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O-MINIMALITY ON TWISTED UNIVERSAL TORSORS AND MANIN'S CONJECTURE OVER NUMBER FIELDS

BY CHRISTOPHER FREI AND MARTA PIEROPAN

ABSTRACT. – Manin's conjecture predicts the distribution of rational points on Fano varieties. Using explicit parameterizations of rational points by integral points on universal torsors and lattice-point-counting techniques, it was proved for several specific varieties over $\mathbb{Q}$, in particular del Pezzo surfaces. We show how this method can be implemented over arbitrary number fields, by proving Manin's conjecture for a singular quartic del Pezzo surface of type $\mathbf{A}_3 + \mathbf{A}_1$. The parameterization step is treated in high generality with the help of twisted integral models of universal torsors. To make the counting step feasible over arbitrary number fields, we deviate from the usual approach over $\mathbb{Q}$ by placing higher emphasis on the geometry of numbers in the framework of o-minimal structures.

RÉSUMÉ. – La conjecture de Manin prédit la répartition des points rationnels sur les variétés de Fano. Elle a été vérifiée pour plusieurs variétés sur $\mathbb{Q}$, en particulier certaines surfaces de del Pezzo, en utilisant des paramétrisations explicites des points rationnels par des points entiers sur des torseurs universels et des techniques de comptage de points de réseaux. On montre comment on peut appliquer cette méthode sur les corps de nombres quelconques, en démontrant la conjecture de Manin pour une surface de del Pezzo singulière de degré quatre et de type $\mathbf{A}_3 + \mathbf{A}_1$. La paramétrisation est présentée d'un point de vue général qui utilise des modèles entiers tordus de torseurs universels. Pour rendre possible le comptage sur les corps de nombres, on dévie de la procédure usuelle sur $\mathbb{Q}$ en mettant l'accent sur la géométrie des nombres dans le cadre des structures o-minimales.

1. Introduction

Let K be a number field and S the anticanonically embedded del Pezzo surface of degree 4 and type $\mathbf{A}_3 + \mathbf{A}_1$ given in $\mathbb{P}^4_K$ by the equations

$$x_0x_3 - x_2x_4 = x_0x_1 + x_1x_3 + x_2^2 = 0. \tag{1.1}$$

Let U be the complement of the lines in S, and let H be the anticanonical height on $S(K)$ induced by the Weil height on $\mathbb{P}^4(K)$,

$$H(x_0 : \cdots : x_4) := \prod_{v \in \Omega_K} \max\{|x_0|_v, \ldots, |x_4|_v\},$$

where Ω_K is the set of places of K and the normalized absolute values $|\cdot|_v$ are given as follows: let w be the place of $\mathbb{Q}$ below v and K_v (resp. $\mathbb{Q}_w$) the completion of K at v (resp. of $\mathbb{Q}$ at w). Then $|\cdot|_v := \left|N_{K_v|\mathbb{Q}_w}(\,\cdot\,)\right|_w$, where $|\cdot|_w$ is the usual real or p-adic absolute value on $\mathbb{Q}_w$. We investigate the counting function

$$N_{U,H}(B) := |\{\mathbf{x} \in U(K) \mid H(\mathbf{x}) \leq B\}|.$$

Generalized versions [6, 45] of Manin's conjecture [28] predict an asymptotic formula

$$N_{U,H}(B) = c_{S,H} B (\log B)^5 (1 + o(1)), \quad \text{as } B \to \infty,$$

with a positive constant $c_{S,H}$, which has been conjecturally interpreted in [43, 6, 45]. Our first main result is a proof of Manin's conjecture for S:

Theorem 1.1. – *Let K be a number field of degree d, let S be given in $\mathbb{P}^4_K$ by* (1.1)*, let U be the complement of the lines in S, and let $\epsilon > 0$. As $B \to \infty$,*

$$N_{U,H}(B) = c_{S,H} B (\log B)^5 + O(B(\log B)^{5-1/d+\epsilon}),$$

with an explicit $c_{S,H} > 0$. This formula agrees with Peyre's refined version of Manin's conjecture [45, Formule empirique 5.1]*. The implicit constant in the error term depends on K and ϵ.*

We describe the constant $c_{S,H}$ explicitly later in this section. The special cases of Theorem 1.1 where $K = \mathbb{Q}$ or K is imaginary quadratic were proved in [20, 23]. A version of Manin's conjecture over arbitrary global function fields was proved for our surface S in [10].

Manin's conjecture is known in some general cases. For complete intersections of large dimension compared to their degree, it follows from an application of the Hardy-Littlewood circle method (cf. [43, 40]). Moreover, it has been proved for certain classes of Fano varieties with additional structure coming from actions of algebraic groups, using Langlands' work on Eisenstein series [28] or harmonic analysis on adelic points (for example, for toric varieties [5] and equivariant compactifications of additive groups [15]).

Other known cases of Manin's conjecture concern specific varieties of low dimension. Del Pezzo surfaces over $\mathbb{Q}$ have received the most attention: some milestones here are the first special cases of Manin's conjecture for (singular or nonsingular) del Pezzo surfaces of degrees 5 [11], 4 [13], 3 [14], and 2 [2] that are not covered by [5] or [15]. The method behind these results and many further proofs of Manin's conjecture for specific varieties over $\mathbb{Q}$ is by now classical. It is usually referred to as the *universal torsor method*.

A major drawback of this method is that almost all of its successful applications are restricted to varieties over $\mathbb{Q}$. Recently, Derenthal and the first-named author started a project with the aim to generalize the universal torsor method to number fields beyond $\mathbb{Q}$. So far, they were able to adapt the basic framework to imaginary quadratic fields [22], and to apply it to some singular del Pezzo surfaces of degrees 4 [23] and 3 [24] over imaginary quadratic fields. To our best knowledge, the only published proofs of Manin's conjecture for varieties over arbitrary number fields that can be interpreted as applications of the universal torsor method concern projective spaces $\mathbb{P}^n_K$ [50] and a specific toric variety [29], which are also covered by [5].

Theorem 1.1 is a first step to overcome this restriction. It is based on the universal torsor approach and is the first proof of Manin's conjecture over arbitrary number fields for a variety that is not included in the general results mentioned above (see [26]). One should note that the surface S is an equivariant compactification of a semidirect product $\mathbb{G}_a \rtimes \mathbb{G}_m$, so recent techniques of Tanimoto and Tschinkel [52] using harmonic analysis could also apply. So far, this was worked out only over $\mathbb{Q}$.

1.1. The universal torsor method

Universal torsors were introduced and studied by Colliot-Thélène and Sansuc [16, 17] to investigate arithmetic properties such as the Hasse principle and weak approximation. Salberger [49] was the first to apply them to Manin's conjecture (see also [44]). After Salberger's pioneering work, the universal torsor method became a prevalent tool to prove special cases of Manin's conjecture over $\mathbb{Q}$.

A typical application of the universal torsor method to a specific del Pezzo surface S consists essentially of two parts:

(a) Parameterizing the rational points on an open subset U by integral points on a universal torsor over a minimal desingularization $\widetilde{S} \to S$, subject to certain coprimality conditions, and lifting the height function to these points.
(b) Counting these integral points of bounded height, essentially replacing sums by integrals and estimating the difference.

A framework covering these parts in some generality was developed over $\mathbb{Q}$ in [20] and generalized in [22] to imaginary quadratic fields.

1.2. Parameterization

The minimal desingularization $\widetilde{S}$ of S is a smooth projective variety over K. For such a variety $\widetilde{S}$ and a torsor Y over $\widetilde{S}$ under an algebraic K-group G, a classical result of Colliot-Thélène and Sansuc [17] shows that there is a partition

$$\widetilde{S}(K) = \bigsqcup_{[\sigma]\in H^1(K,G)} {}_\sigma\pi({}_\sigma Y(K)),$$

for twists ${}_\sigma\pi : {}_\sigma Y \to \widetilde{S}$ of Y. The finest partitions of this kind are achieved if Y is a universal torsor. For quantitative problems such as Manin's conjecture, it is desirable to obtain a parameterization of $\widetilde{S}(K)$ by points with integral coordinates, which allows us to apply lattice-point-counting techniques. Such a parameterization was obtained by Salberger [49] for proper, smooth, split toric varieties X over $\mathbb{Q}$ with globally generated anticanonical sheaf. In this case, the partition induced by a model $\pi : \mathcal{Y} \to \mathcal{X}$ of a universal torsor $Y \to X$ is trivial:

$$X(\mathbb{Q}) = \pi(\mathcal{Y}(\mathbb{Z})). \tag{1.2}$$

Here, the fibers of π are just the orbits under the action of $\mathbb{G}_m^r(\mathbb{Z}) \cong (\mathbb{Z}^\times)^r$, where r is the rank of the Picard group of X. Hence, we obtain a $(2^r : 1)$-parameterization of $X(\mathbb{Q})$ by integral points, which reduces Manin's conjecture to a lattice-point-counting problem. In almost all applications of the universal torsor method to special cases of Manin's conjecture over $\mathbb{Q}$, a parameterization of the form (1.2) is constructed by elementary methods that essentially consist of removing greatest common divisors between existing coordinates by introducing new ones. With some exceptions (e.g., [11, 7]), the connection to universal torsors is usually not made precise.

The first case where it was necessary to consider a partition by integral points on more than one torsor was encountered by de la Bretèche, Browning and Peyre in [12]. In [22], Derenthal and the first-named author observed that similar disjoint unions (over all r-tuples of ideal classes) always appear when considering split del Pezzo surfaces over number fields K of class number $h_K > 1$, even if a trivial partition with just one universal torsor exists over $\mathbb{Q}$. They interpreted this phenomenon as points on one universal torsor, but with coordinates in different ideal classes.

We provide a more conceptual explanation in terms of $\mathcal{O}_K$-points on twisted torsors over the ring of integers $\mathcal{O}_K$. This explanation also gives entirely explicit descriptions, which can be used to apply lattice-point-counting techniques. For split toric varieties, a similar description was found by Robbiani [48], based on ideas of Salberger. In the function field case, the parameterization was treated in high generality by Bourqui [8].

The basic idea is as follows. Let $\mathcal{Y}$ be an $\mathcal{O}_K$-model of a universal torsor Y over $\widetilde{S}$, such that $\mathcal{Y}$ is a torsor over a proper model $\widetilde{\mathcal{S}}$ of $\widetilde{S}$ under a split torus $\mathbb{G}^r_{m,\widetilde{\mathcal{S}}}$. We apply the general theory of torsors and a properness argument to obtain a partition

$$\widetilde{S}(K) = \bigsqcup_{[\sigma]\in H^1_{\text{ét}}(\operatorname{Spec}(\mathcal{O}_K),\mathbb{G}^r_{m,\mathcal{O}_K})} {}_{\sigma}\pi({}_{\sigma}\mathcal{Y}(\mathcal{O}_K)).$$

The identification $H^1_{\text{ét}}(\operatorname{Spec}(\mathcal{O}_K), \mathbb{G}^r_{m,\mathcal{O}_K}) \cong Cl^r_K$ explains the disjoint union over r-tuples of ideal classes appearing in the parameterization in [22]. Under some additional technical conditions, we give an explicit construction of the twists ${}_{\sigma}\pi : {}_{\sigma}\mathcal{Y} \to \widetilde{\mathcal{S}}$ in terms of fractional ideals of $\mathcal{O}_K$ representing the classes corresponding to σ. This is worked out in a general context in Section 2 and summarized in Theorem 2.7.

Explicit descriptions of universal torsors over minimal desingularizations $\widetilde{S}$ of singular del Pezzo surfaces S over $\overline{\mathbb{Q}}$ can be obtained from the descriptions of their Cox rings in [21]. In Section 3, we show how to construct from this data a model $\widetilde{\mathcal{S}}$ of $\widetilde{S}$ and a model $\mathcal{Y}$ of the universal torsor over $\overline{\mathbb{Q}}$. In Theorem 3.3, we give conditions under which $\mathcal{Y}$ is a (universal) torsor over $\widetilde{\mathcal{S}}$ as above.

In Section 4, we present in detail an application of the results from Sections 2 and 3 to the quartic del Pezzo surface given by (1.1) and obtain a parameterization of $U(K)$, where U is the open subset in Theorem 1.1. As summarized in Remark 4.4, analogous arguments apply to all other singular del Pezzo surfaces whose universal torsors are hypersurfaces, classified in [21], allowing us to obtain in each case a good model of the universal torsor and a parameterization.

1.3. Counting integral points

Using the partition described above, we reduce the problem of counting rational points on the open subset U to counting $\mathcal{O}_K$-points in the preimages of $U(K)$ under ${}_{\sigma}\pi$, modulo the action of $\mathbb{G}^r_{m,\widetilde{\mathcal{S}}}(\mathcal{O}_K)$. This action is equivalent to an action of $(\mathcal{O}_K^\times)^r$, which is harmless when the unit group $\mathcal{O}_K^\times$ is finite (i.e., if $K = \mathbb{Q}$ or K is imaginary quadratic). If $\mathcal{O}_K^\times$ is infinite, however, one needs to count integral points in a fundamental domain for this action. The difficulties arising in the treatment of such counting problems are the main reason why the universal torsor method was so far restricted to $\mathbb{Q}$ and imaginary quadratic fields.

To deal with these problems in the case of our specific S given by (1.1), we introduce a major deviation from the usual strategy in part (b). Instead of summing over the coordinates on the twisted torsors one-by-one, we start by considering three coordinates at the same time. The motivation for this departure comes from the specific structure of the action of $\mathbb{G}^r_{m,\widetilde{S}}(\mathcal{O}_K)$. This structure is reflected in the shape of our fundamental domain, which we construct in Section 5.

Let $d := [K\colon \mathbb{Q}]$. The usual embedding $K \to K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^d$ transforms this first summation to the problem of counting lattice points in certain subsets of $\mathbb{R}^{3d}$, depending on the remaining coordinates, with an error term that can be estimated uniformly with respect to the remaining coordinates, see Section 6. These subsets are a priori unbounded, and we need to remove cusps coming from small conjugates of certain coordinates in Section 7.

Even after the removal of the cusps and the exploitation of certain symmetries in Section 8, our sets are of the "long and thin" kind, which makes them resistant to counting arguments that depend on Lipschitz-parameterizations of the boundary, such as [38, Theorem VI.2.2] or [41, Lemma 2]. In principle, Davenport's classical counting theorem [18] would apply, but its error term depends on certain regularity properties which are hard to control uniformly in general. In typical applications of Davenport's theorem, the sets under consideration are semialgebraic, a condition that is not satisfied in our case due to the restriction to a fundamental domain.

A natural generalization of semialgebraic sets is given by the model-theoretic framework of o-minimal structures. The celebrated upper bound by Pila and Wilkie [46] for the number of rational points of bounded height in the transcendental part of sets definable in an o-minimal structure has led to many applications in Diophantine geometry. We apply o-minimality in a new way.

In Section 9, we show that the sets whose lattice points are to be counted form a definable family in Wilkie's [53] o-minimal structure $\langle \mathbb{R}; <, +, \cdot, -, \exp\rangle$. This allows us to apply a recent adaptation of Davenport's counting principle to definable sets by Barroero and Widmer [3].

The error term in Barroero and Widmer's theorem is given, as in Davenport's result, in terms of the volumes of the orthogonal projections of our set to all proper coordinate subspaces of $\mathbb{R}^{3d}$. In Section 10, we establish summable upper bounds for these volumes, which allow us to perform the first summation over three coordinates in Section 11. The proof of Theorem 1.1 is completed in Section 12.

1.4. Description of the leading constant

Let us briefly describe the leading constant $c_{S,H}$ in Theorem 1.1. Let r_1 (resp. r_2) be the number of real (resp. complex) places of K, and let $\Delta_K, R_K, h_K, \mu_K$ denote the discriminant, regulator, class number, and group of roots of unity of K. For any nonarchimedean place v of K, corresponding to a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ of absolute norm $\mathfrak{N}\mathfrak{p}$, we define

$$\omega_v(\widetilde{S}) := \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^6 \left(1 + \frac{6}{\mathfrak{N}\mathfrak{p}} + \frac{1}{\mathfrak{N}\mathfrak{p}^2}\right).$$

For any archimedean place v of K and $(x_0, x_1, x_2) \in K_v^3$, we write

$$N_v(x_0, x_1, x_2) := \max \left\{ \begin{array}{l} |x_0x_1x_2|_v \, , |x_1^3|_v \, , |x_1^2x_2|_v \, , \\ |x_1x_2(x_0+x_2)|_v \, , |x_0x_2(x_0+x_2)|_v \end{array} \right\}, \tag{1.3}$$

and

$$\omega_v(\widetilde{S}) := \begin{cases} \frac{3}{2} \cdot \int_{N_v(x_0,x_1,x_2) \leq 1} \mathrm{d}x_0 \, \mathrm{d}x_1 \, \mathrm{d}x_2 & \text{if } v \text{ is real,} \\ \frac{12}{\pi} \cdot \int_{N_v(x_0,x_1,x_2) \leq 1} \mathrm{d}x_0 \, \mathrm{d}x_1 \, \mathrm{d}x_2 & \text{if } v \text{ is complex,} \end{cases}$$

where the integrals are taken with respect to the usual Lebesgue measure on $K_v \in \{\mathbb{R}, \mathbb{C} \cong \mathbb{R}^2\}$. Then the leading constant in Theorem 1.1 has the form

$$c_{S,H} = \frac{1}{8640} \cdot \left(\frac{2^{r_1}(2\pi)^{r_2} R_K h_K}{|\mu_K|} \right)^6 \cdot \frac{1}{|\Delta_K|^4} \cdot \prod_{v \in \Omega_K} \omega_v(\widetilde{S}).$$

In Section 13, we show that this constant is the one from Peyre's empirical formula [45, formule empirique 5.1].

1.5. More notation

Unless explicitly stated otherwise, the symbol K denotes a number field. Let U_K be the subgroup of $\mathcal{O}_K^\times$ generated by a fixed system of fundamental units. Then U_K is free abelian of rank $q := r_1 + r_2 - 1$, and $\mathcal{O}_K^\times \cong \mu_K \times U_K$. Let $\mathcal{I}_K$ be the monoid of nonzero ideals of $\mathcal{O}_K$, let $\mathcal{P}_K$ be the group of nonzero principal fractional ideals, and Cl_K the class group of $\mathcal{O}_K$. The ideal class of a nonzero fractional ideal $\mathfrak{a}$ is denoted by $[\mathfrak{a}]$.

For any $v \in \Omega_K$, we denote by σ_v the embedding $K \to K_v$, as well as its component-wise extensions $\sigma_v : K^n \to K_v^n$ for $n \in \mathbb{N}$. Moreover, $\sigma : K \to \prod_{v \in \Omega_\infty} K_v$ denotes the embedding $a \mapsto (\sigma_v(a))_{v \in \Omega_\infty}$ or its coordinate-wise continuation $K^n \to \prod_{v \in \Omega_\infty} K_v^n$, $n \in \mathbb{N}$. When it is convenient, we will also write $a^{(v)}$ instead of $\sigma_v(a)$ and $|a|_v$ instead of $|\sigma_v(a)|_v$, for $a \in K$.

For each place $v \in \Omega_K$, lying over a place w of $\mathbb{Q}$, we write $d_v := [K_v : \mathbb{Q}_w]$ for the local degree at v. The set of archimedean (resp. non-archimedean) places of K will be denoted by Ω_∞ (resp. Ω_0). The completion K_v at $v \in \Omega_\infty$ is identified with $\mathbb{R}$ (resp. $\mathbb{C}$) if v is real (resp. complex).

For a prime ideal $\mathfrak{p}$ of a ring A, we write $k(\mathfrak{p})$ for the residue field. Given an inclusion of rings $A \subseteq A'$ and an A-scheme X, we denote by $X_{A'}$ the base change $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A')$. Moreover, $\mathbb{G}_{m,X}$ (resp. $\mathbb{G}_{m,A}$) denotes the multiplicative group scheme over X (resp. over $\mathrm{Spec}(A)$). We denote by $\mathrm{Pic}(X)$ the Picard group of a scheme X. Given an ideal I of a ring R, we denote by $V(I)$ both the closed subset of $\mathrm{Spec}(R)$ defined by I and the closed subscheme $\mathrm{Spec}(R/I)$.

All implied constants in Landau's O-notation and Vinogradov's $\ll$-notation may depend on K. Additional dependencies are indicated by a subscript.

2. Parameterization by integral points on twisted torsors

Torsors over varieties under algebraic groups are known to give partitions of the set of rational points of the variety in terms of images of rational points on twisted torsors (see [17, (2.7.2)] and [51, p. 22]). This phenomenon holds also for torsors over more general schemes as the following proposition shows. For universal torsors of smooth projective split toric schemes over the ring of integers of a number field it was observed in [48, p. 12]. For the definition and basic properties of torsors we refer to [42, §III.4] and [51, § 2.2]. For the notion of twisted torsors see, for example, [51, p. 20].

PROPOSITION 2.1. – *Let Z be a scheme, G an abelian group scheme over Z, X a Z-scheme, and $\pi : Y \to X$ a torsor under $G_X := G \times_Z X$. Assume that the twisted torsors ${}_WY$ exist for all Z-torsors W under G (this is the case, for example, if G is affine over Z). Then*

$$X(Z) = \bigsqcup_{[W] \in H^1_{fppf}(Z,G)} {}_W\pi(({}_WY)(Z)),$$

where $\bigsqcup_{[W] \in H^1_{fppf}(Z,G)}$ is a disjoint union running through a system of representatives for the classes in $H^1_{fppf}(Z,G)$ and ${}_W\pi : {}_WY \to X$ is a twist of Y by $-[W] \in H^1_{fppf}(Z,G)$.

Proof. – The proof given in [51, p. 22] works also with $\operatorname{Spec} k$ replaced by our base scheme Z. □

Our twisted torsors will be given as open subschemes of closed subschemes of twisted affine spaces. Hence, we start with a definition of those.

DEFINITION 2.2. – Let A be a Dedekind domain with fraction field K, and assume that we are given a $\mathbb{Z}^r$-grading on $K[x_1, \dots, x_N]$ defined by $\deg x_i = m^{(i)} \in \mathbb{Z}^r$. For any r-tuple $\underline{\mathfrak{a}} = (\mathfrak{a}_1, \dots, \mathfrak{a}_r)$ of nonzero fractional ideals of A, we define the *$\underline{\mathfrak{a}}$-twisted affine space over A* as the spectrum ${}_{\underline{\mathfrak{a}}}\mathbb{A}^N := \operatorname{Spec}({}_{\underline{\mathfrak{a}}}R)$ of the $\mathbb{Z}^r$-graded ring

$${}_{\underline{\mathfrak{a}}}R := \bigoplus_{m \in \mathbb{Z}^r} \underline{\mathfrak{a}}^{-m} R_m \subseteq K[x_1, \dots, x_N],$$

where $\underline{\mathfrak{a}}^{-m} := \mathfrak{a}_1^{-m_1} \cdots \mathfrak{a}_r^{-m_r}$ if $m = (m_1, \dots, m_r)$, and R_m is the degree-m-part of $A[x_1, \dots, x_N]$.

The twisted affine spaces defined above depend, of course, not only on N and $\underline{\mathfrak{a}}$, but also on the chosen $\mathbb{Z}^r$-grading. Here are some simple properties.

PROPOSITION 2.3. – *The $\underline{\mathfrak{a}}$-twisted affine space over A defined above has the following properties.*

(i) *There is a canonical isomorphism ${}_{\underline{\mathfrak{a}}}\mathbb{A}^N \times_{\operatorname{Spec}(A)} \operatorname{Spec}(K) \cong \mathbb{A}^N_K$.*

(ii) *Let $U = \operatorname{Spec}(A_U)$ be an affine open subset of $\operatorname{Spec}(A)$ such that the fractional ideals $\mathfrak{a}_1 A_U, \dots, \mathfrak{a}_r A_U$ of A_U are principal. Then*

$${}_{\underline{\mathfrak{a}}}\mathbb{A}^N \times_{\operatorname{Spec}(A)} U \cong \mathbb{A}^N_A \times_{\operatorname{Spec}(A)} U.$$

(iii) *Via base extension and the canonical isomorphism from* (i)*, we have*

$${}_{\underline{\mathfrak{a}}}\mathbb{A}^N(A) = \{(a_1, \dots, a_N) \in K^N \mid a_i \in \underline{\mathfrak{a}}^{m^{(i)}} \textit{ for all } 1 \leq i \leq N\}.$$

(iv) ${}_{\underline{\mathfrak{a}}}\mathbb{A}^N$ *depends, up to isomorphism, only on the ideal classes of* $\mathfrak{a}_1, \dots, \mathfrak{a}_r$.

Proof. – The canonical homomorphism ${}_{\underline{\mathfrak{a}}}R \otimes_A K \to K[x_1, \dots, x_N]$ provided by the universal property of the tensor product is an isomorphism, which implies (i). For $j \in \{1, \dots, r\}$, let ρ_j be a generator of $\mathfrak{a}_j A_U$ and, with $m \in \mathbb{Z}^r$, write $\underline{\rho}^m := \rho_1^{m_1} \cdots \rho_r^{m_r}$. Then ${}_{\underline{\mathfrak{a}}}R \otimes_A A_U \cong A_U[\underline{\rho}^{-m^{(1)}} x_1, \dots, \underline{\rho}^{-m^{(N)}} x_N] \cong A_U[x_1, \dots, x_N]$, which implies (ii). For (iii), we observe that every A-homomorphism $\varphi : {}_{\underline{\mathfrak{a}}}R \to A$ extends uniquely to a K-homomorphism $\varphi : K[x_1, \dots, x_N] \to K$. The K-homomorphisms coming from such A-homomorphisms are exactly those with $\varphi({}_{\underline{\mathfrak{a}}}R) \subseteq A$, that is, $\varphi(x_i) \in \underline{\mathfrak{a}}^{m^{(i)}}$ for all $i \in \{1, \dots, N\}$. To prove (iv), let $\underline{b} = (b_1, \dots, b_r) \in (K^\times)^r$ and $\mathfrak{a}'_j := b_j \mathfrak{a}_j$ for $j \in \{1, \dots, r\}$. Then the K-automorphism of $K[x_1, \dots, x_N]$ mapping $x_i \mapsto \underline{b}^{-m^{(i)}} x_i$ induces an A-isomorphism between ${}_{\underline{\mathfrak{a}}}R$ and ${}_{\underline{\mathfrak{a}}'}R$. □

Next, we define twists of open subschemes of closed subschemes of $\mathbb{A}_A^N$ as certain subschemes of twisted affine spaces.

Definition 2.4. – With the hypotheses of Definition 2.2, let I_1, I_2 be $\mathbb{Z}^r$-homogeneous ideals of $A[x_1, \dots, x_N]$, and let Y be the subscheme of $\mathbb{A}_A^N$ defined by $Y := V(I_1) \smallsetminus V(I_2)$. With $I_{j,m}$ denoting the degree-m-part of I_j, we define the ideals

$$ {}_{\underline{\mathfrak{a}}}I_j := \bigoplus_{m \in \mathbb{Z}^r} \underline{\mathfrak{a}}^{-m} I_{j,m} \subseteq {}_{\underline{\mathfrak{a}}}R. $$

The *twist of* Y *by* $\underline{\mathfrak{a}}$ is the subscheme of ${}_{\underline{\mathfrak{a}}}\mathbb{A}^N$ defined by

$$ {}_{\underline{\mathfrak{a}}}Y := V({}_{\underline{\mathfrak{a}}}I_1) \smallsetminus V({}_{\underline{\mathfrak{a}}}I_2). $$

Proposition 2.5. – *The twist of* Y *by* $\underline{\mathfrak{a}}$ *defined above has the following properties.*

(i) *The canonical isomorphism from Proposition 2.3,* (i)*, induces an isomorphism* ${}_{\underline{\mathfrak{a}}}Y \times_{\mathrm{Spec}(A)} \mathrm{Spec}(K) \cong Y_K$.

(ii) *Let* $U = \mathrm{Spec}(A_U)$ *be an affine open subset of* $\mathrm{Spec}(A)$ *such that the fractional ideals* $\mathfrak{a}_1 A_U, \dots, \mathfrak{a}_r A_U$ *of* A_U *are principal. Then*

$$ {}_{\underline{\mathfrak{a}}}Y \times_{\mathrm{Spec}(A)} U \cong Y \times_{\mathrm{Spec}(A)} U. $$

(iii) *Via base extension and the canonical isomorphism from* (i)*, the set of* A*-points* ${}_{\underline{\mathfrak{a}}}Y(A)$ *is the subset of all* $\underline{a} = (a_1, \dots, a_N) \in K^N$ *with* $a_i \in \underline{\mathfrak{a}}^{m^{(i)}}$ *for all* $i \in \{1, \dots, N\}$*, such that*

$$ \sum_{m \in \mathbb{Z}^r} \sum_{f \in I_{2,m}} f(\underline{a}) \underline{\mathfrak{a}}^{-m} = A \tag{2.1} $$

and

$$ g(\underline{a}) = 0 \text{ for all } g \in I_1. \tag{2.2} $$

(iv) ${}_{\underline{\mathfrak{a}}}Y$ *depends, up to isomorphism, only on the ideal classes of* $\mathfrak{a}_1, \dots, \mathfrak{a}_r$.

Proof. – Since the inclusion $A \to K$ is flat and ${}_{\underline{\mathfrak{a}}}I_1 \otimes_A K \cong I_1 \otimes_A K$ under the canonical isomorphism ${}_{\underline{\mathfrak{a}}}R \otimes_A K \cong K[x_1, \dots, x_N]$, we see that $V({}_{\underline{\mathfrak{a}}}I_1) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(K) \cong V(I_1) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(K)$. Let I_2 be generated by homogeneous polynomials $f_1, \dots, f_m \in A[x_1, \dots, x_N]$, and for every $i \in \{1, \dots, m\}$, let $b_{i,1}, \dots, b_{i,t_i}$ be generators of the fractional ideal $\underline{\mathfrak{a}}^{-\deg f_i}$. Then ${}_{\underline{\mathfrak{a}}}I_2$ is generated in ${}_{\underline{\mathfrak{a}}}R$ by the elements $b_{i,j} f_i$, and ${}_{\underline{\mathfrak{a}}}Y$ is covered by affine open subsets $\mathrm{Spec}(({}_{\underline{\mathfrak{a}}}R/{}_{\underline{\mathfrak{a}}}I_1)_{b_{i,j} f_i})$. Moreover,

$$({}_{\underline{\mathfrak{a}}}R/{}_{\underline{\mathfrak{a}}}I_1)_{b_{i,j} f_i} \otimes_A K \cong (A[x_1, \dots, x_N]/I_1)_{f_i} \otimes_A K$$

for every $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, t_i\}$, which shows (i).

For $j \in \{1, \dots, r\}$, let ρ_j be a generator of $\mathfrak{a}_j A_U$ and, with $m \in \mathbb{Z}^r$, write $\underline{\rho}^m := \rho_1^{m_1} \cdots \rho_r^{m_r}$. Let $\varphi_{\underline{\rho}} : A_U[x_1, \dots, x_N] \to A_U[\underline{\rho}^{-m^{(1)}} x_1, \dots, \underline{\rho}^{-m^{(N)}} x_N]$ be the isomorphism that sends $x_i \mapsto \underline{\rho}^{-m^{(i)}} x_i$. For every homogeneous $f \in I_2$ we obtain

$$\begin{aligned}(A[x_1, \dots, x_N]/I_1)_f \otimes_A A_U &\cong (A_U[x_1, \dots, x_N]/(I_1 \otimes_A A_U))_f \\ &\cong \Big(\varphi_{\underline{\rho}}(A_U[x_1, \dots, x_N])/\varphi_{\underline{\rho}}(I_1 \otimes_A A_U)\Big)_{\varphi_{\underline{\rho}}(f)} \\ &\cong \big(({}_{\underline{\mathfrak{a}}}R \otimes_A A_U)/({}_{\underline{\mathfrak{a}}}I_1 \otimes_A A_U)\big)_{\underline{\rho}^{-\deg f} f}.\end{aligned}$$

This proves (ii), since $f \in I_2 \otimes_A A_U$ is equivalent to $\underline{\rho}^{-\deg f} f \in {}_{\underline{\mathfrak{a}}}I_2 \otimes_A A_U$. For (iii), we first consider $V({}_{\underline{\mathfrak{a}}}I_1)(A)$. Via the identification in Proposition 2.3, (iii), these points correspond to K-homomorphisms $\varphi : K[x_1, \dots, x_N] \to K$ with $\varphi(x_i) \in \underline{\mathfrak{a}}^{m^{(i)}}$ whose kernel contains the homogeneous ideal ${}_{\underline{\mathfrak{a}}}I_1 \otimes_A K = I_1 \otimes_A K$, that is, to points $\underline{a} \in K^N$ with $a_i \in \underline{\mathfrak{a}}^{m^{(i)}}$ and satisfying (2.2).

