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David GÉRARD-VARET & Nader MASMOUDI

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

WELL-POSEDNESS FOR THE PRANDTL SYSTEM WITHOUT ANALYTICITY OR MONOTONICITY

by David GÉRARD-VARET and Nader MASMOUDI

ABSTRACT. – It has been thought for a while that the Prandtl system is only well-posed under the Oleinik monotonicity assumption or under an analyticity assumption. We show that the Prandtl system is actually locally well-posed for data that belong to the Gevrey class 7/4 in the horizontal variable x. Our result improves the classical local well-posedness result for data that are analytic in x (that is Gevrey class 1). The proof uses new estimates, based on non-quadratic energy functionals.

RÉSUMÉ. – Il a longtemps été supposé que l'équation de Prandtl n'est bien posée que sous l'hypothèse de monotonie d'Oleinik, ou pour des données analytiques. Nous montrons qu'elle est en fait localement bien posée pour des données appartenant à la classe Gevrey 7/4 en la variable x. Nous améliorons ainsi le résultat classique d'existence locale de solutions analytiques en la variable x (classe Gevrey 1). La preuve repose sur de nouvelles estimations, faisant appel à des fonctionnelles d'énergie non-quadratiques.

1. Introduction

Our concern in this paper is the well-posedness of the Prandtl system. This system, by now classical, was introduced by Prandtl in 1904 to describe an incompressible flow near a wall, at high Reynolds number. Formally, it is derived from the Navier-Stokes equation with no-slip condition:

$$(1.1) \quad \begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \varepsilon \Delta \mathbf{u} = 0, & \mathbf{x} \in \Omega, \\ \nabla \cdot \mathbf{u} = 0, & \mathbf{x} \in \Omega, \\ \mathbf{u}|_{\partial\Omega} = 0, \end{cases}$$

that we consider for simplicity in $\Omega := \mathbb{T} \times \mathbb{R}_+$. We recall that $\mathbf{u}(t, \mathbf{x}) = (u, v)(t, x, y)$ is the velocity field of the fluid, and p its pressure field. The parameter $0 < \varepsilon \ll 1$ is the inverse of the Reynolds number. In the limit case $\varepsilon = 0$, one is left formally with the Euler equation,

for which only the impermeability condition $u \cdot n|_{\partial\Omega} = 0$ can be prescribed. Mathematically, this singular change of boundary condition generates strong gradients of the Navier-Stokes solution $\mathbf{u}^\varepsilon$, as $\varepsilon \to 0$. These gradients correspond to a concentration of the fluid flow in a thin zone near the wall $\partial\Omega$: the so-called *boundary layer*. The understanding of the boundary layer is a great mathematical challenge, that makes the convergence of Navier-Stokes solutions to Euler ones a big open problem, even for smooth data.

To tackle this problem, Prandtl proposed in 1904 an asymptotic model for the flow, based on two different asymptotic expansions of $\mathbf{u}^\varepsilon$, resp. outside and inside the boundary layer:

– outside the boundary layer, no concentration should occur: one should have

$$\mathbf{u}^\varepsilon(t, \mathbf{x}) \sim \mathbf{u}^0(t, \mathbf{x}), \quad \text{the solution of the Euler equation.}$$

– inside the boundary layer, $\mathbf{u}^\varepsilon$ should exhibit strong gradients, transversally to the boundary: more precisely, the asymptotics suggested by Prandtl is

$$u^\varepsilon(t, x, y) \sim u(t, x, y/\sqrt{\varepsilon}), \quad v^\varepsilon(t, x, y) \sim \sqrt{\varepsilon}v(t, x, y/\sqrt{\varepsilon})$$

where $u = u(t, x, Y)$ and $v = v(t, x, Y)$ are *boundary layer profiles*, depending on a rescaled variable $Y = y/\sqrt{\varepsilon}$, $Y > 0$. Note that the scale $\sqrt{\varepsilon}$ is coherent with the parabolic part of (1.1a).

If we plug the expansion above in (1.1) and keep the leading order terms, we derive the famous Prandtl system (denoting Y instead of y):

$$(1.2) \quad \begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p - \partial_y^2 u = 0, \\ \qquad\qquad\qquad\qquad \partial_y p = 0, \\ \qquad\qquad\qquad \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0, \quad \lim_{y \to +\infty} u = U, \quad \lim_{y \to +\infty} p = P, \end{cases}$$

where $U(t, x) := u^0(t, x, 0)$ and $P(t, x) := p^0(t, x, 0)$ are the Euler tangential velocity and pressure at the boundary. We refer to [18] for the formal derivation of the Prandtl system. The condition at $y = +\infty$ in (1.2d) is a matching condition near the boundary between the boundary layer flow and the Euler flow (*matched asymptotics*). Note that, combining (1.2b) with the boundary condition on p, we get $p \equiv P$. Hence, the pressure is not an unknown in the Prandtl model: v is obtained in terms of u by integrating the divergence-free condition (1.2c), so that (1.2a) is a scalar evolution equation on u, which is a priori much simpler than the original Navier-Stokes equation.

However, this appealing formal asymptotics raises strong mathematical issues: well-posedness of the limit Prandtl system on one hand, justification of the Prandtl asymptotics of $\mathbf{u}^\varepsilon$ on the other hand. The difficulty comes from numerous underlying fluid instabilities, that can invalidate the Prandtl model: we refer to [11] for a basic presentation of these aspects.

The aim of the present paper is to investigate this stability problem, from a mathematical viewpoint. We shall focus on the limit Prandtl system, namely on its well-posedness. For simplicity, we shall restrict to homogeneous data: $U = P = 0$. Extension of our results to the case of constant U would not raise any problem. Extension to some $U = U(t, x)$

would require some modifications, see [15] for a similar problem. Hence, we consider here the following system:

$$(1.3) \quad \begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u = 0, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0} = v|_{y=0} = 0, \ \lim_{y=+\infty} u = 0 \end{cases}$$

with initial condition $u|_{t=0} = u_0$.

Before stating our theorem, let us review briefly known results on the existence theory for (1.2). So far, well-posedness has been established in two settings:

- The first results go back to Oleinik [22], who obtained some local well-posedness for *initial data that are monotonic with respect to y: $U > 0, \partial_y u > 0$. For such data, one can use the Crocco transform: in short, using u as an independent variable instead of y and $w := \partial_y u$ as an unknown instead of u, one is left with a nonlinear parabolic equation on w, for which maximum principles are available: see [22] for details. Note that under the extra condition $\partial_x P \leq 0$ (*favorable pressure gradient*), one can go from local to global well-posedness, cf. [26]. From the point of view of physics, this monotonicity assumption is known to be stabilizing: it avoids the *boundary layer separation*, see [11].
- Without monotonicity, well-posedness has been established only locally in time, for *initial data that are analytic with respect to x*. We refer to [25, 17], and to the recent extensions [15, 14]. The assumption of analyticity can be understood as follows. By the divergence-free condition, one obtains $v = -\int_0^y \partial_x u$. Thus, the term $v\, \partial_y u$ in (1.3a) (seen as a functional of u) is first order in x. Moreover, it is not hyperbolic. For instance, let us consider the linearization of the Prandtl equation around a shear flow $\mathbf{u} = (U_s(y), 0)$:

$$(1.4) \qquad \partial_t u + U_s \partial_x u + U_s' v - \partial_y^2 u = 0, \quad \partial_x u + \partial_y v = 0.$$

If we freeze the coefficients at some y_0 and compute the dispersion relation, we obtain the growth rate

$$\sigma(k_x, k_y) = U_s'(y_0)\frac{k_x}{k_y} - k_y^2$$

that increases linearly with the wavenumber k_x. This kind of growth rate would prevent any well-posedness result outside the analytic setting.

However, as discussed in [12], this dispersion relation, formally obtained by freezing the coefficients, is misleading: for instance, the inviscid version of Prandtl (that is removing the $\partial_y^2 u$ term) is locally well-posed in C^k, through the method of characteristics.

In the case of the full Prandtl system (1.3), the situation is even more complex, and was addressed recently by the first author and Emmanuel Dormy in article [7] (see also [8]). This article contains a careful study of the linearized system (1.4), *in the case of a non-monotonic base flow U_s*:

$$\exists a, \quad U_s'(a) = 0, \quad U_s''(a) \neq 0.$$

In short, it is shown in [7][1] that the linear system (1.4) admits approximate solutions with growth rate

(1.5) $$\sigma(k_x) \sim \delta \sqrt{k_x}, \quad \delta > 0, \quad k_x \gg 1.$$

Let us stress that such growing solutions result from an interplay between the lack of monotonicity of U_s and the diffusion term $\partial_y^2 u$. It is therefore coherent with the well-posedness results obtained in the monotonic case, and for the inviscid Prandtl equation.

Of course, the growth rate indicated in (1.5) yields ill-posedness of the Prandtl equation in Sobolev spaces. Still, it leaves room for well-posedness below analytic regularity. *Indeed, the aim of the present paper is to prove that the Prandtl equation is locally well-posed for data that are of Gevrey class* $7/4$ *in variable* x. Precise statements will be given in the next section. Let us already point out that the Gevrey classes $m, m > 1$, contain compactly supported functions, as opposed to the Gevrey class 1 (analytic functions). Hence, stability in the Gevrey context has more physical insight that stability in the analytic context: see [21, 2] for similar issues.

2. Statements of the result

Let $m \geq 1$. We recall that the Gevrey space $G^m(\mathbb{T})$ is the set of functions f satisfying: $\exists C, \tau > 0$ such that

$$|f^{(j)}(x)| \leq C \tau^{-j} (j!)^m, \quad \forall j \in \mathbb{N}, x \in \mathbb{T}.$$

For a reminder on Gevrey spaces, we refer to the paper [6] by Foias and Temam as well as to the papers [16, 5, 23] where these spaces are used. One has in particular $G^m(\mathbb{T}) = \bigcup_{\tau > 0} G_\tau^m(\mathbb{T})$, where

$$G_\tau^m(\mathbb{T}) := \left\{ f \in C^\infty(\mathbb{T}), \quad \sup_j \tau^{j+1} (j!)^{-m} (j+1)^{10} \|f^{(j)}\|_{L^2} < \infty \right\}$$

is a Banach space, stable by multiplication. Note that the extra factor $(j+1)^{10}$ is useful in proving the stability by multiplication [16, 5] (the exponent 10 is arbitrary, any power greater than 1 works as well).

In the context of the Prandtl equation, functions depend not only on x, but also on y. We just require Sobolev regularity and polynomial decay with respect to the y variable. For $s \in \mathbb{N}, \gamma \geq 0$, we define the spaces

$$H_\gamma^s := \left\{ g \in H^s(\mathbb{R}_+), \quad (1+y)^{\gamma+k} g^{(k)} \in L^2(\mathbb{R}_+), k = 0, \dots, s \right\},$$

$$\|g\|_{H_\gamma^s}^2 := \sum_{k=0}^s \|(1+y)^{\gamma+k} g^{(k)}\|_{L^2}^2.$$

We write L_γ^2 instead of H_γ^0. Accordingly, we introduce the space

$$G^m(\mathbb{T}; H_\gamma^s) := \bigcup_{\tau > 0} G_\tau^m(\mathbb{T}; H_\gamma^s)$$

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[1] As mentioned recently by S. Cowley to the first author, the instability mechanism used in [7] had already been described at a formal level in [4].

where

$$G_\tau^m(\mathbb{T}; H_\gamma^s) := \left\{ f \in C^\infty(\mathbb{T}; H_\gamma^s), \quad \sup_j \tau^{j+1} (j!)^{-m} (j+1)^{10} \|\partial_x^j f\|_{L^2(H_\gamma^s)} < \infty \right\}.$$

We shall consider initial data u_0 satisfying

$$u_0 \in G_{\tau_0}^m(\mathbb{T}; H_{\gamma-1}^{s+1}), \quad \omega_0 := \partial_y u_0 \in G_{\tau_0}^m(\mathbb{T}; H_\gamma^s)$$

for some m, s, τ_0 to be fixed later. Our focus will be on data that are non-monotonic with respect to y. More precisely, we shall assume the existence of a single curve of non-degenerate critical points:

(H) $\omega_0(x, y) = 0$ *iff* $y = a_0(x)$, *for some curve* $a_0(x) > 0$, *with* $\partial_y \omega_0(x, a_0(x)) > 0$, $\forall x$.

Without loss of generality, we will assume that $a_0(x) < 3$. Besides, to control the behavior of the flow at large y, we shall need uniform lower and upper bounds on vorticity. We assume the existence of $\sigma > 0$ and $\delta > 0$ such that

(H′)   *For all* $y > 3$, *for all* x, *for all* $\alpha \in \mathbb{N}^2 |\alpha| \leq 2$,

$$|\omega_0(x, y)| \geq \frac{2\delta}{(1+y)^\sigma}, \quad |\partial^\alpha \omega_0(x, y)| \leq \frac{1}{2\delta(1+y)^{\sigma+\alpha_2}}.$$

We can now state our main result:

THEOREM 1. – *Let* $\tau_0 > 0$, $s \geq 8$ *even*, $\gamma \geq 2$, $\sigma \geq \gamma + \frac{1}{2}$, $\delta > 0$. *Let* u_0 *satisfy*

$$u_0 \in G_{\tau_0}^{7/4}(\mathbb{T}; H_{\gamma-1}^{s+1}), \quad \omega_0 := \partial_y u_0 \in G_{\tau_0}^{7/4}(\mathbb{T}; H_\gamma^s),$$

the compatibility condition: $u_0|_{y=0} = 0$, *and assumptions* (H), (H′) *above. Then, there exist* $T > 0$, $0 < \tau \leq \tau_0$ *and a unique solution*

$$u \in L^\infty\big(0, T; G_\tau^{7/4}(\mathbb{T}; H_{\gamma-1}^{s+1})\big), \quad \omega \in L^\infty\big(0, T; G_\tau^{7/4}(\mathbb{T}; H_\gamma^s)\big),$$

of the Prandtl Equation (1.3), *with initial data* u_0.

Remark that a solution u of (1.3) with the regularity above automatically satisfies

$$\partial_t u \in L^\infty\big(0, T; G_{\tau'}^{7/4}(\mathbb{T}; H_{\gamma-1}^{s-1})\big), \quad \forall \tau' < \tau.$$

This yields continuity in time of u with values in $G_{\tau'}^{7/4}(\mathbb{T}; H_{\gamma-1}^{s-1})$, $\tau' < \tau$, giving a meaning to the initial condition.

Let us point out that the solution u of the theorem remains in the Gevrey space $G^{7/4}$ in x, but does not stay a priori in $G_{\tau_0}^{7/4}$: it is likely that the exponent τ deteriorates with time.

The theorem will be the consequence of the control of some well-chosen Gevrey type norms that evolve in time. More precisely, introducing the vorticity $\omega := \partial_y u$, we will control an energy of the form $E(t, \tau(t))$, with

$$E(t, \tau) := \sum_{j=0}^{+\infty} \left(\tau^{j+1} (j!)^{-7/4} (j+1)^{10}\right)^2 \|\partial_x^j \omega(t, \cdot)\|_{L^2(H_\gamma^s)}^2$$

for some positive function $\tau(t)$ decreasing linearly and fast enough with t. Actually, it will be slightly better to consider a variant of the previous energy, namely

$$(2.1) \qquad E_\omega(t, \tau) := \sum_{j=0}^{+\infty} \left(\tau^{j+1} \, (j!)^{-7/4} \, (j+1)^{10} \right)^2 \, \|\omega\|_{\mathcal{H}_\gamma^j}^2,$$

with

$$\|\omega\|_{\mathcal{H}_\gamma^j}^2 := \sum_{\substack{J=(j_1,j_2)\in\mathbb{N}^2 \\ |J|=j, \, 0\leq j_2\leq s}} \|(1+y)^{\gamma+j_2}\partial^J\omega(t,\cdot)\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}^2.$$

We leave it to the reader to check that: $\exists \, c, C > 0$ such that

$$E(t,\tau) \leq E_\omega(t,\tau), \quad E_\omega(t,\tau) \leq C_{\tau,\tau'} E(t,\tau'), \quad \forall \, 0 < \tau < \tau'.$$

Roughly, the choice of E_ω upon E is connected with estimates on the hyperbolic part of (1.3), for which derivatives with respect to x and y have the same cost.

Still, the energy E_ω cannot be controlled in a direct manner. As mentioned in [20], there is a problem with estimating $\partial_x^j \omega$. If the ∂_x^j hits the v in the transport term $v\partial_y\omega$, we end up with $\partial_x^j v \partial_y\omega$, which gives an a priori loss of one full derivative and requires analytic data. This is the main trouble with the Prandtl system, already pointed out in the introduction.

To overcome this difficulty, we shall rely on new estimates, notably inspired by the recent paper [20] by Wong and the second author. This paper is about the well-posedness of the Prandtl equations for monotonic data, that is when u_0 is increasing with y. As we recalled in the introduction, in this monotonic case, well-posedness was obtained by Oleinik in the sixties, using the Crocco transform. The novelty in [20] is to obtain such well-posedness result without using the Crocco transform, namely *performing Sobolev estimates in the original Eulerian formulation* (see also [1] where estimates on the linearized problem were done). The main point in the proof is to avoid the loss of derivative generated by the v-term. One key idea is the following: combining properly the velocity formulation (1.3) and the vorticity formulation ($\omega = \partial_y u$)

$$(2.2) \qquad \partial_t\omega + u\partial_x\omega + v\partial_y\omega - \partial_y^2\omega = 0,$$

one is left with an equation of the form

$$\partial_t g^{mw} + u\partial_x g^{mw} - \partial_y^2 g^{mw} = \text{commutators}$$

on the new nonlinear quantity:

$$g^{mw} := \omega - \frac{\partial_y\omega}{\omega} u = \omega\partial_y\frac{u}{\omega}.$$

The main point with this new equation is that the function v, responsible for the loss of one x-derivative, does not appear. Similarly, one can write down equations on

$$(2.3) \qquad g_j^{mw} := \partial_x^j\omega - \frac{\partial_y\omega}{\omega}\partial_x^j u$$

that do not involve the bad term $\partial_x^j v$. Moreover, broadly speaking, one can show that the control of the family g_j in L^2 amounts to the control of the family $\partial_x^j\omega$ in L^2. Hence, one can expect to derive a Gronwall type inequality, at the Sobolev level, using the g_j's (see [20]).

Of course, the main difference between the present context and the one in [20] is assumption (H): there exists a curve of critical points, meaning that ω vanishes. This critical curve, that reads initially $y = a_0(x)$, with $0 < a_0 < 3$, should evolve into some $y = a(t,x)$, with $0 < a < 3$ for small times. Differentiating the relation $\omega(t,x,a(t,x)) = 0$, one finds that the function a should be governed by the ordinary differential equation in the variable t:

$$(2.4) \qquad \partial_t a(t,x) + \frac{\partial_t \omega(t,x,a(t,x))}{\partial_y \omega(t,x,a(t,x))} = 0, \quad a(0,x) = a_0(x).$$

The quantities g_j^{mw}'s are not suitable in a neighborhood of this curve. As we shall see, even away from it, the $\partial_x^j \omega$'s are not controlled by the g_j^{mw}'s in a suitable way, due to nonlocal phenomena.

Therefore, we need to introduce an additional quantity, that somehow recovers the information lost near the critical curve. We stress that in this region, the flow should not be monotonic anymore. However, due to our non-degeneracy assumption, it should remain convex: $\partial_y^2 u = \partial_y \omega > 0$, for t and $y - a(t,x)$ small enough. It turns out that this kind of convexity assumption has been used in a close context, namely in the study of the hydrostatic Euler equations. These equations, set in $\mathbb{T} \times (0,1)$, read

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p = 0, \partial_y p = 0, \\ \partial_x u + \partial_y v = 0, \\ v|_{y=0} = v|_{y=1} = 0. \end{cases}$$

Again, for this system, there is a possible loss of x-derivative through the function v. As shown in Brenier [3], Grenier [9], the well-posedness of the hydrostatic equations requires some convexity assumption. This fact was emphasized in another recent paper by Wong and the second author [19]. Starting again from the vorticity equation,

$$\partial_t \omega + u\partial_x \omega + v\partial_y \omega = 0,$$

and dividing by $\sqrt{\partial_y \omega}$, one is left with an equation of the type

$$\partial_t h^{mw} + u\partial_x h^{mw} + \sqrt{\partial_y \omega}\, v = \text{commutators}$$

on $h^{mw} := \frac{\omega}{\sqrt{\partial_y \omega}}$. One can then take advantage of the cancellation

$$(2.5) \qquad \int_{\mathbb{T}\times(0,1)} \sqrt{\partial_y \omega}\, v h^{mw} = \int_{\mathbb{T}\times(0,1)} v\partial_y u = -\int_{\mathbb{T}\times(0,1)} \partial_y v u = \int_{\mathbb{T}\times(0,1)} \partial_x \frac{|u|^2}{2} = 0$$

to get rid of the bad terms in v. Let us note that the same idea was used by Grenier in [10] to establish the stability of some characteristic boundary layers. Similar cancellations hold with higher order derivatives in x, through the introduction of

$$(2.6) \qquad h_j^{mw} := \frac{\partial_x^j \omega}{\sqrt{\partial_y \omega}}.$$

Thanks to these quantities, one can obtain local in time Sobolev estimates, like for the Prandtl equations in the monotonic case.

In our context, with regards to the previous remarks, it is tempting to replace the original energy E_ω by a modified one, based on functions g_j and h_j like in (2.3) and (2.6). To be more specific, one could think of combining two local energies: one away from the critical curve, based on the g_j's, and one in a neighborhood of the critical curve, based on the h_j's. With

regards to the recent works [19, 20], one could even expect to obtain stability in Sobolev like spaces. However, such localization process is not straightforward. Indeed, the Prandtl equation, like the hydrostatic one (see recent developments by Renardy [24]) is highly non-local. This non-locality explains the ill-posedness of the Prandtl equation in the Sobolev setting, as can be seen from the mechanism described by the first author and Dormy [7].

At the level of the energy estimates that we will perform, this non-locality will be reflected by some annoying commutator terms. To control such bad commutators, we will use a functional of the following type:

$$(2.7) \qquad \mathcal{E}(\alpha, t, \tau) := \dot{E}_\omega(t, \tau) + E_h(t, \tau) + E_g^1(t, \tau) + \alpha \, E_g^2(t, \tau)$$

where $\alpha > 0$ is a parameter to be chosen later. This functional splits into four parts:

- The first part is a *vorticity energy*

$$(2.8) \qquad \dot{E}_\omega(t, \tau) := \sum_{j \in \mathbb{N}} \left(\tau^{j+1} (j!)^{-7/4} (j+1)^{10} \right)^2 \|\omega\|_{\dot{\mathcal{H}}_\gamma^j}^2$$

with

$$\|\omega\|_{\dot{\mathcal{H}}_\gamma^j}^2 := \sum_{\substack{J=(j_1,j_2)\in\mathbb{N}^2 \\ |J|=j,\, 0<j_2\leq s}} \|(1+y)^{\gamma+j_2} \partial^J \omega(t,\cdot)\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}^2.$$

The difference with the original energy $E_\omega(t, \tau)$ is the restriction $j_2 > 0$ which means that the derivatives $\partial_x^j \omega$ are not included here. For each j this energy will provide a control of all (x, y) derivatives of order j but $\partial_x^j \omega$. For $j = 0$, one has $\|\omega\|_{\dot{\mathcal{H}}_\gamma^j} = 0$, but we keep it in the sum for unity.

- The second part is a *hydrostatic energy*

$$(2.9) \qquad E_h(t, \tau) := \sum_{j \in \mathbb{N}} \left(\tau^{j+1} (j!)^{-7/4} (j+1)^{10} \right)^2 \|h_j(t,\cdot)\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}^2$$

where

$$(2.10) \qquad h_j(t, x, y) = \chi(y - a(t,x)) \frac{\partial_x^j \omega}{\sqrt{\partial_y \omega}}(t, x, y),$$

with $\chi = \chi(p) \in C_c^\infty(\mathbb{R})$ equal to 1 in a neighborhood of $p = 0$. We take χ with small enough support, so that $\chi(y-a)$ is compactly supported in $(0, 3)$, and $\partial_y \omega > 0$ over the support of χ. This truncation function corresponds to the localization near the critical curve mentioned above.

- The third part is a (first) *monotonicity energy*

$$(2.11) \qquad E_g^1(t, \tau) = \sum_{j \in \mathbb{N}} \left(\tau^{j+1} (j!)^{-7/4} (j+1)^{10} \right)^2 \|g_j(t,\cdot)\|_{L^2(L_\gamma^2)}^2$$

where

$$(2.12) \qquad g_j(t, x, y) := \left(\psi(y)\omega(t,x,y) + 1 - \psi(y) \right) \left(\partial_x^j \omega - \frac{\partial_y \omega}{\omega} \partial_x^j u \right)(t, x, y)$$

with $\psi = \psi(y) \in C_c^\infty(\mathbb{R})$, equal to 1 in an open neighborhood of $[0, 3]$.

