denoted by $M[n]$.

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2. Review of Artin-Rees properties and homology theories

2.1. Pro abelian groups and Artin-Rees properties

Here we summarise some results and notation concerning pro abelian group and pro modules which will be used throughout the paper.

If $\mathscr{A}$ is a category, then we denote by Pro $\mathscr{A}$ the category of pro objects of $\mathscr{A}$ indexed over $\mathbb{N}$. That is, an object of Pro $\mathscr{A}$ is an inverse system $M_\infty = \{M_s\}_s =$ "$M_1 \leftarrow M_2 \leftarrow \cdots$", where the objects M_i and the transition maps belong to $\mathscr{A}$; the morphisms are given by

$$\mathrm{Hom}_{\mathrm{Pro}\,\mathscr{A}}(M_\infty, N_\infty) := \varprojlim_r \varinjlim_s \mathrm{Hom}_{\mathscr{A}}(M_s, N_r).$$

If $\mathscr{A}$ is abelian then so is Pro $\mathscr{A}$, and a pro object $M_\infty \in$ Pro $\mathscr{A}$ is isomorphic to zero if and only if it satisfies the trivial Mittag-Leffler condition: that is, that for all $r \geq 1$ there exists $s \geq r$ such that the transition map $M_s \to M_r$ is zero.

We will be particularly interested in the cases $\mathscr{A} = Ab$ and A-mod, where A is a commutative ring, in which case we speak of pro abelian groups and pro A-modules respectively. When it is unlikely to cause any confusion, we will occasionally use ∞ notation in proofs for the sake of brevity; for example, if I is an ideal of a ring A and M is an A-module, then

$$M \otimes_A A/I^\infty = \{M \otimes_A A/I^s\}_s, \quad HH_n(A/I^\infty, M/I^\infty M) = \{HH_n(A/I^s, M/I^s M)\}_s.$$

We now state the fundamental Artin-Rees result which will be used repeatedly, see [1, Prop. 10 & Lem. 11] and [32, Lem. 9.9] (note that (ii) is simply a reformulation of (i)):

THEOREM 2.1 (André, Quillen). – *Let A be a commutative, Noetherian ring, and $I \subseteq A$ an ideal.*

(i) *If M is a finitely generated A-module, then the pro A-module $\{\mathrm{Tor}_n^A(A/I^s, M)\}_s$ vanishes for all $n > 0$.*

(ii) *The "completion" functor $- \otimes_A A/I^\infty$: A-mod $\longrightarrow$ Pro A-mod is exact on the subcategory of finitely generated A-modules.*

COROLLARY 2.2. – *Let A be a commutative, Noetherian ring, $I \subseteq A$ an ideal, and M a finitely generated A-module. Then the pro A-module $\{\mathrm{Tor}_n^A(A/I^s, M/I^s)\}_s$ vanishes for all $n > 0$.*

Proof. – For each $r \geq 1$ we may apply the previous theorem to the module M/I^r to see that there exists $s \geq r$ such that the second of the following arrows is zero:

$$\mathrm{Tor}_n^A(A/I^s, M/I^s) \to \mathrm{Tor}_n^A(A/I^s, M/I^r) \to \mathrm{Tor}_n^A(A/I^r, M/I^r).$$

Hence the composition is zero, completing the proof. □

Corollary 2.3. – *Let A be a commutative, Noetherian ring, $I \subseteq A$ an ideal, M a finitely generated A-module, and G a finite group acting A-linearly on M. Then the canonical map of pro group homologies $\{H_n(G, M) \otimes_A A/I^s\}_s \to \{H_n(G, M/I^s M)\}_s$ is an isomorphism for all $n \geq 0$.*

Proof. – Considering $\mathbb{Z}$ as a left $\mathbb{Z}[G]$-module via the augmentation map, A/I^s as a right A-module, and M as an $A - \mathbb{Z}[G]$-bimodule, there are first quadrant spectral sequences of A-modules with the same abutement by [39, Ex. 5.6.2]:

$$E^2_{ij}(s) = \mathrm{Tor}_i^A(A/I^s, \mathrm{Tor}_j^{\mathbb{Z}[G]}(M, \mathbb{Z})), \quad {}'E^2_{ij}(s) = \mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathrm{Tor}_j^A(A/I^s, M), \mathbb{Z}).$$

These assemble to first quadrant spectral sequences of pro A-modules with the same abutement:

$$E^2_{ij}(\infty) = \{\mathrm{Tor}_i^A(A/I^s, \mathrm{Tor}_j^{\mathbb{Z}[G]}(M, \mathbb{Z}))\}_s, \quad {}'E^2_{ij}(\infty) = \{\mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathrm{Tor}_j^A(A/I^s, M), \mathbb{Z})\}_s.$$

Since $\mathrm{Tor}_j^{\mathbb{Z}[G]}(M, \mathbb{Z})$ is a finitely generated A-module for all $j \geq 0$, Theorem 2.1(i) implies that $E^2_{ij}(\infty) = 0$ for $i > 0$; so the $E(\infty)$-spectral sequence degenerates to the edge map isomorphism. Theorem 2.1(i) similarly implies that $\mathrm{Tor}_j^A(A/I^s, M) = 0$ for $j > 0$, and hence the ${}'E(\infty)$-spectral sequence also degenerates to the edge map isomorphism.

Composing these edge map isomorphisms, we arrive at an isomorphism of pro A-modules $\{\mathrm{Tor}_n^{\mathbb{Z}[G]}(M, \mathbb{Z}) \otimes_A A/I^s\}_s \xrightarrow{\simeq} \{\mathrm{Tor}_n^{\mathbb{Z}[G]}(M/I^s M, \mathbb{Z})\}_s$ for all $n \geq 0$, which is exactly the desired isomorphism. □

Corollary 2.4. – *Let A be a commutative, Noetherian ring, $I \subseteq A$ an ideal, M a finitely generated A-module, and $m \geq 1$. Then the canonical maps*

$$\{M[m] \otimes_A A/I^s\}_s \longrightarrow \{M/I^s M\,[m]\}_s, \quad \{M/mM \otimes_A A/I^s\}_s \longrightarrow \{M/(mM + I^s M)\}_s$$

are isomorphisms of pro A-modules.

Proof. – These isomorphisms follow by Theorem 2.1(ii) from the exact sequence

$$0 \to \{M[m] \otimes_A A/I^s\}_s \to \{M/I^s M\} \xrightarrow{\times m} \{M/I^s M\} \to \{M/mM \otimes_A A/I^s\}_s \to 0. \quad \square$$

2.2. André-Quillen and Hochschild homology

Let k be a commutative ring. We briefly review the André-Quillen and Hochschild homologies of k-algebras, though we assume that the reader has some familiarity with these theories. Let A be a commutative k-algebra, let $P_\bullet \to A$ be a simplicial resolution of A by free commutative k-algebras, and set $\mathbb{L}_{A/k} := \Omega^1_{P_\bullet/k} \otimes_{P_\bullet} A$. Thus $\mathbb{L}_{A/k}$ is a simplicial A-module which is free in each degree; it is called the cotangent complex of the k-algebra A. Given simplicial A-modules $M_\bullet$, $N_\bullet$, the tensor product and alternating powers are the simplicial A-modules defined degreewise. We set $\mathbb{L}^i_{A/k} := \bigwedge^i_A \mathbb{L}_{A/k}$ for each $i \geq 1$. The André-Quillen homology [1, 33, 34] of A, with coefficients in an A-module M, is defined by

$$(n, i \geq 0) \qquad D^i_n(A/k, M) := \pi_n(\mathbb{L}^i_{A/k} \otimes_A M).$$

When $M = A$ the notation is simplified to $D^i_n(A/k) := D^i_n(A/k, A) = \pi_n \mathbb{L}^i_{A/k}$.

For Hochschild homology [24], A can be a possibly non-commutative k-algebra. For an A-bimodule M, the "usual" Hochschild homology of A as a k-algebra with coefficients in M is defined to be $HH^{\text{usual},k}_n(A, M) := \pi_n(C^k_\bullet(A, M))$ for $n \geq 0$, where $C^k_\bullet(A, M) = \{[n] \mapsto M \otimes A^{\otimes_k n}\}$ is the usual simplicial k-module. However, we will work throughout with the derived version of Hochschild homology, which we now explain; more details may be found in [27, §3]. Letting $P_\bullet \to A$ be a simplicial resolution of A by flat k-algebras, let $HH^k(A, M)$ denote the diagonal of the bisimplicial k-module $C^k_\bullet(P_\bullet, M)$; the homotopy type of $HH^k(A, M)$ does not depend on the choice of resolution, and we set

$$(n \geq 0) \qquad HH^k_n(A, M) := \pi_n HH^k(A, M).$$

The canonical map $HH^k_n(A, M) \to HH^{\text{usual},k}_n(A, M)$ is an isomorphism if A is flat over k. When $A = M$ we write $HH^k(A) = HH^k(A, A)$ and $HH^k_n(A) = HH^k_n(A, A)$, and when $k = \mathbb{Z}$ we omit it from the notation.

2.3. Topological Hochschild and cyclic homology

The manipulations of topological Hochschild and cyclic homology contained in this paper are of a mostly algebraic nature, using only the formal properties of the theory. We collect various spectral sequences, long exact sequences, etc. which we need; we hope that the algebraic nature of this exposition will be accessible to non-topologists since the results of this paper will be later applied to problems in arithmetic and algebraic geometry.

2.3.1. *Topological Hochschild homology*. – If A is a ring and M is an A-bimodule then $THH(A, M)$ denotes the associated topological Hochschild homology spectrum, as constructed in, e.g., [8]. Its homotopy groups are the topological Hochschild homology of A with coefficients in M, namely

$$(n \geq 0) \qquad THH_n(A, M) := \pi_n THH(A, M).$$

If A is commutative and M is a symmetric A-module, then these are A-modules. When $M = A$, one writes $THH(A) = THH(A, A)$ and $THH_n(A) = THH_n(A, A)$.

Algebraic properties of THH may be extracted from the following two results:

(i) Pirashvili-Waldhausen's [29, Thm. 4.1] first quadrant spectral sequence of abelian groups (of A-modules if A is commutative and M is a symmetric A-module)

$$E^2_{ij} = HH_i(A, THH_j(\mathbb{Z}, M)) \implies THH_{i+j}(A, M),$$

which compares THH with its algebraic counterpart $HH_n(A, M) = HH^{\mathbb{Z}}_n(A, M)$.

(ii) M. Bökstedt's [8, Thm. 4.1.0.1] calculation of the groups $THH_n(\mathbb{Z}, M)$:

$$THH_n(\mathbb{Z}, M) \cong \begin{cases} M/mM & n = 2m-1 \\ M[m] & n = 2m. \end{cases}$$

For example, if A is a commutative, Noetherian ring for which $HH_n(A)$ is a finitely generated A-module for all $n \geq 0$, then (i) & (ii) easily imply the same for $THH_n(A)$.

2.3.2. *The fixed point spectra TR^r.* – An essential fact is that $THH(A)$ is a cyclotomic spectrum in the sense of [18]. This means that $THH(A)$ is an S^1-spectrum (it carries a "nice" action by the circle group S^1, ensuring the existence of so-called Frobenius and Verschiebung maps), together with additional pro structure ensuring the existence of restriction maps. The intricacies of cyclotomic spectra and the construction of their homotopy fixed points, homotopy orbits, etc. are irrelevant for this paper; we only need certain algebraic consequences which we now list.

Let p be a fixed prime number and, for $r \geq 1$, let $C_{p^{r-1}}$ be the cyclic subgroup of S^1 of order p^{r-1}. The $C_{p^{r-1}}$-fixed point spectrum of $THH(A)$ is denoted by

$$TR^r(A; p) := THH(A)^{C_{p^{r-1}}}$$

(note that $TR^1(A; p) = THH(A)$), and its homotopy groups by $TR^r_n(A; p) := \pi_n TR^r(A; p)$.

Formal algebraic properties of the groups $TR^r_n(A; p)$ may be obtained from the following two facts, which are non-trivial consequences of $THH(A)$ being a cyclotomic spectrum; see, e.g., Lems. 1.4.5 & 2.0.6 of [8, §VI], or Thm. 1.2 and the proof of Prop. 2.3 of [18]:

(i) There is a natural homotopy fibre sequence of spectra

$$THH(A)_{hC_{p^r}} \longrightarrow TR^{r+1}(A; p) \xrightarrow{R} TR^r(A; p)$$

where $THH(A)_{hC_{p^r}}$ is the spectrum of homotopy orbits for the action of C_{p^r} on $THH(A)$, and R is known as the *restriction map*.

(ii) There is a first quadrant spectral sequence

$$E^2_{ij} = H_i(C_{p^r}, THH_j(A)) \implies \pi_{i+j}(THH(A)_{hC_{p^r}}),$$

where $H_i(C_{p^r}, THH_j(A))$ is the homology of C_{p^r} with trivial coefficients $THH_j(A)$.

2.3.3. *Witt structure.* – Assuming that A is commutative, the various aforementioned groups inherit natural algebra or module structures: $THH_n(A)$ is a module over A, and $TR^r_n(A; p)$ and $\pi_n(THH(A)_{hC_{p^{r-1}}})$ are modules over the p-typical Witt ring $W_r(A)$. The Witt vector structure appears in the following way: firstly, $TR^r(A; p)$ is a ring spectrum (and R is a map of ring spectra) and so the homotopy groups $TR^r_n(A; p)$ and $\pi_n(THH(A)_{hC_{p^{r-1}}})$ are modules over the ring $TR^r_0(A; p)$ (this does not require A to be commutative), and secondly it is a theorem of Hesselholt and Madsen [18, Thm. F] that there is a natural isomorphism of rings $W_r(A) \overset{\simeq}{\to} TR^r_0(A; p)$. Moreover, by [15, §1.3] we have the following structure:

(i) The long exact sequence of homotopy groups from 2.3.2(i) above is a long exact sequence of $W_{r+1}(A)$-modules, where the action of $W_{r+1}(A)$ on the $W_r(A)$-module $TR^r_n(A;p)$ is via the restriction map $R: W_{r+1}(A) \to W_r(A)$.
(ii) The group homology spectral sequence from 2.3.2(ii) above is a spectral sequence of $W_{r+1}(A)$-modules, where the action of $W_{r+1}(A)$ on the E^2-page, whose terms are clearly A-modules, is via the r^{th} power of the Frobenius $F^r : W_{r+1}(A) \to A$.

2.3.4. *Topological cyclic homology*. – The homotopy limit over the restriction maps $R\colon TR^{r+1}(A;p) \to TR^r(A;p)$, namely

$$TR(A;p) := \operatorname{holim}_r TR^r(A;p),$$

is a ring spectrum whose homotopy groups $TR_n(A;p) := \pi_n(TR(A;p))$ fit into short exact sequences $0 \to \varprojlim_r^1 TR^r_{n+1}(A;p) \to TR_n(A;p) \to \varprojlim_r TR^r_n(A;p) \to 0$. If A is commutative then the groups $TR_n(A;p)$ are naturally modules over $W(A) = \varprojlim_r W_r(A)$.

The so-called *Frobenius map* $F\colon TR^{r+1}(A;p) \to TR^r(A;p)$ is the inclusion of the C_{p^r}-fixed point spectrum of $THH(A)$ into the $C_{p^{r-1}}$-fixed point spectrum. The Frobenius commutes with the restriction, and thus induces a map $F\colon TR(A;p) \to TR(A;p)$. The p-typical topological cyclic homology spectrum of A is, by definition,

$$TC(A;p) := \operatorname{hofib}(TR(A;p) \xrightarrow{\mathrm{id}-F} TR(A;p)).$$

One may additionally define a p-typical topological cyclic homology spectrum for a fixed level $r \geq 1$ by setting $TC^r(A;p) := \operatorname{hofib}(TR^r(A;p) \xrightarrow{R-F} TR^{r-1}(A;p))$.