Next, let us consider $({}_{\underline{\mathfrak{a}}}\mathbb{A}^N \smallsetminus V({}_{\underline{\mathfrak{a}}}I_2))(A)$. These points correspond to A-homomorphisms $\varphi : {}_{\underline{\mathfrak{a}}}R \to A$ such that ${}_{\underline{\mathfrak{a}}}I_2 \not\subseteq \varphi^{-1}(\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ of A. That is, $\varphi({}_{\underline{\mathfrak{a}}}I_2)A = A$. Keeping in mind that ${}_{\underline{\mathfrak{a}}}I_2$ is generated by its homogeneous elements and using the description of ${}_{\underline{\mathfrak{a}}}\mathbb{A}^N(A)$ from Proposition 2.3, (iii), we see that $({}_{\underline{\mathfrak{a}}}\mathbb{A}^N \smallsetminus V({}_{\underline{\mathfrak{a}}}I_2))(A)$ corresponds to the set of all $\underline{a} \in K^N$ with $a_i \in \underline{\mathfrak{a}}^{m^{(i)}}$ and satisfying (2.1).

To prove (iv), let $\underline{b} = (b_1, \dots, b_r) \in (K^\times)^r$ and $\mathfrak{a}'_j := b_j \mathfrak{a}_j$ for $j \in \{1, \dots, r\}$. Then the K-automorphism of $K[x_1, \dots, x_N]$ mapping $x_i \mapsto \underline{b}^{-m^{(i)}} x_i$ induces an A-isomorphism between ${}_{\underline{\mathfrak{a}}}R$ and ${}_{\underline{\mathfrak{a}}'}R$ which maps ${}_{\underline{\mathfrak{a}}}I_j$ isomorphically onto ${}_{\underline{\mathfrak{a}}'}I_j$, for $j \in \{1, 2\}$. □

Now we can focus on the case where Y is a torsor over an A-scheme X under a split torus $\mathbb{G}_{m,X}^r$. Throughout the rest of this section, we assume the following setup.

Let A be a Dedekind domain with fraction field K, and let there be a $\mathbb{Z}^r$-grading on $K[x_1, \dots, x_N]$ defined by $\deg x_i = m^{(i)} \in \mathbb{Z}^r$.

Let X be a separated scheme of finite type over A that admits an X-torsor $\pi : Y \to X$ under a split torus $\mathbb{G}_{m,X}^r$. We assume that there are $\mathbb{Z}^r$-homogeneous polynomials $f_1, \dots, f_m$, $g_1, \dots, g_s \in A[x_1, \dots, x_N]$ such that $Y = V(g_1, \dots, g_s) \smallsetminus V(f_1, \dots, f_m)$ as subscheme of $\mathbb{A}_A^N$. Moreover, we assume that the action of $\mathbb{G}_{m,X}^r$ on Y is induced by the following action on points:

$$(s_1, \dots, s_r) * (a_1, \dots, a_N) = (\underline{s}^{m^{(1)}} a_1, \dots, \underline{s}^{m^{(N)}} a_N)$$

for all $\underline{s} = (s_1, \dots, s_r) \in \mathbb{G}_{m,X}^r(A)$ and $(a_1, \dots, a_N) \in Y(A)$, where we write $\underline{s}^m := s_1^{m_1} \cdots s_r^{m_r}$ for $m = (m_1, \dots, m_r) \in \mathbb{Z}^r$. Under these assumptions, we now define the twists of $\pi : Y \to X$.

DEFINITION 2.6. – Under the above hypotheses, let $\underline{\mathfrak{a}} = (\mathfrak{a}_1, \dots, \mathfrak{a}_r)$ be an r-tuple of nonzero fractional ideals of A, and let ${}_{\underline{\mathfrak{a}}}Y$ be the twist of Y from Definition 2.4. Then the $\underline{\mathfrak{a}}$-*twist of* $\pi : Y \to X$ is the morphism ${}_{\underline{\mathfrak{a}}}\pi : {}_{\underline{\mathfrak{a}}}Y \to X$ obtained by glueing the following morphisms:

$${}_{\underline{\mathfrak{a}}}\pi_U : {}_{\underline{\mathfrak{a}}}Y \times_{\mathrm{Spec}(A)} U \to X \times_{\mathrm{Spec}(A)} U,$$

where U runs through an open covering of $\mathrm{Spec}(A)$ by affine subschemes $U = \mathrm{Spec}(A_U)$ such that $\mathfrak{a}_1 A_U, \dots, \mathfrak{a}_r A_U$ are principal ideals of A_U, and ${}_{\underline{\mathfrak{a}}}\pi_U$ is defined as composition of π after the isomorphism $\phi_{\underline{\rho}} : {}_{\underline{\mathfrak{a}}}Y \times_{\mathrm{Spec}(A)} U \to Y \times_{\mathrm{Spec}(A)} U$ from Proposition 2.5, (ii), induced by the isomorphism

$$\varphi_{\underline{\rho}} : A_U[x_1, \dots, x_N] \to A_U[\underline{\rho}^{-m^{(1)}} x_1, \dots, \underline{\rho}^{-m^{(N)}} x_N], \quad x_i \mapsto \underline{\rho}^{-m^{(i)}} x_i,$$

where ρ_j is a generator of $\mathfrak{a}_j A_U$ for $j \in \{1, \dots, r\}$, and $\underline{\rho}^m := \rho_1^{m_1} \cdots \rho_r^{m_r}$ for all $m \in \mathbb{Z}^r$. As will be shown in the proof below, the definition of ${}_{\underline{\mathfrak{a}}}\pi$ does not depend on the choice of the open subsets U nor on the choice of the generators ρ.

Now we are ready to state the second main theorem of this article.

THEOREM 2.7. – *The $\underline{\mathfrak{a}}$ twists ${}_{\underline{\mathfrak{a}}}\pi : {}_{\underline{\mathfrak{a}}}Y \to X$ defined above have the following properties.*

(i) *The morphism ${}_{\underline{\mathfrak{a}}}\pi : {}_{\underline{\mathfrak{a}}}Y \to X$ is a torsor over X under $\mathbb{G}_{m,X}^r$ of class $[Y] - [\underline{\mathfrak{a}}]$ belonging to $H^1_{\text{ét}}(X, \mathbb{G}_{m,X}^r)$.*

(ii) *Let $\mathcal{C}$ be a system of representatives for the class group* $\mathrm{Pic}(A)$ *of A. If X is proper over A then, under base extension, the set of rational points on X_K decomposes as a disjoint union*

$$X_K(K) = \bigsqcup_{\underline{\mathfrak{c}} \in \mathcal{C}^r} {}_{\underline{\mathfrak{c}}}\pi({}_{\underline{\mathfrak{c}}}Y(A)).$$

(iii) *As a subset of K^N, the set ${}_{\underline{\mathfrak{c}}}Y(A)$ is equal to the set of all $\underline{a} \in K^N$ whose coordinates a_i lie in the fractional ideals $\underline{\mathfrak{c}}^{m^{(i)}}$, satisfying the coprimality conditions expressed by*

$$\sum_{i=1}^{m} f_i(\underline{a}) \underline{\mathfrak{c}}^{-\deg f_i} = A$$

and the torsor equations

$$g_j(\underline{a}) = 0 \textit{ for all } j \in \{1, \dots, s\}.$$

Proof. – For every choice of affine open subsets U, U' of $\mathrm{Spec}(A)$ as in Definition 2.6, and corresponding r-tuples $\underline{\rho}, \underline{\rho}'$ of generators for the principal fractional ideals over U, resp. U', let $\varphi_{\underline{\rho},\underline{\rho}'} : A_{U \cap U'}[x_1, \dots, x_N] \to A_{U \cap U'}[x_1, \dots, x_N]$ be the isomorphism induced by the automorphism of $K[x_1, \dots, x_N]$ mapping $x_i \mapsto \underline{\rho}^{-m^{(i)}} \underline{\rho}'^{m^{(i)}} x_i$, and $\phi_{\underline{\rho},\underline{\rho}'}$ the automorphism of $Y \times_{\mathrm{Spec}(A)} (U \cap U')$ induced by $\varphi_{\underline{\rho},\underline{\rho}'}$. Then $\phi_{\underline{\rho}} = \phi_{\underline{\rho},\underline{\rho}'} \circ \phi_{\underline{\rho}'}$ on ${}_{\underline{\mathfrak{a}}}Y \times_{\mathrm{Spec}(A)} (U \cap U')$. We observe that $\phi_{\underline{\rho},\underline{\rho}'}$ are the automorphisms induced by the $\mathbb{G}_{m,X}^r$-action of the cocycle $(\rho_1^{-1}\rho_1', \dots, \rho_r^{-1}\rho_r')_{U',U}$ that represents the class $-[\underline{\mathfrak{a}}] \in \mathrm{Pic}(A)^r$. Thus $\pi \circ \phi_{\underline{\rho},\underline{\rho}'} = \pi$ on $Y \times_{\mathrm{Spec}(A)} (U \cap U')$, as $\pi : Y \to X$ is a torsor under $\mathbb{G}_{m,X}^r$, and the morphism ${}_{\underline{\mathfrak{a}}}\pi$ is well defined. Since the automorphisms $\phi_{\underline{\rho},\underline{\rho}'}$ are $\mathbb{G}_{m,X}^r$-equivariant, the X-scheme ${}_{\underline{\mathfrak{a}}}Y$ is endowed with an action of $\mathbb{G}_{m,X}^r$, and the morphism ${}_{\underline{\mathfrak{a}}}\pi$ is an X-torsor under $\mathbb{G}_{m,X}^r$ of class $[Y] - [\underline{\mathfrak{a}}] \in H^1_{\text{ét}}(X, \mathbb{G}_{m,X}^r)$, via the homomorphism of cohomology groups

$$\mathrm{Pic}(A)^r \cong H^1_{\text{ét}}(\mathrm{Spec}(A), \mathbb{G}_{m,A}^r) \to H^1_{\text{ét}}(X, \mathbb{G}_{m,X}^r),$$

where the first isomorphism comes from the fact that étale cohomology commutes with direct sums (see [42, Remark III.3.6 (d)]). Here we used *étale* cohomology groups in place of *fppf* because for $\mathbb{G}_m^r$ they coincide.

We recall that two torsors with the same class in $H^1_{\text{ét}}(X, \mathbb{G}^r_{m,X})$ are X-isomorphic, so the images of their structure morphisms coincide as subsets of X. By the valuative criterion of properness, $X_K(K) = X(A)$ under base extension. Thus, property (ii) follows from Proposition 2.1.

Finally, (iii) was already proved in Proposition 2.5, (iii). □

In the case of universal torsors of smooth projective split toric schemes over the ring of integers of a number field, our Theorem 2.7 recovers the same parameterization via integral points on twisted universal torsors embedded into twisted affine spaces of [48, p. 15].

3. Models of universal torsors

This section is devoted to descent properties of universal torsors of certain projective varieties. Let A be a noetherian integral domain with fraction field K of characteristic 0, and let $\overline{K}$ be an algebraic closure of K.

Given an integral, smooth, projective variety $\overline{X}$ over $\overline{K}$, whose Cox ring $\mathrm{Cox}(\overline{X})$ is finitely generated and defined over A, we construct an A-model X of $\overline{X}$ and an A-model Y of a universal torsor $\overline{Y}$ of $\overline{X}$ contained in the spectrum of $\mathrm{Cox}(\overline{X})$ that turns out to be a universal torsor over X under some additional conditions.

CONSTRUCTION 3.1. – We assume that $\mathrm{Pic}(\overline{X}) \cong \mathbb{Z}^r$, and that the Cox ring of $\overline{X}$ is $\mathrm{Cox}(\overline{X}) = \overline{K}[\eta_1, \dots, \eta_N]/I$, where $\eta_1, \dots, \eta_N$ are $\mathrm{Pic}(\overline{X})$-homogeneous and I is generated by polynomials $g_1, \dots, g_s \in A[\eta_1, \dots, \eta_N]$. We denote by $\overline{Y} \subseteq \mathrm{Spec}(\mathrm{Cox}(\overline{X}))$ the characteristic space defined in [1, Constructions 1.6.1.3, 1.6.3.1]. Then $\overline{Y}$ is a universal torsor of $\overline{X}$ by [1, Proposition 6.1.3.9]. By [1, Corollary 1.6.3.6], we know that $\overline{Y}$ is an open subset of $\mathrm{Spec}(\mathrm{Cox}(\overline{X}))$, whose complement is defined by monic monomial equations

$$f_1, \dots, f_m \in \overline{K}[\eta_1, \dots, \eta_N] \smallsetminus I.$$

Fix an isomorphism between $\mathrm{Pic}(\overline{X}) \cong \mathbb{Z}^r$ given by a basis $\ell_1, \dots, \ell_r$ of $\mathrm{Pic}(\overline{X})$. For $i \in \{1, \dots, N\}$, let $m^{(i)} \in \mathbb{Z}^r$ be the degree of η_i. By [1, Construction 1.6.1.3], the action of $\mathbb{G}^r_{m,\overline{X}}$ on $\overline{Y}$ is induced by the action of $\mathbb{G}^r_{m,\overline{X}}$ on $\mathrm{Spec}(\mathrm{Cox}(\overline{X}))(\overline{K})$ defined by the homomorphism

$$\mathrm{Cox}(\overline{X}) \to \overline{K}[z_1, z_1^{-1}, \dots, z_r, z_r^{-1}] \otimes_{\overline{K}} \mathrm{Cox}(\overline{X}), \quad \eta_j \mapsto \underline{z}^{m^{(j)}} \otimes \eta_j.$$

Without loss of generality, we can assume that $I \cap A[\eta_1, \dots, \eta_N] = (g_1, \dots, g_s)$. Let

$$R := A[\eta_1, \dots, \eta_N]/(g_1, \dots, g_s),$$

and let Y be the complement of the closed subset of $\mathrm{Spec}(R)$ defined by $f_1, \dots, f_m$. For $i \in \{1, \dots, m\}$, let $U_i := \mathrm{Spec}(R[f_i^{-1}])$ and

$$\overline{U}_i := U_i \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\overline{K}) \cong \mathrm{Spec}(\mathrm{Cox}(\overline{X})[f_i^{-1}]).$$

Then $\{U_i\}_{1 \le i \le m}$ is an affine open covering of Y, the family $\{\overline{U}_i\}_{1 \le i \le m}$ is an affine open covering of $\overline{Y}$, and $Y_{\overline{K}} \cong \overline{Y}$.

The $\mathrm{Pic}(\overline{X})$-grading of $\mathrm{Cox}(\overline{X})$ induces a $\mathrm{Pic}(\overline{X})$-grading on R by assigning the degrees of $\eta_1, \dots, \eta_N$. We assume that $(R; f_1, \dots, f_m)$ satisfies the following condition:

(3.1) for every $i, j \in \{1, \dots, m\}$, there is a homogeneous invertible element of $R[f_i^{-1}]$ of degree a multiple of $\deg f_j$.

For $i \in \{1, \dots, m\}$, let R_i be the degree-0-part of the ring $R[f_i^{-1}]$ and $V_i := \mathrm{Spec}(R_i)$. Then $R_i \otimes_A \overline{K}$ is the degree-0-part of $\mathrm{Cox}(\overline{X})[f_i^{-1}]$ for all $i \in \{1, \dots, m\}$, and by construction of the universal torsor structure $\overline{Y} \to \overline{X}$ (see [36, Remark 1.25]), gluing the family of schemes $\{\mathrm{Spec}(R_i \otimes_A \overline{K})\}_{1 \le i \le m}$ yields a variety isomorphic to $\overline{X}$. Let X be the A-scheme obtained by gluing $\{V_i\}_{1 \le i \le m}$. Then X is a model of $\overline{X}$ over A and comes endowed with a natural morphism $\pi : Y \to X$ induced by the inclusions $R_i \to R[f_i^{-1}]$ for $i \in \{1, \dots, m\}$. Since the inclusions $R_i \to R[f_i^{-1}]$ induce surjective morphisms $U_i \to V_i$ for all $i \in \{1, \dots, m\}$, the morphism π is surjective. Moreover, π is of finite presentation because X is noetherian and $R[f_i^{-1}]$ is a finitely generated R_i-algebra for every $i \in \{1, \dots, m\}$. Since $f_1, \dots, f_m$ are $\mathbb{Z}^r$-homogeneous, the homomorphism

$$R \to A[z_1, z_1^{-1}, \dots, z_r, z_r^{-1}] \otimes_A R, \quad \eta_j \mapsto \underline{z}^{m^{(j)}} \otimes \eta_j$$

induces an action of $\mathbb{G}_{m,X}^r$ on Y which is given by

$$\underline{s} * (a_1, \dots, a_N) = (\underline{s}^{m^{(1)}} a_1, \dots, \underline{s}^{m^{(N)}} a_N) \tag{3.2}$$

on A-points $\underline{s} = (s_1, \dots, s_r) \in \mathbb{G}_{m,A}^r(A)$ and $(a_1, \dots, a_N) \in Y(A)$, where $\underline{s}^m := s_1^{m_1} \cdots s_r^{m_r}$ for all $m = (m_1, \dots, m_r) \in \mathbb{Z}^r$.

Moreover, π is an X-torsor under $\mathbb{G}_{m,X}^r$ (compatible with the universal torsor structure of $\overline{Y}$ over $\overline{X}$) if and only if π is flat and the morphism of schemes $\phi : \mathbb{G}_{m,A}^r \times_{\mathrm{Spec}(A)} Y \to Y \times_X Y$ that sends $(\underline{s}, \underline{a}) \mapsto (\underline{s} * \underline{a}, \underline{a})$, obtained by gluing the morphisms

$$\varphi_i : R[f_i^{-1}] \otimes_{R_i} R[f_i^{-1}] \to A[z_1, z_1^{-1}, \dots, z_r, z_r^{-1}] \otimes_A R[f_i^{-1}], \quad \eta_j \otimes \eta_l \mapsto \underline{z}^{m^{(j)}} \otimes \eta_j \eta_l$$

for $1 \le i \le m$, is an isomorphism.

Remark 3.2. – If $A = K$, then $\pi : Y \to X$ is a universal torsor by *fpqc* descent, see [17, §2].

In his proof of Manin's conjecture for toric varieties, Salberger introduced universal torsors for certain schemes defined over noetherian base schemes [49, Definition 5.14]. Under reasonable hypotheses, the following theorem shows that $\pi : Y \to X$ is indeed a universal torsor according to Salberger's definition.

Theorem 3.3. – *Let π be as in Construction 3.1. If $(R; f_1, \dots, f_m)$ satisfies the condition that*

(3.3) *every element of $\mathrm{Pic}(\overline{X})$ is the degree of a homogeneous invertible element of $R[f_i^{-1}]$, for all $i \in \{1, \dots, m\}$,*

then π is an X-torsor under $\mathbb{G}_{m,X}^r$. If we additionally assume that $X(A) \neq \emptyset$, that X is smooth, projective, of constant relative dimension, and with geometrically integral fibers over A, that $\mathrm{Pic}(X_K) = \mathrm{Pic}(\overline{X})$, and that for every prime ideal $\mathfrak{p}$ of A the cohomology groups $H^i(X_{k(\mathfrak{p})}, \mathcal{O}_{X_{k(\mathfrak{p})}})$ vanish for $i \in \{1, 2\}$, then π is a universal torsor of X.

Proof. – Flatness of π is equivalent to flatness of all the inclusions $R_i \to R[f_i^{-1}]$, i.e., to injectiveness of the induced morphisms $J \otimes_{R_i} R[f_i^{-1}] \to R[f_i^{-1}]$ for all ideals J of R_i. Fix $i \in \{1, \dots, m\}$. Let J be an ideal of R_i and $J \otimes_{R_i} R[f_i^{-1}] \to R[f_i^{-1}]$ the induced morphism. A general element in the kernel of this morphism is $h = \sum_{j=1}^n h_j \otimes h'_j$, where $h_j \in J$ has degree 0 and $h'_j \in R[f_i^{-1}]$, and such that $\sum_{j=1}^n h_j h'_j = 0$ in $R[f_i^{-1}]$. Since $R[f_i^{-1}]$ is a graded ring, it is enough to consider homogeneous elements h, i.e., with all h'_j homogeneous of fixed degree $\deg h \in \mathbb{Z}^r$. Since the degrees of the homogeneous invertible elements of $R[f_i^{-1}]$ generate $\mathrm{Pic}(\overline{X}) \cong \mathbb{Z}^r$, there exists $f \in R[f_i^{-1}]^\times$ of degree $\deg h$. Then $h = (\sum_{j=1}^n h_j h'_j f^{-1}) \otimes f = 0$ in $J \otimes_{R_i} R[f_i^{-1}]$.

In order to prove that ϕ is an isomorphism, it suffices to prove that all φ_i are isomorphisms. For every $i \in \{1, \dots, m\}$ and $k \in \{1, \dots, r\}$, let $h_{i,k} \in R[f_i^{-1}]^\times$ be a homogeneous element of degree ℓ_k. Then the morphism

$$\psi_i : A[z_1, z_1^{-1}, \dots, z_r, z_r^{-1}] \otimes_A R[f_i^{-1}] \to R[f_i^{-1}] \otimes_{R_i} R[f_i^{-1}]$$

that sends

$$1 \otimes \eta_j \mapsto 1 \otimes \eta_j \quad \text{and} \quad z_k \otimes 1 \mapsto h_{i,k} \otimes h_{i,k}^{-1}$$

for all $j \in \{1, \dots, N\}$ and $k \in \{1, \dots, r\}$, is well defined and inverse to φ_i, for all $i \in \{1, \dots, m\}$.

By [47, Proposition 2.1] the relative étale Picard functor of X over A is representable by a twisted constant A-group scheme $\mathrm{Pic}_{X/A}$. Since $\mathrm{Pic}(X_K) = \mathrm{Pic}(X_{\overline{K}})$, the group scheme $\mathrm{Pic}_{X/A}$ is constant and represented by $\mathbb{Z}^r$ by étale descent. By [35, Corollary III.12.9], $R^2 f_* \mathcal{O}_X = 0$, where $f : X \to \mathrm{Spec}(A)$ is the structure morphism. Since Y_K is a universal torsor of X_K by Remark 3.2 and the morphism

$$\mathrm{Hom}_A(\widehat{\mathbb{G}_{m,A}^r}, \mathrm{Pic}_{X/A}) \to \mathrm{Hom}_K(\widehat{\mathbb{G}_{m,K}^r}, \mathrm{Pic}_{X_K/K})$$

is injective, the torsor $Y \to X$ is universal, as the exact sequences [49, 5.13] are functorial. □

REMARK 3.4. – A geometric interpretation of (3.3) is the following equivalent formulation: the open subset complement to the support of the effective divisor defined by f_i has trivial Picard group for all $i \in \{1, \dots, m\}$.

The rest of this section provides criteria to check the various hypotheses of Theorem 3.3. We start by showing that the model X of Construction 3.1 is independent of the choice of $f_1, \dots, f_m$ under some conditions.

LEMMA 3.5. – *Let $f'_1, \dots, f'_{m'} \in \overline{K}[\eta_1, \dots, \eta_N] \smallsetminus I$ be monic monomials such that $(R; f_1, \dots, f_m, f'_1, \dots, f'_{m'})$ satisfies the condition* (3.1)*. Let C_A and C'_A be the ideals of $A[\eta_1, \dots, \eta_N]$ generated by $f_1, \dots, f_m, g_1, \dots, g_s$, and $f'_1, \dots, f'_{m'}, g_1, \dots, g_s$, respectively, and assume that $\sqrt{C'_A} = \sqrt{C_A}$. Then X is isomorphic to the A-model X' of $\overline{X}$ constructed using $f'_1, \dots, f'_{m'}$ in Construction 3.1.*

Proof. – For every $i \in \{m+1, \dots, m+m'\}$, let $f_i := f'_{i-m}$, and $V_i := \mathrm{Spec}(R_i)$, where R_i is the degree-0-part of $R[f_i{}^{-1}]$. For every $i, j \in \{1, \dots, m+m'\}$, let $h_{i,j} \in R[f_i^{-1}]^\times$ be a homogeneous element of degree $-n_{i,j} \deg f_j$ for some positive integer $n_{i,j}$, and let $V_{i,j} := \mathrm{Spec}(R_i[(f_j^{n_{i,j}} h_{i,j})^{-1}]) \subseteq V_i$. Since $\sqrt{C'_A} = \sqrt{C_A}$, the ideal of R_i generated by ${f_{m+1}}^{n_{i,m+1}}h_{i,m+1},\dots,{f_{m+m'}}^{n_{i,m+m'}}h_{i,m+m'}$ contains $f_i^n f_i^{-n} = 1$ for some positive integer n. Hence, $V_i = \bigcup_{j=m+1}^{m+m'} V_{i,j}$ for every $i \in \{1,\dots,m\}$. Likewise, $V_i = \bigcup_{j=1}^{m} V_{i,j}$ for every $i \in \{m+1,\dots,m+m'\}$.

The identifications $R_i[(f_j^{n_{i,j}}h_{i,j})^{-1}] = R_j[(f_i^{n_{j,i}}h_{j,i})^{-1}]$ inside $R[(f_if_j)^{-1}]$ induce isomorphisms $V_{i,j} \cong V_{j,i}$, for all $i,j \in \{1,\dots,m+m'\}$, that are compatible on the intersections. The schemes X and X' are the gluing of $\{V_i\}_{1\le i\le m}$, and $\{V_i\}_{m+1\le i\le m+m'}$ respectively, along the isomorphisms mentioned above. Since $\{V_{i,j}\}_{1\le i\le m, m+1\le j\le m+m'}$ is an open covering of X, $\{V_{j,i}\}_{1\le i\le m, m+1\le j\le m+m'}$ is an open covering of X', and all the isomorphisms $V_{i,j} \cong V_{j,i}$ are compatible on the intersections, they glue to a global isomorphism $X \cong X'$. □

The next three propositions provide sufficient conditions for X having geometrically integral fibers, and being smooth and projective over A.

Proposition 3.6. – *If* $\mathrm{Spec}(R) \to \mathrm{Spec}(A)$ *has geometrically integral fibers, then* $X \to \mathrm{Spec}(A)$ *has geometrically integral fibers.*

Proof. – Let $\mathfrak{p}$ be a prime ideal of A, and let k be an algebraic extension of the residue field $k(\mathfrak{p})$. Since $R \otimes_A k$ is an integral domain by hypothesis, the ring $R_i \otimes_A k$ is an integral domain for all $i \in \{1,\dots,m\}$. Thus, X_k is covered by a family of integral open subsets $\{W_i := \mathrm{Spec}(R_i \otimes_A k)\}_{1\le i\le m}$ such that $W_i \cap W_j$ is nonempty for all nonempty W_i and W_j. Indeed, for $i,j \in \{1,\dots,m\}$, the intersection $W_i \cap W_j$ is the spectrum of the degree-0-part of the ring $(R \otimes_A k)[(f_if_j)^{-1}]$, which is nonzero whenever f_i and f_j are nonzero elements of $R \otimes_A k$.

Given any nonempty open subset U of X_k and nonzero sections $s_1, s_2 \in \mathcal{O}_{X_k}(U)$, there exist $i_1, i_2 \in \{1,\dots,m\}$ such that $s_j|_{U\cap W_{i_j}} \neq 0$ for $j \in \{1,2\}$. Therefore, $s_j|_{U\cap W_{i_1}\cap W_{i_2}} \neq 0$ for $j \in \{1,2\}$ as W_{i_1}, W_{i_2} are integral, and $U \cap W_{i_1}$, $W_{i_1} \cap W_{i_2}$ are dense in W_{i_1}. Thus, $(s_1s_2)|_{U\cap W_{i_1}\cap W_{i_2}} \neq 0$ and $s_1s_2 \neq 0$ in $\mathcal{O}_{X_k}(U)$. □

Proposition 3.7. – *Assume that A is a Dedekind domain,* $\mathrm{Spec}(R) \to \mathrm{Spec}(A)$ *has geometrically integral fibers, and π is flat (the last holds, for example, if* (3.3) *is satisfied). If the Jacobian matrix*

$$\left(\frac{\partial g_i}{\partial \eta_j}(\underline{a})\right)_{\substack{1\le i\le s\\ 1\le j\le N}}$$

has rank $N - \dim \overline{X} - r$ for all $\underline{a} \in Y(\overline{k(\mathfrak{p})})$ and $\mathfrak{p} \in \mathrm{Spec}(A)$, where $\overline{k(\mathfrak{p})}$ is an algebraic closure of the residue field $k(\mathfrak{p})$, then X is smooth over A.

Proof. – We prove first that Y is smooth over A. By [34, Proposition 17.8.2], the scheme Y is smooth over A if and only if $Y \to \mathrm{Spec}(A)$ is flat and $Y_{k(\mathfrak{p})}$ is smooth over $k(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$. Since $\mathrm{Cox}(\overline{X})$ is an integral domain (see [1, Theorem 1.5.1.1]) and $I \cap A[\eta_1,\dots,\eta_N] = (g_1,\dots,g_s)$, the ring R is an integral domain. Moreover, $A \to R$ is injective. Thus, R is a flat A-algebra by [39, Corollary 1.2.14] as A is a Dedekind domain, and in particular Y is flat over A.

Since $\overline{\pi} : \overline{Y} \to \overline{X}$ is a torsor under $\mathbb{G}_{m,\overline{X}}^r$, the fiber $\overline{Y}_x$ of $\overline{\pi}$ at a point $x \in \overline{X}$ is a trivial $k(x)$-torsor under $\mathbb{G}_{m,k(x)}^r$, where $k(x)$ is the residue field of $\overline{X}$ at x, (see [42, Corollary III.4.7 and Lemma III.4.10]). Hence, $\overline{Y}_x \cong \mathbb{G}^r_{m,k(x)}$ has dimension r for all $x \in \overline{X}$, and $\overline{Y}$ has dimension $\dim \overline{X} + r$ by [35, Exercise II.3.22]. Then $\dim Y_{k(\mathfrak{p})} \geq \dim \overline{X} + r$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$ by [33, Lemme 13.1.1]. Let $\mathfrak{p} \in \operatorname{Spec}(A)$. By the assumptions on the Jacobian matrix and [35, Theorem I.3.2 (c), Theorem I.5.1 and Proposition I.5.2A], we conclude that $\dim Y_{k(\mathfrak{p})} = \dim \overline{X} + r$ and $Y_{\overline{k(\mathfrak{p})}}$ is regular at all its closed points. Then $Y_{k(\mathfrak{p})}$ is smooth over $k(\mathfrak{p})$.

Therefore, X is smooth over A by [34, Proposition 17.7.7], as Y is smooth over A and π is flat and surjective. □

Proposition 3.8. – *Assume that $f_1, \dots, f_m$ have all the same degree $[D]$, which is an ample class in* $\operatorname{Pic}(\overline{X})$. *Let $C_{\overline{K}}$ and C_A be the ideals of $\overline{K}[\eta_1, \dots, \eta_N]$ and $A[\eta_1, \dots, \eta_N]$ generated by $f_1, \dots, f_m, g_1, \dots, g_s$. If $\sqrt{C_{\overline{K}}} \cap A[\eta_1, \dots, \eta_N] = \sqrt{C_A}$ then X is projective over A.*

Proof. – Since R is a finitely generated A-algebra, the subring $\bigoplus_{n \in \mathbb{N}} R_{n[D]}$, where $R_{n[D]}$ denotes the degree $n[D]$-part of R, is a finitely generated A-algebra by [1, Proposition 1.1.2.4]. By [32, Lemme 2.1.6], there exists a positive integer d such that $R' := \bigoplus_{n \in \mathbb{N}} R_{nd[D]}$ is generated by $R_{d[D]}$ as A-algebra. Let $f'_1, \dots, f'_{m'}$ be generators of the A-module $R_{d[D]}$.