Note that the truncation ψ makes the quantity g_j well-defined for all y, even in the neighborhood of $y = a(t,x)$. Indeed, for large y, it amounts to the original Definition (2.3), whereas near the critical curve, it reads

$$g_j(t,x,y) = \omega\partial_x^j\omega - \partial_y\omega\partial_x^j u.$$

Note that the first term at the right-hand side vanishes near the critical curve, which leads to a loss of control of $\partial_x^j\omega$ in terms of g_j. As explained before, this is why we add the hydrostatic energy to the energy functional. More precisely, we will show that the sum of the vorticity energy, the hydrostatic energy, and the first monotonicity energy controls the original functional $E_\omega(t,\tau)$. However, we are not able to obtain a closed estimate on this sum.

– Hence, we need to add a second *monotonicity energy*

$$(2.13) \qquad E_g^2(t,\tau) = \sum_{j\in\mathbb{N}} \left(\tau^{j+1}\,(j!)^{-7/4}\,(j+1)^{10}\right)^2 (j+1)^{3/2}\|\tilde{g}_j(t,\cdot)\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}^2$$

where

$$(2.14) \qquad \tilde{g}_j(t,x,y) := \partial_x^{j-5}\left(\omega\partial_x^5\omega - \partial_y\omega\partial_x^5 u\right)$$

(with convention $\partial_x^k = 0$ for $k < 0$). Let us remark that $\tilde{g}_j$ is close to g_j: for instance, in the region $\psi = 1$, one has $\tilde{g}_j = g_j$ up to commutator terms. Indeed, the replacement of g_j by $\tilde{g}_j$ is only a technical issue, that will be explained in due course. The real key-point in the definition of this second monotonicity energy is the extra factor $(j+1)^{3/2}$, creating an anisotropy in the total energy $\mathscr{E}(\alpha,t,\tau)$. *Such a choice of anisotropic energy is the main feature that will allow us to prove stability estimates in the Gevrey setting, below the analytic case.*

3. A priori estimates

The key for Gevrey well-posedness is some a priori estimate on the anisotropic energy $\mathscr{E}(\alpha,t,\tau)$ introduced in the previous section. This energy involves notably the functions g_j, see (2.12), that contains the factor $\partial_y\omega/\omega$. To control their behavior for large y, a lower bound on ω and an upper bound on $\partial_y\omega$ are needed. More precisely, we shall work with vorticities ω satisfying: *for all $y > 3$, for all t, x, for all $\alpha \in \mathbb{N}^2$, $|\alpha| \leq 2$,*

$$(3.1) \qquad |\omega(t,x,y)| \geq \frac{\delta}{(1+y)^\sigma}, \quad |\partial^\alpha\omega(t,x,y)| \leq \frac{1}{\delta(1+y)^{\sigma+\alpha_2}}.$$

Except for a factor 2, this condition is the "all-time" version of condition (H′), the latter dealing only with the initial time. Of course, in the end, we will only assume (H′): we will build solutions through an approximation scheme, and doing so, we will show that the approximations satisfy (3.1) over a uniform time interval.

THEOREM 2 (Main a priori estimate). – *Let $T > 0$, $0 < \tau \leq \tau_0$, s, γ, σ as in Theorem 1. Let u be a smooth solution of the Prandtl Equation (1.3) over $(0,T)$, with a single curve of non-degenerate critical points: $y = a(t,x)$, $\inf a > 0$, $\sup a < 3$. Assume that the vorticity ω satisfies (3.1) over $]0,T]$, and that*

$$(3.2) \qquad E_\omega(t,\tau) \leq M, \quad M > 0, \quad \forall t \in (0,T).$$

Then, there exists $0 < \alpha \leq 1$, $C > 0$ such that for all $t \in (0, T)$:

$$\partial_t \mathcal{E}(\alpha, t, \tau) \leq C \, \partial_\tau \mathcal{E}(\alpha, t, \tau).$$

REMARK 1. – *One will see through the proof that the constant C depends on the following parameters: τ, M, $\inf a$, $\sup a$, the δ in (3.1),*

$$\theta_0 := \inf_{t,x,y=a} |\partial_y \omega|,$$

$$\theta_1 := \inf_{\substack{t,x, \\ |y-a| \geq \varepsilon, \\ y \leq 3}} |\omega|, \quad \text{for } \varepsilon := \min\left(\frac{\inf a}{4}, \frac{3 - \sup a}{4}, \frac{\inf_{t,x,y=a} |\partial_y \omega|}{4 \sup_{t,x,y} |\partial_y^2 \omega|} \right), \text{and}$$

$$\theta_2 := \inf_{\substack{t,x, \\ |y-a| \geq \mu, \\ y \leq 3}} |\omega|,$$

where μ is a constant depending on $M, \tau \, \theta_0$, and ε_0 (this constant is introduced in Paragraph 3.4, see "estimate of D_j"). More precisely, C can be taken continuous and increasing in M and $\sup a$, decreasing in the other parameters. Let us mention again that in the upper bound $a < 3$ or in the condition $y > 3$ in (3.1), the choice of the value 3 is purely arbitrary.

The constant $\alpha \in]0, 1]$ depends also on τ, M, $\inf a$, $\sup a$, the δ in (3.1), and $\theta_0, \theta_1, \theta_2$.

The estimate of the theorem will yield the Gevrey stability, playing on the radius τ of Gevrey regularity. Indeed, taking some time-dependent $\tau(t)$, we observe that

$$\partial_t \mathcal{E}(\alpha, t, \tau(t)) = \partial_t \mathcal{E}(\alpha, t, \tau(t)) + \tau'(t)\partial_\tau \mathcal{E}(\alpha, t, \tau(t)) \leq (C + \tau'(t))\mathcal{E}(\alpha, t, \tau(t)) < 0$$

if τ is decreasing fast enough with time.

We insist that all this section is about a priori estimates. The construction of solutions will require a further approximation scheme, on which similar estimates will be shown to hold. This will be detailed in later sections.

Notations. – As many constants will be involved in the calculations, we shall use the following notations for brevity:

- We shall call *absolute constant* any constant that depends only on the fixed quantities $\tau_0, s, \gamma, \sigma$.
- We shall denote $a \lesssim_\tau b$, respectively $a \lesssim_{\tau,M} b$ whenever $a \leq C\,b$ for some constant C depending on τ, respectively on τ and M.
- We shall denote $a \lesssim b$ whenever $a \leq C\,b$ for some constant C depending on τ, M, $\inf a$, $\sup a$, the δ in (3.1), and θ_0, θ_1, *but not on θ_2*, see Remark 1 for definitions.
- We shall denote $a \lesssim_{\text{all}} b$, whenever $a \leq C\,b$ for some constant C depending on τ, M, $\inf a$, $\sup a$, the δ in (3.1), and $\theta_0, \theta_1, \theta_2$, see Remark 1 for definitions.

3.1. Preliminaries

In order to prove Theorem 2, we need some extra notations. For any $p \in [1, +\infty[$, we introduce the weighted l^p space

$$l^p(\tau) := \left\{ (a_j)_{j \in \mathbb{N}}, \quad \sum_{j=0}^{+\infty} \left(\tau^{j+1} (j!)^{-7/4} (j+1)^{10} \right)^2 |a_j|^p < +\infty \right\},$$

with norm

$$\|a_j\|_{l^p(\tau)} := \left(\sum_{j=0}^{+\infty} \left(\tau^{j+1} (j!)^{-7/4} (j+1)^{10} \right)^2 |a_j|^p \right)^{1/p}.$$

In particular, we have $\dot{E}_\omega(t, \tau) = \left\| \|\omega\|_{\dot{\mathscr{H}}_\gamma^j} \right\|_{l^2(\tau)}^2$ and $\partial_\tau \dot{E}_\omega(t, \tau) = \frac{2}{\tau} \left\| (j+1)^{1/2} \|\omega\|_{\dot{\mathscr{H}}_\gamma^j} \right\|_{l^2(\tau)}^2$.

We shall make repeated use of the following inequality:

LEMMA 1. – *For all $m \leq 5$,*

$$\| \sum_{k=0}^{\frac{j}{2}} \binom{j}{k} a_{k+m} \, b_{j-k} \|_{l^2(\tau)} \lesssim_\tau \|a_j\|_{l^2(\tau)} \|b_j\|_{l^2(\tau)}$$

and symmetrically,

$$\| \sum_{k=\frac{j}{2}}^{j} \binom{j}{k} a_k \, b_{j-k+m} \|_{l^2(\tau)} \lesssim_\tau \|a_j\|_{l^2(\tau)} \|b_j\|_{l^2(\tau)}.$$

REMARK 2. – *In this lemma and in all the text, the notation $\frac{j}{2}$ that appears as an index in the sums is slightly abusive: it stands for the integer part of $\frac{j}{2}$.*

Proof. – Denoting $\alpha_j(\tau) := \tau^j (j!)^{-7/4} (j+1)^{10}$, we find

$$\| \sum_{k=0}^{\frac{j}{2}} \binom{j}{k} a_{k+m} b_{j-k} \|_{l^2(\tau)}$$

$$\lesssim_\tau \| \sum_{k=0}^{\frac{j}{2}} \binom{j}{k}^{-3/4} \left(\frac{(k+m)!}{k!} \right)^{7/4} (k+m+1)^{-10} \alpha_{k+m}(\tau) a_{k+m} \, \alpha_{j-k}(\tau) b_{j-k} \|_{l^2}$$

$$\lesssim_\tau \| \sum_{k=0}^{+\infty} (k+m+1)^{\frac{7}{4}m-10} \alpha_{k+m}(\tau) a_{k+m} \, \alpha_{j-k}(\tau) b_{j-k} \|_{l^2}.$$

By a standard convolution inequality for discrete sums, we get

$$\| \sum_{k=0}^{\frac{j}{2}} \binom{j}{k} a_{k+m} b_{j-k} \|_{l^2(\tau)} \lesssim_\tau \| (j+1)^{\frac{7}{4}m-10} (\alpha_j(\tau) a_j) \|_{l^1} \| \alpha_j(\tau) b_j \|_{l^2}.$$

As $m \leq 5$, we have $\frac{7}{4}M - 10 \leq -1$, and we can use the Cauchy-Schwartz inequality to bound the first factor, which yields the result. □

The next lemma emphasizes some important relations between our energy functionals.

LEMMA 2. – *Under the assumptions of Theorem 2*

$$(3.3) \qquad E_\omega(t,\tau) - \dot{E}_\omega(t,\tau) \lesssim E_g^1(t,\tau) + E_h(t,\tau) \lesssim E_\omega(t,\tau) - \dot{E}_\omega(t,\tau)$$

and

$$(3.4) \qquad \partial_\tau E_\omega(t,\tau) - \partial_\tau \dot{E}_\omega(t,\tau) \lesssim \partial_\tau E_g^1(t,\tau) + \partial_\tau E_h(t,\tau) \lesssim \partial_\tau E_\omega(t,\tau) - \partial_\tau \dot{E}_\omega(t,\tau).$$

These relations, to be proved in the appendix, follow from a direct but crucial representation of $\partial_x^j u$ in terms of g_j (see (2.12) for the definition of g_j):

LEMMA 3. – *One can write:*
$$(3.5)$$
$$\partial_x^j u(t,x,y) = \begin{cases} \omega(t,x,y) \displaystyle\int_3^y \left(\psi + \dfrac{(1-\psi)}{\omega} \right)^{-1} \dfrac{g_j}{\omega^2} + C_j(t,x)\omega(t,x,y), & y > a(t,x), \\[4mm] \omega(t,x,y) \displaystyle\int_0^y \left(\psi + \dfrac{(1-\psi)}{\omega} \right)^{-1} \dfrac{g_j}{\omega^2}, & y < a(t,x) \end{cases}$$

with $C_j := -\partial_x^j u(t,x,3)/\omega(t,x,3)$.

Proof. – The proof is trivial: integrate the relation

$$(3.6) \qquad \partial_y \frac{\partial_x^j u}{\omega} = \left(\psi + \frac{(1-\psi)}{\omega} \right)^{-1} \frac{g_j}{\omega^2}$$

from 0 to y and from 3 to y respectively. We point out that the choice 3 is arbitrary.

We stress again that u is assumed to be a smooth solution of the Prandtl equation, notably smooth across the critical curve $y = a(t,x)$. Note also that (3.6) formally yields the relation

$$\partial_x^j u(t,x,y) = \omega(t,x,y) \int_0^y \left(\psi + \frac{(1-\psi)}{\omega} \right)^{-1} \frac{g_j}{\omega^2}$$

for all y. However, the right-hand side does not make sense for $y > a(t,x)$: as ω degenerates near $y = a$, the integral is not properly defined. One cannot use it to bound nicely $\partial_x^j u$ in terms of g_j, even away from $y = a$. This is why we substitute to this formula the decomposition

$$\partial_x^j u(t,x,y) = \omega(t,x,y) \int_3^y \left(\psi + \frac{(1-\psi)}{\omega} \right)^{-1} \frac{g_j}{\omega^2} + C_j(t,x)\omega(t,x,y)$$

with a first term that depends nicely on g_j away from $y = a$, and a second term that, broadly speaking, will be controlled by the hydrostatic energy. $\qquad \square$

Let us conclude this paragraph by a trivial remark: for any energy functional $E = E_\omega, E_h, \ldots$ that will be used in the text, one has $E \leq \frac{2}{\tau}\partial_\tau E$. This will be used many times without mentioning.

3.2. Estimate of the vorticity energy

We focus here on the energy $\dot{E}_\omega(t,\tau)$ defined in (2.8). We also introduce the functional

$$(3.7) \qquad \dot{D}_\omega(t,\tau) := \sum_{j=0}^{+\infty} \left(\tau^{j+1} \, (j!)^{-7/4} \, (j+1)^{10} \right)^2 \, \|\partial_y \omega(t,\cdot)\|_{\dot{\mathcal{H}}_\gamma^j}^2$$

which will appear due to the viscous term in the Prandtl equation. We shall establish

PROPOSITION 1 (Estimate on the vorticity energy). − *Let $T > 0$, $\tau \le \tau_0$, s, γ, σ be as in Theorem 1. Let u be a smooth solution of the Prandtl equation over $]0, T]$. Assume that the vorticity ω satisfies (3.1) over $]0, T]$, and that*

$$E_\omega(t,\tau) \le M, \quad M > 0, \quad \forall t \in]0, T].$$

Then, for all $t \in]0, T]$:

$$\partial_t \dot{E}_\omega(t,\tau) + \dot{D}_\omega(t,\tau) \lesssim_{M,\tau} \partial_\tau E_\omega(t,\tau).$$

The starting point is to write down an equation on $\omega_J := \partial^J \omega$, $J = (j_1, j_2) \in \mathbb{N}^2$, $0 < j_2 \le s$. Differentiating the vorticity Equation (2.2), we find

$$\partial_t \omega_J + u \partial_x \omega_J + v \partial_y \omega_J - \partial_y^2 \omega_J = -[\partial^J, u] \partial_x \omega - [\partial^J, v] \partial_y \omega.$$

After multiplication by $(1 + y)^{2\gamma + 2j_2} \omega_J$, integration over $\mathbb{T} \times \mathbb{R}_+$ and standard integration by parts, we get

$$\frac{1}{2} \frac{d}{dt} \|(1+y)^{\gamma+j_2} \omega_J(t,\cdot)\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}^2 + \|(1+y)^{\gamma+j_2} \partial_y \omega_J(t,\cdot)\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}^2$$

$$= -\int_{\mathbb{T}\times\mathbb{R}_+} (1+y)^{2\gamma+2j_2} [\partial^J, u] \partial_x \omega \, \omega_J - \int_{\mathbb{T}\times\mathbb{R}_+} (1+y)^{2\gamma+2j_2} [\partial^J, v] \partial_y \omega \, \omega_J$$

$$- (2\gamma + 2j_2) \int_{\mathbb{T}\times\mathbb{R}_+} (1+y)^{2\gamma+2j_2-1} (\partial_y \omega_J - v \omega_J) \, \omega_J + \int_{\mathbb{T}\times\{0\}} \partial_y \omega_J \, \omega_J.$$

Multiplying by $\left(\tau^{|J|+1} (|J|!)^{-7/4} (|J|+1)^{10} \right)^2$ and summing over $J \in \mathbb{N} \times [\![1, s]\!]$, we obtain

$$\partial_t \dot{E}_\omega(t,\tau) + \dot{D}_\omega(t,\tau)$$

$$\le \big(\|A_j\|_{l^2(\tau)} + \|B_j\|_{l^2(\tau)} + (2\gamma + 2s)\|C_j\|_{l^2(\tau)}$$

$$+ (2\gamma + 2s)\|D_j\|_{l^2(\tau)} \big) \|(j+1)^{1/2}\|\omega\|_{\dot{\mathcal{H}}_\gamma^j}\|_{l^2(\tau)} + \|E_j\|_{l^1(\tau)}$$

$$\lesssim_\tau \big(\|A_j\|_{l^2(\tau)} + \|B_j\|_{l^2(\tau)} + \|C_j\|_{l^2(\tau)} + \|D_j\|_{l^2(\tau)} \big) \sqrt{\partial_\tau \dot{E}_\omega(t,\tau)} + \|E_j\|_{l^1(\tau)}$$

where

$$(3.8) \qquad A_j^2 := \frac{1}{j+1} \sum_{\substack{J=(j_1,j_2)\in\mathbb{N}^2 \\ |J|=j,\, 0<j_2\le s}} \|(1+y)^{\gamma+j_2}[\partial^J,u]\partial_x\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}^2,$$

$$(3.9) \qquad B_j^2 := \frac{1}{j+1} \sum_{\substack{J=(j_1,j_2)\in\mathbb{N}^2 \\ |J|=j,\, 0<j_2\le s}} \|(1+y)^{\gamma+j_2}[\partial^J,v]\partial_y\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}^2,$$

$$(3.10) \qquad C_j^2 := \frac{1}{j+1} \sum_{\substack{J=(j_1,j_2)\in\mathbb{N}^2 \\ |J|=j,\, 0<j_2\le s}} \|(1+y)^{\gamma+j_2-1}v\,\omega_J\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}^2,$$

$$(3.11) \qquad D_j^2 := \frac{1}{j+1} \sum_{\substack{J=(j_1,j_2)\in\mathbb{N}^2 \\ |J|=j,\, 0<j_2\le s}} \|(1+y)^{\gamma+j_2-1}\partial_y\omega_J\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}^2,$$

$$(3.12) \qquad E_j = \sum_{\substack{J=(j_1,j_2)\in\mathbb{N}^2 \\ |J|=j,\, 0<j_2\le s}} \int_{\mathbb{T}\times\{0\}} \partial_y\omega_J\,\omega_J.$$

Note that the first term at the right-hand side comes from Cauchy-Schwarz inequality, first in $L^2(\mathbb{T}\times\mathbb{R}_+)$, then in $l^2(\mathbb{N}\times[\![1,s]\!])$.

Estimate on A_j. – We first write, for $J\in\mathbb{N}\times[\![1,s]\!]$, $|J|=j$:

$$(1+y)^{\gamma+j_2}[\partial^J,u]\partial_x\omega = \sum_{\substack{K\le J \\ |K|>0}} \binom{J}{K}(1+y)^{k_2}\partial^K u\,(1+y)^{\gamma+j_2-k_2}\partial^{J-K}\partial_x\omega.$$

As $\gamma\ge 1$, we obtain

$$(1+y)^{\gamma+j_2}\big|[\partial^J,u]\partial_x\omega\big| \le \sum_{\substack{K\le J \\ |K|>0}} \binom{J}{K}\big|(1+y)^{\gamma+k_2-1}\partial^K u\big|\,\big|(1+y)^{\gamma+j_2-k_2}\partial^{J-K}\partial_x\omega\big|.$$

Then, we write

$$(3.13) \quad \|(1+y)^{\gamma+j_2}[\partial^J,u]\partial_x\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}$$

$$\le \sum_{\substack{K\le J \\ \frac{|J|}{2}\ge|K|>0}} \binom{J}{K}a_K^1\,b_{J-K}^1 + \sum_{\substack{K\le J \\ J\ge|K|>\frac{|J|}{2}}} \binom{J}{K}a_K^2\,b_{J-K}^2$$

where

$$a_K^1 := \|(1+y)^{\gamma+k_2-1}\partial^K u\|_{L^\infty(\mathbb{T}\times\mathbb{R}_+)}, \quad b_{J-K}^1 := \|(1+y)^{\gamma+j_2-k_2}\partial^{J-K}\partial_x\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}$$

$$a_K^2 := \|(1+y)^{\gamma+k_2-1}\partial^K u\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}, \quad b_{J-K}^2 := \|(1+y)^{\gamma+j_2-k_2}\partial^{J-K}\partial_x\omega\|_{L^\infty(\mathbb{T}\times\mathbb{R}_+)}.$$

Treatment of the first sum. – We first notice that $\binom{J}{K}\le\binom{|J|}{|K|}$, uniformly for $J\in\mathbb{N}\times[\![1,s]\!]$. Then, using Sobolev and Hardy inequalities (A.1), (A.2), we obtain easily that

$$a_K^1 \le C_a\left(\|\omega\|_{\mathscr{H}_\gamma^{|K|}} + \|\omega\|_{\mathscr{H}_\gamma^{|K|+1}}\right),$$

for an absolute constant C_a (only depending on γ). Also,

$$b_{J-K}^1 \leq \|\omega\|_{\mathscr{H}_\gamma^{|J-K|+1}}.$$

Denoting $k := |K|$, $j := |J|$, we obtain for some absolute constant C

$$\frac{1}{(j+1)^{1/2}} \sum_{\substack{K \leq J \\ \frac{|J|}{2} \geq |K| > 0}} \binom{J}{K} a_K^1 \, b_{J-K}^1$$

$$\leq \frac{C}{(j+1)^{1/2}} \sum_{k=1}^{\frac{j}{2}} \binom{j}{k} \|\omega\|_{\mathscr{H}_\gamma^k} \|\omega\|_{\mathscr{H}_\gamma^{j-k+1}} + \frac{C}{(j+1)^{1/2}} \sum_{k=1}^{\frac{j}{2}} \binom{j}{k} \|\omega\|_{\mathscr{H}_\gamma^{k+1}} \|\omega\|_{\mathscr{H}_\gamma^{j-k+1}}$$

$$\leq \frac{C}{(j+1)^{1/2}} \sum_{k=0}^{\frac{j}{2}-1} \binom{j}{k+1} \|\omega\|_{\mathscr{H}_\gamma^{k+1}} \|\omega\|_{\mathscr{H}_\gamma^{j-k}} + \frac{C}{(j+1)^{1/2}} \sum_{k=0}^{\frac{j}{2}-1} \binom{j}{k+1} \|\omega\|_{\mathscr{H}_\gamma^{k+2}} \|\omega\|_{\mathscr{H}_\gamma^{j-k}}$$

$$\leq C \sum_{k=0}^{\frac{j}{2}} \binom{j}{k} \|\omega\|_{\mathscr{H}_\gamma^{k+1}} \left((j-k)^{1/2} \|\omega\|_{\mathscr{H}_\gamma^{j-k}} \right) + C \sum_{k=0}^{\frac{j}{2}} \binom{j}{k} \|\omega\|_{\mathscr{H}_\gamma^{k+2}} \left((j-k)^{1/2} \|\omega\|_{\mathscr{H}_\gamma^{j-k}} \right).$$

We finally control the $l^2(\tau)$ norm of the right-hand side using Lemma 1 (we take $m = 1$ for the first term, $m = 2$ for the second term). We end up with
(3.14)

$$\left\| \frac{1}{(j+1)^{1/2}} \sum_{\substack{J=(j_1,j_2) \in \mathbb{N}^2 \\ |J|=j, \, 0 < j_2 \leq s}} \sum_{\substack{K \leq J \\ \frac{|J|}{2} \geq |K| > 0}} \binom{J}{K} a_K^1 \, b_{J-K}^1 \right\|_{l^2(\tau)} \lesssim_\tau \sqrt{E_\omega(t,\tau)} \sqrt{\partial_\tau E_\omega(t,\tau)}$$

$$\lesssim_\tau \sqrt{M} \sqrt{\partial_\tau E_\omega(t,\tau)}.$$

Treatment of the second sum. – We proceed as for the first sum, reversing the role of factors a and b: more precisely, we apply the Hardy inequality (A.2) to a_K^2:

$$a_K^2 \leq C_a \|\omega\|_{\mathscr{H}_\gamma^{|K|}}$$

and the Sobolev bound (A.1) to b_{J-K}^2. Note that, due to the last two terms at the right-hand side of (A.1), the quantities $\partial_y \partial^{J-K} \partial_x \omega$ and $\partial_y \partial^{J-K} \partial_x^2 \omega$ are needed to control b_{J-K}^2. In the special case where $j_2 = s$, $k_2 = 0$, they involve $s + 1$ ∂_y-derivatives. Thus, they cannot be controlled by $\|\omega\|_{\mathscr{H}_\gamma^{|J-K|+m}}$, similarly to what we did for a_K^1 (the definition of $\mathscr{H}_\gamma^j$ spaces involves only $j_2 \leq s$ ∂_y-derivatives). Hence, in this special case, we rather use the inequality

$$\|(1+y)^{\gamma+j_2-k_2} \partial_y \partial^{J-K} \partial_x^m \omega\|_{L^2} \leq C \|\partial_y \omega\|_{\dot{\mathscr{H}}^{|J-K|+m}}, \quad m = 1, 2.$$