2.4. Finite coefficients and p-completions

Given a prime number p and a simplicial abelian group $M_\bullet$, its *(derived) p-completion* is by definition the simplicial abelian group $(M_\bullet)^\wedge_p := \operatorname{holim}_v(M_\bullet \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^v)$. We write $\pi_n(M_\bullet;\mathbb{Z}/p^v) = \pi_n(M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^v)$ and $\pi_n(M_\bullet;\mathbb{Z}_p) = \pi_n((M_\bullet)^\wedge_p)$, and recall the short exact seqences

$$0 \longrightarrow \pi_n(M_\bullet) \otimes_{\mathbb{Z}} \mathbb{Z}/p^v \longrightarrow \pi_n(M_\bullet;\mathbb{Z}/p^v) \longrightarrow \pi_{n-1}(M_\bullet)[p^v] \longrightarrow 0$$

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, \pi_n(M_\bullet)) \to \pi_n(M_\bullet;\mathbb{Z}_p) \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, \pi_{n-1}(M_\bullet)) \to 0.$$

Similarly, if X is a spectrum then its *p-completion* is by definition $X^\wedge_p := \operatorname{holim}_v(X \wedge S/p^v)$, where S/p^r denotes the $p^{r\,th}$ Moore spectrum; the same short exact sequences as for a simplicial abelian group apply, and we point out that $H((M_\bullet)^\wedge_p) = H(M_\bullet)^\wedge_p$ for any simplicial abelian group $M_\bullet$, where $H(-)$ denotes the Eilenberg-Maclane construction.

We remark that if M is an abelian group then $M^\wedge_p$ denotes the usual p-adic completion of M, namely $M^\wedge_p = \varprojlim_v M/p^vM$, and not the derived p-completion of M as a constant simplicial abelian group.

Now let A be a commutative ring, and M an A-module. We will use the notation

$$HH(A,M;\mathbb{Z}/p^v) := HH(A,M) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^v \qquad HH(A,M;\mathbb{Z}_p) := HH(A,M)^\wedge_p$$
$$THH(A,M;\mathbb{Z}/p^v) := THH(A,M) \wedge S/p^v \qquad THH(A,M;\mathbb{Z}_p) := THH(A,M)^\wedge_p$$
$$TR^r(A;\mathbb{Z}/p^v) := TR^r(A;p) \wedge S/p^v \qquad TR^r(A;p,\mathbb{Z}_p) := TR(A;p)^\wedge_p.$$

and similarly for TR, TC^r, and TC; the homotopy groups are denoted in the obvious manner. To make the already overburdened notation more manageable, we have chosen to write $TR^r(A;\mathbb{Z}/p^v)$, $TC^r(A;\mathbb{Z}/p^v)$, etc., rather than $TR^r(A;p,\mathbb{Z}/p^v)$, $TC^r(A;p,\mathbb{Z}/p^v)$, etc.

There is an exact sequence of simplicial A-modules $0\to\Sigma A[p^v]\to A\otimes^{\mathbb{L}}_{\mathbb{Z}}\mathbb{Z}/p^v\to A/p^vA\to 0$ (where Σ denotes suspension, i.e., -1 shift in the language of complexes), and this induces a long exact sequence of A-modules

$$\cdots\longrightarrow HH_{n-1}(A,A[p^v])\longrightarrow HH_n(A;\mathbb{Z}/p^v)\longrightarrow HH_n(A,A/p^vA)\longrightarrow\cdots,$$

and likewise for THH.

Smashing the homotopy fibre sequence of Section 2.3.2(i) with S/p^v yields a new homotopy fibre sequence, and hence a long exact sequence of the homotopy groups with finite coefficients. Moreover $THH(A)_{hC_{p^r}}\wedge S/p^v\simeq(THH(A)\wedge S/p^v)_{hC_{p^r}}$, and hence there is a homotopy orbit spectral sequence, as in Section 2.3.2(ii), with finite coefficients.

Next, $HH(A;\mathbb{Z}/p^v)$ is a simplicial module over $HH(A)$; $THH(A;\mathbb{Z}/p^v)$ is a module spectrum over $THH(A)$; and $TR^r(A;\mathbb{Z}/p^v)$ is a module spectrum over $TR^r(A;p)$. Hence their homotopy groups are naturally modules over A, A, and $W_r(A)$ respectively. The Witt structure outlined in Section 2.3.3 thus remains true with finite coefficients.

2.5. Finite generation of p-completions

We explain how finite generation results with finite coefficients lead to similar finite generation statements after p-completing. We claim no originality for these results, but could not find such algebraic statements summarised in the literature.

LEMMA 2.5. – *Let A be a commutative, Noetherian ring, and M an A-module. Then* (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv)*, where*

(i) *M is flat or finitely generated over A.*
(ii) *The p-power torsion of M is bounded, i.e., there exists $c\geq 1$ such that any p-power torsion element of M is killed by p^c.*
(iii) *The pro A-module $\{M[p^r]\}_r$ vanishes.*
(iv) *$\varprojlim^1_r M[p^r]$ and $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p,M)$ are zero.*

Proof. – Suppose first that M is a finitely generated A-module. Then M satisfies the ascending chain condition on submodules, so the chain $M[p]\subseteq M[p^2]\subseteq M[p^3]\subseteq\cdots$ is eventually constant, meaning exactly that all p-power torsion in M is killed by p^c for some fixed $c\geq 1$. This proves (ii) for finitely generated M.

In particular, taking $M=A$, there exists $c\geq 1$ such that $A[p^r]=A[p^c]$ for all $r\geq c$; in other words, the sequence $0\to A[p^c]\to A\xrightarrow{\times p^r}A$ is exact. So, if M is flat, then the sequence $0\to A[p^c]\otimes_A M\to M\xrightarrow{\times p^r}M$ is also exact, whence $p^cM[p^r]=0$ for all $r\geq c$; this proves (ii) for flat M.

(ii)$\Rightarrow$(iii): Let $c\geq 1$ be as in part (ii). Then, for each $r\geq 1$, the transition map $M[p^{r+c}]\xrightarrow{\times p^c}M[p^r]$ is zero, as required to prove the vanishing of $\{M[p^r]\}_r$. Using the identification $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p,M)=\varprojlim_r M[p^r]$, (iii)$\Rightarrow$(iv) is immediate. □

LEMMA 2.6. – *Let A be a commutative, Noetherian ring, and M an A-module. If M/pM (resp. $M[p]$) is a finitely generated A-module, then* $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, M)$ *(resp.* $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, M)$*) is a finitely generated $A^{\widehat{}}_p$-module.*

Proof. – (i): Assume M/pM is finitely generated. There exists a finitely generated A-submodule $N \subseteq M$ such that $N/pN \to M/pM$ is surjective; let $\Lambda = M/N$, which is p-divisible. The long exact $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, -)$ sequence for $0 \to N \to M \to \Lambda \to 0$ contains

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, N) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, M) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, \Lambda),$$

and we claim that the final term vanishes. Indeed, it is equivalent to show that both $\varprojlim_r^1 \Lambda[p^r]$ and $\Lambda^{\widehat{}}_p$ vanish, and this easily follows from the p-divisibility of Λ.

Moreover, it follows from Lemma 2.5 that $\varprojlim_r^1 N[p^r] = 0$, whence $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, N) = N^{\widehat{}}_p$, which is a finitely generated $A^{\widehat{}}_p$-module; since $N^{\widehat{}}_p$ surjects onto $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, M)$, we deduce that the latter is also finitely generated.

(ii): Assume $M[p]$ is finitely generated. The map

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, M) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \hookrightarrow M[p] \qquad f \otimes 1 \mapsto f(1/p)$$

is injective, whence the left side is finitely generated over $A^{\widehat{}}_p$. Applying part (i) to the module $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, M)$, we therefore deduce that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, M))$ is a finitely generated $A^{\widehat{}}_p$-module, hence that its quotient $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, M)^{\widehat{}}_p$ is also a finitely generated $A^{\widehat{}}_p$-module. But $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, M) = \varprojlim_r M[p^r]$ is evidently already p-adically complete, so that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, M)^{\widehat{}}_p = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, M)$. □

PROPOSITION 2.7. – *Let A be a commutative, Noetherian ring, and p a prime number. Suppose X is a connective ring spectrum and that $A \to \pi_0(X)$ is a ring homomorphism such that the homotopy groups of $X \wedge S/p$ are finitely generated A-modules; then $\pi_n(X; \mathbb{Z}_p)$ is naturally a finitely generated $A^{\widehat{}}_p$-module for all $n \geq 0$. Similarly for simplicial rings.*

Proof. – The p-completion $X^{\widehat{}}_p$ is again a ring spectrum and so the groups $\pi_n(X; \mathbb{Z}_p)$ are naturally $\pi_0(X; \mathbb{Z}_p)$-modules. Moreover, there is a canonical ring homomorphism

$$A^{\widehat{}}_p = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p; A) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p; \pi_0(X)) = \pi_0(X; \mathbb{Z}_p),$$

where the first equality follows from Lemma 2.5 with $M = A$, and the second from the fact that X is connective. Hence $\pi_n(X; \mathbb{Z}_p)$ is naturally an $A^{\widehat{}}_p$-module, and

$$(\dagger) \quad 0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, \pi_n(X)) \longrightarrow \pi_n(X; \mathbb{Z}_p) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}_p/\mathbb{Z}_p, \pi_{n-1}(X))) \longrightarrow 0$$

is an exact sequence of $A^{\widehat{}}_p$-modules. By assumption $\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ and $\pi_n(X)[p]$ are finitely generated A-modules for all $n \geq 0$, so the previous lemma implies that the outer terms of (†) are finitely generated $A^{\widehat{}}_p$-modules; this completes the proof. □

3. Preliminaries on Witt rings and F-finiteness

3.1. Witt rings

We establish some preliminary results on Witt rings of commutative rings; some similar material may be found in [10]. We will use the language of both big Witt rings $\mathbb{W}_S(A)$ associated to truncation sets $S \subseteq \mathbb{N}$, and of p-typical Witt rings $W_r(A) = \mathbb{W}_{\{1,p,p^2,\dots,p^{r-1}\}}(A)$ when a particular prime number p is clear from the context. The p-typical case is classical, while the language of truncation sets is due to [19]; a more detailed summary, to which we will refer for various Witt ring identities, may be found in [35, Appendix].

Given an inclusion of truncation sets $S \supseteq T$, there are the associated Restriction and Teichmüller maps

$$R_T : \mathbb{W}_S(A) \to \mathbb{W}_T(A), \quad [-]_S : A \to \mathbb{W}_S(A),$$

which are a ring homomorphism and multiplicative respectively. If $m \geq 1$ is an integer one defines the truncation set $S/m := \{s \in S : sm \in S\}$ and writes R_m instead of $R_{S/m}$; then there are moreover the associated Frobenius and Verschiebung maps

$$F_m : \mathbb{W}_S(A) \to \mathbb{W}_{S/m}(A), \quad V_m : \mathbb{W}_{S/m}(A) \to \mathbb{W}_S(A),$$

which are a ring homomorphism and additive respectively. In the p-typical case we follow the standard abuse of notation, writing $R,\ F : W_r(A) \to W_{r-1}(A)$ and $V : W_{r-1}(A) \to W_r(A)$ in place of R_p, F_p, and V_p.

If S is finite then each element of $\mathbb{W}_S(A)$ may be written uniquely as $\sum_{i\in S} V_i[a_i]_{S/i}$ for some $a_i \in A$; we will often use this to reduce questions to the study of terms of the form $V_i[a]_{S/i}$, which we will abbreviate to $V_i[a]$ when the truncation set S is clear.

If I is an ideal of A, then $\mathbb{W}_S(I)$ denotes the ideal of $\mathbb{W}_S(A)$ defined as the kernel of the quotient map $\mathbb{W}_S(A) \twoheadrightarrow \mathbb{W}_S(A/I)$. Alternatively, $\mathbb{W}_S(I)$ is the Witt vectors of the non-unital ring I. An element $\alpha \in \mathbb{W}_S(A)$ lies in $\mathbb{W}_S(I)$ if and only if, in its expansion $\alpha = \sum_{i\in S} V_i[a_i]$, the coefficients $a_i \in A$ all belong to I.

Lemma 3.1. – *Let A be a ring, $I, J \subseteq A$ ideals, and S a finite truncation set. Then:*

(i) $\mathbb{W}_S(I)\mathbb{W}_S(J) \subseteq \mathbb{W}_S(IJ)$.
(ii) $\mathbb{W}_S(I)^N \subseteq \mathbb{W}_S(I^N)$ *for all* $N \geq 1$.
(iii) $\mathbb{W}_S(I) + \mathbb{W}_S(J) = \mathbb{W}_S(I+J)$.
(iv) *Assume I is a finitely generated ideal; then for any $N \geq 1$ there exists $M \geq 1$ such that* $\mathbb{W}_S(I^M) \subseteq \mathbb{W}_S(I)^N$.

Proof. – (i): It is enough to show that $\alpha\beta \in \mathbb{W}_S(IJ)$ in the case that $\alpha = V_i[a]$ and $\beta = V_j[b]$ for some $i, j \in S$, $a \in I$, and $b \in J$, since such terms additively generate $\mathbb{W}_S(I)$ and $\mathbb{W}_S(J)$. But this follows from the standard Witt ring identity $V_i[a]\,V_j[b] = gV_{ij/g}[a^{i/g}b^{j/g}]$, where $g := \gcd(i,j)$ [35, A.4(v)]. Now (ii) follows from (i) by induction.

(iii): The surjection $J \to \frac{I+J}{I}$ induces a surjection $\mathbb{W}_S(J) \twoheadrightarrow \mathbb{W}_S(\frac{I+J}{I}) \cong \frac{\mathbb{W}_S(I+J)}{\mathbb{W}_S(I)}$, whence $\mathbb{W}_S(I+J) \subseteq \mathbb{W}_S(I) + \mathbb{W}_S(J)$. The reverse inclusion is obvious.

(iv): By assumption we have $I = \langle t_1, \dots, t_m\rangle$ for some $t_1, \dots, t_m \in A$. For any $M \geq 1$, we will write $I^{(M)} := \langle t_1^M, \dots, t_m^M\rangle \subseteq I^M$. Note that $I^{(M)} \supseteq I^{m(M-1)+1}$, so it is enough to find $M \geq 1$ such that $\mathbb{W}_S(I^{(M)}) \subseteq \mathbb{W}_S(I)^N$; we claim that $M = N\ell$ suffices, where ℓ is the least common multiple of all elements of S. To prove the claim we first use (iii) to see that $\mathbb{W}_S(I^{(M)}) = \mathbb{W}_S(At_1^M) + \cdots + \mathbb{W}_S(At_m^M)$, and then we note that $\mathbb{W}_S(At_j^M)$ is additively generated by terms $V_i[at_j^M]$ where $i \in S$ and $a \in A$; so it is enough to prove the claim for such terms. Writing $M = N\ell = N'i$ for some $N' \geq N$, this claim follows from the standard Witt ring identity [35, A.4(vi)]

$$V_i[at_j^M] = V_i[at_j^{N'i}] = [t_j]^{N'} V_i[a] \in \mathbb{W}_S(I)^{N'} \subseteq \mathbb{W}_S(I)^N. \qquad \square$$

REMARK 3.2. – The proof of part (iv) of Lemma 3.1 establishes a stronger result: namely that for any $N \geq 1$ and any set of generators $t_1, \ldots, t_m$ of I, there exists $M \geq 1$ such that $\mathbb{W}_S(I^M) \subseteq \langle [t_1], \ldots, [t_m] \rangle^N$. In particular, if $f : A \to B$ is a ring homomorphism, then we have $\mathbb{W}_S(f(I^M)B) \subseteq f(\mathbb{W}_S(I)^N)\mathbb{W}_S(B)$.

