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FROM A KAC-LIKE PARTICLE SYSTEM TO THE LANDAU EQUATION FOR HARD POTENTIALS AND MAXWELL MOLECULES

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ABSTRACT. – We prove a quantitative result of convergence of a conservative stochastic particle system to the solution of the homogeneous Landau equation for hard potentials. There are two main difficulties: (i) the conservativeness of the particle system is an obstacle for approximate independence, as is the case for *true* physical particle systems; (ii) the known stability results for this class of Landau equations concern *regular* solutions and seem difficult to extend to study the rate of convergence of some empirical measures. Due to (i), we have to use a double-coupling. We first couple our particle system with some *non independent* nonlinear processes, of which law solves, in some sense, the Landau equation. We then introduce a second coupling to prove that these nonlinear processes are not so far from being independent. To overcome (ii), we prove a new stability result for the Landau equation for hard potentials concerning very general measure solutions. Using finally some ideas of Rousset [26], we show that in the case of Maxwell molecules, the convergence of the particle system is uniform in time.

RÉSUMÉ. – Nous prouvons des résultats quantitatifs de convergence d'un système de particules conservatif vers la solution de l'équation de Landau homogène pour des potentiels durs. Il y a deux principales difficultés : (i) le caractère conservatif du système est un obstacle pour obtenir de l'indépendance (même approchée), comme c'est le cas pour de *vrais* systèmes de particules physiques ; (ii) les résultats connus de stabilité pour ces équations de Landau concernent des solutions régulières et paraissent difficiles à étendre pour étudier la vitesse de convergence de mesures empiriques. Pour le point (i), nous procédons à un double couplage. Nous couplons d'abord notre système avec des processus non linéaires *non indépendants* dont la loi résout en un certain sens l'équation de Landau. Nous construisons ensuite un second couplage afin de montrer que ces processus non linéaires ne sont pas loin d'être indépendants. Pour résoudre (ii), nous établissons de nouveaux résultats de stabilité pour l'équation de Landau pour des potentiels durs et des solutions de type mesure très générales. Finalement, en utilisant des idées de Rousset [26], nous montrons que dans le cas des molécules maxwelliennes, la convergence du système de particules est uniforme en temps.

1. Introduction and main results

1.1. The Landau equation

The homogeneous Landau equation reads

$$(1) \qquad \partial_t f_t(v) = \frac{1}{2} \mathrm{div}_v \Big(\int_{\mathbb{R}^3} a(v - v_*)[f_t(v_*) \nabla f_t(v) - f_t(v) \nabla f_t(v_*)] \, dv_* \Big).$$

The unknown $f_t : \mathbb{R}^3 \mapsto \mathbb{R}$ stands for the velocity-distribution in a plasma and the initial condition f_0 is given. We denote by S_3^+ the set of symmetric nonnegative 3×3 matrices. The function $a : \mathbb{R}^3 \mapsto S_3^+$ is given, for some $\gamma \in [-3, 1]$, by

$$a(v) = |v|^{2+\gamma} \Pi_{v\perp}, \quad \text{where} \quad \Pi_{v\perp} = \mathbf{I}_3 - \frac{v \otimes v}{|v|^2}$$

is the projection matrix onto $v^\perp$. The only physically relevant case, namely $\gamma = -3$ which corresponds to a Coulomb interaction, is unfortunately the most difficult. For mathematical results, let us mention the recent papers by Desvillettes [9] and Carrapatoso, Desvillettes and He [7] and the references therein. The other cases are interesting mathematically and numerically. In particular, the Landau equation can be seen as an approximation of the Boltzmann equation in the asymptotic of *grazing collisions*, as rigorously shown by Villani [31] for all values of $\gamma \in [-3, 1]$. We are concerned here with Maxwell molecules ($\gamma = 0$) and hard potentials ($\gamma \in (0, 1]$). The well-posedness, regularization properties and large-time behavior of the Landau equation have been studied in great details by Villani [32] for Maxwell molecules and by Desvillettes and Villani [10, 11] for hard potentials. We finally refer to the long reviews paper of Villani [33] and Alexandre [1] on the Boltzmann and Landau models.

1.2. Notation

We denote by $\mathscr{P}(\mathbb{R}^3)$ the set of probability measures on $\mathbb{R}^3$. When $f \in \mathscr{P}(\mathbb{R}^3)$ has a density, we also denote by $f \in L^1(\mathbb{R}^3)$ this density.

For $q > 0$, we set $\mathscr{P}_q(\mathbb{R}^3) = \{ f \in \mathscr{P}(\mathbb{R}^3) \ : \ m_q(f) < \infty \}$, where $m_q(f) = \int_{\mathbb{R}^3} |v|^q f(dv) < \infty$. For $\alpha > 0$ and $f \in \mathscr{P}(\mathbb{R}^3)$, we put $\mathscr{E}_\alpha(f) = \int_{\mathbb{R}^3} \exp(|v|^\alpha) f(dv)$. The entropy of $f \in \mathscr{P}(\mathbb{R}^3)$ is defined by $H(f) = \int_{\mathbb{R}^3} f(v) \log f(v) dv$ if f has a density and by $H(f) = \infty$ else.

We will use the Wasserstein distance defined as follows. For $f, g \in \mathscr{P}_2(\mathbb{R}^3)$, we introduce $\mathscr{H}(f, g) = \{ R \in \mathscr{P}(\mathbb{R}^3 \times \mathbb{R}^3) \ : \ R \text{ has marginals } f \text{ and } g \}$ and we set

$$\mathscr{W}_2(f, g) = \inf \Big\{ \Big(\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - w|^2 R(dv, dw) \Big)^{1/2} \ : \ R \in \mathscr{H}(f, g) \Big\}.$$

See Villani [34] for many details on this distance.

We also define, for $v \in \mathbb{R}^3$,

$$b(v) = \mathrm{div} \, a(v) = -2|v|^\gamma v \quad \text{and} \quad \sigma(v) = [a(v)]^{1/2} = |v|^{1+\gamma/2} \Pi_{v\perp}.$$

For $f \in \mathscr{P}(\mathbb{R}^3)$ and $v \in \mathbb{R}^3$, we set

$$b(f, v) := \int_{\mathbb{R}^3} b(v - v_*) f(dv_*), \quad a(f, v) := \int_{\mathbb{R}^3} a(v - v_*) f(dv_*), \quad a^{1/2}(f, v) := \big[a(f, v) \big]^{1/2}.$$

More generally, we will write $\varphi(f, v) = \int_{\mathbb{R}^3} \varphi(v - v_*) f(dv_*)$ when $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}$. We emphasize that $a^{1/2}(f, v)$ is $[a(f, v)]^{1/2}$ and is not $\int_{\mathbb{R}^3} a^{1/2}(v - v_*) f(dv_*)$.

Finally, for A and B two 3×3 matrices, we put $\|A\|^2 = \mathrm{Tr}(AA^*)$ and $\langle\!\langle A, B \rangle\!\rangle = \mathrm{Tr}(AB^*)$.

1.3. Well-posedness

We will use the following notion of weak solutions.

DEFINITION 1. – *Let $\gamma \in [0,1]$. We say that $f = (f_t)_{t \geq 0}$ is a weak solution to (1) if it belongs to $L^\infty_{\mathrm{loc}}([0,\infty), \mathscr{P}_{2+\gamma}(\mathbb{R}^3))$ and if for all $\varphi \in C^2_b(\mathbb{R}^3)$, all $t \geq 0$,*

$$(2) \qquad \int_{\mathbb{R}^3} \varphi(v) f_t(dv) = \int_{\mathbb{R}^3} \varphi(v) f_0(dv) + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} L\varphi(v, v_*) f_s(dv) f_s(dv_*) ds,$$

where

$$L\varphi(v, v_*) := \frac{1}{2} \sum_{k,l=1}^3 a_{kl}(v - v_*) \partial^2_{kl} \varphi(v) + \sum_{k=1}^3 b_k(v - v_*) \partial_k \varphi(v).$$

A weak solution f is conservative *if it conserves momentum and energy, that is $\int_{\mathbb{R}^3} v f_t(dv) = \int_{\mathbb{R}^3} v f_0(dv)$ and $m_2(f_t) = m_2(f_0)$ for all $t \geq 0$.*

An important remark is that $|L\varphi(v, v_*)| \leq C_\varphi (1 + |v| + |v_*|)^{2+\gamma}$ for $\varphi \in C^2_b(\mathbb{R}^3)$ and since $f \in L^\infty_{\mathrm{loc}}([0,\infty), \mathscr{P}_{2+\gamma}(\mathbb{R}^3))$, every term makes sense in (2). Our first result concerns well-posedness and stability.

THEOREM 2. – *(i) If $\gamma = 0$, then for any $f_0 \in \mathscr{P}_2(\mathbb{R}^3)$, (1) has a unique weak solution $f = (f_t)_{t \geq 0}$ starting from f_0. This solution is conservative. If moreover $H(f_0) < \infty$, then $H(f_t) \leq H(f_0)$ for all $t \geq 0$. If $f_0 \in \mathscr{P}_q(\mathbb{R}^3)$ for some $q > 2$, then $\sup_{[0,\infty)} m_q(f_t) < \infty$. Finally, for any other weak solution $g = (g_t)_{t \geq 0}$ to (1), it holds that $\mathscr{W}_2(f_t, g_t) \leq \mathscr{W}_2(f_0, g_0)$ for all $t \geq 0$.*

(ii) If $\gamma \in (0,1]$, consider $f_0 \in \mathscr{P}_2(\mathbb{R}^3)$ with $\mathscr{E}_\alpha(f_0) < \infty$ for some $\alpha \in (\gamma, 2)$. Then (1) has a unique weak solution $f = (f_t)_{t \geq 0}$ starting from f_0. Moreover, this solution is conservative and $\sup_{t \geq 0} \mathscr{E}_\alpha(f_t) < \infty$. If $H(f_0) < \infty$, then $H(f_t) \leq H(f_0)$ for all $t \geq 0$. Finally, for all $\eta \in (0,1)$, all $T > 0$ and any other weak solution to $g = (g_t)_{t \geq 0}$ to (1), it holds that $\sup_{[0,T]} \mathscr{W}_2(f_t, g_t) \leq C_{\eta,T} (\mathscr{W}_2(f_0, g_0))^{1-\eta}$, the constant $C_{\eta,T}$ depending only on η, T, γ, α and on (upper bounds of) $\mathscr{E}_\alpha(f_0)$ and $\sup_{[0,T]} m_{2+\gamma}(g_t)$.

Point (i) is well-known, even if we found no precise reference for all the claims of the statement. The well-posedness, propagation of moments and entropy dissipation have been checked by Villani [32] when f_0 has a density and the well-posedness when $f_0 \in \mathscr{P}_2(\mathbb{R}^3)$ has been established by Guérin [20]. The noticeable fact that $\mathscr{W}_2$ decreases along solutions was discovered by Tanaka [30] for the Boltzmann equation for Maxwell molecules, see also Carrapatoso [6, Lemma 4.15].

Similarly, the existence part in point (ii) is more or less standard: the well-posedness, propagation of moments and entropy dissipation can be found in [10] when $H(f_0) < \infty$, but $H(f_0) < \infty$ is mainly assumed for simplicity. The propagation of exponential moments seems to be new, but far from surprising: it is well-known (and more complicated) for the Boltzmann equation for hard potentials, as was discovered by Bobylev [5], see also Alonso *et al.* [2].

On the contrary, the uniqueness/stability part in point (ii) seems to be new and rather interesting. As far as we know, the best available uniqueness result is the one of Desvillettes

and Villani [10, Theorem 7], where $f_0 \in \mathscr{P}_2(\mathbb{R}^3)$ is assumed to have a density satisfying $\int_{\mathbb{R}^3} f_0^2(v)(1 + |v|^s)dv < \infty$ for some $s > 15 + 5\gamma$. Thus, we assume much less regularity, but much more localization. Furthermore, our stability result holds *in the class of all weak solutions*. Actually, a stability result in the class of all weak solutions (at least with finite entropy) can also be derived using the ideas of Desvillettes and Villani, but this would use the regularization properties of the equation which guarantee that *any* weak solution is smooth. On the contrary, we use no such regularization. This is crucial for propagation of chaos, since then the approximate solution consists of empirical measures which, by nature, are not smooth. Similarly, it is very important for us that the stability result does not involve any exponential moment of the second solution g, because we are not able to propagate the exponential moments of our particle system.

1.4. The particle system

We now introduce an approximating particle system, in the spirit of Kac [22], who was dealing with the Boltzmann equation. More precisely, consider N particles entirely characterized by their velocities $(v_1, \dots, v_N) \in (\mathbb{R}^3)^N$, where each pair of particles (with velocities v_i and v_j) collide at some rate proportional to $|v_i - v_j|^\gamma$. Assume that these collisions are elastic. Then in the limit of *grazing collisions*, that is, when collisions occur very frequently but generate very small deviations, we find, at least formally, the following diffusive system. See Carrapatoso [6] when $\gamma = 0$ and Miot, Pulvirenti and Saffirio [24] when $\gamma = -3$.

We fix $N \geq 2$ and consider an exchangeable $(\mathbb{R}^3)^N$-valued random variable $(V_0^{i,N})_{i=1,\dots,N}$, independent of a family $(B_t^{ij})_{1 \leq i < j < N, t \geq 0}$ of i.i.d. 3D Brownian motions. For $1 \leq j < i \leq N$, we set $B_t^{ij} = -B_t^{ji}$. We also put $B_t^{ii} = 0$ for all $i = 1, \dots, N$ and we consider the system

(3)
$$V_t^{i,N} = V_0^{i,N} + \frac{1}{N} \sum_{j=1}^N \int_0^t b(V_s^{i,N} - V_s^{j,N})ds + \frac{1}{\sqrt{N}} \sum_{j=1}^N \int_0^t \sigma(V_s^{i,N} - V_s^{j,N})dB_s^{ij}, \quad i = 1, \dots, N.$$

PROPOSITION 3. – *Fix $\gamma \in [0,1]$ and $N \geq 2$. The system (3) has a pathwise unique solution $(V_t^{i,N})_{i=1,\dots,N,t\geq 0}$, which is furthermore exchangeable. The system is conservative: a.s., for all $t \geq 0$, it holds that $\sum_1^N V_t^{i,N} = \sum_1^N V_0^{i,N}$ and $\sum_1^N |V_t^{i,N}|^2 = \sum_1^N |V_0^{i,N}|^2$.*

In [13], Fontbona, Guérin and Méléard consider, when $\gamma = 0$, the same system of equations, but with a fully i.i.d. family $(B_t^{ij})_{1 \leq i,j \leq N, t \geq 0}$ of Brownian motions. Such a system also approximates the Landau equation, but is not conservative (one only has $\mathbb{E}[\sum_1^N V_t^{i,N}] = \mathbb{E}[\sum_1^N V_0^{i,N}]$ and $\mathbb{E}[\sum_1^N |V_t^{i,N}|^2] = \mathbb{E}[\sum_1^N |V_0^{i,N}|^2]$) and thus physically less relevant.

1.5. Propagation of chaos

The main result of the paper is the following.

THEOREM 4. – *Fix $\gamma \in [0,1]$ and $f_0 \in \mathscr{P}_2(\mathbb{R}^3)$. If $\gamma \in (0,1]$, assume moreover that $\mathscr{E}_\alpha(f_0) < \infty$ for some $\alpha \in (\gamma, 2)$. Consider the unique weak solution $(f_t)_{t\geq 0}$ to (1) built in Theorem 2. For each $N \geq 2$, consider an exchangeable $(\mathbb{R}^3)^N$-valued random variable $(V_0^{i,N})_{i=1,\dots,N}$ and the corresponding unique solution $(V_t^{i,N})_{i=1,\dots,N,t\geq 0}$ to (3). Set $\mu_t^N = N^{-1} \sum_1^N \delta_{V_t^{i,N}}$. Assume that for all $p \geq 2$, $M_p := m_p(f_0) + \sup_{N \geq 2} \mathbb{E}[|V_0^{1,N}|^p] < \infty$.*

(i) *If $\gamma = 0$, then for all $\eta \in (0,1)$, there is a constant C_η depending only on η, on (some upper bounds of) $\{M_p, p \geq 2\}$ and on (some upper bound of) $H(f_0)$ when $H(f_0) < \infty$ such that*

$$\sup_{[0,\infty)} \mathbb{E}[\mathcal{W}_2^2(\mu_t^N, f_t)] \leq \begin{cases} C_\eta(\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] + N^{-1/4})^{1-\eta} & \text{in general,} \\ C_\eta(\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] + N^{-1/3})^{1-\eta} & \text{if } H(f_0) < \infty. \end{cases}$$

(ii) *If $\gamma \in (0,1]$, then for all $T > 0$, all $\eta \in (0,1)$, there is a constant $C_{\eta,T}$ depending only on η, T, γ, α, on (some upper bounds of) $\mathcal{E}_\alpha(f_0)$ and $\{M_p, p \geq 2\}$ and on (some upper bound of) $H(f_0)$ when $H(f_0) < \infty$ such that*

$$\sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^N, f_t)] \leq \begin{cases} C_{\eta,T}(\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] + N^{-1/4})^{1-\eta} & \text{in general,} \\ C_{\eta,T}(\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] + N^{-1/3})^{1-\eta} & \text{if } H(f_0) < \infty. \end{cases}$$

As already mentioned in the abstract, there are two main difficulties. First, and this is the main issue, the conservativeness of the particle system is an obstacle for approximate independence. This is an important point because even if we deal here with a toy model, one of the main difficulties when trying to derive the Landau (or Boltzmann) equation from *true* physical particle systems, conservative collisions (by pairs) produce the intrinsic correlations, and the main challenge is to show that these correlations are small. To overcome this difficulty, we use a double coupling. Next, in the case of hard potentials, we really need the new stability result established in Theorem 2-(ii): the previous available results were concerning *regular* solutions, which is not well-adapted to the study of empirical measures.

If $(V_0^{i,N})_{i=1,\dots,N} \sim f_0^{\otimes N}$, then we know from [16, Theorem 1] that $\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] \leq CN^{-1/2}$ and that $N^{-1/2}$ is generally the best rate we can hope for when comparing an empirical measure of an i.i.d. sample to the common distribution. Here we obtain a rate in $N^{-1/3}$ (or $N^{-1/4}$ without entropy), up to an arbitrarily small loss. Let us mention that the double coupling is responsible for this loss, because we have to apply a (quantitative) law of large numbers for non independent variables, see Subsection 8.5.

Let us finally mention that in point (i), the time uniformity really uses that we are in dimension $d > 2$.

1.6. References on propagation of chaos

Showing the convergence of a toy particle system to the Boltzmann equation was proposed by Kac [22] as a step to its rigorous derivation. He called *propagation of chaos* such a convergence. Getting some *uniform in time* convergence is quite relevant, since then the large time behavior of the PDE indeed describes that of the particle system. Another important motivation is the numerical resolution of the Boltzmann equation without cutoff: indeed, it may be relevant to replace *grazing* collisions by a diffusive Landau-like term. Choosing the right threshold level requires to know quite well the rates of convergence. See [15] for a complete study, in this spirit, of the 1D Kac equation.

To our knowledge, the only result *directly* comparable to Theorem 4 is the one of Carrapatoso [6, Theorem 4.2] which concerns Maxwell molecules ($\gamma = 0$): he obtains (under some different conditions on f_0), a uniform in time rate of convergence in (almost) $N^{-1/972}$ for another distance, strictly controlled by $\sup_{[0,\infty)} \mathbb{E}[\mathcal{W}_2^2(\mu_t^N, f_t)]^{1/2}$, which we can bound by (almost) $N^{-1/6}$.

In the much more difficult Coulomb case $\gamma = -3$, Miot, Pulvirenti and Saffirio [24] have studied a similar conservative particle system and obtained a weak result, namely, up to extraction of subsequences, the convergence to the Landau hierarchy. Also, Bobylev, Pulvirenti and Saffirio [4] study some *true* physical particle systems, expected to converge to the Landau equation with Coulomb interaction, and prove some convergence results in very small time (more precisely, concerning the time derivative at time 0).

Concerning the non-conservative particle system approximating the Landau equation, Maxwell molecules have been studied by Fontbona, Guérin and Méléard [13] (there it is proved that $\sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^N, f_t)] \leq C_T N^{-2/7}$), see also [14]. Moderately soft potentials are investigated in the companion paper [17] ($\sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^N, f_t)] \leq C_T N^{-1/2}$ when $\gamma \in (-1/4, 0)$, a less good rate when $\gamma \in (-1, -1/4]$ and a convergence without rate when $\gamma \in (-2, -1]$). As compared to [17], the present situation is simpler, because hard potentials are rather easier than soft potentials, but more complicated, because we study the conservative particle system.

Sznitman [29] was the first to prove the convergence (without rate) of Kac's conservative particle system to the Boltzmann equation for hard spheres ($\gamma = 1$). Some recent progresses have been made by Mischler and Mouhot [25] (from which [6] is inspired) where, using an abstract and purely analytic method, a uniform in time quantitative convergence of Kac's particle system was derived, for the Boltzmann equation for Maxwell molecules ($\gamma = 0$, with a rate in $N^{-\varepsilon}$ for some very small ε) and hard spheres ($\gamma = 1$, with a rate in $(\log N)^{-\varepsilon}$ for some very small ε). Even if these rates are clearly far from being sharp, these results are impressive. However, the method uses some smoothness of the solution $(f_t)_{t \geq 0}$ with respect to f_0 (something like one or two derivatives, in some sense, required), which is closely related to uniqueness/stability theory. Such a theory is completely understood only for Maxwell molecules (where the kinetic cross section is constant) and hard spheres (where the angular cross section is integrable). Finally, let us mention the paper of Cortez and Fontbona [8], who considered the simplest model (the 1D Kac equation), but who obtained by coupling methods a *good* rate of convergence (although probably not optimal, in $N^{-1/3}$) for a *conservative* particle system [(1)].

1.7. Scheme of the proofs

Interpreting a solution $(f_t)_{t \geq 0}$ to a kinetic equation in terms of the time-marginals of a 3D process $(V_t)_{t \geq 0}$ solving some *nonlinear* Poisson SDE was initiated by Tanaka [30] for the Boltzmann equation for Maxwell molecules. Roughly, $(V_t)_{t \geq 0}$ represents the time-evolution of the velocity of a *typical* particle. A similar process was proposed by Guérin [20] for the Landau equation, with a white noise-driven SDE. Here and in [17], we rather use a Brownian SDE. We show that for any weak solution $(f_t)_{t \geq 0}$ and for $V_0 \sim f_0$, the SDE $V_t = V_0 + \int_0^t [b(f_s, V_s)ds + a^{1/2}(f_s, V_s)dB_s]$ is well-posed and $V_t \sim f_t$ for all $t \geq 0$. We call $(V_t)_{t \geq 0}$ a $(f_t)_{t \geq 0}$-Landau process. To prove uniqueness/stability, we will consider two weak solutions $(f_t)_{t \geq 0}$ and $(g_t)_{t \geq 0}$ and we will couple a $(f_t)_{t \geq 0}$-Landau process and a $(g_t)_{t \geq 0}$-Landau process in such a way that they remain as close as possible. Using the

[(1)] These authors told us that, putting together the ideas of [8] and of [18], they are now treating the case of Kac's conservative system for the $3D$ Boltzmann equation for hard potentials.

same Brownian motion for both processes, sometimes called synchronous coupling, does not provide sufficiently good estimates. We will use a finer coupling, based on some ideas of Givens and Shortt [19] about the optimal coupling of (multidimensional) Gaussian random variables (for $\mathcal{W}_2$ distance). Such a finer coupling is crucial, in particular to obtain a stability result that requires exponential moments of only one of the two solutions. As already mentioned, this is important because we are not able to propagate exponential moments of the particle system.

Similarly, we will finely couple our particle system $(V^{i,N})_{i=1,\dots,N}$ with a family $(W^{i,N})_{i=1,\dots,N}$ of $(f_t)_{t\geq 0}$-Landau processes. The conservativeness of our particle system implies that the family $(W_t^{i,N})_{i=1,\dots,N}$ is not independent. But we will use a second coupling to show that for $1 \ll K \ll N$, $(W_t^{i,N})_{i=1,\dots,K}$ are approximately independent. The idea of using two couplings is already present in the paper by Cortez and Fontbona [8].

Let us mention that the *finer* coupling would also be crucial if studying some non conservative system.

The time uniformity we obtain in the case of Maxwell molecules relies on a recent noticeable argument of Rousset [26] for the Boltzmann equation. For two solutions $(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$, Tanaka's theorem [30] tells us (roughly) that $(d/dt)\mathcal{W}_2(f_t, g_t) \leq 0$. Rousset manages to prove, in dimension $d \geq 3$, something like $(d/dt)\mathcal{W}_2(f_t, g_t) \leq -\kappa_\varepsilon \mathcal{W}_2^{1+\varepsilon}(f_t, g_t)$ for all $\varepsilon > 0$. This implies that f_t tends to a unique equilibrium as $t \to \infty$ at some arbitrarily fast polynomial speed. Much better, he gets a similar result for the particle system, uniformly in N. Previously, the large time behavior of the Landau equation for Maxwell molecules was already well understood, see Villani [32], but the only known uniform in N convergence to equilibrium for the particle system had been obtained by Carrapatoso [6], with some weaker distance and, more importantly, some much lower rate (namely, *some* very small polynomial rate). Again, extending this strategy to the Landau equation really uses a fine coupling with suitable different Brownian motions.