For all $i \in \{1, \dots, m'\}$, denote by R'_i the degree-0-part of $R'[f_i'^{-1}]$, which is generated by $f'_1 f_i'^{-1}, \dots, f'_{m'} f_i'^{-1}$ and coincides with the degree-0-part of $R[f_i'^{-1}]$. We recall that $\operatorname{Proj}(R')$ is defined as gluing of $\{V'_i := \operatorname{Spec}(R'_i)\}_{1 \leq i \leq m'}$ along the isomorphisms on principal open subsets induced by the identifications $R'_i[f'_i f_j'^{-1}] = R'_j[f'_j f_i'^{-1}]$ inside $R[(f'_i f'_j)^{-1}]$ for $1 \leq i, j \leq m'$.

Let C'_A be the ideal of $A[\eta_1, \dots, \eta_N]$ generated by $f'_1, \dots, f'_{m'}, g_1, \dots, g_s$. Since $\sqrt{C_A} = \sqrt{C_A^d}$ and $C_A^d \subseteq C'_A$ by construction, there is an inclusion of radical ideals $\sqrt{C_A} \subseteq \sqrt{C'_A}$. By [1, Corollary 1.6.3.6], the polynomials $f'_1, \dots, f'_{m'}$ and $g_1, \dots, g_s$ generate an ideal of $\overline{K}[\eta_1, \dots, \eta_N]$ whose radical is $\sqrt{C_{\overline{K}}}$. Hence, $\sqrt{C'_A} \subseteq \sqrt{C_A}$. Since $(R; f_1, \dots, f_m, f'_1, \dots, f'_{m'})$ satisfies the condition (3.1), there is an isomorphism $X \cong \operatorname{Proj}(R')$ by Lemma 3.5. □

In the applications that we have in mind, $\overline{X}$ is obtained from $\mathbb{P}^2_{\overline{K}}$ by a chain of blowing-ups at closed points. The next proposition provides some conditions that make Construction 3.1 compatible with such blowing-ups. This can be used to verify the cohomology conditions of Theorem 3.3.

In the situation of Construction 3.1, we assume that $\overline{X}$ is a surface. We assume that the effective divisor on $\overline{X}$ corresponding to the section η_i is an integral curve E_i for all $i \in \{1, \dots, N\}$, and that E_N is a (-1)-curve on $\overline{X}$. Let $b : \overline{X} \to \overline{X}'$ be a birational morphism that contracts exactly E_N according to Castelnuovo's criterion. For every $i \in \{1, \dots, N-1\}$, let $E'_i = b(E_i)$. Assume that $x = b(E_N)$ belongs to E'_i exactly for $i \in \{1, 2\}$, and $E_1 \cap E_2 = \varnothing$. Then $\operatorname{Cox}(\overline{X}') \cong \operatorname{Cox}(\overline{X})/(\eta_N - 1)$ by [37, Proposition 2.2], and the canonical pull-back of sections is defined by

$$b^* : \operatorname{Cox}(\overline{X}') \to \operatorname{Cox}(\overline{X}), \quad \eta_i \mapsto \begin{cases} \eta_i \eta_N & \text{if } i \in \{1, 2\}; \\ \eta_i & \text{otherwise.} \end{cases}$$

Let $\overline{Y}' \subseteq \operatorname{Spec}(\operatorname{Cox}(\overline{X}'))$ be the characteristic space of $\overline{X}'$, and let $f'_1, \dots, f'_{m'}$ belonging to $\overline{K}[\eta_1, \dots, \eta_{N-1}]$ be monic monomials that define the closed subset of $\operatorname{Spec}(\operatorname{Cox}(\overline{X}'))$ complement to $\overline{Y}'$. Let I' be the ideal of $\overline{K}[\eta_1, \dots, \eta_N]$ generated by $g_1, \dots, g_s$ and $\eta_N - 1$. Assume that $I' \cap A[\eta_1, \dots, \eta_N] = (g_1, \dots, g_s, \eta_N - 1)$. Let $R' = R/(\eta_N - 1)$, and let $Y' \to X'$ be the A-model of the universal torsor $\overline{Y}' \to \overline{X}'$ defined in Construction 3.1. Let C_A and C'_A be the ideals of $A[\eta_1, \dots, \eta_N]$ generated by $f_1, \dots, f_m, g_1, \dots, g_s$, and $b^*(f'_1)\eta_1, \dots, b^*(f'_{m'})\eta_1, b^*(f'_1)\eta_2, \dots, b^*(f'_{m'})\eta_2, g_1, \dots, g_s$, respectively. We assume that $\sqrt{C_A} = \sqrt{C'_A}$, that (3.3) holds for $(R; f_1, \dots, f_m)$ and $(R'; f'_1, \dots, f'_{m'})$, and that (η_1, η_2) is a prime ideal in $\operatorname{Cox}(\overline{X}')$.

Proposition 3.9. – *Under the hypotheses listed above, X is a blowing-up of X' with center the closed subscheme defined by η_1, η_2.*

Proof. – Let $f' \in \{f'_1, \dots, f'_{m'}\}$ and $f := b^*(f')$. Since $\operatorname{Pic} \overline{X} \cong \operatorname{Pic}(\overline{X}') \oplus \mathbb{Z}[E_N]$ and $\deg \eta_j = \deg b^*\eta_j - [E_N]$ for $j \in \{1,2\}$, the degrees of the homogeneous invertible elements of $R[(f\eta_j)^{-1}]$ generate $\operatorname{Pic}(\overline{X})$ for $j \in \{1,2\}$. Hence, (3.3) holds for $(R; b^*(f'_1)\eta_1, \dots, b^*(f'_{m'})\eta_1, b^*(f'_1)\eta_2, \dots, b^*(f'_{m'})\eta_2)$. Let $X'_{f'}$ be the spectrum of the degree-0-part R'_0 of the ring $R'[f'^{-1}]$, and let $X_{f\eta_j}$ be the spectrum of the degree-0-part $R[(f\eta_j)^{-1}]_0$ of the ring $R[(f\eta_j)^{-1}]$ for $j \in \{1,2\}$. Let $\overline{X}'_{f'}$ be the complement in $\overline{X}'$ of the support of the effective divisor corresponding to the section f', analogously we define $\overline{X}_f$ and $\overline{X}_{f\eta_j}$ for $j \in \{1,2\}$. By [1, Corollary 1.6.3.5], $\overline{X}'_{f'} = \operatorname{Spec}(R'_0 \otimes_A \overline{K})$. Since $E_1 \cap E_2 = \varnothing$ in $\overline{X}$, $\overline{X}_f = \overline{X}_{f\eta_1} \cup \overline{X}_{f\eta_2}$.

Let $h_1, h_2 \in R'[f'^{-1}]^\times$ be homogeneous elements of degrees $-\deg \eta_1, -\deg \eta_2$ respectively. Then $(\eta_1 h_1, \eta_2 h_2)$ is the ideal of $R'_0 \otimes_A \overline{K}$ defining $\{x\} \cap \overline{X}'_{f'}$.

If $f' \in (\eta_1, \eta_2)$ in R', then $x \notin \overline{X}'_{f'}$, and b induces an isomorphism between $\overline{X}_f = b^{-1}(\overline{X}'_{f'})$ and $\overline{X}'_{f'}$. That is, b^* induces an isomorphism between the degree-0-part of $\operatorname{Cox}(\overline{X}')[f'^{-1}]$ and the degree-0-part of $\operatorname{Cox}(\overline{X})[f^{-1}]$ that descends to an isomorphism between R'_0 and the degree-0-part $R[f^{-1}]_0$ of $R[f^{-1}]$ with the quotient morphism as inverse. Moreover, $X_{f\eta_j}$ is the spectrum of the degree-0-part of $R[(f\eta_j\eta_N)^{-1}]$ for $j \in \{1,2\}$, as f is a multiple of η_N in R. Then $X_{f\eta_1} \cup X_{f\eta_2} = \operatorname{Spec}(R[f^{-1}]_0)$, as $1 \in (\eta_1\eta_N b^*(h_1), \eta_2\eta_N b^*(h_2))$ in $R[f^{-1}]_0$.

If $f' \notin (\eta_1, \eta_2)$ in R', then $x \in \overline{X}'_{f'}$, and $\overline{X}_f = b^{-1}(\overline{X}'_{f'})$ is the blowing-up of $\overline{X}'_{f'}$ with center x. The blowing-up of $X'_{f'}$ with center $V(\eta_1 h_1, \eta_2 h_2)$ is covered by two open subsets that are the spectra of the degree-0-parts of the localizations of $\bigoplus_{d \geq 0}(\eta_1 h_1, \eta_2 h_2)^d$ at its degree-1-elements $\eta_j h_j$ for $j \in \{1,2\}$, respectively. Such an open covering is isomorphic to the gluing of the spectra of $R'_0[\eta_i h_i(\eta_j h_j)^{-1}]$, for $\{i,j\} = \{1,2\}$. Since, for $\{i,j\} = \{1,2\}$, b^* induces an isomorphism $R'_0[\eta_i h_i(\eta_j h_j)^{-1}] \to R[(f\eta_j)^{-1}]_0$ with the quotient morphism as inverse, the gluing of $X_{f\eta_1}$ and $X_{f\eta_2}$ is the blowing-up of $X'_{f'}$ with center $V(\eta_1 h_1, \eta_2 h_2)$.

By Lemma 3.5, the scheme X is isomorphic to the gluing of $X_{b^*(f')\eta_j}$ for $f' \in \{f'_1, \dots, f'_{m'}\}$ and $j \in \{1,2\}$. Hence, it is a blowing-up of X' with center the closed subscheme defined by η_1, η_2. $\square$

4. Parameterization of rational points on S

We recall that S is the anticanonically embedded del Pezzo surface of degree 4 and type $\mathbf{A}_3 + \mathbf{A}_1$ given by (1.1). Let $\overline{K}$ be an algebraic closure of K, and $\widetilde{S}_{\overline{K}}$ the minimal desingularization of $S_{\overline{K}}$ as in [21].

The aim of this section is to apply Theorem 2.7 to an $\mathcal{O}_K$-model of a universal torsor of $\widetilde{S}_{\overline{K}}$ obtained by Construction 3.1 in order to get a parameterization of the set of K-rational points on the open subset U complement of the lines in S via integral points on twisted torsors. An elementary application of the results in [22] would lead to the same parameterization.

We start by describing the universal torsor of $\widetilde{S}_{\overline{K}}$ inside the spectrum of its Cox ring. By the data provided in [21, § 3.4], $\widetilde{S}_{\overline{K}}$ is a blowing-up of $\mathbb{P}^2_{\overline{K}}$ in five points in almost general position with Picard group $\mathrm{Pic}(\widetilde{S}_{\overline{K}}) \cong \mathbb{Z}^6$, and the Cox ring of $\widetilde{S}_{\overline{K}}$ is a $\mathrm{Pic}(\widetilde{S}_{\overline{K}})$-graded $\overline{K}$-algebra with nine generators and one homogeneous relation:

$$\mathrm{Cox}(\widetilde{S}_{\overline{K}}) = \overline{K}[\eta_1, \dots, \eta_9]/(\eta_1\eta_9 + \eta_2\eta_8 + \eta_3\eta_4^2\eta_5^3\eta_7).$$

For $i \in \{1, \dots, 9\}$, the degree of η_i is $[E_i] \in \mathrm{Pic}(\widetilde{S}_{\overline{K}})$, where $[E_i]$ are the divisor classes listed below. Let $\ell_0, \dots, \ell_5$ be the basis of $\mathrm{Pic}(\widetilde{S}_{\overline{K}})$ given in [21]. Then the intersection form is defined by $\ell_0^2 = 1$, $\ell_i^2 = -1$ for $1 \leq i \leq 5$, and $\ell_i.\ell_j = 0$ for all $0 \leq i < j \leq 5$. The classes

$$[E_1] = \ell_5, \quad [E_2] = \ell_4, \quad [E_5] = \ell_3$$

are the (-1)-curves on $\widetilde{S}_{\overline{K}}$,

$$[E_3] = \ell_1 - \ell_2, \quad [E_4] = \ell_2 - \ell_3, \quad [E_6] = \ell_0 - \ell_1 - \ell_4 - \ell_5, \quad [E_7] = \ell_0 - \ell_1 - \ell_2 - \ell_3$$

are the (-2)-curves on $\widetilde{S}_{\overline{K}}$, and

$$[E_8] = \ell_0 - \ell_4, \quad [E_9] = \ell_0 - \ell_5.$$

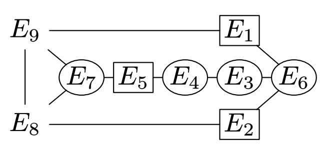

FIGURE 1. Configuration of curves on $\widetilde{S}_{\overline{K}}$.

The Dynkin diagram in Figure 1 encodes the configuration of curves on $\widetilde{S}_{\overline{K}}$. For any $i \neq j$ the number of edges between E_i and E_j is the intersection number $[E_i].[E_j]$.

To construct an $\mathcal{O}_K$-model of the universal torsor $\overline{Y} \to \widetilde{S}_{\overline{K}}$ which is a universal torsor over a projective $\mathcal{O}_K$-model of $\widetilde{S}_{\overline{K}}$, we consider the following monomials. For all $1 \leq i < j \leq 9$, let $A_{i,j} := \prod_{l \in \{1,\dots,9\} \smallsetminus \{i,j\}} \eta_l$, and $A_{7,8,9} := \eta_1\eta_2\eta_3\eta_4\eta_5\eta_6$. Let J be the ideal of $\mathrm{Cox}(\widetilde{S}_{\overline{K}})$ generated by the following monomials:

(4.1) $\qquad A_{7,8,9}, \quad A_{1,6}, \quad A_{1,9}, \quad A_{2,6}, \quad A_{2,8}, \quad A_{3,4}, \quad A_{3,6}, \quad A_{4,5}, \quad A_{5,7},$

which are obtained from the Dynkin diagram in Figure 1 by considering the subsets of vertices that are pairwise connected by at least one edge.

Since $E_7 \cap E_8 \cap E_9 \neq \varnothing$ by [21], the open subscheme $\overline{Y}$ complement to $V(J)$ in $\mathrm{Spec}(\mathrm{Cox}(\widetilde{S}_{\overline{K}}))$ is a universal torsor of $\widetilde{S}_{\overline{K}}$ by [9, Remark 6].

We denote by $f_1, \dots, f_9$ the monomials in (4.1). Let

$$R := \mathcal{O}_K[\eta_1, \dots, \eta_9]/(\eta_1\eta_9 + \eta_2\eta_8 + \eta_3\eta_4^2\eta_5^3\eta_7)$$

and let $\mathcal{Y} \to \widetilde{\mathcal{S}}$ be the $\mathcal{O}_K$-model of the universal torsor $\overline{Y} \to \widetilde{S}_{\overline{K}}$ defined by $f_1, \dots, f_9$ in Construction 3.1. Some properties of this model are described in the following proposition, which is an application of the results of Section 3.

Proposition 4.1. – (i) *The scheme $\widetilde{\mathcal{S}}$ is smooth, projective, and with geometrically integral fibers over $\mathcal{O}_K$.*

(ii) *For every prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, the fibre $\widetilde{\mathcal{S}}_{k(\mathfrak{p})}$ is obtained from $\mathbb{P}^2_{k(\mathfrak{p})}$ by a chain of 5 blowing-ups at $k(\mathfrak{p})$-points.*

(iii) *The morphism $\mathcal{Y} \to \widetilde{\mathcal{S}}$ is a universal torsor under $\mathbb{G}^6_{m,\widetilde{\mathcal{S}}}$.*

Proof. – We start by proving that the model $\widetilde{\mathcal{S}}$ is obtained from $\mathbb{P}^2_{\mathcal{O}_K}$ by a chain of five blowing-ups. By the data provided in [21], $\widetilde{S}_{\overline{K}}$ is a blowing-up of a split toric $\overline{K}$-variety $S'_{\overline{K}}$ at a closed point and with exceptional divisor corresponding to the section $\eta_1 \in \mathrm{Cox}(\widetilde{S}_{\overline{K}})$. The center of the blowing-up $b : \widetilde{S}_{\overline{K}} \to S'_{\overline{K}}$ is the intersection of the prime divisors of $S'_{\overline{K}}$ corresponding to the sections $\eta_6, \eta_9 \in \mathrm{Cox}(S'_{\overline{K}})$ under the identification

$$\mathrm{Cox}(S'_{\overline{K}}) \cong \mathrm{Cox}(\widetilde{S}_{\overline{K}})/(\eta_1 - 1) \cong \overline{K}[\eta_2, \dots, \eta_9]/(\eta_9 + \eta_2\eta_8 + \eta_3\eta_4^2\eta_5^3\eta_7)$$

provided by [37, Proposition 2.2]. The rays of the fan Δ defining $S'_{\overline{K}}$ correspond to the generators $\eta_2, \dots, \eta_8$ of $\mathrm{Cox}(S'_{\overline{K}})$. We denote them by $\rho_2, \dots, \rho_8$ (see Figure 2). Let $\mathcal{S}'$ be

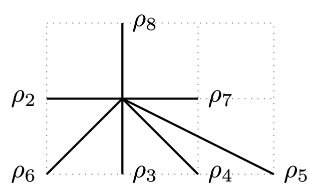

Figure 2. The fan Δ.

the $\mathcal{O}_K$-toric scheme defined by Δ as in [49, Remarks 8.6 (b)], and

$$R' := \mathcal{O}_K[\eta_2, \dots, \eta_9]/(\eta_9 + \eta_2\eta_8 + \eta_3\eta_4^2\eta_5^3\eta_7) \cong \mathcal{O}_K[\eta_2, \dots, \eta_8].$$

The fan Δ has seven maximal cones. For $1 \leq i \leq 7$, let f'_i be the product $\prod \eta_j$ running over the indices $j \in \{2, \dots, 8\}$ such that the ray ρ_j does not belong to the i-th maximal cone. By [49, § 8], the monomials $f'_1, \dots, f'_7$ define the complement of the universal torsor of $S'_{\overline{K}}$ contained in $\mathrm{Spec}(\mathrm{Cox}(S'_{\overline{K}}))$.

For every $i \in \{1, \dots, 7\}$, the open affine toric subvariety of $S'_{\overline{K}}$ corresponding to the i-th maximal cone has trivial Picard group, and its complement consists of the effective divisor defined by the section f'_i. Hence, $(R'; f'_1, \dots, f'_7)$ satisfies (3.3) by Remark 3.4, and $\mathcal{S}'$ is the $\mathcal{O}_K$-model of S' defined by Construction 3.1.

Recall the notation before Proposition 3.9. Since the radical of the ideal of $\mathcal{O}_K[\eta_1, \dots, \eta_9]$ generated by $f_1, \dots, f_9$ is the radical of the ideal generated by $b^*(f'_1)\eta_6, \dots, b^*(f'_7)\eta_6, b^*(f'_1)\eta_9, \dots, b^*(f'_7)\eta_9$, the model $\widetilde{\mathcal{S}}$ is a blowing-up of $\mathcal{S}'$ with center the closed subscheme defined by η_6, η_9 by Proposition 3.9. We observe that $\mathcal{S}'$ is obtained from $\mathbb{P}^2_{\mathcal{O}_K}$ by a chain of four toric blowing-ups, which correspond to adding the rays $\rho_2, \rho_3, \rho_4, \rho_5$, respectively, to the fan of $\mathbb{P}^2_{\mathcal{O}_K}$ with rays ρ_6, ρ_7, ρ_8. Hence, the model $\widetilde{\mathcal{S}}$ is obtained from $\mathbb{P}^2_{\mathcal{O}_K}$ by a chain of five blowing-ups, and it is projective (cf. [31, Proposition 13.96].

To prove (ii), let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$, and $\overline{k(\mathfrak{p})}$ an algebraic closure of the residue field $k(\mathfrak{p})$. Since the closed subscheme of $\mathcal{S}'$ defined by η_6, η_9 is flat over $\mathcal{O}_K$, the variety $\widetilde{\mathcal{S}}_{k(\mathfrak{p})}$ is the blowing-up of $\mathcal{S}'_{k(\mathfrak{p})}$ in the $k(\mathfrak{p})$-point defined by η_6, η_9. Moreover, $\mathcal{S}'_{k(\mathfrak{p})}$ is the split toric $k(\mathfrak{p})$-variety defined by Δ, which is obtained from $\mathbb{P}^2_{k(\mathfrak{p})}$ by four toric blowing-ups at $k(\mathfrak{p})$-points. Therefore, $\mathcal{S}$ has geometrically integral fibers over $\mathcal{O}_K$, and $H^i(\widetilde{\mathcal{S}}_{k(\mathfrak{p})}, \mathcal{O}_{\widetilde{\mathcal{S}}_{k(\mathfrak{p})}}) = 0$ for $i \in \{1, 2\}$ by [35, Proposition V.3.4] and [30, p. 74].

Simple computations show that the degrees of the variables η_j appearing in f_i generate $\mathrm{Pic}(\widetilde{S}_{\overline{K}})$ for all $i \in \{1, \dots, 9\}$. Since these η_j are invertible in $R[{f_i}^{-1}]$, the condition (3.3) holds for $(R; f_1, \dots, f_9)$.

The Jacobian matrix $(\partial g / \partial \eta_i)_{1 \le i \le 9}$ is

$$(\eta_9,\ \eta_8,\ \eta_4^2\eta_5^3\eta_7,\ 2\eta_3\eta_4\eta_5^3\eta_7,\ 3\eta_3\eta_4^2\eta_5^2\eta_7,\ 0,\ \eta_3\eta_4^2\eta_5^3,\ \eta_2,\ \eta_1),$$

and has rank 1 on $\mathcal{Y}(\overline{k(\mathfrak{p})})$ because the monomials $f_1, \dots, f_9$ belong to the ideal generated by η_1, η_2. Then $\widetilde{\mathcal{S}}$ is smooth by Proposition 3.7. Hence, all the hypotheses of Theorem 3.3 are satisfied. $\square$

REMARK 4.2. – The projectiveness and integrality of the fibers of the $\mathcal{O}_K$-model $\widetilde{\mathcal{S}}$ can be verified also using Propositions 3.6 and 3.8, which are not restricted to the case of surfaces. To show a concrete application, we verify their hypotheses for our surface $\widetilde{S}$. Since $g := \eta_1\eta_9 + \eta_2\eta_8 + \eta_3\eta_4^2\eta_5^3\eta_7$ is irreducible in $\overline{k(\mathfrak{p})}[\eta_1, \dots, \eta_9]$ for all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$, the $\mathcal{O}_K$-scheme $\widetilde{\mathcal{S}}$ has geometrically integral fibers by Proposition 3.6. To verify the hypotheses of Proposition 3.8, we define $C'_{\overline{K}}$ and $C'_{\mathcal{O}_K}$ as the ideals of $\overline{K}[\eta_1, \dots, \eta_9]$ and $\mathcal{O}_K[\eta_1, \dots, \eta_9]$, respectively, generated by $f_1, \dots, f_9$ and g. One can check that $C'_{\overline{K}}$ has a Gröbner basis $\{h_1, \dots, h_l\} \subseteq C'_{\mathcal{O}_K}$ consisting of polynomials whose coefficients are all equal to 1. This implies that $C'_{\overline{K}} \cap \mathcal{O}_K[\eta_1, \dots, \eta_9] = (h_1, \dots, h_l) = C'_{\mathcal{O}_K}$. According to [21, § 3.4], the surface $\widetilde{S}_{\overline{K}}$ is a blowing-up of $\mathbb{P}^2_{\overline{K}}$ in five points. Such a description of $\widetilde{S}_{\overline{K}}$ allows us to determine the irreducible curves and the intersection pairing on $\widetilde{S}_{\overline{K}}$ (see [35, §V.3]), and to show that the divisor class

$$[D] := 9\ell_0 - 3\ell_1 - 2\ell_2 - \ell_3 - \ell_4 - \ell_5$$

is ample by the Nakai-Moishezon criterion. Let $C_{\overline{K}}$ and $C_{\mathcal{O}_K}$ be the ideals of $\overline{K}[\eta_1, \dots, \eta_9]$ and $\mathcal{O}_K[\eta_1, \dots, \eta_9]$, respectively, generated by g and by the following monomials of degree $[D]$:

$$\begin{aligned}&\eta_1^8\eta_2^8\eta_3^6\eta_4^4\eta_5^3\eta_6^9,\ \eta_2^3\eta_3\eta_4^3\eta_5^6\eta_7^4\eta_8^4\eta_9,\ \eta_2^5\eta_3\eta_4^2\eta_5^4\eta_6\eta_7^3\eta_8^5,\ \eta_1^3\eta_3\eta_4^3\eta_5^6\eta_7^4\eta_8\eta_9^4,\ \eta_1^5\eta_3\eta_4^2\eta_5^4\eta_6\eta_7^3\eta_9^5,\\ &\eta_1^5\eta_2\eta_5\eta_6\eta_7^2\eta_8\eta_9^5,\ \eta_1^2\eta_2^2\eta_4\eta_5^3\eta_7^3\eta_8^3\eta_9^3,\ \eta_1^6\eta_2^3\eta_3\eta_6^3\eta_7\eta_8\eta_9^4,\ \eta_1^7\eta_2^6\eta_3^3\eta_4\eta_6^6\eta_8\eta_9^2.\end{aligned} \tag{4.2}$$

We observe that these monomials are obtained from $f_1, \dots, f_9$ by increasing the exponents of some variables. Hence, the $\mathcal{O}_K$-model of the universal torsor $\overline{Y} \to \widetilde{S}_{\overline{K}}$ defined by the monomials (4.2) in Construction 3.1 is the same as the one defined by $f_1, \dots, f_9$, namely $\widetilde{\mathcal{S}}$, and the radical of the ideal $C_{\overline{K}}$ (resp. $C_{\mathcal{O}_K}$) coincides with the radical of $C'_{\overline{K}}$ (resp. of $C'_{\mathcal{O}_K}$). Thus, $\widetilde{\mathcal{S}}$ is projective over $\mathcal{O}_K$ by Proposition 3.8.

The action of $\mathbb{G}_{m,\mathcal{O}_K}^6(\mathcal{O}_K) \cong (\mathcal{O}_K^\times)^6$ on $\mathcal{Y}(\mathcal{O}_K)$ is given by (3.2), where $m^{(1)}, \dots, m^{(9)} \in \mathbb{Z}^6$ denote the degrees of $\eta_1, \dots, \eta_9$, respectively, under the identification $\operatorname{Pic}(\widetilde{S}_{\overline{K}}) \cong \mathbb{Z}^6$ provided by the basis $\ell_0, \dots, \ell_5$. Namely,

$$\begin{aligned}
&m^{(1)} = (0,0,0,0,0,1), && m^{(2)} = (0,0,0,0,1,0), && m^{(3)} = (0,1,-1,0,0,0),\\
&m^{(4)} = (0,0,1,-1,0,0), && m^{(5)} = (0,0,0,1,0,0), && m^{(6)} = (1,-1,0,0,-1,-1),\\
&m^{(7)} = (1,-1,-1,-1,0,0), && m^{(8)} = (1,0,0,0,-1,0), && m^{(9)} = (1,0,0,0,0,-1).
\end{aligned}$$

Before we apply Theorem 2.7 to obtain a parameterization of $U(K)$ by integral points on twists of $\mathcal{Y}$, we describe the preimage of U inside the universal torsor, and we fix some more notation.

Let $\widetilde{S} := \widetilde{\mathcal{S}}_K$, $Y := \mathcal{Y}_K$, and $\pi : Y \to \widetilde{S}$ the base change of the torsor morphism under the inclusion $\mathcal{O}_K \subseteq K$. We observe that π is a universal torsor of $\widetilde{S}$ by Remark 3.2.

Let $\overline{\Psi} : \overline{Y} \to S_{\overline{K}}$ be the composition of the universal torsor morphism $\overline{Y} \to \widetilde{S}_{\overline{K}}$ with the minimal desingularization morphism $\widetilde{S}_{\overline{K}} \to S_{\overline{K}}$. According to [21, §3.4], the map $\overline{\Psi} : \overline{Y}(\overline{K}) \to S_{\overline{K}}(\overline{K})$ sends a point $(a_1, \dots, a_9) \in \overline{Y}(\overline{K})$ to the point

$$(a_2a_3a_4a_5a_6a_7a_8 : a_1^2a_2^2a_3^2a_4a_6^3 : a_1a_2a_3^2a_4^2a_5^2a_6^2a_7 : a_1a_3a_4a_5a_6a_7a_9 : a_7a_8a_9) \tag{4.3}$$

in $S_{\overline{K}}(\overline{K}) \subseteq \mathbb{P}^4(\overline{K})$. Since $\overline{\Psi}$ is defined over K, it induces a morphism $\Psi : Y \to S \subseteq \mathbb{P}^4_K$ which is given by (4.3) on K-rational points.

Since $\pi : Y \to \widetilde{S}$ is a geometric quotient, the invariant morphism Ψ factors through a minimal desingularization $\gamma : \widetilde{S} \to S$, which is a model of the minimal desingularization $\widetilde{S}_{\overline{K}} \to S_{\overline{K}}$.

We recall that U is defined as the complement of the lines in S. By [21, Table 6], the surface $S_{\overline{K}}$ contains exactly three lines of $\mathbb{P}^4_{\overline{K}}$. These are defined over K and an easy computation shows that

$$S \smallsetminus U = \{x_0x_1 = x_0x_3 = x_1x_3 = x_2 = 0\}.$$

Then $\Psi^{-1}(S \smallsetminus U) = \{\eta_1\eta_2\eta_3\eta_4\eta_5\eta_6\eta_7 = 0\}$, and

$$\Psi^{-1}(U(K)) = Y(K) \cap (K^\times)^7 \times K^2. \tag{4.4}$$

From now on, $\mathcal{C}$ refers to a fixed system of integral representatives for Cl_K, that is, it contains exactly one integral ideal from each class. For any given $\underline{\mathfrak{c}} = (\mathfrak{c}_0, \dots, \mathfrak{c}_5) \in \mathcal{C}^6$, we denote by ${}_{\underline{\mathfrak{c}}}\pi : {}_{\underline{\mathfrak{c}}}\mathcal{Y} \to \widetilde{S}$ the twist of $\mathcal{Y}$ constructed as in Definition 2.6. We write

$$u_{\underline{\mathfrak{c}}} := \mathfrak{N}(\mathfrak{c}_0^3\mathfrak{c}_1^{-1}\cdots\mathfrak{c}_5^{-1}), \qquad \mathcal{O}_j := \underline{\mathfrak{c}}^{m^{(j)}} \text{ for } 1 \le j \le 9,$$

$$\text{and} \quad \mathcal{O}_{j*} := \begin{cases} \mathcal{O}_j^{\neq 0} & \text{if } j \in \{1,\dots,7\},\\ \mathcal{O}_j & \text{if } j \in \{8,9\}.\end{cases}$$

For $a_j \in \mathcal{O}_j$, let

$$\mathfrak{a}_j := a_j\mathcal{O}_j^{-1}.$$

For $v \in \Omega_K$ and $(x_1,\dots,x_8) \in K_v^8$ with $x_1 \neq 0$, we write

$$\tilde{N}_v(x_1,\dots,x_8) := \max\left\{\begin{array}{l} \left|x_2x_3x_4x_5x_6x_7x_8\right|_v, \left|x_1^2x_2^2x_3^2x_4x_6^3\right|_v,\\ \left|x_1x_2x_3^2x_4^2x_5^2x_6^2x_7\right|_v,\\ \left|x_3x_4x_5x_6x_7(x_3x_4^2x_5^3x_7+x_2x_8)\right|_v,\\ \left|\dfrac{x_2x_7x_8^2+x_3x_4^2x_5^3x_7^2x_8}{x_1}\right|_v \end{array}\right\}.$$

Let $\mathcal{F}$ be a fundamental domain for the action

$$\text{of} \quad U_K \times (\mathcal{O}_K^\times)^5 \quad \text{on} \quad (K^\times)^7 \times K^2,$$

induced by (3.2), where $\underline{u} = (u_0,\dots,u_5)$ maps $(a_1,\dots,a_9)$ to

$$(\underline{u}^{m^{(1)}}\cdot a_1,\dots,\underline{u}^{m^{(9)}}\cdot a_9). \tag{4.5}$$

After all these preparations, we define $M_{\underline{\mathfrak{c}}}(B)$ as the set of all

$$(a_1,\dots,a_9) \in (\mathcal{O}_{1*}\times\cdots\times\mathcal{O}_{9*}) \cap \mathcal{F}$$

that satisfy the *height conditions*

$$\prod_{v\in\Omega_\infty} \tilde{N}_v(\sigma_v(a_1,\dots,a_8)) \le u_{\underline{\mathfrak{c}}}B, \tag{4.6}$$

the *torsor equation*

$$a_1a_9 + a_2a_8 + a_3a_4^2a_5^3a_7 = 0, \tag{4.7}$$

and the *coprimality conditions*

$$\mathfrak{a}_j + \mathfrak{a}_k = \mathcal{O}_K \text{ for all distinct nonadjacent vertices } E_j, E_k \text{ in Figure 1.} \tag{4.8}$$

We can now reduce our counting problem to counting the $M_{\underline{\mathfrak{c}}}(B)$.