LEMMA 3.3. – *Let A be a ring, $I \subseteq A$ a finitely generated ideal, and S a finite truncation set; let $\widehat{A} := \varprojlim_s A/I^s$ be the I-adic completion of A. Then the canonical maps*

$$\varprojlim_s \mathbb{W}_S(A)/\mathbb{W}_S(I)^s \longrightarrow \varprojlim_s \mathbb{W}_S(A)/\mathbb{W}_S(I^s) \longleftarrow \mathbb{W}_S(\widehat{A})$$

are isomorphisms.

Proof. – The left arrow is an isomorphism since the two chains of ideals $\mathbb{W}_S(I^s)$ and $\mathbb{W}_S(I)^s$ are intertwined by Lemma 3.1. Regarding the right arrow, note that $\mathbb{W}_S$ commutes with arbitrary inverse systems of rings (since $\mathbb{W}_S(-)$, as a functor from rings to sets, is simply $R \mapsto R^S$) and so, in particular,

$$\mathbb{W}_S(\widehat{A}) = \mathbb{W}_S(\varprojlim_s A/I^s) \xrightarrow{\simeq} \varprojlim_s \mathbb{W}_S(A/I^s) = \varprojlim_s \mathbb{W}_S(A)/\mathbb{W}_S(I^s). \qquad \square$$

The p-adic completion of a ring R is denoted by $R^\wedge_p := \varprojlim_s R/p^s R$.

LEMMA 3.4. – *Let A be a ring, p a prime number, and $r \geq 1$. Then there is a natural isomorphism of rings $W_r(A)^\wedge_p \cong W_r(A^\wedge_p)$.*

Proof. – By Lemma 3.3, it is enough to show that the ideals $pW_r(A)$ and $W_r(pA)$ each contain a power of the other. It is well-known that $W_r(\mathbb{F}_p) = \mathbb{Z}/p^r\mathbb{Z}$, whence $W_r(A/pA)$ is a $\mathbb{Z}/p^r\mathbb{Z}$-algebra; in other words, $p^r W_r(A) \subseteq W_r(pA)$. By Remark 3.2 there exists $M \geq 1$ such that $W_r(p^M A) \subseteq [p]^p W_r(A)$; so $W_r(pA)^M \subseteq W_r(p^M A) \subseteq [p]^p W_r(A)$, where the first inclusion is by Lemma 3.1(ii). Hence we can complete the proof by showing that $[p]^2 \in pW_r(A)$. Since $R^{r-1}([p]) = p \in A$, and since there is a short exact sequence

$$0 \longrightarrow W_{r-1}(A) \xrightarrow{V} W_r(A) \xrightarrow{R^{r-1}} A \longrightarrow 0,$$

we deduce that $[p] - p \in VW_{r-1}(A)$, whence $[p]^2 \in pW_r(A) + (VW_{r-1}(A))^2$. Finally, it follows from standard Witt vector identities that the square of the ideal $VW_{r-1}(A)$ lies inside $pW_r(A)$, e.g., [35, Prop. A.4(v)]. $\square$

Now we turn to the Frobenius:

LEMMA 3.5. – *Let A be a ring, $I \subseteq A$ a finitely generated ideal, and $r \geq 1$.*

(i) *The ideal of A generated by $F^{r-1}W_r(I)$ contains I^M for $M \gg 0$.*

(ii) *The natural ring homomorphisms* $A \otimes_{W_r(A)} W_r(A/I^s) \to A/I^s$*, induced by the commutative diagrams of rings*

$$\begin{tikzcd} W_r(A) \arrow[r] \arrow[d, "F^{r-1}"'] & W_r(A/I^s) \arrow[d, "F^{r-1}"] \\ A \arrow[r] & A/I^s, \end{tikzcd}$$

induce an isomorphism of pro rings $\{A \otimes_{W_r(A)} W_r(A/I^s)\}_s \xrightarrow{\simeq} \{A/I^s\}_s$.

Proof. – (i): If I is generated by $t_1, \dots, t_m \in A$, then $I^{(M)} = \langle t_1^M, \dots, t_m^M \rangle$ contains $I^{m(M-1)+1}$; so it is enough to show that $I^{(M)} \subseteq \langle F^{r-1} W_r(I) \rangle$ for $M \gg 0$. But $M = p^{r-1}$ clearly has this property, since for any $a \in I$ we have $F^{r-1}[a] = a^{p^{r-1}}$.

(ii): Since $W_r(A) \to W_r(A/I^s)$ is surjective with kernel $W_r(I^s)$, the tensor product $A \otimes_{W_r(A)} W_r(A/I^s)$ is simply $A/\langle F^{r-1} W_r(I^s) \rangle$. Thus the claimed isomorphism of pro rings is the statement that the chains of ideals I^s and $F^{r-1} W_r(I^s) A$, for $s \geq 1$, are intertwined; one inclusion is obvious and the other is (i). □

To say more we will focus on $\mathbb{Z}_{(p)}$-algebras A which are F-finite in the sense of Definition 1.1. The following results of Langer and Zink may be found in [23, appendix]:

THEOREM 3.6 (Langer-Zink). – *Let A be an F-finite $\mathbb{Z}_{(p)}$-algebra and $r \geq 1$. Then:*

(i) *The Frobenius $F : W_{r+1}(A) \to W_r(A)$ is a finite ring homomorphism; i.e., $W_r(A)$ is finitely generated as a module over its subring $F W_{r+1}(A)$.*
(ii) *If B is a finitely generated A-algebra, then B is also F-finite and $W_r(B)$ is a finitely generated $W_r(A)$-algebra.*
(iii) *If A is Noetherian then $W_r(A)$ is also Noetherian.*

We now reach the main result of this section, relating the Frobenius with pro completion; this is our primary algebraic tool for extending results from THH to TR^r:

THEOREM 3.7. – *Let A be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra, $I \subseteq A$ an ideal, and $r \geq 1$. Consider the "completion" functor* $\Phi_r \colon W_r(A)\text{-mod} \longrightarrow \text{Pro}\, W_r(A)\text{-mod}$ *given by* $\Phi_r(M) := M \otimes_{W_r(A)} W_r(A/I^\infty) = \{M \otimes_{W_r(A)} W_r(A/I^s)\}_s$*. Then:*

(i) *The functor Φ_r is exact on the subcategory of finitely generated $W_r(A)$-modules.*
(ii) *The following diagram commutes up to natural isomorphism:*

$$\begin{tikzcd} A\text{-mod} \arrow[r, "\Phi_1"] \arrow[d, "(F^{r-1})^*"'] & \text{Pro}\, A\text{-mod} \arrow[d, "(F^{r-1})^*"] \\ W_r(A)\text{-mod} \arrow[r, "\Phi_r"] & \text{Pro}\, W_r(A)\text{-mod}. \end{tikzcd}$$

In other words, if M is an A-module, viewed as a $W_r(A)$-module via the Frobenius $F^{r-1} : W_r(A) \to A$, then there is a natural isomorphism $\Phi_r(M) \cong \{M \otimes_A A/I^s\}_s$.

Proof. – (i): We must prove that the pro abelian group $\{\mathrm{Tor}_n^{W_r(A)}(W_r(A/I^s), M)\}_s$ vanishes for any finitely generated $W_r(A)$-module M and integer $n > 0$. According to Lemma 3.1(ii)+(iv), the chain of ideals $W_r(I^s)$ is intertwined with the chain $W_r(I)^s$, so it is sufficient to prove that the pro abelian group $\{\mathrm{Tor}_n^{W_r(A)}(W_r(I)^s, M)\}_s$ vanishes. But according to Langer-Zink, $W_r(A)$ is Noetherian, so this vanishing claim is covered by the Artin-Rees Theorem 2.1(i).

(ii) is a restatement of Lemma 3.5(ii). □

3.2. F-finiteness

In this section we prove some basic properties surrounding F-finiteness (Definition 1.1), for which we claim no originality but for which we know of no suitable reference. We fix a prime number p for the next three lemmas.

LEMMA 3.8. – *Let A be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra. Then the following are also Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebras:*

(i) *Any finitely generated A-algebra.*
(ii) *Any localisation of A at a multiplicative system.*
(iii) *The completion of A at any ideal.*
(iv) *The Henselisation of A at any ideal.*
(v) *The strict Henselisation of A at any maximal ideal.*

Proof. – The claim that operations (i)–(v) preserve the Noetherian property is standard commutative algebra, so we need only prove the F-finiteness assertion, for which we may replace A by A/pA and therefore assume that A is an $\mathbb{F}_p$-algebra. Let $a_1, \dots, a_n$ generate A as an algebra over its p-th powers A^p.

(i): If $B = A[b_1, \dots, b_m]$ is a finitely generated A-algebra, then it is generated by $a_1, \dots, a_n$, $b_1, \dots, b_m$ over its p-th powers, hence is F-finite. (ii): If B is a localisation of A then B is also generated by $a_1, \dots, a_n$ over its p-th powers. (iii): Let $\widehat{A}$ be the completion of A at an ideal $I \subseteq A$. Picking generators $t_1, \dots, t_d \in I$ for the ideal I, there is a resulting surjection $A[[X_1, \dots, X_d]] \to \widehat{A}$ and so the F-finiteness of $\widehat{A}$ follows from that of $A[[X_1, \dots, X_d]]$ (it is generated by $a_1, \dots, a_n, X_1, \dots, X_d$ over its p-th powers).

We omit the proofs of (iv) and (v) as they are not required in the paper. □

LEMMA 3.9. – *Let A be a $\mathbb{Z}_{(p)}$-algebra and $r \geq 1$. Then:*

(i) *$W_r(A)$ is a $\mathbb{Z}_{(p)}$-algebra.*
(ii) *If p is nilpotent in A then p is nilpotent in $W_r(A)$.*
(iii) *If A is F-finite then $W_r(A)$ is also F-finite.*

Proof. – (i): If an integer is invertible in A then it is also invertible in $\mathbb{W}_S(A)$ for any truncation set S; see e.g., [16, Lem. 1.9].

(ii): It is well-known that $W_r(\mathbb{F}_p) = \mathbb{Z}/p^r\mathbb{Z}$, whence $p^r = 0$ in $W_r(A/pA)$; since the kernel of $W_r(A) \to W_r(A/pA)$ is nilpotent by Lemma 3.1(ii) and the assumption, it follows that p is also nilpotent in $W_r(A)$.

(iii): The formula $FV = p$, e.g., [35, A.4(vi)], implies that the Frobenius F induces a ring homomorphism $A \cong W_{r+1}(A)/VW_r(A) \xrightarrow{F} W_r(A)/pW_r(A)$, which is a finite morphism by Langer-Zink; since A is F-finite by assumption, it follows from Lemma 3.8(i) that $W_r(A)/pW_r(A)$ is also F-finite. □

Finally we observe that F-finiteness is a sufficient (and, in fact, necessary) condition for the finite generation of Kähler differentials; this fact underlies the finite generation of André-Quillen homology which we will prove in Theorem 4.6:

Lemma 3.10. – *Let A be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra and $n \geq 0$. Then $\Omega^n_A \otimes_A A/pA$ is a finitely generated A-module. If furthermore p is nilpotent in A, then Ω^n_A is a finitely generated A-module.*

Proof. – It is enough to treat the case $n = 1$. Let $a_1, \dots, a_m$ generate A/pA as a module over its p-th powers. Then any $b \in A$ may be written as a sum $b = pb' + \sum_i b_i^p a_i$ for some $b', b_1, \dots, b_m \in A$, and so we deduce that

$$db = p\,db' + \sum_i (b_i^p\,da_i + pb_i^{p-1} a_i\,db_i) \equiv \sum_i b_i^p\,da_i \mod p.$$

That is, $\Omega^1_{A/k} \otimes_A A/pA$ is generated by $da_1, \dots, da_m$. Multiplying by p^e, it follows that $p^e\Omega^1_A/p^{e+1}\Omega^1_A$ is also finitely generated; so if p is nilpotent then $p^e\Omega^1_A$, $e \geq 1$, defines a finite filtration on Ω^1_A with finitely generated steps, whence Ω^1_A is finitely generated. □

4. Finite generation results for HH, THH, and TR^r

The primary aim of this section is Theorem 4.7, which states that the algebraic and topological Hochschild homology groups, with finite coefficients, of a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra A are finitely generated over A, and that the groups $TR^r_n(A; \mathbb{Z}/p^v)$ are finitely generated over $W_r(A)$. From this we deduce additional finite generation results for the p-completed theories, and for rings in which p is nilpotent. We believe these are the first general finite generation results for topological Hochschild homology.

The key step is similar finite generation results for the André-Quillen homology of A; indeed, the following lemma reduces the problem to André-Quillen homology:

Lemma 4.1. – *Let A be a Noetherian ring and $I \subseteq A$ an ideal. Then* (i) $\Longleftrightarrow$ (i′) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (ii′) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iii′)*, where*

(i) *$D^i_n(A/\mathbb{Z}, A/I)$ is a finitely generated A-module for all $n, i \geq 0$.*
(i′) *$D^i_n(A/\mathbb{Z}, M)$ is a finitely generated A-module for all $n, i \geq 0$ and all finitely generated A-modules M killed by a power of I.*
(ii) *$HH_n(A, A/I)$ is a finitely generated A-module for all $n \geq 0$.*
(ii′) *$HH_n(A, M)$ is a finitely generated A-module for all $n \geq 0$ and all finitely generated A-modules M killed by a power of I.*
(iii) *$THH_n(A, A/I)$ is a finitely generated A-module for all $n \geq 0$.*
(iii′) *$THH_n(A, M)$ is a finitely generated A-module for all $n \geq 0$ and all finitely generated A-modules M killed by a power of I.*

Proof. – The implications (i)$\Rightarrow$(i′), (ii)$\Rightarrow$(ii′), and (iii)$\Rightarrow$(iii′) (whose converses are trivial) follow in the usual way using universal coefficient spectral sequences and long exact homology sequences; we will demonstrate the argument in the case of André-Quillen homology. Let M be a finitely generated A-module killed by a power of I; by induction on the power of I killing M, and using the long exact sequence

$$\cdots \longrightarrow D_n^i(A/\mathbb{Z}, IM) \longrightarrow D_n^i(A/\mathbb{Z}, M) \longrightarrow D_n^i(A/\mathbb{Z}, M/IM) \longrightarrow \cdots,$$

we may clearly assume that $IM = 0$. Thus $M \otimes_A A/I = M$ and so there is a universal coefficient spectral sequence

$$E_{st}^2 = \mathrm{Tor}_s^A(M, D_t^i(A/\mathbb{Z}, A/I)) \Longrightarrow D_{s+t}^i(A/\mathbb{Z}, M).$$

Assuming (i), the A-modules on the left are finitely generated, hence the abutment is also finitely generated, proving (i′).

Implication (i)$\Rightarrow$(ii) is an immediate consequence of the André-Quillen-to-Hochschild-homology spectral sequence $E_{ij}^2 = D_i^j(A/\mathbb{Z}, A/I) \Rightarrow HH_{i+j}(A, A/I)$.

