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A PROOF OF THE LANDAU-GINZBURG/ CALABI-YAU CORRESPONDENCE VIA THE CREPANT TRANSFORMATION CONJECTURE

BY Yuan-Pin LEE, Nathan PRIDDIS AND Mark SHOEMAKER

Abstract. – We establish a new relationship (the MLK correspondence) between twisted FJRW theory and local Gromov-Witten theory in all genera. As a consequence, we show that the Landau-Ginzburg/Calabi-Yau correspondence is implied by the crepant transformation conjecture for Fermat type in genus zero. We use this to then prove the Landau-Ginzburg/Calabi-Yau correspondence for Fermat type, generalizing the results of A. Chiodo and Y. Ruan in [7].

Résumé. – Nous établissons une nouvelle relation (la correspondance MLK) entre la théorie FJRW twistée et la théorie de Gromov-Witten en tout genre. Cela nous permet de montrer que la conjecture de la transformation crépante pour le type de Fermat en genre zéro implique la correspondance de Landau-Ginzburg/Calabi-Yau. Nous nous servons de ce résultat pour prouver la correspondance de Landau-Ginzburg/Calabi-Yau pour le type de Fermat, généralisant les résultats de A. Chiodo et Y. Ruan de [6].

0. Introduction

The crepant transformation conjecture ([17, 18]) describes a relationship between the Gromov-Witten theories of K-equivalent varieties in terms of analytic continuation and symplectic transformation. Another conjecture inspired by physics, the Landau-Ginzburg/Calabi-Yau (LG/CY) correspondence ([34, 23, 7]), proposes a similar relationship between the Gromov-Witten theory of a Calabi-Yau variety and the FJRW theory of a singularity. The primary goal of this paper is to relate these two conjectures.

0.1. FJRW theory and its relatives

FJRW theory was constructed by Fan, Jarvis and Ruan ([19]) as a “Landau-Ginzburg (LG) A model” to verify a conjecture of Witten ([33]). The construction gives a cohomological field theory defined by a virtual class on a cover of the moduli space of curves. It may be viewed as an analogue of Gromov-Witten theory, yielding invariants of a singularity rather than a smooth variety. Roughly, the input of the theory is an *LG pair* (Q, G) where

Q is a quasi-homogeneous polynomial $Q : \mathbb{C}^N \to \mathbb{C}$, and G an *admissible group* of diagonal automorphisms of Q (See Section 1.1). The moduli space is defined to be N-tuples of line bundles on curves, $\mathscr{L}_i \to \mathscr{C}$, such that for each monomial $Q_s = Q_s(x_1, \dots, x_N)$ in Q, $Q_s(\mathscr{L}_1, ..., \mathscr{L}_N) \cong \omega_{\mathscr{C},\log}$, the log-canonical bundle. The most difficult part of the construction is to define the virtual classes. This was done in the analytic category by Fan-Jarvis-Ruan in [19] and in the algebraic category by Polishchuk-Vaintrob in [29].

This paper begins with the observation that in the moduli problem for FJRW theory, one may replace the log-canonical bundle with a power of the log-canonical bundle. This yields new collection of moduli spaces, and for any given power of the log-canonical bundle one may construct a corresponding cohomological field theory. For instance, if one takes the zeroth power, i.e., if we consider the trivial line bundle, then one recovers the orbifold GW theory for the abelian quotient stack $[\mathbb{C}^N/G]$.

It turns out that many of the cohomological field theories associated to the various choices of power of $\omega_{\mathscr{C},\log}$ are equivalent in a precise sense. We dub this the “multiple log-canonical” or *MLK correspondence* (Theorem 5.5). The first important example of this correspondence is the following: restrict to LG pairs (Q, G) where Q is a quasi-homogeneous polynomial of Fermat type i.e., $Q = \sum_{i=1}^N x_i^{d/c_i}$. In this case the MLK correspondence has the following interesting implication, that the genus zero FJRW theory of (Q, G) is determined completely (and explicitly) from the genus zero orbifold GW theory of $[\mathbb{C}^N/G]$ (Theorem 5.11).

0.2. The crepant transformation conjecture and the LG/CY correspondence

The crepant transformation conjecture (CTC) (see [17, 18]) predicts that the Gromov-Witten invariants of K-equivalent varieties should be related. The relationship is complicated: generating functions of the respective invariants are identified via analytic continuation and symplectic transformation by an element Givental’s symplectic loop group ([21, 27]). At this point the CTC is considered well-understood, at least in genus zero, and has been established for a wide class of examples (see, e.g., [17, 16, 28]).

The LG/CY correspondence is a similar type of conjecture, this time connecting the Gromov-Witten theory of certain Calabi-Yau varieties to the FJRW theory a related singularity. The first mathematical proof of the LG/CY correspondence was given by Chiodo and Ruan for the quintic threefold ([7]). When Q is the Fermat quintic in five variables and $G = \operatorname{diag}(\mu_5)$, they proved that the genus zero FJRW theory of (Q, G) is equivalent to genus zero Gromov-Witten theory of the quintic hypersurface $Z(Q) = \{Q = 0\}$ in $\mathbb{P}^4$. The identification of the two theories is striking in that it takes exactly the same form as in the CTC, namely, analytic continuation and symplectic transformation.

Via the MLK correspondence, we explain this similarity by proving that a general form of the LG/CY correspondence can be deduced from the CTC. Symbolically we write:

$$\text{CTC} \Rightarrow \text{LG/CY}.$$

More precisely, let $\mathbb{P}(G) := [\mathbb{P}(c_1, \dots, c_N)/\bar{G}]$, where $\bar{G}$ is the quotient of G by those elements acting trivially on $[\mathbb{P}(c_1, \dots, c_N)]$. Let $K_{\mathbb{P}(G)}$ denote the total space of the canonical bundle over $\mathbb{P}(G)$ and let $Z(Q) \subset \mathbb{P}(G)$ be the Calabi-Yau orbifold defined by Q. Then the

LG/CY correspondence may be established by a special case of the CTC. The relationship is summarized in the following diagram:

$$\begin{tikzcd} GW_0(K_{P(G)}) \arrow[r, "\text{QSD}"] \arrow[d, leftrightarrow, "\text{CTC}"] & GW_0(Z(Q)) \arrow[d, leftrightarrow, "\text{LG/CY}"] \\ GW_0([\mathbb{C}^N/G]) \arrow[r, "\text{MLK}"] & \mathrm{FJRW}_0(Q,G). \end{tikzcd} \tag{0.2.1}$$

In the upper right corner is the genus zero GW theory for the Calabi-Yau orbifold Z(Q). The lower right corner is the genus zero FJRW theory associated to the LG pair (Q, G). The right vertical arrow is the LG/CY correspondence discussed above. The left vertical arrow is the CTC relating the genus zero orbifold GW theory of $[\mathbb{C}^N/G]$ with the genus zero GW theory of its crepant partial resolution $K_{\mathbb{P}(G)}$. The upper horizontal arrow is quantum Serre duality ([15]), which relates the GW theory of the total space of a line bundle with the GW theory of the hypersurface defined by a section of the line bundle. The MLK correspondence, established in Section 5.4.3, completes the square.

0.3. A remark on proofs

The LG/CY correspondence has previously been proven in genus zero in all cases where the Calabi-Yau is a hypersurface in projective space ([6]) as well as for the mirror quintic ([30]). In addition to generalizing the previous results, we hope that the proof described above provides a useful conceptual framework in which to view the LG/CY correspondence.

In the physics literature, the LG/CY correspondence is understood as an example of a *phase transition* ([34]). From this perspective, the origin of the correspondence is quite natural, and completely analogous to the wall crossing phenomenon arising in the CTC (see [34] and [23] for more information). However the mathematical proofs of the LG/CY correspondence to-date have been computational in nature, and do not shed light on why such a correspondence should hold. The present result then, which shows that the LG/CY correspondence in fact follows from the CTC, may be viewed as a mathematical justification for the relationship between these two conjectures, and a verification of the physical insight which predicts such a correspondence.

Beyond simply verifying the physical description of the LG/CY correspondence, the structure in (0.2.1) may be useful for better understanding the relationship in higher genus. Recall that both the CTC and LG/CY correspondence are given in terms of analytic continuation and symplectic transformation by an element of Givental's symplectic loop group ([21, 27]).

Due to the existence of stabilizers in the action of the symplectic loop group, relating the genus zero FJRW and GW theory in this correspondence requires one to make a choice of symplectic transformation. Crucially, two symplectic transformations which have the same effects on the genus zero theory might have quantizations which act differently on higher genus theories. Therefore, any correspondence in higher genus requires a canonical way of choosing the symplectic transformation relating the genus zero invariants.

In the case of the crepant transformation between K-equivalent varieties, there is indeed such a canonical choice, coming from the Fourier-Mukai functor in K-theory (see [26]). In the case of the LG/CY correspondence however, there is a priori no canonical choice of symplectic transformation. However we may obtain one via diagram (0.2.1). The symplectic transformation used for the CTC may be transported from the left hand side of (0.2.1) to the right hand side, yielding a canonical choice of transformation for the LG/CY correspondence. We predict that (the quantization of) this choice of transformation is the one which should be used to identify the higher genus FJRW and GW invariants.

0.4. Contents of the paper

In Section 1 we give a general construction of a cohomological field theory defined as a twisted theory over a generalization of the moduli of r-spin curves. In Section 2 we show how in special cases of the above construction one recovers the Gromov-Witten theory of local affine quotients as well as the genus zero FJRW theory of Fermat LG pairs. Section 3 gives a brief summary of Givental's symplectic formalism which we use in Section 4 to compute the cohomological field theories introduced earlier. In Section 5 we are able to state and prove the MLK correspondence, which relates the genus zero Gromov-Witten theory of affine quotients to the FJRW theory of Fermat LG pairs. We then apply this correspondence in Section 6 to show that the crepant transformation conjecture implies the LG/CY correspondence in a large number of cases. Finally in Section 7 we prove a version of the LG/CY correspondence for the cases of interest to us.

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1. Twisted invariants

1.1. W structures

Definition 1.1 ([7, Definition A.1]). – Let d be a non-negative integer. A *d-stable n-pointed genus h curve* is an n-pointed stable orbi-curve such that all marked points and nodes have cyclic stabilizers of order d and no other non-trivial stabilizers.

Notation 1.2. – Let $\overline{\mathscr{M}}^d_{h,n}$ denote the moduli space of d-stable n-pointed genus h curves. A d-stable curve (or a family of such) will always be denoted by $\mathscr{C}$. Let $\overline{\mathscr{A}}^{(d,c)}_{h,n}$ denote the moduli space of d-th roots of $\omega^{\otimes c}_{\mathscr{C}/\overline{\mathscr{M}},\log}$.

Let $Q : \mathbb{C}^N \to \mathbb{C}$ be a nondegenerate quasi-homogeneous polynomial, i.e., for $\alpha \in \mathbb{C}^*$,

$$Q(\alpha^{c_1} x_1, \ldots, \alpha^{c_N} x_N) = \alpha^d Q(x_1, \ldots, x_N),$$

where the c_j's are positive integers. We assume that $\gcd(c_1, \ldots, c_N) = 1$. The polynomial Q is said to have degree d with integer weights $c_1, \ldots, c_N$.

Let
$$G_Q = G_Q^{\max} := (\mathbb{C}^*)^N \cap \mathrm{Aut}(Q)$$
denote the (maximal) group of diagonal automorphisms of Q. We define a distinguished element $\mathfrak{j} \in G_Q$, the *grading* element, by
$$\mathfrak{j} := \big(\exp\big(2\pi i \frac{c_1}{d}\big), \ldots, \exp\big(2\pi i \frac{c_N}{d}\big)\big).$$
Let $\bar{d}$ denote the *period* of G_Q, defined as
$$\bar{d} := \max\left\{|g|\,\middle|\, g \in G\right\},$$
and let $\bar{c}_j = c_j\bar{d}/d$. Then we may write
$$\mathfrak{j} = \big(\exp\big(2\pi i \frac{\bar{c}_1}{\bar{d}}\big), \ldots, \exp\big(2\pi i \frac{\bar{c}_N}{\bar{d}}\big)\big),$$
which will be convenient since we will work on $\bar{d}$-stable curves.

Definition 1.3. – On a marked $\bar{d}$-stable curve $\mathscr{C}$, a W^c *structure* is the data of N $\bar{d}$-th roots of the log canonical bundle
$$(\mathscr{L}_j, \phi_j : \mathscr{L}_j^{\otimes \bar{d}} \xrightarrow{\cong} \omega_{\mathscr{C},\log}^{\otimes(c\cdot\bar{c}_j)})$$
which satisfy
$$Q_s(\mathscr{L}_1, \ldots, \mathscr{L}_N) \cong \omega_{\mathscr{C},\log}^{\otimes c} \tag{1.1.1}$$
for each monomial Q_s in Q.

Remark 1.4. – The "W" in W^c structures stands for E. Witten, whose ideas initiated the study of such moduli ([33]).

Definition 1.5. – The *moduli space* $W^c_{h,n}$ *of* W^c *structures* of Q is the open and closed substack of the fiber product
$$\overline{\mathscr{A}}^{(\bar{d},c\cdot\bar{c}_1)}_{h,n} \times_{\overline{\mathscr{M}}^{\bar{d}}_{h,n}} \cdots \times_{\overline{\mathscr{M}}^{\bar{d}}_{h,n}} \overline{\mathscr{A}}^{(\bar{d},c\cdot\bar{c}_N)}_{h,n}$$
consisting of those N-tuples
$$(\mathscr{L}_j, \phi_j : \mathscr{L}_j^{\otimes \bar{d}} \xrightarrow{\cong} \omega_{\mathscr{C},\log}^{\otimes(c\cdot\bar{c}_j)})$$
which satisfy (1.1.1) for all monomials Q_s in Q.

We now add the information of a group of automorphisms into the definition of our moduli space. A group $G \leq G_Q$ is *admissible* if $\mathfrak{j} \in G$. (See [19, Definition 2.3.2 and Proposition 2.3.5] for an alternative equivalent definition.)

Definition 1.6. – A *(gauged) Landau-Ginzburg (LG) pair* is a pair (Q, G) where Q is a nondegenerate quasi-homogeneous polynomial and G is an admissible subgroup of G_Q.

Notation 1.7. – Given $g \in G$, let $m_j(g)$ denote the *multiplicity* of g on the jth factor of $\mathbb{C}^N$. In other words, g acts on $\mathbb{C}^N$ via $G \subset (\mathbb{C}^*)^N$ by
$$(\exp(2\pi i m_1(g)), \ldots, \exp(2\pi i m_N(g)))$$
such that $0 \leq m_j(g) < 1$. Let $N_g := \dim(\mathbb{C}^N)^g = \#\{j | m_j(g) = 0\}$.

Let Q' denote a degree d Laurent polynomial with different monomials than Q such that the group $G_{Q+Q'}$ of diagonal automorphisms of $Q+Q'$ is exactly G.

DEFINITION 1.8. – The *moduli space $W^c_{h,n,G}$ of W^c structures of* (Q,G) is defined to be the moduli space of W^c structures of $Q+Q'$, where Q' is as above.

It is easy to show such Q' exists and that the above definition does not depend on a choice of Q'.

As a consequence of the definition of $W^c_{h,n,G}$, at each marked point p_i, the isotropy acts on fibers of $\bigoplus_{j=1}^N \mathscr{L}_j$ by an element of G. One may therefore break $W^c_{h,n,G}$ into open and closed substacks based on the action of the corresponding isotropy group. Let

$$W^c_{h,n,G}(g_1,\dots,g_n)$$

denote the substack where the isotropy at p_i acts by g_i. The following fact will be used later.

LEMMA 1.9 ([19]). – *Assume $n>0$, the stack $W^c_{h,n,G}$ splits into a disjoint union of open and closed substacks*

$$W^c_{h,n,G} = \coprod_{g_1,\dots,g_n\in G} W^c_{h,n,G}(g_1,\dots,g_n).$$

Furthermore $W^c_{h,n,G}(g_1,\dots,g_n)$ is nonempty if and only if

$$\frac{cc_j}{d}(2h-2+n) - \sum_{i=1}^n m_j(g_i) \in \mathbb{Z} \qquad 1\le j\le N. \tag{1.1.2}$$

The first statement is easy to see. The second statement is essentially proven in [19, Proposition 2.2.8]. The numerical condition (1.1.2) is established using the observation that the corresponding line bundles $|\mathscr{L}_j|$ on the coarse moduli have integral degrees, plus the calculation

$$\deg(|\mathscr{L}_j|) = \frac{cc_j}{d}(2h-2+n) - \sum_{i=1}^n m_j(g_i). \tag{1.1.3}$$

1.2. "Untwisted" theories

There is a map

$$W^c_{h,n,G} \to \overline{\mathscr{M}}_{h,n}$$

obtained by forgetting the line bundles $\mathscr{L}_j$ as well as the orbifold structure of the underlying curve. By pulling back ψ-classes from $\overline{\mathscr{M}}_{h,n}$ we obtain tautological classes on $W^c_{h,n,G}$. We can integrate these classes over the moduli space to obtain invariants.

Given a W^c structure of (Q,G), we introduce the W^c *state space* as a vector space formally generated by basis vectors ϕ^c_g for each $g\in G$,

$$H^c := \bigoplus_{g\in G} \mathbb{C}\phi^c_g.$$

Define the *untwisted W^c invariant*

$$\left\langle \psi^{a_1}\phi^c_{g_1},\dots,\psi^{a_n}\phi^c_{g_n}\right\rangle^c_{h,n} := \int_{W^c_{h,n,G}(g_1\mathfrak{j}^c,\dots,g_n\mathfrak{j}^c)} \prod_{i=1}^n \psi_i^{a_i}. \tag{1.2.1}$$

Note the shifting by $\mathfrak{j}^c$ in this definition. There is a pairing given by

$$\langle \phi^c_{g_1}, \phi^c_{g_2}\rangle^c := \big\langle \phi^c_{g_1}, \phi^c_{g_2}, \phi^c_e\big\rangle^c_{0,3}$$

where e is the identity element in G.

Although the definition of H^c looks somewhat contrived, the corresponding invariants should not. As it stands, H^c should be viewed as giving "place-holders" for the various connected components of the moduli space. The geometric meaning will be clear after we establish the relationship to Gromov-Witten and FJRW theory.

1.3. Twisted theories

Let $\mathbb{C}^*$ act on a W^c structure by acting on each line bundle. This induces an action on $W^c_{h,n,G}$.

NOTATION 1.10. – Let λ denote the equivariant parameter, and let $-\lambda_j$ denote the character of the action on the jth bundle (i.e., λ_j is a multiple of λ). We assume always that each character is nontrivial.