LEMMA 4.3. – *With the sets $M_{\underline{\mathfrak{c}}}(B)$ defined as above and $N_{U,H}(B)$ as in Theorem 1.1, we have*

$$N_{U,H}(B) = \frac{1}{|\mu_K|}\sum_{\underline{\mathfrak{c}}\in\mathcal{C}^6} |M_{\underline{\mathfrak{c}}}(B)|.$$

Proof. – Since U is contained in the smooth locus of S, the minimal desingularization $\gamma : \widetilde{S} \to S$ induces an isomorphism $\gamma^{-1}(U) \to U$, so

$$N_{U,H}(B) = |\{x \in \gamma^{-1}(U)(K) \mid H(\gamma(x)) \le B\}|.$$

By Theorem 2.7, (ii), there is a disjoint union

$$\gamma^{-1}(U)(K) = \bigsqcup_{\underline{\mathfrak{c}} \in \mathcal{C}^6} {}_{\underline{\mathfrak{c}}}\pi({}_{\underline{\mathfrak{c}}}\mathcal{Y}(\mathcal{O}_K) \cap \Psi^{-1}(U(K))).$$

Let $\underline{\mathfrak{c}} \in \mathcal{C}^6$. By (4.4) and Theorem 2.7, (iii), we see that ${}_{\underline{\mathfrak{c}}}\mathcal{Y}(\mathcal{O}_K) \cap \Psi^{-1}(U(K))$ is the set of all

$$(a_1, \dots, a_9) \in (\mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{9*})$$

that satisfy (4.7) and

$$\sum_{i=1}^{9} f_i(\underline{\mathfrak{a}}) = \mathcal{O}_K. \tag{4.9}$$

By $f_i(\underline{\mathfrak{a}})$, we mean the ideal $\mathfrak{a}_1^{e_1} \cdots \mathfrak{a}_9^{e_9}$, if f_i is the monomial $\eta_1^{e_1} \cdots \eta_9^{e_9}$. Let us show that (4.9) is equivalent to the coprimality conditions (4.8). These are certainly equivalent to

$$\prod_{\substack{1 \le i < j \le 9 \\ [E_i].[E_j]=0}} (\mathfrak{a}_i + \mathfrak{a}_j) = \mathcal{O}_K. \tag{4.10}$$

The ideal

$$I = \prod_{\substack{1 \le i < j \le 9 \\ [E_i].[E_j]=0}} (\eta_i, \eta_j)$$

in $K[\eta_1, \dots, \eta_9]$ is generated by $t := 2^{25}$ monic monomials $h_1, \dots, h_t$, obtained by choosing for each of the 25 factors (η_i, η_j) one of the generators η_i, η_j and multiplying them all. Due to the distributive property of ideals, condition (4.10) is equivalent to

$$\sum_{i=1}^{t} h_i(\underline{\mathfrak{a}}) = \mathcal{O}_K. \tag{4.11}$$

The radical of I is generated by $f_1, \dots, f_9$. Since the f_i and h_i are monic monomials, this implies that for each h_i there is an f_j such that a power h_i^m is divisible by f_j, and vice versa. Therefore, (4.11) is equivalent to (4.9).

Next, we consider the height condition. Let $(a_1, \dots, a_9) \in {}_{\underline{\mathfrak{c}}}\mathcal{Y}(\mathcal{O}_K)$. Using the torsor equation (4.7) to eliminate a_9, we see that

$$H(\Psi(a_1, \dots, a_9)) = \prod_{v \in \Omega_K} \tilde{N}_v(\sigma_v(a_1, \dots, a_8)).$$

Moreover, the coprimality conditions (4.8) imply that

$$\prod_{v \in \Omega_0} \tilde{N}_v(\sigma_v(a_1, \dots, a_8)) = \mathfrak{N}(a_2a_3a_4a_5a_6a_7a_8\mathcal{O}_K + \cdots + a_7a_8a_9\mathcal{O}_K)^{-1} = u_{\underline{\mathfrak{c}}}^{-1}.$$

Thus the condition $H(\Psi(a_1, \dots, a_9)) \le B$ is equivalent to our height conditions (4.6).

Let $\mathcal{F}' \subseteq \mathcal{F}$ be a fundamental domain for the action of $\mathbb{G}_{m,\mathcal{O}_K}^6(\mathcal{O}_K) = (\mathcal{O}_K^\times)^6$ on $(K^\times)^7 \times K^2$ and consider the set $M'_{\underline{\mathfrak{c}}}(B)$ of all

$$(a_1, \dots, a_9) \in (\mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{9*}) \cap \mathcal{F}'$$

that satisfy (4.6), (4.7) and (4.8).

Since the action of $(\mathcal{O}_K^\times)^6$ on ${}_{\underline{\mathfrak{c}}}\mathcal{Y}(\mathcal{O}_K)$ is free, each orbit is the union of $|\mu_K|$ orbits of the induced action of $U_K \times (\mathcal{O}_K^\times)^5$. Each of these orbits has exactly one representative in $\mathcal{F}$, so $|M_{\underline{\mathfrak{c}}}(B)| = |\mu_K| \cdot |M'_{\underline{\mathfrak{c}}}(B)|$.

Finally, we observe that the fibers of ${}_{\underline{\mathfrak{c}}}\pi$ are the orbits of the action of $\mathbb{G}^6_{m,\widetilde{\mathcal{S}}}$ on ${}_{\underline{\mathfrak{c}}}\mathcal{Y}$. Hence, there is a bijection between the sets $U(K)$ and $\bigsqcup_{\underline{\mathfrak{c}}\in\mathcal{C}^6}({}_{\underline{\mathfrak{c}}}\mathcal{Y}(\mathcal{O}_K)\cap\Psi^{-1}(U(K))\cap\mathcal{F}')$, so

$$N_{U,H}(B) = \sum_{\underline{\mathfrak{c}}\in\mathcal{C}^6} |M'_{\underline{\mathfrak{c}}}(B)| = \frac{1}{|\mu_K|}\sum_{\underline{\mathfrak{c}}\in\mathcal{C}^6} |M_{\underline{\mathfrak{c}}}(B)|. \qquad \square$$

REMARK 4.4. – Our surface S is one of 30 types of del Pezzo surfaces (over $\bar{\mathbb{Q}}$) whose universal torsors are hypersurfaces, classified in [21]. It is not hard to adapt our arguments to the remaining 29 cases. Indeed, in each case we checked that applying Construction 3.1 to the universal torsor given by the monomials obtained from the Dynkin diagram (like the monomials (4.1) for S) provides us with $\mathcal{O}_K$-models of the surface and the universal torsor, for which the analogues of Proposition 4.1 and Lemma 4.3 hold. That is, these monomials satisfy condition (3.3), the equation defining the Cox ring as a hypersurface is irreducible modulo every prime, and the hypotheses of Proposition 3.7 and of Proposition 3.9 are satisfied. Regarding the last one, we recall that blowing-down a (-1)-curve on a weak del Pezzo surface produces another weak del Pezzo surface of smaller degree. Hence, we checked the hypotheses of Proposition 3.9 going down step by step for the chain of blowing-ups of $\mathbb{P}^2$ that define the del Pezzo surfaces in [21].

The verification of these facts in each case requires straightforward but lengthy computations entirely analogous to the arguments from this section. Since they are not needed for our main result, we omit the details here.

During the verification of our claims, we found the following two misprints in [21]: in the description of the del Pezzo surface of degree 5 of type $\mathbf{A}_2$ at page 656 the class of the curve E_1 is $\bar{E}_1 = \ell_0 - \ell_1 - \ell_3 - \ell_4$, and in the description of the del Pezzo surface of degree 2 of type $\mathbf{D}_5 + \mathbf{A}_1$ at page 671 the curves E_5 and E_6 must be exchanged in the second and third line.

5. Construction of a fundamental domain

In this section, we choose our fundamental domain $\mathcal{F}$ for the action (4.5). Our main objective is to find a fundamental domain that lends itself well to lattice point counting. In a much simpler case, such a fundamental domain was constructed by Schanuel [50]. Our notation is inspired by [41].

Let Σ be the hyperplane in $\mathbb{R}^{\Omega_\infty}$ where the sum of the coordinates vanishes, and $\delta := (d_v)_{v\in\Omega_\infty} \in \mathbb{R}^{\Omega_\infty}$. By Dirichlet's unit theorem, the usual logarithmic embedding $l : \mathcal{O}_K^\times \to \Sigma$,

$$l(u) := (\log|\sigma_v(u)|_v)_{v\in\Omega_\infty},$$

has μ_K as its kernel and a lattice $l(\mathcal{O}_K^\times) = l(U_K)$ in Σ as its image.

Let us fix, once and for all, a fundamental parallelotope F for this lattice and denote the vector sum $F + \mathbb{R}\delta$ by $F(\infty)$. Then

(5.1) $\qquad F(\infty)$ is a fundamental domain for the additive action of $l(U_K)$ on $\mathbb{R}^{\Omega_\infty}$.

For fixed $\underline{a}' := (a_1, \ldots, a_5) \in (K^\times)^5$, we define $\tilde{N}_v(\underline{a}'; \cdot) : (K_v^\times)^2 \times K_v \to (0, \infty)$ by

$$\tilde{N}_v(\underline{a}'; x_6, x_7, x_8) := \tilde{N}_v(a_1^{(v)}, \ldots, a_5^{(v)}, x_6, x_7, x_8).$$

Let $S_F(\underline{a}'; \infty)$ be the set of all

$$(x_{jv})_{\substack{j \in \{6,7,8\} \\ v \in \Omega_\infty}} \in \prod_{v \in \Omega_\infty} ((K_v^\times)^2 \times K_v)$$

such that

$$\frac{1}{3} \cdot (\log \tilde{N}_v(\underline{a}'; x_{6v}, x_{7v}, x_{8v}))_{v \in \Omega_\infty} \in F(\infty).$$

Since all terms of the maximum in $\tilde{N}_v$ are homogeneous of degree 3 in x_6, x_7, x_8, the relation

$$(\log \tilde{N}_v(\underline{a}'; u^{(v)} \cdot (x_{6v}, x_{7v}, x_{8v})))_{v \in \Omega_\infty} = 3l(u) + (\log \tilde{N}_v(\underline{a}'; x_{6v}, x_{7v}, x_{8v}))_{v \in \Omega_\infty}$$

holds for all $u \in U_K$. Due to this and (5.1), the set

$$\mathcal{F}_0(\underline{a}') := \{(a_6, a_7, a_8) \in (K^\times)^2 \times K \mid \sigma(a_6, a_7, a_8) \in S_F(\underline{a}'; \infty)\}$$

is a fundamental domain for the action of U_K on $(K^\times)^2 \times K$ by scalar multiplication.

Let $\mathcal{F}_1$ be a fundamental domain for the multiplicative action of $\mathcal{O}_K^\times$ on $K^\times$, chosen in such a way that

$$N(a)^{d_v/d} \ll |a|_v \ll N(a)^{d_v/d} \tag{5.2}$$

holds for all $a \in \mathcal{F}_1$ and all $v \in \Omega_\infty$. By ignoring the last coordinate, the action described by (4.5) induces an action of $U_K \times (\mathcal{O}_K^\times)^5$ on $(K^\times)^7 \times K$. Basic linear algebra with the exponents $m^{(1)}, \ldots, m^{(8)}$ shows that this action is free and has the fundamental domain

$$\mathcal{F}' := \left\{ (a_1, \ldots, a_8) \in (K^\times)^7 \times K \;\middle|\; \begin{array}{l} \underline{a}' \in \mathcal{F}_1^5, \\ (a_6, a_7, a_8) \in \mathcal{F}_0(\underline{a}') \end{array} \right\}.$$

Therefore, we may choose

$$\mathcal{F} := \mathcal{F}' \times K$$

as our fundamental domain for the action of $U_K \times (\mathcal{O}_K^\times)^5$ on $(K^\times)^7 \times K^2$.

The main advantage of this fundamental domain is that it allows a natural incorporation of the height conditions (4.6). Indeed, let

$$F(B) := F + (-\infty, \log B] \cdot \delta.$$

It follows immediately from the definitions that a tuple $(x_{jv})_{j,v} \in S_F(\underline{a}', \infty)$ satisfies

$$\prod_{v \in \Omega_\infty} \tilde{N}_v(\underline{a}'; x_{6v}, x_{7v}, x_{8v}) \leq B$$

if and only if it is an element of the subset

$$S_F(\underline{a}'; B) := \left\{ (x_{jv})_{j,v} \;\middle|\; \frac{1}{3} \cdot (\log \tilde{N}_v(\underline{a}'; x_{6v}, x_{7v}, x_{8v}))_{v \in \Omega_\infty} \in F(B^{1/(3d)}) \right\}$$

of $\prod_{v \in \Omega_\infty} ((K_v^\times)^2 \times K_v)$. Let $\mathcal{F}_0(\underline{a}'; u_{\mathfrak{c}} B)$ be the set of all $(a_6, a_7, a_8) \in \mathcal{F}_0(\underline{a}')$ that satisfy (4.6). Then

$$\mathcal{F}_0(\underline{a}'; u_{\mathfrak{c}} B) = \{(a_6, a_7, a_8) \in (K^\times)^2 \times K \mid \sigma(a_6, a_7, a_8) \in S_F(\underline{a}'; u_{\mathfrak{c}} B)\}.$$

The following observation will be crucial for all our upcoming error estimates. Our construction of $F(\infty)$ implies that

$$\tilde{N}_v(\underline{a}'; x_{6v}, x_{7v}, x_{8v})^{1/d_v} \ll \tilde{N}_w(\underline{a}'; x_{6w}, x_{7w}, x_{8w})^{1/d_w} \ll \tilde{N}_v(\underline{a}'; x_{6v}, x_{7v}, x_{8v})^{1/d_v}$$

for all $(x_{jv})_{j,v} \in S_F(\underline{a}'; \infty)$ and all $v, w \in \Omega_\infty$. In particular,

$$\tilde{N}_v(\underline{a}'; x_{6v}, x_{7v}, x_{8v}) \ll B^{d_v/d} \tag{5.3}$$

holds for all $(x_{jv})_{j,v} \in S_F(\underline{a}'; u_{\mathfrak{c}} B)$ and all $v \in \Omega_\infty$.

If we identify $\mathbb{C}$ with $\mathbb{R}^2$ then $\prod_{v \in \Omega_\infty} K_v^3 = \mathbb{R}^{3d}$. Hence, we define the volume of a (measurable) subset of $\prod_{v \in \Omega_\infty} K_v^3$ as its usual Lebesgue measure. As one would expect, the volume of $S_F(\underline{a}'; u_{\mathfrak{c}} B)$ will appear at a later point as part of an asymptotic formula. Therefore, we compute it here.

Lemma 5.1. – *For $B \geq 0$, the set $S_F(\underline{a}'; B)$ is measurable with volume*

$$\operatorname{vol}(S_F(\underline{a}'; B)) = \frac{1}{3} \cdot 2^{r_1} \cdot \left(\frac{\pi}{4}\right)^{r_2} \cdot \left(\prod_{v \in \Omega_\infty} \omega_v(\widetilde{S})\right) \cdot R_K \cdot \frac{B}{|N(a_2 a_3 a_4 a_5)|}.$$

Proof. – First of all, we observe that $S_F(\underline{a}'; B) = B^{1/(3d)} S_F(\underline{a}'; 1)$ is homogeneously expanding, so it suffices to compute $\operatorname{vol}(S_F(\underline{a}'; 1))$. For $v \in \Omega_\infty$, we define a scaling factor $l_v := \left|a_1 a_2 a_3 a_4^2 a_5^3\right|_v$. We transform the coordinates x_{jv} in $S_F(\underline{a}'; 1)$ to

$$\begin{aligned} y_{0v} &= l_v^{-1/(3d_v)} \sigma_v(a_2) \cdot x_{8v} \\ y_{1v} &= l_v^{-1/(3d_v)} \sigma_v(a_1 a_2 a_3 a_4 a_5) \cdot x_{6v} \\ y_{2v} &= l_v^{-1/(3d_v)} \sigma_v(a_3 a_4^2 a_5^3) \cdot x_{7v}, \end{aligned}$$

for $v \in \Omega_\infty$. The Jacobi determinant of this transformation has absolute value $|N(a_2 a_3 a_4 a_5)|^{-1}$, and we easily verify that $\tilde{N}_v(\underline{a}'; x_{6v}, x_{7v}, x_{8v}) = N_v(y_{0v}, y_{1v}, y_{2v})$. Thus,

$$\operatorname{vol}(S_F(\underline{a}'; 1)) = \frac{1}{|N(a_2 a_3 a_4 a_5)|} \int_{\substack{(y_{jv})_{j,v} \in \prod_{v \in \Omega_\infty} (K_v \times (K_v^\times)^2) \\ 1/3 \cdot (\log N_v(y_{0v}, y_{1v}, y_{2v}))_{v \in \Omega_\infty} \in F(1)}} \prod_{\substack{v \in \Omega_\infty \\ j \in \{0,1,2\}}} \mathrm{d} y_{jv}.$$

Let $f : \prod_{v \in \Omega_\infty} K_v^3 \to \mathbb{R}_{\geq 0}^{\Omega_\infty}$ be given by

$$f((y_{jv})_{j,v}) := (N_v(y_{0v}, y_{1v}, y_{2v}))_{v \in \Omega_\infty}.$$

Then f is Lebesgue-measurable and

$$\operatorname{vol}(S_F(\underline{a}'; 1)) = \frac{1}{|N(a_2 a_3 a_4 a_5)|} \cdot f_*(\operatorname{vol})(\exp(3F(1))),$$

where exp is the coordinate-wise exponential map. Let us compute the pushforward measure $f_*(\operatorname{vol})$. For $T \geq 0$, let $S_v(T) := \{(y_0, y_1, y_2) \in K_v^3 \mid N_v(y_0, y_1, y_2) \leq T\}$. Since

$$N_v(t \cdot (y_0, y_1, y_2)) = |t|_v^3 \, N_v(y_0, y_1, y_2)$$

for all $t \in K_v$, we have $S_v(T) = T^{1/(3d_v)} S_v(1)$, and $\operatorname{vol}(S_v(T)) = T \operatorname{vol}(S_v(1))$. Hence, for any cell $E := \prod_{v \in \Omega_\infty} (\alpha_v, \beta_v]$ in $\mathbb{R}_{\geq 0}^{\Omega_\infty}$, we have

$$(\operatorname{vol} \circ f^{-1})(E) = \prod_{v \in \Omega_\infty} \int_{N_v(y_0, y_1, y_2) \in (\alpha_v, \beta_v]} \mathrm{d} y_0 \, \mathrm{d} y_1 \, \mathrm{d} y_2 = \prod_{v \in \Omega_\infty} (\beta_v - \alpha_v) \operatorname{vol}(S_v(1)).$$

We conclude that

$$f_*(\mathrm{vol}) = \prod_{v\in\Omega_\infty} \mathrm{vol}(S_v(1)) \cdot \mathrm{vol} = \left(\frac{2}{3}\right)^{r_1} \left(\frac{\pi}{12}\right)^{r_2} \prod_{v\in\Omega_\infty} (\omega_v(\widetilde{S})) \cdot \mathrm{vol}\,.$$

To finish the proof, we need to compute $\mathrm{vol}(\exp(3F(1)))$. To this end, choose $w \in \Omega_\infty$ and transform the coordinates by

$$\begin{aligned} x_v &= e^{3y_v+3d_v t}, \text{ for } v \in \Omega_\infty \smallsetminus \{w\},\\ x_w &= e^{-3(\sum_{v\in\Omega_\infty\smallsetminus\{w\}} y_v)+3d_w t}, \end{aligned}$$

with Jacobi determinant $3^{|\Omega_\infty|} d \prod_{v\in\Omega_\infty} x_v = 3^{|\Omega_\infty|} d e^{3dt}$. We obtain

$$\int_{\exp(3F(1))} \prod_{v\in\Omega_\infty} \mathrm{d}x_v = \int_F \prod_{v\in\Omega_\infty\smallsetminus\{w\}} \mathrm{d}y_v \int_{-\infty}^0 3^{|\Omega_\infty|} d e^{3d\cdot t}\,\mathrm{d}t = 3^{|\Omega_\infty|-1} R_K. \qquad \square$$

6. Möbius inversions

In this section, we fix $\underline{\mathfrak{c}} \in \mathcal{C}^6$ and reduce the task of counting, for fixed $a_1, \ldots, a_5$, the set of all (a_6, a_7, a_8, a_9) with $(a_1, \ldots, a_9) \in M_{\underline{\mathfrak{c}}}(B)$ to a lattice point problem. The main job here is to deal with the coprimality conditions (4.8) for a_6, a_7, a_8, a_9. We write

$$\begin{aligned} \underline{a}' &:= (a_1, \ldots, a_5)\\ \mathcal{O}_*' &:= \mathcal{O}_{1*} \times \cdots \times \mathcal{O}_{5*}\\ \underline{\mathfrak{a}}' &:= (\mathfrak{a}_1, \ldots, \mathfrak{a}_5). \end{aligned}$$

To encode the coprimality conditions (4.8) for $\underline{a}' \in \mathcal{O}_*'$, we define the function $\theta_0(\underline{\mathfrak{a}}') := \prod_{\mathfrak{p}} \theta_{0,\mathfrak{p}}(J_{\mathfrak{p}}(\underline{\mathfrak{a}}'))$, where $J_{\mathfrak{p}}(\underline{\mathfrak{a}}') := \{j \in \{1, \ldots, 5\} \ : \ \mathfrak{p} \mid \mathfrak{a}_j\}$ and

$$\theta_{0,\mathfrak{p}}(J) := \begin{cases} 1 & \text{if } J = \varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{3,4\}, \{4,5\},\\ 0 & \text{otherwise.} \end{cases}$$

The product over $\mathfrak{p}$ runs over all nonzero prime ideals of $\mathcal{O}_K$. Clearly, $\theta_0(\underline{\mathfrak{a}}') = 1$ if and only if (4.8) holds for all $j, k \in \{1, \ldots, 5\}$, and $\theta_0(\underline{\mathfrak{a}}') = 0$ otherwise. We rewrite the coprimality conditions (4.8) as follows.

LEMMA 6.1. – *Let* $(a_1, \ldots, a_9) \in \mathcal{O}_1 \times \cdots \times \mathcal{O}_9$ *satisfy the torsor equation* (4.7). *Then the coprimality conditions* (4.8) *hold if and only if the following conditions are satisfied:*

$$\theta_0(\underline{\mathfrak{a}}') = 1 \tag{6.1}$$

$$\mathfrak{a}_6 + \mathfrak{a}_4\mathfrak{a}_5 = \mathcal{O}_K \tag{6.2}$$

$$\mathfrak{a}_7 + \mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4 \cdot \mathfrak{a}_6 = \mathcal{O}_K \tag{6.3}$$

$$\mathfrak{a}_8 + \mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5 \cdot \mathfrak{a}_6 = \mathcal{O}_K \tag{6.4}$$

$$\mathfrak{a}_9 + \mathfrak{a}_6 = \mathcal{O}_K. \tag{6.5}$$

Proof. – The conditions (4.8) are equivalent to (6.1), (6.2), (6.3), and

$$\mathfrak{a}_8 + \mathfrak{a}_1\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5 \cdot \mathfrak{a}_6 = \mathcal{O}_K \tag{6.6}$$

$$\mathfrak{a}_9 + \mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5 \cdot \mathfrak{a}_6 = \mathcal{O}_K. \tag{6.7}$$

We show that the torsor equation (4.7) and conditions (6.1), (6.3), and (6.4) already imply (6.6). Assume that $\mathfrak{p} \mid \mathfrak{a}_8 + \mathfrak{a}_1$. Then in particular,

$$a_2a_8\mathfrak{c}_0^{-1} = \mathfrak{a}_2\mathfrak{a}_8 \subseteq \mathfrak{p} \text{ and } a_1a_9\mathfrak{c}_0^{-1} = \mathfrak{a}_1\mathfrak{a}_9 \subseteq \mathfrak{p}.$$

Using (4.7),

$$\mathfrak{a}_3\mathfrak{a}_4^2\mathfrak{a}_5^3\mathfrak{a}_7 = a_3a_4^2a_5^3a_7\mathfrak{c}_0^{-1} = (a_2a_8 + a_1a_9)\mathfrak{c}_0^{-1} \subseteq a_2a_8\mathfrak{c}_0^{-1} + a_1a_9\mathfrak{c}_0^{-1} \subseteq \mathfrak{p}.$$

We conclude that $\mathfrak{p} \mid \mathfrak{a}_1 + \mathfrak{a}_3\mathfrak{a}_4^2\mathfrak{a}_5^3\mathfrak{a}_7$, which contradicts (6.1) or (6.3). Similarly, one can show that (4.7), (6.1), (6.3), (6.4), and (6.5) imply (6.7). □

We use the following notation for certain 6-tuples of nonzero ideals:

$$\underline{\mathfrak{d}} := (\mathfrak{d}_{67}, \mathfrak{d}_{68}, \mathfrak{d}_{69}, \mathfrak{d}_6, \mathfrak{d}_7, \mathfrak{d}_8),$$
$$\mu_K(\underline{\mathfrak{d}}) := \mu_K(\mathfrak{d}_{67})\mu_K(\mathfrak{d}_{68})\mu_K(\mathfrak{d}_{69})\mu_K(\mathfrak{d}_6)\mu_K(\mathfrak{d}_7)\mu_K(\mathfrak{d}_8),$$

where $\mu_K(\mathfrak{a})$ is the Möbius function for nonzero ideals $\mathfrak{a}$ of $\mathcal{O}_K$. In the next lemma, $\underline{\mathfrak{d}}$ runs over all 6-tuples of nonzero ideals satisfying the conditions (depending on $\underline{\mathfrak{a}}'$):

$$\begin{aligned} \mathfrak{d}_{67} + \mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5 &= \mathcal{O}_K, \\ \mathfrak{d}_{68} + \mathfrak{a}_1\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5 &= \mathcal{O}_K, \\ \mathfrak{d}_{69} + \mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5 &= \mathcal{O}_K, \\ \mathfrak{d}_{68} + \mathfrak{d}_{69} &= \mathcal{O}_K, \end{aligned} \tag{6.8}$$

and

$$\begin{aligned} &\mathfrak{d}_6 \mid \mathfrak{a}_4\mathfrak{a}_5, \\ &\mathfrak{d}_7 \mid \mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4, \\ &\mathfrak{d}_8 \mid \mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5. \end{aligned} \tag{6.9}$$

For any fixed $\underline{\mathfrak{c}}, \underline{\mathfrak{a}}', \underline{\mathfrak{d}}$ satisfying $\theta_0(\underline{\mathfrak{a}}') = 1$ and the above conditions, we define the fractional ideals

$$\begin{aligned} \mathfrak{b}_6 &:= \mathfrak{d}_6(\mathfrak{d}_{67} \cap (\mathfrak{d}_{68}\mathfrak{d}_{69}))\mathcal{O}_6 \\ \mathfrak{b}_7 &:= \mathfrak{d}_7\mathfrak{d}_{67}\mathcal{O}_7 \\ \mathfrak{b}_8 &:= \mathfrak{a}_1\mathfrak{d}_8\mathfrak{d}_{68}\mathfrak{d}_{69}\mathcal{O}_8. \end{aligned}$$

Conditions (6.8) and (6.9), together with $\theta_0(\underline{\mathfrak{a}}') = 1$, imply that

$$\mathfrak{a}_2\mathfrak{d}_8\mathfrak{d}_{68}\mathfrak{c}_1\mathfrak{c}_2\mathfrak{c}_3 + \mathfrak{a}_1\mathfrak{d}_{69}\mathfrak{c}_1\mathfrak{c}_2\mathfrak{c}_3 = \mathfrak{c}_1\mathfrak{c}_2\mathfrak{c}_3.$$

Moreover, $a_3a_4^2a_5^3\mathcal{O}_K = \mathfrak{a}_3\mathfrak{a}_4^2\mathfrak{a}_5^3\mathcal{O}_3\mathcal{O}_4^2\mathcal{O}_5^3 = \mathfrak{a}_3\mathfrak{a}_4^2\mathfrak{a}_5^3\mathfrak{c}_1\mathfrak{c}_2\mathfrak{c}_3$, so $a_3a_4^2a_5^3 \equiv 0 \bmod \mathfrak{c}_1\mathfrak{c}_2\mathfrak{c}_3$. Unique ideal factorization in $\mathcal{O}_K$ allows us to apply the Chinese remainder theorem with not necessarily coprime moduli to conclude that the congruence

$$\gamma_8^* \equiv \begin{cases} 0 & \bmod \mathfrak{a}_2\mathfrak{d}_8\mathfrak{d}_{68}\mathfrak{c}_1\mathfrak{c}_2\mathfrak{c}_3, \\ -a_3a_4^2a_5^3 & \bmod \mathfrak{a}_1\mathfrak{d}_{69}\mathfrak{c}_1\mathfrak{c}_2\mathfrak{c}_3 \end{cases} \tag{6.10}$$

has a solution $\gamma_8^*(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}) \in \mathcal{O}_K$, which is unique modulo

$$\mathfrak{a}_2\mathfrak{d}_8\mathfrak{d}_{68}\mathfrak{c}_1\mathfrak{c}_2\mathfrak{c}_3 \cap \mathfrak{a}_1\mathfrak{d}_{69}\mathfrak{c}_1\mathfrak{c}_2\mathfrak{c}_3 = \mathfrak{a}_1\mathfrak{a}_2\mathfrak{d}_8\mathfrak{d}_{68}\mathfrak{d}_{69}\mathfrak{c}_1\mathfrak{c}_2\mathfrak{c}_3 = a_2\mathcal{O}_7^{-1}\mathfrak{b}_8.$$

Let $\gamma_8 := \gamma_8^*/a_2$. Then we define $\mathcal{G}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}})$ as the additive subgroup of K^3 consisting of all (a_6, a_7, a_8) with

$$\begin{aligned} a_6 &\in \mathfrak{b}_6 \\ a_7 &\in \mathfrak{b}_7 \\ a_8 &\in \gamma_8 \cdot a_7 + \mathfrak{b}_8. \end{aligned}$$

Note that $\mathcal{G}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}})$ does not depend on the choice of γ_8^*.