1.8. Plan of the paper

In the next section, we quickly prove the existence part of Theorem 2. In Section 3, we study the regularity of b, a, σ and $b(f, \cdot), a(f, \cdot), a^{1/2}(f, \cdot)$. We prove Proposition 3 (well-posedness of the particle system) and the well-posedness of the Landau process in Section 4. Section 5 is devoted to the proof of a central inequality, which is used a first time in Section 6 to prove the uniqueness/stability part of Theorem 2. We next show in Section 7 that all the moments of the particle system propagate, uniformly in N and in time. This allows us to handle the proof of Theorem 4 (propagation of chaos) in Section 8, based on a second use of our central inequality, except the time-uniformity (when $\gamma = 0$) which is verified in Section 9.

2. Existence, moments and exponential moments

As we will use several times in the paper, the explicit expressions of a and b yield

$$(4) \qquad \begin{cases} \operatorname{Tr} a(v - v_*) = 2|v - v_*|^{2+\gamma}, \\ a(v - v_*)v \cdot v = |v - v_*|^\gamma (|v|^2|v_*|^2 - (v \cdot v_*)^2), \\ b(v - v_*) \cdot v = -2|v - v_*|^\gamma (|v|^2 - v \cdot v_*). \end{cases}$$

The existence part of Theorem 2 is, as already mentioned, more or less well-known.

PROPOSITION 5. – *Let $\gamma \in [0,1]$ be fixed and let $f_0 \in \mathscr{P}_{2+2\gamma}(\mathbb{R}^3)$. Then there exists a conservative weak solution $(f_t)_{t\geq 0}$ in the sense of Definition 1 enjoying the following properties.*

(i) *If $H(f_0) < \infty$, then $H(f_t) \leq H(f_0)$ for all $t \geq 0$.*

(ii) *If $m_q(f_0) < \infty$ for some $q > 2$, then $\sup_{[0,\infty)} m_q(f_t) \leq C_q$, for some finite constant depending only on γ, q and on (an upper bound of) $m_q(f_0)$.*

(iii) *If $\gamma \in (0,1]$ and $\mathscr{E}_\alpha(f_0) < \infty$ for some $\alpha \in (0,2)$, then $\sup_{[0,\infty)} \mathscr{E}_\alpha(f_t) \leq C_\alpha$, for some finite constant C_α depending only on α, γ and on (an upper bound of) $\mathscr{E}_\alpha(f_0)$.*

Proof. – If $\gamma = 0$, the existence (and uniqueness) of a weak solution $(f_t)_{t\geq 0}$ to (1) has been checked by Guérin [20, Corollaries 6 and 7]. Point (i) is proved by Villani [32, Section 8] as well as point (ii) (see [32, Theorem 1]): he assumes additionally but does not use that $f_0 \in L^1(\mathbb{R}^3)$.

If $\gamma \in (0,1]$ and if $f_0 \in \mathscr{P}_{2+\gamma}(\mathbb{R}^3)$ with $H(f_0) < \infty$, then we know from Desvillettes and Villani [10, Theorems 1 and 3] that (1) has a weak solution $(f_t)_{t\geq 0}$ satisfying points (i) and (ii). If we only know that $f_0 \in \mathscr{P}_{2+2\gamma}(\mathbb{R}^3)$, we introduce $f_0^n = f_0 \star G_n$, with $G_n(v) = (n/2\pi)^{3/2} \exp(-n|x|^2/2)$. Then $H(f_0^n) < \infty$ and we consider a corresponding weak solution $(f_t^n)_{t\geq 0}$, satisfying points (i) and (ii). In particular, we have $\sup_{n\geq 1} \sup_{[0,\infty)} m_{2+2\gamma}(f_t^n) < \infty$. We thus infer from (2) that for all $\varphi \in C_b^2(\mathbb{R}^3)$, $\sup_{n\geq 1} \sup_{[0,\infty)} |(d/dt) \int_{\mathbb{R}^3} \varphi(v) f_t^n(dv)| < \infty$: the family $\{(f_t^n)_{t\geq 0}, n \geq 1\} \subset C([0,\infty), \mathscr{P}(\mathbb{R}^3))$ is equicontinuous (with $\mathscr{P}(\mathbb{R}^3)$ endowed with the topology of weak convergence). We thus can find $(f_t)_{t\geq 0} \in C([0,\infty), \mathscr{P}(\mathbb{R}^3))$ so that, up to extraction of a subsequence, $\lim_n \sup_{[0,T]} |\int_{\mathbb{R}^3} \varphi(v)(f_t^n - f_t)(dv)| = 0$ for all $\varphi \in C_b(\mathbb{R}^3)$ and all $T > 0$. This function $(f_t)_{t\geq 0}$ also satisfies point (ii), because point (ii) is satisfied by $(f_t^n)_{t\geq 0}$ uniformly in n. Thus $(f_t)_{t\geq 0} \in L^\infty([0,\infty), \mathscr{P}_{2+2\gamma}(\mathbb{R}^3))$. Finally, it is not difficult to pass to the limit, for each $\varphi \in C_b^2(\mathbb{R}^3)$, each $t \geq 0$, in the equation $\int_{\mathbb{R}^3} \varphi(v) f_t^n(dv) = \int_{\mathbb{R}^3} \varphi(v) f_0^n(dv) + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} L\varphi(v,v_*) f_s^n(dv) f_s^n(dv_*)$, to deduce that $(f_t)_{t\geq 0}$ is a weak solution to (1): the only difficulty is that $L\varphi$ is not bounded, but this problem is fixed using that $|L\varphi(v,v_*)| \leq C_\varphi(1 + |v| + |v_*|)^{2+\gamma}$ and that $\sup_{n\geq 1} \sup_{[0,\infty)} m_{2+2\gamma}(f_t + f_t^n) < \infty$.

We now assume that $\gamma \in (0,1]$, we fix $\alpha \in (0,2)$, and we give a formal proof of point (iii) without justifying the computations: this probably does not prove that *every* weak solution propagates exponential moments, but certainly shows that it is possible to build such weak solutions. We consider $\varphi(v) = \exp((1 + |v|^2)^{\alpha/2})$, we set $\widetilde{\mathscr{E}}_\alpha(f) = \int_{\mathbb{R}^3} \varphi(v) f(dv)$ and we observe that $\mathscr{E}_\alpha(f) \leq \widetilde{\mathscr{E}}_\alpha(f) \leq e\mathscr{E}_\alpha(f)$. It holds that $\partial_k \varphi(v) = \alpha v_k(1+|v|^2)^{\alpha/2-1}\varphi(v)$ and $\partial_{kl}\varphi(v) = \alpha[(1+|v|^2)^{\alpha/2-1}\mathbb{I}_{\{k=l\}} + (\alpha-2)v_k v_l(1+|v|^2)^{\alpha/2-2} + \alpha v_k v_l(1+|v|^2)^{\alpha-2}]\varphi(v)$, whence

$$L\varphi(v,v_*) = \frac{\alpha}{2}\Big[2(1+|v|^2)^{\alpha/2-1}v \cdot b(v-v_*) + (1+|v|^2)^{\alpha/2-1}\operatorname{Tr} a(v-v_*)$$
$$+ \big((\alpha-2)(1+|v|^2)^{\alpha/2-2} + \alpha(1+|v|^2)^{\alpha-2}\big)a(v-v_*)v \cdot v\Big]\varphi(v).$$

Recalling (4), we find

$$L\varphi(v, v_*) = \frac{\alpha}{2}|v - v_*|^\gamma (1 + |v|^2)^{\alpha/2-2}\Big[-2(1 + |v|^2)|v|^2 + 2(1 + |v|^2)|v_*|^2$$
$$+ \big((\alpha - 2) + \alpha(1 + |v|^2)^{\alpha/2}\big)\big(|v|^2|v_*|^2 - (v \cdot v_*)^2\big)\Big]\varphi(v).$$

Using that $|v - v_*|^\gamma \geq |v|^\gamma - |v_*|^\gamma$ and that $|v - v_*|^\gamma \leq |v|^\gamma + |v_*|^\gamma$, we deduce that

$$L\varphi(v, v_*) \leq -\alpha\Big[(1 + |v|^2)^{\alpha/2-1}|v|^{2+\gamma} - (1 + |v|^2)^{\alpha/2-1}|v|^2|v_*|^\gamma\Big]\varphi(v)$$
$$+ \frac{\alpha}{2}(|v|^\gamma + |v_*|^\gamma)(1 + |v|^2)^{\alpha/2-2}\Big[2(1 + |v|^2)|v_*|^2$$
$$+ \big((\alpha - 2) + \alpha(1 + |v|^2)^{\alpha/2}\big)\big(|v|^2|v_*|^2 - (v \cdot v_*)^2\big)\Big]\varphi(v)$$
$$\leq -\alpha(1 + |v|^2)^{\alpha/2-1}|v|^{2+\gamma}\varphi(v)$$
$$+ C\big((1 + |v|^2)^{\alpha/2} + (1 + |v|^2)^{\gamma/2+\alpha-1}\big)(1 + |v_*|^{2+\gamma})\varphi(v)$$

for some constant C depending only on γ, α. By the weak formulation of (1), we get

$$\frac{d}{dt}\widetilde{\mathcal{E}}_\alpha(f_t) \leq \int_{\mathbb{R}^3}\Big[-\alpha(1 + |v|^2)^{\alpha/2-1}|v|^{2+\gamma}$$
$$+ C\big((1 + |v|^2)^{\alpha/2} + (1 + |v|^2)^{\gamma/2+\alpha-1}\big)(1 + m_{2+\gamma}(f_t))\Big]\varphi(v)f_t(dv).$$

But we know from point (ii) that $\sup_{[0,\infty)} m_{2+\gamma}(f_t)$ is bounded by some constant depending only on γ and $m_{2+\gamma}(f_0)$ (which is itself controlled by $\mathcal{E}_\alpha(f_0)$). We end with

$$\frac{d}{dt}\widetilde{\mathcal{E}}_\alpha(f_t) \leq \int_{\mathbb{R}^3}\Big[-\alpha(1 + |v|^2)^{\alpha/2-1}|v|^{2+\gamma} + C(1 + |v|^2)^{\alpha/2} + C(1 + |v|^2)^{\gamma/2+\alpha-1}\Big]\varphi(v)f_t(dv).$$

For large values of $|v|$, we have $(1 + |v|^2)^{\alpha/2-1}|v|^{2+\gamma} \simeq |v|^{\alpha+\gamma}$ and $(1 + |v|^2)^{\alpha/2} + (1+|v|^2)^{\gamma/2+\alpha-1} \simeq |v|^{\max\{\alpha, \gamma+2\alpha-2\}}$. But $\alpha+\gamma > \alpha$ (because $\gamma > 0$) and $\alpha+\gamma > \gamma+2\alpha-2$ (because $\alpha < 2$), so that we can find some constants $K, L \geq 0$ so that for all $v \in \mathbb{R}^3$,

$$-\alpha(1 + |v|^2)^{\alpha/2-1}|v|^{2+\gamma} + C(1 + |v|^2)^{\alpha/2} + C(1 + |v|^2)^{\gamma/2+\alpha-1} \leq -1 + K\mathbb{1}_{\{|v|\leq L\}}.$$

Consequently,

$$\frac{d}{dt}\widetilde{\mathcal{E}}_\alpha(f_t) \leq -\widetilde{\mathcal{E}}_\alpha(f_t) + K\int_{\mathbb{R}^3}\mathbb{1}_{\{|v|\leq L\}}\varphi(v)f_t(dv) \leq -\widetilde{\mathcal{E}}_\alpha(f_t) + K\varphi(L).$$

We classically deduce that $\sup_{[0,\infty)}\widetilde{\mathcal{E}}_\alpha(f_t) \leq \max\{\widetilde{\mathcal{E}}_\alpha(f_0), K\varphi(L)\}$ as desired. □

3. Regularity estimates

The following estimates can be found in [14, Lemma 11] (with $C = 1$, but with another norm). Let S_3^+ be the set of symmetric nonnegative 3×3-matrices with real entries.

LEMMA 6. – *There is a constant C such that for any $A, B \in S_3^+$,*

$$\|A^{1/2} - B^{1/2}\| \leq C\|A - B\|^{1/2} \quad and \quad \|A^{1/2} - B^{1/2}\| \leq C(\|A^{-1}\| \wedge \|B^{-1}\|)^{1/2}\|A - B\|.$$

We will sometimes need the ellipticity estimate of Desvillettes and Villani [10, Proposition 4].

LEMMA 7. – *Let $\gamma \in [0,1]$. For all $A > 0$, there is C_A depending only on A and γ such that for all $f \in \mathscr{P}_2(\mathbb{R}^3)$ satisfying $H(f) \leq A$ and $m_2(f) \leq A$, for all $v \in \mathbb{R}^3$, $\|[a(f,v)]^{-1}\| \leq C_A(1+|v|)^{-\gamma}$.*

We next observe that the coefficients a, b and σ are locally Lipschitz continuous.

LEMMA 8. – *Fix $\gamma \in [0,1]$. There is C depending only on γ such that for all $v, w \in \mathbb{R}^3$,*

$$|b(v) - b(w)| \leq C|v-w|(|v|^\gamma + |w|^\gamma), \qquad \|\sigma(v) - \sigma(w)\| \leq C|v-w|(|v|^{\gamma/2} + |w|^{\gamma/2}),$$

$$and \quad \|a(v) - a(w)\| \leq C|v-w|(|v|^{1+\gamma} + |w|^{1+\gamma}).$$

Proof. – Since $b(v) = -2|v|^\gamma v$, since $\sigma(v) = |v|^{\gamma/2+1}\Pi_{v^\perp} = |v|^{\gamma/2+1}(\mathbf{I}_3 - |v|^{-2}v \otimes v)$ and since $a(v) = |v|^{2+\gamma}(\mathbf{I}_3 - |v|^{-2}v \otimes v)$, one easily checks that $|Db(v)| \leq C|v|^\gamma$, that $|D\sigma(v)| \leq C|v|^{\gamma/2}$ and that $|Da(v)| \leq C|v|^{1+\gamma}$, from which the results follow. $\qquad\square$

Our main results are based on the use of a SDE of which we now study roughly the coefficients.

LEMMA 9. – *Fix $\gamma \in [0,1]$. There is C depending only on γ such that for every $f \in \mathscr{P}_{2+\gamma}(\mathbb{R}^3)$ and every $v, w \in \mathbb{R}^3$,*
  (i) $|b(f,v)| \leq C(|v|^{1+\gamma} + m_{1+\gamma}(f))$,
  (ii) $|b(f,v) - b(f,w)| \leq C|v-w|(|v|^\gamma + |w|^\gamma + m_\gamma(f))$,
  (iii) $\|a(f,v)\| \leq C(|v|^{2+\gamma} + m_{2+\gamma}(f))$,
  (iv) $\|a(f,v) - a(f,w)\| \leq C|v-w|(|v|^{1+\gamma} + |w|^{1+\gamma} + m_{1+\gamma}(f))$,
  (v) $\|a^{1/2}(f,v)\|^2 \leq C(|v|^{2+\gamma} + m_{2+\gamma}(f))$,
  (vi) $\|a^{1/2}(f,v) - a^{1/2}(f,w)\|^2 \leq C|v-w|^2(1 + m_{2+\gamma}(f))(1 + |v|^2 + |w|^2)$.

Proof. – First, we have $|b(f,v)| \leq 2\int_{\mathbb{R}^3} |v-w|^{1+\gamma} f(dw) \leq C(|v|^{1+\gamma} + m_{1+\gamma}(f))$ and $\|a(f,v)\| \leq \|a^{1/2}(f,v)\|^2 = \operatorname{Tr} a(f,v) = \int_{\mathbb{R}^3} \operatorname{Tr} a(v-w)f(dw) = 2\int_{\mathbb{R}^3} |v-w|^{2+\gamma} f(dw) \leq C(|v|^{2+\gamma} + m_{2+\gamma}(f))$.

Next, $|b(f,v) - b(f,w)| = |\int_{\mathbb{R}^3}(b(v-z) - b(w-z))f(dz)|$, so that by Lemma 8,

$$|b(f,v) - b(f,w)| \leq C|v-w| \int_{\mathbb{R}^3}(|v-z|^\gamma + |w-z|^\gamma)f(dz) \leq C|v-w|(|v|^\gamma + |w|^\gamma + m_\gamma(f)).$$

With the same arguments, one finds $\|a(f,v) - a(f,w)\| \leq C|v-w|\int_{\mathbb{R}^3}(|v-z|^{1+\gamma} + |w-z|^{1+\gamma})f(dz) \leq C|v-w|(|v|^{1+\gamma} + |w|^{1+\gamma} + m_{1+\gamma}(f))$.

Point (vi) is more difficult, although probably far from being optimal. Stroock and Varadhan [28, Theorem 5.2.3] state that there is $C > 0$ such that for all $A : \mathbb{R}^3 \mapsto S_3^+$, $\|D(A^{1/2})\|_\infty \leq C\|D^2A\|_\infty^{1/2}$, which we apply to $A(v) = (1+|v|^2)^{-\gamma/2}a(f,v)$. Observing that $\|a(z)\| \leq C|z|^{2+\gamma}$, that $\|Da(z)\| \leq C|z|^{1+\gamma}$ and that $\|D^2a(z)\| \leq C|z|^\gamma$, we find

$$\|D^2A(v)\| \leq C\Big[(1+|v|^2)^{-\gamma/2}\int_{\mathbb{R}^3}|v-z|^\gamma f(dz) + (1+|v|^2)^{-\gamma/2-1/2}\int_{\mathbb{R}^3}|v-z|^{1+\gamma} f(dz)$$

$$+ (1+|v|^2)^{-\gamma/2-1}\int_{\mathbb{R}^3}|v-z|^{2+\gamma} f(dz)\Big]$$

$$\leq C(1+m_{2+\gamma}(f)).$$

Thus $\|D(A^{1/2})\|_\infty^2 \leq C(1+m_{2+\gamma}(f))$ and

$$\|(A(v))^{1/2} - (A(w))^{1/2}\|^2 \leq C(1+m_{2+\gamma}(f))|v-w|^2.$$

We now write, using that $(A(v))^{1/2} = (1 + |v|^2)^{-\gamma/4} a^{1/2}(f, v)$,

$$\|a^{1/2}(f, v) - a^{1/2}(f, w)\|^2 \leq 2(1 + |v|^2)^{\gamma/2} \|(A(v))^{1/2} - (A(w))^{1/2}\|^2$$
$$+ 2(1 + |w|^2)^{-\gamma/2} |(1 + |v|^2)^{\gamma/4} - (1 + |w|^2)^{\gamma/4}|^2 \|a^{1/2}(f, w)\|^2.$$

Recalling (v) and using that $|(1 + |w|^2)^{\gamma/4} - (1 + |v|^2)^{\gamma/4}| \leq C|v - w|$, we get

$$\|a^{1/2}(f, v) - a^{1/2}(f, w)\|^2$$
$$\leq C|v - w|^2 \big[(1 + |v|^2)^{\gamma/2} (1 + m_{2+\gamma}(f)) + (1 + |w|^2)^{-\gamma/2} (|w|^{2+\gamma} + m_{2+\gamma}(f)) \big].$$

This can be bounded by $C|v - w|^2 (1 + m_{2+\gamma}(f))(1 + |v|^2 + |w|^2)$ as desired. $\qquad\square$

4. Well-posedness of the particle system and of the Landau process

We first verify that the particle system (3) is well-posed.

Proof of Proposition 3. – Since b and σ are locally Lipschitz continuous by Lemma 8, the system classically admits a pathwise unique *local* solution $(V_t^{i,N})_{i=1,\dots,N, t \in [0,\tau)}$ with $\tau = \sup_{k \geq 1} \tau_k$ and $\tau_k = \inf\{t \geq 0 : \sum_1^N |V_t^{i,N}|^2 \geq k\}$. We now show that a.s., $\sum_1^N V_t^{i,N} = \sum_1^N V_0^{i,N}$ and $\sum_1^N |V_t^{i,N}|^2 = \sum_1^N |V_0^{i,N}|^2$ for all $t \in [0, \tau)$. This will of course imply that $\tau = \infty$ and thus end the proof.

Summing (3) over $i = 1, \dots, N$, using that $b(-x) = -b(x)$, that $\sigma(-x) = \sigma(x)$, that $\sigma(0) = 0$ and that $B^{ij} = -B^{ji}$ for all $i \neq j$, we immediately find that $\sum_1^N V_t^{i,N} = \sum_1^N V_0^{i,N}$ for all $t \in [0, \tau)$. We next apply the Itô formula, which is licit on $[0, \tau)$, to get, using that $\sigma(x)\sigma^*(x) = a(x)$,

$$\sum_{i=1}^N |V_t^{i,N}|^2 = \sum_{i=1}^N |V_0^{i,N}|^2 + \frac{1}{N} \sum_{i,j=1}^N \int_0^t [2V_s^{i,N} \cdot b(V_s^{i,N} - V_s^{j,N}) + \mathrm{Tr}\, a(V_s^{i,N} - V_s^{j,N})] ds$$

$$+ \frac{2}{\sqrt{N}} \sum_{i,j=1}^N \int_0^t V_s^{i,N} \cdot \sigma(V_s^{i,N} - V_s^{j,N}) dB_s^{ij}.$$

But since $b(x) = -2|x|^\gamma x$ and $\mathrm{Tr}\, a(x) = 2|x|^{\gamma+2}$,

$$\sum_{i,j=1}^N [2V_s^{i,N} \cdot b(V_s^{i,N} - V_s^{j,N}) + \mathrm{Tr}\, a(V_s^{i,N} - V_s^{j,N})]$$

$$= \sum_{i,j=1}^N [(V_s^{i,N} - V_s^{j,N}) \cdot b(V_s^{i,N} - V_s^{j,N}) + \mathrm{Tr}\, a(V_s^{i,N} - V_s^{j,N})] = 0.$$

Using next that $\sigma(-x) = \sigma(x)$ and that $B^{ij} = -B^{ji}$, we also have

$$\sum_{i,j=1}^N V_s^{i,N} \cdot \sigma(V_s^{i,N} - V_s^{j,N}) dB_s^{ij} = \sum_{1 \leq i < j \leq N} (V_s^{i,N} - V_s^{j,N}) \cdot \sigma(V_s^{i,N} - V_s^{j,N}) dB_s^{ij},$$

which a.s. vanishes because $\sigma(x) = |x|^{1+\gamma/2} \Pi_{x^\perp}$ (so that $x^* \sigma(x) = 0$). We conclude that $\sum_1^N |V_t^{i,N}|^2 = \sum_1^N |V_0^{i,N}|^2$ on $[0, \tau)$, which ends the proof. $\qquad\square$

We next build our Landau process.

PROPOSITION 10. – *Fix $\gamma \in [0, 1]$ and $f = (f_t)_{t \geq 0} \in L^\infty_{\mathrm{loc}}([0, \infty), \mathscr{P}_{2+\gamma}(\mathbb{R}^3))$, as well as $g_0 \in \mathscr{P}_2(\mathbb{R}^3)$ and a g_0-distributed random variable V_0 independent of a 3D Brownian motion $(B_t)_{t \geq 0}$.*

(i) *The SDE $V_t = V_0 + \int_0^t b(f_s, V_s)ds + \int_0^t a^{1/2}(f_s, V_s)dB_s$ has a pathwise unique solution.*

(ii) *If f is a weak solution to (1) and if $g_0 = f_0$, then V_t is f_t distributed for all $t \geq 0$.*

Proof. – We start with point (i). Since the coefficients $v \mapsto b(f_s, v)$ and $v \mapsto a^{1/2}(f_s, v)$ are locally Lipschitz continuous (uniformly on compact time intervals) by Lemma 9 and because $f \in L^\infty_{\mathrm{loc}}([0, \infty), \mathscr{P}_{2+\gamma}(\mathbb{R}^3))$ by assumption, the SDE under study classically has a pathwise unique *local* solution $(V_t)_{t \in [0, \tau)}$, where $\tau = \sup_{k \geq 1} \tau_k$ and $\tau_k = \inf\{t \geq 0 : |V_t| \geq k\}$. We thus only have to verify that $\tau = \infty$ a.s. Using the Itô formula and taking expectations, one easily checks that for all $k \geq 1$, all $t \geq 0$, $\mathbb{E}[|V_{t \wedge \tau_k}|^2] = \mathbb{E}[|V_0|^2] + \mathbb{E}[\int_0^{t \wedge \tau_k} \kappa(s, V_s)ds]$, where $\kappa(s, v) = 2x \cdot b(f_s, v) + \mathrm{Tr}\, a(f_s, v)$. Recalling that $b(v) = -2|v|^\gamma v$ and $\mathrm{Tr}\, a(v) = 2|v|^{2+\gamma}$, we find that $\kappa(s, v) = 2 \int_{\mathbb{R}^3} |v - w|^\gamma (|w|^2 - |v|^2) f_s(dw) \leq 2 \int_{\mathbb{R}^3} |v - w|^\gamma |w|^2 f_s(dw)$. It is not hard to deduce that $\kappa(s, v) \leq C(1 + m_{2+\gamma}(f_s))(1 + |v|^2)$ and then that

$$\mathbb{E}[|V_{t \wedge \tau_k}|^2] \leq m_2(g_0) + C \int_0^t (1 + m_{2+\gamma}(f_s))\mathbb{E}[1 + |V_{s \wedge \tau_k}|^2]ds$$

for all $t \geq 0$ and all $k \geq 1$. Since $m_{2+\gamma}(f_s)$ is locally bounded by assumption, the Gronwall lemma implies that for all $T > 0$, $C_T := \sup_{k \geq 1} \sup_{[0, T]} \mathbb{E}[|V_{t \wedge \tau_k}|^2] < \infty$. Hence for all T, $\mathrm{Pr}(\tau_k \leq T) = k^{-2}\mathbb{E}[|V_{\tau_k}|^2 \mathbb{I}_{\tau_k \leq T}] \leq \mathbb{E}[|V_{T \wedge \tau_k}|^2] \leq C_T k^{-2} \to 0$ as $k \to \infty$. We conclude that $\tau = \infty$ a.s.