We may express an invertible multiplicative characteristic class as

$$\mathbf{s} : \oplus \mathscr{L}_j \mapsto \exp\left(\sum_{j=1}^{N}\sum_{k\geq 0} s^j_k \operatorname{ch}_k(\mathscr{L}_j)\right),$$

where

$$\exp(s^j_0),\ s^j_k \in \mathbb{C}[\lambda, \lambda^{-1}] \text{ for } 1 \leq j \leq N,\ k > 0.$$

We define the $\mathbf{s}$-*twisted virtual class* on $W^c_{h,n,G}$ as the class

$$[W^c_{h,n,G}]^{\mathbf{s}} := \mathbf{s}(R\pi_* \bigoplus_{j=1}^{N} \mathscr{L}_j) \cap [W^c_{h,n,G}]$$

and the *twisted invariants*

$$\big\langle \psi^{a_1}\phi^c_{g_1}, \dots, \psi^{a_n}\phi^c_{g_n}\big\rangle^{c,\mathbf{s}}_{h,n} := \int_{W^c_{h,n,G}(g_1\mathfrak{j}^c,\dots,g_n\mathfrak{j}^c)} \mathbf{s}(R\pi_* \bigoplus_{j=1}^{N}\mathscr{L}_j)\prod_{i=1}^{n}\psi_i^{a_i}. \tag{1.3.1}$$

We note that the shifting by $\mathfrak{j}^c$ is consistent with the definition of the untwisted invariants in (1.2.1).

There is an $\mathbf{s}$-twisted pairing given by

$$\begin{aligned}\langle \phi^c_{g_1}, \phi^c_{g_2}\rangle^{c,\mathbf{s}} &:= \big\langle \phi^c_{g_1}, \phi^c_{g_2}, \phi^c_e\big\rangle^{c,\mathbf{s}}_{0,3} \\ &= \exp\left(\sum_{j=1}^{N}\chi\big(R\pi_*(\mathscr{L}_j)\big)s^j_0\right)\delta_{g_1g_2=\mathfrak{j}^{-2c}}/\bar{d}^N\end{aligned} \tag{1.3.2}$$

defined on $H^c[\lambda, \lambda^{-1}]$. The last equality follows easily from the definition and the fact that $\overline{\mathscr{M}}_{0,3}$ is a point. This definition of the pairing is chosen to give a *Frobenius algebra* structure on H^c.

LEMMA 1.11. – 1. *When $s_k^j = 0$ for all j and k we recover the untwisted W^c invariants. In this case the pairing is simply*

$$\langle \phi^c_{g_1\mathfrak{j}^{-c}}, \phi^c_{g_2\mathfrak{j}^{-c}} \rangle^{c,0} = \frac{\delta_{g_1=g_2^{-1}}}{\bar{d}^N}.$$

2. *More generally,*

$$\langle \phi^c_{g_1\mathfrak{j}^{-c}}, \phi^c_{g_2\mathfrak{j}^{-c}} \rangle^{c,\mathbf{s}} = \exp\left(\sum_{j=1}^N \Big(\lfloor 1 - m_j(g_1) \rfloor + \lfloor \tfrac{cc_j}{d} \rfloor \Big) s_0^j \right) \frac{\delta_{g_1=g_2^{-1}}}{\bar{d}^N}.$$

In particular, when $cc_j < d$,

$$\langle \phi^c_{g_1\mathfrak{j}^{-c}}, \phi^c_{g_2\mathfrak{j}^{-c}} \rangle^{c,\mathbf{s}} = \exp\Big(\sum_{j=1}^N \big(\lfloor 1 - m_j(g_1) \rfloor \, s_0^j \big) \Big) \frac{\delta_{g_1=g_2^{-1}}}{\bar{d}^N}. \tag{1.3.3}$$

The condition $cc_j < d$ holds in particular for the cases $c = 0$ or 1.

Proof. – If $g_1 \neq g_2^{-1}$, the pairing is zero. (1) follows from (1.3.2). (2) follows from a simple orbifold Riemann-Roch calculation. From Equation (1.1.3), if $g_1 = g_2^{-1}$, we have

$$\begin{aligned} \deg(|\mathcal{L}_j|) &= \lfloor \tfrac{cc_j}{d} \rfloor - m_j(g_1) - m_j(g_2) \\ &= \begin{cases} \lfloor \frac{cc_j}{d} \rfloor & \text{if } m_j(g_1) = 0 \quad (\text{and } m_j(g_2) = 0), \\ \lfloor \frac{cc_j}{d} \rfloor - 1 & \text{if } m_j(g_1) \neq 0 \quad (\text{and } m_j(g_2) \neq 0). \end{cases} \end{aligned}$$

Then

$$\chi\big(R\pi_*(\mathcal{L}_j)\big) = \begin{cases} \lfloor \frac{cc_j}{d} \rfloor + 1 & \text{if } m_j(g_1) = 0, \\ \lfloor \frac{cc_j}{d} \rfloor & \text{if } m_j(g_1) \neq 0. \end{cases} \qquad \square$$

The data of the vector space $H^c[\lambda, \lambda^{-1}]$ together with the $\mathbf{s}$-twisted pairing are called the (equivariant) *twisted state space*, denoted by $H^{c,\mathbf{s}}$. The twisted state space and the $\mathbf{s}$-twisted W^c invariants give an *axiomatic Gromov-Witten theory*. See Section 3 for the definition.

2. Relations to other invariants

Twisted invariants of W^c structures give a general setting in which to describe other better known invariants. The first example is local invariants of a quotient of affine space and the second is the (genus zero) FJRW theory of Fermat-type Landau-Ginzburg pairs.

2.1. Local invariants of $[\mathbb{C}^N/G]$

Given a pair (Q, G) as before, we may define *local GW invariants* of $[\mathbb{C}^N/G]$. For $g \in G$, let $\mathbb{1}_g \in H^*_{CR}(BG)$ denote the fundamental class of the g-twisted sector of the inertia stack $I(BG)$ (see [4, 3, 1] for details on Chen-Ruan cohomology and orbifold Gromov-Witten theory). We view

$$[\mathbb{C}^N/G] \to BG$$

as a rank N equivariant vector bundle, where $\mathbb{C}^*$ acts on the jth factor with character $-\lambda_j$.

We define genus-h local invariants of $[\mathbb{C}^N/G]$ as

$$\begin{aligned}(2.1.1)\qquad \langle \psi^{a_1}\mathbb{1}_{g_1},\ldots,\psi^{a_n}\mathbb{1}_{g_n}\rangle_{h,n}^{[\mathbb{C}^N/G]} &:= \int_{[\overline{\mathscr{M}}_{h,n}([\mathbb{C}^N/G])]^{vir}} \prod_{i=1}^{n} \psi_i^{a_i} \cup ev_i^*(\mathbb{1}_{g_i}) \\ &:= \int_{\overline{\mathscr{M}}_{h,n}(BG)} \frac{\prod_{i=1}^{n} \psi_i^{a_i} \cup ev_i^*(\mathbb{1}_{g_i})}{e_{\mathbb{C}^*}\left(R\pi_* f^*[\mathbb{C}^N/G]\right)},\end{aligned}$$

where f is the universal map and π the universal curve

$$\begin{array}{ccc}\mathscr{C} & \xrightarrow{\ f\ } & BG \\ \Big\downarrow{\scriptstyle \pi} & & \\ \overline{\mathscr{M}}_{h,n}(BG). & & \end{array}$$

Note that the product $\prod_{i=1}^{n} ev_i^*(\mathbb{1}_{g_i})$ simply specifies an open and closed substack of $\overline{\mathscr{M}}_{h,n}(BG)$ over which to integrate. We would like to compare these integrals to $\mathbf{s}$-twisted W^c invariants.

First, observe that although $\overline{\mathscr{M}}_{h,n}(BG)$ consists of *representable* morphisms $\mathscr{C} \to BG$, one may also consider the moduli space $\overline{\mathscr{M}}^{\bar{d}}_{h,n}(BG)$ consisting of morphisms $\mathscr{C} \to BG$ from a $\bar{d}$-stable curve which are not necessarily representable.

LEMMA 2.1. – *There is a map*

$$\rho : \overline{\mathscr{M}}^{\bar{d}}_{h,n}(BG) \to \overline{\mathscr{M}}_{h,n}(BG)$$

where $\bar{d}$ is the period of G. Furthermore, ρ is an isomorphism over the open and dense locus consisting of non-nodal domain curves.

Proof. – By the r-th root construction and in particular [2, Theorem 4.1], there is a unique way of adding the $\mu_{\bar{d}}$ orbifold structure at the marked points. The lemma follows. □

Thus we may instead define local invariants as integrals over $\overline{\mathscr{M}}^{\bar{d}}_{h,n}(BG)$, where the integrand from (2.1.1) is pulled back via ρ.

LEMMA 2.2. – *There exists a natural morphism*

$$\pi : W^0_{h,n,G} \to \overline{\mathscr{M}}^{\bar{d}}_{h,n}(BG).$$

This map is a $\prod_{j=1}^{N}(\mu_{\bar{d}})$-gerbe.

Proof. – We first show the existence of the morphism $\pi : W^0_{g,n,G} \to \overline{\mathscr{M}}^{\bar{d}}_{g,n}(BG)$. Given $(\mathscr{C}, \mathscr{L}_1, \ldots, \mathscr{L}_N) \in W^0_{g,n,G}$, by construction there is a well defined G-action on each fiber of $\bigoplus_{j=1}^{N} \mathscr{L}_j$, coming from the inclusion $G < \prod_j \mu_{\bar{d}}$. The fact that the associated principal bundle is a G-bundle follows from the definition of the W^0 structure in Definition 1.8. This defines the morphism π. The fact that π is a gerbe can be seen from unraveling the definitions. □

REMARK 2.3. – Alternatively, one can verify the degree of π by the following observations. Firstly, the degree of $W^0_{h,n,G} \to \overline{\mathscr{M}}^{\bar{d}}_{h,n}$ is $|G|^{2h-1+n}/(\bar{d}^N)$. In fact the fiber is $|G|^{2h-1+n}$ copies of $\prod_{j=1}^N B\mu_{\bar{d}}$, with the automorphisms coming from automorphisms of each line bundle $\mathscr{L}_j$. Secondly, the moduli space $\overline{\mathscr{M}}^{\bar{d}}_{h,n}(BG)$ parameterizes curves $\mathscr{C}$ in $\overline{\mathscr{M}}^{\bar{d}}_{h,n}$ together with a homomorphism $\pi_1^{orb}(\mathscr{C}) \to G$. Thus the fiber is given by $|G|^{2h-1+n}$ points, which parameterize maps $\pi_1^{orb}(\mathscr{C}) \to G$. Combining above degree counts, one gets the degree count for $\pi : W^0_{h,n,G} \to \overline{\mathscr{M}}^{\bar{d}}_{h,n}(BG)$. In fact, a detailed analysis of the above two steps gives another verification of the second statement of Lemma 2.2.

Thus by the projection formula, integrals over $\overline{\mathscr{M}}^{d}_{h,n}(BG)$ coincide with those over $W^0_{h,n,G}$ up to a factor of $\bar{d}^N$. If we consider $\mathbf{s}$-twisted invariants with

$$e^{s_0^j} = -\frac{1}{\lambda_j} \text{ and } s_k^j = (k-1)!/\lambda_j^k \text{ for } 1 \le j \le N,\ k > 0,$$

then $\mathbf{s}([\mathbb{C}^N/G]) = 1/e_{\mathbb{C}^*}([\mathbb{C}^N/G])$. We finally arrive at the following relation.

COROLLARY 2.4. – *We have*

$$\bar{d}^N \left\langle \psi^{a_1}\phi^0_{g_1}, \ldots, \psi^{a_n}\phi^0_{g_n} \right\rangle^{0,\mathbf{s}}_{h,n} = \langle \psi^{a_1}\mathbb{1}_{g_1}, \ldots, \psi^{a_n}\mathbb{1}_{g_n} \rangle^{[\mathbb{C}^N/G]}_{h,n}.$$

In this way $\mathbf{s}$-twisted invariants of W^0 structures specialize to the local Gromov-Witten theory of a point.

2.2. FJRW invariants of Fermat polynomials

Given a Landau-Ginzburg pair (Q, G), Fan, Jarvis and Ruan have constructed a cohomological field theory called FJRW theory. The corresponding numerical invariants are likewise called FJRW invariants.

DEFINITION 2.5. – Given a Landau-Ginzburg pair (Q, G), the *narrow FJRW state space* is given by

$$\mathscr{H}_{FJRW}(Q, G) := \bigoplus_{g \in \hat{G}} \mathbb{C}\varphi_g,$$

where

$$\hat{G} := \{g \in G | gj \text{ fixes only the origin in } \mathbb{C}^N\}$$

and φ_g is a vector formally associated to $g \in \hat{G}$.

REMARK 2.6. – There is a larger FJRW state space which includes the so-called *broad sectors*, (subspaces corresponding to those $g \notin \hat{G}$) but we will restrict ourselves here to the narrow state space without loss of information. In fact all invariants involving broad sectors vanish due to the so called *Ramond vanishing* property. See Remark 2.3.2 of [7].

Similar to the case of local GW theory, one may specialize $\mathbf{s}$-twisted invariants of W^1 structures to recover genus zero FJRW invariants. When $c = 1$, there is a birational map from $W^1_{h,n,G}$ to the FJRW moduli space, denoted $\mathscr{W}_{h,n,G}$. Again we may define FJRW invariants as integrals over $W^1_{h,n,G}$ by pulling back classes on $\mathscr{W}_{h,n,G}$ via this map.

The construction of the FJRW virtual cycle is in general quite complicated, but in case Q is a Fermat polynomial and the genus is zero the situation simplifies greatly. In this case one can prove [19] that

$$R^0\pi_*(\bigoplus_{j=1}^N \mathscr{L}_j) = 0$$

and

$$-R\pi_*(\bigoplus_{j=1}^N \mathscr{L}_j) = R^1(\bigoplus_{j=1}^N \mathscr{L}_j)[-1]$$

is a vector bundle. Then by axiom (5a) of [19, Theorem 4.1.8],

(2.2.1)

$$\begin{aligned}\langle \psi^{a_1}\varphi_{0,g_1}, \ldots, \psi^{a_n}\varphi_{0,g_n}\rangle_{0,n}^{(Q,G)} &:= \bar{d}^N \int_{W_{0,n,G}(g_1\mathrm{j},\ldots,g_n\mathrm{j})} \frac{\prod_{i=1}^n \psi_i^{a_i}}{e\left(R\pi_*(\bigoplus_{j'=1}^N \mathscr{L}_{j'})^\vee\right)} \\ &= (-1)^{\chi(\oplus\mathscr{L}_{j'})}\bar{d}^N \int_{W_{0,n,G}(g_1\mathrm{j},\ldots,g_n\mathrm{j})} \frac{\prod_{i=1}^n \psi_i^{a_i}}{e\left(R\pi_*(\bigoplus_{j'=1}^N \mathscr{L}_{j'})\right)}\end{aligned}$$

for $g_i \in \hat{G}$.

Similar to the case of local GW theory, consider $\mathbf{s}'$-twisted invariants with

$$e^{s_0^{j\,\prime}} = \frac{1}{\lambda_j} \text{ and } s_k^{j\,\prime} = (k-1)!/\lambda_j^k \text{ for } 1 \le j \le N,\ k > 0. \tag{2.2.2}$$

Note that $e^{s_0^{j\,\prime}}$ differs from $e^{s_0^j}$ by a sign, this will alter the overall sign of our invariants by $(-1)^{\chi(\oplus\mathscr{L}_j)}$. We obtain a relation between the nonequivariant limit of $\mathbf{s}'$-twisted invariants and FJRW invariants:

Corollary 2.7. – *If $g_i \in \hat{G}$ for all i,*

$$\langle \psi^{a_1}\varphi_{g_1}, \ldots, \psi^{a_n}\varphi_{g_n}\rangle_{0,n}^{(Q,G)} = \lim_{\lambda\mapsto 0} \bar{d}^N \left\langle \psi^{a_1}\phi_{g_1}^0, \ldots, \psi^{a_n}\phi_{g_n}^0\right\rangle_{0,n}^{1,\mathbf{s}'}.$$

The inner product on the narrow state space (Definition 2.5) is defined as in (1.3.2). Due to narrowness condition, the pairing will not degenerate at the non-equivariant limit.

3. Givental's symplectic formalism

Motivated by the common structures in Gromov-Witten theory, Givental [22] has developed a formalism for dealing with "Gromov-Witten-like" theories, which we shall refer to as axiomatic Gromov-Witten theories (Definition 3.1). Although we will not give a complete description of such a theory here, we collect below several of the important facts which shall be used in the what follows. We refer the interested reader to [22] for more information.

Let $\square$ denote the data of a state space $\left(H^\square, \langle -, -\rangle_\square\right)$ and invariants

$$\langle \psi^{a_1}\beta_{i_1}, \ldots, \psi^{a_n}\beta_{i_n}\rangle_{g,n}^\square$$

for $\{\beta_i\}_{i\in I}$ a basis of $H^\square$. The examples of $\square$ to have in mind are Gromov-Witten theory, FJRW theory, or that of $\mathbf{s}$-twisted W^c structures.

We may define formal generating functions of $\square$ invariants. Let $\boldsymbol{t} = \sum_{i\in I} t^i\beta_i$ represent a point of $H^\square$ written in terms of the basis. For notational convenience denote the formal series $\sum_{k\geq 0} \boldsymbol{t}_k\psi^k$ as $\boldsymbol{t}(\psi)$. Define the genus g generating function by

$$\mathcal{F}_g^\square := \sum_n \frac{1}{n!}\langle \boldsymbol{t}(\psi), \ldots, \boldsymbol{t}(\psi)\rangle_{g,n}^\square.$$

Let $\mathcal{D}$ denote the *total genus descendent potential*,

$$\mathcal{D}^\square := \exp\left(\sum_{g\geq 0} \hbar^{g-1}\mathcal{F}_g^\square\right).$$

GW theory, FJRW theory, and $\mathbf{s}$-twisted W^c invariants all share a similar structure. In particular, their genus-g generating functions satisfy three differential equations, the so–called *string equation* (SE), *dilation equation* (DE), and *topological recursion relation* (TRR). (See [27] for an explicit description of each.)

Definition 3.1. – We call $\square$ an *axiomatic GW theory* if the correlators satisfy the SE, DE, and TRR.

Remark 3.2. – For the proof that Gromov-Witten theory satisfies the above equations see [32], in the case of FJRW theory see [19]. That $\mathbf{s}$-twisted W^c structure invariants give an axiomatic GW theory follows from Theorem 4.3 and the corresponding statement for untwisted invariants.

We can use this extra structure to rephrase the genus zero data in terms of Givental's *overruled Lagrangian cone*. For a more detailed exposition of what follows we refer the reader to Givental's original paper on the subject ([20]).