Implication (ii′)$\Rightarrow$(iii) follows from the results recalled in Section 2.3.1. Indeed, the Pirashvili-Waldhausen spectral sequence $E_{ij}^2 = HH_i(A, THH_j(\mathbb{Z}, A/I)) \Rightarrow THH_{i+j}(A, A/I)$ will prove this implication if we know that $THH_j(\mathbb{Z}, A/I)$ is a finitely generated A-module for all j; fortunately Bökstedt's calculation shows that this is indeed the case:

$$THH_j(\mathbb{Z}, A/I) \cong \begin{cases} A/I & j = 0 \\ A/(I + mA) & j = 2m - 1 \\ (A/I)[m] & j = 2m > 0. \end{cases}$$

It remains to prove (iii′)$\Rightarrow$(ii′), although we will not need this implication in the remainder of the paper. To proceed by induction, fix $n \geq 0$ and suppose we have shown that $HH_i(A, M)$ is finitely generated for all $i < n$ and for all finitely generated A-modules M killed by a power of I; note that the base case of the induction is covered by the identity $HH_0(A, M) = M$. To prove the desired implication, it is now enough to fix an A-module M killed by a power of I and to show that $HH_n(A, M)$ is finitely generated. The inductive hypothesis and Bökstedt's calculation shows that, in the Pirashvili-Waldhausen spectral sequence

$$E_{ij}^2 = HH_i(A, THH_j(\mathbb{Z}, M)) \Longrightarrow THH_{i+j}(A, M),$$

the A-modules E_{ij}^2 are finitely generated for all $i < n$. Since $THH_n(A, M)$ is finitely generated by assumption, it easily follows, e.g., by working in the Serre quotient category of A-modules modulo the finitely generated modules, that $E_{n\,0}^2 = HH_n(A, M)$ is also finitely generated, as required. □

Given a commutative ring A, let sA-mod denote the category of simplicial A-modules, which is equivalent via the Dold-Kan correspondence to the category of chain complexes vanishing in negative degrees. We will be particularly interested in those simplicial A-modules having finitely generated homotopy over A; i.e., those $M_\bullet \in sA$-mod for which $\pi_n(M_\bullet)$ is a finitely generated A-module for all $n \geq 0$.

LEMMA 4.2. – *Let A be a Noetherian ring. Then:*

(i) *If $L_\bullet \to M_\bullet \to N_\bullet$ is a homotopy cofibre sequence of simplicial A-modules such that two of $L_\bullet$, $M_\bullet$, $N_\bullet$ have finitely generated homotopy over A, then so does the third.*

(ii) *If $L_\bullet \in sA$-mod consists of flat A-modules in each degree and has finitely generated homotopy, and B is a Noetherian A-algebra, then $L_\bullet \otimes_A B$ has finitely generated homotopy over B.*

(iii) *If $L_\bullet, M_\bullet \in sA$-mod have finitely generated homotopy, and $L_\bullet$ consists of flat A-modules in each degree, then $L_\bullet \otimes_A M_\bullet$ has finitely generated homotopy.*

Proof. – (i) is an obvious consequence of the long exact sequence of A-modules

$$\cdots \longrightarrow \pi_n(L_\bullet) \longrightarrow \pi_n(M_\bullet) \longrightarrow \pi_n(N_\bullet) \longrightarrow \cdots .$$

(ii): The flatness assumption implies that there is a spectral sequence of B-modules $E^2_{ij} = \operatorname{Tor}^A_i(\pi_j(L_\bullet), B) \Rightarrow \pi_{i+j}(L_\bullet \otimes_A B)$, so it is enough to show that the Tors are finitely generated B-modules; since $\pi_j(L_\bullet)$ is finitely generated over A by assumption, we may pick a resolution $P_\bullet$ of it by finitely generated, projective A-modules, and then $\operatorname{Tor}^A_i(\pi_j(L_\bullet), B) = H_i(P_\bullet \otimes_A B)$ is evidently finitely generated over B.

(iii): The flatness assumption implies that there is a spectral sequence of A-modules $E^1_{ij} = L_i \otimes_A \pi_j(M_\bullet) \Rightarrow \pi_{i+j}(L_\bullet \otimes_A M_\bullet)$, whose E^2-page is $E^2_{ij} = \pi_i(L_\bullet \otimes_A \pi_j(M_\bullet))$. But these terms on the E^2-page are finitely generated thanks to our assumption and the spectral sequences ${}'E^2_{st} = \operatorname{Tor}^A_s(\pi_t(L_\bullet), \pi_j(M_\bullet)) \Rightarrow \pi_{s+t}(L_\bullet \otimes_A \pi_j(M_\bullet))$ for each $j \geq 0$. ☐

The second tool to prove the required results about finite generation of André-Quillen homology is a spectral sequence due originally to C. Kassel and A. Sletsjøe [21, Thm. 3.2]. We state it here as a filtration on the cotangent complexes, rather than as the resulting spectral sequence:

LEMMA 4.3 (Kassel-Sletsjøe). – *Let $A \to B \to C$ be homomorphisms of rings. Then it is possible to choose the cotangent complexes $\mathbb{L}_{C/A}$, $\mathbb{L}_{C/B}$, and $\mathbb{L}_{B/A}$ to be degree-wise projective modules and, for all $i \geq 0$, such that $\mathbb{L}^i_{C/A}$ has a natural filtration*

$$\mathbb{L}^i_{C/A} = \mathrm{F}^0\mathbb{L}^i_{C/A} \supseteq \mathrm{F}^1\mathbb{L}^i_{C/A} \supseteq \cdots \supseteq \mathrm{F}^i\mathbb{L}^i_{C/A} = \mathbb{L}^i_{B/A} \otimes_B C \supseteq \mathrm{F}^{i+1}\mathbb{L}^i_{C/A} = 0$$

of length i by simplicial C-modules, with graded pieces

$$(j = 0, \dots, i) \qquad \mathrm{gr}^j\, \mathbb{L}^i_{C/A} \cong (\mathbb{L}^j_{B/A} \otimes_B C) \otimes_C \mathbb{L}^{i-j}_{C/B}.$$

Sketch of proof. – One chooses simplicial resolutions in the usual way [33, Thm. 5.1] to ensure that the Jacobi-Zariski sequence

$$0 \to \mathbb{L}_{B/A} \otimes_B C \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B} \to 0$$

is actually a short exact sequence of simplicial C-modules which are projective in each degree. Then observe that whenever $0 \to L \to M \to N \to 0$ is a short exact sequence of projective modules over C, there is a resulting filtration on $\bigwedge^i M$ with graded pieces $\bigwedge^j L \otimes_C \bigwedge^{i-j} N$. ☐

DEFINITION 4.4. – Let $m \geq 0$ be an integer. We say that a homomorphism $k \to A$ of Noetherian rings is *m-AQ-finite*, or that A is m-AQ-finite over k, if and only if the André-Quillen homology groups $D_n^i(A/k, A/mA)$ are finitely generated A-modules for all $n, i \geq 0$. In the case $m = 0$ we omit the "0-"; so A is AQ-finite over k if and only if $D_n^i(A/k)$ is finitely generated for all $n, i \geq 0$. In the case $k = \mathbb{Z}$ we omit the "over $\mathbb{Z}$", and say simply that A is m-AQ-finite or AQ-finite.

LEMMA 4.5. – *Assume all rings in the following are Noetherian, and let $m \geq 0$. Then:*

(i) *A finite type morphism is m-AQ-finite.*
(ii) *Localising at a multiplicative system is m-AQ-finite.*
(iii) *A composition of two m-AQ-finite morphisms is again m-AQ-finite.*

Proof. – Since AQ-finiteness implies m-AQ-finiteness (this follows from the implication (i)⇒(i′) of Lemma 4.1), it is enough to prove (i) and (ii) in the case $m = 0$.

Then (i) is a result of Quillen, obtained by constructing a simplicial resolution of the finitely generated k-algebra A by finitely generated, free k-algebras; see [33, Prop. 4.12]. Claim (ii) is another result of Quillen: if S is a multiplicative system in k, then [33, Thm. 5.4] states that $D_n^i(S^{-1}k/k) = 0$.

(iii): Let $A \to B \to C$ be homomorphisms of Noetherian rings such that $A \to B$ and $B \to C$ are m-AQ-finite. Pick the cotangent complexes according to Lemma 4.3, and fix $i \geq 0$. Since all the simplicial modules appearing in the statement of Lemma 4.3 are degree-wise projective, the description of the filtration remains valid after tensoring by any C-module. In particular, we deduce that $\mathbb{L}^i_{C/A} \otimes_C C/mC$ has a decreasing filtration with graded pieces

$$\begin{aligned}\mathrm{gr}^j(\mathbb{L}^i_{C/A} \otimes_C C/mC) &\cong (\mathbb{L}^j_{B/A} \otimes_B C) \otimes_C \mathbb{L}^{i-j}_{C/B} \otimes_C C/mC \\ &\cong ((\mathbb{L}^j_{B/A} \otimes_B B/mB) \otimes_{B/mB} C/mC) \otimes_{C/mC} (\mathbb{L}^{i-j}_{C/B} \otimes_C C/mC)\end{aligned}$$

for $j = 0, \dots, i$. By the m-AQ-finiteness assumption, $\mathbb{L}^j_{B/A} \otimes_B B/mB$ and $\mathbb{L}^{i-j}_{C/B} \otimes_C C/mC$ have finitely generated homotopy over B and C, respectively; so Lemma 4.2(ii)+(iii) imply that the above graded pieces have finitely generated homotopy. Applying Lemma 4.2(i) i times, it follows that $\mathbb{L}^i_{C/A} \otimes_C C/mC$ has finitely generated homotopy, as desired. □

The following finite generation result is the first main result of the paper; it states, in particular, that if A is a Noetherian, F-finite $\mathbb{F}_p$-algebra, then the André-Quillen homology groups $D_n^i(A/\mathbb{F}_p)$ are finitely generated A-modules for all $n, i \geq 0$:

THEOREM 4.6. – *Let p be a prime number and $e \geq 1$.*

(i) *Any Noetherian, F-finite $\mathbb{Z}/p^e\mathbb{Z}$-algebra is AQ-finite over $\mathbb{Z}/p^e\mathbb{Z}$ and over $\mathbb{Z}$.*
(ii) *Any Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra is p-AQ-finite over $\mathbb{Z}_{(p)}$ and over $\mathbb{Z}$.*

Proof. – First let A be a Noetherian, F-finite $\mathbb{F}_p$-algebra. Let $F : A \to A$ be the absolute Frobenius $x \mapsto x^p$. The homomorphism F may be factored as a composition of $\mathbb{F}_p$-algebra homomorphisms $A \xrightarrow{\pi} A^p \xrightarrow{e} A$, where A^p is the subring of A consisting of p-th powers, $\pi(x) := x^p$, and e is the natural inclusion. Our proof will be based on the following observations. Firstly, π and e are finite type homomorphisms between Noetherian rings (indeed, A^p is a quotient of the Noetherian ring A, and the fact that e is of finite type is exactly our assumption that A is F-finite). Secondly, the absolute Frobenius $F : \mathbb{L}^i_{A/\mathbb{F}_p} \to \mathbb{L}^i_{A/\mathbb{F}_p}$ is zero for $i \geq 1$ (indeed, in any $\mathbb{F}_p$-algebra, we have $F(d\beta) = d\beta^p = p\beta^{p-1}\, d\beta = 0$).

We will now prove the following statement by induction on $i \geq 0$:

> If A is a Noetherian, F-finite $\mathbb{F}_p$-algebra, then $\mathbb{L}^i_{A/\mathbb{F}_p}$ has finitely generated homotopy over A.

The claim is trivial for $i = 0$ since $\mathbb{L}^0_{A/\mathbb{F}_p} \simeq A$, so assume $i \geq 1$. We remark that, in the inductive proof that follows, we will need to choose different models of the cotangent complex $\mathbb{L}_{A/\mathbb{F}_p}$. Apply Lemma 4.3 to the composition $\mathbb{F}_p \to A^p \xrightarrow{e} A$ to see that it is possible to pick the cotangent complexes $\mathbb{L}_{A/\mathbb{F}_p}$, $\mathbb{L}_{A/A^p}$, and $\mathbb{L}_{A^p/\mathbb{F}_p}$ in such a way that $\mathbb{L}^i_{A/\mathbb{F}_p}$ has a descending filtration $\mathrm{F}^\bullet\mathbb{L}^i_{A/\mathbb{F}_p}$ with graded pieces

$$\mathrm{gr}^j\, \mathbb{L}^i_{A/\mathbb{F}_p} \cong (\mathbb{L}^j_{A^p/\mathbb{F}_p} \otimes_{A^p} A) \otimes_A \mathbb{L}^{i-j}_{A/A^p} \tag{$\dagger$}$$

for $j = 0, \ldots, i$. For each $j = 0, \ldots, i-1$, the simplicial A^p-module $\mathbb{L}^j_{A^p/\mathbb{F}_p}$ has finitely generated homotopy over A^p by the inductive hypothesis (note that A^p is also a Noetherian, F-finite $\mathbb{F}_p$-algebra since it is a quotient of A); meanwhile, $\mathbb{L}^{i-j}_{A/A^p}$ has finitely generated homotopy over A by Lemma 4.5(i). Applying Lemma 4.2(ii)+(iii) we deduce that the right side of ($\dagger$) has finitely generated homotopy over A for $j = 0, \ldots, i-1$; from Lemma 4.2(i) it follows that $X := \mathbb{L}^i_{A/\mathbb{F}_p}/\mathrm{F}^i\mathbb{L}^i_{A/\mathbb{F}_p}$ has finitely generated homotopy. In summary, we have a homotopy cofibre sequence

$$\mathrm{gr}^i\, \mathbb{L}^i_{A/\mathbb{F}_p} = \mathbb{L}^i_{A^p/\mathbb{F}_p} \otimes_{A^p} A \xrightarrow{e} \mathbb{L}^i_{A/\mathbb{F}_p} \longrightarrow X,$$

where X has finitely generated homotopy over A.

We now apply a similar argument to the composition $\mathbb{F}_p \to A \xrightarrow{\pi} A^p$, after first picking new models for the cotangent complexes $\mathbb{L}_{A^p/\mathbb{F}_p}$, $\mathbb{L}_{A^p/A}$, and $\mathbb{L}_{A/\mathbb{F}_p}$ such that $\mathbb{L}^i_{A^p/\mathbb{F}_p}$ has a descending filtration $\mathrm{F}^\bullet\mathbb{L}^i_{A^p/\mathbb{F}_p}$ with graded pieces

$$\mathrm{gr}^j\, \mathbb{L}^i_{A^p/\mathbb{F}_p} \cong (\mathbb{L}^j_{A/\mathbb{F}_p} \otimes_A A^p) \otimes_{A^p} \mathbb{L}^{i-j}_{A^p/A}$$

for $j = 0, \ldots, i$. The inductive hypothesis implies that $\mathbb{L}^j_{A/\mathbb{F}_p}$ has finitely generated homotopy over A for $j = 0, \ldots, i-1$, and Lemma 4.5(i) implies that $\mathbb{L}^{i-j}_{A^p/A}$ has finitely generated homotopy over A^p. So by the same argument as in the previous paragraph, $\mathbb{L}^i_{A^p/\mathbb{F}_p}/\mathrm{F}^i\mathbb{L}^i_{A^p/\mathbb{F}_p}$ has finitely generated homotopy over A^p. Using Lemma 4.2(ii) to base change along $A^p \xrightarrow{e} A$, we deduce that there is a homotopy cofibre sequence

$$\mathbb{L}^i_{A/\mathbb{F}_p} \otimes_A A^p \otimes_{A^p} A \xrightarrow{\pi} \mathbb{L}^i_{A^p/\mathbb{F}_p} \otimes_{A^p} A \longrightarrow Y$$

of simplicial A-modules, where Y has finitely generated homotopy over A.