We now prove (ii). For $t \geq 0$ and $\varphi \in C_b^2(\mathbb{R}^3)$, we introduce

$$\mathscr{A}_t\varphi(v) = \int_{\mathbb{R}^3} L\varphi(v, v_*)f_t(dv_*) = (1/2)\sum_{k,l=1}^{3} a_{kl}(f_t, v)\partial_{kl}^2\varphi(v) + \sum_{k=1}^{3} b_k(f_t, v)\partial_k\varphi(v).$$

Then $g_t = \mathscr{L}(V_t)$ solves

$$(5) \qquad \int_{\mathbb{R}^3} \varphi(v)g_t(dv) = \int_{\mathbb{R}^3} \varphi(v)\mu(dv) + \int_0^t \int_{\mathbb{R}^3} \mathscr{A}_s\varphi(v)g_s(dv)ds \quad \text{for all } \varphi \in C_c^2(\mathbb{R}^3),$$

with $\mu = g_0$. But $(f_t)_{t \geq 0}$, being a weak solution to (1), also solves (5) with $\mu = f_0$. Horowitz and Karandikar [21, Theorem B.1], who generalize Ethier and Kurtz [12, Chapter 4, Theorem 7.1], tell us that (5) has a unique solution (for any given $\mu \in \mathscr{P}(\mathbb{R}^3)$). Since $f_0 = g_0$ by assumption, we thus have $(f_t)_{t \geq 0} = (g_t)_{t \geq 0}$.

To apply [21, Theorem B.1], we need to verify the following conditions:

(a) $C_c^2(\mathbb{R}^3)$ is dense in $C_0(\mathbb{R}^3)$ (the set of continuous functions vanishing at infinity) for the uniform convergence;

(b) for each $\varphi \in C_c^2(\mathbb{R}^3)$, $(t, v) \mapsto \mathscr{A}_t\varphi(v)$ is measurable;

(c) for each $t \geq 0$, if $\varphi \in C_c^2$ attains its maximum at v_0, then $\mathscr{A}_t\varphi(v_0) \leq 0$;

(d) there is a countable family $\{\varphi_k\}_{k \geq 1} \subset C_c(\mathbb{R}^3)$ such that for all $t \geq 0$, $\{(\varphi_k, \mathscr{A}_t\varphi_k)\}_{k \geq 1}$ is dense in $\{(\varphi, \mathscr{A}_t\varphi), \varphi \in C_c^2(\mathbb{R}^3)\}$ for the bounded-pointwise convergence;

(e) for any deterministic $(t_0, v_0) \in [0, \infty) \times \mathbb{R}^3$, there exists a unique (in law) continuous $\mathbb{R}^3$-valued process $(X_t)_{t \geq t_0}$ such that $X_{t_0} = v_0$ and for all $\varphi \in C_c^2(\mathbb{R}^3)$, the process $\varphi(X_t) - \int_{t_0}^t \mathscr{A}_s\varphi(X_s)ds$ is a martingale.

Points (a) and (b) are obvious, as well as point (c) (simply because $\nabla\varphi(v_0) = 0$, because the Hessian $(\partial_{kl}\varphi(v_0))_{kl}$ is non-positive and because $a(f_t, v_0)$ is nonnegative). Point (e) is equivalent to the existence and uniqueness in law, for each $(t_0, v_0) \in [0, \infty) \times \mathbb{R}^3$, for the SDE $V_t = v_0 + \int_{t_0}^t b(f_s, V_s)ds + \int_{t_0}^t a^{1/2}(f_s, V_s)dB_s$. If $t_0 = 0$, this follows from point (i) (choose $g_0 = \delta_{v_0}$). The generalization to all positive values of t_0 is clearly not an issue. For (d), consider a countable family $\{\varphi_k\}_{k\geq 1} \subset C_c(\mathbb{R}^3)$ so that for any $\varphi \in C_c^2(\mathbb{R}^3)$ with, say Supp $\varphi \subset B(0, R)$, there is a subsequence $(k_n)_{n\geq 1}$ so that Supp $\varphi_{k_n} \subset B(0, 2R)$ and $\lim_n[|\varphi_{k_n} - \varphi|_\infty + |\nabla\varphi_{k_n} - \nabla\varphi|_\infty + |D^2\varphi_{k_n} - D^2\varphi|_\infty] = 0$. Then for each $t \geq 0$, we clearly have $\lim_n \|\mathscr{A}_t\varphi_{k_n} - \mathscr{A}_t\varphi\|_\infty = 0$. $\qquad\square$

5. A central inequality

As already explained in Subsection 1.7, our uniqueness, stability and propagation of chaos results are based on some coupling between SDEs, and using similar Brownian motions is not sufficient to our purposes. We recall the following fact: the best coupling between two multidimensional Gaussian distributions $\mathcal{N}(0, \Sigma_1)$ and $\mathcal{N}(0, \Sigma_2)$ does not, in general, consist in setting $X_1 = \Sigma_1^{1/2}Y$ and $X_2 = \Sigma_2^{1/2}Y$ for the same Y with law $\mathcal{N}(0, \mathbf{I}_3)$. As shown by Givens and Shortt [19], the optimal coupling is obtained when setting $X_1 = \Sigma_1^{1/2}Y$ and $X_2 = \Sigma_2^{1/2}U(\Sigma_1, \Sigma_2)Y$, where

$$(6) \qquad U(\Sigma_1, \Sigma_2) = \Sigma_2^{-1/2}\Sigma_1^{-1/2}(\Sigma_1^{1/2}\Sigma_2\Sigma_1^{1/2})^{1/2}$$

is an orthogonal matrix. Point (i) below, proved in [17], is an immediate consequence of [19].

LEMMA 11. – (i) *Let m be a probability measure on some measurable space F, consider a pair of measurable families of 3×3 matrices $(\sigma_1(x))_{x \in F}$ and $(\sigma_2(x))_{x \in F}$ and set $\Sigma_i = \int_F \sigma_i(x)\sigma_i^*(x)m(dx)$. If Σ_1 and Σ_2 are invertible,*

$$\left\|\Sigma_1^{1/2} - \Sigma_2^{1/2}U(\Sigma_1, \Sigma_2)\right\|^2 \leq \int_F \|\sigma_1(x) - \sigma_2(x)\|^2 m(dx).$$

(ii) *Let $\varepsilon \in (0, 1)$. With the same notation as in (i) but without assuming that Σ_1 and Σ_2 are invertible, setting $U_\varepsilon(\Sigma_1, \Sigma_2) = U(\Sigma_1 + \varepsilon\mathbf{I}_3, \Sigma_2 + \varepsilon\mathbf{I}_3)$,*

$$\left\|\Sigma_1^{1/2} - \Sigma_2^{1/2}U_\varepsilon(\Sigma_1, \Sigma_2)\right\|^2 \leq C\sqrt{\varepsilon}(1 + \|\Sigma_1 + \Sigma_2\|^{1/2}) + \int_F \|\sigma_1(x) - \sigma_2(x)\|^2 m(dx),$$

where C is a universal constant.

(iii) *For each $\varepsilon \in (0, 1)$, the map $(\Sigma_1, \Sigma_2) \mapsto U_\varepsilon(\Sigma_1, \Sigma_2)$ is locally Lipschitz continuous on $S_3^+ \times S_3^+$.*

Of course, we introduced U_ε to avoid some technical problems, because we will generally not be able to control the invertibility of the matrices we will study.

Proof. – Point (i) is nothing but [17, Lemma 3.1] and point (iii) is obvious. To check (ii), we introduce the space $F' = F \cup \{\Delta\}$ (where $\Delta \notin F$ is some abstract point), the probability

measure $m' = (1-\varepsilon)\mathbb{I}_F m + \varepsilon\delta_\Delta$ on F', and the maps $\sigma_i' = (1-\varepsilon)^{-1/2}\sigma_i\,\mathbb{I}_F + \mathbf{I}_3\,\mathbb{I}_{\{\Delta\}}$ from F' to $\mathcal{M}_{3\times3}(\mathbb{R})$. It holds that $\int_{F'}\sigma_i'(\sigma_i')^*dm' = \Sigma_i + \varepsilon\mathbf{I}_3$, so that point (i) yields

$$\left\|(\Sigma_1+\varepsilon\mathbf{I}_3)^{1/2}-(\Sigma_2+\varepsilon\mathbf{I}_3)^{1/2}U_\varepsilon(\Sigma_1,\Sigma_2)\right\|^2$$
$$\leq \int_{F'}\|\sigma_1'(x)-\sigma_2'(x)\|^2 m'(dx) = \int_F \|\sigma_1(x)-\sigma_2(x)\|^2 m(dx).$$

It then easily follows, using that $U_\varepsilon(\Sigma_1,\Sigma_2)$ is orthogonal (whence $\|U_\varepsilon(\Sigma_1,\Sigma_2)\|^2 = \mathrm{Tr}\,\mathbf{I}_3 = 3$) and Lemma 6 (which gives $\|(\Sigma_i+\varepsilon\mathbf{I}_3)^{1/2}-\Sigma_i^{1/2}\| \leq C\sqrt{\varepsilon}$), that

$$\left\|\Sigma_1^{1/2}-\Sigma_2^{1/2}U_\varepsilon(\Sigma_1,\Sigma_2)\right\| \leq C\sqrt{\varepsilon} + \left\|(\Sigma_1+\varepsilon\mathbf{I}_3)^{1/2}-(\Sigma_2+\varepsilon\mathbf{I}_3)^{1/2}U_\varepsilon(\Sigma_1,\Sigma_2)\right\|$$
$$\leq C\sqrt{\varepsilon} + \left(\int_F \|\sigma_1(x)-\sigma_2(x)\|^2 m(dx)\right)^{1/2}.$$

The conclusion follows: it suffices to take squares and to note that $\int_F \|\sigma_1(x)-\sigma_2(x)\|^2 m(dx) \leq 2\int_F(\|\sigma_1(x)\|^2+\|\sigma_2(x)\|^2)m(dx) = 2\mathrm{Tr}(\Sigma_1+\Sigma_2) \leq C\|\Sigma_1+\Sigma_2\|$. $\qquad\square$

The following proposition, to be used several times for both uniqueness and propagation of chaos, plays a central role in the paper. The ε present in the statement is here only for technical reasons and may be disregarded at first read.

PROPOSITION 12. – *Let* $\gamma \in [0,1]$ *be fixed, let* $f,g \in \mathscr{P}_{2+\gamma}(\mathbb{R}^3)$ *and* $R \in \mathscr{H}(f,g)$. *For* $\varepsilon \in (0,1)$, *let*

$$\Gamma_\varepsilon(R) = \int_{\mathbb{R}^3\times\mathbb{R}^3}\left(\|a^{1/2}(f,v)-a^{1/2}(g,w)U_\varepsilon(a(f,v),a(g,w))\|^2\right.$$
$$\left. + 2(v-w)\cdot(b(f,v)-b(g,w))\right)R(dv,dw).$$

(i) *If* $\gamma = 0$, *there is a universal constant* C *such that* $\Gamma_\varepsilon(R) \leq C\sqrt{\varepsilon}(1+m_2(f+g))^{1/2}$.

(ii) *If* $\gamma \in (0,1]$, *then we fix* $\alpha > \gamma$. *There are some constants* $\kappa > 0$ *and* C *depending only on* γ,α, *such that for all* $M > 0$,

$$\Gamma_\varepsilon(R) \leq C\sqrt{\varepsilon}(1+m_{2+\gamma}(f+g))^{1/2}$$
$$+ M\int_{\mathbb{R}^3\times\mathbb{R}^3}|v-w|^2 R(dv,dw) + C(1+m_{2+\gamma}(g)+\mathcal{E}_\alpha(f))e^{-\kappa M^{\alpha/\gamma}}.$$

As already mentioned, it is important that no exponential moment of g is required in (ii).

Proof. – We thus fix $\gamma \in [0,1]$, $f,g \in \mathscr{P}_{2+\gamma}(\mathbb{R}^3)$, $R \in \mathscr{H}(f,g)$ and $\varepsilon \in (0,1)$.

Step 1. We first verify that for all $x,y \in \mathbb{R}^3$,

$$\|\sigma(x)-\sigma(y)\|^2 \leq 2|x|^{2+\gamma}+2|y|^{2+\gamma}-4(|x||y|)^{\gamma/2}(x\cdot y).$$

Recall that $\sigma(x) = |x|^{1+\gamma/2}\Pi_{x\perp}$ and that $\|\sigma(x)\|^2 = \mathrm{Tr}\,a(x) = 2|x|^{2+\gamma}$: we have to check that $\langle\!\langle\sigma(x),\sigma(y)\rangle\!\rangle \geq 2(|x||y|)^{\gamma/2}(x\cdot y)$, *i.e.,* that $\langle\!\langle\Pi_{x\perp},\Pi_{y\perp}\rangle\!\rangle \geq 2(x\cdot y)/(|x||y|)$. A computation shows that $\Pi_{x\perp}\Pi_{y\perp} = \mathbf{I}_3 - |x|^{-2}xx^* - |y|^{-2}yy^* + (x.y)|x|^{-2}|y|^{-2}xy^*$ and thus $\langle\!\langle\Pi_{x\perp},\Pi_{y\perp}\rangle\!\rangle = \mathrm{Tr}\,\Pi_{x\perp}\Pi_{y\perp} = 1 + (x\cdot y)^2/(|x|^2|y|^2)$. The conclusion follows.

Step 2. We fix v and w and we apply Lemma 11-(ii) with $F = \mathbb{R}^3\times\mathbb{R}^3$, with $m = R(dy,dz)$, with $\sigma_1(y,z) = \sigma(v-y)$ and $\sigma_2(y,z) = \sigma(w-z)$. It holds that

$\int_F \sigma_1 \sigma_1^* dm = \int_{\mathbb{R}^3 \times \mathbb{R}^3} a(v-y)R(dy,dz) = a(f,v)$ (because $\sigma(x)\sigma^*(x) = a(x)$ and $R \in \mathscr{H}(f,g)$) and $\int_F \sigma_2\sigma_2^* dm = a(g,w)$. We thus find

$$\|a^{1/2}(f,v) - a^{1/2}(g,w)U_\varepsilon(a(f,v),a(g,w))\|^2 \leq C\sqrt{\varepsilon}(1 + \|a(f,v) + a(g,w)\|)^{1/2}$$
$$+ \int_{\mathbb{R}^3 \times \mathbb{R}^3} \|\sigma(v-y) - \sigma(w-z)\|^2 R(dy,dz).$$

Next, it holds that $b(f,v) - b(g,w) = \int_{\mathbb{R}^3 \times \mathbb{R}^3}(b(v-y) - b(w-z))R(dy,dz)$ simply because $R \in \mathscr{H}(f,g)$. Recalling finally that $\|a(f,v)\| \leq C(m_{2+\gamma}(f) + |v|^{2+\gamma})$ by Lemma 9, we get

$$\Gamma_\varepsilon(R) \leq C\sqrt{\varepsilon} \int_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{2+\gamma} + |w|^{2+\gamma} + m_{2+\gamma}(f+g))^{1/2} R(dv,dw)$$
$$+ \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \Delta(v,y,w,z)R(dy,dz)R(dv,dw)$$
$$\leq C\sqrt{\varepsilon}(1 + m_{2+\gamma}(f+g))^{1/2} + \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \Delta(v,y,w,z)R(dy,dz)R(dv,dw)$$

where

$$\Delta(v,y,w,z) = \|\sigma(v-y) - \sigma(w-z)\|^2 + 2(v-w) \cdot (b(v-y) - b(w-z)).$$

Step 3. The goal of this step is to check that $\Delta(v,y,w,z) = \Delta_1(v,y,w,z) + \Delta_2(v,y,w,z)$, where

$$\Delta_1(v,y,w,z) = (v - w + y - z) \cdot (b(v-y) - b(w-z))$$

is antisymmetric (i.e., $\Delta_1(y,v,z,w) = -\Delta_1(v,y,w,z)$) and where

$$\Delta_2(v,y,w,z) \leq \begin{cases} 0 & \text{if } \gamma = 0, \\ 4(|v-w|^2 + |y-z|^2)|v-y|^\gamma & \text{if } \gamma \in (0,1]. \end{cases}$$

We introduce the shortened notation $\Delta_2 = \Delta_2(v,y,w,z)$, $X = v - y$ and $Y = w - z$. By definition, we have $\Delta_2 = (X - Y) \cdot (b(X) - b(Y)) + \|\sigma(X) - \sigma(Y)\|^2$. Using that $b(X) = -2|X|^\gamma X$ and Step 1, we find

$$\Delta_2 \leq -2(X-Y) \cdot (|X|^\gamma X - |Y|^\gamma Y) + 2|X|^{2+\gamma} + 2|Y|^{2+\gamma} - 4(|X||Y|)^{\gamma/2}(X \cdot Y)$$
$$= 2(X \cdot Y)(|X|^{\gamma/2} - |Y|^{\gamma/2})^2.$$

If $\gamma = 0$, this gives $\Delta_2 \leq 0$. If now $\gamma \in (0,1]$, we use that $(x \vee y)^{1-\gamma/2}|x^{\gamma/2} - y^{\gamma/2}| \leq |x-y|$ (for $x, y \geq 0$) to write

$$\Delta_2 \leq 2|X||Y|(|X|^{\gamma/2} - |Y|^{\gamma/2})^2 \leq 2|X||Y|(|X| \vee |Y|)^{\gamma-2}(|X| - |Y|)^2 \leq 2(|X| \wedge |Y|)^\gamma |X - Y|^2.$$

We conclude noting that $(|X| \wedge |Y|)^\gamma \leq |X|^\gamma = |v-y|^\gamma$ and $|X-Y|^2 \leq 2(|v-w|^2 + |y-z|^2)$.

Step 4. We now observe that $L := \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \Delta_1(v,y,w,z)R(dy,dz)R(dv,dw) = 0$. Indeed, Δ_1 being antisymmetric, we have

$$L = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \Delta_1(y,v,z,w)R(dy,dz)R(dv,dw) = -L.$$

Step 5. When $\gamma = 0$, it suffices to gather Steps 2, 3, 4 to conclude the proof.

Step 6. Finally, gathering Steps 2, 3, 4 when $\gamma \in (0, 1]$ yields

$$\Gamma_\varepsilon(R) \le C \sqrt{\varepsilon}(1 + m_{2+\gamma}(f + g))^{1/2}$$

$$+ 4 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} (|v - w|^2 + |y - z|^2)|v - y|^\gamma R(dy, dz)R(dv, dw)$$

$$= C \sqrt{\varepsilon}(1 + m_{2+\gamma}(f + g))^{1/2}$$

$$+ 8 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - w|^2 |v - y|^\gamma f(dy)R(dv, dw).$$

For the last equality, we used a symmetry argument and that the first marginal of R is f. Finally, we recall that $\alpha > \gamma$ is fixed and we write, for any $M > 0$,

$$8 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} |v - w|^2 |v - y|^\gamma f(dy)R(dv, dw) \le M \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - w|^2 R(dv, dw) + I_M,$$

where

$$I_M = 8 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} |v - w|^2 |v - y|^\gamma \mathbb{1}_{\{8|v-y|^\gamma \ge M\}} f(dy)R(dv, dw)$$

$$\le 16 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} (|v|^2 + |w|^2)(|v|^\gamma + |y|^\gamma)[\mathbb{1}_{\{|v|^\gamma \ge M/16\}} + \mathbb{1}_{\{|y|^\gamma \ge M/16\}}] f(dy)R(dv, dw).$$

We then write, for $a > 0$ to be chosen later,

$$I_M \le 16 e^{-a(M/16)^{\alpha/\gamma}} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} (|v|^2 + |w|^2)(|v|^\gamma + |y|^\gamma)[e^{a|v|^\alpha} + e^{a|y|^\alpha}] f(dy)R(dv, dw)$$

$$\le C e^{-a(M/16)^{\alpha/\gamma}} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |w|^2)[e^{2a|v|^\alpha} + e^{2a|y|^\alpha}] f(dy)R(dv, dw)$$

$$\le C e^{-a(M/16)^{\alpha/\gamma}} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3} \left(1 + |w|^{2+\gamma} + e^{\frac{4+2\gamma}{1+\gamma}a|v|^\alpha} + e^{\frac{4+2\gamma}{1+\gamma}a|y|^\alpha}\right) f(dy)R(dv, dw)$$

by the Young inequality. Choosing $a = (1 + \gamma)/(4 + 2\gamma)$, setting $\kappa = a/16^{\alpha/\gamma}$ and using that $R \in \mathcal{H}(f, g)$, we conclude that

$$I_M \le C e^{-\kappa M^{\alpha/\gamma}}(1 + m_{2+\gamma}(g) + \mathcal{E}_\alpha(f))$$

as desired.   □

6. Well-posedness

We now have all the weapons to give the

Proof of Theorem 2. – We fix $\gamma \in [0, 1]$. If $\gamma = 0$, we assume that $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ and consider the weak solution $(f_t)_{t \ge 0}$ to (1) built in Proposition 5, which indeed satisfies all the properties of the statement. If $\gamma \in (0, 1]$, we assume that $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ satisfies $\mathcal{E}_\alpha(f_0) < \infty$ for some $\alpha \in (\gamma, 2)$ and consider the weak solution $(f_t)_{t \ge 0}$ to (1) built in Proposition 5, which also satisfies all the properties of the statement. In particular, $\sup_{t \ge 0} \mathcal{E}_\alpha(f_t) < \infty$ depends only on γ, α and on (an upper bound of) $\mathcal{E}_\alpha(f_0)$. We consider another weak solution $(g_t)_{t \ge 0}$ to (1), only assumed to lie in $L^\infty_{\text{loc}}([0, \infty), \mathcal{P}_{2+\gamma}(\mathbb{R}^3))$.

Step 1. We consider $V_0 \sim f_0$ and $W_0 \sim g_0$ such that $\mathbb{E}[|V_0 - W_0|^2] = \mathcal{W}_2^2(f_0, g_0)$ and a 3D Brownian motion $(B_t)_{t \geq 0}$, independent of (V_0, W_0). We consider the pathwise unique solution to

$$V_t = V_0 + \int_0^t b(f_s, V_s)ds + \int_0^t a^{1/2}(f_s, V_s)dB_s,$$

see Proposition 10, and we know that $V_t \sim f_t$ for all $t \geq 0$. Next, we recall that the matrix U_ε was introduced in Lemma 11-(ii) and is locally Lipschitz continuous, so that it is not difficult to verify, as in the proof of Proposition 10-(i), that the SDE (with stochastic parameter $(V_s)_{s \geq 0}$)

$$(7) \qquad W_t^\varepsilon = W_0 + \int_0^t b(g_s, W_s^\varepsilon)ds + \int_0^t a^{1/2}(g_s, W_s^\varepsilon)U_\varepsilon(a(f_s, V_s), a(g_s, W_s^\varepsilon))dB_s$$

has a pathwise unique local solution. But the matrix $U_\varepsilon(a(f_s, V_s), a(g_s, W_s^\varepsilon))$ being a.s. orthogonal for all $s \geq 0$, the process $B_t^\varepsilon = \int_0^t U_\varepsilon(a(f_s, V_s), a(g_s, W_s^\varepsilon))dB_s$ is a 3D Brownian motion. We conclude that the SDE (7) is, in law, equivalent to the SDE $W_t = W_0 + \int_0^t b(g_s, W_s)ds + \int_0^t a^{1/2}(g_s, W_s)dB_s$. We know from Proposition 10-(i) that such a process does not explode in finite time, so that the unique solution to (7) is global, and from Proposition 10-(ii) that $W_t^\varepsilon \sim g_t$ for all $t \geq 0$. Consequently, we have $\mathcal{W}_2^2(f_t, g_t) \leq \mathbb{E}[|V_t - W_t^\varepsilon|^2]$ for all values of $t \geq 0$ and $\varepsilon \in (0, 1)$.