Let $\mathcal{V}^\square$ denote the vector space $H^\square((z^{-1}))$, equipped with the symplectic pairing

$$\Omega_\square(f_1, f_2) := \mathrm{Res}_{z=0}\langle f_1(-z), f_2(z)\rangle_\square.$$

$\mathcal{V}^\square$ admits a natural polarization $\mathcal{V}^\square = \mathcal{V}_+^\square \oplus \mathcal{V}_-^\square$ defined in terms of powers of z:

$$\mathcal{V}_+^\square = H^\square[z],$$
$$\mathcal{V}_-^\square = z^{-1}H^\square[[z^{-1}]].$$

We obtain Darboux coordinates $\{q_k^i, p_{k,i}\}$ with respect to the polarization on $\mathcal{V}^\square$ by representing each element of $\mathcal{V}^\square$ in the form

$$\sum_{k\geq 0}\sum_{i\in I} q_k^i\beta_i z^k + \sum_{k\geq 0}\sum_{i\in I} p_{k,i}\beta^i(-z)^{-k-1}.$$

One can view $\mathcal{F}_0^\square$ as the generating function of a Lagrangian subspace $\mathcal{L}^\square$ of $\mathcal{V}^\square$. Let β_0 denote the unit in $H^\square$, and make the change of variables (the so–called Dilaton shift)

$$q_1^0 = t_1^0 - 1 \quad q_k^i = t_k^i \text{ for } (k, i) \neq (1, 0). \tag{3.0.1}$$

Then the set

$$\mathcal{L}^\square := \left\{\mathbf{p} = d_{\mathbf{q}}\mathcal{F}_0^\square\right\} \tag{3.0.2}$$

defines a Lagrangian subspace. More explicitly, $\mathscr{L}^{\square}$ contains the points of the form

$$(3.0.3)\qquad -\beta_0 z+\sum_{\substack{k\geq 0\\ i\in I}} t_k^i\beta_i z^k+\sum_{\substack{a_1,\dots,a_n,a\geq 0\\ i_1,\dots,i_n,i\in I}}\frac{t_{a_1}^{i_1}\cdots t_{a_n}^{i_n}}{n!(-z)^{a+1}}\langle\psi^a\beta_i,\psi^{a_1}\beta_{i_1},\dots,\psi^{a_n}\beta_{i_n}\rangle_{0,n+1}^{\square}\beta^i.$$

Because $\mathscr{F}_0^{\square}$ satisfies the SE, DE, and TRR, $\mathscr{L}^{\square}$ will take a special form. In fact, $\mathscr{L}^{\square}$ is a cone satisfying the condition that for all $f\in\mathscr{V}^{\square}$,

$$(3.0.4)\qquad \mathscr{L}^{\square}\cap L_f=zL_f$$

where L_f is the tangent space to $\mathscr{L}^{\square}$ at f. Equation (3.0.4) justifies the term overruled, as each tangent space L_f is filtered by powers of z:

$$L_f\supset zL_f\supset z^2L_f\supset\cdots$$

and $\mathscr{L}^{\square}$ itself is ruled by the various zL_f. The codimension of zL_f in L_f is equal to $\dim(H^{\square})$.

A generic slice of $\mathscr{L}^{\square}$ parameterized by $H^{\square}$, i.e.,

$$\{f(\boldsymbol{t})|\boldsymbol{t}\in H^{\square}\}\subset\mathscr{L}^{\square},$$

will be transverse to the ruling. Given such a slice, we can reconstruct $\mathscr{L}^{\square}$ as

$$(3.0.5)\qquad \mathscr{L}^{\square}=\left\{zL_{f(\boldsymbol{t})}|\boldsymbol{t}\in H^{\square}\right\}.$$

Givental's J-function is defined in terms of the intersection

$$\mathscr{L}^{\square}\cap-\beta_0 z\oplus H\oplus\mathscr{V}^-.$$

More explicitly, the J-function is given by

$$J^{\square}(\boldsymbol{t},-z)=-\beta_0 z+\boldsymbol{t}+\sum_{n\geq 0}\sum_{i\in I}\frac{1}{n!}\left\langle\frac{\beta_i}{-z-\psi},\boldsymbol{t},\dots,\boldsymbol{t}\right\rangle_{0,n+1}^{\square}\beta^i.$$

In other words, we obtain the J--function by setting $t_k^i=0$ in (3.0.3) whenever $k>0$.

In [22] it is shown that the image of $J^{\square}(\boldsymbol{t},-z)$ is transverse to the ruling of $\mathscr{L}^{\square}$, so $J^{\square}(\boldsymbol{t},-z)$ is a function satisfying (3.0.5). Thus the ruling at $J^{\square}(\boldsymbol{t},-z)$ is spanned by the derivatives of $J^{\square}$, i.e.,

$$(3.0.6)\qquad zL_{J^{\square}(\boldsymbol{t},-z)}=\left\{J^{\square}(\boldsymbol{t},-z)+z\sum c_i(z)\frac{\partial}{\partial t^i}J^{\square}(\boldsymbol{t},-z)|c_i(z)\in\mathbb{C}[z]\right\}.$$

By the string equation, $z\frac{\partial}{\partial t^0}J^{\square}(\boldsymbol{t},z)=J^{\square}(\boldsymbol{t},z)$, so (3.0.6) simplifies to

$$(3.0.7)\qquad zL_{J^{\square}(\boldsymbol{t},-z)}=\left\{z\sum c_i(z)\frac{\partial}{\partial t^i}J^{\square}(\boldsymbol{t},-z)|c_i(z)\in\mathbb{C}[z]\right\}.$$

4. Twisted theory from untwisted theory

Here a correspondence between **0**-twisted W^c invariants and **s**-twisted W^c invariants is presented using the language of Givental's symplectic formalism.

4.1. Grothendieck-Riemann-Roch for r-spin curves

We recall A. Chiodo's Grothendieck-Riemann-Roch calculation for r-spin curves [5] which will then be adapted to the setting of W^c structures. First, we set notation.

Let

$$\mathtt{m}_1, \dots, \mathtt{m}_n \in \left\{ \frac{0}{r}, \dots, \frac{r-1}{r} \right\}$$

be multiples of $1/r$ with $0 \leq \mathtt{m}_i < 1$. Let

$$\overline{\mathscr{A}}^{(r,c)}_{h,n}(\mathtt{m}_1, \dots, \mathtt{m}_n)$$

denote the component of $\overline{\mathscr{A}}^{(r,c)}_{h,n}$ such that the multiplicity of the isotropy at p_i is $\mathtt{m}_i$. By (1.1.2), if $(c/r)(2h-2+n) - \sum_{i=1}^n \mathtt{m}_i \in \mathbb{Z}$ then $\overline{\mathscr{A}}^{(r,c)}_{h,n}(\mathtt{m}_1, \dots, \mathtt{m}_n)$ will be nonempty. Let Sing denote the stack classifying nodal curves equipped with an rth root, along with a choice of node. By specifying a branch at the node, we obtain a double cover $\mathrm{Sing}' \to \mathrm{Sing}$. The stack Sing maps to $\overline{\mathscr{A}}^{(r,c)}_{h,n}(\mathtt{m}_1, \dots, \mathtt{m}_n)$, composing we obtain

$$\iota : \mathrm{Sing}' \to \overline{\mathscr{A}}^{(r,c)}_{h,n}(\mathtt{m}_1, \dots, \mathtt{m}_n).$$

The stack Sing' decomposes as a disjoint union of substacks

$$\mathrm{Sing} := \bigsqcup_{0 \leq q < (r-1)/r} \mathrm{Sing}'_q,$$

determined by the multiplicity at the node. Namely, given a point $p \in \mathrm{Sing}'$, let $\mathscr{L} \to \mathscr{C}$ denote the corresponding rth root. The isotropy at the distinguished node acts on the restriction of $\mathscr{L}$ to the first branch. Let $q(p)$ denote the multiplicity of this action. This multiplicity is constant on connected components, so we define Sing'_q to be the subset of Sing' where the multiplicity is q. We further denote by

$$\iota_q : \mathrm{Sing}'_q \to \overline{\mathscr{A}}^{(r,c)}_{h,n}(\mathtt{m}_1, \dots, \mathtt{m}_n)$$

the restriction of the map ι. There are line bundles over Sing' whose fibers are the cotangent space of first branch of the coarse curve at the node and the cotangent space of the second branch of the coarse curve at the node. Let $\psi, \hat{\psi} \in H^2(\mathrm{Sing}', \mathbb{Q})$ denote their respective first Chern classes. Finally, define the class

$$\gamma_r = \sum_{i+j=r} (-\psi)^i \hat{\psi}^j.$$

Let $B_k(x)$ denote the kth *Bernoulli polynomial* defined by

$$\sum_{k \geq 0} B_k(x) z^k / k! = z e^{zx} / (e^z - 1).$$

Chiodo proves the following generalization of Mumford's Grothendieck-Riemann-Roch calculation.

THEOREM 4.1 ([5]). – *Let $\mathscr{L}$ denote the universal line bundle over the universal curve $\pi : \mathscr{C} \to \overline{\mathscr{A}}^{(r,c)}_{h,n}(\mathtt{m}_1, \dots, \mathtt{m}_n)$. Then*

$$\mathrm{ch}(R\pi_* \mathscr{L}) = \sum_{k \geq 0} \left(\frac{B_{k+1}(c/r)}{(k+1)!} \kappa_k - \sum_{i=1}^n \frac{B_{k+1}(\mathtt{m}_i)}{(k+1)!} \psi_i^k + \frac{1}{2} \sum_{q=0}^{r-1} \frac{r B_{k+1}(q)}{(k+1)!} (\iota_q)_* (\gamma_{k-1}) \right),$$

where κ_k are the κ classes (cf. (4.2.4)*).*

4.2. Twisted from untwisted invariants

An important application of Givental's symplectic formalism is that it enables one to systematically relate twisted and untwisted invariants. This has been used to great effect in Gromov-Witten theory by Coates and Givental [15]. We will adapt that method to the setting of W^c structures.

In the spirit of [15] and [8], the above theorem gives an explicit relationship between twisted and untwisted W^c invariants, which may be expressed most neatly in the language of Givental's symplectic formalism. *We will assume without further comment in what follows that we are in the situation $cc_j < d$ for all j.*

NOTATION 4.2. – For the purposes of more clearly stating the main theorem, we let $m_j(\phi_g^c)$ denote the multiplicity corresponding to *an insertion of* ϕ_g^c. Due to the shifting in (1.3.1), an $\mathbf{s}$-twisted invariant with a ϕ_g^c insertion corresponds to a marked point where the action of the isotropy on the fiber is by $g\mathfrak{j}^c$. Thus $m_j(\phi_g^c) = m_j(g\mathfrak{j}^c)$, the multiplicity of $g\mathfrak{j}^c$ on the jth line bundle.

THEOREM 4.3. – *For any Landau-Ginzburg pair (Q, G), let $\Delta^c : \mathscr{V}^{c,\mathbf{0}} \to \mathscr{V}^{c,\mathbf{s}}$ be the symplectic transformation defined by*

$$\Delta^c := \bigoplus_{g \in G} \prod_{j=1}^{N} \exp\left(\sum_{k \geq 0} s_k^j \frac{B_{k+1}(m_j(\phi_g^c))}{(k+1)!} z^k\right)$$

and let $\widehat{\Delta}^c$ denote the quantization of Δ^c, as defined in [15] *(or* [27]*). Then,*

1. *$\widehat{\Delta}^c$ relates the twisted and untwisted total descendent potentials (of all genera)*

$$\mathscr{D}^{c,\mathbf{s}} = \widehat{\Delta}^c \mathscr{D}^{c,\mathbf{0}}. \tag{4.2.1}$$

2. *Δ^c relates the twisted and untwisted Lagrangian cones*

$$\mathscr{L}^{c,\mathbf{s}} = \Delta^c \mathscr{L}^{c,\mathbf{0}}.$$

REMARK 4.4. – In the above notation, the direct sum means simply that we act on the $\mathbb{C}((z))$-span of ϕ_g^c by multiplication by

$$\prod_{j=1}^{N} \exp\left(\sum_{k \geq 0} s_k^j \frac{B_{k+1}(m_j(\phi_g^c))}{(k+1)!} z^k\right).$$

REMARK 4.5. – In the r-spin ($Q = x^r, G = \langle \mathfrak{j} \rangle, c = 1$) case, restricting to narrow sectors, the above result was proven in [8]. A similar generalization to the above was given in the case of the Fermat quintic in [7], with a slight difference due to their definition of the W^c moduli space at broad sectors.

Proof. – The proof follows the method first used in [15], and is a straightforward generalization of [8]. We first remark that (1) implies (2), as (2) is nothing but a semi-classical limit of (1) (see [15] and [10]). Therefore, it is enough to show (1).

Viewing both sides of (4.2.1) as functions with respect to the formal parameters s_k^j for $k \geq 0$, it suffices to show that they satisfy the same differential equation with respect to s_k^j. Note first that both sides of (4.2.1) satisfy the same initial condition, i.e., when $\mathbf{s} = 0$ the two are equal. We next claim that both sides satisfy

$$\frac{\partial \Phi}{\partial s_k^j} = P_k^{(j)} \Phi, \tag{4.2.2}$$

where

$$P_k^{(j)} = \frac{B_{k+1}(m_j(\phi_e^c))}{(k+1)!} \frac{\partial}{\partial t_{k+1}^e} - \sum_{\substack{a \geq 0 \\ g \in G}} \frac{B_{k+1}(m_j(\phi_g^c))}{(k+1)!} t_a^g \frac{\partial}{\partial t_{a+k}^g} + \frac{\hbar}{2} \sum_{\substack{a+a'=k-1 \\ g,g' \in G}} (-1)^{a'} \eta^{g,g'} \frac{B_{k+1}(m_j(\phi_g^c))}{(k+1)!} \frac{\partial^2}{\partial t_a^g \partial t_{a'}^{g'}}.$$

Here $\eta_{g,g'} = \langle \phi_g^c, \phi_{g'}^c \rangle^{c,\mathbf{s}}$ is the pairing matrix, and upper indices denote the corresponding coordinate of the dual matrix.

That the right side of (4.2.1) satisfies this equation is a direct consequence of the definition of $\widehat{\Delta}^c$ (See for instance [8] or [10] for the definition. We remark that first term in the equation is due to the Dilation shift (3.0.1): $q_1^e = t_1^e - 1$.). The proof of Theorem 4.3 will be complete after we show that the left side of (4.2.1) also satisfies (4.2.2).

By Theorem 4.1, differentiating $\mathscr{F}^{c,\mathbf{s}}$ with respect to s_k^j has the effect of adding a factor of

$$\left(\frac{B_{k+1}(m_j(\phi_e^c))}{(k+1)!} \kappa_k - \sum_{i=1}^{n} \frac{B_{k+1}(m_j(\phi_{g_i}^c))}{(k+1)!} \psi_i^k + \frac{1}{2} \sum_{g \in G} \frac{\bar{d}^N B_{k+1}(m_j(\phi_g^c))}{(k+1)!} (\iota_g)_* (\gamma_{k-1}) \right) \tag{4.2.3}$$

to each integrand in the generating function $\mathscr{F}^{c,\mathbf{s}}$.

We will investigate the contribution of each term in the above expression to $\frac{\partial}{\partial s_k^j} \mathscr{F}^{c,\mathbf{s}}$.

Step 1: Recall the class κ_k is defined as the pushforward of ψ_{n+1}^{k+1} under the map

$$\coprod_{g_1,\ldots,g_n \in G} W_{h,n+1}^c(g_1, \ldots, g_n, \mathrm{j}^c) \to \coprod_{g_1,\ldots,g_n \in G} W_{h,n}^c(g_1, \ldots, g_n). \tag{4.2.4}$$

By the projection formula

$$\begin{aligned} &\int_{W_{h,n,G}^c(g_1\mathrm{j}^c,\ldots,g_n\mathrm{j}^c)} \kappa_k \cup \mathbf{s}(R\pi_* \bigoplus_{j=1}^{N} \mathscr{L}_j) \prod_{i=1}^{n} \psi_i^{a_i} \\ &= \int_{W_{h,n+1,G}^c(g_1\mathrm{j}^c,\ldots,g_n\mathrm{j}^c,\mathrm{j}^c)} \psi_{n+1}^{k+1} \cup \mathbf{s}(R\pi_* \bigoplus_{j=1}^{N} \mathscr{L}_j) \prod_{i=1}^{n} \psi_i^{a_i} \\ &= \langle \psi^{a_1} \phi_{g_1}^c, \ldots, \psi^{a_n} \phi_{g_n}^c, \psi^{k+1} \phi_e^c \rangle_{h,n}^{c,\mathbf{s}}. \end{aligned}$$

Thus the first term in (4.2.3) contributes a summand

$$\frac{B_{k+1}(m_j(\phi_e^c))}{(k+1)!}\frac{\partial}{\partial t_{k+1}^e}\mathcal{F}^{c,\mathbf{s}}$$

to $\frac{\partial}{\partial s_k^j}\mathcal{F}^{c,\mathbf{s}}$.

Step 2: It can be seen immediately that adding a factor of

$$\sum_{i=1}^{n}\frac{B_{k+1}(m_j(\phi_{g_i}^c))}{(k+1)!}\psi_i^k$$

to the integrand of

$$\int_{W_{h,n,G}^c(g_1\mathrm{j}^c,\ldots,g_n\mathrm{j}^c)}\mathbf{s}(R\pi_*\bigoplus_{j=1}^{N}\mathcal{L}_j)\prod_{i=1}^{n}\psi_i^{a_i}$$

for every such integral in $\mathcal{F}^{c,\mathbf{s}}$ is equivalent to acting on $\mathcal{F}^{c,\mathbf{s}}$ by

$$\sum_{a,g}\frac{B_{k+1}(m_j(\phi_g^c))}{(k+1)!}t_a^g\frac{\partial}{\partial t_{a+k}^g}.$$

Thus the second term in (4.2.3) contributes a summand

$$-\sum_{\substack{a\geq 0\\ g\in G}}\frac{B_{k+1}(m_j(\phi_g^c))}{(k+1)!}t_a^g\frac{\partial}{\partial t_{a+k}^g}\mathcal{F}^{c,\mathbf{s}}$$

to $\frac{\partial}{\partial s_k^j}\mathcal{F}^{c,\mathbf{s}}$.