We insist that the r.h.s involves the homogeneous space $\dot{\mathscr{H}}^{|J-K|+m}$, because $j_2 - k_2 \neq 0$ ($j_2 = s$, $k_2 = 0$). This is important to use the dissipation term $\dot{D}_\omega$. Eventually, we find

$$b_{J-K}^2 \leq C_b \sum_{m=1}^{3} \left(\|\omega\|_{\mathscr{H}_\gamma^{|J-K|+m}} + \|\partial_y \omega\|_{\dot{\mathscr{H}}_\gamma^{|J-K|+m}} \right)$$

for some absolute constant C_b. Then,

$$\frac{1}{(j+1)^{1/2}} \sum_{\substack{K \leq J \\ J \geq |K| > \frac{|J|}{2}}} \binom{J}{K} a_K^2 \, b_{J-K}^2$$

$$\leq \frac{C}{(j+1)^{1/2}} \sum_{m=1}^{3} \sum_{k=\frac{j}{2}}^{j} \binom{j}{k} \|\omega\|_{\mathcal{H}_\gamma^k} \left(\|\omega\|_{\mathcal{H}_\gamma^{j-k+m}} + \|\partial_y \omega\|_{\dot{\mathcal{H}}_\gamma^{j-k+m}} \right)$$

for an absolute constant C. We crudely bound $(j+1)^{-1/2}$ by 1, and apply Lemma 1, to find

$$\left\| \frac{1}{(j+1)^{1/2}} \sum_{\substack{J=(j_1,j_2)\in\mathbb{N}^2 \\ |J|=j,\, 0<j_2\leq s}} \sum_{\substack{K \leq J \\ J \geq |K| > \frac{|J|}{2}}} \binom{J}{K} a_K^2 \, b_{J-K}^2 \right\|_{l^2(\tau)}$$

$$(3.15) \qquad\qquad \lesssim_\tau \sqrt{E_\omega(t,\tau)} \left(\sqrt{E_\omega(t,\tau)} + \sqrt{\dot{D}_\omega(t,\tau)} \right)$$

$$\lesssim_\tau \sqrt{M} \left(\sqrt{E_\omega(t,\tau)} + \sqrt{\dot{D}_\omega(t,\tau)} \right).$$

Gathering of the bounds (3.14)-(3.15) leaves us with

$$\|A_j\|_{l^2(\tau)} \lesssim_{\tau,M} \left(\sqrt{E_\omega(t,\tau)} + \sqrt{\partial_\tau E_\omega(t,\tau)} + \sqrt{\dot{D}_\omega(t,\tau)} \right)$$

$$(3.16) \qquad\qquad \lesssim_{\tau,M} \left(\sqrt{\partial_\tau E_\omega(t,\tau)} + \sqrt{\dot{D}_\omega(t,\tau)} \right).$$

Estimate on B_j. – Let us first point out that $v = -\int_0^y \partial_x u$ does not decay at $y = \infty$. When $K = (k,0)$, one has

(3.17)

$$\|\partial^K v\|_{L^\infty} = \|\partial_x^k v\|_{L^\infty} \leq \left\| \int_0^{+\infty} |\partial_x^{k+1} u| \, dy \right\|_{L^\infty(\mathbb{T})}$$

$$\leq C_v \|\partial_x^{k+1} u\|_{L^\infty(L^2_{\gamma-1})} \leq C_v' \left(\|\partial_x^{k+1} u\|_{L^2(L^2_{\gamma-1})} + \|\partial_x^{k+2} u\|_{L^2(L^2_{\gamma-1})} \right)$$

$$\leq C_v'' \left(\|\partial_x^{k+1} \omega\|_{L^2(L^2_\gamma)} + \|\partial_x^{k+2} \omega\|_{L^2(L^2_\gamma)} \right)$$

where the last line comes from (A.3) (constants C_v, C_v', C_v'' are absolute). When $K = (k_1, k_2)$ with $k_2 > 0$, we use that $\gamma \geq 2$, which gives

$$(3.18) \quad \|(1+y)^{k_2} \partial^K v\|_{L^\infty} \leq \|(1+y)^{\gamma+k_2-2} \partial^K v\|_{L^\infty} = \|(1+y)^{\gamma+k_2-2} \partial_x^{k_1+1} \partial_y^{k_2-1} u\|_{L^\infty}$$

which, as seen before, can be controlled by $\|\omega\|_{\mathcal{H}_\gamma^{|K|}} + \|\omega\|_{\mathcal{H}_\gamma^{|K|+1}}$ through (A.1) and (A.2).

Proceeding as for A_j, we get

$$(3.19) \quad \|(1+y)^{\gamma+j_2} [\partial^J, v] \partial_y \omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}$$

$$\leq \sum_{\substack{K \leq J \\ \frac{|J|}{2} \geq |K| > 0}} \binom{J}{K} a_K^1 \, b_{J-K}^1 + \sum_{\substack{K \leq J, \\ j_2-k_2 \leq s-2 \\ J \geq |K| > \frac{|J|}{2}}} \binom{J}{K} a_K^2 \, b_{J-K}^2 + \sum_{\substack{K \leq J, \\ s-1 \leq j_2-k_2 \leq s \\ J \geq |K| > \frac{|J|}{2}}} \binom{J}{K} a_K^3 \, b_{J-K}^3.$$

where this time

$$a^1_K := \|(1+y)^{k_2}\partial^K v\|_{L^\infty(\mathbb{T}\times\mathbb{R}_+)}, \quad b^1_{J-K} := \|(1+y)^{\gamma+j_2-k_2}\partial^{J-K}\partial_y\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}$$

$$a^2_K := \|(1+y)^{k_2-1}\partial^K v\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}, b^2_{J-K} := \|(1+y)^{\gamma+j_2-k_2+1}\partial^{J-K}\partial_y\omega\|_{L^\infty(\mathbb{T}\times\mathbb{R}_+)},$$

$$a^3_K := \|(1+y)^{k_2}\partial^K v\|_{L^\infty(\mathbb{T}\times\mathbb{R}_+)}, \quad b^3_{J-K} := \|(1+y)^{\gamma+j_2-k_2}\partial^{J-K}\partial_y\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}.$$

Treatment of the first sum. – Thanks to (3.17) and (3.18), we find (for an absolute constant C_a only depending on γ)

$$a^1_K \leq C_a \sum_{m=0}^{1} \|\omega\|_{\mathscr{H}^{|K|+m}_\gamma}.$$

Regarding b^1_{J-K}, one must take care of the special case: $j_2 = s$, $k_2 = 0$, for which $\partial^{J-K}\partial_y\omega$ involves $s+1$ derivatives with respect to y. Hence, we write

$$b^1_{J-K} \leq \|\omega\|_{\mathscr{H}^{|J-K|+1}_\gamma} + \|\partial_y\omega\|_{\dot{\mathscr{H}}^{|J-K|}_\gamma}$$

where the last term at the right-hand side accounts for the special case. Proceeding as for A_j, relying on Lemma 1, we find

$$(3.20) \quad \|\frac{1}{(j+1)^{1/2}} \sum_{\substack{J=(j_1,j_2)\in\mathbb{N}^2 \\ |J|=j,\, 0<j_2\leq s}} \sum_{\substack{K\leq J \\ \frac{|J|}{2}\geq|K|>0}} \binom{J}{K} a^1_K\, b^1_{J-K}\|_{l^2(\tau)}$$

$$\lesssim_\tau \sqrt{M}\left(\sqrt{\partial_\tau E_\omega(t,\tau)} + \sqrt{\dot{D}_\omega(t,\tau)}\right).$$

Treatment of the second sum. – We handle the second sum similarly: we apply Hardy inequality to a^2_K, Sobolev inequality to b^2_K. Remark that we put factor $(1+y)^{k_2-1}$ in front of $\partial^K v$, in the definition of a^2_K. It allows the L^2 norm to be finite even in the case $k_2 = 0$. More precisely, we apply in this case (A.3), and then (A.2). We get

$$(3.21) \quad \begin{aligned} a^2_K = a^2_{(k,0)} &= \|(1+y)^{-1}\partial^k_x v\|_{L^2} \\ &\leq C_a \|\partial^k_x\partial_y v\|_{L^2} = C_a\|\partial^{k+1}_x u\|_{L^2} \leq C'_a\|(1+y)\partial^{k+1}_x\omega\|_{L^2} \leq C'_a\|\omega\|_{\mathscr{H}^{k+1}_\gamma}. \end{aligned}$$

For $k_2 > 0$, we can use directly (A.2) to find

$$(3.22) \quad a^2_K = \|(1+y)^{k_2-1}\partial^{k_1+1}_x\partial^{k_2-1}_y u\|_{L^2} \leq C_a\|\omega\|_{\mathscr{H}^k_\gamma}$$

(for another absolute constant C_a).

Regarding b^2_K, the factor $(1+y)^{k_2-1}$ in front of $\partial^K v$ forces a factor $(1+y)^{\gamma+j_2-k_2+1}$ in front of $\partial^{J-K}\partial_y\omega$, but it is harmless thanks to the extra ∂_y. Moreover, as we restrict here to indices satisfying $j_2 - k_2 \leq s-2$, the control of b^2_K by the inequality (A.1) (which involves for instance $\partial^{J-K}\partial^2_y\omega$) only requires terms with less than s derivatives in y. We leave the details to the reader. We get $b^2_{J-K} \leq C_b \sum_{m=1}^3 \|\omega\|_{\mathscr{H}^{j-k+m}_\gamma}$.

Let us stress that the estimate (3.21) for $k_2 = 0$ is not so good as estimate (3.22) for $k_2 > 0$. Indeed, the upper bound for $k_2 = 0$ involves one more x derivative of ω. But thanks to the assumption $j_2 > 0$ (together with $K \leq J$), one always have $k+1 \leq j_1 + 1 \leq j$: one never

needs more than j x-derivatives. From a practical point of view, this allows to make a change of index $k' := k + 1$ in the sum corresponding to indices $k_2 = 0$. Namely, we get

$$
\left\| \frac{1}{(j+1)^{1/2}} \sum_{\substack{J=(j_1,j_2)\in\mathbb{N}^2 \\ |J|=j, \\ 0<j_2\leq s}} \sum_{\substack{K\leq J, k_2=0 \\ j_2-k_2\leq s-2 \\ J\geq|K|>\frac{|J|}{2}}} \binom{J}{K} a_K^2 \, b_{J-K}^2 \right\|_{l^2(\tau)}
$$

(3.23)
$$
\leq C \sum_{k'=\frac{j}{2}}^{j} \binom{j}{k'} (k')^{1/2} \|\omega\|_{\mathscr{H}_\gamma^{k'}} \sum_{m=2}^{4} \|\omega\|_{\mathscr{H}_\gamma^{j-k'+m}}
$$
$$
\lesssim_\tau \sqrt{\partial_\tau E_\omega(t,\tau)} \sqrt{E_\omega(t,\tau)} \lesssim_\tau \sqrt{M} \sqrt{\partial_\tau E_\omega(t,\tau)}.
$$

Meanwhile,
(3.24)
$$
\left\| \frac{1}{(j+1)^{1/2}} \sum_{\substack{J=(j_1,j_2)\in\mathbb{N}^2 \\ |J|=j, \\ 0<j_2\leq s}} \sum_{\substack{K\leq J, k_2>0 \\ j_2-k_2\leq s-2 \\ J\geq|K|>\frac{|J|}{2}}} \binom{J}{K} a_K^2 \, b_{J-K}^2 \right\|_{l^2(\tau)} \lesssim_\tau E_\omega(t,\tau) \lesssim_\tau \sqrt{M} \sqrt{E_\omega(t,\tau)},
$$

and finally

(3.25)
$$
\left\| \frac{1}{(j+1)^{1/2}} \sum_{\substack{J=(j_1,j_2)\in\mathbb{N}^2 \\ |J|=j, \\ 0<j_2\leq s}} \sum_{\substack{K\leq J, \\ j_2-k_2\leq s-2 \\ J\geq|K|>\frac{|J|}{2}}} \binom{J}{K} a_K^2 \, b_{J-K}^2 \right\|_{l^2(\tau)} \lesssim_\tau \sqrt{M} \sqrt{\partial_\tau E_\omega(t,\tau)}.
$$

Treatment of the third sum. – Note that the third sum is empty except when $j_2 = s$ (in which case $k_2 \in \{0,1\}$) or when $j_2 = s - 1$ (in which case $k_2 = 0$). In both cases, the important thing to notice is that

(3.26) $|K| = k_1 + k_2 \leq j_1 + k_2 \leq |J| - s + 1,$ and $\binom{J}{K} \leq \binom{|J|}{|K|+m}$ $\forall m \leq s - 1.$

Using once again (3.17), (3.18) and (A.1), we end up with

$$
a_K^3 \leq C_a \sum_{m=0}^{1} \|\omega\|_{\mathscr{H}_\gamma^{|K|+m}}
$$

whereas

$$
b_K^3 \leq \|\partial_y \omega\|_{\dot{\mathscr{H}}_\gamma^{|J-K|}}.
$$

We get, taking (3.26) into account:

$$
\frac{1}{(j+1)^{1/2}} \sum_{\substack{K\leq J, \\ s-1\leq j_2-k_2\leq s \\ J\geq|K|>\frac{|J|}{2}}} \binom{J}{K} a_K^3 \, b_{J-K}^3
$$
$$
\leq \frac{C}{(j+1)^{1/2}} \sum_{m=0}^{1} \sum_{k=\frac{j}{2}+1}^{j-s+1} \binom{j}{k+m} \|\omega\|_{\mathscr{H}_\gamma^{k+m}} \|\partial_y \omega\|_{\dot{\mathscr{H}}_\gamma^{j-k}}.
$$

We can bound $\frac{1}{(j+1)^{1/2}}$ by 1, and make a change of index: $k' = k + m$. Then, application of Lemma 1 leads to

$$
\left\| \frac{1}{(j+1)^{1/2}} \sum_{\substack{K \leq J, \\ s-1 \leq j_2 - k_2 \leq s \\ J \geq |K| > \frac{|J|}{2}}} \binom{J}{K} a_K^3 \, b_{J-K}^3 \right\|_{l^2(\tau)} \lesssim_\tau \sqrt{E_\omega(t,\tau)} \sqrt{\dot{D}_\omega(t,\tau)}
$$

(3.27)

$$
\lesssim_\tau \sqrt{M} \sqrt{\dot{D}_\omega(t,\tau)}.
$$

Gathering of the bounds (3.20)-(3.25)-(3.27) leads to

$$
\|B_j\|_{l^2(\tau)} \lesssim_{\tau,M} \left(\sqrt{E_\omega(t,\tau)} + \sqrt{\partial_\tau E_\omega(t,\tau)} + \sqrt{\dot{D}_\omega(t,\tau)} \right)
$$

(3.28)

$$
\lesssim_{\tau,M} \left(\sqrt{\partial_\tau E_\omega(t,\tau)} + \sqrt{\dot{D}_\omega(t,\tau)} \right).
$$

Estimate on C_j. – By a combination of (A.1) and (A.3), we get that

$$
\left\| \frac{v}{1+y} \right\|_{L^\infty} \leq C_0 \sum_{m=1}^{2} \|(1+y)\partial_x^m \omega\|_{L^2(\mathbb{T} \times \mathbb{R}_+)}
$$

for some absolute constant C_0. It follows that

$$
|C_j| \leq \left\| \frac{v}{(1+y)} \right\|_{L^\infty} \|\omega\|_{\dot{\mathcal{H}}_\gamma^j} \lesssim_\tau \sqrt{E_\omega(t,\tau)} \, \|\omega\|_{\dot{\mathcal{H}}_\gamma^j} \lesssim_\tau \sqrt{M} \sqrt{E_\omega(t,\tau)}.
$$

Thus,

(3.29)
$$
\|C_j\|_{l^2(\tau)} \lesssim_{\tau,M} \sqrt{E_\omega(t,\tau)}.
$$

Estimate on D_j. – Clearly,

(3.30)
$$
\|D_j\|_{l^2(\tau)} \leq \sqrt{\dot{D}_\omega(t,\tau)}.
$$

Estimate on E_j. – To handle the boundary term, a simple application of the trace theorem is not enough. One shall adapt ideas from [20]. The main point is to reduce the number of derivatives in the boundary term $\partial_y \omega_I \, \omega_I|_{y=0}$, thanks to the equation. For instance, one can observe that

$$
\partial_y \omega|_{y=0} = (\partial_t u + u\partial_x u + v\partial_y u)|_{y=0} = 0.
$$

Then,

(3.31)
$$
\partial_y^3 \omega|_{y=0} = \partial_y (\partial_t \omega + u\partial_x \omega + v\partial_y \omega)|_{y=0} = \omega \partial_x \omega|_{y=0}.
$$

For higher derivatives, one has

LEMMA 4 (from [20, Lemma 5.9]). – *For $j_2 \geq 4$ an even number, $\partial_y \omega_J$ is a linear combination of terms of the form*

$$
\partial_x^{j_1} \left(\prod_{l=1}^{N} \partial_x^{\alpha_l} \partial_y^{\beta_l} \omega|_{y=0} \right)
$$

with

$$2 \le N \le \frac{j_2}{2}, \quad \begin{cases} \displaystyle\sum_{l=1}^{N} 3\alpha_l + \beta_l = j_2 + 1, \\[2mm] \displaystyle\sum_{l=1}^{N} \alpha_l \le \frac{j_2}{2} - 1, \quad \sum_{l=1}^{N} \beta_l \le j_2 - 2, \\[2mm] \alpha_l + \beta_l \le j_2 - 1, \quad \forall l = 1, \dots, N. \end{cases}$$

Besides this lemma, we need a slight generalization of Lemma 1, whose proof is left to the reader:

LEMMA 5. – *Let* $N \ge 2$, $m_2, \dots, m_N \le 5$. *Let*

$$K_1(j) := \left\{ (k_1, \dots, k_N) \in \mathbb{N}^N, \quad s.t. \ k_1 + \cdots + k_N = j, \quad k_1 \ge \frac{j}{N} \right\}$$

for all $j \in \mathbb{N}$. *Then, for sequences* a_j^l, $j \in \mathbb{N}$, $l = 1, \dots, N$, *one has*

$$\left\| \sum_{(k_1, \dots, k_N) \in K_1(j)} \frac{j!}{\prod_{l=1}^{N} k_l!} a_{k_1}^1 \prod_{l=2}^{N} a_{k_l + m_l}^l \right\|_{l^2(\tau)} \le C_{\tau, N} \prod_{l=1}^{N} \|a_j^l\|_{l^2(\tau)}.$$

We now write

$$E_j = \sum_{\substack{J \in \mathbb{N}^* \times 2\mathbb{N}, \\ |J| = j, \, 0 < j_2 \le s}} \fint_{\mathbb{T} \times \{0\}} \partial_y \omega_J \, \omega_J + \sum_{\substack{J \in \mathbb{N}^* \times (2\mathbb{N}+1), \\ |J| = j, \, 0 < j_2 \le s}} \fint_{\mathbb{T} \times \{0\}} \partial_y \omega_J \, \omega_J$$

$$+ \mathbf{1}_{[\![1, \dots, s]\!]}(j) \fint_{\mathbb{T} \times \{0\}} \partial_y^{j+1} \omega \, \partial_y^j \omega := E_j^1 + E_j^2 + E_j^3.$$

Study of E_j^1. – Let $J \in \mathbb{N}^* \times 2\mathbb{N}$, $|J| = j$, $0 < j_2 \le s$. We write

$$\left| \fint_{\mathbb{T} \times \{0\}} \partial_y \omega_J \, \omega_J \right| \le \|\partial_y \omega_J|_{y=0}\|_{L^2(\mathbb{T})} \|\omega_J|_{y=0}\|_{L^2(\mathbb{T})}$$

$$\le \|\partial_y \omega_J|_{y=0}\|_{L^2(\mathbb{T})} \|\omega_J\|_{L^2(\mathbb{T} \times \mathbb{R}_+)}^{1/2} \|\partial_y \omega_J\|_{L^2(\mathbb{T} \times \mathbb{R}_+)}^{1/2} \le \|\partial_y \omega_J|_{y=0}\|_{L^2(\mathbb{T})} \|\partial_y \omega\|_{\dot{\mathcal{H}}_\gamma^j}$$

applying the trace theorem and then the Hardy inequality (A.2) to the second factor. Regarding the first factor, one must use the reductions seen above. We focus on the case $j_2 \ge 4$ (and even), which is treated thanks to Lemma 4. The case $j_2 = 2$, involving (3.31), is simpler. The L^2 norm of $\|\partial_y \omega_J|_{y=0}\|_{L^2(\mathbb{T})}$ can be bounded by a finite number of terms of the type $\|\partial_x^{j_1} \left(\prod_{l=1}^{N} \partial_x^{\alpha_l} \partial_y^{\beta_l} \omega|_{y=0} \right)\|_{L^2(\mathbb{T})}$, where N and the (α_l, β_l)'s satisfy the conditions of the lemma.

As usual, we use the Lebnitz formula to write

$$\|\partial_x^{j_1} \left(\prod_{l=1}^{N} \partial_x^{\alpha_l} \partial_y^{\beta_l} \omega|_{y=0} \right)\|_{L^2(\mathbb{T})} = \| \sum_{k_1 + \cdots + k_n = j_1} \frac{j_1!}{\prod_{l=1}^{N} k_l!} \prod_{l=1}^{N} \partial_x^{\alpha_l + k_l} \partial_y^{\beta_l} \omega|_{y=0} \|_{L^2(\mathbb{T})}$$

$$\le \sum_{l'=1}^{N} \| \sum_{(k_1, \dots, k_n) \in K_{l'}(j_1)} \frac{j_1!}{\prod_{l=1}^{N} k_l!} \prod_{l=1}^{N} \partial_x^{\alpha_l + k_l} \partial_y^{\beta_l} \omega|_{y=0} \|_{L^2(\mathbb{T})} := \sum_{l'=1}^{N} \mathcal{N}_{l'}$$

where

$$K_l(j_1) := \left\{ (k_1, \dots, k_N) \in \mathbb{N}^N, \quad \text{s.t. } k_1 + \cdots + k_N = j_1, \quad k_l \geq \frac{j_1}{N} \right\}.$$

We now bound $\mathscr{N}_1$, the other terms being treated in the same way. We write

$$\mathscr{N}_1 \leq \sum_{(k_1, \dots, k_N) \in K_1(j_1)} \frac{j_1!}{\prod_{l=1}^{N} k_l!} \|\partial_x^{\alpha_1 + k_1} \partial_y^{\beta_1} \omega|_{y=0}\|_{L^2(\mathbb{T})} \prod_{l=2}^{N} \|\partial_x^{\alpha_l + k_l} \partial_y^{\beta_l} \omega\|_{L^\infty(\mathbb{T} \times \mathbb{R}_+)}.$$

Note that

$$\begin{cases} \beta_l + 2 \leq j_2 \leq s \quad \text{for all } l, \\ \alpha_1 + \beta_1 + k_1 + 1 \leq j, \quad \alpha_l + \beta_l + k_l + 2 \leq j, \quad l = 2, \dots, N \end{cases}$$

($k_l \leq j_1 - 1$ for $l \geq 2$, as $k_1 \geq \frac{j_1}{N}$). We use the trace theorem with the first factor:

$$\|\partial_x^{\alpha_1 + k_1} \partial_y^{\beta_1} \omega|_{y=0}\|_{L^2(\mathbb{T})} \leq C_0 \|\partial_x^{\alpha_1 + k_1} \partial_y^{\beta_1} \omega\|_{L^2(\mathbb{T} \times \mathbb{R}_+)}^{1/2} \|\partial_x^{\alpha_1 + k_1} \partial_y^{\beta_1 + 1} \omega\|_{L^2(\mathbb{T} \times \mathbb{R}_+)}^{1/2}$$

$$\leq C_0' \|(1+y)\partial_x^{\alpha_1 + k_1} \partial_y^{\beta_1 + 1} \omega\|_{L^2(\mathbb{T} \times \mathbb{R}_+)}$$

for absolute constants C_0, C_0'. The last bound comes from (A.2). Using the Sobolev imbedding (A.1) with the second factor, we get:

$$\mathscr{N}_1 \leq C_1 \sum_{(k_1, \dots, k_N) \in K_1(j_1)} \frac{j_1!}{\prod_{l=1}^{N} k_l!} \|\omega\|_{\dot{\mathscr{H}}_\gamma^{\alpha_1 + \beta_1 + k_1 + 1}} \prod_{l=2}^{N} \sum_{m=0}^{2} \|\omega\|_{\dot{\mathscr{H}}_\gamma^{\alpha_l + \beta_l + k_l + m}}$$

$$\leq C_1 \sum_{(k_1', \dots, k_N') \in K_1(j')} \frac{j'!}{\prod_{l=1}^{N} k_l'!} \|\omega\|_{\dot{\mathscr{H}}_\gamma^{k_1'}} \prod_{l=2}^{N} \sum_{m=0}^{2} \|\omega\|_{\dot{\mathscr{H}}_\gamma^{k_l' + m}}$$

for an absolute constant C_1. The last inequality involved the change of index

$$k_1' := \alpha_1 + \beta_1 + k_1 + 1, \quad k_l' := \alpha_l + \beta_l + k_l, \quad l \geq 2, \quad j' := j_1 + \sum_{l=1}^{N}(\alpha_l + \beta_l) + 1.$$

From Lemma 5, it follows that

$$\left(\sum_{j'} \left(\tau^{j'+1} (j'!)^{-7/4} (j'+1)^{10} \right)^2 |\mathscr{N}_1|^2 \right)^{1/2} \lesssim_\tau \left(\|\|\omega\|_{\dot{\mathscr{H}}_\gamma^j}\|_{l^2(\tau)} \right)^N.$$

Eventually, we point out that $j' \leq j$. Indeed, from the conditions expressed in Lemma 4, it is easily deduced that one of the α_l's is non-zero, and from there that $\sum(\alpha_l + \beta_l) \leq j_2 - 1$. It follows that

$$\tau^{j+1} (j!)^{-7/4} (j+1)^{10} \lesssim_\tau \tau^{j'+1} (j'!)^{-7/4} (j'+1)^{10}$$

for all j. Hence,

$$\left(\sum_{j} \left(\tau^{j+1} (j!)^{-7/4} (j+1)^{10} \right)^2 |\mathscr{N}_1|^2 \right)^{1/2} \lesssim_\tau \sqrt{E_\omega(t, \tau)}^N \lesssim_{\tau, M} \sqrt{E_\omega(t, \tau)}.$$

By symmetry, the same bound applies to $\mathscr{N}_2, \dots, \mathscr{N}_N$. Thus,

$$\|E_j^1\|_{l^1(\tau)} \lesssim_{\tau, M} \sqrt{\dot{E}_\omega(t, \tau)} \sqrt{\dot{D}_\omega(t, \tau)} \lesssim_{\tau, M} \eta \dot{D}_\omega(t, \tau) + \frac{1}{\eta} \dot{E}_\omega(t, \tau)$$

for all $\eta \in {]0,1]}$.