Step 2. We set $u_t^\varepsilon = \mathbb{E}[|V_t - W_t^\varepsilon|^2]$. By the Itô formula, we find, for some martingale $(M_t^\varepsilon)_{t \geq 0}$,

$$|V_t - W_t^\varepsilon|^2 = |V_0 - W_0|^2 + \int_0^t 2(V_s - W_s^\varepsilon) \cdot (b(f_s, V_s) - b(g_s, W_s^\varepsilon))ds + M_t^\varepsilon$$

$$+ \int_0^t \mathrm{Tr}\Big(\big[a^{1/2}(f_s, V_s) - a^{1/2}(g_s, W_s^\varepsilon)U_\varepsilon(a(f_s, V_s), a(g_s, W_s^\varepsilon))\big]$$

$$\big[a^{1/2}(f_s, V_s) - a^{1/2}(g_s, W_s^\varepsilon)U_\varepsilon(a(f_s, V_s), a(g_s, W_s^\varepsilon))\big]^*\Big)ds.$$

Recalling that $\|A\|^2 = \mathrm{Tr}[AA^*]$, taking expectations and differentiating the obtained expression with respect to time, we find

$$\frac{d}{dt}u_t^\varepsilon = \mathbb{E}\Big[\|a^{1/2}(f_t, V_t) - a^{1/2}(g_t, W_t^\varepsilon)U_\varepsilon(a(f_t, V_t), a(g_t, W_t^\varepsilon))\|^2$$

$$+ 2(V_t - W_t^\varepsilon) \cdot (b(f_t, V_t) - b(g_t, W_t^\varepsilon))\Big].$$

Denoting by $R_t^\varepsilon \in \mathcal{P}_2(\mathbb{R}^3 \times \mathbb{R}^3)$ the law of (V_t, W_t^ε) and recalling the notation of Proposition 12, we realize that $(d/dt)u_t^\varepsilon = \Gamma_\varepsilon(R_t^\varepsilon)$.

Assume first that $\gamma = 0$. Then Prop. 12 tells us that $(d/dt)u_t^\varepsilon \leq C\sqrt{\varepsilon}(1 + m_2(f_t + g_t))^{1/2}$. Recalling that $f, g \in L_{\mathrm{loc}}^\infty([0, \infty), \mathcal{P}_2(\mathbb{R}^3))$, that $\mathcal{W}_2^2(f_t, g_t) \leq u_t^\varepsilon$ for all $t \geq 0$ and all $\varepsilon \in (0, 1)$, and that $\mathbb{E}[|V_0 - W_0|^2] = \mathcal{W}_2^2(f_0, g_0)$ by construction, we easily deduce that $\mathcal{W}_2^2(f_t, g_t) \leq \mathcal{W}_2^2(f_0, g_0)$ for all $t \geq 0$. Of course, the uniqueness of the weak solution starting from f_0 follows.

When $\gamma \in (0, 1]$, we work on $[0, T]$ for some fixed $T > 0$. By Proposition 12, for all $M > 0$,

$$\frac{d}{dt}u_t^\varepsilon \leq C\sqrt{\varepsilon}(1 + m_{2+\gamma}(f_t + g_t))^{1/2} + Mu_t^\varepsilon + C(1 + m_{2+\gamma}(g_t) + \mathcal{E}_\alpha(f_t))e^{-\kappa M^{\alpha/\gamma}}.$$

For the rest of the step, we call C_T a constant, allowed to vary from line to line, depending only on T, α, γ and on (some upper bounds) of $\sup_{[0,T]} m_{2+\gamma}(g_t)$ and $\mathcal{E}_\alpha(f_0)$. We thus have

$$\frac{d}{dt} u_t^\varepsilon \leq C_T \sqrt{\varepsilon} + M u_t^\varepsilon + C_T e^{-\kappa M^{\alpha/\gamma}},$$

whence $\sup_{[0,T]} u_t^\varepsilon \leq [u_0^\varepsilon + C_T \sqrt{\varepsilon} + C_T e^{-\kappa M^{\alpha/\gamma}}] e^{MT}$. Recalling that $u_0^\varepsilon = \mathcal{W}_2^2(f_0, g_0)$ and that $\mathcal{W}_2^2(f_t, g_t) \leq u_t^\varepsilon$, we may let $\varepsilon \to 0$ and find that

$$\sup_{[0,T]} \mathcal{W}_2^2(f_t, g_t) \leq [\mathcal{W}_2^2(f_0, g_0) + C_T e^{-\kappa M^{\alpha/\gamma}}] e^{MT}.$$

We now choose $M = [\kappa^{-1} \log(1 + 1/\mathcal{W}_2^2(f_0, g_0))]^{\gamma/\alpha}$, which is designed to satisfy $e^{-\kappa M^{\alpha/\gamma}} = \mathcal{W}_2^2(f_0, g_0)/(1 + \mathcal{W}_2^2(f_0, g_0)) \leq \mathcal{W}_2^2(f_0, g_0)$ and we end with

$$\sup_{[0,T]} \mathcal{W}_2^2(f_t, g_t) \leq C_T \mathcal{W}_2^2(f_0, g_0) \exp(T(\kappa^{-1} \log(1 + 1/\mathcal{W}_2^2(f_0, g_0))^{\gamma/\alpha}).$$

We easily conclude, since $\alpha > \gamma$, that for any $\eta \in (0, 1)$, $\sup_{[0,T]} \mathcal{W}_2(f_t, g_t) \leq C_{\eta,T}(\mathcal{W}_2(f_0, g_0))^{1-\eta}$, the constant $C_{\eta,T}$ depending only on η, T, α and on (some upper bounds) of $\sup_{[0,T]} m_{2+\gamma}(g_t)$ and $\mathcal{E}_\alpha(f_0)$. The uniqueness of the weak solution $(f_t)_{t\geq 0}$ starting from f_0 clearly follows. $\qquad \square$

7. Moments of the particle system

The goal of this section is to study the moments of the particle system. The following result uses the fact that the particle system a.s. conserves kinetic energy. Sznitman [29] and Mischler-Mouhot [25] have handled similar computations for the Boltzmann equation for hard spheres.

PROPOSITION 13. – *Fix* $\gamma \in [0, 1]$, $N \geq 2$, *consider an exchangeable* $(\mathbb{R}^3)^N$-*valued random variable* $(V_0^{i,N})_{i=1,...,N}$ *and the corresponding unique solution* $(V_t^{i,N})_{t\geq 0}$ *to* (3). *Then for all* $p > 2$, $\sup_{[0,\infty)} \mathbb{E}[|V_t^{1,N}|^p] \leq C_p (\mathbb{E}[|V_0^{1,N}|^{p+\gamma}])^{p/(p+\gamma)}$, *the constant* C_p *depending only on* p *and* γ.

Proof. – We fix $N \geq 2$ and write $V_t^i = V_t^{i,N}$ for simplicity. We recall from Proposition 3 that a.s., for all $t \geq 0$, $E_t^N := N^{-1} \sum_1^N |V_t^i|^2 = E_0^N$. We fix $p > 2$ and we set $u_t^p = \mathbb{E}[|V_t^1|^p]$.

Step 1. Starting from (3) and applying the Itô formula with $\phi(v) = |v|^p$, for which $\partial_k \phi(v) = p|v|^{p-2} v_k$ and $\partial_{kl} \phi(v) = p[\mathbb{I}_{\{k=l\}}|v|^{p-2} + (p-2)v_k v_l |v|^{p-4}]$, we get, for some martingale $(M_t^p)_{t\geq 0}$,

$$|V_t^1|^p = |V_0^1|^p + \frac{p}{N} \sum_{j=1}^N \int_0^t |V_s^1|^{p-2} V_s^1 \cdot b(V_s^1 - V_s^j) ds + M_t^p$$

$$+ \frac{p}{2N} \sum_{j=1}^N \int_0^t \sum_{k,l=1}^3 [\mathbb{I}_{\{k=l\}}|V_s^1|^{p-2} + (p-2)(V_s^1)_k (V_s^1)_l |V_s^1|^{p-4}] a_{kl}(V_s^1 - V_s^j) ds.$$

Taking expectations and differentiating the obtained expression with respect to time, we find

$$\frac{d}{dt}u_t^p = \frac{p}{2N}\sum_{j=1}^N \mathbb{E}\Big[2|V_t^1|^{p-2}V_t^1 \cdot b(V_t^1 - V_t^j)$$

$$+ |V_t^1|^{p-2}\mathrm{Tr}\, a(V_t^1 - V_t^j) + (p-2)|V_t^1|^{p-4}a(V_t^1 - V_t^j)V_t^1 \cdot V_t^1\Big].$$

Recalling (4), using exchangeability and that everything vanishes when $j = 1$, we find

$$\frac{d}{dt}u_t^p = \frac{p(N-1)}{2N}\mathbb{E}\Big[|V_t^1 - V_t^2|^\gamma \Big(-2|V_t^1|^p + 2|V_t^1|^{p-2}|V_t^2|^2$$

$$+ (p-2)|V_t^1|^{p-4}(|V_t^1|^2|V_t^2|^2 - (V_t^1 \cdot V_t^2)^2)\Big)\Big]$$

$$\leq \frac{p(N-1)}{N}\mathbb{E}\Big[|V_t^1 - V_t^2|^\gamma \Big(-|V_t^1|^p + \frac{p}{2}|V_t^1|^{p-2}|V_t^2|^2\Big)\Big].$$

Step 2. When $\gamma = 0$, we thus have $(d/dt)u_t^p \leq -(p/2)u_t^p + (p^2/2)\mathbb{E}[|V_t^1|^{p-2}|V_t^2|^2]$. We used that $1/2 \leq (N-1)/N \leq 1$ because $N \geq 2$. We then use exchangeability to write

$$\mathbb{E}[|V_t^1|^{p-2}|V_t^2|^2] = \mathbb{E}\Big[|V_t^1|^{p-2}\frac{1}{N-1}\sum_2^N |V_t^i|^2\Big]$$

$$\leq 2\mathbb{E}[|V_t^1|^{p-2}E_t^N] \leq 2(u_t^p)^{(p-2)/p}\mathbb{E}[(E_t^N)^{p/2}]^{2/p}.$$

But $\mathbb{E}[(E_t^N)^{p/2}] = \mathbb{E}[(E_0^N)^{p/2}] \leq \mathbb{E}[|V_0^1|^p]$ by Jensen's inequality and exchangeability. We end with

$$\frac{d}{dt}u_t^p \leq -\frac{p}{2}u_t^p + p^2\mathbb{E}[|V_0^1|^p]^{2/p}[u_t^p]^{1-2/p}.$$

We classically conclude that $\sup_{[0,\infty)} u_t^p \leq \max\{u_0^p, \mathbb{E}[|V_0^1|^p](2p)^{p/2}\} = \mathbb{E}[|V_0^1|^p](2p)^{p/2}$.

Step 3. We suppose next that $\gamma \in (0,1]$. We know from Desvillettes and Villani [10, Lemma 1] that there are some constants $\kappa_p > 0$ and C_p such that for all $x, y \geq 0$,

$$-x^p - y^p + \frac{p}{2}x^2y^{p-2} + \frac{p}{2}y^2x^{p-2} \leq -\kappa_p x^p + C_p(xy^{p-1} + yx^{p-1}).$$

We deduce, using exchangeability, that

$$\frac{d}{dt}u_t^p \leq \frac{p(N-1)}{2N}\mathbb{E}\Big[|V_t^1 - V_t^2|^\gamma \Big(-|V_t^1|^p - |V_t^2|^p + \frac{p}{2}|V_t^1|^2|V_t^2|^{p-2} + \frac{p}{2}|V_t^2|^2|V_t^1|^{p-2}\Big)\Big]$$

$$\leq \frac{p(N-1)}{2N}\mathbb{E}\Big[|V_t^1 - V_t^2|^\gamma \Big(-\kappa_p|V_t^1|^p + C_p|V_t^1||V_t^2|^{p-1} + C_p|V_t^2||V_t^1|^{p-1}\Big)\Big]$$

$$\leq \mathbb{E}\Big[|V_t^1 - V_t^2|^\gamma \Big(-\frac{p\kappa_p}{4}|V_t^1|^p + 2pC_p|V_t^1||V_t^2|^{p-1}\Big)\Big].$$

Changing now the values of $\kappa_p > 0$ and C_p (which still depend only on p) and using that $|v - w|^\gamma \geq \big||v| - |w|\big|^\gamma \geq |v|^\gamma - |w|^\gamma$ and $|v - w|^\gamma \leq |v|^\gamma + |w|^\gamma$, we easily find

$$\frac{d}{dt}u_t^p \leq -\kappa_p\mathbb{E}[|V_t^1|^{p+\gamma}] + C_p\mathbb{E}[|V_t^1|^p|V_t^2|^\gamma + |V_t^1|^{1+\gamma}|V_t^2|^{p-1} + |V_t^1||V_t^2|^{p-1+\gamma}]$$

$$\leq -\kappa_p\mathbb{E}[|V_t^1|^{p+\gamma}] + C_p\mathbb{E}[|V_t^1|^p|V_t^2|^\gamma + |V_t^1|^\gamma|V_t^2|^p].$$

But

$$\mathbb{E}[|V_t^1|^p|V_t^2|^\gamma] = \mathbb{E}\Big[|V_t^1|^p \frac{1}{N-1}\sum_2^N |V_t^i|^\gamma\Big] \le 2\mathbb{E}\Big[|V_t^1|^p \frac{1}{N}\sum_1^N |V_t^i|^\gamma\Big] \le 2\mathbb{E}\Big[|V_t^1|^p (E_t^N)^{\gamma/2}\Big].$$

By Hölder's inequality and since $E_t^N = E_0^N$, we deduce that

$$\mathbb{E}[|V_t^1|^p|V_t^2|^\gamma] \le 2\mathbb{E}[|V_t^1|^{p+\gamma}]^{p/(p+\gamma)}\mathbb{E}[(E_0^N)^{(p+\gamma)/2}]^{\gamma/(p+\gamma)}.$$

A last application of Hölder's inequality shows that $\mathbb{E}[(E_0^N)^{(p+\gamma)/2}] \le \mathbb{E}[|V_0^{1,N}|^{p+\gamma}]$, whence finally

$$\frac{d}{dt}u_t^p \le -\kappa_p\mathbb{E}[|V_t^1|^{p+\gamma}] + C_p\mathbb{E}[|V_0^{1,N}|^{p+\gamma}]^{\gamma/(p+\gamma)}\mathbb{E}[|V_t^1|^{p+\gamma}]^{p/(p+\gamma)}$$

$$\le -\frac{\kappa_p}{2}\mathbb{E}[|V_t^1|^{p+\gamma}] + C_p\mathbb{E}[|V_0^{1,N}|^{p+\gamma}]$$

(8) $$\le -\frac{\kappa_p}{2}(u_t^p)^{(p+\gamma)/p} + C_p\mathbb{E}[|V_0^{1,N}|^{p+\gamma}],$$

the value of C_p depending only on p, γ and changing from line to line. For the second inequality, we used that for $\kappa, a, x \ge 0$, $-\kappa x + ax^{p/(p+\gamma)} \le -(\kappa/2)x + (2/\kappa)^{p/\gamma}a^{(p+\gamma)/\gamma}$: it suffices to separate the cases $\kappa x \ge 2ax^{p/(p+\gamma)}$ and $\kappa x \le 2ax^{p/(p+\gamma)}$. We classically deduce from (8) that $\sup_{[0,\infty)} u_t^p \le \max\{u_0^p, (2C_p\mathbb{E}[|V_0^{1,N}|^{p+\gamma}]/\kappa_p)^{p/(p+\gamma)}\}$. Since $u_0^p = \mathbb{E}[|V_0^{1,N}|^p] \le \mathbb{E}[|V_0^{1,N}|^{p+\gamma}]^{p/(p+\gamma)}$, the proof is complete. $\square$

8. Propagation of chaos

The goal of this section is to check Theorem 4, except the time uniformity in the Maxwell case.

8.1. The setting

We consider, in the whole section, $\gamma \in [0,1]$ fixed and $f_0 \in \mathscr{P}_2(\mathbb{R}^3)$. If $\gamma \in (0,1]$, we assume moreover that $\mathscr{E}_\alpha(f_0) < \infty$ for some $\alpha \in (\gamma, 2)$. We denote by $(f_t)_{t\ge0}$ the unique solution to (1), as well as, for each $N \ge 2$, the unique solution $(V_t^{i,N})_{i=1,\dots,N,t\ge0}$ to (3) starting from a given exchangeable $(\mathbb{R}^3)^N$-valued $(V_0^{i,N})_{i=1,\dots,N}$. We suppose that $M_p := m_p(f_0) + \sup_N \mathbb{E}[|V_0^{1,N}|^p] < \infty$ and we conclude from Theorem 2 and Proposition 13 that for all $p \ge 2$, $\sup_{[0,\infty)} m_p(f_t) + \sup_{N\ge2}\sup_{[0,\infty)} \mathbb{E}[|V_t^{1,N}|^p] < \infty$ and depends only on γ, p and on (some upper bound of) $M_{p+\gamma}$. If $\gamma \in (0,1]$, we know that $\sup_{t\ge0}\mathscr{E}_\alpha(f_t) < \infty$. If finally $H(f_0) < \infty$, then $H(f_t) \le H(f_0)$ for all $t \ge 0$, whence, by Lemma 7,

(9) $$\sup_{t\ge0}\sup_{v\in\mathbb{R}^3}||(a(f_t,v))^{-1}|| < \infty$$

and depends only on γ and on (upper bounds of) $m_2(f_0)$ and $H(f_0)$.

In the whole section, we write C for a constant depending only on γ, α, on (upper bounds of) $\{M_p, p \ge 2\}$ and additionally on (an upper bound of) $\mathscr{E}_\alpha(f_0)$ if $\gamma \in (0,1]$. It is also allowed to depend on (an upper bound of) $H(f_0)$ when the latter is supposed to be finished. Finally, any other dependence will be indicated in subscript.

We fix $N \geq 2$ for the whole section, we recall that $\mu_t^N = N^{-1} \sum_1^N \delta_{V_t^{i,N}}$ and we put $\varepsilon_N = N^{-1}$. By [17, Proposition A.1], we can find $(W_0^{i,N})_{i=1,\ldots,N} \sim f_0^{\otimes N}$ such that

(a) $\{(V_0^{i,N}, W_0^{i,N}), i = 1, \ldots, N\}$ is exchangeable,

(b) $\mathcal{W}_2^2(N^{-1} \sum_1^N \delta_{V_0^{i,N}}, N^{-1} \sum_1^N \delta_{W_0^{i,N}}) = N^{-1} \sum_1^N |V_0^{i,N} - W_0^{i,N}|^2$ a.s.,

(c) denoting by F_N the law of $(V_0^{i,N})_{i=1,\ldots,N}$, $\mathcal{W}_2^2(F_N, f_0^{\otimes N}) = \mathbb{E}[\sum_1^N |V_0^{i,N} - W_0^{i,N}|^2]$.

8.2. A first coupling

We first rewrite suitably the particle system.

LEMMA 14. – *For each $i = 1, \ldots, N$, the process*

$$\beta_t^{i,N} = \frac{1}{\sqrt{N}} \sum_{j=1}^N \int_0^t [a^{1/2}(\mu_s^N, V_s^{i,N})]^{-1} \sigma(V_s^{i,N} - V_s^{j,N}) dB_s^{ij}$$

is a 3D Brownian motion. Furthermore, for all $i = 1, \ldots, N$, all $t \geq 0$,

$$V_t^{i,N} = V_0^i + \int_0^t b(\mu_s^N, V_s^{i,N}) ds + \int_0^t a^{1/2}(\mu_s^N, V_s^{i,N}) d\beta_s^{i,N}.$$

REMARK 15. – *Observe that $a^{1/2}(\mu_s^N, V_s^{i,N}) = [N^{-1} \sum_{j=1}^N a(V_s^{i,N} - V_s^{j,N})]^{1/2}$ with $a(x) = [\sigma(x)]^2$. If $a^{1/2}(\mu_s^N, V_s^{i,N})$ is not invertible, we use Lemma 16 to define $[a^{1/2}(\mu_s^N, V_s^{i,N})]^{-1} \sigma(V_s^{i,N} - V_s^{j,N})$. We thus always have*

(i) *for all $j = 1, \ldots, N$,*

$$a^{1/2}(\mu_s^N, V_s^{i,N})[a^{1/2}(\mu_s^N, V_s^{i,N})]^{-1} \sigma(V_s^{i,N} - V_s^{j,N}) = \sigma(V_s^{i,N} - V_s^{j,N});$$

(ii)

$$N^{-1} \sum_{j=1}^N ([a^{1/2}(\mu_s^N, V_s^{i,N})]^{-1} \sigma(V_s^{i,N} - V_s^{j,N}))([a^{1/2}(\mu_s^N, V_s^{i,N})]^{-1} \sigma(V_s^{i,N} - V_s^{j,N}))^* = \mathbf{I}_3.$$

LEMMA 16. – *For $A_1, \ldots, A_N \in S_3^+$ and $M = N^{-1} \sum_1^N A_j^2$, we can find some matrices $B_1, \ldots, B_N$ such that (a) $M^{1/2} B_j = A_j$ for all $j = 1, \ldots, N$ and (b) $N^{-1} \sum_1^N B_j B_j^* = \mathbf{I}_3$. We write $B_j = M^{-1/2} A_j$, even in the case where M is not invertible.*

Proof. – If M is invertible, it suffices to set $B_j = M^{-1/2} A_j$. If $M = 0$, the choice $B_j = \mathbf{I}_3$ is suitable. Assume now that M has exactly two non-trivial eigenvalues $\lambda_1, \lambda_2 > 0$ (the last case where M has exactly one non-trivial eigenvalue is treated similarly). Consider an orthonormal basis e_1, e_2, e_3 of eigenvectors, that is, $Me_1 = \lambda_1 e_1$, $Me_2 = \lambda_2 e_2$ and $Me_3 = 0$ (so that $M = \lambda_1 e_1 e_1^* + \lambda_2 e_2 e_2^*$) and observe that $A_j e_3 = 0$ for all j. It then suffices to set $B_j = (\lambda_1^{-1/2} e_1 e_1^* + \lambda_2^{-1/2} e_2 e_2^*) A_j + e_3 e_3^*$. $\qquad \square$

We can now give the

Proof of Lemma 14. – For i fixed, the Brownian motions $(B^{ij})_{j \neq i}$ are independent. Hence the (matrix) bracket of the 3D martingale $(\beta_t^{i,N})_{t \geq 0}$ is given by (recall that $\sigma(0) = 0$)

$$\frac{1}{N} \sum_{j=1}^{N} \int_0^t \Big([a^{1/2}(\mu_s^N, V_s^{i,N})]^{-1}\sigma(V_s^{i,N} - V_s^{j,N})\Big)\Big([a^{1/2}(\mu_s^N, V_s^{i,N})]^{-1}\sigma(V_s^{i,N} - V_s^{j,N})\Big)^* ds = \mathbf{I}_3 t,$$

which implies that $(\beta_t^{i,N})_{t \geq 0}$ is a Brownian motion. We used Remark 15-(ii). Rewriting (3) as in the statement is straightforward, using that

$$a^{1/2}(\mu_s^N, V_s^{i,N})[a^{1/2}(\mu_s^N, V_s^{i,N})]^{-1}\sigma(V_s^{i,N} - V_s^{j,N}) = \sigma(V_s^{i,N} - V_s^{j,N})$$

by Remark 15-(i).   □

We next introduce a (non-independent) family of Landau processes. Recall that the matrix U was introduced in (6), that U_ε was defined in Lemma 11-(ii). Denote $\varepsilon_N = N^{-1}$.

LEMMA 17. – *The system of equations (for $i = 1, \ldots, N$)*

$$W_t^{i,N} = W_0^{i,N} + \int_0^t b(f_s, W_s^{i,N})ds + \int_0^t a^{1/2}(f_s, W_s^{i,N})U_{\varepsilon_N}(a(\mu_s^N, V_s^{i,N}), a(v_s^N, W_s^{i,N}))d\beta_s^{i,N},$$

with $v_t^N = N^{-1}\sum_1^N \delta_{W_t^{i,N}}$, has a pathwise unique solution. Furthermore, $W_t^{1,N} \sim f_t$ for all $t \geq 0$ and the family $\{(V_t^{i,N}, W_t^{i,N})_{t \geq 0}, i = 1, \ldots, N\}$ is exchangeable.