Step 3: Let Sing' denote the double cover of the nodal locus $\mathrm{Sing}\subset W_{h,n,G}$ in analogy to the $\bar{d}$-spin case. In this case Sing' splits as a disjoint union $\coprod_{g\in G}\mathrm{Sing}'_g$ where the action of the isotropy group at the node of the distinguished component on the fiber of $\bigoplus_{j=1}^{N}\mathcal{L}_j$ is by $g\mathrm{j}^c$. Let

$$\iota_g:\mathrm{Sing}'\to W_{h,n,G}$$

denote the restriction of ι to Sing'_g. Let

$$\mathcal{F}^{c,\mathbf{s}}:=\sum_{h\geq 0}\hbar^{h-1}\mathcal{F}_h^{c,\mathbf{s}}.$$

The stack Sing' splits into two open and closed subsets based on whether the node is separating or non-separating. Let

$$\iota_{g,\mathrm{irr}}:\mathrm{Sing}'_{g,\mathrm{irr}}\to W_{h,n,G}^c(g_1\mathrm{j}^c,\ldots,g_n\mathrm{j}^c)$$

denote the restriction of ι_g to the non-separating locus. It is easy to see that $\iota_{g,\mathrm{irr}}$ factors through $W_{h-1,n+2,G}^c(g_1\mathrm{j}^c,\ldots,g_n\mathrm{j}^c,g\mathrm{j}^c,g^{-1}\mathrm{j}^{-c})$:

$$\begin{tikzcd}
 & W_{h-1,n+2,G}^c(g_1\mathrm{j}^c,\ldots,g_n\mathrm{j}^c,g\mathrm{j}^c,g^{-1}\mathrm{j}^{-c}) \arrow[d, "\rho"] \\
\mathrm{Sing}'_{g,\mathrm{irr}} \arrow[ur, "\mu_{g,\mathrm{irr}}"] \arrow[r, "\iota_{g,\mathrm{irr}}"] & W_{h,n,G}^c(g_1\mathrm{j}^c,\ldots,g_n\mathrm{j}^c)
\end{tikzcd}$$

induced by normalizing the universal curve. Let

$$\bigoplus_{j=1}^{N} \mathcal{L}_j \to \mathcal{C} \xrightarrow{\pi} W^c_{h,n,G}(g_1\mathfrak{j}^c, \ldots, g_n\mathfrak{j}^c)$$

and

$$\bigoplus_{j=1}^{N} \bar{\mathcal{L}}_j \to \bar{\mathcal{C}} \xrightarrow{\bar{\pi}} W^c_{h-1,n+2,G}(g_1\mathfrak{j}^c, \ldots, g_n\mathfrak{j}^c, g\mathfrak{j}^c, g^{-1}\mathfrak{j}^{-c})$$

be the universal bundles over the universal curves, then there is a natural morphism

$$\nu : \bar{\mathcal{C}} \to \mathcal{C}$$

via the normalization of the curves. The key point is that because the pullback $\nu^*\omega_{\mathcal{C},\log}$ is equal to $\omega_{\bar{\mathcal{C}},\log}$, the line bundle $\mathcal{L}_j$ will pull back to $\bar{\mathcal{L}}_j$. If

$$n : \mathrm{Sing}'_{g,\mathrm{irr}} \to \mathcal{C}$$

is the morphism induced by the nodal locus, the normalization exact sequence yields

$$0 \to R\pi_*\mathcal{L}_j \to R\bar{\pi}_*\nu^*\mathcal{L}_j \to n^*\mathcal{L}_j \to 0.$$

If $g\mathfrak{j}^c$ acts nontrivially on the jth line bundle then $\mathrm{ch}(n^*\mathcal{L}_j) = 0$. Otherwise, $n^*\mathcal{L}_j$ is a root of (a power of) $n^*\omega_{\mathcal{C},\log}$ which is trivial via the residue map. Thus $n^*\mathcal{L}_j$ is rationally trivial and $\mathrm{ch}(n^*\mathcal{L}_j) = 1$. We arrive at the formula

$$\mu_{g,\mathrm{irr}*}\iota^*_{g,\mathrm{irr}}(\mathrm{ch}_k(R\pi_*\mathcal{L}_j)) = \begin{cases} \mathrm{ch}_k(R\bar{\pi}_*\bar{\mathcal{L}}_j) & \text{for } k > 0 \text{ or } m_j(g\mathfrak{j}^c) \neq 0 \\ \mathrm{ch}_0(R\bar{\pi}_*\bar{\mathcal{L}}_j) - 1 & \text{otherwise,} \end{cases}$$

which yields the simple relation (cf. Equation (1.3.3))

$$\begin{aligned}\mu_{g,\mathrm{irr}*}\iota^*_{g,\mathrm{irr}}(\mathbf{s}(R\pi_*\bigoplus_{j=1}^{N}\mathcal{L}_j)) &= \exp(-\sum_{j=1}^{N}\lfloor 1 - m_j(\phi^c_g)\rfloor s^j_0)\mathbf{s}(R\bar{\pi}_*\bigoplus_{j=1}^{N}\bar{\mathcal{L}}_j) \\ &= \frac{\eta^{g,g^{-1}}}{\bar{d}^N}\mathbf{s}(R\bar{\pi}_*\bigoplus_{j=1}^{N}\bar{\mathcal{L}}_j).\end{aligned}$$

Thus integrals involving a pushforward via $\iota_{g,\mathrm{irr}}$ may instead be calculated as integrals over $W^c_{h-1,n+2,G}(g_1\mathfrak{j}^c, \ldots, g_n\mathfrak{j}^c, g\mathfrak{j}^c, g^{-1}\mathfrak{j}^{-c})$.

This implies that

$$\begin{aligned}&\int_{W^c_{h,n,G}(g_1\mathfrak{j}^c,\ldots,g_n\mathfrak{j}^c)} \bar{d}^N \iota_{g,\mathrm{irr}*}(\gamma_{k-1}) \cup \mathbf{s}(R\pi_*\bigoplus_{j=1}^{N}\mathcal{L}_j)\prod_{i=1}^{n}\psi_i^{a_i} \\ &= \eta^{g,g^{-1}}\int_{W^c_{h-1,n,G}(g_1\mathfrak{j}^c,\ldots,g_n\mathfrak{j}^c,g\mathfrak{j}^c,g^{-1}\mathfrak{j}^{-c})} \mu_{g,\mathrm{irr}*}\gamma_{k-1} \cup \mathbf{s}(R\bar{\pi}_*\bigoplus_{j=1}^{N}\mathcal{L}_j)\prod_{i=1}^{n}\psi_i^{a_i} \\ &= \sum_{a+a'=k-1}(-1)^{a'}\eta^{g,g^{-1}}\langle\psi^{a_1}\phi^c_{g_1}, \ldots, \psi^{a_n}\phi^c_{g_n}, \psi^a\phi^c_g, \psi^{a'}\phi^c_{g^{-1}}\rangle^{c,\mathbf{s}}_{h-1,n+2}.\end{aligned}$$

Therefore, the *non-separating* part of the third term in (4.2.3) contributes a summand

$$\frac{\hbar}{2}\sum_{\substack{a+a'=k-1 \\ g,g'\in G}}(-1)^{a'}\eta^{g,g'}\frac{B_{k+1}(m_j(\phi^c_g))}{(k+1)!}\frac{\partial^2}{\partial t^g_a \partial t^{g'}_{a'}}\mathcal{F}^{c,\mathbf{s}}$$

to $\frac{\partial}{\partial s_k^j}\mathscr{F}^{c,\mathbf{s}}$.

A similar argument shows that the *separating* part of the third term in (4.2.3) contributes a summand

$$\frac{\hbar}{2}\sum_{\substack{a+a'=k-1\\ g,g'\in G}}(-1)^{a'}\eta^{g,g'}\frac{B_{k+1}(m_j(\phi_g^c))}{(k+1)!}\frac{\partial}{\partial t_a^g}\mathscr{F}^{c,\mathbf{s}}\frac{\partial}{\partial t_{a'}^{g'}}\mathscr{F}^{c,\mathbf{s}}$$

to $\frac{\partial}{\partial s_k^j}\mathscr{F}^{c,\mathbf{s}}$.

Finally, adding all these contributions, we conclude that

$$\frac{\partial}{\partial s_k^j}\mathscr{D}^{c,\mathbf{s}}=P_k^{(j)}\mathscr{D}^{c,\mathbf{s}}$$

as desired. □

5. Correspondences

5.1. Identification of state spaces

Let

$$\mathscr{L}^{c,\mathbf{0}}\subset\mathscr{V}^{c,\mathbf{0}}\quad\text{and}\quad\mathscr{L}^{c,\mathbf{s}}\subset\mathscr{V}^{c,\mathbf{s}}$$

denote the Lagrangian cones (see (3.0.2)) of untwisted and $\mathbf{s}$-twisted W^c invariants respectively.

We have the following identification of twisted and untwisted state spaces as vector spaces with inner products.

Lemma 5.1. – *If $cc_j<d$ for $1\le j\le N$, the map,*

$$\begin{aligned}i_c: H^{0,\mathbf{s}}&\to H^{c,\mathbf{s}}\\ \phi_g^0&\mapsto\phi_{g\mathfrak{j}^{-c}}^c\end{aligned}$$

is an isomorphism of inner product spaces.

Proof. – This follows immediately from Lemma 1.11. □

Note that in this case i_c extends to an isomorphism of the symplectic spaces $\mathscr{V}^{0,\mathbf{s}}\cong\mathscr{V}^{c,\mathbf{s}}$. Let $\boldsymbol{t}_c:=\sum_{g\in G}t_c^g\phi_g^c$ denote a point in the state space. We will use the notation

$$J^{c,\mathbf{s}}(\boldsymbol{t}_c,z)=\phi_e^cz+\boldsymbol{t}_c+\sum_{n\ge0}\sum_{g\in G}\frac{1}{n!}\left\langle\frac{\phi_g^c}{z-\psi},\boldsymbol{t}_c,\ldots,\boldsymbol{t}_c\right\rangle_{0,n+1}^{c,\mathbf{s}}\phi^{c,g}$$

for the J-function of the twisted theories.

5.2. Untwisted invariants

We will first consider the case $s_k = 0$ for all k (Lemma 1.11). By Equation (3.0.6), the genus zero part of the theory (i.e., the Lagrangian cone) is determined by the J-function. The J-function $J^{c,\mathbf{0}}(\boldsymbol{t}_c, z)$ may be directly calculated.

LEMMA 5.2. – *We have*

$$J^{c,\mathbf{0}}(\boldsymbol{t}_c, z) = \sum_{\{a_g \geq 0\}_{g\in G}} z^{1-\sum a_g} \prod_{g\in G} \frac{(t_c^g)^{a_g}}{a_g!} \phi^c_{\prod g^{a_g}}.$$

Proof. – By pushing forward via the forgetful morphism, the $\mathbf{0}$-twisted invariants may be calculated over $\overline{\mathscr{M}}_{0,n}$. The moduli space $W^c_{0,n,G}(g_1 \mathrm{j}^c, \ldots, g_n \mathrm{j}^c)$ is nonempty if $\prod_{i=1}^n g_i = \mathrm{j}^{-2c}$ by (1.1.2). In this case the degree of the map to $\overline{\mathscr{M}}_{0,n}$ is $1/\bar{d}^N$. Thus by (1.2.1), and the standard formula for integrals of ψ-classes over $\overline{\mathscr{M}}_{0,n}$, the invariant $\left\langle \psi^a \phi^c_{g_0}, \phi^c_{g_1}, \ldots, \phi^c_{g_n} \right\rangle^c_{0,n+1}$ is equal to $1/\bar{d}^N$ exactly when $g_0 = \mathrm{j}^{-2c} \prod_{i=1}^n g_i^{-1}$ and $a = n-2$ and is zero otherwise. By Lemma 1.11, the dual of ϕ_{g_0} is $\bar{d}^N \phi_{g_0^{-1}\mathrm{j}^{-2c}}$. Thus

$$\sum_{\substack{a\geq 0\\ g_0\in G}} z^{-a-1} \left\langle \psi^a \phi^c_{g_0}, \phi^c_{g_1}, \ldots, \phi^c_{g_n} \right\rangle^c_{0,n+1} \phi^{c,g_0} = z^{1-n} \phi^c_{\prod_{i=1}^n g_i}.$$

The lemma follows. □

LEMMA 5.3. – *The transformation i_c identifies derivatives of the two J-functions:*

$$i_c\left(z \frac{\partial}{\partial t_0^{\mathrm{j}^c g'}} J^{0,\mathbf{0}}(\boldsymbol{t}_0, z)\right) = z \frac{\partial}{\partial t_c^{g'}} J^{c,\mathbf{0}}(\boldsymbol{t}_c, z)|_{t_c^g = t_0^g}.$$

In particular,

$$i_c\left(z \frac{\partial}{\partial t_0^{\mathrm{j}^c}} J^{0,\mathbf{0}}(\boldsymbol{t}_0, z)\right) = J^{c,\mathbf{0}}(\boldsymbol{t}_c, z)|_{t_c^g = t_0^g}.$$

Proof. – Observe that

$$\begin{aligned} i_c\left(z \frac{\partial}{\partial t_0^{\mathrm{j}^c g'}} J^{0,\mathbf{0}}(\boldsymbol{t}_0, z)\right) &= i_c\left(\phi^0_{\mathrm{j}^c g'} z + \sum_{g\in G} t_0^g \phi^0_{\mathrm{j}^c g' g} + \sum_{\{a_g\geq 0\}_{g\in G}} z^{\sum 1-a_g} \prod_{g\in G} \frac{(t_0^g)^{a_g}}{a_g!} \phi^0_{\mathrm{j}^c g' \prod_g g^{a_g}} \right) \\ &= \phi^c_{g'} z + \sum_{g\in G} t_0^g \phi^c_{g' g} + \sum_{\{a_g\geq 0\}_{g\in G}} z^{\sum 1-a_g} \prod_{g\in G} \frac{(t_0^g)^{a_g}}{a_g!} \phi^c_{g' \prod_g g^{a_g}} \\ &= z \frac{\partial}{\partial t_c^{g'}} J^{c,\mathbf{0}}(\boldsymbol{t}_c, z)|_{\{t_c^g = t_0^g\}}. \end{aligned}$$

The second statement then follows from the string equation:

$$z \frac{\partial}{\partial t_c^e} J^{c,\mathbf{0}}(\boldsymbol{t}_c, z) = J^{c,\mathbf{0}}(\boldsymbol{t}_c, z).$$ □

PROPOSITION 5.4. – *Under the isomorphism $i_c : \mathscr{V}^{0,\mathbf{0}} \to \mathscr{V}^{c,\mathbf{0}}$, the Lagrangian cones $\mathscr{L}^{0,\mathbf{0}}$ and $\mathscr{L}^{c,\mathbf{0}}$ are identified.*

Proof. – By (3.0.5) and (3.0.7), the Lagrangian cone $\mathscr{L}^{c,\mathbf{0}}$ is spanned by the set of derivatives $\{z\partial/\partial t_c^{g'} J^{c,\mathbf{0}}(\boldsymbol{t}_c, z)\}_{g'\in G}$. From the above lemma we see that i_c identifies the set of derivatives of $J^{0,\mathbf{0}}$ with those of $J^{c,\mathbf{0}}$ and therefore identifies the two Lagrangian cones. □

5.3. The MLK theory correspondence

We are now able to state a relationship between the twisted theories corresponding to different powers c of $\omega_{\mathscr{C},\log}$. As it relates *multiple powers of the log-canonical*, we call it the MLK correspondence.

THEOREM 5.5 (The MLK correspondence). – *The isomorphism* $i_c : \mathscr{V}^{0,\mathbf{s}} \to \mathscr{V}^{c,\mathbf{s}}$ *identifies the* $\mathbf{s}$*-twisted Lagrangian cones* $\mathscr{L}^{0,\mathbf{s}}$ *and* $\mathscr{L}^{c,\mathbf{s}}$.

Proof. – Note that $m_j(\phi_g^c) = m_j(g\mathfrak{j}^c) = m_j(\phi^0_{g\mathfrak{j}^c})$. Thus the action of Δ^0 on the subspace spanned by $\phi^0_{g\mathfrak{j}^c}$ is the same as the action of Δ^c on the subspace spanned by $i_c(\phi^0_{g\mathfrak{j}^c}) = \phi_g^c$. In other words, under the identification given by i_c, Δ^0 and Δ^c are the same symplectic transformation. By Proposition 5.4 identifying the untwisted Lagrangian cones, we see that $i_c\mathscr{L}^{0,\mathbf{s}} = i_c\Delta^0\mathscr{L}^{0,\mathbf{0}} = \Delta^c\mathscr{L}^{c,\mathbf{0}} = \mathscr{L}^{c,\mathbf{s}}$. □

COROLLARY 5.6. – *We have*

$$z\frac{\partial}{\partial t_0^{\mathfrak{j}^c}} i_c\left(J^{0,\mathbf{s}}(\boldsymbol{t}_0, z)\right) = J^{c,\mathbf{s}}(\boldsymbol{t}_c, z)$$

where the change of variables is given by

$$\langle \boldsymbol{t}_c, \phi_g^c\rangle^{c,\mathbf{s}} = \frac{\partial^2}{\partial t_0^{g\mathfrak{j}^c}\partial t_0^{\mathfrak{j}^c}}\mathscr{F}_0^{0,\mathbf{s}}(\boldsymbol{t}_0, 0, 0, \ldots).$$

Proof. – By (3.0.6), and the above corollary, the ruling of the Lagrangian cone $\mathscr{L}^{c,\mathbf{s}}$ at $J^{c,\mathbf{s}}(\boldsymbol{t}_c, -z)$ is in fact spanned by the derivatives of $J^{0,\mathbf{s}}(\boldsymbol{t}_0, -z)$. Thus we have

$$z\sum_{g\in G} C^g(\boldsymbol{t}_0, z)\frac{\partial}{\partial t_0^g} i_c\left(J^{0,\mathbf{s}}(\boldsymbol{t}_0, -z)\right) = J^{c,\mathbf{s}}(\boldsymbol{t}_c, -z)$$

for some functions $C^g(\boldsymbol{t}_0, z)$ and some change of variables between $\boldsymbol{t}_c$ and $\boldsymbol{t}_0$. Equating coefficients of z on either side yields $C^{\mathfrak{j}^c}(\boldsymbol{t}_0, z) = 1$ and all other $C^g(\boldsymbol{t}_0, z)$ equal zero. The change of variables is obtained by then equating coefficients of z^0. □

REMARK 5.7. – The above results should be viewed as akin to quantum Serre duality as given in [20], [15] and [32], and summarized here in Theorem 5.14. Indeed comparing Corollary 5.6 with Theorem 5.14, one sees that they take an almost identical form.

REMARK 5.8. – By applying Teleman's proof ([31]) of Givental's conjecture for semi-simple Frobenius manifolds ([21]) to the above untwisted theories, one may deduce a higher genus correspondence between untwisted theories. Combining this with Theorem 4.3, one obtains a higher genus analogue of the correspondence of Theorem 5.5. This will relate the total genus descendant potentials of the twisted theories via a quantized symplectic operator. We leave the details to the reader.

5.4. Implications to local GW and FJRW theory

In this subsection, we apply the MLK correspondence to prove a relationship between local GW theory and the FJRW theory.

Let (Q, G) be a Landau-Ginzburg pair where Q is Fermat. Recall that in this case $\bar{d} = d$ and we have the relationship mentioned above (Section 2) between the $\mathbf{s}$-twisted theories and both local GW theory and FJRW theory. In this section we fix the specialization of the $\mathbf{s}$ parameter to

$$s_0^j = -\ln(-\lambda_j),\ s_k^j = (k-1)!/\lambda_j^k \text{ for } 1 \le j \le N,\ k > 0. \tag{5.4.1}$$

Recall that under this specialization $\mathbf{s}(V) = 1/e_{\mathbb{C}^*}(V)$. We will still refer to these as $\mathbf{s}$-twisted invariants, where it is understood that we have specialized the s_k^j as above.

5.4.1. *Local GW theory*. – In the case $c = 0$, specializing the $\mathbf{s}$-twisted W^c invariants as above recovers the local GW invariants of $[\mathbb{C}^N/G]$ after multiplying by a factor of d^N. The pairing also differs by this factor. Consider the symplectic transformation

$$\mathscr{V}^{0,\mathbf{s}} \to \mathscr{V}^{[\mathbb{C}^N/G]}$$

induced by

$$\phi_g^0 \mapsto \frac{1}{\sqrt{d^N}} \mathbb{1}_g.$$

Under this transformation, we have the equality

$$\left\langle \frac{\phi_g^0}{z-\psi}, \phi_{g_1}^0, \dots, \phi_{g_n}^0 \right\rangle_{0,n+1}^{0,\mathbf{s}} \phi^{0,g} = \frac{1}{\sqrt{d^N}} \left\langle \frac{\mathbb{1}_g}{z-\psi}, \mathbb{1}_{g_1}, \dots, \mathbb{1}_{g_n} \right\rangle_{0,n+1}^{[\mathbb{C}^N/G]} \mathbb{1}^g,$$

where upper indices denote dual elements with respect to the given basis. Therefore the respective J-functions also differ by an overall factor. We obtain the following lemma.