Study of E_j^2. – Let $J \in \mathbb{N}^* \times (2\mathbb{N}+1)$, $|J| = j$, $0 < j_2 \le s$. Note that $j_2 \le s-1$ (because s is assumed even). We integrate by parts with respect to x:

$$\int_{\mathbb{T}\times\{0\}} \partial_y \omega_J \, \omega_J = -\int_{\mathbb{T}\times\{0\}} \partial_x^{(j_1-1)}\partial_y^{j_2+1}\omega \, \partial_x^{j_1+1}\partial_y^{j_2}\omega.$$

Then, we apply the boundary reduction lemma to the second factor in the integrand. From there, the treatment is exactly the same as in the first case, and leads to

$$\|E_j^2\|_{l^1(\tau)} \lesssim_{\tau,M} \eta \dot{D}_\omega(t,\tau) + \frac{1}{\eta}\dot{E}_\omega(t,\tau)$$

for all $\eta \in {]0,1]}$.

Study of E_j^3. – Note that E_j^3 is non zero only if $j \le s$. When $j = s$, one uses Lemma 4. Otherwise,

$$\sum_{j=0}^{s-1}\left|\int_{\mathbb{T}\times\{0\}}\partial_y^{j+1}\omega\,\partial_y^j\omega\right| \le \sum_{j=0}^{s-1}\|\partial_y^{j+1}\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|\partial_y^j\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}^{1/2}\|\partial_y\partial_y^{j+1}\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}^{1/2}.$$

We end up with

$$\|E_j^3\|_{l^1(\tau)} \lesssim_{\tau,M} \sqrt{\dot{E}_\omega(t,\tau)}\sqrt{\dot{D}_\omega(t,\tau)} \lesssim_{\tau,M} \eta\dot{D}_\omega(t,\tau) + \frac{1}{\eta}\dot{E}_\omega(t,\tau)$$

for all $\eta \in {]0,1]}$.

Eventually,

(3.32) $$\|E_j\|_{l^1(\tau)} \lesssim_{\tau,M} \eta\dot{D}_\omega(t,\tau) + \frac{1}{\eta}\dot{E}_\omega(t,\tau).$$

Combining this last inequality with (3.16)-(3.28)-(3.29)-(3.30), we obtain

$$\partial_t\dot{E}_\omega(t,\tau) + \dot{D}_\omega(t,\tau) \lesssim_{\tau,M} \left(\sqrt{\partial_\tau E_\omega(t,\tau)} + \sqrt{\dot{D}_\omega(t,\tau)}\right)\sqrt{\dot{E}_\omega(t,\tau)} + \eta\dot{D}_\omega(t,\tau)$$
$$+ \frac{1}{\eta}\dot{E}_\omega(t,\tau)$$
$$\lesssim_{\tau,M} \eta\dot{D}_\omega(t,\tau) + \frac{1}{\eta}\partial_\tau E_\omega(t,\tau)$$

for all $\eta \in {]0,1]}$. where C, C_η, C_η'. Taking η small enough yields Proposition 1.

3.3. Estimate of the hydrostatic energy

This section is devoted to the hydrostatic energy $E_h(t,\tau)$ defined in (2.9), with its "viscous" counterpart

(3.33) $$D_h(t,\tau) := \sum_{j=0}^{+\infty}\left(\tau^{j+1}(j!)^{-7/4}(j+1)^{10}\right)^2\|\partial_y h_j(t,\cdot)\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}^2.$$

PROPOSITION 2 (Estimate on the hydrostatic energy). – *Under the same assumptions as in Theorem 2, one has for all $t \in {]0,T]}$:*

(3.34) $$\partial_t E_h(t,\tau) + D_h(t,\tau) \lesssim \left(\partial_\tau E_\omega(t,\tau) + \partial_\tau E_g^2(t,\tau)\right).$$

The first step of the proof is to establish the equation on the function h_j defined in (2.10):

$$h_j = \chi^a \frac{\partial_x^j \omega}{\sqrt{\partial_y \omega}}, \quad \chi^a(t,x,y) = \chi(y - a(t,x)),$$

with χ a truncation function near 0: More precisely, we choose $\chi(z) := \chi_0(z/\varepsilon) \in C_c^\infty(\mathbb{R})$, $0 \leq \chi_0 \leq 1$, satisfying

$$\chi_0 = 1 \text{ on } [-1,1], \quad \chi_0 = 0 \text{ outside } [-2,2],$$

and

(3.35) $$\varepsilon := \min\left(\frac{\inf a}{4}, \frac{3 - \sup a}{4}, \frac{\inf_{t,x,y=a} |\partial_y \omega|}{4 \sup_{t,x,y} |\partial_y^2 \omega|}\right).$$

Let us note that by Sobolev embedding, one has $\sup_{t,x,y} |\partial_y^2 \omega| \lesssim_\tau \sqrt{M}$, so that ε has a lower bound depending only on $M, \tau, \inf a, \sup a$. This fact will be implicitly used in the treatment of derivatives of χ^a, which involve negative powers of ε.

A simple application of Taylor's formula shows that for all $t, x, y \in [0,T] \times \mathbb{T} \times [-2\varepsilon, 2\varepsilon]$, one has

$$|\partial_y \omega(t,x,y)| \geq \frac{1}{2} \inf_{t,x,y=a} |\partial_y \omega| := \frac{1}{2}\theta_0$$

so that h_j is well-defined, with $|h_j| \leq \sqrt{\frac{2}{\theta_0}} |\partial_x^j u|$. Moreover, with our choice, it is compactly supported in $\{0 < y < 3\}$. Starting from the vorticity equation

$$\partial_t \omega + u\partial_x \omega + v\partial_y \omega - \partial_y^2 \omega = 0,$$

we get

$$\partial_t h_j + u\partial_x h_j + v\partial_y - \partial_y^2 h_j = -\frac{\chi^a}{\sqrt{\partial_y \omega}}[\partial_x^j, u]\partial_x \omega - \frac{\chi^a}{\sqrt{\partial_y \omega}}\left([\partial_x^j, v]\partial_y \omega - \partial_x^j v\partial_y \omega\right)$$

$$- [\frac{\chi^a}{\sqrt{\partial_y \omega}}, \partial_t + u\partial_x + v\partial_y - \partial_y^2]\partial_x^j \omega - \frac{\chi^a}{\sqrt{\partial_y \omega}}\partial_x^j v\partial_y \omega.$$

Note that we have singled out the term $\partial_x^j v\partial_y \omega = -\int_0^y \partial_x^{j+1} u\partial_y \omega$ in the commutator with $v\partial_y$. This is the hardest term to control, as it involves $j + 1$ derivatives with respect to x. We shall use a cancellation property similar to (2.5). However, such cancellation will not be enough, and we shall rely on the extra regularity offered by the monotonicity energy E_g^2.

Performing a standard energy estimate on the previous equation, multiplying by $\left(\tau^{j+1}(j!)^{-7/4}(j+1)^{10}\right)^2$, summing over j, we end up with

(3.36) $$\frac{1}{2}\partial_t E_h(t,\tau) + D_h(t,\tau)$$

$$\leq \sqrt{\frac{\tau}{2}}\left(\|A_j\|_{l^2(\tau)} + \|B_j\|_{l^2(\tau)} + \|C_j\|_{l^2(\tau)}\right)\sqrt{\partial_\tau E_h(t,\tau)} + \|D_j\|_{l^1(\tau)} + \|E_j\|_{l^1(\tau)}$$

where

$$A_j := \frac{1}{(j+1)^{1/2}} \| \frac{\chi^a}{\sqrt{\partial_y \omega}} [\partial_x^j, u] \partial_x \omega \|_{L^2},$$

$$B_j := \frac{1}{(j+1)^{1/2}} \| \frac{\chi^a}{\sqrt{\partial_y \omega}} \left([\partial_x^j, v] \partial_y \omega - \partial_x^j v \partial_y \omega \right) \|_{L^2},$$

$$C_j := \frac{1}{(j+1)^{1/2}} \| (\partial_t + u \partial_x + v \partial_y - \partial_y^2)\left(\frac{\chi^a}{\sqrt{\partial_y \omega}} \right) \partial_x^j \omega \|_{L^2},$$

$$D_j := 2 \int_{\mathbb{T} \times \mathbb{R}_+} \partial_y \frac{\chi^a}{\sqrt{\partial_y \omega}} \, \partial_y \partial_x^j \omega \, h_j$$

$$E_j := \int_{\mathbb{T} \times \mathbb{R}_+} \frac{\chi^a}{\sqrt{\partial_y \omega}} \partial_x^j v \, \partial_y \omega \, h_j.$$

Note that h_j is compactly supported in $\{y > 0\}$, so that no boundary integral is present on the right-hand side.

Estimate on A_j. – We proceed here as for the term A_j of Subsection 3.2. The treatment is actually simpler, as only x derivatives of u and ω are involved in the expression of A_j. By use of the Sobolev inequality (A.1), one never encounters more than one y-derivative, in particular never more than s. For brevity, we skip the details. We find

$$(3.37) \qquad \|A_j\|_{l^2(\tau)} \lesssim \sqrt{\partial_\tau E_\omega(t, \tau)}$$

(this upper bound on A_j does not actually involve the constant θ_1, see Remark 1).

Estimate on B_j. – Again, the treatment is parallel to the one of Subsection 3.2. Denoting by K^a the support of χ^a, we find that

$$B_j \lesssim \frac{1}{(j+1)^{1/2}} \sum_{k=1}^{j-1} \binom{j}{k} \|\partial_x^k v \, \partial_y \partial_x^{j-k} \omega\|_{L^2(\mathbb{T} \times K^a)}$$

$$\lesssim \frac{1}{(j+1)^{1/2}} \sum_{k=1}^{\frac{j}{2}} \binom{j}{k} \|\partial_x^k v\|_{L^\infty(\mathbb{T} \times K^a)} \|\partial_y \partial_x^{j-k} \omega\|_{L^2(\mathbb{T} \times K^a)}$$

$$+ \frac{1}{(j+1)^{1/2}} \sum_{k=\frac{j}{2}+1}^{j-1} \binom{j}{k} \|\partial_x^k v\|_{L^2(\mathbb{T} \times K^a)} \|\partial_y \partial_x^{j-k} \omega\|_{L^\infty(\mathbb{T} \times K^a)} := B_j^1 + B_j^2.$$

For the first term, we set $k' := k - 1$ and use (3.17) to get (we drop the prime)

$$B_j^1 \leq \frac{C}{(j+1)^{1/2}} \sum_{k=0}^{\frac{j}{2}-1} \binom{j}{k+1} \sum_{m=2}^{3} \|\omega\|_{\mathcal{H}_\gamma^{k+m}} \|\omega\|_{\mathcal{H}_\gamma^{j-k}}$$

$$\leq C \sum_{k=0}^{\frac{j}{2}} \sum_{m=2}^{3} \binom{j}{k} \|\omega\|_{\mathcal{H}_\gamma^{k+m}} \left((j-k+1)^{1/2} \|\omega\|_{\mathcal{H}_\gamma^{j-k}} \right)$$

for some absolute constant C. The second term B_j^2 is handled in a symmetric way, thanks to the fact that index k stops at $j - 1$. Applying Lemma 1, we eventually derive the bound:

$$(3.38) \qquad \|B_j\|_{l^2(\tau)} \lesssim \sqrt{E_\omega(t, \tau)} \sqrt{\partial_\tau E_\omega(t, \tau)} \lesssim \sqrt{\partial_\tau E_\omega(t, \tau)}.$$

Estimates on C_j, D_j. – Clearly,

$$(3.39) \qquad C_j \lesssim \frac{1}{(j+1)^{1/2}} \|\partial_x^j \omega\|_{L^2(\mathbb{T} \times K^a)}, \quad \|C_j\|_{l^2(\tau)} \lesssim \sqrt{E_\omega(t, \tau)}$$

and after integration by parts

$$D_j \lesssim \frac{1}{(j+1)^{1/2}} \|\partial_x^j \omega\|_{L^2(\mathbb{T} \times K^a)} \left(\|h_j\|_{L^2(\mathbb{T} \times \mathbb{R}_+)} + \|\partial_y h_j\|_{L^2(\mathbb{T} \times \mathbb{R}_+)} \right),$$

$$(3.40) \qquad \|D_j\|_{l^1(\tau)} \lesssim \sqrt{E_\omega(t, \tau)} \left(\sqrt{E_h(t, \tau)} + \sqrt{D_h(t, \tau)} \right)$$

$$\lesssim \eta D_h(t, \tau) + \frac{1}{\eta} E_\omega(t, \tau)$$

for all $\eta \in]0, 1]$. Let us stress that Lemma 2 allowed us to control E_h by E_ω.

Estimates on E_j. – It remains to handle the bad term:

$$E_j = \int_{\mathbb{T} \times \mathbb{R}_+} (\chi^a)^2 \partial_x^j v \partial_x^j \omega = - \int_{\mathbb{T} \times \mathbb{R}_+} \partial_y(\chi^a)^2 \partial_x^j v \partial_x^j u - \int_{\mathbb{T} \times \mathbb{R}_+} (\chi^a)^2 \partial_x^j \partial_y v \partial_x^j u$$

$$= \int_{\mathbb{T} \times \mathbb{R}_+} \partial_y(\chi^a)^2 \partial_x \left(\int_0^y \partial_x^j u \right) \partial_x^j u + \int_{\mathbb{T} \times \mathbb{R}_+} (\chi^a)^2 \partial_x \frac{(\partial_x^j u)^2}{2}$$

$$= - \int_{\mathbb{T} \times \mathbb{R}_+} \partial_y(\chi^a)^2 \int_0^y \partial_x^j u \, \partial_x^{j+1} u - \int_{\mathbb{T} \times \mathbb{R}_+} \left(\partial_{xy}(\chi^a)^2 \int_0^y \partial_x^j u \, \partial_x^j u + \partial_x(\chi^a)^2 \frac{(\partial_x^j u)^2}{2} \right)$$

$$:= E_j^1 + E_j^2.$$

Clearly,

$$(3.41) \qquad \|E_j^2\|_{l^1(\tau)} \lesssim \left\| \|\partial_x^j u\|_{L^2(\mathbb{T} \times K^a)} \right\|_{l^2(\tau)}^2 \lesssim \left\| \|\partial_x^j \omega\|_{L^2(\mathbb{T} \times K^a)} \right\|_{l^2(\tau)}^2 \lesssim E_\omega(t, \tau)$$

with Poincaré inequality allowing to go from u to ω.

We are left with the treatment of E_j^1. Note that for all $t \in [0, T]$, the integrand in $E_j^1 = E_j^1(t)$ is supported in

$$\{|y - a| \geq \varepsilon\} := \{(x, y), |y - a(t, x)| \geq \varepsilon\}, \quad \varepsilon \text{ defined in } (3.35).$$

We then recall the decomposition stated in Lemma 3: for all j,

$$(3.42) \qquad \partial_x^j u = u_g^j + C_j \, 1_{\{y > a\}} \, \omega$$

with

$$u_g^j := \omega(t, x, y) \int_0^y \left(\psi + \frac{(1 - \psi)}{\omega} \right)^{-1} \frac{g_j}{\omega^2} \text{ in } \{y < a\},$$

$$u_g^j := \omega(t, x, y) \int_3^y \left(\psi + \frac{(1 - \psi)}{\omega} \right)^{-1} \frac{g_j}{\omega^2} \text{ in } \{y > a\}.$$

Note that

$$(3.43) \qquad \|u_g^j\|_{L^2(\{\varepsilon \leq |y-a|, y \leq 3\})} \lesssim \|g_j\|_{L^2(\{y \leq 3\})}.$$

For $y < a$, we write

$$(3.44) \qquad \left| \int_{\{y<a\}} \partial_y (\chi^a)^2 \left(\int_0^y \partial_x^j u \right) \partial_x^{j+1} u \right| = \left| \int_{\{y<a\}} \partial_y (\chi^a)^2 \left(\int_0^y u_g^j \right) u_g^{j+1} \right|$$

$$\lesssim \|g_j\|_{L^2(\{y\leq 3\})} \|g_{j+1}\|_{L^2(\{y\leq 3\})}$$

$$\lesssim \|g_j\|_{L^2(\{y\leq 3\})} \|\partial_x^{j+1} \omega\|_{L^2(\{y\leq 3\})}.$$

For $y > a$, we rather write

$$(3.45)$$

$$\left| \int_{\{y>a\}} \partial_y (\chi^a)^2 \left(\int_0^y \partial_x^j u \right) \partial_x^{j+1} u \right|$$

$$= \left| \int_{\{y>a\}} \partial_y (\chi^a)^2 \left(\int_0^y u_g^j \right) \partial_x^{j+1} u + \int_{\{y>a\}} \partial_y (\chi^a)^2 C_j \left(\int_a^y \omega \right) \partial_x^{j+1} u \right|$$

$$\lesssim \| \int_0^y u_g^j \|_{L^2(\{a+\varepsilon\leq y\leq 3\})} \|\partial_x^{j+1} \omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} + \left| \int_{\{y>a\}} \partial_y (\chi^a)^2 C_j \left(\int_a^y \omega \right) \partial_x^{j+1} u \right|.$$

Note that although y is away from a, the function $\int_0^y u_g^j$ involves values of u_g^j near the critical curve $y = a$. As the quantity $\frac{g_j}{\omega^2}$ (involved in the definition of u_g^j) degenerates at $y = a$, a control like (3.43) is not obvious. Still, we claim:

LEMMA 6. −

$$(3.46) \qquad \| \int_0^y u_g^j \|_{L^2(\{a+\varepsilon\leq y\leq 3\})} \lesssim \|g_j\|_{L^2(\{y\leq 3\})} \lesssim \|\partial_x^j \omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}.$$

Furthermore, we have

$$(3.47)$$

$$\|C_j\|_{L^2(\mathbb{T})} \lesssim \|\partial_x^j \omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}, \quad \|\partial_x C_j\|_{L^2(\mathbb{T})} \lesssim \|\partial_x^{j+1} \omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} + \|\partial_x^j \omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}.$$

Finally, using above bounds and the decomposition

$$\partial_x^{j+1} u = \partial_x u_g^j + \partial_x C_j \omega + C_j \partial_x \omega \qquad \text{for } y > a$$

we end up with

$$\left| \int_{\{y>a\}} \partial_y (\chi^a)^2 \left(\int_0^y \partial_x^j u \right) \partial_x^{j+1} u \right|$$

$$\lesssim \|g_j\|_{L^2(\{y\leq 3\})} \|\partial_x^{j+1} \omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} + \left| \int_{\mathbb{T}\times\mathbb{R}_+} \partial_y (\chi^a)^2 C_j \left(\int_a^y \omega \right) \partial_x u_g^j \right|$$

$$+ \left| \int_{\mathbb{T}\times\mathbb{R}_+} \partial_y (\chi^a)^2 \partial_x \frac{C_j^2}{2} \omega \left(\int_0^y \omega \right) \right| + \|\partial_x^j \omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}^2$$

$$\lesssim \|g_j\|_{L^2(\{y\leq 3\})} \|\partial_x^{j+1} \omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} + \|\partial_x^j \omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}^2$$

after integration by parts of the integral terms at the right-hand side. The structure of the second integral term is crucial. Indeed, the integrand

$$\partial_y (\chi^a)^2 (\partial_x C_j \omega)(C_j \int_a^y \omega) = \partial_y (\chi^a)^2 \partial_x \frac{C_j^2}{2} \omega \left(\int_0^y \omega \right)$$

can be integrated by parts. Note that a more general bilinear term in C_j and $\partial_x C_j$ would generically involve an upper bound like $\|C_j\|_{L^2} \|\partial_x C_j\|_{L^2}$. Such bound would ruin our strategy, based on anisotropic energy.

From the previous bound, from inequality (3.44) and Cauchy-Schwarz in $l^2(\tau)$, we get

$$\|E_j^1\|_{l^1(\tau)} \lesssim \left\| \frac{1}{(j+1)^{5/4}} \|\partial_x^{j+1}\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} \right\|_{l^2(\tau)} \left\| (j+1)^{5/4} \|g_j\|_{L^2(\{y\leq 3\})} \right\|_{l^2(\tau)}$$
$$+ \left\| \|\partial_x^j\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} \right\|_{l^2(\tau)}^2.$$

Here, we remark that

(3.48)

$$\left\| \frac{1}{(j+1)^{5/4}} \|\partial_x^{j+1}\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} \right\|_{l^2(\tau)}^2 = \sum_{j=0}^{+\infty} \frac{1}{(j+1)^{5/2}} \left(\tau^{j+1}(j!)^{-7/4}(j+1)^{10} \right)^2 \|\omega\|_{\mathscr{H}_\gamma^{j+1}}^2$$
$$\lesssim \sum_{j=1}^{+\infty} (j+1) \left(\tau^j(j!)^{-7/4}(j+1)^{10} \right)^2 \|\omega\|_{\mathscr{H}_\gamma^j}^2$$
$$\lesssim \left\| \sqrt{j+1} \|\omega\|_{\mathscr{H}_\gamma^j} \right\|_{l^2(\tau)}^2 \lesssim \partial_\tau E_\omega(t,\tau).$$

To conclude, it remains to evaluate the $l^2(\tau)$ norm of $(j+1)^{5/4} \|g_j\|_{L^2(\{y\leq 3\})}$. We use the following bound, to be proved in appendix

LEMMA 7. – *For all $\alpha \geq 0$,*

(3.49) $\left\| \|(j+1)^{\alpha/4}g_j\|_{L^2(\{y\leq 3\})} \right\|_{l^2(\tau)}$
$$\lesssim \left\| \|(j+1)^{\alpha/4}\tilde{g}_j\|_{L^2(\{y\leq 3\})} \right\|_{l^2(\tau)} + \left\| (j+1)^{(\alpha-3)/4}\|\omega\|_{\mathscr{H}_\gamma^j} \right\|_{l^2(\tau)}.$$

For $\alpha = 5$, we find

(3.50) $\left\| \|(j+1)^{5/4}g_j\|_{L^2(\{y\leq 3\})}\|_{l^2(\tau)} \lesssim \sqrt{\partial_\tau E_g^2(t,\tau)} + \sqrt{\partial_\tau E_\omega(t,\tau)}.$

We inject this bound and bound (3.48) in the estimate of $\|E_j^1\|_{l^1(\tau)}$, we get

(3.51) $\|E_j^1\|_{l^1(\tau)} \lesssim \sqrt{\partial_\tau E_\omega(t,\tau)}\sqrt{\partial_\tau E_g^2(t,\tau)} + \partial_\tau E_\omega(t,\tau) \lesssim \partial_\tau E_\omega(t,\tau) + \partial_\tau E_g^2(t,\tau).$

Combination of this inequality and inequality (3.41) yields

(3.52) $$\|E_j\|_{l^1(\tau)} \lesssim \partial_\tau E_\omega(t,\tau) + \partial_\tau E_g^2(t,\tau).$$

Collecting (3.37)-(3.38)-(3.39)-(3.40)-(3.52) leads to

$$\partial_t E_h(t,\tau) + D_h(t,\tau) \lesssim \eta D_h(t,\tau) + \frac{1}{\eta} \partial_\tau E_\omega(t,\tau) + \partial_\tau E_g^2(t,\tau).$$

Taking η small enough yields Proposition 2.