Observe that the fine coupling is necessary to obtain some good results and this is not related to the fact that we deal with a conservative system. Using the naive coupling (without U), only Maxwell molecules can be dealt with, see e.g., [14], and even in this case, the constants increase exponentially fast with time. On the contrary, still in the Maxwell case, the constants increase polynomially (at most like $(1 + t)^{5/2}$) in all the lemmas below and this will easily be compensated by the fast and uniform (in N) convergence to equilibrium of the particle system.

Proof. – As usual, the existence of a pathwise unique local solution follows from the fact that the coefficients are locally Lipschitz continuous (which follows from Lemmas 9 and Lemma 11-(iii)). But for each i, the matrix $U_{\varepsilon_N}(a(\mu_s^N, V_s^{i,N}), a(v_s^N, W_s^{i,N}))$ being a.s. orthogonal for all $s \geq 0$, the process $\int_0^t U_{\varepsilon_N}(a(\mu_s^N, V_s^{i,N}), a(v_s^N, W_s^{i,N}))d\beta_s^{i,N}$ is a 3D Brownian motion. Consequently, the SDE satisfied by $W^{i,N}$ is equivalent (in law) to the SDE $W_t = V_0 + \int_0^t b(f_s, W_s)ds + \int_0^t a^{1/2}(f_s, W_s)dB_s$ (with $V_0 \sim f_0$). We know from Proposition 10-(i) that such a process does not explode in finite time, so that the unique solution is global, and from Proposition 10-(ii) that $W_t^{i,N} \sim f_t$ for all $t \geq 0$. Exchangeability is obvious, using that it holds true at time 0 (see point (a) at the end of Subsection 8.1).   □

8.3. A second coupling

Unfortunately, the processes $(W_t^{i,N})_{t\geq 0}$ are not independent, so we have to show that they are *almost* independent in some sense.

LEMMA 18. – *For all* $K = 1,\ldots,N$, *we can find an i.i.d. family of processes* $(Z_t^{i,N,K})_{i=1,\ldots,K,t\geq 0}$ *such that* $Z_t^{i,N,K} \sim f_t$ *for all* $t \geq 0$, *all* $i = 1,\ldots,K$ *and such that for all* $\eta \in (0,1)$, *all* $T > 0$,

$$(10) \qquad \sup_{i=1,\ldots,K}\ \sup_{[0,T]} \mathbb{E}[|W_t^{i,N} - Z_t^{i,N,K}|^2] \leq C_{\eta,T}\frac{K}{N^{1-\eta}}.$$

Moreover, the constant $C_{\eta,T}$ *is of the form* $C_\eta T$ *if* $\gamma = 0$.

Proof. – Let $K \in \{1,\ldots,N\}$ and $\eta \in (0,1/2)$ be fixed for the whole proof. We also put $\delta = (K/N)^2 > 0$. For simplicity, we write $V_s^i = V_s^{i,N}$, $W_s^i = W_s^{i,N}$ and $Z_s^i = Z_s^{i,N,K}$.

Step 1. We recall that the Brownian motions $(B^{ij})_{1\leq i<j\leq N}$ are independent, that $B^{ij} = -B^{ji}$ and we introduce a new family $(\tilde{B}^{ij})_{1\leq i,j\leq N}$ of independent Brownian motions (also independent of everything else). We recall that the Brownian motions $\beta_t^{i,N}$ were defined in Lemma 14 and we introduce, for $i = 1,\ldots,K$,

$$\tilde{\beta}_t^{i,N} = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\int_0^t [a^{1/2}(\mu_s^N, V_s^{i,N})]^{-1}\sigma(V_s^{i,N} - V_s^{j,N})d[\mathbb{1}_{\{j\leq K\}}\tilde{B}_s^{ij} + \mathbb{1}_{\{j>K\}}B_s^{ij}].$$

One easily checks, using Remark 15-(ii), that the continuous 3D martingales $\tilde{\beta}^{1,N},\ldots,\tilde{\beta}^{K,N}$ satisfy $\langle\tilde{\beta}^{i,N},\tilde{\beta}^{j,N}\rangle_t = \mathbf{I}_3 t\,\mathbb{1}_{\{i=j\}}$, so that they are independent 3D Brownian motions. We next claim that the system of equations (for $i = 1,\ldots,K$)

$$Z_t^i = W_0^i + \int_0^t b(f_s, Z_s^i)ds + \int_0^t a^{1/2}(f_s, Z_s^i)X_s^i U_s^i d\tilde{\beta}_s^{i,N},$$

where we have set $U_s^i = U_{\varepsilon_N}(a(\mu_s^N, V_s^i), a(v_s^N, W_s^i))$ and $X_s^i = U_\delta(a(f_s, W_s^i), a(f_s, Z_s^i))$ for simplicity, has a pathwise unique solution (with the same arguments as usual, see the proof of Lemma 17) and that for each $i = 1,\ldots,K$, $Z_t^i \sim f_t$ for all $t \geq 0$. Furthermore, the Brownian motions $\int_0^t X_s^i U_s^i d\tilde{\beta}_s^{i,N}$ being independent (as orthogonal martingales with deterministic brackets), as well as the initial conditions W_0^i, the pathwise uniqueness stated in Proposition 10-(i) implies that the processes $(Z_t^i)_{t\geq 0}$, for $i = 1,\ldots,K$, are independent. It only remains to prove (10) and, by exchangeability, it suffices to study $\mathbb{E}[|W_t^1 - Z_t^1|^2]$.

Step 2. Here we verify that, denoting by R_t^N the law of (W_t^1, Z_t^1), of which the two marginals equal f_t and using the notation of Proposition 12, we have

$$\frac{d}{dt}\mathbb{E}[|W_t^1 - Z_t^1|^2] \leq C_\eta K N^{\eta-1} + \Gamma_\delta(R_t^N).$$

Recalling the equations satisfied by W^1 (see Lemma 17) and Z^1, as well as the expressions of $\beta^{1,N}$ (see Lemma 14) and $\tilde{\beta}^{1,N}$, we see that

$$
W_t^1 - Z_t^1 = \int_0^t [b(f_s, W_s^1) - b(f_s, Z_s^1)]ds
$$

$$
+ \frac{1}{\sqrt{N}} \sum_{j=K+1}^N \int_0^t \left[a^{1/2}(f_s, W_s^1) - a^{1/2}(f_s, Z_s^1)X_s^1 \right]
$$

$$
U_s^1 [a^{1/2}(\mu_s^N, V_s^{1,N})]^{-1} \sigma(V_s^{1,N} - V_s^{j,N}) dB_s^{1j}
$$

$$
+ \frac{1}{\sqrt{N}} \sum_{j=1}^K \int_0^t a^{1/2}(f_s, W_s^1) U_s^1 [a^{1/2}(\mu_s^N, V_s^{1,N})]^{-1} \sigma(V_s^{1,N} - V_s^{j,N}) dB_s^{1j}
$$

$$
- \frac{1}{\sqrt{N}} \sum_{j=1}^K \int_0^t a^{1/2}(f_s, Z_s^1) X_s^1 U_s^1 [a^{1/2}(\mu_s^N, V_s^{1,N})]^{-1} \sigma(V_s^{1,N} - V_s^{j,N}) d\tilde{B}_s^{1j}.
$$

All the Brownian motions appearing in this formula are independent. By the Itô formula, we find $(d/dt)\mathbb{E}[|W_t^1 - Z_t^1|^2] = \mathbb{E}[I_1 + I_2 + I_3 + I_4]$, with

$$
I_1 = 2(W_t^1 - Z_t^1) \cdot [b(f_t, W_t^1) - b(f_t, Z_t^1)],
$$

$$
I_2 = \frac{1}{N} \sum_{j=K+1}^N \left\| [a^{1/2}(f_t, W_t^1) - a^{1/2}(f_t, Z_t^1)X_t^1] U_t^1 [a^{1/2}(\mu_t^N, V_t^{1,N})]^{-1} \sigma(V_t^{1,N} - V_t^{j,N}) \right\|^2,
$$

$$
I_3 = \frac{1}{N} \sum_{j=1}^K \left\| [a^{1/2}(f_t, W_t^1) U_t^1 [a^{1/2}(\mu_t^N, V_t^{1,N})]^{-1} \sigma(V_t^{1,N} - V_t^{j,N}) \right\|^2,
$$

$$
I_4 = \frac{1}{N} \sum_{j=1}^K \left\| [a^{1/2}(f_t, Z_t^1) U_t^1 X_t^1 [a^{1/2}(\mu_t^N, V_t^{1,N})]^{-1} \sigma(V_t^{1,N} - V_t^{j,N}) \right\|^2.
$$

Using that $N^{-1} \sum_{j=1}^N [\sigma(V_t^{1,N} - V_t^{j,N})]^2 = [a^{1/2}(\mu_t^N, V_t^{1,N})]^2$ and that $\|A\|^2 = \mathrm{Tr}\, AA^*$, we find

$$
I_2 \leq \frac{1}{N} \sum_{j=1}^N \left\| [a^{1/2}(f_t, W_t^1) - a^{1/2}(f_t, Z_t^1)X_t^1] U_t^1 [a^{1/2}(\mu_t^N, V_t^{1,N})]^{-1} \sigma(V_t^{1,N} - V_t^{j,N}) \right\|^2
$$

$$
= \left\| [a^{1/2}(f_t, W_t^1) - a^{1/2}(f_t, Z_t^1)X_t^1] U_t^1 \right\|^2
$$

$$
= \left\| a^{1/2}(f_t, W_t^1) - a^{1/2}(f_t, Z_t^1)X_t^1 \right\|^2
$$

because U_t^1 is a.s. an orthogonal matrix. Recalling the notation of Proposition 12 and that $X_t^1 = U_\delta(a(f_t, W_t^i), a(f_t, Z_t^i))$, we conclude that $\mathbb{E}[I_1 + I_2] \leq \Gamma_\delta(R_t^N)$.

By exchangeability, we have, for $q > 1$ and $q' = q/(q-1)$, by Hölder's inequality,

$$
\mathbb{E}[I_3] \leq \frac{K}{N} \mathbb{E}\left[\left\| a^{1/2}(f_t, W_t^1) \right\|^2 \left\| [a^{1/2}(\mu_t^N, V_t^{1,N})]^{-1} \sigma(V_t^{1,N} - V_t^{2,N}) \right\|^2 \right]
$$

$$
\leq \frac{K}{N} \mathbb{E}\left[\left\| a^{1/2}(f_t, W_t^1) \right\|^{2q} \right]^{1/q} \mathbb{E}\left[\left\| [a^{1/2}(\mu_t^N, V_t^{1,N})]^{-1} \sigma(V_t^{1,N} - V_t^{2,N}) \right\|^{2q'} \right]^{1/q'}.
$$

By Lemma 9, since $W_t^1 \sim f_s$ and since $\sup_{[0,\infty)} m_{(2+\gamma)q}(f_s) < \infty$ (see Subsection 8.1), we have $\mathbb{E}[\|a^{1/2}(f_t, W_t^1)\|^{2q}]^{1/q} \le C_q$. Next, we have a.s.

$$\|[a^{1/2}(\mu_t^N, V_t^{1,N})]^{-1}\sigma(V_t^{1,N} - V_t^{2,N})\|^2 \le \sum_{j=1}^N \|[a^{1/2}(\mu_t^N, V_t^{1,N})]^{-1}\sigma(V_t^{1,N} - V_t^{j,N})\|^2 = N$$

by Remark 15-(ii) and, by exchangeability,

$$\mathbb{E}[\|[a^{1/2}(\mu_t^N, V_t^{1,N})]^{-1}\sigma(V_t^{1,N} - V_t^{2,N})\|^2]$$

$$= N^{-1}\mathbb{E}[\sum_{j=1}^N \|[a^{1/2}(\mu_t^N, V_t^{1,N})]^{-1}\sigma(V_t^{1,N} - V_t^{j,N})\|^2] = 1.$$

Consequently,

$$\mathbb{E}[I_3] \le C_q \frac{K}{N}\mathbb{E}\Big[\Big\|[a^{1/2}(\mu_t^N, V_t^{1,N})]^{-1}\sigma(V_t^{1,N} - V_t^{2,N})\Big\|^2 N^{2(q'-1)}\Big]^{1/q'} = C_q K N^{2(q'-1)/q'-1}.$$

Choosing $q = 2/\eta$, we find that $2(q'-1)/q' = \eta$, whence $\mathbb{E}[I_3] \le C_\eta K N^{\eta-1}$. Finally, I_4 is treated exactly as I_3 and this ends the step.

Step 3. If $\gamma = 0$, by Proposition 12-(i), $\Gamma_\delta(R_t^N) \le C\sqrt{\delta}(1 + m_2(f_t))^{1/2} \le C\sqrt{\delta}$, so that we end with $(d/dt)\mathbb{E}[|W_t^1 - Z_t^1|^2] \le C\sqrt{\delta} + C_\eta K N^{\eta-1} \le C_\eta K N^{\eta-1}$ (because $\delta = (K/N)^2$). Since $W_0^1 = Z_0^1$, we conclude that $\mathbb{E}[|W_t^1 - Z_t^1|^2] \le C_\eta K N^{\eta-1}T$ as desired.

Step 4. Assume next that $\gamma \in (0, 1]$. We then have $\sup_{[0,\infty)}[m_{2+\gamma}(f_t) + \mathcal{E}_\alpha(f_t)] < \infty$, see Subsection 8.1. We thus infer from Proposition 12-(ii) that for all $M > 0$,

$$\Gamma_\delta(R_t^N) \le C\sqrt{\delta} + M\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - w|^2 R_t^N(dv, dw) + Ce^{-\kappa M^{\alpha/\gamma}}.$$

But $\sqrt{\delta} = KN^{-1}$ and $\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - w|^2 R_t^N(dv, dw) = \mathbb{E}[|W_t^1 - Z_t^1|^2]$, so that we have proved that $(d/dt)\mathbb{E}[|W_t^1 - Z_t^1|^2] \le C_\eta K N^{\eta-1} + M\mathbb{E}[|W_t^1 - Z_t^1|^2] + Ce^{-\kappa M^{\alpha/\gamma}}$ and thus

$$\sup_{[0,T]} \mathbb{E}[|W_t^1 - Z_t^1|^2] \le [C_\eta T K N^{\eta-1} + CTe^{-\kappa M^{\alpha/\gamma}}]e^{MT}.$$

Choosing $M = [\kappa^{-1}\log(1 + K^{-1}N^{1-\eta})]^{\gamma/\alpha}$, for which $e^{-\kappa M^{\alpha/\gamma}} = 1/(1 + K^{-1}N^{1-\eta}) \le KN^{\eta-1}$,

$$\sup_{[0,T]} \mathbb{E}[|W_t^1 - Z_t^1|^2] \le C_\eta T K N^{\eta-1} \exp\Big(T[\kappa^{-1}\log(1 + K^{-1}N^{1-\eta})]^{\gamma/\alpha}\Big).$$

Since $\gamma < \alpha$, this is easily bounded by $C_{\eta,T}(KN^{\eta-1})^{1-\eta} \le C_{\eta,T}KN^{-(1-\eta)^2} \le C_{\eta,T}KN^{2\eta-1}$. $\square$

A first consequence of the previous lemma is the following quantitative law of large numbers.

LEMMA 19. – *Consider a function* $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}$ *satisfying* $|\varphi(x) - \varphi(y)| \le C|x - y|(1 + |x|^q + |y|^q)$ *for some* $q \ge 2$. *As usual, we set* $\varphi(\mu, x) = \int_{\mathbb{R}^3} \varphi(x - y)\mu(dy)$ *for any probability measure* μ *on* $\mathbb{R}^3$. *Then for all* $T > 0$, *all* $\eta \in (0, 1/2)$,

$$\sup_{[0,T]} \mathbb{E}\big[|\varphi(\nu_t^N, W_t^{1,N}) - \varphi(f_t, W_t^{1,N})|^2\big] \le C_{\eta,T,\varphi} N^{\eta-1/2}.$$

Moreover, the constant $C_{\eta,T,\varphi}$ is of the form $C_{\eta,\varphi}\sqrt{T}$ if $\gamma = 0$.

Proof. – Using exchangeability, we write

$$\mathbb{E}\big[|\varphi(v_t^N, W_t^{1,N}) - \varphi(f_t, W_t^{1,N})|^2\big] = \frac{1}{N^2}\mathbb{E}\Big[\Big(\sum_{i=1}^N [\varphi(W_t^{1,N} - W_t^{i,N}) - \varphi(f_t, W_t^{1,N})]\Big)^2\Big]$$

$$= \frac{1}{N^2}\Big(I_1 + 2(N-1)I_2 + (N-1)I_3 + (N-1)(N-2)I_4\Big),$$

with (we develop the squared sum and separate the cases (a) $i = j = 1$, (b) $i = 1$ and $j \neq 1$ or $i \neq 1$ and $j = 1$, (c) $i = j \neq 1$, (d) $i \neq j$, $i \neq 1$, $j \neq 1$)

$$I_1 = \mathbb{E}[(\varphi(0) - \varphi(f_t, W_t^{1,N}))^2],$$
$$I_2 = \mathbb{E}[(\varphi(0) - \varphi(f_t, W_t^{1,N}))(\varphi(W_t^{1,N} - W_t^{2,N}) - \varphi(f_t, W_t^{1,N}))]$$
$$I_3 = \mathbb{E}[(\varphi(W_t^{1,N} - W_t^{2,N}) - \varphi(f_t, W_t^{1,N}))^2]$$
$$I_4 = \mathbb{E}[(\varphi(W_t^{1,N} - W_t^{2,N}) - \varphi(f_t, W_t^{1,N}))(\varphi(W_t^{1,N} - W_t^{3,N}) - \varphi(f_t, W_t^{1,N}))].$$

Using only that φ has at most polynomial growth, that $W_t^{1,N} \sim W_t^{2,N} \sim f_t$ and that all the moments of f_t are uniformly (in time) bounded, we easily verify that $I_1 + I_2 + I_3 \leq C_\varphi$, whence $N^{-2}(I_1 + 2(N-1)I_2 + (N-1)I_3) \leq C_\varphi N^{-1}$. We next use Lemma 18 with $K = 3$ to write $I_4 \leq J_1 + J_2 + J_3$, with

$$J_1 = \mathbb{E}[(\varphi(Z_t^{1,N,3} - Z_t^{2,N,3}) - \varphi(f_t, Z_t^{1,N,3}))(\varphi(Z_t^{1,N,3} - Z_t^{3,N,3}) - \varphi(f_t, Z_t^{1,N,3}))],$$
$$J_2 = \mathbb{E}[(|\varphi(W_t^{1,N} - W_t^{2,N}) - \varphi(Z_t^{1,N,3} - Z_t^{2,N,3})| + |\varphi(f_t, W_t^{1,N}) - \varphi(f_t, Z_t^{1,N,3})|)$$
$$\times |\varphi(Z_t^{1,N,3} - Z_t^{3,N,3}) - \varphi(f_t, Z_t^{1,N,3})|],$$
$$J_3 = \mathbb{E}[|\varphi(W_t^{1,N} - W_t^{2,N}) - \varphi(f_t, W_t^{1,N})|$$
$$\times (|\varphi(W_t^{1,N} - W_t^{3,N}) - \varphi(Z_t^{1,N,3} - Z_t^{3,N,3})| + |\varphi(f_t, W_t^{1,N}) - \varphi(f_t, Z_t^{1,N,3})|)].$$

But $J_1 = 0$ because $Z_t^{1,N,3}, Z_t^{2,N,3}, Z_t^{3,N,3}$ are independent and f_t-distributed: it suffices to first take the conditional expectation knowing $Z_t^{1,N,3}$ and to observe that $\mathbb{E}[\varphi(Z_t^{1,N,3} - Z_t^{2,N,3})|Z_t^{1,N,3}] = \varphi(f_t, Z_t^{1,N,3})$. Next, using that all the variables $W_t^{1,N}, W_t^{2,N},$ $W_t^{3,N}, Z_t^{1,N,3}, Z_t^{2,N,3}, Z_t^{3,N,3}$ are f_t-distributed, that all the moments of f_t are uniformly bounded, that φ has at most polynomial growth, the local Lipschitz property of φ, and that $|\varphi(f_t, w) - \varphi(f_t, z)| \leq C_\varphi |w - z|(1 + m_q(f_t) + |w|^q + |z|^q)$, we easily get convinced that, by exchangeability and the Cauchy-Schwarz inequality,

$$\mathbb{E}[J_2 + J_3] \leq C_\varphi \mathbb{E}[|W_t^{1,N} - Z_t^{1,N,3}|^2]^{1/2}.$$

This is bounded by $C_{\eta,T,\varphi} N^{\eta-1/2}$ by Lemma 18 with $K = 3$, and the constant $C_{\eta,T,\varphi}$ is of the form $C_{\eta,\varphi}\sqrt{T}$ in the case where $\gamma = 0$.   □

8.4. Computation of the error

We now handle the main computation of the proof.

LEMMA 20. – *For all $T > 0$, all $\eta \in (0, 1/4)$, all $t \in [0, T]$,*

$$\mathbb{E}[|V_t^{1,N} - W_t^{1,N}|^2]$$

$$\leq \begin{cases} C_\eta(1 + T)^{5/2}\big(\mathbb{E}[|V_0^{1,N} - W_0^{1,N}|^2] + N^{\eta-1/4}\big) & \text{if } \gamma = 0, \\ C_\eta(1 + T)^{5/2}\big(\mathbb{E}[|V_0^{1,N} - W_0^{1,N}|^2] + N^{\eta-1/2}\big) & \text{if } \gamma = 0 \text{ and } H(f_0) < \infty, \\ C_{\eta,T}\big(\mathbb{E}[|V_0^{1,N} - W_0^{1,N}|^2] + N^{-1/4}\big)^{1-\eta} & \text{if } \gamma \in (0, 1], \\ C_{\eta,T}\big(\mathbb{E}[|V_0^{1,N} - W_0^{1,N}|^2] + N^{-1/2}\big)^{1-\eta} & \text{if } \gamma \in (0, 1] \text{ and } H(f_0) < \infty. \end{cases}$$

Proof. – For simplicity, we write $V_t^i = V_t^{i,N}$, $W_t^i = W_t^{i,N}$ and $U_t^i = U_{\varepsilon_N}(a(\mu_t^N, V_t^{i,N}), a(v_t^N, W_t^{i,N}))$. Also, we set $u_t^N = \mathbb{E}[|V_t^{1,N} - W_t^{1,N}|^2]$. For each $t \geq 0$, we define $\zeta_t^N = N^{-1} \sum_1^N \delta_{(V_t^{i,N}, W_t^{i,N})}$, which a.s. belongs to $\mathscr{H}(\mu_t^N, v_t^N)$. We fix $T > 0$ and we work on $[0, T]$. We also fix $\eta \in (0, 1/9)$: it of course suffices to check the estimates for small values of $\eta > 0$.

Step 1. Recalling the equations satisfied by V^1 (see Lemma 14) and W^1 (see Lemma 17), the Itô formula leads us to

$$\frac{d}{dt}u_t^N = \mathbb{E}\Big[2(V_t^1 - W_t^1) \cdot (b(\mu_t^N, V_t^1) - b(f_t, W_t^1)) + \|a^{1/2}(\mu_t^N, V_t^1) - a^{1/2}(f_t, W_t^1)U_t^1\|^2\Big]$$

$$= \mathbb{E}\Big[2(V_t^1 - W_t^1) \cdot (b(\mu_t^N, V_t^1) - b(v_t^N, W_t^1)) + \|a^{1/2}(\mu_t^N, V_t^1) - a^{1/2}(v_t^N, W_t^1)U_t^1\|^2\Big]$$

$$+ \mathbb{E}\Big[2(V_t^1 - W_t^1) \cdot (b(v_t^N, W_t^1) - b(f_t, W_t^1))\Big]$$

$$+ \mathbb{E}\Big[\|(a^{1/2}(v_t^N, W_t^1) - a^{1/2}(f_t, W_t^1))U_t^1\|^2\Big]$$

$$+ 2\mathbb{E}\Big[\langle\!\langle a^{1/2}(\mu_t^N, V_t^1) - a^{1/2}(v_t^N, W_t^1)U_t^1, (a^{1/2}(v_t^N, W_t^1) - a^{1/2}(f_t, W_t^1))U_t^1\rangle\!\rangle\Big].$$

Using now that U_t^1 is an orthogonal matrix and the Cauchy-Schwarz inequality, we find that

$$\frac{d}{dt}u_t^N \leq \mathbb{E}[I_t^N] + 2\sqrt{u_t^N\mathbb{E}[J_t^N]} + \mathbb{E}[K_t^N] + 2\mathbb{E}\Big[\sqrt{L_t^N K_t^N}\Big],$$

where

$$I_t^N = 2(V_t^1 - W_t^1) \cdot (b(\mu_t^N, V_t^1) - b(v_t^N, W_t^1)) + \|a^{1/2}(\mu_t^N, V_t^1) - a^{1/2}(v_t^N, W_t^1)U_t^1\|^2,$$

$$J_t^N = |b(v_t^N, W_t^1) - b(f_t, W_t^1)|^2,$$

$$K_t^N = \|a^{1/2}(v_t^N, W_t^1) - a^{1/2}(f_t, W_t^1)\|^2,$$

$$L_t^N = \|a^{1/2}(\mu_t^N, V_t^1) - a^{1/2}(v_t^N, W_t^1)U_t^1\|^2.$$

Step 2. We first prove that $\mathbb{E}[I_t^N] = \mathbb{E}[\Gamma_{\varepsilon_N}(\zeta_t^N)]$. Using exchangeability,

$$\mathbb{E}[I_t^N] = \mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^N [2(V_t^i - W_t^i) \cdot (b(\mu_t^N, V_t^i) - b(v_t^N, W_t^i))$$

$$+ \|a^{1/2}(\mu_t^N, V_t^i) - a^{1/2}(v_t^N, W_t^i)U_t^i\|^2\Big].$$

It then suffices to recall that $\zeta_t^N = N^{-1}\sum_1^N \delta_{(V_t^{i,N}, W_t^{i,N})}$, of which the marginals are μ_t^N and v_t^N, that $U_t^i = U_{\varepsilon_N}(a(\mu_t^N, V_t^i), a(v_t^N, W_t^i))$ and the notation of Proposition 12.