LEMMA 5.9. – *After specializing the* s_k^j *as in* (5.4.1)*,*

$$\mathscr{L}^{0,\mathbf{s}} = \mathscr{L}^{[\mathbb{C}^N/G]}$$

under the identification $\phi_g^0 \mapsto \frac{1}{\sqrt{d^N}} \mathbb{1}_g$.

Proof. – Let $\boldsymbol{t} = \sum_{g \in G} t^g \mathbb{1}_g$. By the above we see that

$$J^{0,\mathbf{s}}(\boldsymbol{t}_0, z) = \frac{1}{\sqrt{d^N}} J^{[\mathbb{C}^N/G]}(\boldsymbol{t}, z)$$

after the change of variables $t_0^g = t^g$. In particular they generate the same Lagrangian cone. □

5.4.2. *FJRW theory.* – Let $\mathbf{s}$ be as in (5.4.1). A similar identification results between the $\mathbf{s}$-twisted J-function at $c = 1$ and the corresponding FJRW J-function. In this case we must account for the factor of plus or minus one, determined by the parity of $\chi(\bigoplus_{j=1}^N \mathscr{L}_j)$ (see (2.2.1)). Adjusting our specialization of $s_0^j = -\ln(-\lambda_j)$ to ${s_0^j}' = -\ln(-\lambda_j) + \pi\sqrt{-1}$ as in (2.2.2) will have the effect of modifying our twisted invariants by this sign. This alters the symplectic transformation of Theorem 4.3 by

$$\begin{aligned}\bigoplus_{g\in G}\prod_{j=1}^N \exp\left(B_1(m_j(\phi_g^1))\pi\sqrt{-1}\right) &= \bigoplus_{g\in G}\exp\left(B_1\left(\sum_{j=1}^N m_j(\phi_g^1)\right)\pi\sqrt{-1}\right)\\ &\sim \bigoplus_{g\in G}\exp\left(\sum_{j=1}^N m_j(g\mathfrak{j})\pi\sqrt{-1}\right)\\ &\sim \bigoplus_{g\in G}(-1)^{\sum_{j=1}^N m_j(g\mathfrak{j})}\end{aligned}$$

where $\sim$ means equal up to a constant factor (which will not effect the Lagrangian cone). Define

$$\Delta' := \bigoplus_{g\in G}(-1)^{\sum_{j=1}^N m_j(g\mathfrak{j})}.$$

LEMMA 5.10. – *Given the specialization of s_k^j as in* (5.4.1)*, let $F^{1,\mathbf{s}}$ be a point on $\mathscr{L}^{1,\mathbf{s}}$ such that the nonequivariant limit $\lim_{\lambda\mapsto 0} F^{1,\mathbf{s}}$ is both well defined and supported in the subset $\left(\bigoplus_{g,\in\hat{G}}\mathbb{C}\cdot\phi_g^1\right)[z]\oplus\mathscr{V}_-^{1,\mathbf{s}}\subset\mathscr{V}^{1,\mathbf{s}}$. Then $\lim_{\lambda\mapsto 0}\Delta'(F^{1,\mathbf{s}})$ lies in $\mathscr{L}^{(Q,G)}$ under the identification $\phi_g^1\mapsto\frac{1}{\sqrt{d^N}}\varphi_g$.*

Proof. – The symplectic transformation Δ' maps $\mathscr{L}^{1,\mathbf{s}}$ to $\mathscr{L}^{1,\mathbf{s}'}$ where ${s_k^j}' = s_k^j$ for $k > 0$ and ${s_0^j}' = s_0^j + \pi\sqrt{-1}$ as in (2.2.2). By (3.0.3), $\lim_{\lambda\mapsto 0}\Delta'(F^{1,\mathbf{s}})$ may be written as

$$-z\phi_0^1 + \sum_{\substack{k\geq 0\\ g\in\hat{G}}} t_k^g\phi_g^1 z^k + \lim_{\lambda\mapsto 0}\sum_{\substack{a_1,\dots,a_n,a\geq 0\\ g_1,\dots,g_n\in\hat{G}\\ g\in G}}\frac{t_{a_1}^{g_1}\cdots t_{a_n}^{g_n}}{n!(-z)^{a+1}}\langle\psi^a\phi_g^1,\psi^{a_1}\phi_{g_1}^1,\dots,\psi^{a_n}\phi_{g_n}^1\rangle_{0,n+1}^{1,\mathbf{s}'}\phi^{1,g}.$$

Recall Lemma 1.11, which with the specialization $\mathbf{s}'$ implies that

$$\phi^{1,g} = d^N\prod_{j=1}^N(\lambda_j)^{\lfloor 1-m_j(g\mathfrak{j})\rfloor}\phi_{g^{-1}\mathfrak{j}^{-2}}.$$

Therefore, for $g\notin\hat{G}$, the power of λ in this expression for $\phi^{1,g}$ is positive. Note also that $g\in\hat{G}$ if and only if $g^{-1}\mathfrak{j}^{-2}\in\hat{G}$. By (1.1.3), one calculates that

$$\langle\psi^a\phi_g^1,\psi^{a_1}\phi_{g_1}^1,\dots,\psi^{a_n}\phi_{g_n}^1\rangle_{0,n+1}^{1,\mathbf{s}'}\in\mathbb{C}[\lambda].$$

Thus in the non-equivariant limit of the above expression all terms containing the insertion ϕ_g^1 for $g\notin\hat{G}$ vanish.

After applying $\phi_g^1 \mapsto \frac{1}{\sqrt{d^N}}\varphi_g$ and recalling (2.2.1), the above expression becomes

$$\frac{1}{\sqrt{d^N}}\Bigg(-z\varphi_0 + \sum_{\substack{k\geq 0\\ g\in\hat{G}}} t_k^g \varphi_g z^k + \sum_{\substack{a_1,\ldots,a_n,a\geq 0\\ g_1,\ldots,g_n,g\in\hat{G}}} \frac{t_{a_1}^{g_1}\cdots t_{a_n}^{g_n}}{n!(-z)^{a+1}}\langle \psi^a\varphi_g, \psi^{a_1}\varphi_{g_1},\ldots,\psi^{a_n}\varphi_{g_n}\rangle_{0,n+1}^{(Q,G)}\varphi^g\Bigg),$$

which gives a point on $\mathscr{L}^{(Q,G)}$. □

5.4.3. *The correspondence*. – Let Δ° denote the symplectic transformation given by

$$\Delta^\circ : \mathbb{1}_g \mapsto (-1)^{\sum_{j=1}^N m_j(g)}\varphi_{g\mathrm{j}^{-1}}.$$

The previous two lemmas together with Corollary 5.6 allow us to determine the FJRW J-function from that of $[\mathbb{C}^N/G]$.

Theorem 5.11 (the Landau-Ginzburg/local GW correspondence)
We have the relationship

$$\lim_{\lambda\mapsto 0}\Delta^\circ\left(z\frac{\partial}{\partial t^{\mathrm{j}}}\left(J^{[\mathbb{C}^N/G]}(\boldsymbol{t},z)\right)\right) = J^{(Q,G)}(\boldsymbol{t}',z)$$

with the substitution given by

$$\langle \boldsymbol{t}', \varphi_g\rangle = \lim_{\lambda\mapsto 0}\frac{\partial^2}{\partial t^{g\mathrm{j}}\partial t^{\mathrm{j}}}\mathscr{F}_0^{[\mathbb{C}^N/G]}(\boldsymbol{t},0,0,\ldots)$$

for any $g\in\hat{G}$.

Proof. – Consider the function $z\frac{\partial}{\partial t^{\mathrm{j}}}\left(J^{[\mathbb{C}^N/G]}(\boldsymbol{t},z)\right) = z\mathbb{1}_{\mathrm{j}} + \mathscr{O}(1)$. The terms with non-positive z-coefficient are of the form

$$z^{-k}\left\langle \mathbb{1}_g\psi^k, \mathbb{1}_{\mathrm{j}}, \boldsymbol{t},\ldots,\boldsymbol{t}\right\rangle \prod_{j=1}^N(-\lambda_j)^{\lfloor 1-m_j(g\mathrm{j})\rfloor}\mathbb{1}_{g^{-1}}.$$

Due to the insertion of $\mathbb{1}_{\mathrm{j}}$, the universal line bundles over the relevant moduli space have negative degree, and thus $\left\langle \mathbb{1}_g\psi^k, \mathbb{1}_{\mathrm{j}}, \boldsymbol{t},\ldots,\boldsymbol{t}\right\rangle \in \mathbb{C}[\lambda]$. We see that the coefficient of $\mathbb{1}_g$ in $z\frac{\partial}{\partial t^{\mathrm{j}}}\left(J^{[\mathbb{C}^N/G]}(\boldsymbol{t},z)\right)$ is a $\mathbb{C}[\lambda]$-multiple of λ^{N_g}. Therefore, $i_1\left(z\frac{\partial}{\partial t^{\mathrm{j}}}\left(J^{[\mathbb{C}^N/G]}(\boldsymbol{t},z)\right)\right)$ satisfies the hypotheses of $F^{1,\mathbf{s}}$ from the previous lemma. We conclude that

$$\lim_{\lambda\mapsto 0}\Delta^\circ\left(-z\frac{\partial}{\partial t^{\mathrm{j}}}\left(J^{[\mathbb{C}^N/G]}(\boldsymbol{t},-z)\right)\right) = \lim_{\lambda\mapsto 0}\Delta'\circ i_1\left(-z\frac{\partial}{\partial t^{\mathrm{j}}}\left(J^{[\mathbb{C}^N/G]}(\boldsymbol{t},-z)\right)\right)$$

lies on $\mathscr{L}^{(Q,G)}$. The result then follows by examining the coefficients of z^1 and z^0 in the above expression. □

The following more general statement will prove useful for applications. The proof is the same argument as above.

Theorem 5.12. – *Let $F^{[\mathbb{C}^N/G]}(\boldsymbol{t},z)$ be a function lying on $\mathscr{L}^{[\mathbb{C}^N/G]}$ such that the projection of the non-equivariant limit $\lim_{\lambda\mapsto 0}F^{[\mathbb{C}^N/G]}(\boldsymbol{t},z)$ to $\mathscr{V}_+^{[\mathbb{C}^N/G]}$ is both well defined and is supported in the span of $\mathbb{1}_g$ such that g fixes only the origin in $\mathbb{C}^N$. Then $\lim_{\lambda\mapsto 0}\Delta^\circ(F^{[\mathbb{C}^N/G]}(\boldsymbol{t},z))$ lies on $\mathscr{L}^{(Q,G)}$.*

REMARK 5.13. – Theorem 5.11 should extend more generally to the setting of *hybrid theories*, where the moduli spaces $W_{g,n,G}(\mathcal{X})$ parameterize stable maps from curves into a target $\mathcal{X}$ together with roots of certain universal bundles. In this setting the $c = 0$ case would correspond to local GW theory over $\mathcal{X}$ and the $c = 1$ case to a hybrid theory. See [9] and also [8] for more details on this setting.

5.5. Quantum Serre duality

Here we recall the statement of quantum Serre duality. The purpose of this section is twofold. First, we wish to emphasize the analogy between quantum Serre duality and the MLK correspondence given above. Second, we will use these results in the next section to relate the crepant transform conjecture to the LG/CY correspondence.

Let $\mathcal{X}$ be a smooth projective orbifold and let $E \to \mathcal{X}$ be a vector bundle over $\mathcal{X}$ which is pulled back from the coarse underlying space. Given an invertible multiplicative characteristic class

$$\mathbf{s} : V \mapsto \exp\left(\sum_{k\geq 0} s_k \operatorname{ch}_k(V)\right),$$

we may define the $\mathbf{s}$-twisted GW invariants of $\mathcal{X}$ in a manner akin to (1.3.1) (see [15] for details). We will denote these invariants and their corresponding generating functions with the superscript $E, \mathbf{s}$.

Quantum Serre duality gives a relation between invariants twisted with respect to the vector bundle E and those twisted with respect to the dual bundle $E^\vee$. The main statement in genus zero is given below. This is Corollary 10 of [15], and follows in the orbifold case from Theorem 6.1.1 in [32].

Let $\{\gamma_i\}_{i\in I}$ be a basis for $H^*_{CR}(\mathcal{X})$. Let $\mathbf{s}^* : V \mapsto \exp\left(\sum_{k\geq 0}(-1)^{k+1} s_k \operatorname{ch}_k(V)\right)$, so that $\mathbf{s}^*(E^\vee) = \frac{1}{\mathbf{s}(E)}$.

THEOREM 5.14 (Quantum Serre duality). – *Define the (symplectic) transformation $i_{E^\vee} : \mathcal{V}^{E^\vee,\mathbf{s}^*} \to \mathcal{V}^{E,\mathbf{s}}$ by $\gamma_i \mapsto \gamma_i/\mathbf{s}(E)$. Then $i_{E^\vee}(\mathcal{L}^{E^\vee,\mathbf{s}^*}) = \mathcal{L}^{E,\mathbf{s}}$. Furthermore we have the identification*

$$z\frac{\partial}{\partial t^{\mathbf{s}(E)}_{E^\vee}} i_{E^\vee}\left(J^{E^\vee,\mathbf{s}^*}(\boldsymbol{t}_{E^\vee}, z)\right) = J^{E,\mathbf{s}}(\boldsymbol{t}_E, z)$$

where the change of variables is given by

$$\langle \boldsymbol{t}_E, \gamma_i\rangle = \sum_{i\in I} \frac{\partial^2}{\partial t^i_{E^\vee}\partial t^{\mathbf{s}(E)}_{E^\vee}} \mathcal{F}_0^{E^\vee,\mathbf{s}^*}(\boldsymbol{t}_{E^\vee}, 0, 0, \ldots).$$

5.5.1. *Complete intersections and local invariants.* – Consider the special case where E is a direct sum of convex line bundles ($H^1_{CR}(\mathcal{C}, f^*(E)) = 0$ for all maps f from a curve $\mathcal{C}$ into $\mathcal{X}$) and $\mathbf{s}$ is the equivariant Euler characteristic:

$$s_0 = \ln(\lambda),\ s_k = (-1)^k (k-1)!/\lambda^k \text{ for } k > 0. \tag{5.5.1}$$

In this case, the genus-zero $\mathbf{s}$-twisted invariants with respect to E are related to invariants of the hypersurface $\mathcal{Z}$ cut out by a generic section of E by the so-called *quantum Lefschetz principle* ([15], [32]). Coates has recently rephrased this relationship in terms of Lagrangian cones. Let $i : \mathcal{Z} \to \mathcal{X}$ denote the inclusion.

THEOREM 5.15 ([11]). – *Assume that the vector bundle E is pulled back from a vector bundle over the coarse space of $\mathcal{X}$. After specializing the s_k as above, let $F^{E,\mathbf{s}}$ be a point in $\mathscr{L}^{E,\mathbf{s}}$ with a well defined non-equivariant limit. Then* $\lim_{\lambda\mapsto 0} i^*(F^{E,\mathbf{s}})$ *lies on* $\mathscr{L}^{\mathcal{Z}}$.

REMARK 5.16. – Although the above theorem was proven only for the case of $\mathcal{X}$ a smooth variety, the proof extends to orbifolds provided we assume that E is pulled back from a vector bundle $|E| \to |\mathcal{X}|$ over the coarse underlying space of $\mathcal{X}$.

On the other hand, if we specialize to

$$s_0' = -\ln(-\lambda),\ s_k' = (k-1)!/\lambda^k \text{ for } k > 0$$

as in (5.4.1), the $\mathbf{s}^*$-twisted invariants with respect to $E^\vee$ give the local invariants of the total space of $E^\vee$. For $\gamma_1, \ldots, \gamma_n \in H^*_{CR}(\mathcal{X})$,

$$\langle \psi^{a_1}\gamma_1, \ldots, \psi^{a_n}\gamma_n\rangle^{\mathrm{Tot}(E^\vee)}_{h,n,d} = \langle \psi^{a_1}\gamma_1, \ldots, \psi^{a_n}\gamma_n\rangle^{E^\vee,\mathbf{s}^*}_{h,n,d}. \tag{5.5.2}$$

Theorem 5.14 implies a relation between the local invariants of $\mathrm{Tot}(E^\vee)$ and the invariants of the hypersurface $\mathcal{Z}$. For our purposes, it is most useful to phrase the relationship in a manner analogous to Theorem 5.12. Let $\Delta^\diamond$ denote the symplectic transformation

$$\Delta^\diamond : \phi_i \mapsto e^{\pi\sqrt{-1}c_1(E)/z}\frac{(-1)^{\mathrm{rk}(E)}}{e_{\mathbb{C}^*}(E)}\phi_i.$$

THEOREM 5.17. – *Let $F^{Tot(E^\vee)}(\boldsymbol{t}, z)$ be a function lying on $\mathscr{L}^{Tot(E^\vee)}$. Assume further that $F^{Tot(E^\vee)}(\boldsymbol{t}, z)$ takes the form*

$$F^{Tot(E^\vee)}(\boldsymbol{t}, z) = e_{\mathbb{C}^*}(E)\widetilde{F}^{Tot(E^\vee)}(\boldsymbol{t}, z),$$

and that $\widetilde{F}^{Tot(E^\vee)}(\boldsymbol{t}, z)$ has a well defined non-equivariant limit.

Then $\lim_{\lambda\mapsto 0} i^* \circ \Delta^\diamond(F^{Tot(E^\vee)}(\boldsymbol{t}, z))$ *lies on* $\mathscr{L}^{\mathcal{Z}}$.

Proof. – The symplectic transformation $\Delta^\diamond$ may be written as $\Delta'' \circ i_{E^\vee}$, where $i_{E^\vee}$ is as in Theorem 5.14, and $\Delta'' = e^{\pi\sqrt{-1}c_1(E)/z}$. The map Δ'' compensates for the fact that with our given specializations, s_0' does not equal $-s_0$ as in the relationship between s_0^* and s_0 in Theorem 5.14, but rather $s_0' = -s_0 - \sqrt{-1}\pi$ (see the remark after Theorem 1′ in [15] for details). By Theorem 5.14, $\Delta^\diamond(F^{Tot(E^\vee)}(\boldsymbol{t}, z))$ lies in $\mathscr{L}^{E,\mathbf{s}}$ with the specialization (5.5.1). The specific assumptions on $F^{Tot(E^\vee)}(\boldsymbol{t}, z)$ guarantee that $\Delta^\diamond(F^{Tot(E^\vee)}(\boldsymbol{t}, z))$ has a well defined non-equivariant limit. Theorem 5.15 then implies the result. □

6. The CTC and the LG/CY correspondence

In this section we give an application of Theorem 5.11. In particular we use it together with its analogue, quantum Serre duality (Theorem 5.17), to relate two well known conjectures from Gromov-Witten theory. We show that in genus zero, the well known *crepant transformation conjecture* (also known as the crepant resolution conjecture) from [12, 18] implies the more recent *LG/CY correspondence* of [7].