In conclusion, we have a diagram in the homotopy category of simplicial A-modules

$$\begin{tikzcd} & Y \\ & \mathbb{L}^i_{A^p/\mathbb{F}_p} \otimes_{A^p} A \arrow[u] \arrow[r, "e"] & \mathbb{L}^i_{A/\mathbb{F}_p} \arrow[r] & X \\ & \mathbb{L}^i_{A/\mathbb{F}_p} \otimes_A A^p \otimes_{A^p} A \arrow[u, "\pi"] \end{tikzcd}$$

in which the row and column are both homotopy cofibre sequences, and in which X and Y have finitely generated homotopy. There is therefore a resulting homotopy cofibre sequence

$$Y \longrightarrow \mathrm{hocofib}(\mathbb{L}^i_{A/\mathbb{F}_p} \otimes_A A^p \otimes_{A^p} A \xrightarrow{e\circ\pi} \mathbb{L}^i_{A/\mathbb{F}_p}) \longrightarrow X,$$

and so Lemma 4.2(i) implies that the simplicial A-module in the centre of this sequence also has finitely generated homotopy. But $e\circ\pi = F$ is nulhomotopic on the cotangent complexes, so this means that both $\mathbb{L}^i_{A/\mathbb{F}_p} \otimes_A A^p \otimes_{A^p} A$ and $\mathbb{L}^i_{A/\mathbb{F}_p}$ must have finitely generated over homotopy over A, which completes the proof of the inductive step.

We have proved that any Noetherian, F-finite $\mathbb{F}_p$-algebra is AQ-finite over $\mathbb{F}_p$; the remaining claims of the theorem will all follow from this. Firstly let k denote either $\mathbb{Z}/p^e\mathbb{Z}$ or $\mathbb{Z}_{(p)}$, and note that any Noetherian, F-finite $\mathbb{F}_p$-algebra is also AQ-finite over k; this follows from Lemma 4.5.

Now let A be a Noetherian, F-finite k-algebra. We will prove by induction on $i \geq 0$ that $\mathbb{L}^i_{A/k} \otimes_A A/pA$ has finitely generated homotopy over A/pA. We apply Lemma 4.3 to the composition $k \to A \to A/pA$ and note the following, similar to the earlier part of the proof: $\mathbb{L}^i_{(A/pA)/k}$ has finitely generated homotopy by what we have already proved, while $\mathrm{gr}^j\,\mathbb{L}^i_{(A/pA)/k} \cong (\mathbb{L}^j_{A/k} \otimes_A A/pA) \otimes_{A/pA} \mathbb{L}^{i-j}_{(A/pA)/A}$ has finitely generated homotopy over A/pA for $j = 0, \ldots, i-1$ by Lemma 4.2 and the inductive hypothesis. By Lemma 4.2 again, it follows that the remaining part of the filtration, namely $\mathrm{gr}^i\,\mathbb{L}^i_{(A/pA)/k} \cong \mathbb{L}^i_{A/k} \otimes_A A/pA$, also has finitely generated homotopy over A/pA, as required.

We have proved that any Noetherian, F-finite k-algebra is p-AQ-finite over k, hence also over $\mathbb{Z}$ by Lemma 4.5.

From now on $k = \mathbb{Z}/p^e\mathbb{Z}$ for some $e \geq 1$. All that remains to be shown is that if a Noetherian k-algebra A is p-AQ-finite over k, then it is actually AQ-finite over k. By implication (i)⇒(i′) of Lemma 4.1 with $I = pA$, it follows that $D^i_n(A/k, M)$ is finitely generated for any finitely generated A-module M which is killed by a power of p; in particular, taking $M = A = A/p^eA$ we deduce that $D^i_n(A/k)$ is finitely generated for all $n, i \geq 0$, as required. □

The following is our main finite generation theorem for HH, THH, and TR^r, from which others will follow:

THEOREM 4.7. – *Let A be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra, and $v > 0$. Then:*

(i) *$HH_n(A; \mathbb{Z}/p^v)$ and $THH_n(A; \mathbb{Z}/p^v)$ are finitely generated A-modules for all $n \geq 0$.*
(ii) *$TR^r_n(A; \mathbb{Z}/p^v)$ is a finitely generated $W_r(A)$-module for all $n \geq 0$ and $r \geq 1$.*

Proof. – (i): As explained in Section 2.4, there is a long exact sequence of A-modules

$$\cdots \longrightarrow HH_{n-1}(A, A[p^v]) \longrightarrow HH_n(A; \mathbb{Z}/p^v) \longrightarrow HH_n(A, A/p^v A) \longrightarrow \cdots.$$

The A-modules $HH_n(A, A[p^v])$ and $HH_n(A, A/p^v A)$ are finitely generated for all $n \geq 0$ by Theorem 4.6 and Lemma 4.1, whence $HH_n(A; \mathbb{Z}/p^v)$ is also finitely generated. The argument for THH is verbatim equivalent.

(ii): The fundamental long exact sequence

$$\cdots \longrightarrow \pi_n(THH(A; \mathbb{Z}/p^v)_{hC_{p^r}}) \longrightarrow TR_n^{r+1}(A; \mathbb{Z}/p^v) \longrightarrow TR_n^r(A; \mathbb{Z}/p^v) \to \cdots$$

is one of $W_{r+1}(A)$-modules, as explained in Sections 2.3.3 and 2.4. Since the ring $W_{r+1}(A)$ is Noetherian by Langer-Zink (Theorem 3.6), it is enough by the five lemma and induction (recall that $TR_n^1(A; \mathbb{Z}/p^v) = THH_n(A; \mathbb{Z}/p^v)$ to start the induction) to show that $\pi_n(THH(A; \mathbb{Z}/p^v)_{hC_{p^r}})$ is a finitely generated $W_{r+1}(A)$-module for all $n, r \geq 0$.

To show this, recall the group homology spectral sequence

$$E_{ij}^2 = H_i(C_{p^r}, THH_j(A; \mathbb{Z}/p^v)) \implies \pi_{i+j}(THH(A; \mathbb{Z}/p^v)_{hC_{p^r}}),$$

which is a spectral sequence of $W_{r+1}(A)$-modules, where $W_{r+1}(A)$ acts on the A-modules on the E^2-page via $F^r : W_{r+1}(A) \to A$. But each A-module E_{st}^2 is finitely generated by part (i), hence is also finitely generated as a $W_{r+1}(A)$-module since F^r is a finite morphism (again by Langer-Zink). Thus the abutment of the spectral sequence is also finitely generated over $W_{r+1}(A)$, as required. □

Corollary 4.8. – *Let A be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra, and $n \geq 0$, $r \geq 1$. Then $HH_n(A; \mathbb{Z}_p)$ and $THH_n(A; \mathbb{Z}_p)$ are finitely generated $\hat{A_p}$-modules, and $TR_n^r(A; p, \mathbb{Z}_p)$ is a finitely generated $W_r(\hat{A_p})$-module.*

Proof. – Letting $v \to \infty$, this follows from Theorem 4.7 via Proposition 2.7; note that $W_r(A)\hat{_p} \cong W_r(\hat{A_p})$ by Lemma 3.4. □

In the case in which p is nilpotent in A, for example when A is an $\mathbb{F}_p$-algebra, it is evidently not necessary to p-complete:

Corollary 4.9. – *Let A be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra in which p is nilpotent, and $n \geq 0$, $r \geq 1$. Then $HH_n(A)$ and $THH_n(A)$ are finitely generated A-modules, and $TR_n^r(A; p)$ is a finitely generated $W_r(A)$-module.*

Proof. – Since p is nilpotent in A, it is also nilpotent in $W_r(A)$ by Lemma 3.9(ii); hence $HH_n(A)$, $THH_n(A)$, and $TR_n^r(A; p)$ are all groups of bounded p-torsion. It follows that $HH_n(A) = HH_n(A; \mathbb{Z}_p)$, similarly for THH and TR, and that $\hat{A_p} = A$. So the claim follows from Corollary 4.8 (or it can be deduced directly from Theorem 4.7 without passing via the p-completion). □

Remark 4.10. – The assertions of Corollary 4.9 are true whenever A is a Noetherian, F-finite, AQ-finite $\mathbb{Z}_{(p)}$-algebra (e.g., an essentially finite type $\mathbb{Z}_{(p)}$-algebra).

Firstly, the HH and THH assertions follow immediately from Lemmas 4.5 and 4.1. To prove the TR^r assertion, one then repeats the proof of Theorem 4.7(ii), except working directly with $THH(A)$ and $TR^r(A; p)$ instead of $THH(A; \mathbb{Z}/p^v)$ and $TR^r(A; \mathbb{Z}/p^v)$.

5. Continuity and the pro HKR theorems

Let A be a Noetherian ring and $I \subseteq A$ an ideal. In [4] it was asked when the canonical map of pro A-modules

$$\{THH_n(A) \otimes_A A/I^s\}_s \longrightarrow \{THH_n(A/I^s)\}_s,$$

is an isomorphism. It was shown by Geisser and Hesselholt [12, §1] to be an isomorphism in the case that $A = R[X_1, \ldots, X_d]$ is a polynomial algebra over any (even non-Noetherian) ring R and $I = \langle X_1, \ldots, X_d \rangle$. Under our usual hypotheses including F-finiteness, we will show in Section 5.1 that it is an isomorphism for *any* ideal I, at least working with finite coefficients.

5.1. Degree-wise continuity and continuity of HH, THH, TR^r, etc.

A key tool in our forthcoming proof of continuity properties is the following restriction spectral sequence for Hochschild homology. The case of topological Hochschild homology was established by M. Brun [3, Thm. 6.2.10]; the case of derived Hochschild homology is new but unsurprising, and may be proved in a purely algebraic way via simplicial methods (see [27, Prop. 3.4] for the details of the proof).

Proposition 5.1. – *Let k be a commutative ring, $A \to B$ a morphism of k-algebras, and M a B-bimodule. Then there is a natural spectral sequence of k-modules*

$$E^2_{ij} = HH^k_i(B, \mathrm{Tor}^A_j(B, M)) \Longrightarrow HH^k_{i+j}(A, M),$$

and similarly for THH.

Combining Proposition 5.1 with our earlier finite generation results and the Artin-Rees theorem, we may now establish our "degree-wise continuity" results, starting with the following lemma:

Lemma 5.2. – *Let A be a Noetherian ring, $I \subseteq A$ an ideal, M a finitely generated A-module, and $n \geq 0$. Consider the canonical maps:*

$$\{HH_n(A, M) \otimes_A A/I^s\}_s \xrightarrow{\text{(ii)}} \{HH_n(A, M/I^sM)\}_s \xrightarrow{\text{(i)}} \{HH_n(A/I^s, M/I^sM)\}_s$$

$$\{THH_n(A, M) \otimes_A A/I^s\}_s \xrightarrow{\text{(ii)}} \{THH_n(A, M/I^sM)\}_s \xrightarrow{\text{(i)}} \{THH_n(A/I^s, M/I^sM)\}_s.$$

The maps (i) are isomorphisms. If A is furthermore assumed to be m-AQ-finite for some $m \geq 0$, and M is annihilated by m, then the maps (ii) are also isomorphisms.

Proof. – (i): By Proposition 5.1 there is a first quadrant spectral sequence of A-modules

$$E^2_{ij}(s) = HH_i(A/I^s, \mathrm{Tor}^A_j(A/I^s, M/I^sM)) \Longrightarrow HH_{i+j}(A, M/I^sM)$$

for each $s \geq 1$. These assemble to a spectral sequence of pro A-modules

$$E^2_{ij}(\infty) = \{HH_i(A/I^s, \mathrm{Tor}^A_j(A/I^s, M/I^sM))\}_s \Longrightarrow \{HH_{i+j}(A, M/I^sM)\}_s.$$

By Corollary 2.2 $\{\mathrm{Tor}^A_j(A/I^s, M/I^sM)\}_s = 0$ for all $j \geq 1$, so this spectral sequence collapses to edge isomorphisms $\{HH_n(A, M/I^sM)\}_s \xrightarrow{\simeq} \{HH_n(A/I^s, M/I^sM)\}_r$ of pro A-modules for all $n \geq 0$; this proves isomorphism (i) for HH. The proof for THH is the same.

(ii): Now assume further that A is m-AQ-finite and that M is killed by m. Universal coefficient spectral sequences for HH assemble to a spectral sequence of pro A-modules

$$'E^2_{ij}(\infty) = \{\mathrm{Tor}_i^A(A/I^s, HH_j(A, M))\}_s \implies \{HH_{i+j}(A, M/I^s M)\}_s.$$

But the A-modules $HH_j(A, M)$ are finitely generated for all $j \geq 0$ by assumption and Lemma 4.1, so $\{\mathrm{Tor}_i^A(A/I^s, HH_j(A, M))\}_s = 0$ for $i \geq 1$ by the Artin-Rees Theorem 2.1(i). Thus we again obtain edge map isomorphisms $\{HH_n(A, M) \otimes_A A/I^s\}_s \xrightarrow{\simeq} \{HH_n(A, M/I^s M)\}_s$, completing the proof of isomorphism (ii) for HH. The proof for THH is verbatim equivalent. □

We now reach our main degree-wise continuity results, which establishes Theorem 1.4 of the Introduction:

Theorem 5.3. – *Let A be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra, and $I \subseteq A$ an ideal. Then the canonical maps*

(i) $\{HH_n(A; \mathbb{Z}/p^v) \otimes_A A/I^s\}_s \longrightarrow \{HH_n(A/I^s; \mathbb{Z}/p^v)\}_s$
(ii) $\{TR_n^r(A; \mathbb{Z}/p^v) \otimes_{W_r(A)} W_r(A/I^s)\}_s \longrightarrow \{TR_n^r(A/I^s; \mathbb{Z}/p^v)\}_s$

are isomorphisms for all $n \geq 0$ and $v, r \geq 1$.

Proof. – (i): This follows from the HH part of Lemma 5.2 in a relatively straightforward way: to keep the proof clear we will use ∞ notation for all the pro A-modules. As in the proof of Theorem 4.7(i), there is a long exact sequence of A-modules

$$\cdots \longrightarrow HH_{n-1}(A, A[p^v]) \longrightarrow HH_n(A; \mathbb{Z}/p^v) \longrightarrow HH_n(A, A/p^v A) \longrightarrow \cdots,$$

all of which are finitely generated. Hence we may base change by A/I^∞, as in the Artin-Rees Theorem 2.1(ii), to obtain a long exact sequence of pro A-modules

$$(1) \quad \cdots \to HH_{n-1}(A, A[p^v]) \otimes_A A/I^\infty \\ \to HH_n(A; \mathbb{Z}/p^v) \otimes_A A/I^\infty \to HH_n(A, A/p^v A) \otimes_A A/I^\infty \to \cdots.$$

Replacing A by A/I^∞, and using the isomorphisms of Corollary 2.4 with $M = A$, there is also a long exact sequence of pro A-modules

$$(2) \quad \cdots \to HH_{n-1}(A/I^\infty, A[p^v] \otimes_A A/I^\infty) \\ \to HH_n(A/I^\infty; \mathbb{Z}/p^v) \to HH_n(A/I^\infty, A/p^v A \otimes_A A/I^\infty) \to \cdots.$$

The obvious map from (1) to (2) induces an isomorphism on all the terms with coefficients in $A[p^v]$ and $A/p^v A$, by Theorem 4.6 and Lemma 5.2, hence also induces an isomorphism on the $\mathbb{Z}/p^v$ terms, as desired.

(ii): If $r = 1$ then the claim is that the canonical map $\{THH_n(A; \mathbb{Z}/p^v) \otimes_A A/I^s\}_s \to \{THH_n(A/I^s; \mathbb{Z}/p^v)\}_s$ is an isomorphism; the proof of this is verbatim equivalent to part (i). So now assume $r > 1$ and proceed by induction.