Step 3. Using Lemma 19 and the Lipschitz property of b checked in Lemma 8, we immediately get that $\mathbb{E}[J_t^N] \leq C_{\eta,T} N^{\eta-1/2}$, with moreover $C_{\eta,T} = C_\eta T^{1/2}$ if $\gamma = 0$.

Step 4. Here we verify that
(i) we always have $\mathbb{E}[K_t^N] \leq C_{\eta,T} N^{\eta-1/4}$, with moreover $C_{\eta,T} = C_\eta T^{1/4}$ if $\gamma = 0$;
(ii) if $H(f_0) < \infty$, then $\mathbb{E}[K_t^N] \leq C_{\eta,T} N^{\eta-1/2}$, with moreover $C_{\eta,T} = C_\eta T^{1/2}$ if $\gamma = 0$.
For (i), we use the first inequality of Lemma 6 to write

$$K_t^N = \|a^{1/2}(v_t^N, W_t^1) - a^{1/2}(f_t, W_t^1)\|^2 \leq C\|a(v_t^N, W_t^1) - a(f_t, W_t^1)\|.$$

We then apply Lemma 19, which is licit thanks to the Lipschitz property of a checked in Lemma 8, to get $\mathbb{E}[K_t^N] \leq C_{\eta,T} N^{\eta-1/4}$, with $C_{\eta,T} = C_\eta T^{1/4}$ if $\gamma = 0$.

For point (ii), we use the second inequality of Lemma 6 and then the ellipticity estimate (9) to write $K_t^N \leq C\|[a(f_t, W_t^1)]^{-1}\| \|a(v_t^N, W_t^1) - a(f_t, W_t^1)\|^2 \leq \|a(v_t^N, W_t^1) - a(f_t, W_t^1)\|^2$. Again, Lemma 19 implies that $\mathbb{E}[K_t^N] \leq C_{\eta,T} N^{\eta-1/2}$, with moreover $C_{\eta,T} = C_\eta T^{1/2}$ if $\gamma = 0$.

Step 5. We now check that $\mathbb{E}[\sqrt{K_t^N L_t^N}] \leq \mathbb{E}[I_t^N] + \mathbb{E}[K_t^N] + C_\eta \sqrt{u_t^N} \mathbb{E}[K_t^N]^{(1-\eta)/2}$. We first observe that by Lemma 8,

$$L_t^N = I_t^N - 2(V_t^1 - W_t^1) \cdot (b(\mu_t^N, V_t^1) - b(v_t^N, W_t^1))$$

$$= I_t^N - 2(V_t^1 - W_t^1) \cdot \frac{1}{N} \sum_{i=1}^N (b(V_t^1 - V_t^i) - b(W_t^1 - W_t^i))$$

$$\leq I_t^N + C|V_t^1 - W_t^1| \frac{1}{N} \sum_{i=1}^N (|V_t^1 - W_t^1| + |V_t^i - W_t^i|)(1 + |V_t^1| + |W_t^1| + |V_t^i| + |W_t^i|)^\gamma$$

$$\leq I_t^N + C|V_t^1 - W_t^1| \left(\frac{1}{N} \sum_{i=1}^N (|V_t^1 - W_t^1|^2 + |V_t^i - W_t^i|^2)\right)^{1/2} \sqrt{H_t^N}$$

$$\leq I_t^N + CM_t^N \sqrt{H_t^N},$$

where we have set

$$H_t^N = \frac{1}{N} \sum_{i=1}^N (1 + |V_t^1| + |W_t^1| + |V_t^i| + |W_t^i|)^{2\gamma} \quad \text{and} \quad M_t^N = |V_t^1 - W_t^1|^2 + \frac{1}{N} \sum_{i=1}^N |V_t^i - W_t^i|^2.$$

Since $\sqrt{x(y+z)} \leq \sqrt{xy} + \sqrt{xz} \leq x + y + \sqrt{xz}$, we conclude that

$$\mathbb{E}[\sqrt{K_t^N L_t^N}] \leq \mathbb{E}[I_t^N] + \mathbb{E}[K_t^N] + C\mathbb{E}[(K_t^N)^{1/2}(M_t^N)^{1/2}(H_t^N)^{1/4}]$$

$$= \mathbb{E}[I_t^N] + \mathbb{E}[K_t^N] + C\mathbb{E}[(M_t^N)^{1/2}(K_t^N)^{(1-\eta)/2}((K_t^N)^{\eta/2}(H_t^N)^{1/4})]$$

$$\leq \mathbb{E}[I_t^N] + \mathbb{E}[K_t^N] + C\mathbb{E}[M_t^N]^{1/2}\mathbb{E}[K_t^N]^{(1-\eta)/2}\mathbb{E}[((K_t^N)^{\eta/2}(H_t^N)^{1/4})^{2/\eta}]^{\eta/2},$$

where we used the triple Hölder inequality with $p = 2$, $q = 2/(1-\eta)$ and $r = 2/\eta$ for the last inequality. But it holds that $\mathbb{E}[M_t^N] = 2u_t^N$ by exchangeability. To complete the step, it only remains to prove that $\mathbb{E}[((K_t^N)^{\eta/2}(H_t^N)^{1/4})^{2/\eta}] \leq C_\eta$.

But $K_t^N \leq 2\|a^{1/2}(v_t^N, W_t^1)\|^2 + 2\|a^{1/2}(f_t, W_t^1)\|^2 \leq C(m_{2+\gamma}(f_t + v_t^N) + |W_t^1|^2)$ by Lemma 9 and $H_t^N \leq 1 + |V_t^1|^{2\gamma} + |W_t^1|^{2\gamma} + m_{2\gamma}(\mu_t^N + v_t^N)$. Observing that $\mathbb{E}[(m_{2\gamma}(\mu_t^N + v_t^N))^p] \leq \mathbb{E}[|V_t^1|^{2\gamma p} + |W_t^1|^{2\gamma p}]$ by Hölder's inequality (if $p \geq 1$), that

$W_t^{1,N} \sim f_t$ and recalling that $\sup_{[0,\infty)} m_p(f_t) + \sup_{N \geq 2} \sup_{[0,\infty)} \mathbb{E}[|V_t^{1,N}|^p] < \infty$ for all $p > 2$ (see Subsection 8.1), we conclude that K_t^N and H_t^N have uniformly bounded moments of all orders, so that finally, $\mathbb{E}[((K_t^N)^{\eta/2}(H_t^N)^{1/4})^{2/\eta}] \leq C_\eta$.

Step 6. From Steps 1 and 5, $(d/dt)u_t^N \leq 3\mathbb{E}[I_t^N] + 3\mathbb{E}[K_t^N] + C_\eta \sqrt{u_t^N(\mathbb{E}[J_t^N] + \mathbb{E}[K_t^N]^{1-\eta})}$. Using Steps 3 and 4, we see that $\mathbb{E}[J_t^N] + \mathbb{E}[K_t^N] + \mathbb{E}[K_t^N]^{1-\eta} \leq \delta_{\eta,T,N}$, where (i) $\delta_{\eta,T,N} = C_{\eta,T}(N^{\eta-1/4})^{1-\eta} \leq C_{\eta,T}N^{2\eta-1/4}$ in general (with $C_{\eta,T} = C_\eta(1+T)^{1/2}$ if $\gamma = 0$); (ii) $\delta_{\eta,T,N} = C_{\eta,T}(N^{\eta-1/2})^{1-\eta} \leq N^{2\eta-1/2}$ if $H(f_0) < \infty$ (with $C_{\eta,T} = C_\eta(1+T)^{1/2}$ if $\gamma = 0$). Using finally Step 2, we end with

$$\frac{d}{dt}u_t^N \leq 3\mathbb{E}[\Gamma_{\varepsilon_N}(\zeta_t^N)] + \delta_{\eta,T,N} + C\sqrt{u_t^N \delta_{\eta,T,N}}.$$

Step 7. Assume that $\gamma = 0$. By Proposition 12-(i) (recall that $\varepsilon_N = N^{-1}$),

$$\mathbb{E}[\Gamma_{\varepsilon_N}(\zeta_t^N)] \leq C\sqrt{\varepsilon_N}\mathbb{E}[1 + m_2(\mu_t^N + \nu_t^N)]^{1/2}$$
$$= C\sqrt{\varepsilon_N}\mathbb{E}[|V_t^{1,N}|^2 + |W_t^{1,N}|^2]^{1/2} \leq C\sqrt{\varepsilon_N} \leq C\delta_{\eta,T,N}.$$

Thus $(d/dt)u_t^N \leq C\delta_{\eta,T,N} + C\sqrt{u_t^N \delta_{\eta,T,N}} \leq C\sqrt{\delta_{\eta,T,N}^2 + u_t^N \delta_{\eta,T,N}}$. Integrating this differential inequality, we deduce that $\sup_{[0,T]} u_t^N \leq C(1+T)^2(u_0^N + \delta_{\eta,T,N})$, from which the conclusion follows.

Step 8. Assume next that $\gamma \in (0,1]$ and recall that $\alpha > \gamma$. By Proposition 12-(ii), for all $M > 0$,

$$\Gamma_{\varepsilon_N}(\zeta_t^N) \leq C\sqrt{\varepsilon_N}(1 + m_{2+\gamma}(\mu_t^N + \nu_t^N))^{1/2} + M\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v-w|^2 \zeta_t^N(dv,dw)$$
$$+ Ce^{-\kappa M^{\gamma/\alpha}}(m_{2+\gamma}(\mu_t^N) + \mathcal{E}_\alpha(\nu_t^N)).$$

We have $\mathbb{E}[m_{2+\gamma}(\mu_t^N + \nu_t^N)] = \mathbb{E}[|V_t^1|^{\gamma+2}] + m_{2+\gamma}(f_t) \leq C$, see Subsection 8.1. Also, it holds that $\mathbb{E}[\mathcal{E}_\alpha(\nu_t^N)] = \mathcal{E}_\alpha(f_t) \leq C$. Finally, $\mathbb{E}[\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v-w|^2 \zeta_t^N(dv,dw)] = u_t^N$. All in all, we have checked that $\mathbb{E}[\Gamma_{\varepsilon_N}(\zeta_t^N)] \leq C\sqrt{\varepsilon_N} + Ce^{-\kappa M^{\gamma/\alpha}} + Mu_t^N$. Recalling Step 6, using that $\sqrt{\varepsilon_N} \leq C\delta_{\eta,T,N}$ and that $\sqrt{xy} \leq x + y$, we conclude that

$$\frac{d}{dt}u_t^N \leq C\delta_{\eta,T,N} + (M+C)u_t^N + Ce^{-\kappa M^{\gamma/\alpha}},$$

whence $u_t^N \leq [u_0^N + CT\delta_{\eta,T,N} + CTe^{-\kappa M^{\gamma/\alpha}}]e^{(M+C)T}$. As usual, we make the choice $M = [\kappa^{-1}\log(1 + 1/[u_0^N + \delta_{\eta,T,N}])]^{\gamma/\alpha}$, for which $e^{-\kappa M^{\alpha/\gamma}} \leq u_0^N + \delta_{\eta,T,N}$, and this leads us to

$$u_t^N \leq C_T[u_0^N + \delta_{\eta,T,N}]\exp(T[\kappa^{-1}\log(1 + 1/[u_0^N + \delta_{\eta,T,N}])]^{\gamma/\alpha} + CT) \leq C_{\eta,T}[u_0^N + \delta_{\eta,T,N}]^{1-\eta},$$

because $\gamma < \alpha$. We conclude that $u_t^N \leq C_{\eta,T}(u_0^N + N^{2\eta-1/4})^{1-\eta} \leq C_{\eta,T}(u_0^N + N^{-1/4})^{1-9\eta}$ in general and $u_t^N \leq C_{\eta,T}(u_0^N + N^{2\eta-1/2})^{1-\eta} \leq C_{\eta,T}(u_0^N + N^{-1/2})^{1-5\eta}$ if $H(f_0) < \infty$. $\quad\square$

8.5. A quantified law of large numbers for non independent variables

Here we check the following result, to be applied soon to the family $W_t^{i,N}$.

LEMMA 21. – *Let $N \geq 2$, $\mu \in \mathcal{P}_5(\mathbb{R}^3)$, $\eta \in (0,1)$ and $\kappa > 0$. Consider an exchangeable family $W_1, \ldots, W_N$ of $\mathbb{R}^3$-valued random variables such that for all $K = 1, \ldots, N$, there are some i.i.d. μ-distributed random variables $Z_1^K, \ldots, Z_K^K$ such that $\max_{i=1,\ldots,K} \mathbb{E}[|W_i - Z_i^K|^2] \leq \kappa K N^{\eta-1}$. Then*

$$\mathbb{E}\left[\mathcal{W}_2^2\left(\frac{1}{N}\sum_1^N \delta_{W_i}, \mu\right)\right] \leq \frac{C(1 + m_5(\mu)^{2/5} + \kappa)}{N^{(1-\eta)/3}},$$

where C is a universal constant.

Proof. – We divide the proof into four steps.

Step 1. We recall the well-known fact that for $f, f', g, g' \in \mathcal{P}_2(\mathbb{R}^3)$ and $\lambda \in (0,1)$, it holds that $\mathcal{W}_2^2(\lambda f + (1-\lambda)g, \lambda f' + (1-\lambda)g') \leq \lambda \mathcal{W}_2^2(f,g) + (1-\lambda)\mathcal{W}_2^2(f',g')$. Indeed, consider $X \sim f$ and $Y \sim g$ such that $\mathbb{E}[|X - Y|^2] = \mathcal{W}_2^2(f,g)$, $X' \sim f'$ and $Y' \sim g'$ such that $\mathbb{E}[|X' - Y'|^2] = \mathcal{W}_2^2(f',g')$, and $U \sim \text{Bernoulli}(\lambda)$, with $(X,Y), (X',Y'), U$ independent. Then $Z := UX + (1-U)Y \sim \lambda f + (1-\lambda)g$, $Z' := UX' + (1-U)Y' \sim \lambda f' + (1-\lambda)g'$, and one easily verifies that $\mathbb{E}[|Z - Z'|^2] = \lambda \mathbb{E}[|X - Y|^2] + (1-\lambda)\mathbb{E}[|X' - Y'|^2] = \lambda \mathcal{W}_2^2(f,g) + (1-\lambda)\mathcal{W}_2^2(f',g')$.

Step 2. For $K \in \{1, \ldots, N\}$, we set $\mu_K = K^{-1}\sum_{i=1}^K \delta_{W_i}$. We prove in this step that

$$\mathbb{E}[\mathcal{W}_2^2(\mu_N, \mu)] \leq \mathbb{E}[\mathcal{W}_2^2(\mu_K, \mu)] + \frac{(6m_2(\mu) + 4\kappa)K}{N}.$$

To this end, we set $R = \lfloor N/K \rfloor$ and we assume that $RK < N$, the other case being easier (no need to introduce ν_{R+1}^N). We introduce, for $k = 1, \ldots, R$, $\nu_k^N = K^{-1}\sum_{i=(k-1)K+1}^{kK} \delta_{W_i}$, as well as $\nu_{R+1}^N = (N - RK)^{-1}\sum_{i=RK+1}^N \delta_{W_i}$.

We then write $\mu_N = KN^{-1}\sum_{k=1}^R \nu_k^N + (N - RK)N^{-1}\nu_{R+1}^N$ and we use Step 1 to obtain $\mathcal{W}_2^2(\mu_N, \mu) \leq KN^{-1}\sum_{k=1}^R \mathcal{W}_2^2(\nu_k^N, \mu) + (N - RK)N^{-1}\mathcal{W}_2^2(\nu_{R+1}^N, \mu)$. By exchangeability, we thus find

$$\mathbb{E}[\mathcal{W}_2^2(\mu_N, \mu)] \leq \frac{RK}{N}\mathbb{E}[\mathcal{W}_2^2(\nu_1^N, \mu)] + \frac{N - RK}{N}\mathbb{E}[\mathcal{W}_2^2(\nu_{R+1}^N, \mu)].$$

The conclusion follows, because $RK \leq N$, because $\nu_1^N = \mu_K$, because $N - RK \leq K$ and because $\mathbb{E}[\mathcal{W}_2^2(\nu_{R+1}^N, \mu)] \leq 2m_2(\mu) + 2\mathbb{E}[|W_1|^2] \leq 2m_2(\mu) + 4\mathbb{E}[|Z_1^1|^2] + 4\mathbb{E}[|W_1 - Z_1^1|^2] \leq 6m_2(\mu) + 4\kappa$.

Step 3. We then introduce $\zeta_K = K^{-1}\sum_{i=1}^K \delta_{Z_i^K}$. Since the Z_i^K's are i.i.d. and μ-distributed, we know from [16, Theorem 1] (with $p = 2$, $d = 3$ and $q = 5$) that for all $K = 1, \ldots, N$, $\mathbb{E}[\mathcal{W}_2^2(\zeta_K, \mu)] \leq C(m_5(\mu))^{2/5} K^{-1/2}$.

Next, we have $\mathbb{E}[\mathcal{W}_2^2(\mu_K, \zeta_K)] \leq K^{-1}\sum_1^K \mathbb{E}[|W_i - Z_i^K|^2] \leq \kappa K N^{\eta-1}$.

Consequently, $\mathbb{E}[\mathcal{W}_2^2(\mu_K, \mu)] \leq C(m_5(\mu))^{2/5} K^{-1/2} + 2\kappa K N^{\eta-1}$.

Step 4. Gathering Steps 2 and 3, we find that for all $K \in \{1, \dots, N\}$,

$$\mathbb{E}[\mathcal{W}_2^2(\mu_N, \mu)] \leq \frac{C(m_5(\mu))^{2/5}}{\sqrt{K}} + \frac{2\kappa K}{N^{1-\eta}} + \frac{(6m_2(\mu) + 4\kappa)K}{N}$$
$$\leq C(1 + (m_5(\mu))^{2/5} + \kappa)\left[\frac{1}{\sqrt{K}} + \frac{K}{N^{1-\eta}}\right].$$

Choosing $K = \lfloor N^{2(1-\eta)/3} \rfloor$ completes the proof. $\qquad\square$

8.6. Conclusion

We now have all the weapons to prove Theorem 4, except the time uniformity in the Maxwell case. We start with the case of hard potentials.

Proof of Theorem 4-(ii). – We thus assume that $\gamma \in (0, 1]$ and we fix $T > 0$ and $\eta \in (0, 1)$. We recall that $\mu_t^N = N^{-1}\sum_1^N \delta_{V_t^{i,N}}$ and $\nu_t^N = N^{-1}\sum_1^N \delta_{W_t^{i,N}}$ and we write $\mathcal{W}_2^2(\mu_t^N, f_t) \leq 2\mathcal{W}_2^2(\mu_t^N, \nu_t^N) + 2\mathcal{W}_2^2(\nu_t^N, f_t)$. Lemma 18 (together with the fact that $\sup_{t\geq 0} m_5(f_t) < \infty$) allows us to apply Lemma 21 to obtain $\sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\nu_t^N, f_t)] \leq C_{\eta,T} N^{(\eta-1)/3} \leq C_{\eta,T} N^{\eta-1/3}$.

Next, we write $\mathbb{E}[\mathcal{W}_2^2(\mu_t^N, \nu_t^N)] \leq N^{-1}\sum_1^N \mathbb{E}[|V_t^{i,N} - W_t^{i,N}|^2] = \mathbb{E}[|V_t^{1,N} - W_t^{1,N}|^2]$. We conclude from Lemma 20 that $\sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^N, \nu_t^N)]$ is controlled by $C_{\eta,T}(\mathbb{E}[|V_0^{1,N} - W_0^{1,N}|^2] + N^{-1/4})^{1-\eta}$ in general and by $C_{\eta,T}(\mathbb{E}[|V_0^{1,N} - W_0^{1,N}|^2] + N^{-1/2})^{1-\eta}$ if $H(f_0) < \infty$. By points (a) and (b) stated at the end of Subsection 8.1, $\mathbb{E}[|V_0^{1,N} - W_0^{1,N}|^2] = \mathbb{E}[N^{-1}\sum_1^N |V_0^{i,N} - W_0^{i,N}|^2] = \mathbb{E}[\mathcal{W}_2^2(\mu_0^N, \nu_0^N)] \leq 2\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] + 2\mathbb{E}[\mathcal{W}_2^2(\nu_0^N, f_0)]$. And $\mathbb{E}[\mathcal{W}_2^2(\nu_0^N, f_0)] \leq CN^{-1/2}$ by [16, Theorem 1], because $(W_0^{i,N})_{i=1,\dots,N} \sim f_0^{\otimes N}$ and $m_5(f_0) < \infty$.

All in all, we can bound $\sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^N, f_t)]$ by $C_{\eta,T}(\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] + N^{-1/4})^{1-\eta}$, and even by $C_{\eta,T}(\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] + N^{-1/3})^{1-\eta}$ if $H(f_0) < \infty$. $\qquad\square$

Proceeding similarly, we find the following weak version of Theorem 4-(i).

THEOREM 22. – *Assume that* $\gamma = 0$. *Fix* $f_0 \in \mathscr{P}_2(\mathbb{R}^3)$ *and consider the corresponding unique weak solution* $(f_t)_{t\geq 0}$ *to* (1). *For each* $N \geq 2$, *consider an exchangeable* $(\mathbb{R}^3)^N$-*valued random variable* $(V_0^{i,N})_{i=1,\dots,N}$ *and the corresponding unique solution* $(V_t^{i,N})_{i=1,\dots,N,t\geq 0}$ *to* (3). *Set* $\mu_t^N = N^{-1}\sum_1^N \delta_{V_t^{i,N}}$. *Assume that for all* $p \geq 2$, $M_p := m_p(f_0) + \sup_{N\geq 2} \mathbb{E}[|V_0^{1,N}|^p] < \infty$. *For all* $\eta \in (0, 1/4)$, *there is a constant* C_η *depending only on* η, *on (some upper bounds of)* $\{M_p, p \geq 2\}$ *and on (some upper bound of)* $H(f_0)$ *when it is finite such that*

$$\mathbb{E}[\mathcal{W}_2^2(\mu_t^N, f_t)] \leq \begin{cases} C_\eta(1+t)^{5/2}(\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] + N^{\eta-1/4}) & \text{in general,} \\ C_\eta(1+t)^{5/2}(\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] + N^{\eta-1/3}) & \text{if } H(f_0) < \infty. \end{cases}$$

9. Uniform convergence to equilibrium in the Maxwell case

We now prove, when $\gamma = 0$, the uniform (in N) convergence to equilibrium of the particle system, following the arguments of Rousset [26]. We will easily deduce the time-uniformity of the propagation of chaos.

In the whole section, we assume that $\gamma = 0$. For $N \geq 2$ and for F_N an exchangeable law on $(\mathbb{R}^3)^N$, we call $\mathbf{L}^N(F_N) \in \mathscr{P}(C([0, \infty), (\mathbb{R}^3)^N)$ the law of the solution $(V_t^{i,N})_{i=1,\ldots,N, t\geq 0}$ to (3) with $(V_0^{i,N})_{i=1,\ldots,N} \sim F_N$. We also put $\mathbf{L}_t^N(F_N) = \mathscr{L}((V_t^{i,N})_{i=1,\ldots,N}) \in \mathscr{P}((\mathbb{R}^3)^N)$ for each $t \geq 0$. We introduce

$$\mathscr{S}_N = \Big\{(v_1, \ldots, v_N) \in (\mathbb{R}^3)^N \ : \ N^{-1}\sum_1^N v_i = 0, \ N^{-1}\sum_1^N |v_i|^2 = 1\Big\}.$$

REMARK 23. – *The uniform distribution on S_N is invariant: $\mathbf{L}_t^N(\mathscr{U}(S_N)) = \mathscr{U}(S_N)$ for all $t \geq 0$.*

This observation is classical and actually holds true for any value of $\gamma \in [0, 1]$. To give a precise reference, let us mention that in [6, Theorem 4.2-(ii)], Carrapatoso shows that under some conditions on $F_N \in \mathscr{P}(S_N)$, $\mathscr{W}_1(\mathbf{L}_t^N(F_N), \mathscr{U}(S_N))$ tends to 0 as $t \to \infty$, which implies that $\mathscr{U}(S_N)$ is invariant.