6.1. The Conjectures

Let (Q,G) be an LG pair with $Q=\sum_{j=1}^N x_j^{d/c_j}$ a Fermat polynomial and G an admissible subgroup of $SL_N(\mathbb{C})$. We assume always that $\gcd(c_1,\dots,c_N) = 1$. Let $\bar{G}$ denote the quotient $G/\langle \mathfrak{j}\rangle$. Note that Q may be viewed as a homogeneous function on the stack quotient $\mathbb{P}(G)=[\mathbb{P}(c_1,\dots,c_N)/\bar{G}]$.

Definition 6.1. – We say a quasi-homogeneous polynomial satisfies the *Calabi-Yau condition* if $\sum_{i=1}^N c_i = d$. This is equivalent to requiring that Q give a global section of the anticanonical bundle of $\mathbb{P}(G)$, which, by the adjunction formula, implies that $\{Q=0\}$ defines a Calabi-Yau variety.

Assume from here forward that Q satisfies the Calabi-Yau condition. Let $\mathcal{X}$ denote the stack quotient $[\mathbb{C}^N/G]$ and let $\mathcal{Y}$ denote the total space of the canonical bundle $K=K_{\mathbb{P}(G)}$.

Lemma 6.2. – *The space $\mathcal{Y}$ gives a toric crepant partial resolution of $\mathcal{X}$.*

Proof. – Let $M\cong\mathbb{Z}^N$ denote a lattice and let $\Sigma\subset M$ be a fan such that $X_\Sigma=\mathbb{P}(G)$. Let $p_1,\dots,p_N\in M$ be the primitive generators of the N rays of Σ. The cones of Σ are exactly those whose extremal rays are generated by $\{p_{j_1},\dots,p_{j_k}\}$ where $\{j_1,\dots,j_k\}$ is a strict subset of $\{1,\dots,N\}$. Abusing notation, we will identify a cone with its ray generators.

Let $\widetilde{M}$ denote the augmented lattice $M\oplus\mathbb{Z}$, and define $\tilde{p}_j := (p_j,1)\in\widetilde{M}$. Define Σ' as the fan in $\widetilde{M}$ consisting of the cones $\{\tilde{p}_{j_1},\dots,\tilde{p}_{j_k}\}$ for $\{j_1,\dots,j_k\}$ *any* subset of $[[1,N]]$. Define $\widetilde{\Sigma}$ as the star subdivision of Σ' after adding the ray generated by $(\mathbf{0},1)$ where $\mathbf{0}$ is the origin in M. One may check using simple toric arguments that $\mathcal{X}$ is equal to the toric stack $X_{\Sigma'}$, and $\mathcal{Y}$ is $X_{\widetilde{\Sigma}}$. It is apparent from this description that $\mathcal{Y}$ is a toric partial resolution of $\mathcal{X}$. Furthermore note that all ray generators of Σ' are at height one in the augmented coordinate, as is the added ray $(\mathbf{0},1)$ defining the resolution. This implies that $\mathcal{Y}\to|\mathcal{X}|$ is crepant. $\square$

The inertia orbifold $I\mathcal{X}$ is a disjoint union of components $\mathcal{X}_g$ indexed by $g\in G$. There is a natural choice of basis for the equivariant cohomology of $\mathcal{X}$ given by $\{\mathbb{1}_g\}_g\in G$, where $\mathbb{1}_g$ is the fundamental class of $\mathcal{X}_g$.

The components of the inertia orbifold $I_{\mathcal{Y}}$ are indexed by those $g\in G$ which fix a positive-dimensional subspace of $\mathbb{C}^N$, i.e., $N_g>0$. For notational convenience we will write $I\mathcal{Y}=\coprod_{g\in G}\mathcal{Y}_g$, with the understanding that $\mathcal{Y}_g$ is empty unless $N_g>0$. An equivariant basis for the Chen-Ruan cohomology of $\mathcal{Y}$ is given by

$$\bigcup_{g\in G}\{\tilde{\mathbb{1}}_g,\tilde{\mathbb{1}}_gH,\dots,\tilde{\mathbb{1}}_gH^{(N_g-1)}\},$$

where $\tilde{\mathbb{1}}_g$ is the fundamental class of $\mathcal{Y}_g$ and $\tilde{\mathbb{1}}_gH^k$ denotes the pullback of the kth power of the hyperplane class from the course space of $\mathcal{Y}_g$. Here again we use the convention that $\mathbb{1}_g$ is zero if $\mathcal{Y}_g$ is empty.

Gromov-Witten theory for local toric targets is usually defined in terms of equivariant cohomology. In our case we use a $\mathbb{C}^*$-action on $\mathcal{X}$ with weight $-c_j$ on the jth component, in other words $\lambda_j = c_j\lambda$. The corresponding action on $\mathcal{Y}$ is by multiplication in the fiber direction with character $-d\lambda$. We now give (a refined version of) the genus zero crepant transformation conjecture. Let $\mathcal{L}^{\mathcal{X}}\subset\mathcal{V}^{\mathcal{X}}$ and $\mathcal{L}^{\mathcal{Y}}\subset\mathcal{V}^{\mathcal{Y}}$ denote the Lagrangian cones corresponding to the equivariant GW theory of $\mathcal{X}$ and $\mathcal{Y}$ respectively. We distinguish two coordinates in the respective J-functions. Let $t = t^{\mathfrak{j}}$ denote the dual coordinate to $\mathbb{1}_{\mathfrak{j}}$ in $H^*_{CR}(\mathcal{X})$. Let q denote the *exponential* of the dual coordinate to the hypersurface H in $H^*(\mathcal{Y}) \subset H^*_{CR}(\mathcal{Y})$. By the divisor equation, the function $J^{\mathcal{Y}}$ (and therefore the Lagrangian cone $\mathscr{L}^{\mathcal{Y}}$) is a well defined function of q ([1]). Let us assume further that there exists a function $I^{\mathcal{Y}}(\boldsymbol{t}, z)$ which generates $\mathscr{L}^{\mathcal{Y}}$ in the sense of (3.0.6) and is in fact analytic in a neighborhood of $q = 0$. Then via the change of variables

$$q = t^{-d}$$

and analytic continuation, we can view $I^{\mathcal{Y}}(\boldsymbol{t}, z)$ as a function of t. Thus it makes sense to analytically continue $\mathscr{L}^{\mathcal{Y}}$ from $q = 0$ to $t = 0$.

Conjecture 6.3 (The crepant transformation conjecture for $\mathcal{Y} \dashrightarrow \mathcal{X}$, [17, 18])
The analytic continuation of $\mathscr{L}^{\mathcal{Y}}$ converges in a neighborhood of $t = 0$, and there exists a symplectic transformation $\mathbb{U} : \mathcal{V}^{\mathcal{X}} \to \mathcal{V}^{\mathcal{Y}}$ which identifies $\mathscr{L}^{\mathcal{X}}$ with the analytic continuation of $\mathscr{L}^{\mathcal{Y}}$.

In our case we deal with local targets, here one may refine the above conjecture to take into account the equivariant nature of the theory.

Conjecture 6.4 (The refined crepant transformation conjecture)
Conjecture 6.3 holds. In addition the following conditions are satisfied:

1. *$\mathbb{U}$ has coefficients in $\mathbb{C}[\lambda, z, z^{-1}]$. In the non-equivariant limit, $\mathbb{U}$ restricts to an isomorphism between the subspaces of $\mathcal{V}^{\mathcal{X}}_c \subset \mathcal{V}^{\mathcal{X}}$ and $\mathcal{V}^{\mathcal{Y}}_c \subset \mathcal{V}^{\mathcal{Y}}$ spanned by classes of compact support.*
2. *We have*
$$\mathbb{U}(\mathbb{1}_g) = C_0(\lambda)\tilde{\mathbb{1}}_g + \sum_{b=1}^{d-1} (\lambda + H) \cdot C_b(\lambda)\tilde{\mathbb{1}}_{g\mathfrak{j}^b}$$
where $C_b(\lambda) \in H^*(\mathcal{Y})[\lambda]((z^{-1}))$. In particular, the restriction, $\mathbb{U}_c$, of $\mathbb{U}$ to $\mathcal{V}^{\mathcal{X}}_c$ has image in the $\mathbb{C}((z^{-1}))$-span of $(\lambda + H) \cdot H^*_{CR}(\mathcal{Y})[\lambda]$.

Remark 6.5. – Conditions (1) and (2) above are very natural. It is generally believed that the symplectic transformation $\mathbb{U}$ should be induced by a Fourier-Mukai transform between equivariant K-groups, in the sense of [25]. In this case, $\mathbb{U}$ will automatically be symplectic, because the Fourier-Mukai transform is a category equivalence and preserves the categorical Euler pairing. Furthermore the Fourier-Mukai transform has a nonequivariant limit and preserves the compactly supported part of the K-groups, which induces the corresponding properties in $\mathbb{U}$. See [25] for more details. In the next section we give further evidence for these conditions.

The Landau-Ginzburg/Calabi-Yau (LG/CY) correspondence takes a similar form to Conjecture 6.3. Given an LG pair (Q, G) as in the previous section, let $\mathcal{Z}$ denote the Calabi-Yau variety $\{Q = 0\} \subset \mathbb{P}(G)$. Let $i : \mathcal{Z} \to \mathbb{P}(G)$ denote the inclusion. The LG/CY correspondence relates the FJRW theory of (Q, G) to the GW theory of $\mathcal{Z}$ in a similar fashion to the crepant transformation conjecture. In particular, in genus zero the conjecture states that there is a symplectic transformation identifying the respective Lagrangian cones.

CONJECTURE 6.6 (The LG/CY correspondence for (Q,G)). – *There exists a symplectic transformation* $\mathbb{V} : \mathscr{V}^{(Q,G)} \to \mathscr{V}^{Z}$ *which identifies* $\mathscr{L}^{(Q,G)}$ *with the analytic continuation of* $\mathscr{L}^{Z}$.

6.2. rCTC implies LG/CY

In this section we give an explanation of the similarity between these two correspondences. Namely we show that the refined crepant transformation conjecture implies the LG/CY correspondence.

LEMMA 6.7. – *Assuming Conjecture 6.4, define the map* $\mathbb{V}$ *by*

$$\mathbb{V} := \left(i^* \circ \Delta^\diamond \circ \mathbb{U}_c \circ (\Delta^\circ)^{-1}\right)|_{\lambda=0}.$$

Then $\mathbb{V}$ *is symplectic.*

Proof. – We will use conditions (1) and (2) from Conjecture 6.4. First, for any compactly supported class $\alpha \in H^*_{CR,c}(\mathcal{X})((z^{-1})) \subset H^*_{CR}(\mathcal{X})((z^{-1}))$ with a well-defined non-equivariant limit, $\mathbb{U}_c(\alpha) = \mathbb{U}(\alpha)$ may be written as

$$(\lambda + H) \cdot \sum_{\substack{g\in G \\ 0\le k\le N_g-1}} C^{\alpha}_{g,k}(\lambda)\mathbb{1}_g H^k,$$

where $C^{\alpha}_{g,k}(\lambda) \in \mathbb{C}[\lambda]((z^{-1}))$. Note that in the non-equivariant limit of $\mathbb{U}_c(\alpha)$, the terms of the form $(\lambda + H)C^{\alpha}_{g,N_g-1}(\lambda)\mathbb{1}_g H^{N_g-1}$ vanish. By condition (1), the map

$$(\mathbb{U}_c)|_{\lambda=0} : \alpha \mapsto H \cdot \sum_{\substack{g\in G \\ 0\le k\le N_g-2}} C^{\alpha}_{g,k}(0) i^*(\mathbb{1}_g H^k)$$

is an isomorphism.

For $\alpha, \beta \in H^*_{CR,c}(\mathcal{X})((z^{-1}))$, the pairing $\langle \alpha, \beta\rangle \in \mathbb{C}$. Since $\mathbb{U}$ and $\Delta^\diamond$ are symplectic this implies that

$$\langle \Delta^\diamond \circ \mathbb{U}(\alpha), \Delta^\diamond \circ \mathbb{U}(\beta)\rangle = \langle \Delta^\diamond \circ \mathbb{U}_c(\alpha), \Delta^\diamond \circ \mathbb{U}_c(\beta)\rangle \in \mathbb{C}$$

and thus

$$\langle (\Delta^\diamond \circ \mathbb{U}_c)|_{\lambda=0}(\alpha), (\Delta^\diamond \circ \mathbb{U}_c)|_{\lambda=0}(\beta)\rangle = \lim_{\lambda \mapsto 0} \langle \Delta^\diamond \circ \mathbb{U}_c(\alpha), \Delta^\diamond \circ \mathbb{U}_c(\beta)\rangle = \langle \alpha, \beta\rangle .$$

Therefore

$$(\Delta^\diamond \circ \mathbb{U}_c)|_{\lambda=0} : \alpha \mapsto \sum_{\substack{g\in G \\ 0\le k\le N_g-1}} C^{\alpha}_{g,k}(0) i^*(\mathbb{1}_g H^k)$$

is in fact a symplectic isomorphism. On the other hand, examining the pairing given by the twisted theory of $(-K, \mathbf{s})$, where $\mathbf{s}$ is as in (5.5.1), we see that when terms of the form $\mathbb{1}_g H^{N_g-1}$ are paired with elements of $H^*_{CR}(\mathcal{Y})[\lambda]$, the result lies in $\mathscr{O}(\lambda)$. Therefore these terms do not contribute to the pairing $\langle (\Delta^\diamond \circ \mathbb{U}_c)|_{\lambda=0}(\alpha), (\Delta^\diamond \circ \mathbb{U}_c)|_{\lambda=0}(\beta)\rangle$ for $\alpha, \beta \in H^*_{CR,c}(\mathcal{X})((z^{-1}))$. So in fact we conclude that the map

$$\alpha \mapsto \sum_{\substack{g\in G \\ 0\le k\le N_g-2}} C^{\alpha}_{g,k}(0) i^*(\mathbb{1}_g H^k)$$

is symplectic.

Note that $\lim_{\lambda\mapsto 0} i^* \circ \Delta^\diamond \circ \mathbb{U}_c(\alpha)$ is equal to

$$i^*\Big(\sum_{\substack{g\in G\\ 0\le k\le N_g-2}} C^\alpha_{g,k}(0)(\mathbb{1}_g H^k)\Big).$$

Since $i^* : \mathscr{V}^{-K,\mathbf{s}} \to \mathscr{V}^{Z}$ is a symplectic isomorphism when restricted to the span of $\{\mathbb{1}_g H^k\}_{g\in G,\, 0\le k\le N_g-2}$, the map $\alpha \mapsto \lim_{\lambda\mapsto 0} i^* \circ \Delta^\diamond \circ \mathbb{U}_c(\alpha)$ will be a symplectic isomorphism.

Δ° is a symplectic isomorphism when restricted to the span of elements of compact support. Thus $\mathbb{V}$, defined as the composition of the above map with $(\Delta^\circ)^{-1}$, is as well. □

LEMMA 6.8. – *Assume there exists a function $I^{\mathcal{X}}(\boldsymbol{t},z)$ lying on $\mathscr{L}^{\mathcal{X}}$ such that for any group element g, the coefficient of $\mathbb{1}_g$ in $\frac{\partial}{\partial t} I^{\mathcal{X}}(\boldsymbol{t},z)$ lies in $\lambda^{N_g}\mathbb{C}[[\boldsymbol{t}]]((z^{-1}))[\lambda]$. Then the function*

$$I^{(Q,G)}(\boldsymbol{t},z) := \lim_{\lambda\mapsto 0} \Delta^\circ\left(z\frac{\partial}{\partial t}\left(I^{\mathcal{X}}(\boldsymbol{t},z)\right)\right)$$

lies on $\mathscr{L}^{(Q,G)}$, and the symplectic transformation $\mathbb{V}$ maps $I^{(Q,G)}(\boldsymbol{t},z)$ to the analytic continuation of $\mathscr{L}^{Z}$.

Proof. – Note that by assumption we can apply Theorem 5.12 to deduce that $I^{(Q,G)}(\boldsymbol{t},z)$ lies on $\mathscr{L}^{(Q,G)}$.

To prove the second part of the lemma, we first claim that

$$\lim_{\lambda\mapsto 0} i^* \circ \Delta^\diamond \circ \mathbb{U}\left(z\frac{\partial}{\partial t}\left(I^{\mathcal{X}}(\boldsymbol{t},z)\right)\right)$$

lies on the analytic continuation of $\mathscr{L}^{Z}$.

To see this let $\widetilde{I^{\mathcal{Y}}(\boldsymbol{t},z)}$ denote $\mathbb{U}(I^{\mathcal{X}}(\boldsymbol{t},z))$. Conjecture 6.4 implies that $\widetilde{I^{\mathcal{Y}}(\boldsymbol{t},z)}$ lies on the analytic continuation of $\mathscr{L}^{\mathcal{Y}}$, and therefore by (3.0.4) so does $z\frac{\partial}{\partial t}\widetilde{I^{\mathcal{Y}}(\boldsymbol{t},z)} = \mathbb{U}(z\frac{\partial}{\partial t} I^{\mathcal{X}}(\boldsymbol{t},z))$. Therefore the strategy is to show that $z\frac{\partial}{\partial t}\widetilde{I^{\mathcal{Y}}(\boldsymbol{t},z)}$ satisfies the conditions of Theorem 5.17, i.e., that $z\frac{\partial}{\partial t}\widetilde{I^{\mathcal{Y}}(\boldsymbol{t},z)}$ may by written as

$$e_{\mathbb{C}^*}(-K)\widetilde{F}(\boldsymbol{t},z),$$

where $\widetilde{F}(\boldsymbol{t},z)$ has a well defined non-equivariant limit. For each $g\in G$, we will show that $\mathbb{U}$ maps the part of $z\frac{\partial}{\partial t} I^{\mathcal{X}}(\boldsymbol{t},z)$ supported on $\mathcal{X}_g$ to something divisible by $e_{\mathbb{C}^*}(-K) = d(\lambda+H)$. For g such that $N_g = 0$, the statement follows immediately by condition (2) of Conjecture 6.4 because $\tilde{\mathbb{1}}_g = 0$. For g such that $N_g > 0$, $\tilde{\mathbb{1}}_g \neq 0$, but nevertheless we compute

$$\frac{\lambda^{N_g}\tilde{\mathbb{1}}_g}{(\lambda+H)} = \sum_{i=0}^{N_g-1} \lambda^{N_g-1-i}(-H)^i\,\tilde{\mathbb{1}}_g.$$

So combining condition (2) of Conjecture 6.4 with the assumptions of the lemma implies the claim.