We may compare the fundamental long exact sequences with finite coefficients for both A and A/I^s as follows:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(THH(A;\mathbb{Z}/p^v)_{hC_{p^r}}) \arrow[r] \arrow[d] & TR_n^{r+1}(A;\mathbb{Z}/p^v) \arrow[r] \arrow[d] & TR_n^r(A;\mathbb{Z}/p^v) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \pi_n(THH(A/I^s;\mathbb{Z}/p^v)_{hC_{p^r}}) \arrow[r] & TR_n^{r+1}(A/I^s;\mathbb{Z}/p^v) \arrow[r] & TR_n^r(A/I^s;\mathbb{Z}/p^v) \arrow[r] & \cdots
\end{tikzcd}$$

As we saw in the proof of Theorem 4.7(ii), the top row consists of finitely generated $W_{r+1}(A)$-modules; so by Theorem 3.7(i), it remains exact after base changing by the pro $W_{r+1}(A)$-algebra $W_{r+1}(A/I^\infty)$. Simultaneously assembling the bottom row into pro $W_{r+1}(A)$-modules therefore yields a map of long exact sequences of pro $W_{r+1}(A)$-modules; by the five lemma and the inductive hypothesis, it is therefore enough to prove that

$$\pi_n(THH(A;\mathbb{Z}/p^v)_{hC_{p^r}}) \otimes_{W_{r+1}(A)} W_{r+1}(A/I^\infty) \longrightarrow \pi_n(THH(A/I^\infty;\mathbb{Z}/p^v)_{hC_{p^r}})$$

is an isomorphism for all $n \geq 0$. Both sides are the abutment of natural group homology spectral sequences, so it is now enough to check that the map of spectral sequence is an isomorphism on the second page, namely that the canonical map

$$(\dagger)\quad H_i(C_{p^r}, THH_j(A;\mathbb{Z}/p^v)) \otimes_{W_{r+1}(A)} W_{r+1}(A/I^\infty) \longrightarrow H_i(C_{p^r}, THH_j(A/I^\infty;\mathbb{Z}/p^v))$$

is an isomorphism for all $i, j \geq 0$. Since $H_i(C_{p^r}, THH_j(A;\mathbb{Z}/p^v))$ is a finitely generated A-module, the left side of ($\dagger$) is precisely $H_i(C_{p^r}, THH_j(A;\mathbb{Z}/p^v)) \otimes_A A/I^\infty$ by Theorem 3.7(ii); meanwhile, the right side is $H_i(C_{p^r}, THH_j(A;\mathbb{Z}/p^v) \otimes_A A/I^\infty)$ by the isomorphism for THH which has already been established.

Therefore it is finally enough to prove that the map

$$H_i(C_{p^r}, THH_j(A;\mathbb{Z}/p^v)) \otimes_A A/I^\infty \longrightarrow H_i(C_{p^r}, THH_j(A;\mathbb{Z}/p^v) \otimes_A A/I^\infty)$$

is an isomorphism; this follows from the finite generation of $THH_j(A;\mathbb{Z}/p^v)$ and Corollary 2.3. $\square$

Corollary 5.4. – *Let A be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra, and $I \subseteq A$ an ideal; let $\widehat{A}$ denote the I-adic completion of A. Then all of the following maps (not just the compositions) are isomorphisms for all $n \geq 0$ and $v, r \geq 1$:*

$$HH_n(A;\mathbb{Z}/p^v) \otimes_A \widehat{A} \longrightarrow HH_n(\widehat{A};\mathbb{Z}/p^v) \longrightarrow \varprojlim_s HH_n(A/I^s;\mathbb{Z}/p^v)$$

$$TR_n^r(A;\mathbb{Z}/p^v) \otimes_{W_r(A)} W_r(\widehat{A}) \longrightarrow TR_n^r(\widehat{A};\mathbb{Z}/p^v) \longrightarrow \varprojlim_s TR_n^r(A/I^s;\mathbb{Z}/p^v).$$

Proof. – We claim that each of the following canonical maps is an isomorphism:

$$HH_n(A;\mathbb{Z}/p^v) \otimes_A \widehat{A} \longrightarrow \varprojlim_s HH_n(A;\mathbb{Z}/p^v) \otimes_A A/I^s \longrightarrow \varprojlim_s HH_n(A/I^s;\mathbb{Z}/p^v).$$

Firstly, $HH_n(A;\mathbb{Z}/p^v)$ is a finitely generated A-module by Theorem 4.7, and A is Noetherian, so standard commutative algebra, e.g., [25, Thm. 8.7], implies that the first map is an isomorphism. Secondly, Theorem 5.3(i) implies that the second map is an isomorphism.

However, Lemma 3.8 implies that $\widehat{A}$ is also a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra, so applying the same argument to $\widehat{A}$ with respect to the ideal $I\widehat{A}$ we obtain another composition of isomorphisms

$$HH_n(\widehat{A};\mathbb{Z}/p^v)\otimes_{\widehat{A}}\widehat{A}\longrightarrow \varprojlim_s HH_n(\widehat{A};\mathbb{Z}/p^v)\otimes_{\widehat{A}}\widehat{A}/I^s\widehat{A}\longrightarrow \varprojlim_s HH_n(\widehat{A}/I^s\widehat{A};\mathbb{Z}/p^v).$$

Since $\widehat{A}/I^s\widehat{A}\cong A/I^s$ and $HH_n(\widehat{A})\otimes_{\widehat{A}}\widehat{A}\cong HH_n(\widehat{A})$, the desired isomorphisms for HH follow.

The proofs of the isomorphisms for TR^r are exactly the same as for HH, except that for TR^r one must also note that $W_r(A)$ is Noetherian by Langer-Zink (Theorem 3.6) and use Lemma 3.3. □

Whereas the previous two continuity results have concerned individual groups, we now prove the spectral continuity of THH, TR^r, etc. under our usual hypotheses; this establishes Theorem 1.5 of the Introduction:

Theorem 5.5. – *Let A be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra, and $I\subseteq A$ an ideal; assume that A is I-adically complete. Then, for any $r\geq 1$, the canonical map of spectra*

$$TR^r(A;p)\longrightarrow \operatorname{holim}_s TR^r(A/I^s;p)$$

is a weak equivalence after p-completion. Similarly for THH, TR, TC^r, and TC.

Proof. – To prove the desired result for TR^r we must show that the map $TR^r(A;\mathbb{Z}/p^v)\to \operatorname{holim}_s TR^r(A/I^s;\mathbb{Z}/p^v\mathbb{Z})$ is a weak equivalence for all $v\geq 1$. Firstly, fixing $r\geq 1$, the homotopy groups of $\operatorname{holim}_s TR^r(A/I^s;\mathbb{Z}/p^v)$ fit into short exact sequences

$$0\to \varprojlim_s{}^1 TR^r_{n+1}(A/I^s;\mathbb{Z}/p^v)\to \pi_n(\operatorname{holim}_s TR^r(A/I^s;\mathbb{Z}/p^v))\to \varprojlim_s TR^r_n(A/I^s;\mathbb{Z}/p^v)\to 0.$$

Theorem 5.3(i) implies that the left-most term is $\varprojlim^1_s TR^r_{n+1}(A;\mathbb{Z}/p^v)\otimes_{W_r(A)}W_r(A/I^s)$, which vanishes because of the surjectivity of the transition maps in the pro abelian group $\{TR^r_{n+1}(A;\mathbb{Z}/p^v)\otimes_{W_r(A)}W_r(A/I^s)\}_s$. In conclusion, the natural map

$$\pi_n(\operatorname{holim}_s TR^r(A/I^s;\mathbb{Z}/p^v))\longrightarrow \varprojlim_s TR^r_n(A/I^s;\mathbb{Z}/p^v)$$

is an isomorphism for all $n\geq 0$. But since A is already I-adically complete, Corollary 5.4 states that $TR^r_n(A;\mathbb{Z}/p^v)\to\varprojlim_s TR^r_n(A/I^s;\mathbb{Z}/p^v)$ is also an isomorphism for all $n\geq 0$. So the map $TR^r(A;\mathbb{Z}/p^v)\to \operatorname{holim}_s TR^r(A/I^s;\mathbb{Z}/p^v)$ induces an isomorphism on all homotopy groups, as required.

The claims for TR, TC^r, and TC then follow since homotopy limits commute. □

In the remainder of this section we consider straightforward consequences of the previous three results in special situations. We begin with the case in which p is nilpotent in A:

COROLLARY 5.6. – *Let A be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra in which p is nilpotent, and $I \subseteq A$ an ideal; let $\widehat{A}$ denote the I-adic completion of A. Then all of the following maps (not just the compositions) are isomorphisms for all $n \geq 0$ and $r \geq 1$:*

$$\{HH_n(A) \otimes_A A/I^s\}_s \longrightarrow \{HH_n(A/I^s)\}_s$$

$$\{TR^r_n(A; p) \otimes_{W_r(A)} W_r(A/I^s)\}_s \longrightarrow \{TR^r_n(A/I^s; p)\}_s$$

$$HH_n(A) \otimes_A \widehat{A} \longrightarrow HH_n(\widehat{A}) \longrightarrow \varprojlim_s HH_n(A/I^s)$$

$$TR^r_n(A; p) \otimes_{W_r(A)} W_r(\widehat{A}) \longrightarrow TR^r_n(\widehat{A}; p) \longrightarrow \varprojlim_s TR^r_n(A/I^s; p).$$

Moreover, the weak equivalences of Theorem 5.5 hold without p-completing.

Proof. – Note that p is also nilpotent in $W_r(A)$, by Lemma 3.9(ii). So, fixing $r \geq 1$, we may pick $v \gg 0$ such that the groups

$$HH_n(A),\ HH_n(A/I^s),\ THH_n(A),\ THH_n(A/I^s),\ TR^r_n(A; p),\ TR^r_n(A/I^s; p)$$

are annihilated by p^v for all $n \geq 0, s \geq 1$. Hence the spectra appearing in Theorem 5.5 are all p-complete, and the isomorphisms follow from Theorem 5.3 and Corollary 5.4 by examining the short exact sequences for homotopy groups with finite coefficients, in the usual way. □

REMARK 5.7. – Some of the statements of Corollary 5.6 hold for rings other than Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebras in which p is nilpotent. In particular, if A is a Noetherian ring which is AQ-finite over $\mathbb{Z}$ (e.g., an essentially finite type $\mathbb{Z}$-algebra) and $I \subseteq A$ is an ideal, then the pro HH and $THH = TR^1$ isomorphisms of Corollary 5.6 hold; indeed, this follows immediately from Lemma 5.2 with $m = 0$ and $M = A$.

Suppose now, in addition to be being Noetherian and AQ-finite, that A is an F-finite $\mathbb{Z}_{(p)}$-algebra (e.g., an essentially finite type $\mathbb{Z}_{(p)}$-algebra). Then the pro TR^r isomorphisms of Corollary 5.6 hold: this is proved by verbatim repeating the proof of Theorem 5.3(ii) integrally instead of with finite coefficients.

Now, in stark contrast with the case in which p is nilpotent, we consider the case where $I = pA$; here our methods yield a new proof, albeit under different hypotheses, of a result of Geisser and Hesselholt, as we will discuss in Remark 5.9:

COROLLARY 5.8. – *Let A be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra. Then the canonical map*

$$HH_n(A; \mathbb{Z}/p^v) \longrightarrow \{HH_n(A/p^s A; \mathbb{Z}/p^v)\}_s$$

is an isomorphism for all $n \geq 0$ and $v \geq 1$. Similarly for THH, TR^r, and TC^r.

Moreover, for all $r \geq 1$, the canonical maps (not just the composition) of spectra

$$TR^r(A; p) \longrightarrow TR^r(\widehat{A_p}; p) \longrightarrow \operatorname*{holim}_s TR^r(A/p^s A; p)$$

are weak equivalences after p-completion. Similarly for THH, TR, TC^r, and TC.

Proof. – The groups $HH_n(A;\mathbb{Z}/p^v)$ and $TR^r_n(A;\mathbb{Z}/p^v)$ are annihilated by p^v. Since we proved in Lemma 3.4 that $p^v W_r(A)$ contains $W_r(p^s A)$ for $s \gg 0$, we deduce that

$$HH_n(A;\mathbb{Z}/p^v) \xrightarrow{\simeq} \{HH_n(A;\mathbb{Z}/p^v) \otimes_A A/p^s A\}_s,$$
$$TR^r_n(A;\mathbb{Z}/p^v) \xrightarrow{\simeq} \{TR^r_n(A;\mathbb{Z}/p^v) \otimes_{W_r(A)} W_r(A/p^s A)\}_s.$$

Hence the desired pro HH and TR^r isomorphisms follow from Theorem 5.3. The pro TC^r isomorphism then follows in the usual way by applying the five lemma to the long exact sequence relating TC^r, TR^r, and TR^{r-1}.

Since $A^\wedge_p$ is also a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra by Lemma 3.8, and since $A^\wedge_p/p^s A^\wedge_p \cong A/p^s A$, applying the pro TR isomorphism to both A and $A^\wedge_p$ yields

$$TR^r_n(A;\mathbb{Z}/p^v) \cong \{TR^r_n(A/p^s;\mathbb{Z}/p^v)\}_s \cong TR^r_n(A^\wedge_p;\mathbb{Z}/p^v)$$

for any integer $r \geq 1$. This proves that $TR^r(A;p)^\wedge_p \simeq TR^r(A^\wedge_p;p)^\wedge_p$, and the same follows for TR, TC^r, and TC by taking homotopy limits.

Finally, $TR^r(A^\wedge_p;p) \to \operatorname{holim}_s TR^r(A/p^s A;p)$ is a weak equivalence after p-completion by Theorem 5.5, and similarly for TR, TC^r, and TC. □

Remark 5.9. – The pro isomorphisms of Corollary 5.8 were proved by Geisser and Hesselholt [11, §3] for any (possibly non-commutative, non-Noetherian) ring A in which p is a non-zero divisor. The key assertion is that $HH_n(A;\mathbb{Z}/p^v) \xrightarrow{\simeq} \{HH_n(A/p^s A;\mathbb{Z}/p^v)\}_s$, which can also be proved using our methods as follows: mimicking the proof of Lemma 5.2 via Proposition 5.1 and the Artin-Rees vanishing result $\{\operatorname{Tor}^A_n(A/I^s, A/I^s)\}_s = 0$, it is enough to show that $\{\operatorname{Tor}^A_n(A/p^s A, A/p^s A)\}_s = 0$ for all $n > 0$ whenever p is a non-zero divisor of a possibly non-commutative ring A. But in such a situation we may calculate Tor using the projective resolution $0 \to A \xrightarrow{\times p^s} A \to A/p^s A \to 0$ of $A/p^s A$, and from this calculation it easily follows that the map $\operatorname{Tor}^A_1(A/p^{2s}A, A/p^{2s}A) \to \operatorname{Tor}^A_1(A/p^s A, A/p^s A)$ is zero, as required.

5.2. The pro Hochschild-Kostant-Rosenberg theorems

Given a geometrically regular (e.g., smooth) morphism $k \to A$ of Noetherian rings, the classical Hochschild-Kostant-Rosenberg theorem [24, Thm. 3.4.4], combined with Néron-Popescu desingularision (see the following remark), states that the antisymmetrisation map $\Omega^n_{A/k} \to HH^k_n(A)$ is an isomorphism of A-modules for all $n \geq 0$.

Remark 5.10. – Since the notion of a geometrically regular morphism may not be familiar to all readers, here we offer a brief explanation. A good reference is R. Swan's exposition of Néron-Popescu desingularisation [37].

If k is a field, then a Noetherian k-algebra A is said to be *geometrically regular* over k, or that $k \to A$ is a geometrically regular morphism, if and only if $A \otimes_k k'$ is a regular ring for all finite field extensions k'/k. If k is perfect then this is equivalent to A being a regular ring, which is equivalent to A being smooth over k if we moreover assume that A is essentially of finite type over k. If k is no longer necessarily a field, then $k \to A$ is said to be geometrically regular if and only if it is flat and $k(\mathfrak{p}) \to A \otimes_k k(\mathfrak{p})$ is geometrically regular in the previous sense for all prime ideals $\mathfrak{p} \subseteq k$, where $k(\mathfrak{p}) = k_{\mathfrak{p}}/\mathfrak{p}k_{\mathfrak{p}}$.