Lemma 6.2. – *Let $\underline{\mathfrak{c}} \in \mathcal{C}^6$ and $B \geq 0$. With $\mathcal{G}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}})$ as defined above and $\mathcal{F}_1, \mathcal{F}_0(\underline{a}'; u_{\underline{\mathfrak{c}}}B)$ as defined in Section 5,*

$$|M_{\underline{\mathfrak{c}}}(B)| = \sum_{\underline{a}' \in \mathcal{F}_1^5 \cap \mathcal{O}'_*} \theta_0(\underline{\mathfrak{a}}') \sum_{\substack{\underline{\mathfrak{d}} \\ (6.8),(6.9)}} \mu_K(\underline{\mathfrak{d}}) |\mathcal{G}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}) \cap \mathcal{F}_0(\underline{a}'; u_{\underline{\mathfrak{c}}}B)|.$$

Proof. – For now, let us fix $\underline{a}'$ with $\theta_0(\underline{\mathfrak{a}}') = 1$ and write $\mathcal{F}_0 := \mathcal{F}_0(\underline{a}'; u_{\underline{\mathfrak{c}}}B)$. We define

$$\tilde{M} = \tilde{M}(\underline{\mathfrak{c}}, \underline{a}', B) := |\{(a_6, a_7, a_8, a_9) \mid (a_1, \dots, a_9) \in M_{\underline{\mathfrak{c}}}(B)\}|.$$

Möbius inversion for the coprimality condition (6.5) shows that

$$\tilde{M} = \sum_{\mathfrak{d}_{69} \in \mathcal{I}_K} \mu_K(\mathfrak{d}_{69}) \left| \left\{ \begin{aligned} (a_6, a_7, a_8, a_9) \in ((\mathfrak{d}_{69}\mathcal{O}_6 \times \mathcal{O}_7 \times \mathcal{O}_8) \cap \mathcal{F}_0) \times \mathfrak{d}_{69}\mathcal{O}_9) \\ : (4.7), (6.2) - (6.4) \end{aligned} \right\} \right|.$$

There is a one-to-one correspondence between (a_6, a_7, a_8, a_9) as above satisfying the torsor equation (4.7) and triples $(a_6, a_7, a_8) \in (\mathfrak{d}_{69}\mathcal{O}_6 \times \mathcal{O}_7 \times \mathcal{O}_8) \cap \mathcal{F}_0$ satisfying the congruence

$$a_3 a_4^2 a_5^3 a_7 + a_2 a_8 \equiv 0 \mod a_1\mathfrak{d}_{69}\mathcal{O}_9 = \mathfrak{d}_{69}\mathfrak{a}_1\mathfrak{c}_0. \tag{6.11}$$

Moreover, (6.11) and (6.2)–(6.4) imply that $\mathfrak{d}_{69} + \mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5 = \mathcal{O}_K$. We apply Möbius inversion to resolve the coprimality condition $\mathfrak{a}_6 + \mathfrak{a}_8 = \mathcal{O}_K$ resulting from (6.4). As a result, $\tilde{M}$ is equal to

$$\sum_{\substack{\mathfrak{d}_{68}, \mathfrak{d}_{69} \in \mathcal{I}_K \\ \mathfrak{d}_{69} + \mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5 = \mathcal{O}_K}} \mu_K(\mathfrak{d}_{68}, \mathfrak{d}_{69}) \left| \left\{ \begin{aligned} (a_6, a_7, a_8) \in ((\mathfrak{d}_{68} \cap \mathfrak{d}_{69})\mathcal{O}_6 \times \mathcal{O}_7 \times \mathfrak{d}_{68}\mathcal{O}_8) \cap \mathcal{F}_0 \\ : (6.11), (6.2), (6.3), \mathfrak{a}_8 + \mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5 = \mathcal{O}_K \end{aligned} \right\} \right|,$$

where $\mu_K(\mathfrak{d}_{68}, \mathfrak{d}_{69}) := \mu_K(\mathfrak{d}_{68})\mu_K(\mathfrak{d}_{69})$. Clearly, the summand is 0 whenever $\mathfrak{d}_{68} + \mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5 \neq \mathcal{O}_K$. Moreover, due to (6.11) and (6.3), we see that $\mathfrak{d}_{68} + \mathfrak{d}_{69}\mathfrak{a}_1 = \mathcal{O}_K$. One further application of Möbius inversion to resolve the coprimality condition between $\mathfrak{a}_6$ and $\mathfrak{a}_7$ resulting from (6.3) shows that

$$\tilde{M} = \sum_{\substack{\mathfrak{d}_{67}, \mathfrak{d}_{68}, \mathfrak{d}_{69} \in \mathcal{I}_K \\ \mathfrak{d}_{68} + \mathfrak{a}_1\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5 = \mathcal{O}_K \\ \mathfrak{d}_{69} + \mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5 = \mathcal{O}_K \\ \mathfrak{d}_{68} + \mathfrak{d}_{69} = \mathcal{O}_K}} \mu_K(\mathfrak{d}_{67}, \mathfrak{d}_{68}, \mathfrak{d}_{69}) \cdot$$

$$\left| \left\{ \begin{aligned} (a_6, a_7, a_8) \in ((\mathfrak{d}_{67} \cap (\mathfrak{d}_{68}\mathfrak{d}_{69}))\mathcal{O}_6 \times \mathfrak{d}_{67}\mathcal{O}_7 \times \mathfrak{d}_{68}\mathcal{O}_8) \cap \mathcal{F}_0 \\ : (6.11), (6.2), \mathfrak{a}_7 + \mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4 = \mathcal{O}_K, \mathfrak{a}_8 + \mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5 = \mathcal{O}_K \end{aligned} \right\} \right|,$$

with $\mu_K(\mathfrak{d}_{67}, \mathfrak{d}_{68}, \mathfrak{d}_{69})$ defined similarly as above. Clearly, we may add the condition $\mathfrak{d}_{67} + \mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5 = \mathcal{O}_K$ under the sum. After three more applications of Möbius inversion to resolve the remaining coprimality conditions,

$$\tilde{M} = \sum_{\substack{\underline{\mathfrak{d}} \\ (6.8),(6.9)}} \mu_K(\underline{\mathfrak{d}}) \left| \left\{ \begin{array}{r} (a_6, a_7, a_8) \in (\mathfrak{b}_6 \times \mathfrak{b}_7 \times \mathfrak{d}_8\mathfrak{d}_{68}\mathcal{O}_8) \cap \mathcal{F}_0 \\ : (6.11) \end{array} \right\} \right|.$$

The conditions $a_8 \in \mathfrak{d}_8\mathfrak{d}_{68}\mathcal{O}_8$ and (6.11) are equivalent to the system of congruences

$$\begin{aligned} a_2a_8 &\equiv 0 && \bmod \mathfrak{a}_2\mathfrak{d}_8\mathfrak{d}_{68}\mathfrak{c}_0 \\ a_2a_8 &\equiv -a_3a_4^2a_5^3a_7 && \bmod \mathfrak{d}_{69}\mathfrak{a}_1\mathfrak{c}_0. \end{aligned} \tag{6.12}$$

Recall that γ_8^* is a solution to the system (6.10). Multiplying by a_7, we see that $a_2a_8 = \gamma_8^*a_7$ is a solution to (6.12). Hence, (6.12) is equivalent to

$$a_2a_8 \equiv \gamma_8^*a_7 \mod (\mathfrak{a}_2\mathfrak{d}_8\mathfrak{d}_{68}\mathfrak{c}_0 \cap \mathfrak{d}_{69}\mathfrak{a}_1\mathfrak{c}_0) = \mathfrak{a}_1\mathfrak{a}_2\mathfrak{d}_8\mathfrak{d}_{68}\mathfrak{d}_{69}\mathfrak{c}_0.$$

Dividing by a_2 proves the lemma. □

7. Small conjugates

From the conditions $a_6, a_7 \neq 0$ in $\mathcal{F}_0(\underline{a}'; u_{\underline{\mathfrak{c}}}B)$, we see that every (a_6, a_7, a_8) belonging to $\mathcal{G}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}) \cap \mathcal{F}_0(\underline{a}'; u_{\underline{\mathfrak{c}}}B)$ satisfies $N(a_6) \geq \mathfrak{N}\,\mathfrak{b}_6$ and $N(a_7) \geq \mathfrak{N}\,\mathfrak{b}_7$. We would like to replace these by the stronger conditions

$$|a_6|_v \geq \mathfrak{N}\,\mathfrak{b}_6^{d_v/d} \text{ and } |a_7|_v \geq \mathfrak{N}\,\mathfrak{b}_7^{d_v/d} \text{ for all } v \in \Omega_\infty.$$

If $|\Omega_\infty| = 1$ then there is nothing to do. Let us first prove some auxiliary results.

Lemma 7.1. – *Let $\mathfrak{a}$ be a nonzero fractional ideal of K and $b \in K$. For each $v \in \Omega_\infty$, let $y_v \in K_v$ and $c_v > 0$. Define*

$$\mathcal{B} := \{a \in K \mid |a^{(v)} + y_v|_v \leq c_v \text{ for all } v \in \Omega_\infty\}.$$

Then

$$|(b + \mathfrak{a}) \cap \mathcal{B}| \ll \frac{1}{\mathfrak{N}\,\mathfrak{a}} \left(\prod_{v \in \Omega_\infty} c_v \right) + 1.$$

Proof. – Replacing y_v by $y_v + b^{(v)}$, we may assume that $b = 0$. Denote by $B_{-y_v}(c_v)$ the closed ball in K_v (with respect to $|\cdot|_v$) with center $-y_v$ and radius c_v. Let $M := \prod_{v \in \Omega_\infty} B_{-y_v}(c_v)$. Let $c := (\prod_{v \in \Omega_\infty} c_v)^{1/d}$, and let $\tau : \prod_{v \in \Omega_\infty} K_v \to \prod_{v \in \Omega_\infty} K_v$ be the $\mathbb{R}$-linear transformation of determinant 1 given by $\tau(x_v) = cc_v^{-1/d_v} \cdot x_v$. Clearly,

$$|\mathfrak{a} \cap \mathcal{B}| = |\sigma(\mathfrak{a}) \cap M| = |\tau(\sigma(\mathfrak{a})) \cap \tau(M)|.$$

With the usual identification $\prod_{v \in \Omega_\infty} K_v = \mathbb{R}^d$, the boundary of $\tau(M)$ is Lipschitz-parameterizable with Lipschitz constant $\ll c$ (cf. [41], [38, V, §2]) and $\tau(\sigma(\mathfrak{a}))$ is a lattice in $\mathbb{R}^d$ with determinant $2^{-s}\,\mathfrak{N}\,\mathfrak{a}\sqrt{|\Delta_K|}$ and first successive minimum $\lambda_1 \geq \mathfrak{N}\,\mathfrak{a}^{1/d}$ (cf. [29, Lemma 5.1]). A classical counting argument (cf. [41, Lemma 2]) shows that

$$|\tau(\sigma(\mathfrak{a})) \cap \tau(M)| \ll \frac{c^d}{\mathfrak{N}\,\mathfrak{a}} + 1.$$

□

LEMMA 7.2. – *Let $\mathfrak{a}$ be a nonzero fractional ideal of K and $c_v > 0$ for all $v \in \Omega_\infty$. For $\alpha \in [0,1)$,*

$$\sum_{\substack{0 \neq a \in \mathfrak{a} \\ |a|_v \leq c_v \ \forall v \in \Omega_\infty}} \frac{1}{N(a)^\alpha} \ll_\alpha \frac{1}{\mathfrak{N}\,\mathfrak{a}} \prod_{v \in \Omega_\infty} c_v^{(1-\alpha)}.$$

For $\alpha > 1$,

$$\sum_{\substack{0 \neq a \in \mathfrak{a} \\ |a|_v \geq c_v \ \forall v \in \Omega_\infty}} \frac{1}{N(a)^\alpha} \ll_\alpha \frac{1}{\mathfrak{N}\,\mathfrak{a}} \prod_{v \in \Omega_\infty} c_v^{(1-\alpha)}.$$

Proof. – Write $c := \prod_{v \in \Omega_\infty} c_v$. Let us start by proving the assertion for $\alpha \in [0,1)$. We have

$$\sum_{\substack{0 \neq a \in \mathfrak{a} \\ |a|_v \leq c_v \ \forall v \in \Omega_\infty}} \frac{1}{N(a)^\alpha} = \sum_{\substack{\mathfrak{b} \in \mathcal{P}_K \\ \mathfrak{b} \subseteq \mathfrak{a} \\ \mathfrak{N}\,\mathfrak{b} \leq c}} \frac{n(\mathfrak{b})}{\mathfrak{N}\,\mathfrak{b}^\alpha},$$

with

$$n(\mathfrak{b}) := |\{a \in K^\times \mid a\mathcal{O}_K = \mathfrak{b},\ |a|_v \leq c_v \text{ for all } v \in \Omega_\infty\}|.$$

Let us find an upper bound for $n(\mathfrak{b})$. Let $b \in K^\times$ with $b\mathcal{O}_K = \mathfrak{b}$, write $q := |\Omega_\infty| - 1$, and let $u_1, \dots, u_q$ be a system of fundamental units of $\mathcal{O}_K$. Then

$$\begin{aligned} n(\mathfrak{b}) &= |\{u \in \mathcal{O}_K^\times \mid |ub|_v \leq c_v \text{ for all } v \in \Omega_\infty\}| \\ &= |\mu_K| \cdot |\{(e_1, \dots, e_q) \in \mathbb{Z}^q \mid \left|u_1^{e_1} \cdots u_q^{e_q}\right|_v \leq c_v / |b|_v \text{ for all } v \in \Omega_\infty\}. \end{aligned}$$

Choose $w \in \Omega_\infty$. Taking logarithms and using the fact that $|N(u_j)| = 1$ to express the condition for w, we see that $n(\mathfrak{b})$ is $|\mu_K|$ times the number of solutions $(e_1, \dots, e_q) \in \mathbb{Z}^q$ of the system

$$e_1 \log|u_1|_v + \cdots + e_q \log|u_q|_v \leq \log(c_v / |b|_v), \qquad \text{for } v \in \Omega_\infty \smallsetminus \{w\} \text{ and}$$

$$e_1 \left(-\sum_{v \in \Omega_\infty \smallsetminus \{w\}} \log|u_1|_v \right) + \cdots + e_q \left(-\sum_{v \in \Omega_\infty \smallsetminus \{w\}} \log|u_q|_v \right) \leq \log(c_w / |b|_w).$$

For every $v \in \Omega_\infty \smallsetminus \{w\}$, we add the first inequality for all $v_0 \neq v$ to the second one and obtain

$$\log(c_v / |b|_v) - \log(c / \mathfrak{N}\,\mathfrak{b}) \leq e_1 \log|u_1|_v + \cdots + e_q \log|u_q|_v \leq \log(c_v / |b|_v).$$

The fundamental units $u_1, \dots, u_q$ can be chosen such that the $(\log|u_i|_v)_{v \in \Omega_\infty \smallsetminus \{w\}}$, $1 \leq i \leq q$, are a basis of a lattice in $\mathbb{R}^q$ of determinant and first successive minimum $\gg 1$. Hence, we need to estimate the number of lattice points in a box of edge-length $\log(c / \mathfrak{N}\,\mathfrak{b})$. With $\epsilon := (1-\alpha)/2 > 0$, we have

$$n(\mathfrak{b}) \ll \log(c / \mathfrak{N}\,\mathfrak{b})^q + 1 \ll_\alpha (c / \mathfrak{N}\,\mathfrak{b})^\epsilon.$$

We conclude that

$$\sum_{\substack{\mathfrak{b} \in \mathcal{P}_K \\ \mathfrak{b} \subseteq \mathfrak{a} \\ \mathfrak{N}\,\mathfrak{b} \leq c}} \frac{n(\mathfrak{b})}{\mathfrak{N}\,\mathfrak{b}^\alpha} \ll_\alpha \sum_{\substack{\mathfrak{b} \in \mathcal{P}_K \\ \mathfrak{b} \subseteq \mathfrak{a} \\ \mathfrak{N}\,\mathfrak{b} \leq c}} \frac{c^\epsilon}{\mathfrak{N}\,\mathfrak{b}^{\alpha+\epsilon}} = \frac{c^\epsilon}{\mathfrak{N}\,\mathfrak{a}^{\alpha+\epsilon}} \sum_{\substack{\mathfrak{b} \in [\mathfrak{a}^{-1}] \cap \mathcal{I}_K \\ \mathfrak{N}\,\mathfrak{b} \leq c / \mathfrak{N}\,\mathfrak{a}}} \frac{1}{\mathfrak{N}\,\mathfrak{b}^{\alpha+\epsilon}} \ll_\alpha \frac{c^\epsilon}{\mathfrak{N}\,\mathfrak{a}^{\alpha+\epsilon}} \cdot \left(\frac{c}{\mathfrak{N}\,\mathfrak{a}} \right)^{1-\alpha-\epsilon}.$$

The second assertion follows analogously. This time,

$$\sum_{\substack{0\neq a\in\mathfrak{a}\\ |a|_v\geq c_v\ \forall v\in\Omega_\infty}} \frac{1}{N(a)^\alpha} = \sum_{\substack{\mathfrak{b}\in\mathcal{P}_K\\ \mathfrak{b}\subseteq\mathfrak{a}\\ \mathfrak{N}\,\mathfrak{b}\geq c}} \frac{n^*(\mathfrak{b})}{\mathfrak{N}\,\mathfrak{b}^\alpha},$$

with

$$n^*(\mathfrak{b}) := |\{a\in K^\times \mid a\mathcal{O}_K=\mathfrak{b},\ |a|_v\geq c_v \text{ for all } v\in\Omega_\infty\}|.$$

The same argument as before with reversed inequalities shows that $n^*(\mathfrak{b})$ can be estimated by the number of lattice points in a box of edge-length $\log(\mathfrak{N}\,\mathfrak{b}/c)$, so $n^*(\mathfrak{b}) \ll_\alpha (\mathfrak{N}\,\mathfrak{b}/c)^\epsilon$ for $\epsilon := (a-1)/2>0$. Thus,

$$\sum_{\substack{\mathfrak{b}\in\mathcal{P}_K\\ \mathfrak{b}\subseteq\mathfrak{a}\\ \mathfrak{N}\,\mathfrak{b}\geq c}} \frac{n(\mathfrak{b})}{\mathfrak{N}\,\mathfrak{b}^\alpha} \ll_\alpha \sum_{\substack{\mathfrak{b}\in\mathcal{P}_K\\ \mathfrak{b}\subseteq\mathfrak{a}\\ \mathfrak{N}\,\mathfrak{b}\geq c}} \frac{c^{-\epsilon}}{\mathfrak{N}\,\mathfrak{b}^{\alpha-\epsilon}} = \frac{c^{-\epsilon}}{\mathfrak{N}\,\mathfrak{a}^{\alpha-\epsilon}} \sum_{\substack{\mathfrak{b}\in[\mathfrak{a}^{-1}]\cap\mathcal{I}_K\\ \mathfrak{N}\,\mathfrak{b}\geq c/\,\mathfrak{N}\,\mathfrak{a}}} \frac{1}{\mathfrak{N}\,\mathfrak{b}^{\alpha-\epsilon}} \ll_\alpha \frac{c^{-\epsilon}}{\mathfrak{N}\,\mathfrak{a}^{\alpha-\epsilon}}\cdot\left(\frac{c}{\mathfrak{N}\,\mathfrak{a}}\right)^{1-\alpha+\epsilon}.$$

□

The following technical lemma provides conditions under which certain error terms are summable. We use it in our error estimates here and later in Sections 10 and 11. Recall the definitions of $\underline{\mathfrak{d}}$ and of the $\mathfrak{b}_j$ from Section 6. For $\beta\in\mathbb{R}^{\neq 0}$, let $\operatorname{sgn}(\beta)\in\{\pm1\}$ be its sign.

LEMMA 7.3. – *Let* $\underline{\mathfrak{c}}\in\mathcal{C}^6$*, let* $\epsilon,\alpha_6,\alpha_7,\alpha_8>0$*, and* $\alpha\in[0,1]$. *Let* $e_1,\dots,e_5\in\mathbb{Z}$*, not all equal to* 0*, and* $\beta\in\mathbb{R}^{\neq0}$. *Consider norm conditions*

$$N(a_1^{e_1}\cdots a_5^{e_5})^{\operatorname{sgn}(\beta)} \ll B^{\operatorname{sgn}(\beta)} \text{ and } \quad N(a_j)\ll B \text{ for all } j\in\{1,\dots,5\}. \tag{7.1}$$

If

$$\alpha\alpha_6+\alpha_7>1 \quad \text{and} \quad (1-\alpha)\alpha_6+\alpha_8>1 \tag{7.2}$$

then the sum

$$\sum_{\substack{\underline{a}'\in\mathcal{F}_1^5\cap\mathcal{O}'_*\\ (7.1)}} \sum_{\substack{\underline{\mathfrak{d}}\\ (6.8),(6.9)}} \frac{|\mu_K(\underline{\mathfrak{d}})|}{\mathfrak{N}\,\mathfrak{b}_6^{\alpha_6}\,\mathfrak{N}\,\mathfrak{b}_7^{\alpha_7}\,\mathfrak{N}\,\mathfrak{b}_8^{\alpha_8}} \cdot \frac{B}{|N(a_1)|^{1-\alpha_8}|N(a_2a_3a_4a_5)|}\cdot\left(\frac{B}{|N(a_1^{e_1}\cdots a_5^{e_5})|}\right)^{-\beta}$$

is $\ll B(\log B)^{4+\epsilon}$*, for* $B\geq3$. *The implicit constant depends on* K, ϵ, α_6, α_7, α_8, α, $e_1,\dots,e_5$, β *and on the implicit constants in* (7.1).

Proof. – From the definitions of the $\mathfrak{b}_j$, we see that

$$\mathfrak{N}\,\mathfrak{b}_6^{\alpha_6}\,\mathfrak{N}\,\mathfrak{b}_7^{\alpha_7}\,\mathfrak{N}\,\mathfrak{b}_8^{\alpha_8} \gg \mathfrak{N}\,\mathfrak{d}_6^{\alpha_6}\,\mathfrak{N}\,\mathfrak{d}_7^{\alpha_7}\,\mathfrak{N}\,\mathfrak{d}_8^{\alpha_8}\,\mathfrak{N}\,\mathfrak{d}_{67}^{\alpha\alpha_6+\alpha_7}\,\mathfrak{N}(\mathfrak{d}_{68}\mathfrak{d}_{69})^{(1-\alpha)\alpha_6+\alpha_8}N(a_1)^{\alpha_8}.$$

By (7.2),

$$\sum_{\mathfrak{d}_{67}\in\mathcal{I}_K} \frac{1}{\mathfrak{N}\,\mathfrak{d}_{67}^{\alpha\alpha_6+\alpha_7}} \ll 1,$$

and similar estimates hold for the sums over $\mathfrak{d}_{68},\mathfrak{d}_{69}$. To sum over the $\mathfrak{d}_j$, we use the assumption that $\alpha_j>0$. Let $\epsilon' := \min\{\epsilon/35, 1\}$. For $j=6$, for example, we have

$$\sum_{\mathfrak{d}_6\mid\mathfrak{a}_4\mathfrak{a}_5} \frac{|\mu_K(\mathfrak{d}_6)|}{\mathfrak{N}\,\mathfrak{d}_6^{\alpha_6}} = \prod_{\mathfrak{p}\mid\mathfrak{a}_4\mathfrak{a}_5}(1+\mathfrak{N}\,\mathfrak{p}^{-\alpha_6}) \ll \prod_{\mathfrak{p}\mid\mathfrak{a}_4\mathfrak{a}_5}(1+\epsilon') = (1+\epsilon')^{\omega_K(\mathfrak{a}_4\mathfrak{a}_5)},$$

where $\omega_K(\mathfrak{a})$ is the number of distinct prime ideals dividing the ideal $\mathfrak{a}$. Similar bounds hold for $j = 7, 8$. Since

$$(1+\epsilon')^{\omega_K(\mathfrak{a}_4\mathfrak{a}_5)+\omega_K(\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4)+\omega_K(\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5)} \le (1+\epsilon/5)^{\omega_K(\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5)},$$

the sum in the lemma is

$$\ll \sum_{\substack{\underline{a}'\in\mathcal{F}_1^5\cap\mathcal{O}'_*\\(7.1)}} \frac{(1+\epsilon/5)^{\omega_K(\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5)}B}{|N(a_1a_2a_3a_4a_5)|}\left(\frac{B}{|N(a_1^{e_1}\cdots a_5^{e_5})|}\right)^{-\beta}.$$

If a_j runs through $\mathcal{F}_1\cap\mathcal{O}_{j*}$ then $\mathfrak{a}_j = a_j\mathcal{O}_j^{-1}$ runs through all nonzero ideals in the class of $\mathcal{O}_j^{-1}$. Moreover, $\mathfrak{N}\,\mathfrak{a}_j \ll N(a_j) \ll \mathfrak{N}\,\mathfrak{a}_j$, so the above sum is

$$\ll \sum_{\substack{\underline{\mathfrak{a}}'\in\mathcal{I}_K^5\\(7.4),(7.5)}} \frac{(1+\epsilon/5)^{\omega_K(\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5)}B}{N(\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5)}\left(\frac{B}{N(\mathfrak{a}_1^{e_1}\cdots\mathfrak{a}_5^{e_5})}\right)^{-\beta}, \tag{7.3}$$

where $\underline{\mathfrak{a}}'$ runs through all 5-tuples of ideals $(\mathfrak{a}_1,\dots,\mathfrak{a}_5)\in\mathcal{I}_K^5$ with

$$N(\mathfrak{a}_1^{e_1}\cdots\mathfrak{a}_5^{e_5})^{\operatorname{sgn}(\beta)} \ll B^{\operatorname{sgn}(\beta)} \text{ and} \tag{7.4}$$

$$N(\mathfrak{a}_j) \ll B \text{ for all } j\in\{1,\dots,5\}. \tag{7.5}$$

All of the following summations are elementary applications of partial summation. For details, see [22, Lemma 2.9, Lemma 2.4].

Let us first consider the case where $\beta > 0$. If $e_j \le 0$ for all j then the sum in (7.3) is $\ll B^{1-\beta}(\log B)^{4+4\epsilon/5}$. Hence, we may assume without loss of generality that $e_1 > 0$. Using (7.4) to sum over $\mathfrak{a}_1$, we see that the sum in (7.3) is

$$\ll \sum_{\substack{\mathfrak{a}_2,\dots,\mathfrak{a}_5\\(7.5)}} \frac{(1+\epsilon/5)^{\omega_K(\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5)}B(\log B)^{\epsilon/5}}{N(\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5)} \ll B(\log B)^{4+\epsilon}. \tag{7.6}$$

Now we assume that $\beta < 0$. If $e_j \le 0$ for all j then the sum in (7.3) is $\ll 1$. If $e_1 > 0$ then we may use (7.4) again to sum over $\mathfrak{a}_1$ and obtain (7.6). □

For $w\in\Omega_\infty$, let

$$\mathcal{R}_6^{(w)}(\underline{\mathfrak{c}},\underline{a}',\underline{\mathfrak{D}};B) := \{(a_6,a_7,a_8)\in\mathcal{G}(\underline{\mathfrak{c}},\underline{a}',\underline{\mathfrak{D}})\cap\mathcal{F}_0(\underline{a}';u_{\underline{\mathfrak{c}}}B) \mid |a_6|_w < \mathfrak{N}\,\mathfrak{b}_6^{d_w/d}\},$$
$$\mathcal{R}_7^{(w)}(\underline{\mathfrak{c}},\underline{a}',\underline{\mathfrak{D}};B) := \{(a_6,a_7,a_8)\in\mathcal{G}(\underline{\mathfrak{c}},\underline{a}',\underline{\mathfrak{D}})\cap\mathcal{F}_0(\underline{a}';u_{\underline{\mathfrak{c}}}B) \mid |a_7|_w < \mathfrak{N}\,\mathfrak{b}_7^{d_w/d}\}.$$

We show that the contribution of all $\mathcal{R}_6^{(w)}$ and $\mathcal{R}_7^{(w)}$ to $|\mathcal{G}(\underline{\mathfrak{c}},\underline{a}',\underline{\mathfrak{D}})\cap\mathcal{F}_0(\underline{a}';u_{\underline{\mathfrak{c}}}B)|$ is insignificant. To this end, we note some conditions satisfied by all (a_6,a_7,a_8) belonging to $\mathcal{G}(\underline{\mathfrak{c}},\underline{a}',\underline{\mathfrak{D}})\cap\mathcal{F}_0(\underline{a}';u_{\underline{\mathfrak{c}}}B)$, which follow from (5.3) and the definition of $\tilde{N}_v$. For all $v\in\Omega_\infty$, we have

$$\left|a_2a_7a_8^2 + a_3a_4^2a_5^3a_7^2a_8\right|_v \ll |a_1|_v\,B^{d_v/d}, \tag{7.7}$$

$$\left|a_3^2a_4^3a_5^4a_6a_7^2\right|_v \ll B^{d_v/d}, \tag{7.8}$$

$$\left|a_1^2a_2^2a_3^2a_4a_6^3\right|_v \ll B^{d_v/d}. \tag{7.9}$$

Moreover, any $\underline{a}', \underline{\mathfrak{d}}$ with $\mathcal{G}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}) \cap \mathcal{F}_0(\underline{a}'; u_{\underline{\mathfrak{c}}}B) \neq \varnothing$ satisfies

$$N(a_j) \ll B \text{ for all } j \in \{1, \dots, 5\}, \tag{7.10}$$

$$N(a_1^2 a_2^2 a_3^2 a_4) \ll B. \tag{7.11}$$

In our calculations, we will encounter the new height condition

$$N(a_3^2 a_4^4 a_5^6) \leq N(a_1 a_2) v_{\underline{\mathfrak{c}}} B, \tag{7.12}$$

with $v_{\underline{\mathfrak{c}}} := \mathfrak{N}(\mathcal{O}_3^2 \mathcal{O}_4^4 \mathcal{O}_5^6 \mathcal{O}_1^{-1} \mathcal{O}_2^{-1})$, so $1 \ll v_{\underline{\mathfrak{c}}} \ll 1$.

Lemma 7.4. – *Let $\underline{\mathfrak{c}} \in \mathcal{C}^6$ and $\epsilon > 0$. Then, for $B \geq 3$,*

$$\sum_{\underline{a}' \in \mathcal{F}_1^5 \cap \mathcal{O}_*'} \theta_0(\underline{\mathfrak{a}}') \sum_{\substack{\underline{\mathfrak{d}} \\ (6.8),(6.9)}} |\mu_K(\underline{\mathfrak{d}})| \cdot |\mathcal{R}_6^{(w)}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; B)| \ll_\epsilon B(\log B)^{4+\epsilon}.$$

Proof. – Let us first fix $\underline{a}', \underline{\mathfrak{d}}, a_6, a_7$ and find an upper bound for the number of a_8 with $(a_6, a_7, a_8) \in \mathcal{R}_6^{(w)}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; B)$. Condition (7.7) implies that, for all $v \in \Omega_\infty$, one of

$$|a_8|_v \ll \frac{B^{d_v/(2d)} |a_1|_v^{1/2}}{|a_2 a_7|_v^{1/2}} \quad \text{or} \quad \left| a_8 + \frac{a_3 a_4^2 a_5^3 a_7}{a_2} \right|_v \ll \frac{B^{d_v/(2d)} |a_1|_v^{1/2}}{|a_2 a_7|_v^{1/2}}$$

holds. By Lemma 7.1, the number of such a_8 in $\gamma_8 a_7 + \mathfrak{b}_8$ is

$$\ll \frac{1}{\mathfrak{N}\,\mathfrak{b}_8} \left(\frac{BN(a_1)}{N(a_2 a_7)} \right)^{1/2}. \tag{7.13}$$

If there is an a_8 with $(a_6, a_7, a_8) \in \mathcal{R}_6^{(w)}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; B)$ then this expression is indeed $\gg 1$: Since $\mathfrak{d}_{68}\mathfrak{d}_{69} \mid \mathfrak{a}_6$ and $\mathfrak{d}_8 \mid \mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5$, we have $\mathfrak{N}(\mathfrak{d}_8\mathfrak{d}_{68}\mathfrak{d}_{69}) \ll N(a_3 a_4 a_5 a_6)$. Thus,

$$\mathfrak{N}\,\mathfrak{b}_8^2 N(a_1^{-1} a_2 a_7) \ll N(a_1 a_2 a_7)\, \mathfrak{N}(\mathfrak{d}_8\mathfrak{d}_{68}\mathfrak{d}_{69})^2 \ll N(a_1 a_2 a_3^2 a_4^2 a_5^2 a_6^2 a_7) \ll B,$$

by (4.6).