3.4. Estimate of the (second) monotonicity energy

The control of E_h, performed in the previous subsection, has relied on the so-called monotonicity energy E_g^2. Its time variations $\partial_t E_g^2$ need in turn to be controlled by $\partial_\tau \mathcal{E}$, in order to obtain a closed estimate. The main difficulty comes from the extra factor $(j+1)^{3/2}$ in the definition of E_g^2: indeed, naive energy bounds would involve the $l^2(\tau)$ norm of $\left((j+1)^{5/2} \, \|\omega\|_{\mathcal{H}_\gamma^j} \right)_{j \in \mathbb{N}}$, which is not controlled by $\sqrt{\partial_\tau \mathcal{E}}$.

To get a good estimate, we must take advantage of cancellation properties in the equation for

$$\tilde{g}_j(t,x,y) = \partial_x^{j-5} \left(\omega \partial_x^5 \omega - \partial_y \omega \partial_x^5 u \right).$$

As mentioned in Section 2, this kind of cancellations was used for the first time in paper [20], in the case of monotonic data: this paper shows indeed some Sobolev stability, with special role played by

$$g_j := \omega \partial_x^j \omega - \partial_y \omega \partial_x^j u.$$

However, in our setting, there is a technical difficulty with using g_j (or the g_j defined in (2.12), better suited to large y). Broadly speaking, the equation for g_j involves the commutator term $j \, \partial_x^{j-1} v \, \partial_x \partial_y \omega$, with a factor j in front. This factor is harmless at the Sobolev level (finite j), but is annoying at the Gevrey level, for which behavior at large j is important. This is why we rather use $\tilde{g}_j$ than g_j: $\mathcal{C}_j$ is somehow replaced by $5 \, \partial_x^{j-1} v \, \partial_x \partial_y \omega$, with no bad factor. Roughly, instead of differentiating j times the velocity and vorticity equations, and combining them to obtain cancellations, we go the reverse way: we first combine the equations and then differentiate, so as to benefit from "earlier" cancellations.

We shall prove

PROPOSITION 3. – *Under the assumptions of Theorem 2, we have*

$$(3.53) \qquad \partial_t E_g^2(t,\tau) + D_g^2(t,\tau) \lesssim_{\mathrm{all}} \partial_\tau E_\omega(t,\tau) + \partial_\tau E_g^2(t,\tau) + D_h(t,\tau).$$

We recall that D_h was defined in (3.33), whereas

$$(3.54) \qquad D_g^2(t,\tau) := \sum_{j \in \mathbb{N}} \left(\tau^{j+1} \, (j!)^{-7/4} \, (j+1)^{10} \right)^2 (j+1)^{3/2} \|\partial_y \tilde{g}_j(t,\cdot)\|_{L^2(\mathbb{T} \times \mathbb{R}_+)}^2.$$

As usual, the starting point is to derive an equation for $\tilde{g}_j$, $j \geq 5$. First, we apply $\omega \partial_5^x$ to the vorticity equation, $\partial_y \omega \partial_x^5$ to the velocity equation, and subtract one from the other: we obtain

$$(3.55) \quad \partial_t \tilde{g}_5 + u \partial_x \tilde{g}_5 + v \partial_y \tilde{g}_5 - \partial_y^2 \tilde{g}_5 = -\sum_{k=1}^{5} \binom{5}{k} \partial_x^k u \, \bar{g}_{5-k+1} - \sum_{k=1}^{4} \binom{5}{k} \partial_x^k v \, \hat{g}_{5-k+1}$$

$$- (\partial_t + u \partial_x + v \partial_y - \partial_y^2) \partial_y \omega \, \partial_x^5 u + 2 \partial_y^2 \omega \, \partial_x^5 \omega - 2 \partial_y \omega \, \partial_y \partial_x^5 \omega := \sum_{i=1}^{5} \mathcal{C}_i$$

with

$$(3.56) \qquad \begin{aligned} \bar{g}_k &:= \omega \partial_x^k \omega - \partial_y \omega \partial_x^k u, \\ \hat{g}_k &:= \omega \partial_x^{k-1} \partial_y \omega - \partial_y \omega \partial_x^{k-1} \omega, \quad k \geq 1. \end{aligned}$$

We finally apply ∂_x^{j-5} to the equation, which yields

$$(3.57) \qquad \partial_t \tilde{g}_j + u\partial_x \tilde{g}_j + v\partial_y \tilde{g}_j - \partial_y^2 \tilde{g}_j = [u\partial_x, \partial_x^{j-5}]\tilde{g}_5 + [v\partial_y, \partial_x^{j-5}]\tilde{g}_5 + \sum_{i=1}^{5} \partial_x^{j-5}\mathscr{C}_i.$$

We perform an energy estimate: we multiply by $(j+1)^{3/2}\left(\tau^{j+1}(j!)^{-7/4}(j+1)^{10}\right)^2$ and sum over j:

(3.58)

$$\partial_t E_g^2(t,\tau) + D_g^2(t,\tau) \leq \sqrt{\frac{2}{\tau}}\left(\|A_j\|_{l^2(\tau)} + \|B_j\|_{l^2(\tau)} + \sum_{i=1}^{4}\|C_{i,j}\|_{l^2(\tau)}\right)\sqrt{\partial_\tau E_g^2(t,\tau)}$$
$$+ \|\max(D_j,0)\|_{l^1(\tau)}$$

with

$$A_j := (j+1)^{1/4}\|[u\partial_x, \partial_x^{j-5}]\tilde{g}_5\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}, \quad B_j := (j+1)^{1/4}\|[v\partial_y, \partial_x^{j-5}]\tilde{g}_5\|_{L^2(\mathbb{T}\times\mathbb{R}_+)},$$

$$C_{i,j} := (j+1)^{1/4}\|\partial_x^{j-5}\mathscr{C}_i\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}, \quad D_j := (j+1)^{3/2}\int_{\mathbb{T}\times\mathbb{R}_+}\partial_x^{j-5}\mathscr{C}_5\tilde{g}_j.$$

Note that

$$\partial_y \tilde{g}_j|_{y=0} = \partial_x^{j-5}\left(\partial_y\omega\partial_x^5\omega - \partial_y^2\omega\partial_x^5 u\right)|_{y=0} = 0.$$

Indeed, $\partial_x^5 u|_{y=0} = 0$, and $\partial_y\omega|_{y=0} = \partial_y^2 u|_{y=0} = 0$, evaluating the Prandtl equation at $y = 0$. In particular, there is no boundary term due to the integration by parts.

Estimate on A_j. – We find

$$[u\partial_x, \partial_x^{j-5}]\tilde{g}_5 = \sum_{k=1}^{j-5}\binom{j-5}{k}\partial_x^k u\,\tilde{g}_{j-k+1}.$$

Thus,

$$A_j \leq (j+1)^{1/4}\sum_{k=1}^{j/2}\binom{j-5}{k}\|\partial_x^k u\|_{L^\infty}\|\tilde{g}_{j-k+1}\|_{L^2} + (j+1)^{1/4}\sum_{k=j/2}^{j-5}\|\partial_x^k u\|_{L^2}\|\tilde{g}_{j-k+1}\|_{L^\infty} := A_j^1 + A_j^2.$$

Using like before (A.3) and (A.1), we get

$$\|\partial_x^k u\|_{L^\infty} \leq C_u \sum_{m=0}^{1}\|\omega\|_{\mathscr{H}_\gamma^{k+m}},$$

$$\|\tilde{g}_{j-k+1}\|_{L^\infty} \leq C_g\left(\sum_{m=1}^{2}\|\tilde{g}_{j-k+m}\|_{L^2} + \sum_{m=2}^{3}\|\partial_y\tilde{g}_{j-k-1+m}\|_{L^2}\right).$$

Hence (set $k' := k-1$, drop the prime)

$$A_j^1 \leq C_u (j+1)^{1/4}\sum_{k=0}^{j/2-1}\binom{j-5}{k+1}\sum_{m=1}^{2}\|\omega\|_{\mathscr{H}_\gamma^{k+m}}\|\tilde{g}_{j-k}\|_{L^2}$$

$$\leq C_u'\sum_{k=0}^{j/2}\binom{j}{k}\sum_{m=1}^{2}\|\omega\|_{\mathscr{H}_\gamma^{k+m}}(j-k)^{1/4}\|\tilde{g}_{j-k}\|_{L^2}$$

using that for $0 \leq k \leq j/2$, one has

$$\binom{j-5}{k+1} = \frac{j-5}{k+1}\binom{j-6}{k} \leq 2\binom{j}{k} \quad \text{and} \quad (j+1)^{1/4} \leq 2^{1/4}(j-k+1)^{1/4}.$$

Hence, by Lemma 1

$$(3.59) \qquad \begin{aligned} \|A_j^1\|_{l^2(\tau)} &\lesssim_\tau \|\|\omega\|_{\mathcal{H}_\gamma^j}\|_{l^2(\tau)} \|(j+1)^{1/4}\|\tilde{g}_j\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)} \\ &\lesssim_\tau \sqrt{E_\omega(t,\tau)}\sqrt{\partial_\tau E_g^2(t,\tau)} \lesssim_\tau \sqrt{M}\sqrt{\partial_\tau E_g^2(t,\tau)}. \end{aligned}$$

Finally,

$$A_j^2 \leq C_g(j+1)^{1/4}\sum_{k=j/2}^{j-5}\binom{j-5}{k}\|\partial_x^k u\|_{L^2}\sum_{m=1}^{2}\|\tilde{g}_{j-k+m}\|_{L^2} + \sum_{m=2}^{3}\|\partial_y\tilde{g}_{j-k-1+m}\|_{L^2}$$

$$\leq C_g'\sum_{k=j/2}^{j}\binom{j}{k}k^{1/4}\|\omega\|_{\mathcal{H}_\gamma^k}\left(\sum_{m=1}^{2}\|\tilde{g}_{j-k+m}\|_{L^2} + \sum_{m=2}^{3}\|\partial_y\tilde{g}_{j-k-1+m}\|_{L^2}\right)$$

using that $\binom{j-5}{k} \leq \binom{j}{k}$ and that for $j/2 \leq k \leq j-5$, $(j+1)^{1/4} \leq 2^{1/4}(k+1)^{1/4}$. Hence,

$$\|A_j^2\|_{l^2(\tau)} \lesssim_\tau \left(\|\|\tilde{g}_j\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)} + \|\|\partial_y\tilde{g}_{j-1}\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)}\right)\left\|(j+1)^{1/4}\|\omega\|_{\mathcal{H}_\gamma^j}\right\|_{l^2(\tau)}.$$

We conclude by using

LEMMA 8. – *For all $\alpha \geq 0$, for all $l = 0,1,2$,*

$$\|(j+1)^\alpha \|(1+y)\partial_y^l\tilde{g}_{j-l}\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)} \lesssim \|(j+1)^\alpha\|\omega\|_{\mathcal{H}_\gamma^j}\|_{l^2(\tau)}, \quad \alpha \geq 0.$$

For a proof, see the appendix. We get

$$(3.60) \qquad \|A_j^2\|_{l^2(\tau)} \lesssim \sqrt{E_\omega(t,\tau)}\sqrt{\partial_\tau E_\omega(t,\tau)} \lesssim \sqrt{\partial_\tau E_\omega(t,\tau)}$$

and combining with (3.59),

$$(3.61) \qquad \|A_j\|_{l^2(\tau)} \lesssim \sqrt{\partial_\tau E_g^2(t,\tau)} + \sqrt{\partial_\tau E_\omega(t,\tau)}.$$

Estimate on B_j. – Leibniz formula gives

$$[v\partial_y, \partial_x^{j-5}]\tilde{g}_5 = \sum_{k=1}^{j-5}\binom{j-5}{k}\partial_x^k v\,\partial_y\tilde{g}_{j-k}.$$

Proceeding as before, we get

$$B_j \leq (j+1)^{1/4}\sum_{k=1}^{j/2}\binom{j-5}{k}\|\partial_x^k v\|_{L^\infty}\|\partial_y\tilde{g}_{j-k}\|_{L^2}$$

$$+ (j+1)^{1/4}\sum_{k=j/2+1}^{j-5}\|\frac{\partial_x^k v}{1+y}\|_{L^2}\|(1+y)\partial_y\tilde{g}_{j-k}\|_{L^\infty} := B_j^1 + B_j^2.$$

For B_j^1, we use (3.17), and the basic inequalities $\binom{j-5}{k} \leq \binom{j}{k}$, $(j+1)^{1/4} \leq 2^{1/4}(j-k+1)^{1/4}$ for $0 \leq k \leq j/2$, we find

$$B_j^1 \leq C_1\sum_{k=0}^{j/2}\binom{j}{k}\sum_{m=1}^{2}\|\omega\|_{\mathcal{H}_\gamma^{k+m}}(j-k+1)^{1/4}\|\partial_y\tilde{g}_{j-k}\|_{L^2}$$

(with C_1 an absolute constant) so that

$$(3.62) \quad \|B_j^1\|_{l^2(\tau)} \lesssim_\tau \sqrt{E_\omega(t,\tau)}\sqrt{\|(j+1)^{1/4}\|\partial_y \tilde{g}_j\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)}} \lesssim_\tau \sqrt{M}\sqrt{D_g^2(t,\tau)}.$$

For B_j^2, we use (3.21), (A.1), (A.2) (and a change of index $k' = k+1$) to get

$$B_j^2 \le C_2\,(j+1)^{1/4} \sum_{k'=j/2+1}^{j-4} \binom{j-5}{k'-1}\|\omega\|_{\mathscr{H}_\gamma^{k'}}$$
$$\cdot\left(\sum_{m=2}^3 \|(1+y)\partial_y \tilde{g}_{j-k'-1+m}\|_{L^2} + \sum_{m=3}^4 \|(1+y)\partial_y^2 \tilde{g}_{j-k'-2+m}\|_{L^2}\right)$$
$$\le C_2' \sum_{k=j/2+1}^{j} \binom{j}{k}\left((k+1)^{1/4}\|\omega\|_{\mathscr{H}_\gamma^{k'}}\right)$$
$$\cdot\left(\sum_{m=2}^3 \|(1+y)\partial_y \tilde{g}_{j-k'-1+m}\|_{L^2} + \sum_{m=3}^4 \|(1+y)\partial_y^2 \tilde{g}_{j-k'-2+m}\|_{L^2}\right)$$

using that $\binom{j-5}{k-1} \le \binom{j}{k}$, $(j+1)^{1/4} \le 2^{1/4}(k+1)^{1/4}$ in the range $j/2+1 \le k \le j-4$. Hence,

$$\|B_j^2\|_{l^2(\tau)} \lesssim_\tau \sqrt{\partial_\tau E_\omega(t,\tau)}\big(\|\,\|(1+y)\partial_y \tilde{g}_{j-1}\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)}$$
$$(3.63) \qquad\qquad + \|\,\|(1+y)\partial_y^2 \tilde{g}_{j-2}\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)}\big)$$
$$\lesssim \sqrt{\partial_\tau E_\omega(t,\tau)}\sqrt{E_\omega(t,\tau)} \lesssim \sqrt{\partial_\tau E_\omega(t,\tau)}$$

where we have applied Lemma 8 to go from the first to the second line. Gathering the estimates on B_j^1 and B_j^2,

$$(3.64) \qquad\qquad \|B_j\|_{l^2(\tau)} \lesssim \sqrt{D_g^2(t,\tau)} + \sqrt{\partial_\tau E_\omega(t,\tau)}.$$

Estimates on $C_{i,j}$, $1 \le i \le 2$. – The quantities $C_{1,j}$ and $C_{2,j}$ involve commutators with the nonlinearities. As we have manipulated several times such commutators, we shall not detail the computation of their $l^2(\tau)$ norms. One can check that

$$\|C_{1,j}\|_{l^2(\tau)} \lesssim_\tau \|\,\|\omega\|_{\mathscr{H}_\gamma^j}\|_{l^2(\tau)} \sum_{m=1}^5 \|(j+1)^{1/4}\|\partial_x^{j-m}\bar{g}_m\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)}$$
$$(3.65) \qquad + \|(j+1)^{1/4}\|\omega\|_{\mathscr{H}_\gamma^j}\|_{l^2(\tau)} \sum_{m=1}^5 \big(\|\,\|\partial_x^{j-m}\bar{g}_m\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)}$$
$$\qquad\qquad + \|\,\|\partial_y \partial_x^{j-m-1}\bar{g}_m\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)}\big)$$
$$\lesssim \sqrt{\partial_\tau E_\omega(t,\tau)}.$$

Here, we use implicitly the bound

$$(3.66) \quad \|(j+1)^\alpha\|(1+y)\partial_y^l \partial_x^{j-m-l}\bar{g}_m\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)} \lesssim \|(j+1)^\alpha \|\omega\|_{\mathscr{H}_\gamma^j}\|_{l^2(\tau)}$$

which is valid for all $1 \le m \le 5$, $\alpha \ge 0$ and $0 \le l \le 1$. Note that the case $m=5$ is included in Lemma 8. We leave the other very similar cases to the reader.

Regarding $C_{2,j}$,
(3.67)

$$\|C_{2,j}\|_{l^2(\tau)} \lesssim_\tau \|\|\omega\|_{\mathscr{H}_\gamma^j}\|_{l^2(\tau)} \sum_{m=1}^{4} \|(j+1)^{1/4}\|\partial_x^{j-m}\hat{g}_m\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)}$$

$$+ \|(j+1)^{1/4}\|\omega\|_{\mathscr{H}_\gamma^j}\|_{l^2(\tau)} \sum_{m=2}^{5} \left(\|\|(1+y)\partial_x^{j-m}\hat{g}_m\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)} \right.$$

$$\left. + \|\|(1+y)\partial_y\partial_x^{j-m-1}\hat{g}_m\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)}\right)$$

$$\lesssim \sqrt{\partial_\tau E_\omega(t,\tau)}.$$

Again, we have applied inequalities of type (3.66) with $\bar{g}$ replaced by $\hat{g}$.

Estimate on $C_{4,j}$. – Once more, one can follow closely the previous ideas, and derive

(3.68) $\qquad \|C_{4,j}\|_{l^2(\tau)} \lesssim_\tau \sqrt{E_\omega(t,\tau)}\|(j+1)^{1/4}\|\omega\|_{\mathscr{H}_\gamma^j}\|_{l^2(\tau)} \lesssim_{\tau,M} \sqrt{\partial_\tau E_\omega(t,\tau)}.$

Estimate on $C_{3,j}$. – We remind that

$$(\partial_t + u\partial_x + v\partial_y - \partial_y^2)\omega = 0$$

so that

$$(\partial_t + u\partial_x + v\partial_y - \partial_y^2)\partial_y\omega = -\partial_y u\partial_x\omega - \partial_y v\partial_y\omega = -\partial_y u\partial_x\omega + \partial_x u\partial_y\omega.$$

Hence,

$$\partial_x^{j-5}\mathscr{C}_{3,j} = \sum_{k=0}^{j-5}\binom{j-5}{k}\partial_x^k(-\partial_y u\partial_x\omega + \partial_x u\partial_y\omega)\partial_x^{j-k}u.$$

Denoting $a_j := \|\partial_x^j(-\partial_y u\partial_x\omega + \partial_x u\partial_y\omega)\|_{H^2}$, we obtain (with (A.1), (A.2)),

$$C_{3,j} \le C(j+1)^{1/4}\left(\sum_{k=0}^{j/2}\binom{j-5}{k}a_k\|\omega\|_{\mathscr{H}_\gamma^{j-k}} + C\sum_{k=j/2+1}^{j-5}\binom{j-5}{k}a_k\sum_{m=0}^{1}\|\omega\|_{\mathscr{H}_\gamma^{j-k+m}}\right)$$
$$:= C_{3,j}^1 + C_{3,j}^2$$

for some absolute constant C. We get (for another absolute constant C)

$$C_{3,j}^1 \le C\sum_{k=0}^{j/2}\binom{j}{k}a_{(k-3)+3}(j-k)^{1/4}\|\omega\|_{\mathscr{H}_\gamma^{j-k}}$$

because $\binom{j-5}{k} \le \binom{j}{k}$ and $(j+1)^{1/4} \le 2^{1/4}(j-k+1)^{1/4}$ for $0 \le k \le j/2$. We then use Lemma 1 (with $m=3$) to obtain

(3.69) $\qquad \|C_{3,j}^1\|_{l^2(\tau)} \lesssim_\tau \|a_{j-3}\|_{l^2(\tau)}\|(j+1)^{1/4}\|\omega\|_{\mathscr{H}_\gamma^j}\|_{l^2(\tau)}.$

Then, we write

$$C_{3,j}^2 = (j+1)^{1/4}\sum_{k=j/2+3}^{j-2}\binom{j-5}{k-3}a_{k-3}\sum_{m=3}^{4}\|\omega\|_{\mathscr{H}_\gamma^{j-k+m}}$$

$$\le C\sum_{k=j/2+1}^{j}\binom{j}{k}(k+1)^{1/4}a_{k-3}\sum_{m=3}^{4}\|\omega\|_{\mathscr{H}_\gamma^{j-k}}.$$

because $\binom{j-5}{k-3} \leq \binom{j}{k}$ and $(j+1)^{1/4} \leq 2(k+1)^{1/4}$ for $j/2 + 1 \leq k \leq j$. Hence,

$$(3.70) \quad \|C_{3,j}^2\|_{l^2(\tau)} \lesssim_\tau \|(j+1)^{1/4} a_{j-3}\|_{l^2(\tau)} \|\|\omega\|_{\mathcal{H}_\gamma^j}\|_{l^2(\tau)} \lesssim_{\tau,M} \|(j+1)^{1/4} a_{j-3}\|_{l^2(\tau)}.$$

Finally, applying the same type of arguments, it is shown that

$$\|a_{j-3}\|_{l^2(\tau)} \lesssim_\tau \|\|\omega\|_{\mathcal{H}_\gamma^j}\|_{l^2(\tau)} \lesssim_\tau \sqrt{E_\omega(t,\tau)},$$

$$\|(j+1)^{1/4} a_{j-3}\|_{l^2(\tau)} \lesssim_\tau \|(j+1)^{1/4}\|\omega\|_{\mathcal{H}_\gamma^j}\|_{l^2(\tau)} \lesssim_\tau \sqrt{\partial_\tau E_\omega(t,\tau)}.$$

Inserting these bounds into (3.69)-(3.70) and summing:

$$(3.71) \quad \|C_{3,j}\|_{l^2(\tau)} \lesssim_{\tau,M} \sqrt{\partial_\tau E_\omega(t,\tau)}.$$

Estimate on D_j. – We now turn to the most delicate term

$$D_j = -2(j+1)^{3/2} \int_{\mathbb{T}\times\mathbb{R}_+} \partial_x^{j-5} \mathscr{C}_5 \, \tilde{g}_j = -2(j+1)^{3/2} \int_{\mathbb{T}\times\mathbb{R}_+} \partial_y\omega \partial_x^j \partial_y\omega \, \tilde{g}_j$$

$$- 2(j+1)^{3/2} \int_{\mathbb{T}\times\mathbb{R}_+} [\partial_x^{j-5}, \partial_y\omega] \partial_y \partial_x^5 \omega \, \tilde{g}_j := D_j^1 + D_j^2.$$

Study of D_j^2. – We integrate by parts to find

$$D_j^2 = 2(j+1)^{3/2} \int_{\mathbb{T}\times\mathbb{R}_+} [\partial_x^{j-5}, \partial_y\omega]\partial_x^5\omega \, \partial_y\tilde{g}_j + 2(j+1)^{3/2} \int_{\mathbb{T}\times\mathbb{R}_+} [\partial_x^{j-5}, \partial_y^2\omega]\partial_x^5\omega \, \tilde{g}_j := D_j^{2'} + D_j^{2''}.$$