The Néron-Popescu desingularisation theorem [30, 31] states that if A is a k-algebra, with both rings Noetherian, then A is geometrically regular over k if and only if it is a filtered colimit of smooth, finite-type k-algebras.

The following establishes the pro Hochschild-Kostant-Rosenberg theorem for algebraic Hochschild homology in full generality:

THEOREM 5.11 (Pro HKR theorem for Hochschild homology). – *Let $k \to A$ be a geometrically regular morphism of Noetherian rings, and $I \subseteq A$ an ideal. Then the canonical map of pro A-modules*

$$\{\Omega^n_{(A/I^s)/k}\}_s \longrightarrow \{HH^k_n(A/I^s)\}_s$$

is an isomorphism for all $n \geq 0$.

Proof. – Consider the following commutative diagram of pro A-modules, in which the vertical arrows are the antisymmetrisation maps:

$$\begin{array}{ccccc} \{A/I^s \otimes_A HH^k_n(A)\}_s & \xrightarrow{(1)} & \{HH^k_n(A, A/I^s)\}_s & \xrightarrow{(2)} & \{HH^k_n(A/I^s)\}_s \\ \uparrow & & & & \uparrow \\ \{A/I^s \otimes_A \Omega^n_{A/k}\}_s & & \xrightarrow{\qquad\qquad(3)\qquad\qquad} & & \{\Omega^n_{(A/I^s)/k}\}_s. \end{array}$$

As recalled above, the HKR theorem implies that the antisymmetrization map $\Omega^j_{A/k} \to HH^k_j(A)$ is an isomorphism for all $j \geq 0$. So the left vertical arrow is an isomorphism. Moreover, Néron-Popescu desingularisation implies that A is a filtered colimit of smooth, finite type k-algebras, and so $\Omega^j_{A/k} \cong HH^k_j(A)$ is a filtered colimit of free A-modules, hence is a flat A-module.

For any A-module M, the universal coefficient spectral sequence $\mathrm{Tor}^A_i(M, HH^k_j(A)) \Rightarrow HH_{i+j}(A, M)$ therefore collapses to edge map isomorphisms $M \otimes_A \Omega^n_{A/k} \xrightarrow{\simeq} HH^k_n(A, M)$. In particular, taking $M = A/I^s$ shows that arrow (1) is an isomorphism.

Next, Lemma 5.2(i) states that arrow (2) is an isomorphism (to be precise, Lemma 5.2(i) was stated only for the ground ring $\mathbb{Z}$, but the proof worked verbatim for any ground ring k). Finally, arrow (3) is easily seen to be an isomorphism using the inclusion $d(I^{2s}) \subseteq I^s\Omega^n_{A/k}$.

It follows that the right vertical arrow is also an isomorphism, as desired. □

COROLLARY 5.12 (Pro HKR Theorem for cyclic homology). – *Let $k \to A$ be a geometrically regular morphism of Noetherian rings, and $I \subseteq A$ an ideal. Then there is a natural spectral sequence of pro k-modules*

$$E^2_{pq} = \begin{cases} \{\Omega^q_{(A/I^s)/k} / d\Omega^{q-1}_{(A/I^s)/k}\}_s & p = 0 \\ \{H^{q-p}_{dR}((A/I^s)/k)\}_s & p > 0 \end{cases} \Longrightarrow \{HC^k_{p+q}(A/I^s)\}_s.$$

If k contains $\mathbb{Q}$ then this degenerates with naturally split filtration, yielding

$$\{HC^k_n(A/I^s)\}_s \cong \{\Omega^n_{(A/I^s)/k} / d\Omega^{n-1}_{(A/I^s)/k}\}_s \oplus \bigoplus_{0 \leq p < \frac{n}{2}} \{H^{n-2p}_{dR}((A/I^s)/k)\}_s.$$

Proof. – This follows by combining Theorem 5.11 with standard arguments in cyclic homology. □

Remark 5.13. – In the special case of certain finite type algebras over fields, the pro HKR theorem for algebraic Hochschild homology was established by G. Cortiñas, C. Haesemeyer, and C. Weibel [5, Thm. 3.2]. The full version of the pro HKR theorem presented here has recently been required in the study of the infinitesimal deformation of algebraic cycles [2, 26].

Next we turn to topological Hochschild homology, for which we must first briefly review the de Rham-Witt complex. Given an $\mathbb{F}_p$-algebra A, the existence and theory of the p-typical de Rham-Witt complex $W_r\Omega^\bullet_A$, which is a pro differential graded $W(A)$-algebra, is due to S. Bloch, P. Deligne, and L. Illusie; see especially [20, Def. I.1.4]. It was later extended by Hesselholt and Madsen to $\mathbb{Z}_{(p)}$-algebras with p odd, and by V. Costeanu [6] to $\mathbb{Z}_{(2)}$-algebras; see the introduction to [16] for further discussion. We will only require the classical formulation for $\mathbb{F}_p$-algebras, with which we assume the reader has some familiarity.

If A is an $\mathbb{F}_p$-algebra, then the pro graded ring $\{TR^r_\bullet(A;p)\}_r$ is a p-typical Witt complex with respect to its operators F, V, R; by universality of the de Rham-Witt complex, there are therefore natural maps of graded $W_r(A)$-algebras [15, Prop. 1.5.8]

$$W_r\Omega^\bullet_A \longrightarrow TR^r_\bullet(A;p)$$

for $r \geq 0$, which are compatible with the Frobenius, Verschiebung, and Restriction maps (in other words, a morphism of p-typical Witt complexes). Since there is also a natural map of graded $W_r(\mathbb{F}_p)$-algebras $TR^r_\bullet(\mathbb{F}_p;p) \to TR^r_\bullet(A;p)$, we may tensor these algebra maps to obtain a natural morphism of graded $W_r(A)$-algebras

$$W_r\Omega^\bullet_A \otimes_{W_r(\mathbb{F}_p)} TR^r_\bullet(\mathbb{F}_p;p) \longrightarrow TR^r_\bullet(A;p)$$

which by Hesselholt's HKR theorem [15, Thm. B] is an isomorphism for all $r \geq 1$.

We now prove the pro HKR theorem for THH and TR^r; since infinite direct sums do not commute with the formation of pro-abelian groups, we must state it degree-wise:

Theorem 5.14 (Pro HKR Theorem for THH and TR^r). – *Let A be a regular, F-finite $\mathbb{F}_p$-algebra, and $I \subseteq A$ an ideal. Then the canonical map*

$$\bigoplus_{i=0}^n \{W_r\Omega^i_{A/I^s} \otimes_{W_r(\mathbb{F}_p)} TR^r_{n-i}(\mathbb{F}_p;p)\}_s \longrightarrow \{TR^r_n(A/I^s;p)\}_s$$

of pro $W_r(A)$-modules is an isomorphism for all $n \geq 0$ and $r \geq 1$.

Proof. – Consider the following commutative diagram of pro $W_r(A)$-modules:

$$\begin{tikzcd} \bigoplus_{i=0}^n \{W_r(A/I^s) \otimes_{W_r(A)} W_r\Omega^i_A \otimes_{W_r(\mathbb{F}_p)} TR^r_{n-i}(\mathbb{F}_p;p)\}_s \arrow[r] \arrow[d] & \{W_r(A/I^s) \otimes_{W_r(A)} TR^r_n(A;p)\}_s \arrow[d] \\ \bigoplus_{i=0}^n \{W_r\Omega^i_{A/I^s} \otimes_{W_r(\mathbb{F}_p)} TR^r_{n-i}(\mathbb{F}_p;p)\}_s \arrow[r] & \{TR^r_n(A/I^s;p)\}_s. \end{tikzcd}$$

Hesselholt's HKR theorem implies that the top horizontal arrow is an isomorphism. Corollary 5.6 implies that the right vertical arrow is an isomorphism. Since we wish to establish that the bottom horizontal arrow is an isomorphism, it is now enough to show that the left vertical arrow is an isomorphism, for which it suffices to prove that the canonical map

$$(\dagger) \qquad \{W_r(A/I^s) \otimes_{W_r(A)} W_r\Omega^i_A\}_s \longrightarrow \{W_r\Omega^i_{A/I^s}\}_s$$

of pro $W_r(A)$-modules is an isomorphism for all $i \geq 0$. To show this, note that the maps in (†) are surjective, so one needs only to show that the pro abelian group arising from the kernels is zero. This is an easy consequence of Lemma 3.1(iv) and the Leibniz rule; see, e.g., [12, Prop. 2.5]. □

Next we let $r \to \infty$ to prove the pro HKR theorem for $\{TR^r\}_r$. This takes the form of an isomorphism of *pro pro abelian groups*; that is, an isomorphism in the category Pro(Pro Ab). We note that, by taking the diagonal in the result, we also obtain a weaker isomorphism of pro abelian groups $\{W_r\Omega^n_{A/I^r}\}_r \xrightarrow{\simeq} \{TR^r_n(A/I^r; p)\}_r$.

Corollary 5.15 (Pro HKR Theorem for TR). – *With notation as in the previous theorem, the canonical map of pro pro abelian groups*

$$\big\{\{W_r\Omega^n_{A/I^s}\}_s\big\}_r \longrightarrow \big\{\{TR^r_n(A/I^s; p)\}_s\big\}_r$$

is an isomorphism for all $n \geq 0$.

Proof. – By [15, Thm. B] or [18, Thm. 5.5], there are isomorphisms $W_r(\mathbb{F}_p)[\sigma_r] \cong TR^r_\bullet(\mathbb{F}_p; p)$ of graded $W_r(\mathbb{F}_p)$-algebras, where the polynomial variable σ_r has degree 2 and $R(\sigma_r) = p\lambda_r\sigma_{r-1}$ for some unit $\lambda_r \in W_{r-1}(\mathbb{F}_p)$ (it fact, it is now known that one may take $\lambda_r = 1$ [17, Rmk. 4.3]). In particular, it follows that $R^r : TR^{2r}_n(\mathbb{F}_p; p) \to TR^r_n(\mathbb{F}_p; p)$ is zero for all $n, r \geq 1$. Thus the inverse system of pro abelian groups

$$\cdots \xrightarrow{R} \{W_{r+1}\Omega^i_{A/I^s} \otimes_{W_{r+1}(\mathbb{F}_p)} TR^{r+1}_{n-i}(\mathbb{F}_p; p)\}_s \xrightarrow{R} \{W_r\Omega^i_{A/I^s} \otimes_{W_r(\mathbb{F}_p)} TR^r_{n-i}(\mathbb{F}_p; p)\}_s \xrightarrow{R} \cdots$$

is trivial Mittag-Leffler unless $i = n$. So the isomorphisms of Theorem 5.14 assemble over $r \geq 1$ to the desired isomorphism. □

6. Proper schemes over an affine base

In this section we extend the finite generation results of Section 4 and the continuity results of Section 5 to proper schemes over an affine base. The key idea is to combine the already-established results with Zariski descent and Grothendieck's formal function theorem for coherent cohomology, which we will recall in Theorem 6.5 for convenience.

For a quasi-compact, quasi-separated scheme X, the spectra $THH(X)$, $TR^r(X; p)$, $TC^r(X; p)$, $TR(X; p)$, and $TC(X; p)$ were defined by Geisser and Hesselholt in [9] in such a way that all these presheaves of spectra satisfy Zariski descent (see the proof of [9, Corol. 3.3.3]). In particular, assuming that X has finite Krull dimension, there is a bounded spectral sequence

$$E_2^{ij} = H^i(X, \mathcal{THH}_{-j}(X)) \implies THH_{-i-j}(X)$$

where the caligraphic notation $\mathcal{THH}_n(X)$ denotes the sheafification of the Zariski presheaf on X given by $U \mapsto THH_n(\mathcal{O}_X(U))$; the same applies to $TR^r(-; p)$ and $TC^r(-; p)$, and we will always use caligraphic notation to denote such Zariski sheafifications, including when working with finite coefficients.

Moreover, the relations between the theories in the affine case explained in Section 2.3 continue to hold for schemes [9, Prop. 3.3.2], and analogous comments also apply to algebraic Hochschild homology, thanks to Weibel [40, 38].

We recall the scheme-theoretic version of Witt vectors; further details may be found in the appendix of [23]. Given a ring A and an element $f \in A$, there is a natural isomorphism $W_r(A_f) \cong W_r(A)_{[f]}$ where $[f]$ is the Teichmüller lift of f; this localisation result means that, for any scheme X, we may define a new scheme $W_r(X)$ by applying W_r locally. The restriction map induces a closed embedding of schemes $R^{r-1} : X \hookrightarrow W_r(X)$, which is an isomorphism of the underlying topological spaces if p is nilpotent on X. In the presence of F-finiteness (a $\mathbb{Z}_{(p)}$-scheme is said to be F-finite if and only if it has a finite cover by spectra of F-finite $\mathbb{Z}_{(p)}$-algebras), many properties of X are inherited by $W_r(X)$; see Prop. A.1 – Corol. A.7 of [23]: in particular, if X is separated (resp. Noetherian and F-finite), then $W_r(X)$ is separated (resp. Noetherian), and if $X \to Y$ of finite type and Y F-finite (resp. proper and Y F-finite), then $W_r(X) \to W_r(Y)$ is of finite type (resp. proper).

Our first aim is to prove that the sheaves arising from HH, THH, and TR^r are quasi-coherent; this is a standard result for HH and THH, but we could not find a reference covering the TR^r sheaves. We note that the following lemma remains true when working with finite $\mathbb{Z}/p^v$-coefficients; this follows from the lemma as stated using the usual short exact sequences for finite coefficients.

LEMMA 6.1. – *Let X be a quasi-compact, quasi-separated scheme and $n \geq 0$, $r \geq 1$. Then $\mathcal{HH}_n(X)$ and $\mathcal{THH}_n(X)$ are quasi-coherent sheaves on X, and $R_*^{r-1}\,\mathcal{TR}_n^r(X;p)$ is a quasi-coherent sheaf on $W_r(X)$.*

Proof. – Let $\operatorname{Spec} R \subseteq X$ be any affine open subscheme of X; we must show that for any $f \in R$, the canonical maps

$$HH_n(R) \otimes_R R_f \to HH_n(R_f), \qquad TR_n^r(R;p) \otimes_{W_r(R)} W_r(R_f) \to TR_n^r(R_f;p)$$

are isomorphisms for all $n \geq 0$, $r \geq 1$. Firstly, R_f is flat over R, so the spectral sequence of Prop. 5.1 degenerates to edge map isomorphisms $HH_n(R, R_f) \xrightarrow{\simeq} HH_n(R_f, R_f \otimes_R R_f)$. But $R_f \otimes_R R_f = R_f$, and its flatness over R implies that $HH_n(R, R_f) = HH_n(R) \otimes_R R_f$; this proves the claim for HH. The proof for $THH = TR^1$ is similar.