Next, we still fix $\underline{a}', \underline{\mathfrak{d}}, a_6$ and sum the expression in (7.13) over all $a_7 \in \mathfrak{b}_7^{\neq 0}$ with (7.8) for all $v \in \Omega_\infty$. By Lemma 7.2, the result is

$$\ll \frac{1}{\mathfrak{N}\,\mathfrak{b}_8} \left(\frac{BN(a_1)}{N(a_2)} \right)^{1/2} \cdot \frac{1}{\mathfrak{N}\,\mathfrak{b}_7} \left(\frac{B}{N(a_3^2 a_4^3 a_5^4 a_6)} \right)^{1/4}.$$

We use Lemma 7.2 again to sum this over all $a_6 \in \mathfrak{b}_6^{\neq 0}$ with $|a_6|_w \leq \mathfrak{N}\,\mathfrak{b}_6^{d_w/d}$ and (7.9) for all $v \in \Omega_\infty \smallsetminus \{w\}$. Keeping (5.2) in mind, we see that $|\mathcal{R}_6^{(w)}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; B)|$ is

$$\begin{aligned} &\ll \frac{1}{\mathfrak{N}\,\mathfrak{b}_8} \left(\frac{BN(a_1)}{N(a_2)} \right)^{\frac{1}{2}} \cdot \frac{1}{\mathfrak{N}\,\mathfrak{b}_7} \left(\frac{B}{N(a_3^2 a_4^3 a_5^4)} \right)^{\frac{1}{4}} \cdot \frac{1}{\mathfrak{N}\,\mathfrak{b}_6^{1-\frac{3d_w}{4d}}} \left(\frac{B}{N(a_1^2 a_2^2 a_3^2 a_4)} \right)^{\frac{1}{4}-\frac{d_w}{4d}} \\ &\ll \frac{B}{\mathfrak{N}\,\mathfrak{b}_6^{1-\frac{3d_w}{4d}}\, \mathfrak{N}\,\mathfrak{b}_7\, \mathfrak{N}\,\mathfrak{b}_8 N(a_2 a_3 a_4 a_5)} \left(\frac{B}{N(a_1^2 a_2^2 a_3^2 a_4)} \right)^{-\frac{d_w}{4d}}. \end{aligned}$$

Lemma 7.3 with (7.10) and (7.11) now shows the claimed estimate. □

Lemma 7.5. – *Let $\underline{\mathfrak{c}} \in \mathcal{C}^6$ and $\epsilon > 0$. Then, for $B \geq 3$,*

$$\sum_{\underline{a}' \in \mathcal{F}_1^5 \cap \mathcal{O}_*'} \theta_0(\underline{\mathfrak{a}}') \sum_{\substack{\underline{\mathfrak{d}} \\ (6.8),(6.9)}} |\mu_K(\underline{\mathfrak{d}})| \cdot |\mathcal{R}_7^{(w)}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; B)| \ll_\epsilon B(\log B)^{4+\epsilon}.$$

Proof. – If $|\Omega_\infty| = 1$ then the left-hand side is 0. Hence, assume that $|\Omega_\infty| \geq 2$. As in the previous lemma, we start by fixing $\underline{a}', \underline{\mathfrak{d}}, a_6, a_7$ and see that the number of a_8 with $(a_6, a_7, a_8) \in \mathcal{R}_7^{(w)}$ is bounded by (7.13). We apply Lemma 7.2 to sum this over all $a_7 \in \mathfrak{b}_7^{\neq 0}$ with $|a_7|_w \leq \mathfrak{N}\,\mathfrak{b}_7^{d_w/d}$ and (7.8) for all $v \in \Omega_\infty \smallsetminus \{w\}$. This gives the bound

$$\ll \frac{1}{\mathfrak{N}\,\mathfrak{b}_8}\left(\frac{BN(a_1)}{N(a_2)}\right)^{\frac{1}{2}} \cdot \frac{1}{\mathfrak{N}\,\mathfrak{b}_7^{1-\frac{d_w}{2d}}}\left(\frac{B}{N(a_3^2a_4^3a_5^4)}\right)^{\frac{1}{4}-\frac{d_w}{4d}} \cdot \prod_{v\neq w}\frac{1}{|a_6|_v^{\frac{1}{4}}}. \tag{7.14}$$

Our further procedure depends on $\underline{a}'$. We first consider all $\underline{a}'$ that satisfy the additional condition (7.12).

In this case, we note that $\prod_{v\neq w}|a_6|_v^{-1} = N(a_6)^{-1}|a_6|_w$ and estimate $|a_6|_w$ by (7.9) and (5.2). Hence, the expression in (7.14) is

$$\ll \frac{1}{\mathfrak{N}\,\mathfrak{b}_8}\left(\frac{BN(a_1)}{N(a_2)}\right)^{\frac{1}{2}} \cdot \frac{1}{\mathfrak{N}\,\mathfrak{b}_7^{1-\frac{d_w}{2d}}}\left(\frac{B}{N(a_3^2a_4^3a_5^4)}\right)^{\frac{1}{4}-\frac{d_w}{4d}} \cdot \left(\frac{B}{N(a_1^2a_2^2a_3^2a_4)}\right)^{\frac{d_w}{12d}} \cdot \frac{1}{N(a_6)^{\frac{1}{4}}}.$$

Using Lemma 7.2 again to sum this over all $a_6 \in \mathfrak{b}_6^{\neq 0}$ with (7.9), we get the upper bound

$$\ll \frac{B}{\mathfrak{N}\,\mathfrak{b}_6\,\mathfrak{N}\,\mathfrak{b}_7^{1-\frac{d_w}{2d}}\,\mathfrak{N}\,\mathfrak{b}_8 N(a_2a_3a_4a_5)} \cdot \left(\frac{N(a_1a_2)B}{N(a_3^2a_4^4a_5^6)}\right)^{-\frac{d_w}{6d}}.$$

We sum this over all $\underline{a}'$ with (7.10) and (7.12) and all $\underline{\mathfrak{d}}$ using Lemma 7.3.

Now, let us consider all $\underline{a}'$ with the additional condition

$$N(a_3^2a_4^4a_5^6) \gg N(a_1a_2)B. \tag{7.15}$$

We already know that the number of (a_7, a_8) for fixed $\underline{a}', \underline{\mathfrak{d}}, a_6$ is bounded by (7.14). For the existence of an $a_7 \in \mathfrak{b}_7^{\neq 0}$ with $|a_7|_w \leq \mathfrak{N}\,\mathfrak{b}_7^{d_w/d}$ and (7.8) for all $v \neq w$, it is required that

$$1 \ll \left(\frac{B}{N(a_3^2a_4^3a_5^4)}\right)^{\frac{1}{4}-\frac{d_w}{4d}} \cdot \prod_{v\neq w}\frac{1}{|a_6|_v^{\frac{1}{4}}} \ll \left(\left(\frac{B}{N(a_3^2a_4^3a_5^4)}\right)^{\frac{1}{4}-\frac{d_w}{4d}} \cdot \prod_{v\neq w}\frac{1}{|a_6|_v^{\frac{1}{4}}}\right)^{\frac{5d}{4(d-d_w)}}.$$

Hence, we may further estimate the expression in (7.14) by

$$\frac{1}{\mathfrak{N}\,\mathfrak{b}_8}\left(\frac{BN(a_1)}{N(a_2)}\right)^{\frac{1}{2}} \frac{1}{\mathfrak{N}\,\mathfrak{b}_7^{1-\frac{d_w}{2d}}}\left(\frac{B}{N(a_3^2a_4^3a_5^4)}\right)^{\frac{5}{16}}\left(\frac{B}{N(a_1^2a_2^2a_3^2a_4)}\right)^{\frac{5d_w}{48(d-d_w)}} \frac{1}{N(a_6)^{\frac{5d}{16(d-d_w)}}}.$$

The exponent of $N(a_6)$ is in $(0,1)$ since $|\Omega_\infty| \geq 2$ and $d_w \leq 2$. Hence, we can sum this over all $a_6 \in \mathfrak{b}_6^{\neq 0}$ with (7.9) using Lemma 7.2. We obtain the bound

$$\ll \frac{B}{\mathfrak{N}\,\mathfrak{b}_6\,\mathfrak{N}\,\mathfrak{b}_7^{1-\frac{d_w}{2d}}\,\mathfrak{N}\,\mathfrak{b}_8 N(a_2a_3a_4a_5)} \cdot \left(\frac{N(a_1a_2)B}{N(a_3^2a_4^4a_5^6)}\right)^{\frac{1}{24}}.$$

Again, we use Lemma 7.3 to sum this over $\underline{a}'$ satisfying (7.15) and all $\underline{\mathfrak{d}}$. □

To recapitulate the results of Lemma 7.4 and Lemma 7.5, we introduce the sets

$$S_F^*(\mathfrak{c}, \underline{a}', \underline{\mathfrak{d}}; B) := \{(x_{jv})_{j,v} \in S_F(\underline{a}'; B) \mid \forall v : |x_{6v}|_v \geq \mathfrak{N}\,\mathfrak{b}_6^{d_v/d}, |x_{7v}|_v \geq \mathfrak{N}\,\mathfrak{b}_7^{d_v/d}\}$$

and

$$\mathcal{F}_0^*(\mathfrak{c}, \underline{a}', \underline{\mathfrak{d}}; u_{\mathfrak{c}}B) := \{(a_6, a_7, a_8) \in (K^\times)^2 \times K \mid \sigma(a_6, a_7, a_8) \in S_F^*(\mathfrak{c}, \underline{a}', \underline{\mathfrak{d}}; u_{\mathfrak{c}}B)\}.$$

We have just proved that, for $\underline{\mathfrak{c}} \in \mathcal{C}^6$ and $\epsilon > 0$,

$$(7.16)\qquad \begin{aligned}|M_{\underline{\mathfrak{c}}}(B)| = &\sum_{\underline{a}' \in \mathcal{F}_1^5 \cap \mathcal{O}'_*} \theta_0(\underline{\mathfrak{a}}') \sum_{\substack{\underline{\mathfrak{d}} \\ (6.8),(6.9)}} \mu_K(\underline{\mathfrak{d}}) \cdot |\mathcal{G}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}) \cap \mathcal{F}_0^*(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; u_{\underline{\mathfrak{c}}} B)| \\ &+ O_\epsilon(B(\log B)^{4+\epsilon}).\end{aligned}$$

The lower bounds for the $|a_6|_v$, $|a_7|_v$ allow us to introduce (7.12) as an additional height condition:

LEMMA 7.6. – *For $\underline{\mathfrak{c}} \in \mathcal{C}^6$ and $\epsilon > 0$, we have*

$$(7.17)\qquad \begin{aligned}|M_{\underline{\mathfrak{c}}}(B)| = &\sum_{\substack{\underline{a}' \in \mathcal{F}_1^5 \cap \mathcal{O}'_* \\ (7.12)}} \theta_0(\underline{\mathfrak{a}}') \sum_{\substack{\underline{\mathfrak{d}} \\ (6.8),(6.9)}} \mu_K(\underline{\mathfrak{d}}) \cdot |\mathcal{G}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}) \cap \mathcal{F}_0^*(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; u_{\underline{\mathfrak{c}}} B)| \\ &+ O_\epsilon(B(\log B)^{4+\epsilon}).\end{aligned}$$

Proof. – It is enough to prove that

$$(7.18)\qquad \sum_{\substack{\underline{a}' \in \mathcal{F}_1^5 \cap \mathcal{O}'_* \\ (7.15)}} \theta_0(\underline{\mathfrak{a}}') \sum_{\substack{\underline{\mathfrak{d}} \\ (6.8),(6.9)}} |\mu_K(\underline{\mathfrak{d}})| \cdot |\mathcal{G}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}) \cap \mathcal{F}_0^*(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; u_{\underline{\mathfrak{c}}} B)| \ll B(\log B)^{4+\epsilon}.$$

Again, we fix $\underline{a}', \underline{\mathfrak{d}}, a_6, a_7$ and bound the number of a_8 with (a_6, a_7, a_8) belonging to $\mathcal{G}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}) \cap \mathcal{F}_0^*(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; u_{\underline{\mathfrak{c}}} B)$ by (7.13).

We sum this over all $a_6 \in \mathfrak{b}_6^{\neq 0}$ with (7.8) and obtain an upper bound

$$\ll \frac{1}{\mathfrak{N}\,\mathfrak{b}_8} \left(\frac{BN(a_1)}{N(a_2)}\right)^{1/2} \cdot \frac{1}{\mathfrak{N}\,\mathfrak{b}_6} \frac{B}{N(a_3^2 a_4^3 a_5^4)} \cdot \frac{1}{N(a_7)^{5/2}}.$$

By Lemma 7.2, the sum of this expression over all $a_7 \in \mathfrak{b}_7^{\neq 0}$ with $|a_7|_v \geq \mathfrak{N}\,\mathfrak{b}_7^{d_v/d}$ for all v is

$$\ll \frac{B}{\mathfrak{N}\,\mathfrak{b}_6\,\mathfrak{N}\,\mathfrak{b}_7^{5/2}\,\mathfrak{N}\,\mathfrak{b}_8 N(a_2 a_3 a_4 a_5)} \left(\frac{N(a_1 a_2) B}{N(a_3^2 a_4^4 a_5^6)}\right)^{1/2}.$$

We apply Lemma 7.3 with (7.15). □

8. Symmetries

In this section, we consider $\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}$ as fixed. From here on, it will be convenient to write $\mathcal{G} := \mathcal{G}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}})$, $\mathcal{F}_0^* := \mathcal{F}_0^*(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; u_{\underline{\mathfrak{c}}} B)$, and $S_F^* := S_F^*(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; u_{\underline{\mathfrak{c}}} B)$.

In Lemma 7.6, we established that in order to find an asymptotic formula for $|M_{\underline{\mathfrak{c}}}(B)|$, we need to count $|\mathcal{G} \cap \mathcal{F}_0^*|$. The embedding $\sigma : K^3 \to \prod_{v \in \Omega_\infty} K_v^3$ transforms this to

$$|\sigma(\mathcal{G}) \cap S_F^*|.$$

We use some symmetries of S_F^* to facilitate our counting problem. For any $M \subseteq \Omega_\infty$, let $S_F^M = S_F^M(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; u_{\underline{\mathfrak{c}}} B)$ be the set of all $(x_{jv})_{j,v} \in S_F^*$ with

$$\begin{aligned} |\sigma_v(a_2) x_{8v}|_v &\geq \left|\sigma_v(a_2) x_{8v} + \sigma_v(a_3 a_4^2 a_5^3) x_{7v}\right|_v \text{ for all } v \in M \\ |\sigma_v(a_2) x_{8v}|_v &\leq \left|\sigma_v(a_2) x_{8v} + \sigma_v(a_3 a_4^2 a_5^3) x_{7v}\right|_v \text{ for all } v \notin M. \end{aligned}$$

Of these sets, $S_F^{\varnothing}$ is the most convenient to count lattice points in it. Let $\phi_M : \prod_{v\in\Omega_\infty} K_v^3 \to \prod_{v\in\Omega_\infty} K_v^3$ be the $\mathbb{R}$-linear involution given by $\phi_M((x_{jv})_{j,v}) := (x'_{jv})_{j,v}$, with

$$x'_{6v} := x_{6v} \text{ for } v \in \Omega_\infty$$
$$x'_{7v} := \begin{cases} x_{7v} & \text{for } v \in \Omega_\infty \smallsetminus M \\ -x_{7v} & \text{for } v \in M \end{cases}$$
$$x'_{8v} := \begin{cases} x_{8v} & \text{for } v \in \Omega_\infty \smallsetminus M \\ x_{8v} + \sigma_v(a_3a_4^2a_5^3/a_2)\cdot x_{7v} & \text{for } v \in M. \end{cases}$$

Then $|\det \phi_M| = 1$, and one readily verifies that

$$\tilde{N}_v(\underline{a}'; x_{6v}, x_{7v}, x_{8v}) = \tilde{N}_v(\underline{a}'; x'_{6v}, x'_{7v}, x'_{8v})$$

for all $v \in \Omega_\infty$ and all $(x_{6v}, x_{7v}, x_{8v}) \in K_v^3$. Therefore, ϕ_M induces a bijection between S_F^M and $S_F^{\varnothing}$.

Let $\tau : \prod_{v\in\Omega_\infty} K_v^3 \to \prod_{v\in\Omega_\infty} K_v^3$ be the $\mathbb{R}$-endomorphism of determinant $\det\tau = \mathfrak{N}(\mathfrak{b}_6\mathfrak{b}_7\mathfrak{b}_8)^{-1}$ given by

$$\tau((x_{jv})_{j,v}) := (\mathfrak{N}\,\mathfrak{b}_j^{-1/d}\cdot x_{jv})_{j,v}.$$

Define $\Lambda_M = \Lambda_M(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}) := \tau(\phi_M(\sigma(\mathcal{G})))$.

LEMMA 8.1. – *For any $\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}$ as in Lemma 7.6, we have*

$$|\mathcal{G} \cap \mathcal{F}_0^*| = \sum_{M\subseteq\Omega_\infty} |\Lambda_M \cap \tau(S_F^{\varnothing})| + O\left(\max_{\substack{M,N\subseteq\Omega_\infty \\ N\neq\varnothing}} |\Lambda_M \cap \tau(S_F^{\varnothing}\cap S_F^N)|\right).$$

Proof. – Since S_F^* is the union of the sets S_F^M, $M \subseteq \Omega_\infty$, we have

$$|\mathcal{G}\cap\mathcal{F}_0^*| = |\sigma(\mathcal{G})\cap S_F^*| = \sum_{M\subseteq\Omega_\infty} |\sigma(\mathcal{G})\cap S_F^M| + O\left(\max_{M\neq N\subseteq\Omega_\infty} |\sigma(\mathcal{G})\cap S_F^M\cap S_F^N|\right).$$

To prove the lemma, we apply $\tau\circ\phi_M$ to each summand and to the argument of the maximum in the error term. ☐

Let us collect some information about $\Lambda_M(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}})$.

LEMMA 8.2. – *The subset $\Lambda_M(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}) \subseteq \prod_{v\in\Omega_\infty} K_v = \mathbb{R}^{3d}$ is a lattice of rank $3d$ and determinant $(2^{-r_2}\sqrt{|\Delta_K|})^3$. Let λ_1 be its first successive minimum in the sense of Minkowski, with respect to the unit ball in $\mathbb{R}^{3d}$. Then $\lambda_1 \geq 1$.*

Proof. – It is well known that $\sigma(\mathfrak{b}_6\times\mathfrak{b}_7\times\mathfrak{b}_8)$ is a lattice in $\prod_{v\in\Omega_\infty} K_v^3$ of rank $3d$ and determinant $(2^{-r_2}\sqrt{|\Delta_K|})^3\,\mathfrak{N}(\mathfrak{b}_6\mathfrak{b}_7\mathfrak{b}_8)$.

It is clear from the definition of $\mathcal{G}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}})$ that $\sigma(\mathcal{G}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}))$ arises from this lattice via the $\mathbb{R}$-endomorphism $\phi : \prod_{v\in\Omega_\infty} K_v^3 \to \prod_{v\in\Omega_\infty} K_v^3$ of determinant 1 defined by

$$\phi((x_{jv})_{j,v})_{iw} = \begin{cases} x_{iw} & \text{if } i\in\{6,7\}, \\ \gamma_8^{(w)}x_{7w} + x_{8w} & \text{if } i = 8. \end{cases}$$

Hence, $\sigma(\mathcal{G}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}))$ is a lattice of the same rank and determinant. Since $\tau \circ \phi_M$ is a linear transformation with $|\det(\tau \circ \phi)| = \mathfrak{N}(\mathfrak{b}_6\mathfrak{b}_7\mathfrak{b}_8)^{-1} \neq 0$, the set $\Lambda_M(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}})$ is a lattice, and its rank and determinant are as claimed.

We still need to consider λ_1. To this end, let $0 \neq (a_6, a_7, a_8) \in \mathcal{G}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}})$. We show that $(\tau \circ \phi_M \circ \sigma)(a_6, a_7, a_8)$ has length ≥ 1. Assume first that $a_6 \neq 0$. Since $\tau(\phi_M(\sigma(a_6, a_7, a_8)))_{6v} = \mathfrak{N}\,\mathfrak{b}_6^{-1/d} \cdot a_6^{(v)}$ for all $v \in \Omega_\infty$, we have

$$|\tau(\phi_M(\sigma(a_6, a_7, a_8)))| \geq |\,\mathfrak{N}\,\mathfrak{b}_6^{-1/d} \sigma(a_6)| \geq 1.$$

In the second inequality we used the fact that the first successive minimum of $\sigma(\mathfrak{b}_6)$ is at least $\mathfrak{N}\,\mathfrak{b}_6^{1/d}$ (cf. [41, Lemma 5]). A similar argument shows the statement if $a_7 \neq 0$, and if $a_6 = a_7 = 0$ and $a_8 \neq 0$. $\square$

9. Definability in an o-minimal structure

In Lemma 8.1, we reduced our counting problem to controlling the quantities

$$|\Lambda_M \cap \tau(S_F^{\varnothing} \cap S_F^N)|,$$

for $M, N \subseteq \Omega_\infty$. We already know the determinant and a lower bound for the first successive minimum of the lattice $\Lambda_M = \Lambda_M(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}})$.

To count the lattice points in $\tau(S_F^{\varnothing} \cap S_F^N)$, we use a technique going back to Davenport [18], which was recently adapted to the framework of *o*-minimal structures by Barroero and Widmer [3]. We will apply [3, Theorem 1.3], so our sets $\tau(S_F^{\varnothing} \cap S_F^N)$ should be fibers of definable families $Z^{(N)}$ with bounded fibers in an o-minimal structure. For a quick introduction to o-minimal structures, we refer to the survey [54].

By (5.2), there is a constant $c_1 \gg 1$ such that $|a|_v \geq c_1$ for all $v \in \Omega_\infty$ and $a \in \mathcal{F}_1 \cap \mathcal{O}_{j*}$, with $j \in \{1, \dots, 5\}$.

Let $Z^{(N)}$ be the set of all

$$(\beta, \beta_6, \beta_7, \beta_8, (x_{jv})_{\substack{1 \leq j \leq 8 \\ v \in \Omega_\infty}}) \in \mathbb{R}^4 \times \prod_{v \in \Omega_\infty} K_v^8$$

that satisfy the conditions

$$\begin{aligned}
&\beta, \beta_3, \beta_7, \beta_8 > 0,\\
&\qquad |x_{jv}|_v \geq c_1 \text{ for all } j \in \{1, \dots, 5\}, v \in \Omega_\infty,\\
&|x_{6v}|_v, |x_{7v}|_v \geq 1 \text{ for all } v \in \Omega_\infty,\\
&|x_{2v}\beta_8 x_{8v}|_v \leq \left|x_{2v}\beta_8 x_{8v} + x_{3v}x_{4v}^2 x_{5v}^3 \beta_7 x_{7v}\right|_v \text{ for all } v \in \Omega_\infty \smallsetminus N,\\
&|x_{2v}\beta_8 x_{8v}|_v = \left|x_{2v}\beta_8 x_{8v} + x_{3v}x_{4v}^2 x_{5v}^3 \beta_7 x_{7v}\right|_v \text{ for all } v \in N,
\end{aligned}$$

$$\left(\tilde{N}_v(x_{1v}, \dots, x_{5v}, \beta_6 x_{6v}, \beta_7 x_{7v}, \beta_8 x_{8v})^{1/3}\right)_{v \in \Omega_\infty} \in \exp(F(\beta^{1/(3d)})),$$

where $\exp : \mathbb{R}^{\Omega_\infty} \to \mathbb{R}_{>0}^{\Omega_\infty}$ is the coordinate-wise exponential function. For

$$T = (\beta, \beta_3, \beta_7, \beta_8, (x_{jv})_{\substack{1 \leq j \leq 5 \\ v \in \Omega_\infty}}) \in \mathbb{R}^4 \times \prod_{v \in \Omega_\infty} K_v^5,$$

we define the fiber

$$Z_T^{(N)} := \left\{ (x_{jv})_{\substack{j\in\{6,7,8\}\\ v\in\Omega_\infty}} \in \prod_{v\in\Omega_\infty} K_v^3 \;\middle|\; (\beta,\beta_3,\beta_7,\beta_8,(x_{jv})_{\substack{1\leq j\leq 8\\ v\in\Omega_\infty}}) \in Z^{(N)} \right\}.$$

We see immediately from the definitions that $\tau(S_F^{\varnothing}\cap S_F^N)$ is just the fiber $Z_T^{(N)}$, where

$$T := (u_{\underline{\mathfrak{c}}}B, \mathfrak{N}\,\mathfrak{b}_6^{1/d}, \mathfrak{N}\,\mathfrak{b}_7^{1/d}, \mathfrak{N}\,\mathfrak{b}_8^{1/d}, (a_j^{(v)})_{\substack{j\in\{1,\ldots,5\}\\ v\in\Omega_\infty}}).$$

Hence, the following lemma allows us to apply [3, Theorem 1.3]. Recall that we identify $\prod_{v\in\Omega_\infty} K_v^8$ with $\mathbb{R}^{8d}$ by identifying K_v with $\mathbb{R}$ or $\mathbb{R}^2$.

LEMMA 9.1. – *For any $N\subseteq\Omega_\infty$, the subset $Z^{(N)}\subseteq\mathbb{R}^{4+8d}$ is definable in the o-minimal structure $\mathbb{R}_{\exp} = \langle\mathbb{R};<,+,\cdot,-,\exp\rangle$. Moreover, the fibers $Z_T^{(N)}$ are bounded.*

Proof. – O-minimality of the structure $\mathbb{R}_{\exp}$ is a well-known consequence of Wilkie's theorem [53]. After recalling the definitions of F and $F(B)\subseteq\mathbb{R}^{\Omega_\infty}=\mathbb{R}^{q+1}$ from Section 5, it is clear that $Z^{(N)}$ is definable in $\mathbb{R}_{\exp}$. Since $\exp(F(\beta^{1/(3d)}))$ is bounded for any fixed $\beta>0$, boundedness of the fibers follows at once. □

10. Volumes of projections

For any coordinate subspace W of $\prod_{v\in\Omega_\infty} K_v^3 = \mathbb{R}^{3d}$, obtained by equating some coordinates in $\mathbb{R}^{3d}$ to 0, we write $V_W = V_W(\underline{\mathfrak{c}},\underline{a}',\underline{\mathfrak{d}};u_{\underline{\mathfrak{c}}}B)$ for the $(\dim W)$-dimensional volume (i.e., Lebesgue measure) of the orthogonal projection of $\tau(S_F^{\varnothing})$ to W. By convention, the zero-dimensional volume of a point is 1. The following lemma summarizes our progress of the last sections.

LEMMA 10.1. – *For any $\underline{\mathfrak{c}},\underline{a}',\underline{\mathfrak{d}}$ as in Lemma 7.6, we have*

$$|\mathcal{G}\cap\mathcal{F}_0^*| = \frac{2^{3r_2}\operatorname{vol}S_F^*}{|\Delta_K|^{3/2}\,\mathfrak{N}(\mathfrak{b}_6\mathfrak{b}_7\mathfrak{b}_8)} + O\left(\sum_W V_W\right).$$

The implied constant in the error term depends only on K, and W runs over all proper coordinate subspaces of $\mathbb{R}^{3d}$.

Proof. – We start from Lemma 8.1. By the results of the previous section, the sets $\tau(S_F^{\varnothing}\cap S_F^N)$ are fibers of families $Z^{(N)}$ definable in the o-minimal structure $\mathbb{R}_{\exp}$. Hence, by [3, Theorem 1.3] and Lemma 8.2,

$$|\Lambda_M\cap\tau(S_F^{\varnothing}\cap S_F^N)| = \frac{\operatorname{vol}(\tau(S_F^{\varnothing}\cap S_F^N))}{\det\Lambda_M} + O\left(\sum_{j=0}^{3d-1} V_j^{(N)}\right),$$

where $V_j^{(N)}$ is the sum of the j-dimensional volumes of the orthogonal projections of $\tau(S_F^{\varnothing}\cap S_F^N)$ to all j-dimensional coordinate spaces of $\mathbb{R}^{3d}$.

If $N\neq\varnothing$ then $\operatorname{vol}(\tau(S_F^{\varnothing}\cap S_F^N))=0$. Moreover, $\tau(S_F^{\varnothing}\cap S_F^N)\subseteq\tau(S_F^{\varnothing})$, so the same inclusion holds for the projections.

For $N = \varnothing$, we have $\mathrm{vol}(\tau(S_F^\varnothing)) = \mathrm{vol}(S_F^\varnothing)/\mathfrak{N}(\mathfrak{b}_6\mathfrak{b}_7\mathfrak{b}_8) = \mathrm{vol}(S_F^M)/\mathfrak{N}(\mathfrak{b}_6\mathfrak{b}_7\mathfrak{b}_8)$ for all $M \subseteq \Omega_\infty$. Since S_F^* is the union of all S_F^M and the intersection of any two of them has volume zero, the lemma follows immediately. □

Our next goal is to find good estimates for the V_W. Recall that all $(x_{jv})_{j,v} \in S_F^\varnothing$ satisfy

$$\begin{aligned} |x_{6v}|_v &\geq \mathfrak{N}\,\mathfrak{b}_6^{d_v/d},\\ |x_{7v}|_v &\geq \mathfrak{N}\,\mathfrak{b}_7^{d_v/d},\\ |\sigma_v(a_2)x_{8v}|_v &\leq \left|\sigma_v(a_2)x_{8v} + \sigma_v(a_3a_4^2a_5^3)x_{7v}\right|_v,\\ \tilde{N}_v(\underline{a}'; x_{6v}, x_{7v}, x_{8v}) &\ll B^{d_v/d}, \end{aligned}$$

for all $v \in \Omega_\infty$. Let

$$c_6 := \left(\frac{B}{N(a_1^2a_2^2a_3^2a_4)}\right)^{1/3}, \quad c_7 := \left(\frac{B}{N(a_3^2a_4^3a_5^4)}\right)^{1/2}, \quad c_8 := \left(\frac{N(a_1)B}{N(a_2)}\right)^{1/2}.$$

Using (5.2), we see that every $(x_{jv})_{j,v} \in S_F^\varnothing$ satisfies in particular, for $v \in \Omega_\infty$,

$$\mathfrak{N}\,\mathfrak{b}_6^{d_v/d} \ll |x_6|_v \ll c_6^{d_v/d}, \tag{10.1}$$

$$\mathfrak{N}\,\mathfrak{b}_7^{d_v/d} \ll |x_7|_v \ll c_7^{d_v/d} \cdot \frac{1}{|x_6|_v^{1/2}}, \tag{10.2}$$

$$|x_8|_v \ll c_8^{d_v/d} \cdot \frac{1}{|x_7|_v^{1/2}}, \tag{10.3}$$

$$\mathfrak{N}\,\mathfrak{b}_8^{d_v/d} \ll c_8^{d_v/d} \cdot \frac{1}{|x_7|_v^{1/2}}. \tag{10.4}$$

Here, (10.1)–(10.3) follow directly from the properties listed above, and (10.4) follows similarly as in the paragraph after (7.13).

For fixed $\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{D}}$, and $v \in \Omega_\infty$, let $S_F^{(v)} = S_F^{(v)}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{D}}; u_{\underline{\mathfrak{c}}}B)$ be the set of all $(x_6, x_7, x_8) \in K_v^3$ that satisfy (10.1)–(10.4).