Given a function $I^{\mathcal{X}}(\boldsymbol{t},z)$ satisfying the assumptions listed, we have shown that the corresponding $I^{(Q,G)}(\boldsymbol{t},z)$ lies in $\mathscr{L}^{(Q,G)}$ and that $\lim_{\lambda\mapsto 0} i^* \circ \Delta^\diamond \circ \mathbb{U}\left(z\frac{\partial}{\partial t}\left(I^{\mathcal{X}}(\boldsymbol{t},z)\right)\right)$ lies in the analytic continuation $\widetilde{\mathscr{L}^{\mathcal{Z}}}$ of $\mathscr{L}^{\mathcal{Z}}$. Thus to prove that $\mathbb{V}$ sends $I^{(Q,G)}(\boldsymbol{t},z)$ to $\widetilde{\mathscr{L}^{\mathcal{Z}}}$, it suffices to show that the following diagram commutes when applied to $z\frac{\partial}{\partial t}\left(I^{\mathcal{X}}(\boldsymbol{t},z)\right)$:

$$\begin{tikzcd}
\mathscr{L}^{\mathcal{X}} \arrow[r, "\mathbb{U}"] \arrow[d, "\lim_{\lambda\mapsto 0}\Delta^{\circ}"] & \widetilde{\mathscr{L}^{\mathcal{Y}}} \arrow[d, "\lim_{\lambda\mapsto 0} i^*\circ\Delta^{\diamond}"] \\
\mathscr{L}^{(Q,G)} \arrow[r, dashed, "\mathbb{V}"] & \widetilde{\mathscr{L}^{\mathcal{Z}}}.
\end{tikzcd}$$

For g such that $N_g > 0$, terms of $z\frac{\partial}{\partial t}\left(I^{\mathcal{X}}(\boldsymbol{t},z)\right)$ supported in $H^*(\mathcal{X}_g)$ are in the kernel of the left hand map. We need therefore to check that these terms are also in the kernel of the composition of the top map with the right hand map. By the above computation, for g such that $N_g > 0$, the only part of the $\tilde{\mathbb{1}}_g$-coefficient of $z\frac{\partial}{\partial t}\widetilde{I^{\mathcal{Y}}(\boldsymbol{t},z)}$ which survives in the non-equivariant limit is a $\mathbb{C}((z^{-1}))$-multiple of $\tilde{\mathbb{1}}_g H^{N_g-1}$. This class is in the kernel of the map $i^* : H^*_{CR}([\mathbb{P}(c_1,\ldots,c_N)/\bar{G}]) \to H^*_{CR}(\mathcal{Z})$. This implies that the above diagram commutes when applied to $z\frac{\partial}{\partial t}\left(I^{\mathcal{X}}(\boldsymbol{t},z)\right)$ which proves the claim. □

We arrive at the following.

THEOREM 6.9 (rCTC implies LG/CY). – *Given an LG pair (Q,G) as above with Q a Fermat polynomial and G a subgroup of $SL_N(\mathbb{C})$, the refined crepant transformation conjecture for $\mathcal{Y} \dashrightarrow \mathcal{X}$ (Conjecture 6.4) implies the LG/CY correspondence (Conjecture 6.6).*

Proof. – Note first that the J-function $J^{\mathcal{X}}(\boldsymbol{t},-z)$ satisfies the hypothesis of Lemma 6.8, as was shown in the proof of Theorem 5.11. Therefore $\mathbb{V}$ maps $J^{(Q,G)}(\boldsymbol{t},-z)$ to $\mathscr{L}^{\mathcal{Z}}$. This implies that $\mathbb{V}(\mathscr{L}^{(Q,G)}) \subseteq \widetilde{\mathscr{L}^{\mathcal{Z}}}$.

On the other hand, consider the function $J^{\mathcal{Y}}(\boldsymbol{t},z)$. By applying Theorem 5.14 to the particular specializations $\mathbf{s}$ and $\mathbf{s}'$ of Section 5.5.1, we deduce that there exists a choice of $\boldsymbol{t}'$ such that

$$\Delta^{\diamond}\left(z\frac{\partial}{\partial t^{e_{\mathbb{C}^*}(-K)}}J^{\mathcal{Y}}(\boldsymbol{t}',z)\right) = J^{-K,\mathbf{s}}(\boldsymbol{t},z).$$

Standard argument (see e.g., [11], Theorem 1.1) shows that the right hand side has a well defined non-equivariant limit. Since the map $\Delta^{\diamond}$ involves division by $e_{\mathbb{C}^*}(-K)$, we conclude that $z\frac{\partial}{\partial t^{e_{\mathbb{C}^*}(-K)}}J^{\mathcal{Y}}(\boldsymbol{t}',z)$ may be written in the form $e_{\mathbb{C}^*}(-K)\widetilde{F}(\boldsymbol{t},z)$ where $\widetilde{F}(\boldsymbol{t},z)$ has a well defined non-equivariant limit. So by Theorem 5.17,

$$\lim_{\lambda\mapsto 0} i^*\circ\Delta^{\diamond}\left(-z\frac{\partial}{\partial t^{e_{\mathbb{C}^*}(-K)}}J^{\mathcal{Y}}(\boldsymbol{t}',-z)\right) \in \mathscr{L}^{\mathcal{Z}}.$$

Analyzing the non-negative z-coefficients of the right hand side yield that in fact

$$\lim_{\lambda\mapsto 0} i^*\circ\Delta^{\diamond}\left(-z\frac{\partial}{\partial t^{e_{\mathbb{C}^*}(-K)}}J^{\mathcal{Y}}(\boldsymbol{t}',-z)\right) = J^{\mathcal{Z}}(\boldsymbol{t},-z).$$

Note furthermore that the non-equivariant limit of $z\frac{\partial}{\partial t^{e_{\mathbb{C}^*}(-K)}}J^{\mathcal{Y}}(\boldsymbol{t}',z)$ is contained in the span of classes of compact support, thus

$$\lim_{\lambda\mapsto 0}\mathbb{U}^{-1}\left(z\frac{\partial}{\partial t^{e_{\mathbb{C}^*}(E)}}J^{\mathcal{Y}}(\boldsymbol{t}',z)\right)$$

lies in compact support by assumption (1) of Conjecture 6.4. By Theorem 5.12 we have that

$$\lim_{\lambda \mapsto 0} \Delta^{\circ} \circ \mathbb{U}^{-1}\left(z\widetilde{\frac{\partial}{\partial t^{e_{\mathbb{C}^*}(E)}} J^{\mathcal{Y}}(\boldsymbol{t}', z)}\right) \in \mathscr{L}^{(Q,G)},$$

where $\widetilde{(-)}$ denotes analytic continuation.

Next, note that in the non-equivariant limit of $z\widetilde{\frac{\partial}{\partial t^{e_{\mathbb{C}^*}(E)}} J^{\mathcal{Y}}(\boldsymbol{t}', z)}$, the coefficients of $H^{N_g-1}\tilde{\mathbb{1}}_g$ in $\widetilde{F}(\boldsymbol{t}, z)$ do not contribute. In other words the diagram

$$\begin{tikzcd} \mathscr{L}^{\mathcal{X}} \arrow[d, "\lim_{\lambda\mapsto 0} \Delta^{\circ}"] & \widetilde{\mathscr{L}^{\mathcal{Y}}} \arrow[l, "\mathbb{U}^{-1}"'] \arrow[d, "\lim_{\lambda\mapsto 0} i^*\circ\Delta^{\diamond}"] \\ \mathscr{L}^{(Q,G)} & \widetilde{\mathscr{L}^{\mathcal{Z}}}. \arrow[l, dashed, "\mathbb{V}^{-1}"] \end{tikzcd}$$

commutes when applied to $z\widetilde{\frac{\partial}{\partial t^{e_{\mathbb{C}^*}(E)}} J^{\mathcal{Y}}(\boldsymbol{t}', z)}$. Therefore $\mathbb{V}^{-1}\left(\widetilde{J^{\mathcal{Z}}(\boldsymbol{t}, z)}\right)$ lies in $\mathscr{L}^{(Q,G)}$. Thus $\mathbb{V}(\mathscr{L}^{(Q,G)}) \supseteq \widetilde{\mathscr{L}^{\mathcal{Z}}}$. □

7. A proof of the LG/CY correspondence

In this section we verify Conjecture 6.4, the refined crepant transformation conjecture, in the particular case of relevance for the LG/CY correspondence. Although a general proof of the crepant transformation conjecture for toric orbifolds is given in [16], in order to apply such a result towards the LG/CY correspondence, one must verify those specific properties of the transformation $\mathbb{U}$ described in Conjecture 6.4. These properties are not readily apparent in [16], thus we have chosen to explicitly compute $\mathbb{U}$ below. The methods used are well known to the experts and are a straightforward generalization of techniques used in, e.g., [17] and [24], thus they are relegated to the appendix.

7.1. The crepant transformation conjecture

Let the setup be as in Section 6. In particular, we have an LG pair (Q, G) with Q a Fermat polynomial satisfying the Calabi-Yau condition and G an admissible subgroup of $SL_N(\mathbb{C})$.

The crepant transformation conjecture states that there exists a symplectic transformation $\mathbb{U}$ which sends $\mathscr{L}^{\mathcal{X}}$ to the analytic continuation of $\mathscr{L}^{\mathcal{Y}}$. To prove this, we construct two functions $I^{\mathcal{X}}$ and $I^{\mathcal{Y}}$ which generate $\mathscr{L}^{\mathcal{X}}$ and $\mathscr{L}^{\mathcal{Y}}$ respectively, and show they are related by analytic continuation and symplectic transformation.

THEOREM 7.1. – *There is an explicit linear symplectic transformation $\mathbb{U} : \mathcal{V}^{\mathcal{Y}} \to \mathcal{V}^{\mathcal{X}}$ which identifies the I-function $I^{\mathcal{X}}$ with the analytic continuation of $I^{\mathcal{Y}}$. Furthermore, $\mathbb{U}$ is compatible with the Fourier-Mukai transform (in the sense of* [26]*) and satisfies the conditions of Conjecture 6.4.*

7.1.1. *Setting notation.* – Recall that we have a natural choice of basis for the equivariant cohomology of $I\mathcal{X}$ given by $\{\mathbb{1}_g\}_{g\in G}$ where $\mathbb{1}_g$ is the fundamental class of $\mathcal{X}_g$. Let t^g denote the dual coordinate to $\mathbb{1}_g$. As before we distinguish the dual coordinate to $\mathbb{1}_{\mathfrak{j}}$, denoting it as simply t. This will be the analytic continuation coordinate in Theorem 7.1.

NOTATION 7.2. – We let $\{g_s\}_{s\in S}$ denote the set of elements of G which fix at least one coordinate of $\mathbb{C}^N$ ($N_g > 0$).

Due to the fact that $G \leq G_Q$ for a given Fermat polynomial Q, we observe that the group G is generated by $\{g_s\}_{s\in S}$ together with the element $\mathfrak{j}$, $G = \langle\{g_s\}_{s\in S} \cup \{\mathfrak{j}\}\rangle$. For notational convenience we will write $I\mathcal{Y} = \coprod_{g\in G} \mathcal{Y}_g$, with the understanding that $\mathcal{Y}_g$ is empty unless $g \in \{g_s\}_{s\in S}$. We also let t^g denote the dual coordinate of $\tilde{\mathbb{1}}_g$ for $g \in G$, and let q denote the exponential of the dual coordinate to H.

7.1.2. *I-functions.* – Consider the following functions, which lie in $\mathcal{V}^{\mathcal{X}}$ and $\mathcal{V}^{\mathcal{Y}}$ respectively:

$$I^{\mathcal{X}}(t, \boldsymbol{t}, z) = zt^{d\lambda/z} \sum_{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^S} \prod_{s\in S} \frac{(t^{g_s})^{k_s} z^{(\mathrm{age}(g_s)-1)k_s}}{k_s!} \sum_{k_0\geq 0} \frac{t^{k_0}}{z^{\sum_j \langle k_0 c_j/d + a(\mathbf{k})^j\rangle} k_0!}$$
$$\cdot \prod_{j=1}^{N} \frac{\Gamma(1 - c_j\frac{\lambda}{z} - \langle k_0 c_j/d + a(\mathbf{k})^j\rangle)}{\Gamma(1 - c_j\frac{\lambda}{z} - k_0 c_j/d - a(\mathbf{k})^j))} \mathbb{1}_{\mathfrak{j}^{k_0}\prod_s g_s^{k_s}},$$

$$I^{\mathcal{Y}}(q, \boldsymbol{t}, z) = zq^{H/z} \sum_{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^S} \prod_{s\in S} \frac{(t^{g_s})^{k_s} z^{(\mathrm{age}(g_s)-1)k_s}}{k_s!} \sum_{k_0\geq 0} \frac{q^{k_0/d}}{z^{\sum_j \langle k_0 c_j/d - a(\mathbf{k})^j\rangle}} \frac{\Gamma(1 - \frac{d(\lambda+H)}{z})}{\Gamma(1 - k_0 - \frac{d(\lambda+H)}{z})}$$
$$\cdot \prod_{j=1}^{N} \frac{\Gamma(1 + c_j H/z - \langle -k_0 c_j/d + a(\mathbf{k})^j\rangle)}{\Gamma(1 + c_j H/z + k_0 c_j/d - a(\mathbf{k})^j)} \tilde{\mathbb{1}}_{\mathfrak{j}^{-k_0}\prod_s g_s^{k_s}},$$

where $\Gamma(1 + x)$ denotes power series expansion of the Gamma function. The above are referred to as the *I-functions* for $\mathcal{X}$ and $\mathcal{Y}$ respectively. Using the technology of twisted Gromov-Witten theory as in 4.2, on can prove that these lie on the Lagrangian cones $\mathcal{L}^{\mathcal{X}}$ and $\mathcal{L}^{\mathcal{Y}}$.

LEMMA 7.3 (Corollary 5.1 [12], Theorem 21 [14]). – *The function $I^{\mathcal{X}}(t, \boldsymbol{t}, z)$ lies on the Lagrangian cone $\mathcal{L}^{\mathcal{X}}$.* (1) *The function $I^{\mathcal{Y}}(t, \boldsymbol{t}, z)$ lies on the Lagrangian cone $\mathcal{L}^{\mathcal{Y}}$.*

7.1.3. *The transformation.* – To describe the transformation $\mathbb{U}$, we will first need to set some notation.

NOTATION 7.4. – Define the operator

$$Gr : H^*_{CR}(\mathcal{X}) \to H^*_{CR}(\mathcal{X}), \qquad \alpha \mapsto \frac{\deg(\alpha)}{2}\alpha$$

for α of pure degree in $H^*_{CR}(\mathcal{X})$. Here by degree we mean the real Chen-Ruan degree.

In addition, for α a cohomology class of pure degree in $H^*(I\mathcal{X})$ supported on a single connected component, define the function $\deg_0(\alpha)$ to be the *untwisted* degree of α in $H^*(I\mathcal{X})$.

(1) The I-function above is often commonly written without the factor of $t^{d\lambda/z}$. However due to the string equation, multiplication by this factor preserves the cone $\mathcal{L}^{\mathcal{X}}$. See the remark after Corollary 1 in [15] for more details.

For $\mathcal{X}$, define the $\hat{\Gamma}$-class of $\mathcal{X}$ to be

$$\hat{\Gamma}(\mathcal{X}) := \bigoplus_g \prod_{j=1}^N \Gamma(1 - m_j(g) - c_j\lambda),$$

viewed here as an operator on $H^*_{CR}(\mathcal{X})$. Similarly for $\mathcal{Y}$ we define the $\hat{\Gamma}$-class operator as

$$\hat{\Gamma}(\mathcal{Y}) = \bigoplus_g \Gamma(1 - d(H+\lambda)) \prod_{j=1}^N \Gamma(1 - m_j(g) + c_j H).$$

Consider the transformation

$$\overline{\mathbb{U}} : H^*_{CR}(\mathcal{X}) \to H^*_{CR}(\mathcal{Y})$$

given by

$$\overline{\mathbb{U}} : \mathbb{1}_g \mapsto \sum_{0 \le b < d} \frac{e^{d(\lambda+H)} - 1}{d(e^{(\lambda+H)}\xi^b - 1)} \tilde{\mathbb{1}}_{gj^{-b}}. \tag{7.1.1}$$

Definition 7.5. – Define the linear transformation $\mathbb{U} : \mathcal{V}^{\mathcal{X}} \to \mathcal{V}^{\mathcal{Y}}$ by

$$\mathbb{U} := z^{-\mathrm{Gr}} \hat{\Gamma}(\mathcal{Y}) (2\pi i)^{\deg_0/2} \overline{\mathbb{U}} (2\pi i)^{-\deg_0/2} \hat{\Gamma}(\mathcal{X})^{-1} z^{\mathrm{Gr}}.$$

The linear transformation $\mathbb{U}$ will give the desired identification between $\mathcal{L}^{\mathcal{X}}$ and $\mathcal{L}^{\mathcal{Y}}$ from Theorem 7.1. Using this explicit description of $\mathbb{U}$ we can immediately deduce the following.

Proposition 7.6. – *The transformation $\mathbb{U}$ defined above satisfies the conditions of Conjecture 6.4.*

Proof. – From the explicit expression for $\overline{\mathbb{U}}$, it is clear that $\mathbb{U}$ has a well-defined non-equivariant limit. That this limit induces an isomorphism on the restriction to compactly supported classes follows from the fact that it is induced by a Fourier-Mukai transformation ([16]).

To check condition (2) of the conjecture, note that because $z^{-\mathrm{Gr}}\hat{\Gamma}(-)(2\pi i)^{\deg_0}$ acts diagonally on both cohomologies, it is enough to show that the image of $\overline{\mathbb{U}}$ satisfies condition (2).

By the Formula (7.1.1) for $\overline{\mathbb{U}}$, we see that the coefficient of $\tilde{\mathbb{1}}_{gj^{-b}}$ in $\overline{\mathbb{U}}(\mathbb{1}_g)$ is in the $\mathbb{C}[\lambda]$-span of $(\lambda + H)$ unless $b = 0$. When $b = 0$ the coefficient may be expanded as

$$\frac{1}{d}\left(e^{(d-1)(\lambda+H)} + e^{(d-2)(\lambda+H)} + \cdots + e^{(\lambda+H)} + 1\right).$$

This proves the claim. □

Proof of Theorem 7.1. – We show in the appendix that $\mathbb{U}(I^{\mathcal{X}}(t, \boldsymbol{t}, z)) = \widetilde{I^{\mathcal{Y}}}(t, \boldsymbol{t}, z)$, where $\widetilde{I^{\mathcal{Y}}}(t, \boldsymbol{t}, z)$ is the analytic continuation of $I^{\mathcal{Y}}(q, \boldsymbol{t}, z)$ under the substitution $q = t^{-d}$. A similar analysis using *big I*-functions (see [14]) may be done to prove that in fact $\mathbb{U}$ identifies the full Lagrangian cones $\mathcal{L}^{\mathcal{X}}$ and $\widetilde{\mathcal{L}^{\mathcal{Y}}}$. This is verified in Remark 6.5 of [16]. One can check that the function $\mathbb{U}$ defined above agrees with that given in [16]. It is proven in [16] that $\mathbb{U}$ is symplectic and is compatible with a Fourier-Mukai transform in the sense of [26]. □

7.2. The LG/CY correspondence

Let the setup be as above. Namely, we have an LG pair (Q, G) such that Q is a Fermat polynomial satisfying the Calabi-Yau condition and G is an admissible subgroup. Recall that $\mathcal{Z}$ is defined to be the Calabi-Yau hypersurface $\{Q = 0\} \subset \mathbb{P}(G)$. We now verify Conjecture 6.6.

THEOREM 7.7 (The LG/CY correspondence for (Q, G)). – *There exists a linear symplectic transformation* $\mathbb{V} : \mathscr{V}^{(Q,G)} \to \mathscr{V}^{\mathcal{Z}}$ *which identifies* $\mathscr{L}^{(Q,G)}$ *with the analytic continuation of* $\mathscr{L}^{\mathcal{Z}}$.