We prove the claim for TR_n^r by induction on r; we have just established the case $r = 1$. Since $W_r(R_f) = W_r(R)_{[f]}$ is flat over $W_r(R)$, we may base change by $W_{r+1}(R_f)$ the fundamental long exact sequence of Section 2.3.2(i) for $W_{r+1}(R)$, and compare it to the long exact sequence for R_f; this yields a map of long exact sequences, and so by the five lemma and induction on r it is enough to prove that

$$\pi_n(THH(R)_{hC_{p^r}}) \otimes_{W_{r+1}(R)} W_{r+1}(R_f) \longrightarrow \pi_n(THH(R_f)_{hC_{p^r}})$$

is an isomorphism. Moreover, the domain and codomain of this map are compatibly described by group homology spectral sequences, so it is now enough to prove that the canonical map

$$H_i(C_{p^r}, THH_j(R)) \otimes_{W_{r+1}(R)} W_{r+1}(R_f) \longrightarrow H_i(C_{p^r}, THH_j(R_f))$$

is an isomorphism for all $i, j \geq 0$ and $r \geq 1$. Since $W_{r+1}(R_f)$ is flat over $W_{r+1}(R)$, we may identify the left side with $H_i(C_{p^r}, THH_j(R) \otimes_{W_{r+1}(R)} W_{r+1}(R_f))$, and so it remains only to show that the map $THH_j(R) \otimes_{W_{r+1}(R)} W_{r+1}(R_f) \to THH_j(R_f)$ is an isomorphism, where $W_{r+1}(R)$ is acting on $THH_j(R)$ via $F^r : W_{r+1}(R) \to R$. But this follows from the observation that the diagram

$$\begin{tikzcd} W_{r+1}(R) \arrow[r] \arrow[d, "F^r"'] & W_{r+1}(R_f) \arrow[d, "F^r"] \\ R \arrow[r] & R_f \end{tikzcd}$$

is cocartesian: indeed, the pushout along $F^r : W_{r+1}(R) \to R$ of the localisation at $[f]$ is the localisation at $F^r[f] = f^r$. □

As a consequence of the lemma and our earlier finite generation results, we therefore obtain the fundamental coherence results we will need:

COROLLARY 6.2. – *Let A be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra, X an essentially finite-type A-scheme, and $n \geq 0$, $v, r \geq 1$. Then $\mathcal{HH}_n(X; \mathbb{Z}/p^v)$ and $\mathcal{THH}_n(X; \mathbb{Z}/p^v)$ are coherent sheaves on X, and $R_*^{r-1} \mathcal{TR}_n^r(X; \mathbb{Z}/p^v)$ is a coherent sheaf on $W_r(X)$.*

Proof. – The specified sheaves are quasi-coherent by the finite coefficient version of Lemma 6.1, so we must prove that if $\operatorname{Spec} R$ is an affine open subscheme of X, then $HH_n(R; \mathbb{Z}/p^v)$ and $THH_n(R; \mathbb{Z}/p^v)$ are finitely generated R-modules, and that $TR_n^r(R; \mathbb{Z}/p^v)$ is a finitely generated $W_r(R)$-module. But Lemma 3.8 implies that R is also a Noetherian, F-finite, $\mathbb{Z}_{(p)}$-algebra, so this follows from Theorem 4.7. □

We now obtain the generalisation to proper schemes of our earlier finite generation results:

THEOREM 6.3. – *Let A be a Noetherian, F-finite, finite Krull-dimensional $\mathbb{Z}_{(p)}$-algebra, X a proper scheme over A, and $v \geq 1$. Then:*

(i) *$HH_n(X; \mathbb{Z}/p^v)$ and $THH_n(X; \mathbb{Z}/p^v)$ are finitely generated A-modules for all $n \geq 0$.*
(ii) *$TR_n^r(X; \mathbb{Z}/p^v)$ is a finitely generated $W_r(A)$-module for all $n \geq 0$ and $r \geq 1$.*

Proof. – The coherence assertion of Corollary 6.2 and the properness of X over A implies that $H^i(X, \mathcal{HH}_n(X; \mathbb{Z}/p^v))$ is a finitely generated A-module for all $i, n \geq 0$.

Since the presheaf of spectra $HH \wedge S/p^v$ satisfies Zariski descent by the results of Geisser-Hesselholt recalled at the start of the section, and since X has finite Krull dimension by assumption, there is a right half-plane, bounded, Zariski descent spectral sequence $E_2^{ij} = H^i(X, \mathcal{HH}_{-j}(X; \mathbb{Z}/p^v)) \Rightarrow HH_{-i-j}(X; \mathbb{Z}/p^v)$ (with differentials $E_r^{ij} \to E_r^{i+r\, j-r+1}$) of A-modules. This evidently completes the proof for HH.

The proof for TR^r is similar: there is an analogous descent spectral sequence, and the properness of $W_r(X)$ over $W_r(A)$, by [23, Corol. A.7], implies that $H^i(X, \mathcal{TR}_{-j}^r(X; \mathbb{Z}/p^v))$ is a finitely generated $W_r(A)$-module. □

Now, Proposition 2.7 gives that Theorem 6.3 implies Theorem 1.7 of the Introduction:

Corollary 6.4. – *Let A be a Noetherian, F-finite, finite Krull-dimensional $\mathbb{Z}_{(p)}$-algebra, X a proper scheme over A, and $n \geq 0$, $r \geq 1$. Then $HH_n(X;\mathbb{Z}_p)$ and $THH_n(X;\mathbb{Z}_p)$ are finitely generated $\widehat{A_p}$-modules, and $TR^r_n(X;p,\mathbb{Z}_p)$ is a finitely generated $W_r(\widehat{A_p})$-module.*

Next we generalise our degree-wise continuity result of Theorem 5.3 to the case of a proper scheme. Consider a proper scheme X over an affine base A, fix an ideal $I \subseteq X$, and write $X_s := X \times_A A/I^s$. The following theorem of Grothendieck will be required:

Theorem 6.5 (Grothendieck's Formal Functions Theorem [13, Cor. 4.1.7])

Let A be a Noetherian ring, $I \subseteq A$ an ideal, X a proper scheme over A, and N a coherent $\mathcal{O}_X$-module. Then the canonical map of pro A-modules

$$\{H^n(X,N)\otimes_A A/I^s\}_s \xrightarrow{\simeq} \{H^n(X_s, N/I^sN)\}_s$$

is an isomorphism for all $n \geq 0$.

Proof/Remark. – In fact, Grothendieck's theorem is more frequently stated as the isomorphism $\varprojlim_s H^s(X,N)\otimes_A A/I^s \xrightarrow{\simeq} \varprojlim_s H^n(X_s,N/I^sN)$, but a quick examination of the cited proof in EGA shows that the stronger isomorphism of pro A-modules holds. ☐

Combining Grothendieck's formal function theorem with the already-established degree-wise continuity results proves the following formal function theorems for HH, THH, and TR^r:

Theorem 6.6. – *Let A be a Noetherian, F-finite, finite Krull-dimensional $\mathbb{Z}_{(p)}$-algebra, $I \subseteq A$ an ideal, and X a proper scheme over A. Then the canonical maps*

(i) $\{HH_n(X;\mathbb{Z}/p^v)\otimes_A A/I^s\}_s \longrightarrow \{HH_n(X_s,\mathbb{Z}/p^v)\}_s$

(ii) $\{TR^r_n(X;\mathbb{Z}/p^v)\otimes_{W_r(A)} W_r(A/I^s)\}_s \longrightarrow \{TR^r_n(X_s;\mathbb{Z}/p^v)\}_s$

are isomorphisms for all $n \geq 0$ and $v, r \geq 1$.

Proof. – We prove (ii). The proof of (i) is similar, but simpler. Firstly, as at the end of the proof of Theorem 6.3, there is a bounded Zariski descent spectral sequence of finitely generated $W_r(A)$-modules $E_2^{ij} = H^i(X,\mathcal{TR}^r_{-j}(X;\mathbb{Z}/p^v)) \Rightarrow TR^r_{-i-j}(X;\mathbb{Z}/p^v)$. By Theorem 3.7(i), we may base change by $W_r(A/I^\infty)$ to obtain a bounded spectral sequence of pro $W_r(A)$-modules

$$E_2^{ij}(\infty) = \{H^i(X,\mathcal{TR}^r_{-j}(X;\mathbb{Z}/p^v))\otimes_{W_r(A)} W_r(A/I^s)\}_s \Longrightarrow \{TR^r_{-i-j}(X;\mathbb{Z}/p^v)\otimes_{W_r(A)} W_r(A/I^s)\}_s.$$

There is also a bounded Zariski descent spectral sequence for TR^r associated to each scheme X_s, for $s \geq 1$, and these assemble to a spectral sequence of pro $W_r(A)$-modules

$$'E_2^{ij}(\infty) = \{H^i(X_s,\mathcal{TR}^r_{-j}(X_s;\mathbb{Z}/p^v))\}_s \Longrightarrow \{TR^r_{-i-j}(X_s;\mathbb{Z}/p^v)\}_s.$$

The $E(\infty)$-spectral sequence maps to the $'E(\infty)$-spectral sequence, and so to complete the proof it is enough to show that the canonical map of pro $W_r(A)$-modules on the second pages

$$(\dagger)\quad \{H^i(X,\mathcal{TR}^r_{-j}(X;\mathbb{Z}/p^v))\otimes_{W_r(A)} W_r(A/I^s)\}_s \longrightarrow \{H^i(X_s,\mathcal{TR}^r_{-j}(X_s;\mathbb{Z}/p^v))\}_s,$$

is an isomorphism for all $i \geq 0$.

To do this we first transfer the problem to the scheme $W_r(X)$ by rewriting

$$H^i(X, \mathcal{TR}^r_{-j}(X;\mathbb{Z}/p^v)) = H^i(W_r(X), R^{r-1}_* \mathcal{TR}^r_{-j}(X;\mathbb{Z}/p^v))$$

and

$$H^i(X_s, \mathcal{TR}^r_{-j}(X_s;\mathbb{Z}/p^v)) = H^i(W_r(X_s), R^{r-1}_* \mathcal{TR}^r_{-j}(X_s;\mathbb{Z}/p^v))$$

since the cohomology of a sheaf is unchanged after pushing forward along a closed embedding [14, Lem. III.2.10]. We now factor (†) as a composition

$$\begin{aligned}\{H^i(W_r(X), R^{r-1}_* \mathcal{TR}^r_{-j}(X;\mathbb{Z}/p^v)) \otimes_{W_r(A)} W_r(A/I^s)\}_s \\ \longrightarrow \{H^i(W_r(X), R^{r-1}_* \mathcal{TR}^r_{-j}(X;\mathbb{Z}/p^v) \otimes_{W_r(\mathcal{O}_X)} W_r(\mathcal{O}_X/I^s\mathcal{O}_X)\}_s \\ \longrightarrow \{H^i(W_r(X_s), R^{r-1}_* \mathcal{TR}^r_{-j}(X_s;\mathbb{Z}/p^v))\}_s.\end{aligned}$$

Since $W_r(X)$ is a proper scheme over $W_r(A)$, we may apply Grothendieck's formal functions theorem to the coherent (by Corollary 6.2) sheaf $R^{r-1}_* \mathcal{TR}^r_{-j}(X;\mathbb{Z}/p^v)$ and ideal $W_r(I) \subseteq W_r(A)$ (whose powers are intertwined with $W_r(I^s)$, $s \geq 1$, by Lemma 3.1) to deduce that the first arrow is an isomorphism. The second arrow is an isomorphism since the underlying map of pro sheaves is an isomorphism by Theorem 5.3 (note that all the affine open subschemes of X are Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebras, by Lemma 3.8). □

By almost exactly the same proof as of Corollary 5.4 we obtain the following:

Corollary 6.7. – *Let A be a Noetherian, F-finite, finite Krull-dimensional $\mathbb{Z}_{(p)}$-algebra, $I \subseteq A$ an ideal, and X a proper scheme over A. Then all of the following maps (not just the compositions) are isomorphisms for all $n \geq 0$ and $v, r \geq 1$:*

$$HH_n(X;\mathbb{Z}/p^v) \otimes_A \widehat{A} \longrightarrow HH_n(X \times_A \widehat{A};\mathbb{Z}/p^v) \longrightarrow \varprojlim_s HH_n(X_s;\mathbb{Z}/p^v)$$

$$TR^r(X;\mathbb{Z}/p^v) \otimes_{W_r(A)} W_r(\widehat{A}) \longrightarrow TR^r(X \times_A \widehat{A};\mathbb{Z}/p^v) \longrightarrow \varprojlim_s TR^r(X_s;\mathbb{Z}/p^v).$$

We finally reach the scheme-theoretic analogue of Theorem 5.5, namely spectral continuity of THH, TR^r, etc. Again, the proof is identical to that of the affine case.

Theorem 6.8. – *Let A be a Noetherian, F-finite, finite Krull-dimensional $\mathbb{Z}_{(p)}$-algebra, $I \subseteq A$ an ideal, and X a proper scheme over A; assume that A is I-adically complete. Then, for all $r \geq 1$, the canonical map of spectra*

$$TR^r(X;p) \longrightarrow \operatorname{holim}_s TR^r(X_s;p)$$

is a weak equivalence after p-completion. Similarly for THH, TR, TC^r, and TC.

As in the affine case, we now present corollaries of the previous results in the cases in which p is either nilpotent or the generator of I; we begin with the nilpotent case:

Corollary 6.9. – *Let A be a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra in which p is nilpotent, X a proper scheme over A, and $n \geq 0$, $r \geq 1$. Then $HH_n(X)$ and $THH_n(X)$ are finitely generated A-modules, and $TR^r_n(X;p)$ is a finitely generated $W_r(A)$-module.*

Now let $I \subseteq A$ be an ideal. Then all of the following maps are isomorphisms:

$$\{HH_n(X) \otimes_A A/I^s\}_s \longrightarrow \{HH_n(X_s)\}_s,$$
$$\{TR^r_n(X;p) \otimes_{W_r(A)} W_r(A/I^s)\}_s \longrightarrow \{TR^r_n(X_s;p)\}_s,$$
$$HH_n(X) \otimes_A \widehat{A} \longrightarrow HH_n(X \times_A \widehat{A}) \longrightarrow \varprojlim_s HH_n(X_s),$$
$$TR^r_n(X;p) \otimes_{W_r(A)} W_r(\widehat{A}) \longrightarrow TR^r_n(X \times_A \widehat{A};p) \longrightarrow \varprojlim_s TR^r_n(X_s;p).$$

Moreover, the weak equivalences of Theorem 6.8 hold without p-completing.

Proof. – As in the proof of Corollary 5.6, the groups $HH_n(X)$, $THH_n(X)$, and $TR^r_n(X;p)$ are all bounded p-torsion. Hence the spectra are p-complete, and our finite generation and isomorphism claims follow from the already established versions with finite coefficients, namely Theorems 6.3 and 6.6, and Corollary 6.7.

Note that we have not assumed that A has finite Krull dimension. This is because a Noetherian, F-finite $\mathbb{Z}_{(p)}$-algebra in which p is nilpotent automatically has finite Krull dimension: indeed, it suffices to show that A/pA has finite Krull dimension, and this follows from a theorem of E. Kunz [22, Prop. 1.1]. □

Considering the special case $I = pA$ we obtain the following result, proved from Theorem 6.6 by the exact same argument by which Corollary 5.8 was deduced from Theorem 5.3:

COROLLARY 6.10. – *Let A be a Noetherian, F-finite, finite Krull-dimensional $\mathbb{Z}_{(p)}$-algebra, and X a proper scheme over A. Then the canonical map*

$$HH_n(X;\mathbb{Z}/p^v) \longrightarrow \{HH_n(X \times_A A/p^s A;\mathbb{Z}/p^v)\}_s$$

is an isomorphism for all $n \geq 0$ and $v \geq 1$. Similarly for THH, TR^r, and TC^r.

Moreover, for all $r \geq 1$, the maps (not just the composition) of spectra

$$TR^r(X;p) \longrightarrow TR^r(X \times_A \widehat{A_p};p) \longrightarrow \operatorname{holim}_s TR^r(X \times_A A/p^s A;p)$$

are weak equivalences after p-completion. Similarly for THH, TR, TC^r, and TC.

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Bjørn Ian DUNDAS
Universitetet i Bergen
Matematisk institutt
Postboks 7800
5020 Bergen, Norway
E-mail: dundas@math.uib.no

Matthew MORROW
UPMC - Campus Jussieu
4, place Jussieu
Boîte Courrier 247
75252 Paris Cedex 05, France
E-mail: matthieu.morrow@imj-prg.fr