Let $\tau_v : K_v^3 \to K_v^3$, $(x_6, x_7, x_8) \mapsto (\mathfrak{N}\,\mathfrak{b}_6^{-1/d}x_6, \mathfrak{N}\,\mathfrak{b}_7^{-1/d}x_7, \mathfrak{N}\,\mathfrak{b}_8^{-1/d}x_8)$. Then

$$\tau(S_F^\varnothing) \subseteq \prod_{v \in \Omega_\infty} \tau_v(S_F^{(v)}). \tag{10.5}$$

Hence, each projection of $\tau(S_F^\varnothing)$ to a coordinate subspace is contained in a product of projections of the $\tau_v(S_F^{(v)})$ to coordinate subspaces in $K_v^3 = \mathbb{R}^{3d_v}$. Let us investigate these projections. In our estimates, we will use the quantities

$$\begin{aligned}
s_0 &:= \frac{c_8 c_7^{1/2} c_6^{3/4}}{\mathfrak{N}\,\mathfrak{b}_6\,\mathfrak{N}\,\mathfrak{b}_7\,\mathfrak{N}\,\mathfrak{b}_8} = \frac{B}{\mathfrak{N}\,\mathfrak{b}_6\,\mathfrak{N}\,\mathfrak{b}_7\,\mathfrak{N}\,\mathfrak{b}_8 N(a_2a_4a_5a_6)},\\
s_6 &:= \frac{c_8 c_7^{1/2} c_6^{1/4}}{\mathfrak{N}\,\mathfrak{b}_6^{1/2}\,\mathfrak{N}\,\mathfrak{b}_7\,\mathfrak{N}\,\mathfrak{b}_8} = \frac{B}{\mathfrak{N}\,\mathfrak{b}_6^{\frac{1}{2}}\,\mathfrak{N}\,\mathfrak{b}_7\,\mathfrak{N}\,\mathfrak{b}_8 N(a_2a_3a_4a_5)}\left(\frac{B}{N(a_1^2a_2^2a_3^2a_4)}\right)^{-\frac{1}{6}},\\
s_7 &:= \frac{c_8 c_7^{1/4} c_6^{7/8}}{\mathfrak{N}\,\mathfrak{b}_6\,\mathfrak{N}\,\mathfrak{b}_7^{1/2}\,\mathfrak{N}\,\mathfrak{b}_8} = \frac{B}{\mathfrak{N}\,\mathfrak{b}_6\,\mathfrak{N}\,\mathfrak{b}_7^{\frac{1}{2}}\,\mathfrak{N}\,\mathfrak{b}_8 N(a_2a_3a_4a_5)}\left(\frac{N(a_1a_2)B}{N(a_3^2a_4^4a_5^6)}\right)^{-\frac{1}{12}},\\
s_8 &:= \frac{c_8^{1/2} c_7^{3/4} c_6^{5/8}}{\mathfrak{N}\,\mathfrak{b}_6\,\mathfrak{N}\,\mathfrak{b}_7\,\mathfrak{N}\,\mathfrak{b}_8^{1/2}} = \frac{B}{\mathfrak{N}\,\mathfrak{b}_6\,\mathfrak{N}\,\mathfrak{b}_7\,\mathfrak{N}\,\mathfrak{b}_8^{\frac{1}{2}} N(a_1)^{\frac{1}{2}} N(a_2a_3a_4a_5)}\left(\frac{N(a_3a_4^2a_5^3)B}{N(a_1^2a_2^2)}\right)^{-\frac{1}{6}}.
\end{aligned}$$

10.1. Real places

Here, we investigate $\tau_v(S_F^{(v)})$ when v is a real place, so $K_v^3 = \mathbb{R}^3$.

LEMMA 10.2. – *Let $v \in \Omega_\infty$ be a real place. For any $P = (p_6, p_7, p_8) \in \{0,1\}^3$, let V_P be the $(3-(p_6+p_7+p_8))$-dimensional volume of the orthogonal projection of $\tau_v(S_F^{(v)})$ to the coordinate subspace of $\mathbb{R}^3$ given by*

$$x_j = 0 \text{ for all } j \in \{6,7,8\} \text{ with } p_j = 1. \tag{10.6}$$

Then

$$V_P \ll \begin{cases} s_0^{1/d} & \text{if } p_3 = p_7 = p_8 = 0,\\ s_6^{1/d} & \text{if } p_6 = 1,\ p_7 = p_8 = 0,\\ s_7^{1/d} & \text{if } p_7 = 1,\ p_8 = 0,\\ s_8^{1/d} & \text{if } p_8 = 1.\end{cases}$$

REMARK. – The bounds s_j are adapted to the complex case (Lemma 10.3), so the following proof is more complicated than it could be with different bounds.

Proof. – We may assume that $S_F^{(v)} \neq \varnothing$. Let W_P be the projection of $S_F^{(v)}$ to the subspace given by (10.6), which we identify with $\mathbb{R}^{3-p_1-p_2-p_3}$. Since τ_v is just a rescaling of the coordinates,

$$V_P = \frac{\mathrm{vol}(W_P)}{(\mathfrak{N}\,\mathfrak{b}_6^{1-p_6}\,\mathfrak{N}\,\mathfrak{b}_7^{1-p_7}\,\mathfrak{N}\,\mathfrak{b}_8^{1-p_8})^{1/d}}. \tag{10.7}$$

For any $\mathbf{x}' = (x_j)_{\substack{j\in\{6,7,8\}\\ p_j=0}} \in W_P$, we consider the point $\mathbf{y}(\mathbf{x}') = (y_6, y_7, y_8)$ with

$$y_j := \begin{cases} x_j & \text{if } p_j = 0,\\ \mathfrak{N}\,\mathfrak{b}_j^{1/d} & \text{if } p_j = 1.\end{cases}$$

Then it is not hard to see from (10.1)–(10.4) that $\mathbf{y}(\mathbf{x}')$ is an element of the set $D \subseteq \mathbb{R}^3$ defined by the following conditions:

$$\mathfrak{N}\,\mathfrak{b}_6^{1/d} \ll |y_6|_v \ll c_6^{1/d}, \tag{10.8}$$

$$\mathfrak{N}\,\mathfrak{b}_7^{1/d} \ll |y_7|_v \ll \frac{c_7^{1/d}}{|y_6|_v^{1/2}}, \tag{10.9}$$

$$|y_8|_v \ll \frac{c_8^{1/d}}{|y_7|_v^{1/2}}. \tag{10.10}$$

In the following integrals, $\mathrm{d}y_j$ indicates the usual Lebesgue measure on $\mathbb{R}$ if $p_j = 0$, and the Dirac measure at the point $\mathfrak{N}\,\mathfrak{b}_j^{1/d}$ if $p_j = 1$. Then

$$\mathrm{vol}(W_P) \le \int_{\mathbf{y}(\mathbf{x}')\in D} \prod_{p_j=0} \mathrm{d}x_j = \int_{(10.8)-(10.10)} \mathrm{d}y_6\,\mathrm{d}y_7\,\mathrm{d}y_8.$$

If $P = (0,0,0)$, this implies $\mathrm{vol}(W_P) \ll c_8^{1/d} c_7^{1/(2d)} c_6^{3/(4d)}$, which, together with (10.7), proves the lemma in this case. Next, let $p_7 = p_8 = 0$ and $p_6 = 1$. Then

$$\mathrm{vol}(W_P) \le \int_{(\mathfrak{N}\,\mathfrak{b}_6^{1/d}, y_7, y_8)\in D} \mathrm{d}y_7\,\mathrm{d}y_8 \ll \frac{c_8^{1/d} c_7^{1/(2d)}}{\mathfrak{N}\,\mathfrak{b}_6^{1/(4d)}} \ll \frac{c_8^{1/d} c_7^{1/(2d)} c_6^{1/(4d)}}{\mathfrak{N}\,\mathfrak{b}_6^{1/(2d)}}.$$

Again, together with (10.7), this provides the desired bound. Now let us investigate the cases with $p_8 = 0$ and $p_7 = 1$. Here, with D_1 denoting the set of all $(y_6, y_7) \in \mathbb{R}^2$ that satisfy (10.8) and (10.9),

$$\begin{aligned}\mathrm{vol}(W_P) &\ll \frac{c_8^{1/d}}{\mathfrak{N}\,\mathfrak{b}_7^{1/(2d)}} \int_{(y_6, \mathfrak{N}\,\mathfrak{b}_7^{1/d})\in D_1} \mathrm{d}y_6 \ll \frac{c_8^{1/d} c_7^{1/(4d)}}{\mathfrak{N}\,\mathfrak{b}_7^{1/(2d)}} \int_{(10.8)} \frac{1}{|y_6|_v^{1/8}}\,\mathrm{d}y_6 \\ &\ll \frac{c_8^{1/d} c_7^{1/(4d)}}{\mathfrak{N}\,\mathfrak{b}_7^{1/(2d)}} \begin{cases} c_6^{7/(8d)} & \text{if } p_6 = 0, \\ \mathfrak{N}\,\mathfrak{b}_6^{-1/(8d)} & \text{if } p_6 = 1 \end{cases} \ll \frac{c_8^{1/d} c_7^{1/(4d)} c_6^{7/(8d)}}{\mathfrak{N}\,\mathfrak{b}_7^{1/(2d)}\,\mathfrak{N}\,\mathfrak{b}_6^{p_6/d}}.\end{aligned}$$

Finally, let us consider all P with $p_8 = 1$. We have

$$\mathrm{vol}(W_P) \ll \int_{(y_6, y_7, \mathfrak{N}\,\mathfrak{b}_8^{1/d})\in D} \mathrm{d}y_6\,\mathrm{d}y_7 \ll \frac{c_8^{1/(2d)}}{\mathfrak{N}\,\mathfrak{b}_8^{1/(2d)}} \int_{D_1} \frac{1}{|y_7|_v^{1/4}}\,\mathrm{d}y_6\,\mathrm{d}y_7.$$

For fixed y_6,

$$\int_{(10.9)} \frac{1}{|y_7|_v^{1/4}}\,\mathrm{d}y_7 \ll \begin{cases} (c_7^{1/d}/\,|y_6|_v^{1/2})^{3/4} & \text{if } p_7 = 0, \\ \mathfrak{N}\,\mathfrak{b}_7^{-1/(4d)} & \text{if } p_7 = 1 \end{cases} \ll \frac{c_7^{3/(4d)}}{\mathfrak{N}\,\mathfrak{b}_7^{p_7/d}\,|y_6|_v^{3/8}},$$

so

$$\begin{aligned}\mathrm{vol}(W_P) &\ll \frac{c_8^{1/(2d)} c_7^{3/(4d)}}{\mathfrak{N}\,\mathfrak{b}_8^{1/(2d)}\,\mathfrak{N}\,\mathfrak{b}_7^{p_7/d}} \int_{(10.8)} \frac{1}{|y_6|_v^{3/8}}\,\mathrm{d}y_6 \\ &\ll \frac{c_8^{1/(2d)} c_7^{3/(4d)}}{\mathfrak{N}\,\mathfrak{b}_8^{1/(2d)}\,\mathfrak{N}\,\mathfrak{b}_7^{p_7/d}} \begin{cases} c_6^{5/(8d)} & \text{if } p_6 = 0, \\ \mathfrak{N}\,\mathfrak{b}_6^{-3/(8d)} & \text{if } p_6 = 1 \end{cases} \ll \frac{c_8^{1/(2d)} c_7^{3/(4d)} c_6^{5/(8d)}}{\mathfrak{N}\,\mathfrak{b}_8^{1/(2d)}\,\mathfrak{N}\,\mathfrak{b}_7^{p_7/d}\,\mathfrak{N}\,\mathfrak{b}_6^{p_6/d}}.\end{aligned}$$ □

10.2. Complex places

Now, we consider $\tau_v(S_F^{(v)})$ for complex places v. Then $d_v = 2$ and $K_v^3 = \mathbb{C}^3$, which we identify with $\mathbb{R}^6$. The following lemma and its proof are similar to the real case, but more complicated. Recall that $|\cdot|_v = |\cdot|^2$ on $\mathbb{C}$.

LEMMA 10.3. – *Let $v \in \Omega_\infty$ be a complex place. For any $P = (p_6, p_7, p_8) \in \{0,1,2\}^3$, let V_P be the $(6-(p_6+p_7+p_8))$-dimensional volume of the orthogonal projection of $\tau_v(S_F^{(v)})$ to one of the coordinate subspaces of $\mathbb{C}^3 = \mathbb{R}^6$ given as follows: for every $j \in \{6,7,8\}$ with $p_j = 1$, we take one of the equations*

$$\Re x_j = 0 \quad \text{or} \quad \Im x_j = 0,$$

and for every $j \in \{6,7,8\}$ with $p_j = 2$, we take the equation

$$x_j = 0$$

to define the coordinate subspace. Then

$$V_P \ll \begin{cases} s_0^{2/d} & \text{if } p_6 = p_7 = p_8 = 0, \\ s_6^{2/d} & \text{if } p_6 \in \{1,2\},\ p_7 = p_8 = 0, \\ s_7^{2/d} & \text{if } p_7 \in \{1,2\},\ p_8 = 0, \\ s_8^{2/d} & \text{if } p_8 \in \{1,2\}. \end{cases}$$

Proof. – Again, we may assume that $S_F^{(v)} \neq \varnothing$. Clearly, $\tau_v(S_F^{(v)})$ is invariant with respect to swapping real and imaginary parts, so it suffices to consider projections to $\Im x_j = 0$ and $x_j = 0$. Then every $P \in \{0,1,2\}^3$ describes a unique coordinate subspace. Let W_P be the projection of $S_F^{(v)}$ to this subspace, which we identify with $\mathbb{R}^{6-p_6-p_7-p_8}$. Then

$$V_P = \frac{\mathrm{vol}(W_P)}{(\mathfrak{N}\,\mathfrak{b}_6^{2-p_6}\,\mathfrak{N}\,\mathfrak{b}_7^{2-p_7}\,\mathfrak{N}\,\mathfrak{b}_8^{2-p_8})^{1/d}}. \tag{10.11}$$

Let $\mathbf{x}' = (x'_j)_{\substack{j\in\{6,7,8\}\\ p_j\in\{0,1\}}} \in W_P$, i.e., there is an element $(x_6, x_7, x_8) \in S_F^{(v)}$ with

$$x'_j = \begin{cases} x_j \in \mathbb{C} = \mathbb{R}^2 & \text{if } p_j = 0, \\ \Re x_j \in \mathbb{R} & \text{if } p_j = 1. \end{cases}$$

Similarly as in the real case, we consider the point $\mathbf{y}(\mathbf{x}') = (y_6, y_7, y_8)$ with

$$y_j := \begin{cases} x'_j & \text{if } p_j \in \{0,1\}, \\ \mathfrak{N}\,\mathfrak{b}_j^{1/d} & \text{if } p_j = 2. \end{cases}$$

Let $K_j := \mathbb{C}$ if $p_j = 0$ and $K_j := \mathbb{R}$ if $p_j \in \{1,2\}$. Then $\mathbf{y}(\mathbf{x}')$ is an element of the subset $D \subseteq K_6 \times K_7 \times K_8$ defined by

$$\max\{\mathfrak{N}\,\mathfrak{b}_6^{2/d}, |y_6|_v\} \ll c_6^{2/d}, \tag{10.12}$$

$$\max\{\mathfrak{N}\,\mathfrak{b}_7^{2/d}, |y_7|_v\} \ll \frac{c_7^{2/d}}{\max\{\mathfrak{N}\,\mathfrak{b}_6^{1/d}, |y_6|_v^{1/2}\}}, \tag{10.13}$$

$$|y_8|_v \ll \frac{c_8^{2/d}}{\max\{\mathfrak{N}\,\mathfrak{b}_7^{1/d}, |y_7|_v^{1/2}\}}. \tag{10.14}$$

In the following integrals, $\mathrm{d}y_j$ indicates the Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$ if $p_j = 0$, the Lebesgue measure on $\mathbb{R}$ if $p_j = 1$, and the Dirac measure on $\mathbb{R}$ at the point $\mathfrak{N}\,\mathfrak{b}_j^{1/d}$ if $p_j = 2$. Then

$$\mathrm{vol}(W_P) \leq \int_{\mathbf{y}(\mathbf{x}')\in D} \prod_{p_j \in\{0,1\}} \mathrm{d}x'_j = \int_{(10.12)-(10.14)} \mathrm{d}y_6\,\mathrm{d}y_7\,\mathrm{d}y_8.$$

As in Lemma 10.2, we consider first the trivial case $P = (0,0,0)$. Using polar coordinates, we see that $\mathrm{vol}(W_P) \ll (c_8 c_7^{1/2} c_6^{3/4})^{2/d}$, which, together with (10.11), proves the lemma in this case. Next, let $p_7 = p_8 = 0$ and $p_6 \in \{1,2\}$. Here we obtain

$$\begin{aligned}\mathrm{vol}(W_P) &\ll (c_8 c_7^{1/2})^{2/d} \int_{(10.12)} \frac{1}{|y_6|_v^{1/4}}\,\mathrm{d}y_6 \\ &\ll (c_8 c_7^{1/2})^{2/d} \begin{cases} c_6^{1/(2d)} & \text{if } p_6 = 1, \\ \mathfrak{N}\,\mathfrak{b}_6^{-1/(2d)} & \text{if } p_6 = 2 \end{cases} \ll \frac{(c_8 c_7^{1/2} c_6^{1/4})^{2/d}}{\mathfrak{N}\,\mathfrak{b}_6^{(p_6-1)/d}}.\end{aligned}$$

The last estimate holds by (10.12). Together with (10.11), this provides the desired bound.

Now let us investigate all P with $p_8 = 0$ and $p_7 \in \{1,2\}$. Then

$$\mathrm{vol}(W_P) \ll c_8^{2/d} \int_{(10.12),(10.13)} \frac{1}{\max\{\mathfrak{N}\,\mathfrak{b}_7^{1/d}, |y_7|_v^{1/2}\}}\,\mathrm{d}y_6\,\mathrm{d}y_7.$$

For any y_6 with $c_7^{2/d}/\max\{\mathfrak{N}\,\mathfrak{b}_6^{1/d}, |y_6|_v^{1/2}\} \gg 1$, we have

$$\begin{aligned}&\int_{(10.13)} \frac{\mathrm{d}y_7}{\max\{\mathfrak{N}\,\mathfrak{b}_7^{1/d}, |y_7|_v^{1/2}\}} \ll \begin{cases} \log(c_7^{2/d}/\max\{\mathfrak{N}\,\mathfrak{b}_6^{1/d}, |y_6|_v^{1/2}\} + 2) & \text{if } p_7 = 1 \\ \mathfrak{N}\,\mathfrak{b}_7^{-1/d} & \text{if } p_7 = 2 \end{cases} \\ &\ll \frac{1}{\mathfrak{N}\,\mathfrak{b}_7^{(p_7-1)/d}} \left(\frac{c_7^{2/d}}{\max\{\mathfrak{N}\,\mathfrak{b}_6^{1/d}, |y_6|_v^{1/2}\}} \right)^{1/4} \ll \frac{c_7^{1/(2d)}}{\mathfrak{N}\,\mathfrak{b}_7^{(p_7-1)/d}\,\mathfrak{N}\,\mathfrak{b}_6^{p_6/(8d)}} \frac{1}{|y_6|_v^{(2-p_6)/16}}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{vol}(W_P) &\ll \frac{(c_8 c_7^{1/4})^{2/d}}{\mathfrak{N}\,\mathfrak{b}_7^{(p_7-1)/d}\,\mathfrak{N}\,\mathfrak{b}_6^{p_6/(8d)}} \int_{(10.12)} \frac{1}{|y_6|_v^{(2-p_6)/16}}\,\mathrm{d}y_6 \\ &\ll \frac{(c_8 c_7^{1/4})^{2/d}}{\mathfrak{N}\,\mathfrak{b}_7^{(p_7-1)/d}\,\mathfrak{N}\,\mathfrak{b}_6^{p_6/(8d)}} \begin{cases} c_6^{7/(4d)} & \text{if } p_6 = 0, \\ c_6^{7/(8d)} & \text{if } p_6 = 1, \\ 1 & \text{if } p_6 = 2 \end{cases} \ll \frac{(c_8 c_7^{1/4} c_6^{7/8})^{2/d}}{\mathfrak{N}\,\mathfrak{b}_7^{(p_7-1)/d}\,\mathfrak{N}\,\mathfrak{b}_6^{p_6/d}}.\end{aligned}$$

Finally, let us consider all P with $p_8 \in \{1,2\}$. We have

$$\int_{(10.14)} \mathrm{d}y_8 \ll \begin{cases} (c_8^{2/d}/\max\{\mathfrak{N}\,\mathfrak{b}_7^{1/d}, |y_7|_v^{1/2}\})^{1/2} & \text{if } p_8 = 1, \\ 1 & \text{if } p_8 = 2 \end{cases}$$
$$\ll \frac{c_8^{1/d}}{\mathfrak{N}\,\mathfrak{b}_8^{(p_8-1)/d}\max\{\mathfrak{N}\,\mathfrak{b}_7^{1/d}, |y_7|_v^{1/2}\}^{1/2}} \ll \frac{c_8^{1/d}}{\mathfrak{N}\,\mathfrak{b}_8^{(p_8-1)/d}\,N\mathfrak{b}_7^{p_7/(4d)}} \cdot \frac{1}{|y_7|_v^{(2-p_7)/8}},$$

due to (10.14). Then

$$\int_{(10.13)} \frac{1}{|y_7|_v^{(2-p_7)/8}}\,\mathrm{d}y_7 \ll \begin{cases} (c_7^{2/d}/\max\{\mathfrak{N}\,\mathfrak{b}_6^{1/d}, |y_6|_v^{1/2}\})^{3/4} & \text{if } p_7 = 0, \\ (c_7^{2/d}/\max\{\mathfrak{N}\,\mathfrak{b}_6^{1/d}, |y_6|_v^{1/2}\})^{3/8} & \text{if } p_7 = 1, \\ 1 & \text{if } p_7 = 2 \end{cases}$$
$$\ll \frac{c_7^{3/(2d)}}{\mathfrak{N}\,\mathfrak{b}_7^{3p_7/(4d)}\max\{\mathfrak{N}\,\mathfrak{b}_6^{1/d}, |y_6|_v^{1/2}\}^{3/4}} \ll \frac{c_7^{3/(2d)}}{\mathfrak{N}\,\mathfrak{b}_7^{3p_7/(4d)}\,\mathfrak{N}\,\mathfrak{b}_6^{3p_6/(8d)}} \cdot \frac{1}{|y_6|_v^{3(2-p_6)/16}},$$

by (10.13). Hence,

$$\mathrm{vol}(W_P) \ll \frac{(c_8^{1/2}c_7^{3/4})^{2/d}}{\mathfrak{N}\,\mathfrak{b}_8^{(p_8-1)/d}\,\mathfrak{N}\,\mathfrak{b}_7^{p_7/d}\,\mathfrak{N}\,\mathfrak{b}_6^{3p_6/(8d)}} \int_{(10.12)} \frac{1}{|y_6|_v^{3(2-p_6)/16}}\,\mathrm{d}y_6$$
$$\ll \frac{(c_8^{1/2}c_7^{3/4})^{2/d}}{\mathfrak{N}\,\mathfrak{b}_8^{\frac{p_8-1}{d}}\,\mathfrak{N}\,\mathfrak{b}_7^{\frac{p_7}{d}}\,\mathfrak{N}\,\mathfrak{b}_6^{\frac{3p_6}{8d}}} \begin{cases} c_6^{5/(4d)} & \text{if } p_6 = 0, \\ c_6^{5/(8d)} & \text{if } p_6 = 1, \\ 1 & \text{if } p_6 = 2 \end{cases} \ll \frac{(c_8^{1/2}c_7^{3/4}c_6^{5/8})^{2/d}}{\mathfrak{N}\,\mathfrak{b}_8^{\frac{p_8-1}{d}}\,\mathfrak{N}\,\mathfrak{b}_7^{\frac{p_7}{d}}\,\mathfrak{N}\,\mathfrak{b}_6^{\frac{p_6}{d}}}. \qquad \square$$

Next, we use the bounds from Lemma 10.2 and Lemma 10.3 to show that the sum over all $\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}$ of the error term in Lemma 10.1 is sufficiently small. We have already seen in Lemma 7.6 that it suffices to sum over all $\underline{a}'$ with (7.12). Moreover, it is clearly enough to sum the error term over all $\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}$ with

$$S_F^*(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; u_{\underline{\mathfrak{c}}}B) \neq \varnothing, \tag{10.15}$$

since otherwise $|\mathcal{G} \cap \mathcal{F}_0^*|$ and the main term are both 0.

Lemma 10.4. – *Let $\underline{\mathfrak{c}} \in \mathcal{C}^6$. Let W be a proper coordinate subspace of $\prod_{v\in\Omega_\infty} K_v^3 = \mathbb{R}^{3d}$, and let $V_W(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; u_{\underline{\mathfrak{c}}}B)$ be the* $\dim(W)$*-dimensional volume of the orthogonal projection of $\tau(S_F^{\varnothing}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; u_{\underline{\mathfrak{c}}}B))$ to W. For $\epsilon > 0$, we have*

$$\sum_{\substack{\underline{a}' \in \mathcal{F}_1^5 \cap \mathcal{O}_*' \\ (7.12)}} \theta_0(\underline{\mathfrak{a}}') \sum_{\substack{\underline{\mathfrak{d}} \\ (6.8),(6.9) \\ (10.15)}} |\mu_K(\underline{\mathfrak{d}})| \cdot V_W(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; u_{\underline{\mathfrak{c}}}B) \ll_\epsilon B(\log B)^{5-1/d+\epsilon}. \tag{10.16}$$

Proof. – For $i \in \{0,6,7,8\}$, let

$$\Sigma_i := \sum_{\substack{\underline{a}' \in \mathcal{F}_1^5 \cap \mathcal{O}_*' \\ (7.12)}} \sum_{\substack{\underline{\mathfrak{d}} \\ (6.8),(6.9) \\ (10.15)}} |\mu_K(\underline{\mathfrak{d}})| \cdot s_i.$$

Let us start by estimating the Σ_i from above. Condition (10.15) has the consequences

$$N(a_j) \ll B \text{ for all } i, j, \tag{10.17}$$

$$N(a_1^2 a_2^2 a_3^2 a_4) \leq \mathfrak{N}(\mathcal{O}_1^2 \mathcal{O}_2^2 \mathcal{O}_3^2 \mathcal{O}_4) B, \tag{10.18}$$
$$N(a_1^2 a_2^2) \ll N(a_4 a_5^3 a_6^2) B.$$

Using these and (7.12) with Lemma 7.3, we see that $\Sigma_i \ll_\epsilon B(\log B)^{4+\epsilon}$ holds for $i \in \{6,7,8\}$. A simple computation using just (10.17) shows that $\Sigma_0 \ll_\epsilon B(\log B)^{5+\epsilon}$.

Now let us prove (10.16). By (10.5), the projection of $\tau(S_F^{\varnothing}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; u_{\underline{\mathfrak{c}}} B))$ to W is contained in a product of projections of $\tau_v(S_F^{(v)}(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; u_{\underline{\mathfrak{c}}} B))$ to subspaces of K_v^3. The volume of each such projection is bounded by an $s_{i(v)}^{d_v/d}$, so

$$V_W(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; u_{\underline{\mathfrak{c}}} B) \leq \prod_{v \in \Omega_\infty} s_{i(v)}^{d_v/d}.$$

Since W is a proper subspace of $\prod_{v \in \Omega_\infty} K_v^3$, there is at least one $v \in \Omega_\infty$ with $i(v) \neq 0$. Using Hölder's inequality, we see that the sum in (10.16) is

$$\leq \sum_{\substack{\underline{a}' \in \mathcal{F}_1^5 \cap \mathcal{O}'_* \\ (7.12)}} \sum_{\substack{\underline{\mathfrak{d}} \\ (6.8),(6.9) \\ (10.15)}} |\mu_K(\underline{\mathfrak{d}})| \prod_{v \in \Omega_\infty} s_{i(v)}^{d_v/d} \leq \prod_{v \in \Omega_\infty} \Sigma_{i(v)}^{d_v/d}$$
$$\ll_\epsilon (B(\log B)^{5+\epsilon})^{(d-1)/d} (B(\log B)^{4+\epsilon})^{1/d} = B(\log B)^{5-1/d+\epsilon}. \qquad \square$$

11. Completion of the first summation

We have already seen in Lemma 7.6 that we may restrict ourselves to $\underline{\mathfrak{c}}, \underline{a}'$ with (7.12). Moreover, $S_F^*(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; u_{\underline{\mathfrak{c}}} B) = \varnothing$ unless (10.18) holds.

Let us first show that, under these conditions, $S_F^*(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; u_{\underline{\mathfrak{c}}} B)$ is not much smaller than $S_F(\underline{a}'; u_{\underline{\mathfrak{c}}} B)$, whose volume we have already computed in Lemma 5.1.

LEMMA 11.1. – *Let $\underline{\mathfrak{c}} \in \mathcal{C}^6$ and $\epsilon > 0$. Then*

$$\sum_{\substack{\underline{a}' \in \mathcal{F}_1^5 \cap \mathcal{O}'_* \\ (7.12),(10.18)}} \sum_{\substack{\underline{\mathfrak{d}} \\ (6.8),(6.9)}} |\mu_K(\underline{\mathfrak{d}})| \cdot \frac{\mathrm{vol}(S_F(\underline{a}'; u_{\underline{\mathfrak{c}}} B) \smallsetminus S_F^*(\underline{\mathfrak{c}}, \underline{a}', \underline{\mathfrak{d}}; u_{\underline{\mathfrak{c}}} B))}{\mathfrak{N}(\mathfrak{b}_6 \mathfrak{b}_7 \mathfrak{b}_8)} \ll_\epsilon B(\log B)^{4+\epsilon}.$$

Proof. – For $w \in \Omega_\infty$ and fixed $\underline{a}', \underline{\mathfrak{d}}, u_{\underline{\mathfrak{c}}} B$, let $V_w^{(6)}$ (resp. $V_w^{(7)}$) be the volume of the subset of $S_F(\underline{a}'; u_{\underline{\mathfrak{c}}} B)$ where $|x_{6w}|_w < \mathfrak{N}\, \mathfrak{b}_6^{d_w/d}$ (resp. $|x_{7w}|_w < \mathfrak{N}\, \mathfrak{b}_7^{d_w/d}$). Let

$$R_w := \prod_{\substack{v \in \Omega_\infty \\ v \neq w}} \int_{\tilde{N}_v(\underline{a}'; x_{6v}, x_{7v}, x_{8v}) \ll B^{d_v/d}} \mathrm{d}x_{6v}\, \mathrm{d}x_{7v}\, \mathrm{d}x_{8v}.$$

Then (5.3) implies that, for $j \in \{6, 7\}$,

$$V_w^{(j)} \ll R_w \cdot \int_{\substack{\tilde{N}_w(\underline{a}'; x_{6w}, x_{7w}, x_{8w}) \ll B^{d_w/d} \\ |a_{jw}|_w < \mathfrak{N}\, \mathfrak{b}_j^{d_w/d}}} \mathrm{d}x_{6w}\, \mathrm{d}x_{7w}\, \mathrm{d}x_{8w}.$$

Let us bound R_w. For each $v \neq w$, we use the last term in the maximum in $\tilde{N}_v(\underline{a}'; x_{6v}, x_{7v}, x_{8v})$ to bound the integral over x_{7v}, x_{8v} (cf. [20, Lemma 5.1, (5)] for real v and [22, Lemma 3.4, (4)] for complex v) and the second term in the maximum to bound the integral over x_{6v}. With (5.2), this leads to

$$R_w \ll \left(\frac{B}{N(a_2a_3a_4a_5)}\right)^{1-d_w/d}.$$

Recall that $\tilde{N}_w(\underline{a}'; x_{6w}, x_{7w}, x_{8w}) \ll B^{d_w/d}$ implies

$$|x_{7w}|_w \ll \left(\frac{B}{N(a_3^2a_4^3a_5^4)}\right)^{d_w/(2d)} \cdot \frac{1}{|x_{6w}|_w^{1/2}}.$$

Using the first term in the minimum in [20, Lemma 5.1, (4)] resp. [22, Lemma 3.4, (2)] to bound the integral over x_{8w} and the above inequality to bound the integral over x_{7w}, we obtain

$$\int_{\substack{\tilde{N}_w(\underline{a}'; x_{6w}, x_{7w}, x_{8w}) \ll B^{d_w/d} \\ |a_{6w}|_w < \mathfrak{N}\,\mathfrak{b}_6^{d_w/d}}} \mathrm{d}x_{6w}\,\mathrm{d}x_{7w}\,\mathrm{d}x_{8w} \ll \mathfrak{N}\,\mathfrak{b}_6^{\frac{3d_w}{4d}} \left(\frac{B}{N(a_3^2a_4^3a_5^4)}\right)^{\frac{d_w}{4d}} \left(\frac{N(a_1)B}{N(a_2)}\right)^{\frac{d_w}{2d}}.$$

Therefore,

$$\frac{V_w^{(6)}}{\mathfrak{N}(\mathfrak{b}_6\mathfrak{b}_7\mathfrak{b}_8)} \ll \frac{1}{\mathfrak{N}\,\mathfrak{b}_6^{1-3d_w/(4d)}\,\mathfrak{N}(\mathfrak{b}_7\mathfrak{b}_8)} \cdot \frac{B}{N(a_2a_3a_4a_5)} \cdot \left(\frac{B}{N(a_1^2a_2^2a_3^2a_4)}\right)^{-d_w/(4d)}.$$

We sum this over $\underline{a}', \underline{\mathfrak{d}}$ with Lemma 7.3. Similarly,

$$\int_{\substack{\tilde{N}_w(\underline{a}'; x_{6w}, x_{7w}, x_{8w}) \ll B^{d_w/d} \\ |x_{7w}|_w < \mathfrak{N}\,\mathfrak{b}_7^{d_w/d}}} \mathrm{d}x_{6w}\,\mathrm{d}x_{7w}\,\mathrm{d}x_{8w} \ll \left(\frac{B}{N(a_1^2a_2^2a_3^2a_4)}\right)^{\frac{d_w}{3d}} \mathfrak{N}\,\mathfrak{b}_7^{\frac{d_w}{2d}} \left(\frac{BN(a_1)}{N(a_2)}\right)^{\frac{d_w}{2d}},$$

and thus

$$\frac{V_w^{(7)}}{\mathfrak{N}(\mathfrak{b}_6\mathfrak{b}_7\mathfrak{b}_8)} \ll \frac{1}{\mathfrak{N}\,\mathfrak{b}_7^{1-d_w/(2d)}\,\mathfrak{N}(\mathfrak{b}_6\mathfrak{b}_8)} \cdot \frac{B}{N(a_2a_3a_4a_5)} \cdot \left(\frac{N(a_1a_2)B}{N(a_3^2a_4^4a_5^6)}\right)^{-d_w/(6d)}.$$

Again, we sum this up using Lemma 7.3. □

In the rest of this article, a product over $\mathfrak{p}$ runs over non-zero prime ideals of $\mathcal{O}_K$.