Note that there is no boundary term, as $(\partial_x^k \partial_y\omega)|_{y=0} = \partial_x^k \partial_y^2 u|_{y=0} = 0$ for any k, using the Prandtl equation. We have by Cauchy-Schwarz inequality:

$$\|D_j^{2'}\|_{l^1(\tau)} \leq \|(j+1)^{3/4}\|[\partial_x^{j-5}, \partial_y\omega]\partial_x^5\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)} \|(j+1)^{3/4}\|\partial_y\tilde{g}_j\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)}.$$

We find, by now standard arguments:

$$\|(j+1)^{3/4}\|[\partial_x^{j-5}, \partial_y\omega]\partial_x^5\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)} \lesssim_\tau \|(j+1)^{3/4}(j+1)\|\omega\|_{\mathcal{H}_\gamma^{j-1}}\|_{l^2(\tau)}.$$

We notice that $\|(j+1)^{7/4}\|\omega\|_{\mathcal{H}_\gamma^{j-1}}\|_{l^2(\tau)} \lesssim_\tau \|\|\omega\|_{\mathcal{H}_\gamma^j}\|_{l^2(\tau)}$ to conclude that

$$\|D_j^{2'}\|_{l^1(\tau)} \lesssim_\tau \sqrt{E_\omega(t,\tau)} \sqrt{D_g^2(t,\tau)} \lesssim_\tau \eta D_g^2(t,\tau) + \frac{1}{\eta} E_\omega(t,\tau)$$

for all $\eta \in]0,1]$. The treatment of $D_j^{2''}$ is similar: we state

$$\|D_j^{2''}\|_{l^1(\tau)} \lesssim_\tau \sqrt{E_\omega(t,\tau)} \sqrt{E_g^2(t,\tau)} \lesssim E_g^2(t,\tau) + E_\omega(t,\tau).$$

Finally, for all $\eta > 0$,

$$(3.72) \quad \|D_j^2\|_{l^1(\tau)} \lesssim_\tau \eta \, D_g^2(t,\tau) + \frac{1}{\eta} E_\omega(t,\tau) + E_g^2(t,\tau).$$

Study of D_j^1. – We shall use the decomposition of $\partial_x^j u$ recalled in (3.42):

$$\partial_x^j u = C_j \, 1_{\{y>a\}} \, \omega + u_g^j.$$

Let $0 < \mu < \varepsilon$ (with ε defined in Remark 1) to be fixed later, and $\chi^{\mu,a} \in C_c^\infty(\mathbb{R})$ satisfying

$$\chi^{\mu,a} = 1 \quad \text{over } \{|y-a| \le \mu\}, \quad \chi^{\mu,a} = 0 \quad \text{outside } \{|y-a| \ge 2\mu\}.$$

We shall keep track of the μ-dependence in all inequalities below. We write

$$D_j^1 = -2(j+1)^{3/2} \int_{\mathbb{T}\times\mathbb{R}_+} \chi^{\mu,a} \partial_y \omega \, \partial_x^j \partial_y \omega \, \tilde{g}_j - 2(j+1)^{3/2} \int_{\mathbb{T}\times\mathbb{R}_+} (1-\chi^{\mu,a}) \partial_y \omega \, \partial_x^j \partial_y \omega \, \tilde{g}_j$$

$$:= D_j^{1'} + D_j^{1''}.$$

Let $\psi_\omega := \psi \omega + (1-\psi)$, see (2.12). As ω does not vanish away from $y = a$, we can write

$$D_j^{1''} = -2(j+1)^{3/2} \int_{\mathbb{T}\times\mathbb{R}_+} (1-\chi^{\mu,a}) \frac{\partial_y \omega}{\omega} (\partial_y \left(\frac{\omega g_j}{\psi_\omega} \right) - \partial_y^2 \omega \partial_x^j u) \, \tilde{g}_j$$

$$= 2(j+1)^{3/2} \int_{\mathbb{T}\times\mathbb{R}_+} \partial_y \left((1-\chi^{\mu,a}) \frac{\partial_y \omega}{\omega} \right) \frac{\omega}{\psi_\omega} g_j \, \tilde{g}_j$$

$$+ 2(j+1)^{3/2} \int_{\mathbb{T}\times\mathbb{R}_+} (1-\chi^{\mu,a}) \frac{\partial_y \omega}{\psi_\omega} g_j \partial_y \tilde{g}_j$$

$$+ 2(j+1)^{3/2} \int_{\mathbb{T}\times\mathbb{R}_+} (1-\chi^{\mu,a}) \frac{\partial_y \omega}{\omega} \partial_y^2 \omega \partial_x^j u \, \tilde{g}_j.$$

It follows that

$$\|D_j^{1''}\|_{l^1(\tau)} \lesssim \frac{1}{\inf_{t,x|y-a|\ge\mu} |\omega|} \|(j+1)^{3/4}\|g_j\|_{L^2} \|_{l^2(\tau)} \|(j+1)^{3/4}\|\partial_y \tilde{g}_j\|_{L^2}\|_{l^2(\tau)}$$

$$+ \frac{1}{\mu \inf_{t,x|y-a|\ge\mu} |\omega|} \|(j+1)^{1/4}\|\omega\|_{\mathscr{H}_\gamma^j}\|_{l^2(\tau)} \|(j+1)^{5/4}\|\tilde{g}_j\|_{L^2}\|_{l^2(\tau)}.$$

Pondering on Lemma 7, we get

$$\|D_j^{1''}\|_{l^1(\tau)} \lesssim \frac{1}{\inf_{t,x|y-a|\ge\mu} |\omega|} \left(\sqrt{E_g^2(t,\tau)} + \sqrt{E_\omega(t,\tau)} \right) \sqrt{D_g^2(t,\tau)}$$

$$+ \frac{1}{\mu \inf_{t,x|y-a|\ge\mu} |\omega|} \sqrt{\partial_\tau E_\omega(t,\tau)} \sqrt{\partial_\tau E_g^2(t,\tau)}$$

$$\lesssim \eta D_g^2(t,\tau) + \frac{1}{\eta\mu^2 (\inf_{t,x|y-a|\ge\mu})^2 |\omega|} \left(\partial_\tau E_\omega(t,\tau) + \partial_\tau E_g^2(t,\tau) \right).$$

It remains to handle $D_j^{1'}$. Therefore, we introduce

$$\omega_h^j := 1_{y>a} C_j \partial_y \omega, \quad \omega_g^j := \partial_y u_g^j.$$

We emphasize that both ω_h^j and ω_g^j are discontinuous across $y = a$, contrary to $\partial_x^j \omega = \omega_h^j + \omega_g^j$. We find

$$
\begin{aligned}
D_j^{1'} &= -2(j+1)^{3/2} \int_{\{y>a\}} \chi^{\mu,a} \partial_y \omega \, \partial_y \omega_h^j \, \tilde{g}_j - 2(j+1)^{3/2} \int_{\{y>a\}} \chi^{\mu,a} \partial_y \omega \, \partial_y \omega_g^j \, \tilde{g}_j \\
&\quad - 2(j+1)^{3/2} \int_{\{y<a\}} \chi^{\mu,a} \partial_y \omega \, \partial_y \omega_g^j \, \tilde{g}_j \\
&= -2(j+1)^{3/2} \int_{\{y>a\}} \chi^{\mu,a} \partial_y \omega \, \partial_y \omega_h^j \, \tilde{g}_j + 2(j+1)^{3/2} \int_{\mathbb{T} \times \mathbb{R}_+} \partial_y \left(\chi^{\mu,a} \partial_y \omega \right) \omega_g^j \, \tilde{g}_j \\
&\quad + 2(j+1)^{3/2} \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} \partial_y \omega \, \omega_g^j \, \partial_y \tilde{g}_j + 2(j+1)^{3/2} \int_{\{y=a\}} \partial_y \omega \, [\omega_g^j] \, \tilde{g}_j \\
&:= I + II + III + IV.
\end{aligned}
$$

To go from the first to the second equality, we have integrated by parts the last two integrals: gathering the boundary terms at $y = a^+$ and $y = a^-$ we have obtained a jump term

$$
[\omega_g^j] := \omega_g^j|_{y=a^+} - \omega_g^j|_{y=a^-}
$$

across $y = a$. Clearly,

$$
\begin{aligned}
\|I\|_{l^1(\tau)} &\lesssim \| (j+1)^{1/4} \|C_j\|_{L^2(\mathbb{T})} \|_{l^2(\tau)} \| (j+1)^{5/4} \|\tilde{g}_j\|_{L^2(\mathbb{T} \times \mathbb{R}_+)} \|_{l^2(\tau)} \\
&\lesssim \sqrt{\partial_\tau E_\omega(t,\tau)} \sqrt{\partial_\tau E_g^2(t,\tau)} \\
&\lesssim \partial_\tau E_\omega(t,\tau) + \partial_\tau E_g^2(t,\tau)
\end{aligned}
$$

with the last inequality coming from (3.29). Also

$$
\begin{aligned}
\|II\|_{l^1(\tau)} &\lesssim \frac{1}{\mu} \| (j+1)^{1/4} \|\omega_g^j\|_{L^2(\mathbb{T} \times \mathbb{R}_+)} \|_{l^2(\tau)} \| (j+1)^{5/4} \|\tilde{g}_j\|_{L^2(\mathbb{T} \times \mathbb{R}_+)} \|_{l^2(\tau)} \\
&\lesssim \frac{1}{\mu} \left(\| (j+1)^{1/4} \|C_j\|_{L^2(\mathbb{T})} \|_{l^2(\tau)} + \| (j+1)^{1/4} \|\partial_x^j \omega\|_{L^2(\mathbb{T} \times \mathbb{R}_+)} \|_{l^2(\tau)} \right) \\
&\quad \cdot \| (j+1)^{5/4} \|\tilde{g}_j\|_{L^2(\mathbb{T} \times \mathbb{R}_+)} \|_{l^2(\tau)} \\
&\lesssim \frac{1}{\mu} \sqrt{\partial_\tau E_\omega(t,\tau)} \sqrt{\partial_\tau E_g^2(t,\tau)} \lesssim \frac{1}{\mu} \left(\partial_\tau E_\omega(t,\tau) + \partial_\tau E_g^2(t,\tau) \right).
\end{aligned}
$$

Regarding IV, we have

$$
IV = -2(j+1)^{3/2} \int_{\{y=a\}} \partial_y \omega \, [\omega_h^j] \, \tilde{g}_j = -2(j+1)^{3/2} \int_{\{y=a\}} (\partial_y \omega)^2 \, C_j \, \tilde{g}_j
$$

so that

$$
|IV| \lesssim \|j^{1/2} C_j\|_{L^2(\mathbb{T})} \| (j+1)^{5/4} \tilde{g}_j \|_{L^2(\mathbb{T} \times \mathbb{R}_+)}^{1/2} \| (j+1)^{3/4} \partial_y \tilde{g}_j \|_{L^2(\mathbb{T} \times \mathbb{R}_+)}^{1/2}
$$

and finally

$$
\|IV\|_{l^1(\tau)} \lesssim \sqrt{\partial_\tau E_\omega(t,\tau)} \, E_g^2(t,\tau)^{1/4} \, D_g^2(t,\tau)^{1/4} \lesssim \eta D_g^2(t,\tau) + \frac{1}{\eta} \left(\partial_\tau E_\omega(t,\tau) + E_g^2(t,\tau) \right).
$$

To handle III, we split the integral as follows:

$$
\int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} \partial_y \omega \, \omega_g^j \, \partial_y \tilde{g}_j = \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} \partial_y \omega \, \omega_g^j \, \partial_y g_j + \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} \partial_y \omega \, \omega_g^j \, \partial_y (\tilde{g}_j - g_j) = \mathrm{i} + \mathrm{ii}.
$$

As $\mu \leq \varepsilon$, one has on the support of $\chi^{\mu,a}$:

$$g_j = \omega \partial_x^j \omega - \partial_y \omega \partial_x^j u = \omega \omega_g^j - \partial_y \omega u_g^j$$

so that

$$\partial_y g_j = \omega \partial_y \omega_g^j - \partial_y^2 \omega u_g^j.$$

We write

$$\mathrm{i} = \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} \partial_y \omega \, \frac{1}{2} \partial_y (\omega_g^j)^2 - \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} \partial_y \omega \, \partial_y^2 \omega \, \omega_g^j \, u_g^j$$

$$= -\frac{1}{2} \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} (\partial_y \omega)^2 (\omega_g^j)^2 - \frac{1}{2} \int_{\mathbb{T} \times \mathbb{R}_+} \partial_y \chi^{\mu,a} \partial_y \omega \, \omega \, (\omega_g^j)^2 - \frac{1}{2} \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} \partial_y^2 \omega \, \omega \, (\omega_g^j)^2$$

$$+ \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} \partial_y^2 \omega \, \omega_g^j \left(g_j - \omega \omega_g^j \right).$$

As $\mu \leq \varepsilon$, one has on the support of $\chi^{\mu,a}$: $|\partial_y \omega| \geq \frac{\theta_0}{2}$. Hence,

$$-\frac{1}{2} \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} (\partial_y \omega)^2 (\omega_g^j)^2 \leq -\frac{\theta_0^2}{8} \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} (\omega_g^j)^2$$

where $\theta_0 := \inf_{t,x,y=a} |\partial_y \omega|$. On the other hand, as ω vanishes linearly in $y - a$ at the critical curve:

$$\frac{1}{2} \left| \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} \partial_y^2 \omega \, \omega \, (\omega_g^j)^2 \right| \lesssim_{\tau,M} \mu \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} (\omega_g^j)^2.$$

For μ small enough, depending on M, τ, ε and θ_0, we can absorb the latter by the former, resulting in

$$\mathrm{i} \leq -\frac{\theta_0^2}{16} \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} (\omega_g^j)^2 - \frac{1}{2} \int_{\mathbb{T} \times \mathbb{R}_+} \partial_y \chi^{\mu,a} \partial_y \omega \, \omega \, (\omega_g^j)^2 + \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} \partial_y^2 \omega \, \omega_g^j g_j.$$

We fix such a constant μ from now on. Note that $\partial_y \chi^{\mu,a}$ vanishes in a neighborhood of $y = a$ so that, roughly speaking, ω_g^j behaves like g_j over the support of $\partial_y \chi^{\mu,a}$. More precisely,

$$\left| \int_{\mathbb{T} \times \mathbb{R}_+} \partial_y \chi^{\mu,a} \partial_y \omega \, \omega \, (\omega_g^j)^2 \right| \lesssim_{\text{all}} \|g_j\|_{L^2(\mathbb{T} \times \mathbb{R}_+)}^2.$$

Finally, we apply a Young's inequality to the last term:

$$\left| \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} \partial_y^2 \omega \, \omega_g^j g_j \right| \lesssim_{\tau,M} \eta \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} (\omega_g^j)^2 + \frac{1}{\eta} \|g_j\|_{L^2(\mathbb{T} \times \mathbb{R}_+)}^2.$$

We obtain, for $\eta > 0$ small enough

$$\mathrm{i} \lesssim_{\text{all}} -\frac{\theta_0^2}{32} \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} (\omega_g^j)^2 + \|g_j\|_{L^2(\mathbb{T} \times \mathbb{R}_+)}^2.$$

Finally, we apply Young's inequality to ii: for all $\eta' > 0$,

$$\mathrm{ii} \lesssim \eta' \int_{\mathbb{T} \times \mathbb{R}_+} \chi^{\mu,a} (\omega_g^j)^2 + \frac{1}{\eta'} \|\sqrt{\chi^{\mu,a}} \partial_y (\tilde{g}_j - g_j)\|_{L^2(\mathbb{T} \times \mathbb{R}_+)}^2.$$

Combining with the bound on i, we find (for η' small enough)

$$\max(III, 0) = 2(j+1)^{3/2} \max(\mathrm{i} + \mathrm{ii}, 0)$$
$$\lesssim_{\mathrm{all}} (j+1)^{3/2} \|g_j\|_{L^2(\mathbb{T} \times \mathbb{R}_+)}^2 + (j+1)^{3/2} \|\sqrt{\chi^{\mu,a}} \partial_y(\tilde{g}_j - g_j)\|_{L^2(\mathbb{T} \times \mathbb{R}_+)}^2.$$

If we gather this last bound with those on I, II and IV, we conclude that

$$\| \max(D_j^{1'}, 0) \|_{l^1(\tau)} \lesssim_{\mathrm{all}} \eta D_g^2(t,\tau) + C_\eta \left(\partial_\tau E_\omega(t,\tau) + \partial_\tau E_g^2(t,\tau) \right)$$
$$+ \|(j+1)^{3/4} \|g_j\|_{L^2(\mathbb{T} \times \mathbb{R}_+)}\|_{l^2(\tau)}^2$$
$$+ \|(j+1)^{3/4} \|\sqrt{\chi^{\mu,a}} \partial_y(\tilde{g}_j - g_j)\|_{L^2(\mathbb{T} \times \mathbb{R}_+)}\|_{l^2(\tau)}^2.$$

We then use Lemma 7 together with

Lemma 9. – *We have*

$$\|(j+1)^{3/4} \|\sqrt{\chi^{\mu,a}} \partial_y(\tilde{g}_j - g_j)\|_{L^2(\mathbb{T} \times \mathbb{R}_+)}\|_{l^2(\tau)}^2 \lesssim D_h(t,\tau) + E_\omega(t,\tau).$$

This last inequality will be proved in appendix . We obtain

$$\| \max(D_j^{1'}, 0) \|_{l^1(\tau)} \lesssim_{\mathrm{all}} \eta D_g^2(t,\tau) + D_h(t,\tau) + \frac{1}{\eta} \left(\partial_\tau E_\omega(t,\tau) + \partial_\tau E_g^2(t,\tau) \right).$$

In turn, we can combine this bound with those on $D_j^{1''}$, D_j^2, to obtain

$$(3.73) \qquad \| \max(D_j, 0) \|_{l^1(\tau)} \lesssim_{\mathrm{all}} \eta D_g^2(t,\tau) + D_h(t,\tau) + \frac{1}{\eta} \left(\partial_\tau E_\omega(t,\tau) + \partial_\tau E_g^2(t,\tau) \right)$$

for all $\eta \in \,]0,1]$. Then, we collect the estimates on A_j, B_j, $C_{i,j}$. We find

$$\partial_t E_g^2(t,\tau) + D_g^2(t,\tau) \lesssim_{\mathrm{all}} \left(\sqrt{D_g^2(t,\tau)} + \sqrt{\partial_\tau E_g^2(t,\tau)} + \sqrt{\partial_\tau E_\omega(t,\tau)} \right) \sqrt{E_g^2(t,\tau)}$$
$$+ \eta D_g^2(t,\tau) + D_h(t,\tau) + \frac{1}{\eta} \left(\partial_\tau E_\omega(t,\tau) + \partial_\tau E_g^2(t,\tau) \right)$$
$$\lesssim \eta D_g^2(t,\tau) + D_h(t,\tau) + \frac{1}{\eta} \left(\partial_\tau E_\omega(t,\tau) + \partial_\tau E_g^2(t,\tau) \right).$$

This concludes the proof of the proposition (take η small enough).

3.5. Estimate of the (first) monotonicity energy

In the previous paragraphs, we have derived bounds on the time variations of $\dot{E}_\omega$, E_h and E_g^2. These bounds involve the total energy E_ω defined in (2.1). By Lemma 2, this energy is controlled by the sum of $\dot{E}_\omega$, E_h and E_g^1. Hence, to establish our main a priori estimate (Theorem 2), it remains to control $\partial_t E_g^1$. This is the purpose of

Proposition 4. – *Under the same assumptions as in Proposition 4, one has for some $C > 0$ and all $t \in \,]0, T]$:*

$$(3.74) \qquad\qquad \partial_t E_g^1(t,\tau) + D_g^1(t,\tau) \leq C \, \partial_\tau E_\omega(t,\tau).$$

Naturally,

$$D_g^1(t,\tau) := \sum_{j\in\mathbb{N}} \left(\tau^j\,(j!)^{-7/4}\,(j+1)^{10}\right)^2 \|\partial_y g_j(t,\cdot)\|^2_{L^2(L_\gamma^2)}.$$

To prove this proposition, we first write the equation on

$$g_j := \tilde{\psi}\left(\partial_x^j \omega - \frac{\partial_y \omega}{\omega}\partial_x^j u\right), \quad \tilde{\psi} := \psi\omega + (1-\psi).$$

We apply the operator $\tilde{\psi}\partial_x^j$ to the vorticity equation, apply $\tilde{\psi}\frac{\partial_y\omega}{\omega}\partial_x^j$ to the velocity equation, and subtract one from another. A straightforward computation gives

$$\partial_t g_j + u\partial_x g_j + v\partial_y g_j - \partial_y^2 g_j$$

$$= -\sum_{k=1}^{j}\binom{j}{k}\partial_x^k u\, g_{j-k+1} - \sum_{k=1}^{j-1}\binom{j}{k}\partial_x^k v\,\tilde{\psi}\left(\partial_y\partial_x^{j-k}\omega - \frac{\partial_y\omega}{\omega}\partial_x^{j-k}\omega\right)$$

$$+ \left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right)\tilde{\psi}\,\partial_x^j\omega - \left(\partial_t + u\partial_x + v\partial_y - \partial_y^2\right)\left(\tilde{\psi}\frac{\partial_y\omega}{\omega}\right)\partial_x^j u$$

$$- 2\partial_y\tilde{\psi}\partial_y\partial_x^j\omega + 2\partial_y\left(\tilde{\psi}\frac{\partial_y\omega}{\omega}\right)\partial_x^j\omega := \sum_{i=1}^{6}\mathscr{C}_i.$$

Then, one multiplies by $\left(\tau^{j+1}\,(j!)^{-7/4}\,(j+1)^{10}\right)^2(1+y)^{2\gamma}g_j$ and performs a standard energy estimate:
(3.75)

$$\partial_t E_g^1(t,\tau) + D_g^1(t,\tau) \le \left(\|A_j\|_{l^2(\tau)} + \|B_j\|_{l^2(\tau)} + \sum_{i\ne 5}\|C_{i,j}\|_{l^2(\tau)}\right)\sqrt{\partial_\tau E_g^1(t,\tau)} + \|D_j\|_{l^1(\tau)}$$

where:

$$A_j := \frac{2\gamma}{j^{1/2}}\|(1+y)^{\gamma-1}vg_j\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}, \qquad B_j := \frac{2\gamma}{j^{1/2}}\|(1+y)^{\gamma-1}\partial_y g_j\|_{L^2(\mathbb{T}\times\mathbb{R}_+)},$$

$$C_{i,j} := \frac{1}{j^{1/2}}\|\mathscr{C}_i\|_{L^2(L_\gamma^2)}, \quad i=1,2,3,4,6, \qquad D_j := \int_{\mathbb{T}\times\mathbb{R}_+} 2(1+y)^{2\gamma}\partial_y\tilde{\psi}\partial_y\partial_x^j\omega\, g_j.$$

Again, there is no boundary term at $y=0$, as $\partial_y g_j$ vanishes there.

To bound above terms, one can follow the calculations already performed, notably those related to Proposition 1. Note that to control the behavior at infinity of the $\mathscr{C}_i$'s, one must have information on the behavior of the function $\frac{\partial_y\omega}{\omega}$ at infinity. Such information is provided by assumption (3.1). For the sake of brevity, we do not detail the computations. We have

$$\|A_j\|_{l^2(\tau)} \lesssim \sqrt{E_\omega(t,\tau)}, \quad \|B_j\|_{l^2(\tau)} \lesssim \sqrt{D_g^2(t,\tau)}.$$

Regarding the nonlinear terms, we have

$$\|C_{i,j}\|_{l^2(\tau)} \lesssim \sqrt{E_\omega(t,\tau)}\sqrt{\partial_\tau E_\omega(t,\tau)} \lesssim \sqrt{\partial_\tau E_\omega(t,\tau)}, \quad i=1,2,$$

whereas

$$\|C_{i,j}\|_{l^2(\tau)} \lesssim E_\omega(t,\tau), \quad i=3,4,6.$$

Finally, after integration by parts, we find

$$\|D_j\|_{l^1(\tau)} \lesssim C\sqrt{E_\omega(t,\tau)} \left(\sqrt{E_g^1(t,\tau)} + \sqrt{D_g^1(t,\tau)}\right) \lesssim \eta \, D_g^1(t,\tau) + \frac{1}{\eta}E_\omega(t,\tau), \quad \forall \eta > 0.$$

Inserting all these bounds in (3.75) (and taking parameter η small enough) yields the estimate in Proposition 4.