Proof. – From Theorem 7.1, we have that $\mathbb{U}$ identifies $\mathscr{L}^{\mathcal{X}}$ with the analytic continuation of $\mathscr{L}^{\mathcal{Y}}$, and furthermore that the refined crepant transformation conjecture (Conjecture 6.4) is satisfied. The result then follows immediately from Theorem 6.9. □

8. Appendix: Calculating the transformation $\mathbb{U}$

In this section we prove that $\mathbb{U}$ indeed identifies the I-functions for $\mathcal{X}$ and $\mathcal{Y}$ after analytic continuation.

8.0.1. *The I-function of* $\mathcal{X}$. – We consider the J-function of BG, where the domain has been restricted to the span of $\{\mathbb{1}_{\mathfrak{j}}\} \cup \{\mathbb{1}_{g_s}\}_{s \in S}$. By Lemmas 2.1 and 2.2, this coincides with a restriction of $J^{0,\mathbf{0}}(\boldsymbol{t}, z)$ from Lemma 5.2:

$$J^{BG}(t, \boldsymbol{t}, z) = z \sum_{\mathbf{k} \in (\mathbb{Z}_{\geq 0})^S} \prod_{s \in S} \frac{(t^{g_s})^{k_s}}{z^{k_s} k_s!} \sum_{k_0 \geq 0} \frac{t^{k_0}}{z^{k_0} k_0!} \mathbb{1}_{\mathfrak{j}^{k_0} \prod_s g_s^{k_s}}.$$

Using the twisted theory technology, one may alter $J^{BG}(t, \boldsymbol{t}, z)$ by a *hypergeometric modification* (see [12]) to obtain a function $I^{\mathcal{X}}(t, \boldsymbol{t}, z)$ which generates $\mathscr{L}^{\mathcal{X}}$ in the sense of (3.0.5). Let $a(\mathbf{k})^j = \sum_s k_s m_j(g_s)$. Define the modification factor

$$M(k_0, \mathbf{k}) := \prod_{j=1}^{N} \prod_{l=0}^{\lfloor k_0 c_j/d + a(\mathbf{k})^j \rfloor - 1} \Big(- c_j \lambda - (\langle k_0 c_j/d + a(\mathbf{k})^j \rangle + l) z \Big)$$

where $\langle - \rangle$ denotes the fractional part. Then $I^{\mathcal{X}}(t, \boldsymbol{t}, z)$ is defined as

$$I^{\mathcal{X}}(t, \boldsymbol{t}, z) = z t^{d\lambda/z} \sum_{\mathbf{k} \in (\mathbb{Z}_{\geq 0})^S} \prod_{s \in S} \frac{(t^{g_s})^{k_s}}{z^{k_s} k_s!} \sum_{k_0 \geq 0} \frac{M(k_0, \mathbf{k}) t^{k_0}}{z^{k_0} k_0!} \mathbb{1}_{\mathfrak{j}^{k_0} \prod_s g_s^{k_s}}. \tag{8.0.1}$$

The above modification factor is explained in [12], where it is proven that $I^{\mathcal{X}}(t, \boldsymbol{t}, z)$ lies on $\mathscr{L}^{\mathcal{X}}$. We will see below that this matches the formula for $I^{\mathcal{X}}(t, \boldsymbol{t}, z)$ from the previous section.

8.0.2. *The I-function of $\mathcal{Y}$.* – An I-function for projective toric stacks is given in [13]. For the case of $[\mathbb{P}(G)]$ one obtains:

$$I^{[\mathbb{P}(G)]}(q,\boldsymbol{t},z) = zq^{H/z} \sum_{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^S} \prod_{s\in S} \frac{(t^{g_s})^{k_s}}{z^{k_s}k_s!} \sum_{k_0\geq 0} q^{k_0/d} \cdot \prod_{j=1}^{N} \frac{\Gamma(1+c_jH/z - \langle -k_0c_j/d + a(\mathbf{k})^j\rangle)}{\Gamma(1+c_jH/z + k_0c_j/d - a(\mathbf{k})^j)} \tilde{\mathbb{1}}_{\mathrm{j}^{-k_0}\prod_s g_s^{k_s}}.$$

We alter this by another hypergeometric modification to obtain an I-function for $\mathcal{Y}$:

$$I^{\mathcal{Y}}(q,\boldsymbol{t},z) = zq^{H/z} \sum_{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^S} \prod_{s\in S} \frac{(t^{g_s})^{k_s}}{z^{k_s}k_s!} \sum_{k_0\geq 0} q^{k_0/d} \cdot \prod_{l=0}^{k_0-1}(-d(H+\lambda) - lz) \prod_{j=1}^{N} \frac{\Gamma(1+c_jH/z - \langle -k_0c_j/d + a(\mathbf{k})^j\rangle)}{\Gamma(1+c_jH/z + k_0c_j/d - a(\mathbf{k})^j)} \tilde{\mathbb{1}}_{\mathrm{j}^{-k_0}\prod_s g_s^{k_s}}.$$

By Theorem 21 of [14], we see that the function $I^{\mathcal{Y}}(t,\boldsymbol{t},z)$ lies on the Lagrangian cone $\mathscr{L}^{\mathcal{Y}}$.

Next we will show that the above I-functions coincide after analytic continuation and symplectic transformation.

8.0.3. *$\hat{\Gamma}$-classes.* – In order to facilitate our analytic continuation, we will write the I-functions in a different form, motivated by Iritani's integral structure for quantum cohomology [25]. Recall the definition of the $\hat{\Gamma}$-*classes* and the notation introduced in Section 7.1.3.

Consider the modification factor for $I^{\mathcal{X}}(t,\boldsymbol{t},z)$ again.

Using the relation $(x)(x-z)\dots(x-(n-1)z) = z^n\frac{\Gamma(1+x/z)}{\Gamma(1-n+x/z)}$, we obtain

$$M(k_0,\mathbf{k}) = z^{\sum_j \lfloor k_0c_j/d + a(\mathbf{k})^j\rfloor} \prod_{j=1}^{N} \frac{\Gamma(1-c_j\frac{\lambda}{z} - \langle k_0c_j/d + a(\mathbf{k})^j\rangle)}{\Gamma(1-c_j\frac{\lambda}{z} - k_0c_j/d - a(\mathbf{k})^j)}.$$

Via the above expression and the equality

$$\begin{aligned} k_0 + \sum_{s\in S}\mathrm{age}(g_s)k_s &= \sum_{j=1}^{N} k_0c_j/d + a(\mathbf{k})^j \\ &= \sum_{j=1}^{N}\lfloor k_0c_j/d + a(\mathbf{k})^j\rfloor + \sum_{j=1}^{N}\langle k_0c_j/d + a(\mathbf{k})^j\rangle, \end{aligned}$$

$I^{\mathcal{X}}(t,\boldsymbol{t},z)$ simplifies to

$$\begin{aligned} I^{\mathcal{X}}(t,\boldsymbol{t},z) &= zt^{d\lambda/z} \sum_{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^S} \prod_{s\in S} \frac{(t^{g_s})^{k_s} z^{(\mathrm{age}(g_s)-1)k_s}}{k_s!} \sum_{k_0\geq 0} \frac{t^{k_0}}{z^{\sum_j\langle k_0c_j/d + a(\mathbf{k})^j\rangle}k_0!} \\ &\quad\cdot \prod_{j=1}^{N} \frac{\Gamma(1-c_j\frac{\lambda}{z} - \langle k_0c_j/d + a(\mathbf{k})^j\rangle)}{\Gamma(1-c_j\frac{\lambda}{z} - k_0c_j/d - a(\mathbf{k})^j))} \mathbb{1}_{\mathrm{j}^{k_0}\prod_s g_s^{k_s}} \\ &= z^{1-\mathrm{Gr}}\hat{\Gamma}(\mathcal{X})(2\pi i)^{\deg_0/2} H(t,\boldsymbol{t},z), \end{aligned}$$

where

$$
\begin{aligned}
H(t,\boldsymbol{t},z) &= t^{d\lambda/(2\pi i)} \sum_{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^S} \prod_{s\in S} \frac{(t^{g_s})^{k_s} z^{(\mathrm{age}(g_s)-1)k_s}}{k_s!} \\
&\quad \cdot \sum_{k_0\geq 0} \frac{t^{k_0}}{k_0!} \prod_{j=1}^{N} \frac{1}{\Gamma(1-c_j\frac{\lambda}{2\pi i}-k_0c_j/d-a(\mathbf{k})^j)} \mathbb{1}_{\mathrm{j}^{k_0}\prod_s g_s^{k_s}} \\
&= t^{d\lambda/(2\pi i)} \sum_{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^S} \prod_{s\in S} \frac{(t^{g_s})^{k_s} z^{(\mathrm{age}(g_s)-1)k_s}}{k_s!} \\
&\quad \cdot \sum_{0\leq m<d} \sum_{k\geq 0} \frac{t^{m+dk}}{(m+dk)!} \prod_{j=1}^{N} \frac{1}{\Gamma(1-c_j\frac{\lambda}{2\pi i}-kc_j-mc_j/d-a(\mathbf{k})^j)} \mathbb{1}_{\mathrm{j}^{m}\prod_s g_s^{k_s}}.
\end{aligned}
$$

By a similar argument to the previous case, we can also rewrite $I^{\mathcal{Y}}$ in terms of the Gamma class.

$$
\begin{aligned}
I^{\mathcal{Y}}(q,\boldsymbol{t},z) &= zq^{H/z} \sum_{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^S} \prod_{s\in S} \frac{(t^{g_s})^{k_s} z^{(\mathrm{age}(g_s)-1)k_s}}{k_s!} \\
&\quad \cdot \sum_{k_0\geq 0} \frac{q^{k_0/d}}{z^{\sum_j \langle k_0c_j/d-a(\mathbf{k})^j\rangle}} \frac{\Gamma(1-\frac{d(\lambda+H)}{z})}{\Gamma(1-k_0-\frac{d(\lambda+H)}{z})} \\
&\quad \cdot \prod_{j=1}^{N} \frac{\Gamma(1+c_jH/z-\langle -k_0c_j/d+a(\mathbf{k})^j\rangle)}{\Gamma(1+c_jH/z+k_0c_j/d-a(\mathbf{k})^j)} \tilde{\mathbb{1}}_{\mathrm{j}^{-k_0}\prod_s g_s^{k_s}} \\
&= z^{1-\mathrm{Gr}}\hat{\Gamma}(\mathcal{Y})(2\pi i)^{\deg_0/2} H^{\mathcal{Y}}(q,\boldsymbol{t},z),
\end{aligned}
$$

where

$$
\begin{aligned}
H^{\mathcal{Y}}(q,\boldsymbol{t},z) &= q^{H/2\pi i} \sum_{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^S} \prod_{s\in S} \frac{(t^{g_s})^{k_s} z^{(\mathrm{age}(g_s)-1)k_s}}{k_s!} \\
&\quad \cdot \sum_{k_0\geq 0} q^{k_0/d} \frac{\Gamma(k_0+\frac{d(\lambda+H)}{2\pi i})\sin(\pi(k_0+\frac{d(\lambda+H)}{2\pi i}))}{\pi\prod_{j=1}^{N}\Gamma(1+\frac{c_jH}{2\pi i}+k_0c_j/d-a(\mathbf{k})^j)} \tilde{\mathbb{1}}_{\mathrm{j}^{-k_0}\prod_s g_s^{k_s}} \\
&= q^{H/2\pi i} \sum_{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^S} \prod_{s\in S} \frac{(t^{g_s})^{k_s} z^{(\mathrm{age}(g_s)-1)k_s}}{k_s!} \sum_{0\leq b<d} q^{b/d}(-1)^b \frac{\sin(\pi\frac{d(\lambda+H)}{2\pi i})}{\pi} \\
&\quad \cdot \sum_{k\geq 0} q^k(-1)^{dk} \frac{\Gamma(b+dk+\frac{d(\lambda+H)}{2\pi i})}{\prod_{j=1}^{N}\Gamma(1+\frac{c_jH}{2\pi i}-a(\mathbf{k})^j+bc_j/d+k)} \tilde{\mathbb{1}}_{\mathrm{j}^{-b}\prod_s g_s^{k_s}}.
\end{aligned}
$$

In the last equality, we have made the substitution $k_0 = b+dk$ for $0\leq b<d$.

Next we apply the Mellin-Barnes method of analytic continuation to $H^{\mathcal{Y}}(q,\mathbf{t},z)$ to match these two functions. We may rewrite the above expression using residues:

$$H^{\mathcal{Y}}(q,\mathbf{t},z) = q^{H/2\pi i}\sum_{\mathbf{k}\in(\mathbb{Z}_{\geq 0})^S}\prod_{s\in S}\frac{(t^{g_s})^{k_s}z^{(\mathrm{age}(g_s)-1)k_s}}{k_s!}\sum_{0\leq b<d}(-1)^b q^{b/d}\frac{\sin(\frac{d(\lambda+H)}{2i})}{\pi}\tilde{\mathbb{1}}_{\mathrm{j}^{-b}\prod_s g_s^{k_s}}$$
$$-\int_C \frac{e^{-\pi i d s}q^s}{e^{-2\pi i s}-1}\frac{\Gamma(ds+b+\frac{d(\lambda+H)}{2\pi i})}{\prod_j\Gamma(1+c_jH/(2\pi i)-a(\mathbf{k})^j+bc_j/d+c_js)}ds.$$

Here C is a contour going clockwise along the imaginary axis, enclosing the non–negative integers to the right, and enclosing no other poles.

Closing the contour to the left yields the analytic continuation. There are poles at the negative integers due to the exponential, but these vanish due to factors of H. Indeed whenever we are supported on $\mathcal{Y}_{\mathrm{j}^{-b}\prod_s g_s^{k_s}}$, the residue at a negative integer will contribute a factor of c_jH/z for each j which is fixed by $\mathrm{j}^{-b}\prod_s g_s^{k_s}$. There are $N_{\mathrm{j}^{-b}\prod_s g_s^{k_s}} = \dim(\mathcal{Y}_{\mathrm{j}^{-b}\prod_s g_s^{k_s}})+1$ such factors. The other poles are from the Gamma function in the numerator, and occur at

$$s = -(H+\lambda)/(2\pi i) - b/d - m/d \quad \text{for } m\geq 0.$$

The residue of the Gamma function here is

$$\mathrm{Res}_{s=-(H+\lambda)/(2\pi i)-b/d-k/d)}\Gamma(ds+b+\tfrac{d(\lambda+H)}{2\pi i}) = \frac{(-1)^k}{d\cdot k!}.$$

We obtain as the analytic continuation $\widetilde{H^{\mathcal{Y}}}(q,\mathbf{t},z)$:

$$\begin{aligned}
\widetilde{H^{\mathcal{Y}}}(q,\mathbf{t},z) &= 2\pi i q^{H/2\pi i}\sum_{\mathbf{k}}\prod_{s\in S}\frac{(t^{g_s})^{k_s}z^{(\mathrm{age}(g_s)-1)k_s}}{k_s!}\\
&\quad\cdot\sum_{0\leq b<d}(-1)^b q^{b/d}\frac{\sin(d(\lambda+H)/2i)}{\pi}\tilde{\mathbb{1}}_{\mathrm{j}^{-b}\prod_s g_s^{k_s}}\\
&\quad\cdot\sum_{m\geq 0}\frac{e^{\pi i(b+m)}e^{d(H+\lambda)/2}}{e^{2\pi i(b+m)/d}e^{(\lambda+H)}-1}\cdot\frac{(-1)^m}{d\cdot m!}\frac{q^{-(b+m)/d-(\lambda+H)/(2\pi i)}}{\prod_j\Gamma(1-c_j\lambda/z-mc_j/d-a(\mathbf{k})^j)}\\
&= q^{-\lambda/(2\pi i)}\sum_{\mathbf{k}}\prod_{s\in S}\frac{(t^{g_s})^{k_s}z^{(\mathrm{age}(g_s)-1)k_s}}{k_s!}\\
&\quad\cdot\sum_{m\geq 0}\frac{q^{-m/d}}{m!\prod_j\Gamma(1-c_j\lambda/(2\pi i)-mc_j/d-a(\mathbf{k})^j)}\\
&\quad\cdot\sum_{0\leq b<d}\frac{e^{d(\lambda+H)/2}(e^{d(\lambda+H)/2}-e^{-d(\lambda+H)/2})}{d(e^{(\lambda+H)}\xi^{b+m}-1)}\tilde{\mathbb{1}}_{\mathrm{j}^{-b}\prod_s g_s^{k_s}}\\
&= t^{d\lambda/(2\pi i)}\sum_{\mathbf{k}}\prod_{s\in S}\frac{(t^{g_s})^{k_s}z^{(\mathrm{age}(g_s)-1)k_s}}{k_s!}\\
&\quad\cdot\sum_{m\geq 0}\frac{t^m}{m!\prod_j\Gamma(1-c_j\lambda/(2\pi i)-mc_j/d-a(\mathbf{k})^j)}\\
&\quad\cdot\sum_{0\leq b<d}\frac{e^{d(\lambda+H)}-1}{d(e^{(\lambda+H)}\xi^{b+m}-1)}\tilde{\mathbb{1}}_{\mathrm{j}^{-b}\prod_s g_s^{k_s}}
\end{aligned}$$

where we have made the substitution $q = t^{-d}$.

8.0.4. *The transformation.* – Recall the transformation

$$\overline{\mathbb{U}} : H^*_{CR}(\mathcal{X}) \to H^*_{CR}(\mathcal{Y})$$

given by

$$\overline{\mathbb{U}} : \mathbb{1}_g \mapsto \sum_{0 \le b < d} \frac{e^{d(\lambda+H)} - 1}{d(e^{(\lambda+H)}\xi^b - 1)} \tilde{\mathbb{1}}_{gj^{-b}}. \tag{8.0.2}$$

A simple check shows that

$$\overline{\mathbb{U}}(\mathbb{1}_{gj^m}) = \sum_{0 \le b < d} \frac{e^{d(\lambda+H)} - 1}{d(e^{(\lambda+H)}\xi^{b+m} - 1)} \tilde{\mathbb{1}}_{gj^{-b}},$$

from which one sees that $\overline{\mathbb{U}}(H^{\mathcal{X}}(t, \boldsymbol{t}, z)) = \widetilde{H^{\mathcal{Y}}}(t, \boldsymbol{t}, z)$.

Thus if we define the linear transformation $\mathbb{U} : \mathcal{V}^{\mathcal{X}} \to \mathcal{V}^{\mathcal{Y}}$ by

$$\mathbb{U} := z^{-\mathrm{Gr}}\hat{\Gamma}(\mathcal{Y})(2\pi i)^{\deg_0/2}\overline{\mathbb{U}}(2\pi i)^{-\deg_0/2}\hat{\Gamma}(\mathcal{X})^{-1}z^{\mathrm{Gr}},$$

it follows by construction that $\mathbb{U}(I^{\mathcal{X}}(t, \boldsymbol{t}, z)) = \widetilde{I^{\mathcal{Y}}}(t, \boldsymbol{t}, z)$, where $\widetilde{I^{\mathcal{Y}}}(t, \boldsymbol{t}, z)$ is the analytic continuation of $I^{\mathcal{Y}}(q, \boldsymbol{t}, z)$.

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