Let $\theta_1(\underline{\mathfrak{a}}')$ be the arithmetic function defined by $\theta_1(\underline{\mathfrak{a}}') := \prod_{\mathfrak{p}} \theta_{1,\mathfrak{p}}(J_{\mathfrak{p}}(\underline{\mathfrak{a}}'))$, where $J_{\mathfrak{p}}(\underline{\mathfrak{a}}') := \{j \in \{1,\dots,5\} : \mathfrak{p} \mid \mathfrak{a}_j\}$ and

$$\theta_{1,\mathfrak{p}}(J) := \begin{cases} \left(1-\frac{1}{\mathfrak{N}\,\mathfrak{p}}\right)^2\left(1+\frac{2}{\mathfrak{N}\,\mathfrak{p}}\right) & \text{if } J = \varnothing, \\ \left(1-\frac{1}{\mathfrak{N}\,\mathfrak{p}}\right)^2\left(1+\frac{1}{\mathfrak{N}\,\mathfrak{p}}\right) & \text{if } J = \{1\}, \{2\}, \\ \left(1-\frac{1}{\mathfrak{N}\,\mathfrak{p}}\right)^2 & \text{if } J = \{3\}, \{5\}, \\ \left(1-\frac{1}{\mathfrak{N}\,\mathfrak{p}}\right)^3 & \text{if } J = \{4\}, \{3,4\}, \{4,5\}, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 11.2. – *For any* $\underline{\mathfrak{a}}' \in \mathcal{I}_K^5$*, we have*

$$\theta_0(\underline{\mathfrak{a}}') \sum_{\substack{\underline{\mathfrak{d}} \\ (6.8),(6.9)}} \frac{\mu_K(\underline{\mathfrak{d}})}{\mathfrak{N}(\mathfrak{d}_6\mathfrak{d}_7\mathfrak{d}_8\mathfrak{d}_{67}\mathfrak{d}_{68}\mathfrak{d}_{69}(\mathfrak{d}_{67}\cap\mathfrak{d}_{68}\mathfrak{d}_{69}))} = \theta_1(\underline{\mathfrak{a}}').$$

Proof. – For any ideal $\mathfrak{a} \in \mathcal{I}_K$, let $\phi_K^*(\mathfrak{a}) := \prod_{\mathfrak{p}|\mathfrak{a}}(1 + 1/\mathfrak{N}\mathfrak{p})$. For fixed $\underline{\mathfrak{a}}'$, we have

$$\sum_{\substack{\mathfrak{d}_6,\mathfrak{d}_7,\mathfrak{d}_8 \\ (6.9)}} \frac{\mu_K(\mathfrak{d}_6)\mu_K(\mathfrak{d}_7)\mu_K(\mathfrak{d}_8)}{\mathfrak{N}(\mathfrak{d}_6\mathfrak{d}_7\mathfrak{d}_8)} = \phi_K^*(\mathfrak{a}_4\mathfrak{a}_5)\phi_K^*(\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4)\phi_K^*(\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5),$$

and

$$\sum_{\substack{\mathfrak{d}_{67},\mathfrak{d}_{68},\mathfrak{d}_{69} \\ (6.8)}} \frac{\mu_K(\mathfrak{d}_{67})\mu_K(\mathfrak{d}_{68})\mu_K(\mathfrak{d}_{69})}{\mathfrak{N}(\mathfrak{d}_{67}\mathfrak{d}_{68}\mathfrak{d}_{69}(\mathfrak{d}_{67}\cap\mathfrak{d}_{68}\mathfrak{d}_{69}))}$$
$$= \prod_{\substack{\mathfrak{p}|\mathfrak{a}_1 \\ \mathfrak{p}\nmid\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}^2}\right) \prod_{\substack{\mathfrak{p}|\mathfrak{a}_2 \\ \mathfrak{p}\nmid\mathfrak{a}_1\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}^2}\right) \prod_{\mathfrak{p}\nmid\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4\mathfrak{a}_5} \left(1 - \frac{3}{\mathfrak{N}\mathfrak{p}^2} + \frac{2}{\mathfrak{N}\mathfrak{p}^3}\right).$$

A simple comparison of Euler factors proves the lemma. □

LEMMA 11.3. – *Let* $\epsilon > 0$*. Then, for* $B \geq 3$*,*

$$N_{U,H}(B) = \frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{3\cdot|\mu_K|\cdot|\Delta_K|^{3/2}} \left(\prod_{v\in\Omega_\infty}\omega_v(\widetilde{S})\right) \sum_{\substack{\underline{\mathfrak{a}}'\in\mathcal{I}_K^5 \\ (11.1)}} \frac{\theta_1(\underline{\mathfrak{a}}')B}{\mathfrak{N}(\mathfrak{a}_1\cdots\mathfrak{a}_5)}$$
$$+ O_\epsilon(B(\log B)^{5-1/d+\epsilon}),$$

where the sum runs over all 5*-tuples of ideals* $\underline{\mathfrak{a}}' = (\mathfrak{a}_1, \ldots, \mathfrak{a}_5) \in \mathcal{I}_K^5$ *satisfying*

(11.1) $$\mathfrak{N}(\mathfrak{a}_3^2\mathfrak{a}_4^4\mathfrak{a}_5^6) \leq \mathfrak{N}(\mathfrak{a}_1\mathfrak{a}_2)B \quad \textit{and} \quad \mathfrak{N}(\mathfrak{a}_1^2\mathfrak{a}_2^2\mathfrak{a}_3^2\mathfrak{a}_4) \leq B.$$

Proof. – By Lemma 4.3, Lemma 7.6, Lemma 10.1, Lemma 10.4, Lemma 11.1, and Lemma 5.1, the quantity $N_{U,H}(B)$ is

(11.2)
$$\frac{2^{r_1}(2\pi)^{r_2}R_K}{3\cdot|\mu_K|\cdot|\Delta_K|^{3/2}} \left(\prod_{v\in\Omega_\infty}\omega_v(\widetilde{S})\right) \sum_{\underline{\mathfrak{c}}\in\mathcal{C}^6} \sum_{\substack{\underline{a}'\in\mathcal{F}_1^5\cap\mathcal{O}_*' \\ (7.12),(10.18)}} \theta_0(\underline{\mathfrak{a}}') \sum_{\substack{\underline{\mathfrak{d}} \\ (6.8),(6.9)}} \frac{\mu_K(\underline{\mathfrak{d}})u_{\underline{\mathfrak{c}}}B}{\mathfrak{N}(\mathfrak{b}_6\mathfrak{b}_7\mathfrak{b}_8)N(a_2a_3a_4a_5)}$$
$$+ O_\epsilon(B(\log B)^{5-1/d+\epsilon}).$$

From the definitions of the $\mathfrak{b}_j, \mathcal{O}_j$, we see that

$$\frac{\mu_K(\underline{\mathfrak{d}})u_{\underline{\mathfrak{c}}}B}{\mathfrak{N}(\mathfrak{b}_6\mathfrak{b}_7\mathfrak{b}_8)N(a_2a_3a_4a_5)} = \frac{B}{\mathfrak{N}(\mathfrak{a}_1\cdots\mathfrak{a}_5)} \cdot \frac{\mu_K(\underline{\mathfrak{d}})}{\mathfrak{N}(\mathfrak{d}_6\mathfrak{d}_7\mathfrak{d}_8\mathfrak{d}_{67}\mathfrak{d}_{68}\mathfrak{d}_{69}(\mathfrak{d}_{67}\cap\mathfrak{d}_{68}\mathfrak{d}_{69}))}.$$

We evaluate the sum over $\underline{\mathfrak{d}}$ by Lemma 11.2. Moreover, we observe that the $\mathcal{O}_j$, $j \in \{1, \ldots, 5\}$, are independent from $\mathfrak{c}_0$. Hence, the main term in (11.2) is

$$\frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{3\cdot|\mu_K|\cdot|\Delta_K|^{3/2}} \left(\prod_{v\in\Omega_\infty}\omega_v(\widetilde{S})\right) \sum_{\substack{(\mathfrak{c}_1,\ldots,\mathfrak{c}_5) \\ \in\mathcal{C}^5}} \sum_{\substack{\underline{a}'\in\mathcal{F}_1^5\cap\mathcal{O}_*' \\ (7.12),(10.18)}} \frac{\theta_1(\underline{\mathfrak{a}}')B}{\mathfrak{N}(\mathfrak{a}_1\cdots\mathfrak{a}_5)}.$$

When $(\mathfrak{c}_1, \dots, \mathfrak{c}_5)$ runs through all of $\mathcal{C}^5$ then $([\mathcal{O}_1], \dots, [\mathcal{O}_5])$ runs through all 5-tuples of ideal classes. When a_j runs through $\mathcal{F}_1 \cap \mathcal{O}_{j*}$ then $\mathfrak{a}_j = a_j \mathcal{O}_j^{-1}$ runs through all integral ideals in the class $[\mathcal{O}_j]$. Furthermore, the $\mathfrak{a}_j$ satisfy (11.1) if and only if the a_j satisfy (7.12) and (10.18). $\square$

12. The remaining summations

All that remains to be done is the evaluation of the sum over $\underline{\mathfrak{a}}'$ in Lemma 11.3. We proceed as in the proof of [22, Proposition 7.3], except that we start at $r = 5$ instead of $r+1$. Using [22, Proposition 7.2] inductively, we see that

$$\sum_{\substack{\underline{\mathfrak{a}}' \in \mathcal{I}_K^5 \\ (11.1)}} \frac{\theta_1(\underline{\mathfrak{a}}') B}{\mathfrak{N}(\mathfrak{a}_1 \cdots \mathfrak{a}_5)} = \left(\frac{2^{r_1} (2\pi)^{r_2} R_K h_K}{|\mu_K| \sqrt{|\Delta_K|}} \right)^5 \theta_0 V_0(B) + O(B (\log B)^4 \log\log B),$$

where

$$\theta_0 := \mathcal{A}(\theta_1(\underline{\mathfrak{a}}'), \mathfrak{a}_5, \dots, \mathfrak{a}_1)$$

is the iterated "mean value" of $\theta_1(\underline{\mathfrak{a}})$ as defined in [22, Section 2], and

$$V_0(B) := \int_{\substack{t_1, \dots, t_5 \geq 1 \\ t_1^2 t_2^2 t_3^2 t_4 \leq B \\ t_3^2 t_4^4 t_5^6 \leq t_1 t_2 B}} \frac{B}{t_1 \cdots t_5} \, \mathrm{d}t_1 \cdots \mathrm{d}t_5.$$

By [22, Lemma 2.8], we compute

$$\theta_0 = \prod_{\mathfrak{p}} \left(1 - \frac{1}{\mathfrak{N}\mathfrak{p}}\right)^6 \left(1 + \frac{6}{\mathfrak{N}\mathfrak{p}} + \frac{1}{\mathfrak{N}\mathfrak{p}^2}\right).$$

Let $\alpha(\widetilde{S})$ be the constant defined, for example, in [45, Définition 4.8]. We evaluate $V_0(B)$ in terms of $\alpha(\widetilde{S})$. The negative curves $[E_1], \dots, [E_7]$ generate the effective cone of $\widetilde{S}$, and $[-K_{\widetilde{S}}] = [2E_1 + 2E_2 + 2E_3 + E_4 + 3E_6]$, $[E_7] = [E_1 + E_2 - E_4 - 2E_5 + E_6]$. As in the proof of [22, Lemma 8.1], we see that

$$\alpha(\widetilde{S}) = \frac{1}{3} \int_{\substack{x_1, \dots, x_5 \geq 0 \\ 2x_1 + 2x_2 + 2x_3 + x_4 \leq 1 \\ -x_1 - x_2 + 2x_3 + 4x_4 + 6x_5 \leq 1}} \mathrm{d}x_1 \cdots \mathrm{d}x_5.$$

We substitute $t_i = B^{x_i}$ to obtain

$$3\alpha(\widetilde{S}) \cdot B (\log B)^5 = V_0(B).$$

Let us compute the numerical value of $\alpha(\widetilde{S})$. By [25, Theorem 1.3], we have

$$\alpha(\widetilde{S}) = \frac{\alpha(S_0)}{|W(R)|} = \frac{1}{8640},$$

where S_0 is a split smooth del Pezzo surface of degree 4 (with $\alpha(S_0) = 1/180$ by [19, Theorem 4]) and $|W(R)| = 2 \cdot (3 + 1)!$ is the order of the Weyl group of the root system $\mathbf{A}_3 + \mathbf{A}_1$ associated to the singularities of S.

Together with Lemma 11.3, this shows the asymptotic formula in Theorem 1.1, with the constant $c_{S,H}$ described in Subsection 1.4.

13. The leading constant

Let $\gamma : \widetilde{S} \to S$ be the minimal desingularization defined in Section 4. Neither S nor $\widetilde{S}$ are Fano, so the original conjectures of Manin [28, 4] and Peyre [43] do not apply. However, [43, Remarque 2.3.2] already suggested generalizations of Manin's conjecture that cover $\widetilde{S}$, and such a generalization was formulated, for example, by Batyrev and Tschinkel [6]. For our purpose, Peyre's variant [45] is the most convenient. Indeed, $\widetilde{S}$ satisfies [45, Hypothèses 3.3], so we can compare our result to the formula [45, Formule empirique 5.1].

Let S_{reg} be the smooth locus of S, that is, the complement of the rational points $(0:0:0:0:1)$ and $(0:1:0:0:0)$, and let U be as in Theorem 1.1. As we have already observed in the proof of Lemma 4.3,

$$N_{U,H}(B) = |\{x \in \gamma^{-1}(U)(K) \mid H(\gamma(x)) \leq B\}|.$$

We construct an adelic metric $(\|\cdot\|_v)_{v\in\Omega_K}$ on the anticanonical line bundle $\omega_{\widetilde{S}}^{-1}$ in the sense of [45] such that the Arakelov height $(\omega_{\widetilde{S}}^{-1}, (\|\cdot\|_v)_{v\in\Omega_K})$ induces the height function $H \circ \gamma$ on $\widetilde{S}(K)$. We start by relating the canonical sheaves of S and $\widetilde{S}$ to each other. The surface S is a complete intersection defined in $\mathbb{P}^4_K$ by the forms $h_1 := x_0x_3 - x_2x_4$, $h_2 := x_0x_1 + x_1x_3 + x_2^2$. Hence, we may define the canonical sheaf of S as

$$\omega_S := \det(\mathcal{C}_{S|\mathbb{P}^4_K})^\vee \otimes_{\mathcal{O}_S} \Omega^4_{\mathbb{P}^4_K}|_S,$$

where $\mathcal{C}_{S|\mathbb{P}^4_K}$ is the conormal sheaf of the immersion $S \to \mathbb{P}^4_K$ and $\Omega^4_{\mathbb{P}^4_K}$ is the sheaf of differentials of degree 4 of $\mathbb{P}^4_K$ over K (see [39, Definition 6.4.7]). Then ω_S is invertible and is the dualizing sheaf of S. Moreover, ω_S is isomorphic to $\mathcal{O}_S(-1)$. Let us specify such an isomorphism.

We write $S_{(i)}$ for the affine open subset where $x_i \neq 0$, with coordinates $x_j^{(i)} := x_j/x_i$ for $j \neq i$. Then $S_{(i)}$ is defined by the polynomials $h_j^{(i)}(x_0^{(i)}, \dots, \widehat{x_i^{(i)}}, \dots, x_n^{(i)}) := h_j/x_i^2 \in K[x_0^{(i)}, \dots, \widehat{x_i^{(i)}}, \dots, x_n^{(i)}]$, $j \in \{1,2\}$. Observe that $U = S_{(2)}$.

Since S is a complete intersection, we have an isomorphism $\mathcal{O}_S(-2) \oplus \mathcal{O}_S(-2) \to \mathcal{C}_{S|\mathbb{P}^4_K}$ defined on $S_{(i)}$ by $(a_1, a_2) \mapsto a_1h_1 + a_2h_2$, and hence an isomorphism $\mathcal{O}_S(4) \to \det(\mathcal{C}_{S|\mathbb{P}^4_K})^\vee$ defined on $S_{(i)}$ by $x_i^4 \mapsto (h_1^{(i)} \wedge h_2^{(i)})^\vee$. Moreover, we have an isomorphism $\mathcal{O}_S(-5) \to \Omega^4_{\mathbb{P}^4_K}|_S$ given on $S_{(i)}$ by

$$x_i^{-5} \mapsto (-1)^i \,\mathrm{d}x_0^{(i)} \wedge \dots \wedge \widehat{\mathrm{d}x_i^{(i)}} \wedge \dots \wedge \mathrm{d}x_4^{(i)}.$$

This gives an isomorphism $\mathcal{O}_S(-1) \to \omega_S$ defined on $S_{(i)}$ by

$$\frac{1}{x_i} \mapsto s_i := (-1)^i (h_1^{(i)} \wedge h_2^{(i)})^\vee \otimes \mathrm{d}x_0^{(i)} \wedge \dots \wedge \widehat{\mathrm{d}x_i^{(i)}} \wedge \dots \wedge \mathrm{d}x_4^{(i)}.$$

On the smooth locus S_{reg}, we can canonically identify $\omega_{S_{\mathrm{reg}}}$ with $\Omega^2_{S_{\mathrm{reg}}}$ by taking the determinant of the conormal sequence

$$0 \to \mathcal{C}_{S_{\mathrm{reg}}|\mathbb{P}^4_K} \to \Omega^1_{\mathbb{P}^4_K}|_{S_{\mathrm{reg}}} \to \Omega^1_{S_{\mathrm{reg}}} \to 0.$$

For any $k < l \in \{0, \dots, 4\} \smallsetminus \{i\}$, this leads to the identification

$$s_i = (-1)^{i+t} \Delta_{i,k,l}^{-1} \,\mathrm{d}x_k^{(i)} \wedge \mathrm{d}x_l^{(i)} \tag{13.1}$$

in $\omega_{S,\xi} = \omega_{S_{\mathrm{reg}},\xi} = \Omega^2_{K(S)}$, where ξ is the generic point of S, $\Delta_{i,k,l}$ is the determinant of the Jacobian matrix $(\partial h_j^{(i)}/\partial x_n^{(i)})_{j,n}$ with the k-th and l-th columns removed, and $t := k + l$ if $k < i < l$, and $t := k + l - 1$ otherwise.

Since γ is an isomorphism on $\gamma^{-1}(S_{\mathrm{reg}})$, the pullback of differential forms gives an isomorphism $\gamma^*\omega_S|_{\gamma^{-1}(S_{\mathrm{reg}})} \cong \omega_{\widetilde{S}}|_{\gamma^{-1}(S_{\mathrm{reg}})}$. This induces an isomorphism $\omega_{\widetilde{S}} \cong \gamma^*\omega_S \otimes \mathcal{O}_{\widetilde{S}}(P)$, where P is a divisor supported on the complement $\widetilde{S} \smallsetminus \gamma^{-1}(S_{\mathrm{reg}})$. Since $\widetilde{S}$ is split and both singularities of S are rational double points, $[P] = 0$ (see [27, Proposition 8.1.10]). We conclude that

$$\omega_{\widetilde{S}} \cong \gamma^*\omega_S,$$

with an isomorphism whose restriction to $\gamma^{-1}(S_{\mathrm{reg}})$ is given by the pullback of differential forms. In the following, we use this isomorphism to identify $\omega_{\widetilde{S}}$ with $\gamma^*(\omega_S)$ and its dual to identify $\omega_{\widetilde{S}}^{-1}$ with $\gamma^*\omega_S^{-1}$.

For every $i \in \{0, \dots, 4\}$, let τ_i be the global section of $\omega_S^{-1} \cong \mathcal{O}_S(1)$ dual to s_i. Then $\tau_0, \dots, \tau_4$ define the embedding $S \hookrightarrow \mathbb{P}^4_K$. The morphism $\widetilde{S} \to S \hookrightarrow \mathbb{P}^4_K$ is given by the sections $\gamma^*\tau_0, \dots, \gamma^*\tau_4 \in H^0(\widetilde{S}, \omega_{\widetilde{S}}^{-1})$. Consider the Arakelov height $(\omega_{\widetilde{S}}^{-1}, (\|\cdot\|_v)_{v\in\Omega_K})$ defined by these global sections: for all $v \in \Omega_K$, $x \in \widetilde{S}(K_v)$, and $\tau \in \omega_{\widetilde{S}}^{-1}(x)$, we use the v-adic metric

$$\|\tau\|_v := \min_{\substack{0\le i\le 4\\ \gamma^*\tau_i(x)\neq 0}} \left\{ \left| \frac{\tau}{\gamma^*\tau_i(x)} \right|_v \right\}.$$

The corresponding height function on $\widetilde{S}(K)$ (see [45, Définition 2.3]) is $H \circ \gamma$, as desired.

According to [45, Formule empirique 5.1], the leading constant in Theorem 1.1 should hence have the form

$$c_{S,H} = \alpha(\widetilde{S})\beta(\widetilde{S})\tau_H(\widetilde{S}),$$

with $\alpha(\widetilde{S}), \beta(\widetilde{S}), \tau_H(\widetilde{S})$ as in [45, Définition 4.8].

We have already seen at the end of the last section how the factor $\alpha(\widetilde{S})$ appears in our leading constant. Moreover,

$$\beta(\widetilde{S}) := |H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/K), \mathrm{Pic}(\widetilde{S}_{\overline{\mathbb{Q}}}))| = 1,$$

since $\widetilde{S}$ is split.

By [45, Définition 4.6, Définition 4.8], and using the properties of the model $\widetilde{\mathcal{S}}$ proved in Proposition 4.1, the Tamagawa number $\tau_H(\widetilde{S})$ is given as

$$\tau_H(\widetilde{S}) = \lim_{s\to 1}(s-1)^6 L(s, \mathrm{Pic}(\widetilde{S}_{\overline{\mathbb{Q}}})) \frac{1}{|\Delta_K|} \prod_{v\in\Omega_K} \lambda_v^{-1} \boldsymbol{\omega}_{H,v}(\widetilde{S}(K_v)),$$

where

$$\lambda_v := \begin{cases} L_v(1, \mathrm{Pic}(\widetilde{S}_{\overline{\mathbb{Q}}})) & \text{if } v \in \Omega_0, \\ 1 & \text{if } v \in \Omega_\infty. \end{cases}$$

Note that the closure of $\widetilde{S}(K)$ in the set $\widetilde{S}(\mathbf{A}_K)$ of adelic points coincides with $\widetilde{S}(\mathbf{A}_K)$, since the rational variety $\widetilde{S}$ satisfies weak approximation.

By Proposition 4.1, (ii), the Frobenius morphism associated to any non-archimedean place v corresponding to a prime ideal $\mathfrak{p}$ acts trivially on the vector space $\operatorname{Pic}(\widetilde{\mathcal{S}}_{\overline{\mathbb{F}}_\mathfrak{p}}) \otimes \mathbb{Q}$ of dimension 6. Therefore, $L_v(s, \operatorname{Pic}(\widetilde{S}_{\overline{\mathbb{Q}}})) = (1 - \mathfrak{N}\mathfrak{p}^{-s})^{-6}$ and

$$L(s, \operatorname{Pic}(\widetilde{S}_{\overline{\mathbb{Q}}})) = \prod_{v \in \Omega_0} L_v(s, \operatorname{Pic}(\widetilde{S}_{\overline{\mathbb{Q}}})) = \zeta_K(s)^6.$$

By the analytic class number formula,

$$\lim_{s \to 1} (s-1)^6 L(s, \operatorname{Pic}(\widetilde{S}_{\overline{\mathbb{Q}}})) \frac{1}{|\Delta_K|} = \left(\frac{2^{r_1} (2\pi)^{r_2} R_K h_K}{|\mu_K|} \right)^6 \cdot \frac{1}{|\Delta_K|^4}.$$

Let us compute the v-adic measures $\boldsymbol{\omega}_{H,v}(\widetilde{S}(K_v))$ defined in [45, Notations 4.3]. By Proposition 4.1, (i), the base change $\widetilde{S}_{\mathcal{O}_v}$ of our model satisfies the hypotheses of [49, Corollary 2.15] for all $v \in \Omega_0$. If v corresponds to a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, we may thus conclude that

$$\boldsymbol{\omega}_{H,v}(\widetilde{S}(K_v)) = \frac{|\widetilde{S}(k(\mathfrak{p}))|}{\mathfrak{N}\mathfrak{p}^2}.$$

Moreover, Proposition 4.1, (ii) shows that $|\widetilde{S}(k(\mathfrak{p}))| = \mathfrak{N}\mathfrak{p}^2 + 6\,\mathfrak{N}\mathfrak{p} + 1$. Thus, we see that $\lambda_v^{-1}\boldsymbol{\omega}_{H,v}(\widetilde{S}(K_v)) = \omega_v(\widetilde{S})$, with $\omega_v(\widetilde{S})$ as in Subsection 1.4. It remains to compute $\boldsymbol{\omega}_{H,v}(\widetilde{S}(K_v))$ for $v \in \Omega_\infty$.

Since $U = S_{(2)}$ is smooth, its set of rational points $U(K_v)$ has the structure of a K_v-analytic manifold, and $\gamma : \widetilde{S} \to S$ induces an analytic isomorphism $\gamma^{-1}(U)(K_v) \to U(K_v)$, which we again call γ.

The preimage $\gamma^{-1}(S \smallsetminus U)$ is the union of the negative curves on $\widetilde{S}$. As a union of finitely many submanifolds of strictly smaller dimension, $\gamma^{-1}(S \smallsetminus U)(K_v)$ has measure 0, and $\boldsymbol{\omega}_{H,v}(\widetilde{S}(K_v)) = \boldsymbol{\omega}_{H,v}(\gamma^{-1}(U)(K_v))$

The local coordinates $x_0^{(2)} - 1, x_3^{(2)}$ at the rational point $p = (1,-1,0,0)$ of U define an analytic isomorphism $\psi : U(K_v) \to W := \{(z_0, z_3) \in K_v^2 \mid z_0 + z_3 + 1 \neq 0\}$. Since the restriction of γ to $\gamma^{-1}(U)$ is an isomorphism, the functions $y_0 := (x_0^{(2)} - 1) \circ \gamma$, $y_3 := (x_3^{(2)}) \circ \gamma$ form a system of local coordinates at $\gamma^{-1}(p)$, defining an analytic isomorphism $\phi := \psi \circ \gamma : \gamma^{-1}(U)(K_v) \to W$. By definition,

$$\begin{aligned} \boldsymbol{\omega}_{H,v}(\gamma^{-1}(U)(K_v)) &= \int_W \left\| \left(\frac{\partial}{\partial y_0} \wedge \frac{\partial}{\partial y_3} \right) (\phi^{-1}(z_0, z_3)) \right\|_v \mathrm{d}z_0 \, \mathrm{d}z_3 \\ &= \int_W \left\| \gamma^* \left(\frac{\partial}{\partial x_0^{(2)}} \wedge \frac{\partial}{\partial x_3^{(2)}} \right) (\phi^{-1}(z_0, z_3)) \right\|_v \mathrm{d}z_0 \, \mathrm{d}z_3 \\ &= \int_W \min_{\tau_i(\psi^{-1}(z_0,z_3)) \neq 0} \left\{ \left| \frac{\tau_2(\psi^{-1}(z_0, z_3))}{((x_0^{(2)} + x_3^{(2)})\tau_i)(\psi^{-1}(z_0, z_3))} \right|_v \right\} \mathrm{d}z_0 \, \mathrm{d}z_3, \end{aligned}$$

since $\partial/(\partial x_0^{(2)}) \wedge \partial/(\partial x_3^{(2)}) = -\Delta_{2,0,3}^{-1}\tau_2 = -(x_0^{(2)} + x_3^{(2)})^{-1}\tau_2$ due to (13.1). Together with the relation $\tau_i = x_i^{(2)} \cdot \tau_2$, this shows that $\boldsymbol{\omega}_{H,v}(\widetilde{S}(K_v))$ has the explicit form

$$\int_{K_v^2} \frac{\mathrm{d}z_0 \, \mathrm{d}z_3}{\max\{1, |z_0 + z_3|_v, |z_0(z_0 + z_3)|_v, |z_3(z_0 + z_3)|_v, |z_0 z_3(z_0 + z_3)|_v\}}.$$

We assume now that v is a complex place. We transform to variables $t_0 = -z_0, t_1 = z_0+z_1$ and use the identity

$$\frac{1}{s} = \int_{t\geq s} \frac{1}{t^2}\,\mathrm{d}t,$$

for $s \in \mathbb{R} \smallsetminus \{0\}$ to obtain

$$\boldsymbol{\omega}_{H,v}(\widetilde{S}(K_v)) = 4\int_{\substack{(t_0,t_3)\in\mathbb{C}^2\\ t_2\geq\max\{1,|t_1|^2,|t_0t_1|^2,|t_1(t_0+t_1)|^2,|t_0t_1(t_0+t_1)|^2\}}} \frac{\mathrm{d}t_0\,\mathrm{d}t_1\,\mathrm{d}t_2}{t_2^2}.$$

Recall that Peyre normalizes the Haar measure on K_v to be twice the usual Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$, which leads to the factor 4 in front of the integral.

We apply the transformation $t_2 = 1/y_2^3$, $t_0 = y_0/\sqrt{y_2}$, $t_1 = y_1/\sqrt{y_2}$ of Jacobian determinant $-3/y_2^6$ and replace y_2 by a complex variable via polar coordinates to see that $\boldsymbol{\omega}_{H,v}(\widetilde{S}(K_v)) = \omega_v(\widetilde{S})$. By a similar argument, the same equality holds for real places v.

This shows that the constant $c_{S,H}$ in Theorem 1.1 is as expected.

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