Conclusion: Main a priori estimate. – On one hand, from Propositions 1, 2 and 4, we infer that

$$\partial_t\big(\dot{E}_\omega + E_h + E_g^1\big)(t,\tau) + \big(\dot{D}_\omega + D_h + D_g^1\big)(t,\tau) \lesssim \partial_\tau E_\omega(t,\tau) + \partial_\tau E_g^2(t,\tau)$$
$$\lesssim \partial_\tau\left(\dot{E}_\omega + E_h + E_g^1\right)(t,\tau) + \partial_\tau E_g^2(t,\tau),$$

where we have used (3.4) to go from the first to the second line. On the other hand, Proposition 3 yields (still with relation (3.4)):

$$\partial_t E_g^2(t,\tau) + D_g^2(t,\tau) \lesssim_{\text{all}} \partial_\tau\left(\dot{E}_\omega + E_h + E_g^1\right)(t,\tau) + \partial_\tau E_g^2(t,\tau) + D_h(t,\tau).$$

Thus, we can multiply the last equation by a small factor α, and add it to the first equation, to end up with

$$\partial_t\left(\dot{E}_\omega + E_h + E_g^1 + \alpha E_g^2\right)(t,\tau) + \left(\dot{D}_\omega + D_h + D_g^1 + D_g^2\right)(t,\tau)$$
$$\lesssim_{\text{all}} \partial_\tau\left(\dot{E}_\omega + E_h + E_g^1 + \alpha E_g^2\right)(t,\tau).$$

This concludes the proof of Theorem 2.

REMARK 3. – *We insist that we can take (and in fact need to take) $\alpha \leq 1$, that is with an upper bound independent of any parameter. This will be important in the effective construction of solutions.*

REMARK 4. – *In the whole section, we have considered a fixed $\tau \in \,]0, \tau_0]$. Clearly, all the results of the section also apply to a function $\tau : t \mapsto \tau(t) \in \,]0, \tau_0]$, replacing energies like $E_\omega(t,\tau)$ or $E_h(t,\tau)$ by $E_\omega(t,\tau(t))$ or $E_h(t,\tau(t))$.*

4. Existence of a Gevrey solution

We deal here with the existence part of Theorem 1. The core of the argument is the *energy estimate* stated in Theorem 2. Actually, such estimate is derived on a sequence of approximations of the Prandtl system, that we now present.

4.1. Approximate system

We start with an initial data satisfying

$$(4.1) \qquad u_0 \in G_{\tau_0}^{7/4}(\mathbb{T}; H_{\gamma-1}^{s+1}), \quad \omega_0 := \partial_y u_0 \in G_{\tau_0}^{7/4}(\mathbb{T}; H_\gamma^s),$$

as well as assumptions (H) and (H′), see Section 2. Parameters $\tau_0, s, \sigma, \delta$ are as in Theorem 1. To fix ideas, we assume that the curve of non-degenerate critical points satisfies the constraint $0 < a_0 < 3$. The choice of the value 3 is of course arbitrary.

To construct solutions with all the a priori bounds of the previous section, we need to find a good approximating scheme: it should not destroy the energy estimates, and should guarantee that the pointwise bounds (3.1) (which echo those in (H′)) are preserved in small time, uniformly with respect to the approximation parameters.

The same scheme as in [20, Section 4] works here. It goes through the following approximate system (*regularized Prandtl system*)

$$(4.2) \qquad \begin{cases} \partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_y u^\varepsilon - \partial_y^2 u^\varepsilon - \varepsilon \partial_x^2 u^\varepsilon = 0, \\ \partial_x u^\varepsilon + \partial_y v^\varepsilon = 0, \\ u^\varepsilon|_{y=0} = v^\varepsilon|_{y=0} = 0, \ \lim_{y=+\infty} u^\varepsilon = 0. \end{cases}$$

The big difference with the original Prandtl system is the tangential diffusion $-\varepsilon \partial_x^2 u$, which allows to control the loss of x-derivative generated by the v-term. It restores well-posedness in the Sobolev setting. A detailed construction of solutions of (4.2) is performed in [20]. Let us denote

$$H^{s,\gamma}(\mathbb{T} \times \mathbb{R}_+) := \cap_{k=0}^s H_x^k(\mathbb{T}, H_\gamma^{s-k}(\mathbb{R}_+)).$$

As a result of [20], for any initial data

$$u_0 \in H^{s,\gamma-1}(\mathbb{T} \times \mathbb{R}_+), \quad \omega_0 := \partial_y u_0 \in H^{s,\gamma}(\mathbb{T} \times \mathbb{R}_+)$$

there is a unique $T_*^\varepsilon \in \mathbb{R}_+^* \cup \{+\infty\}$, and a unique maximal solution

$$u^\varepsilon \in L_{\text{loc}}^\infty([0, T_*^\varepsilon); H^{s,\gamma-1}(\mathbb{T} \times \mathbb{R}_+)), \quad \omega^\varepsilon := \partial_y u^\varepsilon \in L_{\text{loc}}^2([0, T_*^\varepsilon); H^{s,\gamma}(\mathbb{T} \times \mathbb{R}_+))$$

of (4.2). By maximal, we mean that

$$(4.3) \qquad \text{either } T_*^\varepsilon = +\infty, \quad \text{or } \limsup_{t \to T_*^\varepsilon} \|\omega^\varepsilon(t)\|_{H^{s,\gamma}(\mathbb{T} \times \mathbb{R}_+)} = +\infty.$$

Of course, our Gevrey data u_0 is regular enough to apply this result. *Furthermore, up to some small time (possibly depending on ε), the solution u^ε (resp. ω^ε) remains in $G_{\tilde\tau_0}^{7/4}(H_{\gamma-1}^{s+1})$ (resp. $G_{\tilde\tau_0}^{7/4}(H_\gamma^s)$), for any $\tilde\tau_0 < \tau_0$.*

The fact that system (4.2) preserves Gevrey regularity in small time is somehow classical. One way to show it is to establish Gevrey bounds on $u^{\varepsilon,n}$, solution of the Galerkin type approximation

$$\begin{cases} \partial_t u^{\varepsilon,n} + P^n \left(u^{\varepsilon,n} \partial_x u^{\varepsilon,n} + v^{\varepsilon,n} \partial_y u^{\varepsilon,n} \right) - \partial_y^2 u^{\varepsilon,n} - \varepsilon \partial_x^2 u^{\varepsilon,n} = 0, \\ \partial_x u^{\varepsilon,n} + \partial_y v^{\varepsilon,n} = 0, \\ u^{\varepsilon,n}|_{y=0} = v^{\varepsilon,n}|_{y=0} = 0, \ \lim_{y=+\infty} u^{\varepsilon,n} = 0. \end{cases}$$

P_n is the projection over the Fourier modes $|k| \leq n$ in variable x. One can show for this system uniform (in n) Gevrey bounds for small time T^ε (independent of n). As n goes to

infinity, we get Gevrey bounds for the (unique) solution u^ε of (4.2). We insist that these estimates are much simpler than those of the previous section. No special structure or tricky norm is needed: as the system is fully parabolic, there is no loss of x-derivative. For the sake of brevity, we do not give further details.

Now, let E_ω be the functional defined in (2.1), replacing u by u^ε. Let $M > E_\omega(0, \tilde{\tau}_0)$, and $t \mapsto \tau(t) \in [\frac{\tilde{\tau}_0}{2}, \tilde{\tau}_0]$ a function over $\mathbb{R}$. We denote

$$T^\varepsilon(\tau) := \sup\left\{ T \in [0, T_*^\varepsilon[, \quad E_\omega(t, \tau(t)) \leq M \text{ for all } t \in [0,T] \right\}.$$

By standard continuity arguments, $T^\varepsilon(\tau) > 0$, and $\limsup_{t \to T^\varepsilon(\tau)} E_\omega(t, \tau(t)) = M$.

The point is to show that for a good choice of M and τ (independent of ε), there exists $T^0 > 0$ (independent of ε) satisfying

$$T^\varepsilon(\tau) \geq T^0 \quad \text{uniformly in } \varepsilon.$$

Then, the solution u^ε will exist and have uniform Gevrey bounds on $[0, T^0]$. Finally, standard compactness arguments will allow to let $\varepsilon \to 0$, and obtain a solution of the Prandtl equation with initial data u_0 at the limit.

We now give a few hints on how to construct τ, M, T^0 with the inequality $T^\varepsilon(\tau) \geq T^0$. Let $M > E_\omega(0, \tilde{\tau}_0)$. The key-point is to establish the following: there exists $T^0 > 0$, depending on M, on the initial data (and on $\tilde{\tau}_0$), *but independent of ε*, such that:

 – For all $t \mapsto \tau(t) \in [\frac{\tilde{\tau}_0}{2}, \tilde{\tau}_0]$, for all $t \in [0, \min(T^0, T^\varepsilon(\tau))]$,

(4.4)
$$|\omega^\varepsilon(t,x,y)| \geq \frac{\delta}{(1+y)^\sigma}, \quad |\partial^\alpha \omega^\varepsilon(t,x,y)| \leq \frac{1}{\delta(1+y)^{\sigma+\alpha_2}}$$

   for the same δ as in (H$'$).
 – There exists $\alpha \leq 1$, and a function $t \mapsto \tau(t) \in [\frac{\tilde{\tau}_0}{2}, \tilde{\tau}_0]$ such that:

(4.5)
$$\text{For all } t \in [0, \min(T^0, T^\varepsilon(\tau))], \quad \frac{d}{dt}\mathcal{E}(\alpha, t, \tau(t)) \leq 0,$$

   with

$$\mathcal{E}(\alpha, t, \tau) = \dot{E}_\omega(t,\tau) + E_h(t,\tau) + E_g^1(t,\tau) + \alpha\, E_g^2(t,\tau),$$

   where the energy functionals $\dot{E}_\omega$, $E_h(t,\tau)$, E_g^i are defined respectively in (2.8), (2.9), (2.11) and (2.13), replacing u by u^ε.

We shall discuss the bounds (4.4) and (4.5) in the next paragraph. We explain first how they allow to conclude. From (4.5), we deduce that for all $t \in [0, \min(T^0, T^\varepsilon(\tau))]$,

$$\mathcal{E}(\alpha, t, \tau(t)) \leq \mathcal{E}(\alpha, 0, \tau(0)).$$

As $\alpha \in (0,1)$, this implies in turn

$$\dot{E}_\omega(t,\tau(t)) + E_h(t,\tau(t)) + E_g^1(t,\tau(t)) \leq \dot{E}_\omega(0,\tau(0)) + E_h(0,\tau(0)) + E_g^1(0,\tau(0)) + E_g^2(0,\tau(0)).$$

By Lemma 2, see also Remark 4, we get for all $t \in [0, \min(T^0, T^\varepsilon(\tau))]$

$$E_\omega(t,\tau(t)) \leq C \left(\dot{E}_\omega(t,\tau(t)) + E_h(t,\tau(t)) + E_g^1(t,\tau(t)) \right)$$

$$\leq C \left(\dot{E}_\omega(0,\tau(0)) + E_h(0,\tau(0)) + E_g^1(0,\tau(0)) + E_g^2(0,\tau(0)) \right)$$

for some constant $C > 0$. As stated in Lemma 2, this constant depends on $\tilde{\tau}_0$, δ, M, $\inf a$, $\sup a$, θ_0, θ_1 (see Remark 1, with $T := \min(T^0, T^\varepsilon(\tau))$). A closer look at the proof

of Lemma 2 shows that M is only involved through Sobolev embeddings, to control the L^∞ norms of ω and of some of its low order derivatives, over $[0,T]$:

$$(4.6) \qquad \|\partial^\alpha \omega\|_{L^\infty} \le C_{\alpha,\tilde\tau^0} \sup_{t\in[0,T]} E_\omega(t,\tau(t)) \le C_{\alpha,\tilde\tau^0} M.$$

We claim that by choosing T^0 small enough (depending notably on M and the initial data), one can have C to depend only on the initial data (and on $\tilde\tau_0$). For instance, one can control the L^∞ norm of ω over $[0,T]$ by using the inequality

$$\|\omega(t,\cdot)\|_{L^\infty} \le \|\omega_0\|_{L^\infty} + \|\partial_t\omega\|_{L^\infty}\, t \le \|\omega_0\|_{L^\infty} + C_M\, T^0,$$

the L^∞ norm of $\partial_t \omega$ being estimated using the vorticity equation and Sobolev embeddings of type (4.6). We can then take $T^0 < \frac{\|\omega_0\|_{L^\infty}}{C_M}$ to obtain (for all $t \in [0,T]$)

$$\|\omega(t,\cdot)\|_{L^\infty} \le 2\|\omega_0\|_{L^\infty}.$$

Similarly, one can show that for T^0 small enough,

$$\theta_1 := \inf_{t\in[0,T], x\in\mathbb{T}} \partial_y\omega(t,a(t,x)) \ge \frac{1}{2}\inf_{x\in\mathbb{T}} \partial_y\omega(x,a_0(x)).$$

We leave the details to the reader. Hence, we can take

$$M > 2C\left(\dot E_\omega(0,\tau(0)) + E_h(0,\tau(0)) + E_g^1(t,\tau(0)) + E_g^2(0,\tau(0))\right).$$

This ensures that $\mathcal{E}(\alpha,t,\tau(t)) \le \frac{M}{2}$ over $[0,\min(T^0,T^\varepsilon(\tau))]$, and from there that $T^\varepsilon(\tau) \ge T^0$.

4.2. Uniform bounds

4.2.1. *Maximum principle: proof of* (4.4). – This paragraph is devoted to the proof of (4.4). One must show the upper and lower bounds over a time interval $[0,\min(T^0,T^\varepsilon)]$ for some T^0 independent of ε. Note that the Sobolev embeddings and the bound $E_\omega(t,\tau(t)) \le M$ allow to bound some weighted L^∞ norms of ω^ε and some derivatives. However, the bounds in (4.4) cannot be directly deduced from there, due to the weight $(1+y)^\sigma$, $\sigma \gg \gamma$. Nevertheless, as the regularized Prandtl equation is parabolic, it can be deduced from a maximum principle. This is described in details in article [20]. The proof of the upper bounds

$$|\partial^\alpha \omega^\varepsilon(t,x,y)| \le \frac{1}{\delta(1+y)^{\sigma+\alpha_2}}, \quad \forall |\alpha| \le 2,$$

is exactly the same as in [20]: see inequality (5.5), and Section 5.2. Regarding the lower bound on $(1+y)^\sigma \omega^\varepsilon$, there is only a slight modification. In [20], $\omega \ne 0$ for all $y \ge 0$, so that the minimum principle can be applied over domains of the type $[0,T]\times\mathbb{T}\times\mathbb{R}_+$. In our situation, we must place ourselves above the curve of critical points. More precisely, as $\omega_0(x,a_0(x)) = 0$, $\partial_y\omega_0(x,a_0(x)) \ne 0$, there exists $\eta > 0$, such that $\omega_0(x,a_0(x)+\eta) \ne 0$. One then applies the minimum principle over $[0,T]\times\{(x,y), \quad y = a_0(x)+\eta\}$, T small enough. We refer to [20], inequality (5.6) and Section 5.2.

4.2.2. *Energy estimates: proof of* (4.5). – The point here is to derive the estimates of Theorem 2, uniformly in ε. More precisely, one can show that for any $t \mapsto \tau(t) \in [\frac{\tilde{\tau}_0}{2}, \tilde{\tau}_0]$,

$$(4.7) \qquad \partial_t \mathscr{E}(t, \tau(t)) \leq C \partial_\tau \mathscr{E}(t, \tau(t))$$

with a constant C that does not depend on ε (but depends on M, the initial data, and $\inf \tau = \frac{\tilde{\tau}_0}{2}$). Then,

$$\frac{d}{dt} \mathscr{E}(\alpha, t, \tau(t)) = \partial_t \mathscr{E}(\alpha, t, \tau(t)) + \tau'(t) \partial_\tau \mathscr{E}(\alpha, t, \tau(t)) \leq (C + \tau'(t)) \mathscr{E}(\alpha, t, \tau(t)).$$

Then, choosing $\tau(t) = \tilde{\tau}_0 - Ct$ leads to (4.5) (up to reducing T^0 so that the constraint $\tau \geq \frac{\tilde{\tau}_0}{2}$ remains satisfied).

The proof of (4.7), on the approximate system (4.2), mimics the proof of the a priori estimate in Theorem 2, on the exact Prandtl system. Indeed, the additional term $-\varepsilon \partial_x^2 u$ does not raise any problem. There are only two noticeable differences:

– In the energy estimate for $\dot{E}_\omega$, the estimate of the boundary term

$$E_j := \sum_{\substack{J = (j_1, j_2) \in \mathbb{N}^2 \\ |J| = j,\, 0 < j_2 \leq s}} \int_{\mathbb{T} \times \{0\}} \partial_y \omega_J \, \omega_J$$

changes slightly. Indeed, we recall that the estimate of E_j goes through the Lemma 4: it uses the Prandtl equation to reduce the number of y derivatives in the expression of $\partial_y^k \omega|_{y=0}$, k odd. Due to the regularization, the equation changes, and so the expression: the modified formula is given in [20, Lemma 5.9]. The new terms in the formula do not raise any serious difficulty. For the sake of brevity, we leave the details to the reader.

– In the energy estimate for E_g^2, a new commutator appears, namely

$$2\varepsilon \partial_x^{j-5} (\partial_{xy} \omega \, \partial_x^6 u - 2\varepsilon \partial_x \omega \partial_x^6 \omega).$$

It can be expanded by Leibniz rule: for brevity, we focus on one of the most difficult terms in the sum, that is:

$$\mathscr{C}_j := -2\varepsilon \partial_x \omega \partial_x^{j+1} \omega.$$

Proceeding as in Subsection 3.4, we need to evaluate

$$\|(j+1)^{3/2} \int_{\mathbb{T} \times \mathbb{R}_+} \mathscr{C}_j \, \tilde{g}_j\|_{l^1(\tau)} \lesssim \varepsilon \|(j+1)^{3/4} \|\partial_x^j \omega\|_{L^2}\|_{l^2(\tau)} \|(j+1)^{3/4} \|\partial_x \tilde{g}_j\|_{L^2}\|_{l^2(\tau)}$$

$$+ \varepsilon \|(j+1)^{1/4} \|\partial_x^j \omega\|_{L^2}\|_{l^2(\tau)} \|(j+1)^{5/4} \|\tilde{g}_j\|_{L^2}\|_{l^2(\tau)}$$

$$\lesssim C_\eta \varepsilon \|(j+1)^{3/4} \|\partial_x^j \omega\|_{L^2}\|_{l^2(\tau)}^2$$

$$+ \eta \varepsilon \|(j+1)^{3/4} \|\partial_x \tilde{g}_j\|_{L^2}\|_{l^2(\tau)}^2$$

$$+ \varepsilon \sqrt{\partial_\tau E_\omega(t, \tau)} \sqrt{\partial_\tau E_g^2(t, \tau)}.$$

Note that we have performed an integration by parts to go from the first to the second line. The second term can be absorbed for small η by the tangential dissipative term at

the left-hand side. Regarding the first term, we use that

$$\varepsilon\|(j+1)^{3/4}\|\partial_x^j\omega\|_{L^2}\|_{l^2(\tau)}^2$$

$$\lesssim \varepsilon\Big(\|\|(j+1)^{3/4}\partial_x h_{j-1}\|_{L^2}\|_{l^2(\tau)}^2$$

$$+ \|(j+1)^{3/4}\partial_x g_{j-1}\|_{L^2}\|_{l^2(\tau)}^2 + \|\|(j+1)^{3/4}\partial_x^{j-1}\omega\|_{L^2}\|_{l^2(\tau)}^2\Big)$$

$$\lesssim \varepsilon\Big(\|\|\partial_x h_j\|_{L^2}\|_{l^2(\tau)}^2 + \|\partial_x g_j\|_{L^2}\|_{l^2(\tau)}^2 + \|\|\partial_x^j\omega\|_{L^2}\|_{l^2(\tau)}^2\Big).$$

Such inequality can be obtained along the lines of Lemma 2. Eventually, multiplying by a small constant α the energy estimate on E_g^2, this term will be absorbed by the tangential dissipative terms related to E_h and E_g^1. For the sake of brevity, we skip the details.

5. Uniqueness

In this section, we prove the uniqueness of the solution constructed in the previous section. As usual, uniqueness can be derived by some energy estimate in a space weaker than the space where existence was proved. One can hope to do it in L^2. However, due to the loss of derivative in the equation, we have to do it with the same Gevrey regularity. We will only use a small loss of Sobolev correction.

We assume that we have two solutions u^1 and u^2 of our system (1.3), such that the functionals $\mathcal{E}^1(\alpha^0,t,\tau^0)$ and $\mathcal{E}^2(\alpha^0,t,\tau^0)$, defined as in (2.7) with ω replaced by ω^1 and ω^2, are bounded (in this whole section, ω^2 should not be confused with the square of ω). Here, we can take $\alpha^0 = 1$ and $\tau^0 = \min(\tau^1,\tau^2)$. We also assume the lower bound and the upper bound (3.1) to hold for both solutions ω^1 and ω^2. Finally, we assume the existence of critical curves $a^1(t,x)$ and $a^2(t,x)$ such that for $i=1,2$, we have

$$(5.1) \qquad \partial_t a^i(t,x) + \frac{\partial_t \omega^i(t,x,a(t,x))}{\partial_y \omega^i(t,x,a(t,x))} = 0, \quad a^i(0,x) = a_0(x).$$

We also recall that u^1 and u^2 should remain convex: $\partial_y^2 u^i = \partial_y \omega^i > 0$, for t and $y - a^i(t,x)$ small enough.

Let us denote $u = u^1 - u^2$, $v = v^1 - v^2$ and $\omega = \omega^1 - \omega^2$. Hence, we have

$$(5.2) \qquad \partial_t u + u^1\partial_x u + v^1\partial_y u + u\partial_x u^2 + v\partial_y u^2 - \partial_y^2 u = 0,$$

$$(5.3) \qquad \partial_t \omega + u^1\partial_x \omega + v^1\partial_y \omega + u\partial_x \omega^2 + v\partial_y \omega^2 - \partial_y^2 \omega = 0.$$

We are going to perform the same type of estimates as in the existence part: the relevant energy, still called $\mathcal{E}(\alpha,t,\tau)$, is defined as in (2.7), but the definition of h_j, g_j and $\tilde{g}_j$ used to define E_h, E_g^1 and E_g^2 need to be changed. Indeed, we take

$$(5.4) \qquad h_j(t,x,y) = \chi(y - a^2(t,x))\frac{\partial_x^j\omega}{\sqrt{\partial_y\omega^2}}(t,x,y),$$

$$(5.5) \qquad g_j(t,x,y) := \Big(\psi(y)\omega^2(t,x,y) + 1 - \psi(y)\Big)\Big(\partial_x^j\omega - \frac{\partial_y\omega^2}{\omega^2}\partial_x^j u\Big)(t,x,y),$$

$$(5.6) \qquad \tilde{g}_j(t,x,y) := \partial_x^{j-5}\big(\omega^2\partial_x^5\omega - \partial_y\omega^2\partial_x^5 u\big)$$

with $\psi = \psi(y) \in C_c^\infty(\mathbb{R})$ equal to 1 in an open neighborhood of $[0, \sup |a^2|]$. Also, we replace the constraint $j_2 \leq s$ in the definition of $\dot{E}_\omega$ by $j_2 \leq s - 2$. Finally, in all energies, we replace the Sobolev correction term, namely the factor $(j+1)^{10}$, by $(j+1)^8$.

Now, we follow the strategy used in the proof of the a priori estimates. We do not give all the details, but just mention the main changes we need to make.

5.1. Estimate of the vorticity energy

We differentiate the vorticity Equation (5.3) J times, $J = (j_1, j_2) \in \mathbb{N}^2$, $0 < j_2 \leq s - 2$. We find, for $\omega_J := \partial^J \omega$:

$$\partial_t \omega_J + u^1 \partial_x \omega_J + v^1 \partial_y \omega_J - \partial_y^2 \omega_J + [\partial^J, u^1]\partial_x \omega + [\partial^J, v^1]\partial_y \omega + \partial^J(u \partial_x \omega^2 + v \partial_y \omega^2).$$

Arguing as in the proof of Proposition 1, we deduce that

$$\partial_t \dot{E}_\omega(t, \tau) + \dot{D}_\omega(t, \tau) \lesssim_{\tau, M} \partial_\tau E_\omega(t, \tau).$$

We only point out the presence of the extra terms $\partial^J(u \partial_x \omega_J^2 + v \partial_y \omega_J^2)$ which can be treated along the estimates of A_j and B_j in Paragraph 3.2. Note here that u and v can only be hit by J derivatives. Moreover, due to the difference between the Sobolev correction factors $(j+1)^8$ and $(j+1)^{10}$, we can estimate $\partial^J \partial_x \omega_J^2$ and $\partial^J \partial_y \omega_J^2$ by the new $\mathscr{E}^2$, despite the extra x and y derivative. This is the reason we took a slight loss in Sobolev correction. Also, thanks to the new restriction $j_2 \leq s - 2$, not more than s derivatives with respect to y hit ω^2.