THEOREM 24. – *Fix $N \geq 7$ and some exchangeable $(V_0^{i,N})_{i=1,\ldots,N} \sim F_N \in \mathscr{P}(\mathscr{S}_N)$. For all $p > 0$, there is a constant C_p depending only on p such that if $N \geq 6 + 2p$, for all $t \geq 0$,*

$$\frac{1}{N}\mathscr{W}_2^2(\mathbf{L}_t^N(F_N), \mathscr{U}(S_N)) \leq \min\Big\{\frac{1}{N}\mathscr{W}_2^2(F_N, \mathscr{U}(S_N)), \frac{C_p\mathbb{E}[1 + |V_0^{1,N}|^{8+4p}]^{1/2}}{(1+t)^p}\Big\}.$$

Although we slightly clarify some points and although the coupling is slightly more technical for the Landau equation, the proof closely follows [26]. In the next subsection, we recall some facts about $\mathscr{U}(S_N)$. We build a suitable coupling in Subsection 9.2 and recall Rousset's main inequality in Subsection 9.3. We conclude the proof of Theorem 24 in Subsection 9.4. Finally, we deduce Theorem 4-(i) from Theorems 22 and 24 in Subsection 9.5.

9.1. The uniform law on the sphere

We will need the following facts.

LEMMA 25. – *Let $(X_1^N, \ldots, X_N^N) \sim \mathscr{U}(\mathscr{S}_N)$. Then*
(i) $\mathbb{E}[\mathscr{W}_2^2(N^{-1}\sum_1^N \delta_{X_i^N}, \mathscr{N}(0, 3^{-1}\mathbf{I}_3))] \leq CN^{-1/2}$;
(ii) for all $p \geq 1$, $\mathbb{E}[|X_1^N|^p] \leq C_p$, where C_p depends only on p;
(iii) if $1 \leq p \leq N - 4$, for ρ_N the spectral radius of $M_N = N^{-1}\sum_1^N X_i^N(X_i^N)^*$, we have $\mathbb{E}[(1 - \rho_N)^{-p}] \leq C_p$, where C_p depends only on p.

We will use twice the following observation.

REMARK 26. – *For any $f, g \in \mathscr{P}_2(\mathbb{R}^3)$, $\mathscr{W}_2^2(f, g) \geq [(V_f)^{1/2} - (V_g)^{1/2}]^2 + |m_f - m_g|^2$, where $m_f = \int_{\mathbb{R}^3} vf(dv)$ and $V_f = \int_{\mathbb{R}^3} |v - m_f|^2 f(dv)$.*

Indeed, for any $X \sim f$ and $Y \sim g$, $\mathbb{E}[|X - Y|^2] = \mathbb{E}[|(X - \mathbb{E}[X]) - (Y - \mathbb{E}[Y])|^2] + |\mathbb{E}[X - Y]|^2 \geq V_f + V_g - 2(V_f V_g)^{1/2} + |m_f - m_g|^2$.

Proof of Lemma 25. – Consider an i.i.d. sample $(Y_1, \ldots, Y_N)$ of the $\mathcal{N}(0, 3^{-1}\mathbf{I}_3)$ distribution. Define $m_N = N^{-1} \sum_1^N Y_i$, $E_N = N^{-1} \sum_1^N |Y_i - m_N|^2$ and $X_i^N = E_N^{-1/2}(Y_i - m_N)$. Then it is classical (see e.g., [26, Proof of Lemma 4.3]) that $(X_1^N, \ldots, X_N^N) \sim \mathcal{U}(\mathcal{S}_N)$.

To prove (i), we set $\mu_N = N^{-1} \sum_1^N \delta_{X_i^N}$ and $\nu_N = N^{-1} \sum_1^N \delta_{Y_i}$. We have

$$\mathbb{E}[\mathcal{W}_2^2(\mu_N, \nu_N)] \leq N^{-1} \sum_1^N \mathbb{E}[|Y_i - X_i^N|^2]$$

$$= N^{-1} \sum_1^N \mathbb{E}[|(Y_i - m_N)(1 - 1/\sqrt{E_N}) + m_N|^2]$$

$$= \mathbb{E}[(1 - \sqrt{E_N})^2 + |m_N|^2].$$

By Remark 26 and since $m_{\nu_N} = m_N$ and $V_{\nu_N} = E_N$, we conclude that $\mathbb{E}[\mathcal{W}_2^2(\mu_N, \nu_N)] \leq \mathbb{E}[\mathcal{W}_2^2(\nu_N, \mathcal{N}(0, 3^{-1}\mathbf{I}_3))]$, so that $\mathbb{E}[\mathcal{W}_2^2(\mu_N, \mathcal{N}(0, 3^{-1}\mathbf{I}_3))] \leq 4\mathbb{E}[\mathcal{W}_2^2(\nu_N, \mathcal{N}(0, 3^{-1}\mathbf{I}_3))]$. By [16, Theorem 1], it holds that $\mathbb{E}[\mathcal{W}_2^2(\nu_N, \mathcal{N}(0, 3^{-1}\mathbf{I}_3))] \leq CN^{-1/2}$ and this proves (i).

Point (ii) has been checked by Carrapatoso [6, Lemma 10].

We finally check (iii) (see [26, Lemma 4.4] for a less precise statement), assuming that $N \geq p + 4 \geq 5$. The empirical covariance matrix $A_N = \sum_1^N (Y_i - m_N)(Y_i - m_N)^*$ classically (see Anderson [3, Section 7]) follows a Wishart$(3, N - 1)$-distribution, and $M_N = A_N / \mathrm{Tr}\, A_N$. The eigenvalues $0 \leq L_1^N \leq L_2^N \leq L_3^N$ of A_N are known to have the density (see Anderson [3, Theorem 13.3.2], this uses that $3 \leq N - 1$)

$$g_N(\ell_1, \ell_2, \ell_3) = \kappa_N^{-1}(\ell_1\ell_2\ell_3)^{(N-5)/2}[(\ell_3 - \ell_2)(\ell_3 - \ell_1)(\ell_2 - \ell_1)]e^{-(\ell_1 + \ell_2 + \ell_3)/2} \mathbb{1}_{\{0 < \ell_1 < \ell_2 < \ell_3\}},$$

where $\kappa_N = \pi^{-9/2} 2^{3(N-1)/2} \Gamma((N-1)/2)\Gamma((N-2)/2)\Gamma((N-3)/2)\Gamma(3/2)\Gamma(1)\Gamma(1/2)$. But $1 - \rho_N = (L_1^N + L_2^N)/(L_1^N + L_2^N + L_3^N) \geq 2(L_1^N L_2^N)^{1/2}/(3L_3^N) = 2(L_1^N L_2^N L_3^N)^{1/2}/(3(L_3^N)^{3/2})$. Consequently, for $p \in [1, N - 4]$ (so that $3 \leq N - p - 1$),

$$\mathbb{E}[(1 - \rho_N)^{-p}] \leq \left(\frac{3}{2}\right)^p \int_{\mathbb{R}^3} \ell_3^{3p/2}(\ell_1\ell_2\ell_3)^{-p/2} g_N(\ell_1, \ell_2, \ell_3) d\ell_1 d\ell_2 d\ell_3$$

$$= \left(\frac{3}{2}\right)^p \kappa_N^{-1} \kappa_{N-p} \int_{\mathbb{R}^3} \ell_3^{3p/2} g_{N-p}(\ell_1, \ell_2, \ell_3) d\ell_1 d\ell_2 d\ell_3$$

$$= \left(\frac{3}{2}\right)^p \kappa_N^{-1} \kappa_{N-p} \mathbb{E}[(L_3^{N-p})^{3p/2}].$$

But using that $L_3^N \leq \mathrm{Tr}\, A_N = E_N \sim \chi^2(3N - 3)$, it is not hard to verify that $\mathbb{E}[(L_3^N)^{3p/2}] \leq C_p N^{3p/2}$. We thus end with $\mathbb{E}[(1 - \rho_N)^{-p}] \leq C_p N^{3p/2} \kappa_N^{-1} \kappa_{N-p}$. Using the expression of κ_N and the Stirling formula, we easily conclude that $\sup_{N \geq p+4} \mathbb{E}[(1 - \rho_N)^{-p}] < \infty$ as desired. $\qquad\square$

9.2. The coupling

Recall that U was defined in (6). We need to use $U(a(x), a(y))$, which is unfortunately not well-defined. The lemma below gives some sense to $A(x, y) = \sigma(y)U(a(x), a(y))$.

LEMMA 27. – *Recall that for $x \in \mathbb{R}^3$, $\sigma(x) = |x|\Pi_{x\perp}$ and $a(x) = |x|^2\Pi_{x\perp}$. We can find a measurable family of 3×3 matrices $(A(x, y))_{x,y \in \mathbb{R}^3}$ verifying $A(-x, -y) = A(x, y)$ and* (a) $A(x, y)A^*(x, y) = a(y)$, (b) $\langle\langle \sigma(x), A(x, y) \rangle\rangle = |x||y| + x \cdot y$, (c) $(\sigma(x) - A^*(x, y))(x - y) = 0$.

Proof. – If $x = 0$, it suffices to set $A(x, y) = \sigma(y)$. Else, we consider an orthonormal basis (e_1, e_2, e_3) satisfying $e_1 = x/|x|$ and $e_3 \cdot y = 0$ and we set $A(x, y) = -(y \cdot e_2)e_1e_2^*$ $+(y \cdot e_1)e_2e_2^* + |y|e_3e_3^*$. To check (a), put $y_i = y \cdot e_i$, note that $y_3 = 0$ and that $\mathbf{I}_3 = e_1e_1^* + e_2e_2^* + e_3e_3^*$: direct computations show that both $a(y) = |y|^2\mathbf{I}_3 - yy^*$ and $A(x, y)A^*(x, y)$ equal $y_2^2e_1e_1^* + y_1^2e_2e_2^* + |y|^2e_3e_3^* - y_1y_2(e_1e_2^* + e_2e_1^*)$. For point (b), one starts with $\sigma(x) = |x|\mathbf{I}_3 - |x|^{-1}xx^* = |x|(e_2e_2^* + e_3e_3^*)$, whence $\sigma(x)A^*(x, y) = |x|(-y_2e_2e_1^* + y_1e_2e_2^* + |y|e_3e_3^*)$ and thus $\mathrm{Tr}\, \sigma(x)A^*(x, y) = |x|(y_1 + |y|) = |x||y| + x \cdot y$. For (c), one easily finds that both $\sigma(x)(x - y)$ and $A^*(x, y)(x - y)$ equal $-y_2|x|e_2$.

Since $A(x, y)$ satisfies conditions (a)-(b)-(c) with $-x$ and $-y$, it is possible to handle the construction in such a way that $A(-x, -y) = A(x, y)$. Measurability is not an issue.   □

We now build a suitable coupling.

LEMMA 28. – *Consider two exchangeable laws F_N and G_N in $\mathscr{P}(S_N)$. There exists an exchangeable family $\{(V_t^{i,N}, W_t^{i,N})_{t\geq 0}, i = 1, \ldots, N\}$ satisfying the following properties:*

(i) $\mathbb{E}[\sum_1^N |V_0^{i,N} - W_0^{i,N}|^2] = \mathscr{W}_2^2(F_N, G_N)$;

(ii) *a.s.,* $N^{-1}\sum_1^N |V_0^{i,N} - W_0^{i,N}|^2 = \mathscr{W}_2^2(N^{-1}\sum_1^N \delta_{V_0^{i,N}}, N^{-1}\sum_1^N \delta_{W_0^{i,N}}) \leq 2$;

(iii) $(V_t^{i,N})_{i=1,\ldots,N,t\geq 0} \sim \mathbf{L}^N(F_N)$ *and* $(W_t^{i,N})_{i=1,\ldots,N,t\geq 0} \sim \mathbf{L}^N(G_N)$;

(iv) *a.s., for all $t \geq 0$,*

$$\frac{1}{N}\sum_{i=1}^N |V_t^{i,N} - W_t^{i,N}|^2 = \frac{1}{N}\sum_{i=1}^N |V_0^{i,N} - W_0^{i,N}|^2$$

$$-\frac{2}{N^2}\sum_{i,j=1}^N \int_0^t [|V_s^{i,N} - V_s^{j,N}||W_s^{i,N} - W_s^{j,N}| - (V_s^{i,N} - V_s^{j,N}) \cdot (W_s^{i,N} - W_s^{j,N})]ds.$$

Proof. – The function $A(x, y)$ cannot be continuous and this causes some technical diffi-culties. We write $V_t^i = V_t^{i,N}$ and $W_t^i = W_t^{i,N}$ for simplicity.

Step 1. By [17, Proposition A.1], we can find $H_N \in \mathscr{P}(S_N \times S_N)$ with marginals F_N and G_N such that, for $((V_0^i)_{i=1,\ldots,N}, (W_0^i)_{i=1,\ldots,N}) \sim H_N$, the family $\{(V_0^i, W_0^i), i = 1, \ldots, N\}$ is exchangeable and points (i) and (ii) hold true. Actually, the inequality in (ii) follows from the fact that $\mathscr{W}_2^2(f, g) \leq m_2(f) + m_2(g)$ (choose an independent coupling between f and g) and that $m_2(N^{-1}\sum_1^N \delta_{V_0^i}) = m_2(N^{-1}\sum_1^N \delta_{W_0^i}) = 1$ because both F_N and G_N are carried by S_N.

Step 2. We consider $((V_0^i)_{i=1,\ldots,N}, (W_0^i)_{i=1,\ldots,N}) \sim H_N$ and some families $(B_t^{ij})_{1\leq i < j \leq N,t\geq 0}$, $(\tilde{B}_t^i)_{i=1,\ldots,N,t\geq 0}$, $(\hat{B}_t^i)_{i=1,\ldots,N,t\geq 0}$ of $3D$ Brownian motions, all these objects being indepen-dent. For $1 \leq j < i \leq N$, we set $B_t^{ij} = -B_t^{ji}$. We also put $B_t^{ii} = 0$ for all $i = 1, \ldots, N$ and

consider the system of SDEs

(11)
$$V_t^{i,\varepsilon} = V_0^{i,\varepsilon} + \frac{1}{N}\sum_{j=1}^{N}\int_0^t b(V_s^{i,\varepsilon}-V_s^{j,\varepsilon})ds + \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\int_0^t \sigma(V_s^{i,\varepsilon}-V_s^{j,\varepsilon})dB_s^{ij} + \varepsilon\tilde{B}_t^i,$$

(12)
$$W_t^{i,\varepsilon} = W_0^{i,\varepsilon} + \frac{1}{N}\sum_{j=1}^{N}\int_0^t b(W_s^{i,\varepsilon}-W_s^{j,\varepsilon})ds + \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\int_0^t A(V_s^{i,\varepsilon}-V_s^{j,\varepsilon}, W_s^{i,\varepsilon}-W_s^{j,\varepsilon})dB_s^{ij} + \varepsilon\hat{B}_t^i.$$

This is a $6N$-dimensional stochastic differential equation with measurable coefficients with at most linear growth (because $|b(x)| = 2|x|$ and $\|\sigma(x)\|^2 = \|A(y,x)\|^2 = \operatorname{Tr} a(x) = 2|x|^2$). Thanks to the additional noises, the diffusion matrix is strictly uniformly elliptic. Consequently, we can apply the result of Krylov [23, p 87] (the coefficients are assumed to be bounded in [23], but we can use a standard localization procedure or the results of Rozkosz and Slominski [27]): the system (11)-(12) has at least one (weak) solution.

Step 3. We now prove that

(13)
$$\frac{1}{N}\sum_1^N |V_t^{i,\varepsilon}-W_t^{i,\varepsilon}|^2 = \frac{1}{N}\sum_1^N |V_0^i-W_0^i|^2 + 6\varepsilon^2 t + \frac{2\varepsilon}{N}\sum_{i=1}^N \int_0^t (V_s^{i,\varepsilon}-W_s^{i,\varepsilon})\cdot(d\tilde{B}_s^i - d\hat{B}_s^i)$$
$$- \frac{2}{N^2}\sum_{i,j=1}^N \int_0^t [|V_s^{i,\varepsilon}-V_s^{j,\varepsilon}||W_s^{i,\varepsilon}-W_s^{j,\varepsilon}| - (V_s^{i,\varepsilon}-V_s^{j,\varepsilon})\cdot(W_s^{i,\varepsilon}-W_s^{j,\varepsilon})]ds.$$

This follows from a direct application of the Itô formula, together with the equalities

$$I_s^\varepsilon := \frac{1}{N^2}\sum_{i,j=1}^N \Big(2(V_s^{i,\varepsilon}-W_s^{i,\varepsilon})\cdot(b(V_s^{i,\varepsilon}-V_s^{j,\varepsilon})-b(W_s^{i,\varepsilon}-W_s^{j,\varepsilon}))$$
$$+ \|\sigma(V_s^{i,\varepsilon}-V_s^{j,\varepsilon})-A(V_s^{i,\varepsilon}-V_s^{j,\varepsilon}, W_s^{i,\varepsilon}-W_s^{j,\varepsilon})\|^2\Big)$$
$$= -\frac{2}{N^2}\sum_{i,j=1}^N [|V_s^{i,\varepsilon}-V_s^{j,\varepsilon}||W_s^{i,\varepsilon}-W_s^{j,\varepsilon}| - (V_s^{i,\varepsilon}-V_s^{j,\varepsilon})\cdot(W_s^{i,\varepsilon}-W_s^{j,\varepsilon})]$$

and

$$J_t^\varepsilon := \frac{2}{N\sqrt{N}}\sum_{i,j=1}^N \int_0^t (V_s^{i,\varepsilon}-W_s^{i,\varepsilon})\cdot\Big(\sigma(V_s^{i,\varepsilon}-V_s^{j,\varepsilon})-A(V_s^{i,\varepsilon}-V_s^{j,\varepsilon}, W_s^{i,\varepsilon}-W_s^{j,\varepsilon})\Big)dB_s^{ij} = 0$$

that we now check. Using that $B_s^{ij} = -B_s^{ji}$, $\sigma(-x)=\sigma(x)$ and $A(-x,-y)=A(x,y)$, we see that

$$J_t^\varepsilon = \frac{2}{N\sqrt{N}}\sum_{i<j}\int_0^t \Big((V_s^{i,\varepsilon}-V_s^{j,\varepsilon})-(W_s^{i,\varepsilon}-W_s^{j,\varepsilon})\Big)$$
$$\cdot\Big(\sigma(V_s^{i,\varepsilon}-V_s^{j,\varepsilon})-A(V_s^{i,\varepsilon}-V_s^{j,\varepsilon}, W_s^{i,\varepsilon}-W_s^{j,\varepsilon})\Big)dB_s^{ij}.$$

It then suffices to use that $(v - w)^*(\sigma(v) - A(v, w)) = 0$ by Lemma 27-(c) to conclude that $J_t^\varepsilon = 0$. Using next that $b(-x) = -b(x)$, we write

$$I_s^\varepsilon := \frac{1}{N^2} \sum_{i,j=1}^N \Big(((V_s^{i,\varepsilon} - V_s^{j,\varepsilon}) - (W_s^{i,\varepsilon} - W_s^{j,\varepsilon})) \cdot (b(V_s^{i,\varepsilon} - V_s^{j,\varepsilon}) - b(W_s^{i,\varepsilon} - W_s^{j,\varepsilon}))$$

$$+ \|\sigma(V_s^{i,\varepsilon} - V_s^{j,\varepsilon}) - A(V_s^{i,\varepsilon} - V_s^{j,\varepsilon}, W_s^{i,\varepsilon} - W_s^{j,\varepsilon})\|^2 \Big).$$

But $\|\sigma(x) - A(x, y)\|^2 = \operatorname{Tr} \sigma(x)\sigma^*(x) + \operatorname{Tr} A(x, y)A^*(x, y) - 2\operatorname{Tr} \sigma(x)A^*(x, y) = 2|x|^2 + 2|y|^2 - 2(|x||y| + x \cdot y)$ because $\sigma(x)\sigma^*(x) = a(x)$, $A(x, y)A^*(x, y) = a(y)$ by Lemma 27-(a), because $\operatorname{Tr} a(x) = 2|x|^2$, and because $\operatorname{Tr} \sigma(x)A^*(x, y) = |x||y| + x \cdot y$ by Lemma 27-(b). Also, since $b(x) = -2x$, we have $(x-y)\cdot(b(x)-b(y)) = -2|x|^2 - 2|y|^2 + 4x\cdot y$. All in all, $(x - y) \cdot (b(x) - b(y)) + \|\sigma(x) - A(x, y)\|^2 = -2|x||y| + 2x \cdot y$. This completes the step.

Step 4. The coefficients b, σ, A have at most linear growth and the initial conditions are bounded (for N fixed, since H_N is carried by $S_N \times S_N$). It is thus routine to verify that for all $p \geq 2$, all $T > 0$, $\sup_{\varepsilon \in (0,1)} \mathbb{E}[\sup_{[0,T]} \sum_1^N (|V_t^{i,\varepsilon}|^p + |W_t^{i,\varepsilon}|^p)] < \infty$ and that the family $\{(V_t^{i,\varepsilon}, W_t^{i,\varepsilon})_{i=1,\dots,N,t\geq 0}, \varepsilon \in (0, 1)\}$ is tight in $C([0, \infty), (\mathbb{R}^3)^{2N})$. We thus may consider a limit point $(V_t^i, W_t^i)_{i=1,\dots,N,t\geq 0}$ and we now check that it satisfies all the requirements of the statement. Exchangeability, as well as points (i) and (ii) (which concern only the initial conditions) are of course inherited from the fact that they are satisfied by $(V_t^{i,\varepsilon}, W_t^{i,\varepsilon})_{i=1,\dots,N,t\geq 0}$ for all $\varepsilon \in (0, 1)$. Point (iv) is easily obtained by passing to the limit as $\varepsilon \to 0$ in (13) (this uses only that $\sup_{\varepsilon \in (0,1)} \mathbb{E}[\sup_{[0,T]} \sum_1^N (|V_t^{i,\varepsilon}|^2 + |W_t^{i,\varepsilon}|^2)] < \infty$). Since b, σ are continuous (and even Lipschitz continuous), it is not hard to pass to the limit in (11) and to deduce that $(V_t^i)_{i=1,\dots,N,t\geq 0}$ is a weak solution to (3), whence $(V_t^i)_{i=1,\dots,N,t\geq 0} \sim \mathbf{L}^N(F_N)$. Using finally that $A(x, y)A^*(x, y) = a(y) = \sigma(y)\sigma^*(y)$ and that $A(-x, -y) = A(x, y)$, we deduce that for each $\varepsilon \in (0, 1)$, (12) can be rewritten in the same form as (11) (with another family of Brownian motions B^{ij}). We thus prove as previously that $(W_t^i)_{i=1,\dots,N,t\geq 0} \sim \mathbf{L}^N(G_N)$ and this completes the proof. Observe that although $(V_t^i, W_t^i)_{i=1,\dots,N,t\geq 0}$ satisfies all the required properties, it does not seem possible to check that it solves (11)-(12) with $\varepsilon = 0$. □

9.3. Rousset's inequality

The following lemma summarizes a few results found in [26].

LEMMA 29. – *Consider $f, g \in \mathcal{P}(\mathbb{R}^3)$ and $R \in \mathcal{H}(f, g)$ such that $\int_{\mathbb{R}^3} |v|^2 f(dv) = \int_{\mathbb{R}^3} |v|^2 g(dv) = 1$, $\int_{\mathbb{R}^3} v f(dv) = \int_{\mathbb{R}^3} v g(dv) = 0$ and $\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - w|^2 R(dv, dw) \leq 2$. Denote by $\rho(f)$ the spectral radius of $\int_{\mathbb{R}^3} vv^* f(dv) \in S_3^+$. Observe that $\rho(f) \in (0, 1]$, since $\int_{\mathbb{R}^3} vv^* f(dv)$ has trace 1. For all $q > 1$, there is a constant C_q depending only on q such that*

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - w|^2 R(dv, dw) \leq C_q(1 - \rho(f))^{-1}[m_{2+2q}(f + g)]^{1/q}[D(R)]^{1-1/q}.$$

where $D(R) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} [|v - x||w - y| - (v - x) \cdot (w - y)]R(dv, dw)R(dx, dy)$.

We start with the following lemma [26, Theorem 1.4], of which we sketch the proof for completeness.

LEMMA 30. – *Consider $f, g \in \mathcal{P}(\mathbb{R}^3)$ and $R \in \mathcal{H}(f, g)$ such that $\int_{\mathbb{R}^3} |v|^2 f(dv) = \int_{\mathbb{R}^3} |v|^2 g(dv) = 1$ and $\int_{\mathbb{R}^3} v f(dv) = \int_{\mathbb{R}^3} v g(dv) = 0$. For $\rho(f)$ the spectral radius of $\int_{\mathbb{R}^3} v v^* f(dv)$, it holds that*

$$D'(R) \geq 2(1 - \rho(f))\Big(1 - \Big[\int_{\mathbb{R}^3 \times \mathbb{R}^3} (v \cdot w) R(dv, dw)\Big]^2\Big).$$

where $D'(R) := \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} [|v - x|^2 |w - y|^2 - ((v - x) \cdot (w - y))^2] R(dv, dw) R(dx, dy)$.

Proof. – Consider two independent couples (U, V) and $(\tilde{U}, \tilde{V})$ with law R. Using that $\mathbb{E}[U] = \mathbb{E}[V] = 0$ and $\mathbb{E}[|U|^2] = \mathbb{E}[|V|^2] = 1$, a straightforward tedious computation shows that

$$D'(R) = \mathbb{E}[|U - \tilde{U}|^2 |V - \tilde{V}|^2 - [(U - \tilde{U}) \cdot (V - \tilde{V})]^2] = 2(A + B + C + D),$$

where $A = \mathbb{E}[|U|^2 |V|^2 - (U \cdot V)^2]$, $B = 1 - \mathbb{E}[U \cdot V]^2$, $C = \mathbb{E}[(U \cdot \tilde{U})(V \cdot \tilde{V})] - \mathbb{E}[(U \cdot \tilde{V})^2]$ and $D = \mathbb{E}[(U \cdot \tilde{U})(V \cdot \tilde{V})] - \mathbb{E}[(U \cdot \tilde{V})(\tilde{U} \cdot V)]$. Clearly, $A \geq 0$ and it is not hard to verify that $D = \sum_{k,l=1}^{3} (\mathbb{E}[U_k V_l]^2 - \mathbb{E}[U_k V_l]\mathbb{E}[U_l V_k]) \geq 0$. Next, $C = \sum_{k,l=1}^{3} (\mathbb{E}[U_k V_l]^2 - \mathbb{E}[U_k U_l]\mathbb{E}[V_k V_l])$. Working in an orthonormal basis in which $(\mathbb{E}[U_k U_l])_{k,l}$ is diagonal and using that $\rho(f) = \max_k \mathbb{E}[U_k^2]$,

$$-C \leq \sum_{k=1}^{3} (\mathbb{E}[U_k^2]\mathbb{E}[V_k^2] - \mathbb{E}[U_k V_k]^2) \leq \rho(f) \sum_{k=1}^{3} \Big(\mathbb{E}[V_k^2] - \frac{\mathbb{E}[U_k V_k]^2}{\mathbb{E}[U_k^2]}\Big) = \rho(f)\Big(1 - \sum_{k=1}^{3} \frac{\mathbb{E}[U_k V_k]^2}{\mathbb{E}[U_k^2]}\Big)$$

because $\mathbb{E}[|U|^2] = 1$. But by Cauchy-Scwharz's inequality (and since, again, $\mathbb{E}[|U|^2] = 1$),

$$\mathbb{E}[U \cdot V]^2 = \Big(\sum_{k=1}^{3} \mathbb{E}[U_k V_k]\Big)^2 \leq \Big(\sum_{k=1}^{3} \frac{\mathbb{E}[U_k V_k]^2}{\mathbb{E}[U_k^2]}\Big)\Big(\sum_{k=1}^{3} \mathbb{E}[U_k^2]\Big) = \sum_{k=1}^{3} \frac{\mathbb{E}[U_k V_k]^2}{\mathbb{E}[U_k^2]}.$$

Finally, $-C \leq \rho(f)(1 - \mathbb{E}[U \cdot V]^2)$ and $D'(R) \geq 2B + 2C \geq 2(1 - \rho(f))(1 - \mathbb{E}[U \cdot V]^2)$. □

Proof of Lemma 29. – Using

$$\int_{\mathbb{R}^3} |v|^2 f(dv) = \int_{\mathbb{R}^3} |v|^2 g(dv) = 1 \text{ and } \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - w|^2 R(dv, dw) \leq 2,$$

we deduce that $0 \leq \int_{\mathbb{R}^3 \times \mathbb{R}^3} (v \cdot w) R(dv, dw) \leq 1$ and then that

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - w|^2 R(dv, dw) = 2 - 2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} (v \cdot w) R(dv, dw) \leq 2 - 2\Big(\int_{\mathbb{R}^3 \times \mathbb{R}^3} (v \cdot w) R(dv, dw)\Big)^2.$$

Applying next Lemma 30, we find

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v - w|^2 R(dv, dw) \leq (1 - \rho(f))^{-1} D'(R).$$

Using that (recall that $q > 1$)

$$|X|^2 |Y|^2 - (X \cdot Y)^2 \leq [|X||Y| - (X \cdot Y)][2|X||Y|] \leq [|X||Y| - (X \cdot Y)]^{1-1/q}[2|X||Y|]^{1+1/q}$$

and the Hölder inequality, we see that $D'(R) \leq D(R)^{1-1/q}(K_q(R))^{1/q}$, where we have set $K_q(R) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} [2|v - x||w - y|]^{q+1} R(dv, dw) R(dx, dy)$. To conclude, it suffices to observe that $K_q(R) \leq C_q m_{2q+2}(f + g)$, which immediately follows from the fact that $R \in \mathcal{H}(f, g)$. □

9.4. Conclusion

We can now give the

Proof of Theorem 24. – We fix $N \geq 7$, $p \in [0, (N-6)/2]$ and some exchangeable $F_N \in \mathscr{P}(S_N)$. We put $q = p + 1$. We apply Lemma 28 with $G_N = \mathscr{U}(S_N)$ to build a coupling $(V_t^{i,N}, W_t^{i,N})_{i=1,\dots,N,t\geq 0}$ between $\mathbf{L}^N(F_N)$ and $\mathbf{L}^N(\mathscr{U}(S_N))$. We introduce the notation $U_t^N = N^{-1} \sum_1^N |V_t^{i,N} - W_t^{i,N}|^2$ and $u_t^N = \mathbb{E}[U_t^N]$. We also set $\mu_t^N = N^{-1} \sum_1^N \delta_{V_t^{i,N}}$, $\nu_t^N = N^{-1} \sum_1^N \delta_{W_t^{i,N}}$, as well as $\zeta_t^N = N^{-1} \sum_1^N \delta_{(V_t^{i,N}, W_t^{i,N})}$. Lemma 28-(iv) precisely says that $U_t^N = U_0^N - 2\int_0^t D(\zeta_s^N)ds$, with D defined in Lemma 29. Since $U_0^N \leq 2$ by Lemma 28-(ii), we deduce that a.s., $U_t^N \leq 2$ for all $t \geq 0$.

We now apply Lemma 29 with $R = \zeta_t^N \in \mathscr{H}(\mu_t^N, \nu_t^N)$, which is licit since $\int_{\mathbb{R}^3} v\mu_t^N(dv) = \int_{\mathbb{R}^3} v\nu_t^N(dv) = 0$ and $\int_{\mathbb{R}^3} |v|^2 \mu_t^N(dv) = \int_{\mathbb{R}^3} |v|^2 \nu_t^N(dv) = 1$ (because both F_N and G_N are carried by S_N) and since $U_t^N = \int_{\mathbb{R}^3 \times \mathbb{R}^3} |v-w|^2 \zeta_t^N(dv, dw) \leq 2$: we deduce that

$$U_t^N \leq C_q(1 - \rho(\nu_t^N))^{-1}[m_{2+2q}(\mu_t^N + \nu_t^N)]^{1/q}[D(\zeta_t^N)]^{1-1/q}.$$

Taking expectations and using the triple Hölder inequality (with $2q$, $2q$ and $q/(q-1)$),

$$u_t^N \leq C_q \mathbb{E}[(1-\rho(\nu_t^N))^{-2q}]^{1/(2q)} \mathbb{E}[(m_{2+2q}(\mu_t^N + \nu_t^N))^2]^{1/(2q)} \mathbb{E}[D(\zeta_t^N)]^{1-1/q}.$$

Since $(W_t^{i,N})_{i=1,\dots,N} \sim \mathscr{U}(S_N)$ for each $t \geq 0$ by Remark 23, we infer from Lemma 25-(ii) that $\mathbb{E}[(m_{2+2q}(\nu_t^N))^2] \leq \mathbb{E}[|W_t^{1,N}|^{4q+4}] \leq C_q$ and from Lemma 25-(iii), since $\rho(\nu_t^N)$ is the spectral radius of $\int_{\mathbb{R}^3} vv^*\nu_t^N(dv) = N^{-1} \sum_1^N W_t^{i,N}(W_t^{i,N})^*$ and since $2q \leq N - 4$, that $\mathbb{E}[(1 - \rho(\nu_t^N))^{-2q}] \leq C_q$. Also, we know from Proposition 13 that $\mathbb{E}[(m_{2+2q}(\mu_t^N))^2] \leq \mathbb{E}[|V_t^{1,N}|^{4q+4}] \leq C_q \mathbb{E}[|V_0^{1,N}|^{4q+4}]$. We end with

$$u_t^N \leq C_q \mathbb{E}[1 + |V_0^{1,N}|^{4q+4}]^{1/(2q)} \mathbb{E}[D(\zeta_t^N)]^{1-1/q}.$$

Recalling that $U_t^N = U_0^N - 2\int_0^t D(\zeta_s^N)ds$, we conclude that, for some $c_q > 0$ depending only on q,

$$\frac{d}{dt}u_t^N = -2\mathbb{E}[D(\zeta_t^N)] \leq -c_q \mathbb{E}[1 + |V_0^{1,N}|^{4q+4}]^{-1/(2(q-1))}(u_t^N)^{q/(q-1)}.$$

Integrating this inequality, we find, recalling that $p = q - 1$ and setting $\kappa_p = c_q/(q-1)$,

$$u_t^N \leq \left(\kappa_p \mathbb{E}[1 + |V_0^{1,N}|^{8+4p}]^{-1/(2p)} t + (u_0^N)^{-1/p}\right)^{-p}.$$

By construction, since $\mathbf{L}_t^N(\mathscr{U}(S_N)) = \mathscr{U}(S_N)$ for all $t \geq 0$, we have $N^{-1}\mathcal{W}_2^2(\mathbf{L}_t^N(F_N), \mathscr{U}(S_N)) \leq u_t^N$ and, by Lemma 28-(i)-(ii), $u_0^N = N^{-1}\mathcal{W}_2^2(F_N, \mathscr{U}(S_N)) \leq 2$. We conclude that

$$N^{-1}\mathcal{W}_2^2(\mathbf{L}_t^N(F_N), \mathscr{U}(S_N)) \leq \left(\kappa_p \mathbb{E}[1 + |V_0^{1,N}|^{8+4p}]^{-1/(2p)} t + (N^{-1}\mathcal{W}_2^2(F_N, \mathscr{U}(S_N)))^{-1/p}\right)^{-p}.$$

On the one hand, this implies that $N^{-1}\mathcal{W}_2^2(\mathbf{L}_t^N(F_N), \mathscr{U}(S_N)) \leq N^{-1}\mathcal{W}_2^2(F_N, \mathscr{U}(S_N))$. On the other hand, this gives

$$N^{-1}\mathcal{W}_2^2(\mathbf{L}_t^N(F_N), \mathscr{U}(S_N)) \leq (\kappa_p \mathbb{E}[1 + |V_0^{1,N}|^{8+4p}]^{-1/(2p)} t + 2^{-1/p})^{-p},$$

which is controlled by $C_p \mathbb{E}[1 + |V_0^{1,N}|^{8+4p}]^{1/2}(1+t)^{-p}$. $\qquad\square$

9.5. Uniform propagation of chaos

We start with a consequence of Theorem 24.

COROLLARY 31. – *Assume that* $\gamma = 0$, *fix* $N \geq 2$ *and consider some exchangeable* S_N-*valued initial condition* $(V_0^{i,N})_{i=1,\ldots,N}$, *the corresponding solution* $(V_t^{i,N})_{i=1,\ldots,N,t\geq0}$ *to* (3) *and set* $\mu_t^N = N^{-1} \sum_1^N \delta_{V_t^{i,N}}$, *For all* $p > 0$, *there is a constant* C_p *depending only on* p *such that for all* $t \geq 0$,

$$\mathbb{E}[\mathscr{W}_2^2(\mu_t^N, \mathscr{N}(0, 3^{-1}\mathbf{I}_3))] \leq C_p\left(N^{-1/2} + \mathbb{E}[1 + |V_0^{1,N}|^{8+4p}]^{1/2}(1+t)^{-p}\right).$$

Proof. – Let $p > 0$ be fixed.

If first $N - 6 < 2p$, then we simply use that $\mathscr{W}_2^2(\mu_t^N, \mathscr{N}(0, 3^{-1}\mathbf{I}_3)) \leq 2$ a.s., so that the inequality obviously holds with $C_p = 2\sqrt{2p+6}$.

If next $N - 6 \geq 2p$, we use Theorem 24: for all $t \geq 0$, there is $(X_t^{i,N})_{i=1,\ldots,N} \sim \mathscr{U}(S_N)$ such that $N^{-1} \sum_1^N \mathbb{E}[|V_t^{i,N} - X_t^{i,N}|^2] \leq C_p\mathbb{E}[1 + |V_0^{1,N}|^{8+4p}]^{1/2}(1+t)^{-p}$. We now put $v_t^N = N^{-1} \sum_1^N \delta_{X_t^{i,N}}$ and we know from Lemma 25 that $\mathbb{E}[\mathscr{W}_2^2(v_t^N, \mathscr{N}(0, 3^{-1}\mathbf{I}_3))] \leq CN^{-1/2}$. But $\mathscr{W}_2^2(\mu_t^N, v_t^N) \leq N^{-1} \sum_1^N |V_t^{i,N} - X_t^{i,N}|^2$, whence

$$\mathbb{E}[\mathscr{W}_2^2(\mu_t^N, v_t^N)] \leq C_p\mathbb{E}[1 + |V_0^{1,N}|^{8+4p}]^{1/2}(1+t)^{-p}. \qquad \square$$

We finally give the

Proof of Theorem 4-(i). – Recall that $\gamma = 0$, that $f_0 \in \mathscr{P}_2(\mathbb{R}^3)$ and that $(f_t)_{t\geq0}$ is the unique weak solution to (1). We assume without loss of generality that $\int_{\mathbb{R}^3} vf_0(dv) = 0$ and that $m_2(f_0) = 1$. For each $N \geq 2$, we consider an exchangeable $(\mathbb{R}^3)^N$-valued random variable $(V_0^{i,N})_{i=1,\ldots,N}$ and the solution $(V_t^{i,N})_{i=1,\ldots,N,t\geq0}$ to (3). We set $\mu_t^N = N^{-1} \sum_1^N \delta_{V_t^{i,N}}$. We assume that for all $p \geq 2$, $M_p := m_p(f_0) + \sup_{N\geq2} \mathbb{E}[|V_0^{1,N}|^p] < \infty$. The constants are allowed to depend on upper bounds of $\{M_p, p \geq 2\}$ and on some upper bound of $H(f_0)$ when it is finite. We fix $\eta \in (0, 1/5)$.

Step 1. By Theorem 22, we have
(i) $\mathbb{E}[\mathscr{W}_2^2(\mu_t^N, f_t)] \leq C_\eta(1+t)^{5/2}(\mathbb{E}[\mathscr{W}_2^2(\mu_0^N, f_0)] + N^{\eta-1/4})$ in general;
(ii) $\mathbb{E}[\mathscr{W}_2^2(\mu_t^N, f_t)] \leq C_\eta(1+t)^{5/2}(\mathbb{E}[\mathscr{W}_2^2(\mu_0^N, f_0)] + N^{\eta-1/3})$ if $H(f_0) < \infty$.

Step 2. Here we verify that for any $p > 0$,

$$\mathbb{E}[\mathscr{W}_2^2(\mu_t^N, \mathscr{N}(0, 3^{-1}\mathbf{I}_3))] \leq C_p(1+t)^{-p} + C_pN^{-1/2} + C\mathbb{E}[\mathscr{W}_2^2(\mu_0^N, f_0)].$$

We put $m_0^N = N^{-1} \sum_1^N V_0^{i,N}$ and $E_0^N = N^{-1} \sum_1^N |V_0^{i,N} - m_0^N|^2$.

On the event $\Omega_N = \{E_0^N \geq 1/4\}$, we set $\hat{V}_t^{i,N} = (V_t^{i,N} - m_0^N)/\sqrt{E_0^N}$ and $\hat{\mu}_t^N = N^{-1} \sum_1^N \delta_{\hat{V}_t^{i,N}}$. Conditionally on Ω_N, $(\hat{V}_0^{i,N})_{i=1,\ldots,N}$ is exchangeable and takes values in S_N, so that we can apply Corollary 31:

$$\mathbb{E}[\mathbb{1}_{\Omega_N} \mathscr{W}_2^2(\hat{\mu}_t^N, \mathscr{N}(0, 3^{-1}\mathbf{I}_3))] \leq \frac{C_p}{\sqrt{N}} + \frac{C_p\mathbb{E}[1 + \mathbb{1}_{\Omega_N}|\hat{V}_0^{1,N}|^{8+4p}]^{1/2}}{(1+t)^p} \leq \frac{C_p}{\sqrt{N}} + \frac{C_p}{(1+t)^p}.$$

For the last inequality, we used the fact that $|\hat{V}_0^{1,N}| \leq 4|V_0^{1,N}| + 4|m_0^N|$ on Ω_N, whence $\mathbb{E}[\mathbb{1}_{\Omega_N}|\hat{V}_0^{1,N}|^{8+4p}] \leq C_p\mathbb{E}[|V_0^{1,N}|^{8p+4}] + C_p\mathbb{E}[|m_0^N|^{8p+4}] \leq C_p\mathbb{E}[|V_0^{1,N}|^{8p+4}]$, which is bounded by assumption.

Next, we write

$$\mathscr{W}_2^2(\hat{\mu}_t^N, \mu_t^N) \le \frac{1}{N} \sum_1^N |V_0^{i,N} - \hat{V}_0^{i,N}|^2$$

$$= \frac{1}{N} \sum_1^N \left| (V_t^{i,N} - m_0^N) \frac{\sqrt{E_0^N} - 1}{\sqrt{E_0^N}} + m_0^N \right|^2 = \left(\sqrt{E_0^N} - 1\right)^2 + |m_0^N|^2.$$

By Remark 26, we have $(\sqrt{E_0^N} - 1)^2 + |m_0^N|^2 \le \mathscr{W}_2^2(\mu_0^N, f_0)$, because $m_{\mu_0^N} = m_0^N$, $V_{\mu_0^N} = E_0^N$, $m_{f_0} = 0$ and $V_{f_0} = 1$. At this point, we have proved that

$$\mathbb{E}[\mathbb{I}_{\Omega_N} \mathscr{W}_2^2(\mu_t^N, \mathcal{N}(0, 3^{-1}\mathbf{I}_3))] \le C_p N^{-1/2} + C_p(1 + t)^{-p} + 2\mathbb{E}[\mathscr{W}_2^2(\mu_0^N, f_0)].$$

We next observe that

$$\mathbb{E}[\mathbb{I}_{\Omega_N^c} \mathscr{W}_2^2(\mu_t^N, \mathcal{N}(0, 3^{-1}\mathbf{I}_3))] \le \mathbb{E}[\mathbb{I}_{\{E_0^N < 1/4\}} m_2(\mu_t^N + \mathcal{N}(0, 3^{-1}\mathbf{I}_3))]$$

$$= \mathbb{E}[\mathbb{I}_{\{E_0^N < 1/4\}}(E_0^N + |m_0^N|^2 + 1)] \le (5/4)\Pr(E_0^N < 1/2) + \mathbb{E}[|m_0^N|^2].$$

But, $\Pr(E_0^N < 1/4) \le \Pr(|\sqrt{E_0^N} - 1| > 1/2) \le 4\mathbb{E}[|\sqrt{E_0^N} - 1|^2]$, and we have checked that $\mathbb{E}[\mathbb{I}_{\Omega_N^c} \mathscr{W}_2^2(\mu_t^N, \mathcal{N}(0, 3^{-1}\mathbf{I}_3))] \le 5\mathbb{E}[|\sqrt{E_0^N} - 1|^2] + \mathbb{E}[|m_0^N|^2]$, which is controlled by $5\mathbb{E}[\mathscr{W}_2^2(\mu_0^N, f_0)]$ as seen a few lines above.

Step 3. We deduce that $\mathscr{W}_2^2(f_t, \mathcal{N}(0, 3^{-1}\mathbf{I}_3)) \le C_p(1 + t)^{-p}$: assume (only during this step) that $(V_0^{i,N})_{i=1,\dots,N}$ consists of i.i.d. f_0-distributed random variables, so that $\mathbb{E}[\mathscr{W}_2^2(\mu_0^N, f_0)] \le CN^{-1/2}$ by [16, Theorem 1].
    Write

$$\mathscr{W}_2^2(f_t, \mathcal{N}(0, 3^{-1}\mathbf{I}_3)) \le 2\mathbb{E}[\mathscr{W}_2^2(f_t, \mu_t^N)] + 2\mathbb{E}[\mathscr{W}_2^2(\mu_t^N, \mathcal{N}(0, 3^{-1}\mathbf{I}_3))]$$

$$\le C_\eta(1 + t)^{5/2}(N^{-1/2} + N^{\eta-1/4}) + C_p(1 + t)^{-p} + CN^{-1/2}$$

by Steps 1 and 2. It then suffices to let $N \to \infty$.

Step 4. We now conclude the proof in the general case.
    If $(1 + t)^{5/2} \le (\mathbb{E}[\mathscr{W}_2^2(\mu_0^N, f_0)] + N^{-1/4})^{-\eta}$, then we use Step 1-(i) to write

$$\mathbb{E}[\mathscr{W}_2^2(\mu_t^N, f_t)] \le C_\eta(\mathbb{E}[\mathscr{W}_2^2(\mu_0^N, f_0)] + N^{-1/4})^{-\eta}(\mathbb{E}[\mathscr{W}_2^2(\mu_0^N, f_0)] + N^{\eta-1/4})$$

$$\le C_\eta(\mathbb{E}[\mathscr{W}_2^2(\mu_0^N, f_0)] + N^{-1/4})^{1-5\eta}.$$

If now $(1 + t)^{5/2} > (\mathbb{E}[\mathscr{W}_2^2(\mu_0^N, f_0)] + N^{-1/4})^{-\eta}$, then we use Steps 2 and 3 with $p = 5/(2\eta)$ to write $\mathbb{E}[\mathscr{W}_2^2(\mu_t^N, f_t)] \le C_\eta(1 + t)^{-5/(2\eta)} + C_\eta N^{-1/2} + C\mathbb{E}[\mathscr{W}_2^2(\mu_0^N, f_0)]$. But $(1 + t)^{-5/(2\eta)} \le \mathbb{E}[\mathscr{W}_2^2(\mu_0^N, f_0)] + N^{-1/4}$ and we end with $\mathbb{E}[\mathscr{W}_2^2(\mu_t^N, f_t)] \le C_\eta(\mathbb{E}[\mathscr{W}_2^2(\mu_0^N, f_0)] + N^{-1/4})$.
    Thus $\sup_{[0,\infty)} \mathbb{E}[\mathscr{W}_2^2(\mu_t^N, f_t)] \le C_\eta(\mathbb{E}[\mathscr{W}_2^2(\mu_0^N, f_0)] + N^{-1/4})^{1-5\eta}$ as desired.

Step 5. We finally conclude when $H(f_0) < \infty$.

If $(1+t)^{5/2} \le (\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] + N^{-1/3})^{-\eta}$, then we use Step 1 to write

$$\mathbb{E}[\mathcal{W}_2^2(\mu_t^N, f_t)] \le C_\eta (\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] + N^{-1/3})^{-\eta} (\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] + N^{\eta - 1/3})$$
$$\le C_\eta (\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] + N^{-1/3})^{1-4\eta}.$$

If now $(1+t)^{5/2} > (\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] + N^{-1/3})^{-\eta}$, then we use Steps 2 and 3 with $p = 5/(2\eta)$ to write $\mathbb{E}[\mathcal{W}_2^2(\mu_t^N, f_t)] \le C_\eta (1+t)^{-5/(2\eta)} + C_\eta N^{-1/2} + C\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)]$. But $(1+t)^{-5/(2\eta)} \le \mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] + N^{-1/3}$ and we end with $\mathbb{E}[\mathcal{W}_2^2(\mu_t^N, f_t)] \le C_\eta (\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] + N^{-1/3})$.

We conclude that $\sup_{[0,\infty)} \mathbb{E}[\mathcal{W}_2^2(\mu_t^N, f_t)] \le C_\eta (\mathbb{E}[\mathcal{W}_2^2(\mu_0^N, f_0)] + N^{-1/3})^{1-4\eta}$ as desired. $\square$

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