5.2. Estimate of the hydrostatic energy

As in Subsection (3.3), we write an equation for h_j. It is now:

$$\partial_t h_j + u^1 \partial_x h_j + v^1 \partial_y h_j - \partial_y^2 h_j = \frac{\chi^a}{\sqrt{\partial_y \omega^2}}[\partial_x^j, u^1]\partial_x \omega + \frac{\chi^a}{\sqrt{\partial_y \omega^2}}[\partial_x^j, v^1]\partial_y \omega$$

$$+ [\frac{\chi^a}{\sqrt{\partial_y \omega^2}}, \partial_t + u^1 \partial_x + v^1 \partial_y - \partial_y^2]\partial_x^j \omega - \frac{\chi^a}{\sqrt{\partial_y \omega^2}}\partial_x^j v \partial_y \omega^2$$

$$- \frac{\chi^a}{\sqrt{\partial_y \omega^2}}\partial_x^j(u \partial_x \omega^2) - \frac{\chi^a}{\sqrt{\partial_y \omega^2}}[\partial_x^j, \partial_y \omega^2]v.$$

Performing a standard energy estimate on the previous equation, multiplying by $\left(\tau^j (j!)^{-7/4} j^8\right)^2$, summing over j, we end up with an equation similar to (3.36). The main term now is

$$E_j := \int_{\mathbb{T} \times \mathbb{R}_+} \frac{\chi^a}{\sqrt{\partial_y \omega^2}}\partial_x^j v \, \partial_y \omega^2 \, h_j$$

which can be treated exactly as in Subsection 3.3. Hence, we deduce that the estimate (3.34) holds with the new definition of u, ω and with the change of the Sobolev correction in the energies.

5.3. Estimate of the (second) monotonicity energy

As in Subsection (3.3), we apply $\omega^2 \partial_x^5$ to the vorticity Equation (5.3), $\partial_y \omega^2 \partial_x^5$ to the velocity Equation (5.2), and substract one from the other:

$$
\begin{aligned}
\partial_t \tilde{g}_5 + u^1 \partial_x \tilde{g}_5 + v^1 \partial_y \tilde{g}_5 - \partial_y^2 \tilde{g}_5 = {}&- \sum_{k=1}^{5} \binom{5}{k} \partial_x^k u^1 \, \bar{g}_{5-k+1} - \sum_{k=1}^{5} \binom{5}{k} \partial_x^k v^1 \, \hat{g}_{5-k+1} \\
&- \sum_{k=0}^{5} \binom{5}{k} \partial_x^k u \, \bar{g}_{5-k+1}^2 - \sum_{k=0}^{4} \binom{5}{k} \partial_x^k v \, \hat{g}_{5-k+1}^2 \\
&- (\partial_t + u^1 \partial_x + v^1 \partial_y - \partial_y^2) \partial_y \omega^2 \, \partial_x^5 u \\
&+ 2\partial_y^2 \omega^2 \, \partial_x^5 \omega - 2\partial_y \omega^2 \, \partial_y \partial_x^5 \omega \\
&+ (\partial_t + u^1 \partial_x + v^1 \partial_y - \partial_y^2)\omega^2 \, \partial_x^5 \omega
\end{aligned}
$$

(5.7)

with $\hat{g}_k$, $\bar{g}_k$, $\hat{g}_k^2$, $\bar{g}_k^2$ defined as in (3.56), for instance $\bar{g}_k := \omega^2 \partial_x^k \omega - \partial_y \omega^2 \partial_x^k u$ and $\bar{g}_k^2 := \omega^2 \partial_x^k \omega^2 - \partial_y \omega^2 \partial_x^k u^2$ for $k \leq 5$. Note that the term that caused a loss of one x derivative has disappeared. As in the proof of Proposition 3, we apply ∂_x^{j-5} to the Equation (5.7) and we conclude in a similar way that (3.53) holds.

5.4. Estimate of the (first) monotonicity energy

As in the proof of Proposition 4, we write an equation for g_j defined in (5.5):

$$
\begin{aligned}
&\partial_t g_j + u^1 \partial_x g_j + v^1 \partial_y g_j - \partial_y^2 g_j \\
&= - \sum_{k=1}^{j} \binom{j}{k} \partial_x^k u^1 \, g_{j-k+1} - \sum_{k=1}^{j} \binom{j}{k} \partial_x^k v^1 \, \tilde{\psi} \left(\partial_y \partial_x^{j-k} \omega - \frac{\partial_y \omega^2}{\omega^2} \partial_x^{j-k} \omega \right) \\
&\quad - \sum_{k=0}^{j} \binom{j}{k} \partial_x^k u \, g_{j-k+1}^2 - \sum_{k=0}^{j-1} \binom{j}{k} \partial_x^k v \, \tilde{\psi} \left(\partial_y \partial_x^{j-k} \omega^2 - \frac{\partial_y \omega^2}{\omega^2} \partial_x^{j-k} \omega^2 \right) \\
&\quad + \left(\partial_t + u^1 \partial_x + v^1 \partial_y - \partial_y^2\right) \tilde{\psi} \, \partial_x^j \omega - \left(\partial_t + u^1 \partial_x + v^1 \partial_y - \partial_y^2\right) \left(\tilde{\psi} \frac{\partial_y \omega^2}{\omega^2} \right) \partial_x^j u \\
&\quad - 2\partial_y \tilde{\psi} \partial_y \partial_x^j \omega + 2\partial_y \left(\tilde{\psi} \frac{\partial_y \omega}{\omega} \right) \partial_x^j \omega
\end{aligned}
$$

where $\tilde{\psi} = \left(\psi(y)\omega^2(t,x,y) + 1 - \psi(y) \right)$. Again the main thing here is that in the third line the term that had a potential loss of x derivative, namely the one involving $\partial_x^j v$ was canceled. As above, we get an estimate similar to (3.74).

Putting all these estimates together, we can easily conclude, as in the a priori estimate section, by a Gronwall lemma that $u = 0$.

6. Conclusions

We have proved in this paper the local well-posedness of the Prandtl system for data that are in the Gevrey class $7/4$ in the horizontal variable x. We would like to discuss here some possible extension of this work:

- First, it is unlikely that the Gevrey class 7/4 is optimal. In particular, the analysis and numerics performed in [7] (on a simple linearization) suggest that the optimal exponent may be $s = 2$. However, we believe that extending our result to the Gevrey class 2 or even $2 - \varepsilon$ for all $\varepsilon > 0$ would require some new ideas. Moreover, it remains uncertain that the index $s = 2$ is the critical one since there may be more severe instabilities than those constructed in [7].
- For simplicity, we have considered here homogeneous source terms: $U = 0$ and $P = 0$. However, our work should extend to the case where U is a function of t, x, along the lines of [15]. Also, we have considered data that have polynomial decay when y goes to infinity. One can adapt the proof to data with exponential decay, using some Hardy type inequality with exponential weights.
- An important open question is to prove global existence for small data for the Prandtl system.
- Finally, we recall that the study of the Prandtl system is motivated by the zero viscosity limit in the presence of a boundary. The present work opens the road to a full justification of the Euler-Prandtl description, without analyticity.

Appendix A

Sobolev and Hardy inequalities

We recall classical Sobolev and Hardy inequalities, see [13]: For $f = f(x, y)$, we have

$$(A.1) \qquad \|f\|_{L^\infty} \lesssim \|f\|_{L^2} + \|\partial_x f\|_{L^2} + \|\partial_y f\|_{L^2} + \|\partial_y \partial_x f\|_{L^2}.$$

If $\lambda > -\frac{1}{2}$ and $\lim_{y \to +\infty} f(x, y) = 0$, then

$$(A.2) \qquad \|(1+y)^\lambda f\|_{L^2} \le \frac{2}{2\lambda + 1} \|(1+y)^{\lambda+1} \partial_y f\|_{L^2}.$$

If $\lambda < -\frac{1}{2}$, we have

$$(A.3) \qquad \|(1+y)^\lambda f\|_{L^2} \le \sqrt{-\frac{1}{2\lambda+1}} \|f|_{y=0}\|_{L^2} - \frac{2}{2\lambda+1} \|(1+y)^{\lambda+1} \partial_y f\|_{L^2}.$$

Appendix B

Proof of technical lemmas

B.1. Proof of Lemma 2

The point is to prove that

$$(B.1) \qquad \|\partial_x^j \omega\|_{L^2_\gamma} \lesssim \|h_j\|_{L^2(\mathbb{T} \times \mathbb{R}_+)} + \|g_j\|_{L^2_\gamma} \lesssim \|\partial_x^j \omega\|_{L^2_\gamma}.$$

Inequality on the right. – Clearly, $\|h_j\|_{L^2(\mathbb{T} \times \mathbb{R}_+)} \lesssim \|\partial_x^j \omega\|_{L^2_\gamma}$. Then, by assumption (3.1),

$$(1+y)^\gamma |g_j| \lesssim (1+y)^\gamma |\partial_x^j \omega| + (1+y)^{\gamma-1} |\partial_x^j u|$$

and from Hardy inequality (A.2), we deduce: $\|g_j\|_{L^2_\gamma} \lesssim \|\partial_x^j \omega\|_{L^2_\gamma}$.

Inequality on the left. – First, from the definition of h_j,

(B.2) $$\|\partial_x^j \omega\|_{L^2(\{|y-a|<2\varepsilon\})} \lesssim \|h_j\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}.$$

Then, we use the identities stated in Lemma 3:

(B.3)
$$\partial_x^j u(t,x,y) = \begin{cases} \omega(t,x,y) \displaystyle\int_3^y \left(\psi + \frac{(1-\psi)}{\omega}\right)^{-1} \frac{g_j}{\omega^2} + C_j(t,x)\omega(t,x,y), & y > a(t,x), \\[2mm] \omega(t,x,y) \displaystyle\int_0^y \left(\psi + \frac{(1-\psi)}{\omega}\right)^{-1} \frac{g_j}{\omega^2}, & y < a(t,x). \end{cases}$$

Clearly, as ω does not vanish away from $y = a$, the second identity implies

$$\|\partial_x^j \omega\|_{L^2(\{y-a<-\varepsilon\})} \lesssim \|g_j\|_{L^2(\{y<3\})}.$$

Then, we integrate the first identity for $x \in \mathbb{T}$, $\varepsilon < y - a < 2\varepsilon$. As ω does not vanish in this region, it allows to obtain a control of C_j:

$$\|C_j\|_{L^2(\mathbb{T})} \lesssim \|\partial_x^j u\|_{L^2(\{\frac{\varepsilon}{2}<y-a<\varepsilon\})} + \|g_j\|_{L^2(\{\frac{\varepsilon}{2}<y-a<\varepsilon\})}$$
$$\lesssim \|\partial_x^j \omega\|_{L^2(\{0<y<\varepsilon+a\})} + \|g_j\|_{L^2(\{y<3\})}$$
$$\lesssim \|\partial_x^j \omega\|_{L^2(\{|y-a|<\varepsilon\})} + \|\partial_x^j \omega\|_{L^2(\{y-a<-\varepsilon\})} + \|g_j\|_{L^2(\{y<3\})}$$

(B.4) $$\lesssim \|h_j\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} + \|g_j\|_{L^2(L^2_\gamma)}.$$

Finally, considering identities (B.3) for $|y - a| \geq \varepsilon$, we get

$$\|(1+y)^\gamma \partial_x^j \omega\|_{L^2(\{|y-a|\geq\varepsilon\})} \lesssim \left(\|C_j\|_{L^2(\mathbb{T})} + \|g_j\|_{L^2(L^2_\gamma)}\right) \lesssim \left(\|h_j\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} + \|g_j\|_{L^2(L^2_\gamma)}\right)$$

where the last inequality comes from (B.4). Combining with (B.2) yields the result.

B.2. Proof of Lemma 6

Bound on u_g^j. – We denote $\psi_\omega := \omega\psi + (1-\psi)$. For $a - \varepsilon \leq y \leq 3$

$$\int_0^y u_g^j = \int_0^a \omega \int_0^y \frac{g}{\omega\psi_\omega} + \int_a^y \omega \int_3^y \frac{g}{\omega\psi_\omega}$$
$$= \int_0^a \frac{u|_{y=a} - u}{\omega\psi_\omega} g + \int_a^y \frac{u|_{y=a} - u}{\omega\psi_\omega} g + (u|_{y=a} - u(\cdot,y)) \int_3^y \frac{g}{\omega\psi_\omega}$$

after integration by parts. The third term at the right-hand side does not raise any problem, as the integral only involves values of y ε-away from $y = a$. For the third and second term, we notice that

$$\left|\frac{u - u|_{y=a}}{\omega\psi_\omega}\right| \lesssim 1 \quad \text{for } 0 \leq y \leq 3.$$

Finally, we get

$$\left|\int_0^y u_g^j\right| \lesssim \int_0^3 |g|,$$

and the bound follows easily.

Bound on C_j. – From the proof of Lemma 2 (cf. (B.4)), we know that

$$\|C_j\|_{L^2(\mathbb{T})} \lesssim \|h_j\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} + \|g_j\|_{L^2(L^2_\gamma)} \lesssim \|\partial_x^j \omega\|_{L^2(L^2_\gamma)}.$$

Bound on $\partial_x C_j$. – We write: for $y > a$,

$$(\text{B.5}) \qquad \omega \partial_x C_j = \partial_x^{j+1} u - \partial_x \omega \, C_j - \partial_x \left(\omega \int_3^y \frac{g_j}{\omega \psi_\omega} \right).$$

As for C_j, we integrate for $\varepsilon \leq y \leq 2\varepsilon$, $\varepsilon > 0$. We end up with

$$\|\partial_x C_j\|_{L^2(\mathbb{T})} \lesssim \|C_j\|_{L^2(\mathbb{T})} + \|\partial_x^{j+1}\omega\|_{L^2(L_\gamma^2)} + \|g_j\|_{L^2(\{y<3\})} + \|\partial_x g_j\|_{L^2(\{y<3\})}.$$

Easily, the last two terms are bounded by $\|\partial_x^j \omega\|_{L^2(L_\gamma^2)} + \|\partial_x^{j+1}\omega\|_{L^2(L_\gamma^2)}$. The result follows.

B.3. Proof of Lemma 7

It is enough to prove that for all $\alpha \geq 0$,

$$(\text{B.6}) \qquad \| \, \|(j+1)^{\alpha/4}(\tilde{g}_j - g_j)\|_{L^2(\{y\leq 3\})} \|_{l^2(\tau)} \lesssim \|(j+1)^{(\alpha-3)/4} \|\omega\|_{\mathcal{H}_\gamma^j} \|_{l^2(\tau)}.$$

For $y \in [0, 3]$, we have

$$\tilde{g}_j - g_j = [\partial_x^{j-5}, \omega]\partial_x^5 \omega - [\partial_x^{j-5}, \partial_y \omega]\partial_x^5 u$$

$$= \sum_{k=1}^{j-5} \binom{j-5}{k} \partial_x^k \omega \, \partial_x^{j-k}\omega - \partial_x^k \partial_y \omega \, \partial_x^{j-k} u.$$

This kind of sums has been considered several times throughout the paper. We get here

$$\|\tilde{g}_j - g_j\|_{L^2(\{y\leq 3\})} \leq \sum_{k=1}^{\frac{j-5}{2}} \binom{j-5}{k} \sum_{m=0}^3 \|\omega\|_{\mathcal{H}_\gamma^{k+m}} \|\omega\|_{\mathcal{H}_\gamma^{j-k}}$$

$$+ \sum_{k=\frac{j-5}{2}}^{j-5} \binom{j-5}{k} \left(\|\omega\|_{\mathcal{H}_\gamma^k} + \|\omega\|_{\mathcal{H}_\gamma^{k+1}} \right) \sum_{m=0}^2 \|\omega\|_{\mathcal{H}_\gamma^{j-k+m}}$$

$$:= A_j + B_j.$$

We focus on A_j, as the other term B_j is much better. Through the change of index $k' := k-1$, we find that

$$A_j \leq \sum_{k=0}^{\frac{j-5}{2}-1} \binom{j-5}{k+1} \sum_{m=1}^4 \|\omega\|_{\mathcal{H}_\gamma^{k+m}} \|\omega\|_{\mathcal{H}_\gamma^{j-1-k}}$$

$$\leq 2 \sum_{k=0}^{\frac{j-1}{2}} \binom{j-1}{k} \sum_{m=1}^4 \|\omega\|_{\mathcal{H}_\gamma^{k+m}} (j-k)\|\omega\|_{\mathcal{H}_\gamma^{j-1-k}}$$

using the inequality $\binom{j}{k} \leq \binom{j-1}{k}j \leq 2\binom{j-1}{k}(j-k)$. It follows that

$$\|(j+1)^{\alpha/4} A_j\|_{l^2(\tau)} \lesssim_\tau \|(j+1)^{\alpha/4}(j+1)\|\omega\|_{\mathcal{H}_\gamma^{j-1}}\|_{l^2(\tau)} \lesssim_\tau \|(j+1)^{(\alpha-3)/4}\|\omega\|_{\mathcal{H}_\gamma^j}\|_{l^2(\tau)}.$$

This leads to the expected bound.

B.4. Proof of Lemma 8

We only treat the case $l = 1$:

$$\| (j+1)^\alpha \|(1+y)\partial_y \tilde{g}_{j-1}\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} \|_{l^2(\tau)} \lesssim \|(j+1)^\alpha \|\omega\|_{\mathscr{H}^j_\gamma} \|_{l^2(\tau)}.$$

The other cases $l = 0$ and $l = 2$ are handled in the same way. We compute

$$\partial_y \tilde{g}_{j-1} = \partial_x^{j-6} \left(\omega \, \partial_y \partial_x^5 \omega - \partial_y^2 \omega \, \partial_x^5 u \right)$$

$$= \sum_{k=0}^{j-6} \binom{j-6}{k} \partial_x^k \omega \, \partial_y \partial_x^{j-k-1} \omega - \sum_{k=0}^{j-6} \binom{j-6}{k} \partial_x^k \partial_y^2 \omega \, \partial_x^{j-k-1} u.$$

By now standard computations:

$$\|(1+y)\partial_y \tilde{g}_{j-1}\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} \leq C_\gamma \sum_{k=0}^{\frac{j-6}{2}} \binom{j-6}{k} \sum_{m=0}^{4} \|\omega\|_{\mathscr{H}^{k+m}_\gamma} \left(\|\omega\|_{\mathscr{H}^{j-k}_\gamma} + \|\omega\|_{\mathscr{H}^{j-1-k}_\gamma} \right)$$

$$+ C_\gamma \sum_{\frac{j-6}{2}+1}^{j-6} \binom{j-6}{k} \left(\|\omega\|_{\mathscr{H}^k_\gamma} + \|\omega\|_{\mathscr{H}^{k+2}_\gamma} \right) \sum_{m=-1}^{2} \|\omega\|_{\mathscr{H}^{j-k+m}}$$

$$:= A_j + B_j$$

for some absolute constant C_γ. On one hand, as $\binom{j-6}{k} \leq \binom{j}{k}$, Lemma 1 clearly implies

$$\|(j+1)^\alpha A_j\|_{l^2(\tau)} \lesssim_\tau \|(j+1)^\alpha \|\omega\|_{\mathscr{H}^j_\gamma} \|_{l^2(\tau)}.$$

On the other hand, after the change of index $k' := k + 2$

$$B_j \leq C_\gamma \sum_{k=\frac{j}{2}}^{j} \binom{j}{k} \left(\|\omega\|_{\mathscr{H}^{k-2}_\gamma} + \|\omega\|_{\mathscr{H}^k_\gamma} \right) \sum_{m=1}^{4} \|\omega\|_{\mathscr{H}^{j-k+m}}$$

(notice that $\binom{j-6}{k-2} \leq \binom{j}{k}$). Still by Lemma 1,

$$\|(j+1)^\alpha B_j\|_{l^2(\tau)} \lesssim_\tau \|(j+1)^\alpha \|\omega\|_{\mathscr{H}^j_\gamma} \|_{l^2(\tau)}.$$

Combining the bounds on A_j and B_j yields the result.

B.5. Proof of Lemma 9

For (x, y) in the support of $\chi^{\mu,a}$, we have

$$\partial_y \left(\tilde{g}_j - g_j \right) = \sum_{k=1}^{j-5} \binom{j-5}{k} \left(\partial_x^k \omega \partial_x^{j-k} \partial_y \omega - \partial_x^k \partial_y^2 \omega \, \partial_x^{j-k} u \right).$$

We proceed as we did several times in the paper: we get

$$\|\sqrt{\chi^{\mu,a}}\partial_y(\tilde{g}_j - g_j)\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}$$

$$\leq C_\gamma \sum_{k=1}^{\frac{j-5}{2}} \sum_{m=0}^{4} \|\omega\|_{\mathscr{H}_\gamma^{k+m}} \left(\|\sqrt{\chi^{\mu,a}}\partial_x^{j-k}\partial_y\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} + \|\omega\|_{\mathscr{H}_\gamma^{j-k}} \right)$$

$$+ C_\gamma \sum_{k=\frac{j-5}{2}}^{j-5} \binom{j-5}{k} \left(\|\omega\|_{\mathscr{H}_\gamma^k} \sum_{m=1}^{3} \|\omega\|_{\mathscr{H}_\gamma^{j-k+m}} + \|\omega\|_{\mathscr{H}_\gamma^{k+2}} \sum_{m=0}^{2} \|\omega\|_{\mathscr{H}_\gamma^{j-k+m}} \right)$$

$$:= A_j + B_j.$$

Note that we did not bound $\|\sqrt{\chi^{\mu,a}}\partial_x^{j-k}\partial_y\omega\|_{L^2}$ by $\|\partial_y\omega\|_{\mathscr{H}_\gamma^{j-k}}$: we need to keep track of the localization to handle this term.

Treatment of A_j. – We set $k' := k - 1$. Dropping the prime,

$$A_j \leq C_\gamma \sum_{k=0}^{\frac{j-5}{2}-1} \binom{j-5}{k+1} \sum_{m=1}^{5} \|\omega\|_{\mathscr{H}_\gamma^{k+m}} \left(\|\sqrt{\chi^{\mu,a}}\partial_x^{j-1-k}\partial_y\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} + \|\omega\|_{\mathscr{H}_\gamma^{j-1-k}} \right)$$

$$\leq C_\gamma' \sum_{k=0}^{\frac{j-1}{2}} \binom{j-1}{k} \sum_{m=0}^{5} \|\omega\|_{\mathscr{H}_\gamma^{k+m}} (j-1-k) \left(\|\sqrt{\chi^{\mu,a}}\partial_x^{j-1-k}\partial_y\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} + \|\omega\|_{\mathscr{H}_\gamma^{j-1-k}} \right).$$

Here, we have used $\binom{j-5}{k+1} \leq j\binom{j-1}{k} \leq 2(j-k)\binom{j-1}{k}$. By Lemma 1, we find

$$\|(j+1)^{3/4}A_j\|_{l^2(\tau)} \lesssim \||\omega\|_{\mathscr{H}_\gamma^j}\|_{l^2(\tau)} \left(\|(j+1)^{7/4}\|\sqrt{\chi^{\mu,a}}\partial_x^{j-1}\partial_y\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)} \right.$$

$$+ \|(j+1)^{7/4}\|\omega\|_{\mathscr{H}_\gamma^{j-1}}\|_{l^2(\tau)} \Big)$$

$$\lesssim_\tau \||\omega\|_{\mathscr{H}_\gamma^j}\|_{l^2(\tau)} \left(\||\sqrt{\chi^{\mu,a}}\partial_x^j\partial_y\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}\|_{l^2(\tau)} + \||\omega\|_{\mathscr{H}_\gamma^j}\|_{l^2(\tau)} \right).$$

Finally, we notice that

$$\|\sqrt{\chi^{\mu,a}}\partial_x^j\partial_y\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} \lesssim \|\partial_y h_j\|_{L^2(\mathbb{T}\times\mathbb{R}_+)} + \|\partial_x^j\omega\|_{L^2(\mathbb{T}\times\mathbb{R}_+)}$$

so that

$$\|(j+1)^{3/4}A_j\|_{l^2(\tau)} \lesssim \sqrt{E_\omega(t,\tau)} \left(\sqrt{D_h(t,\tau)} + \sqrt{E_\omega(t,\tau)} \right) \lesssim \sqrt{D_h(t,\tau)} + \sqrt{E_\omega(t,\tau)}.$$

Treatment of B_j. – Through the change of index $k' := k + 1$ (first term) and $k' := k + 3$ (second term), and using that $\binom{j-5}{k-d} \leq \binom{j}{k}$, $d = 1$ or 3, we get:

$$B_j \leq C \sum_{k=\frac{j}{2}}^{j} \binom{j}{k} \|\omega\|_{\mathscr{H}_\gamma^{k-2}} \sum_{m=2}^{5} \|\omega\|_{\mathscr{H}_\gamma^{j-k+m}}.$$

Thus,

$$\|(j+1)^{3/4}B_j\|_{l^2(\tau)} \lesssim \||\omega\|_{\mathscr{H}_\gamma^j}\|_{l^2(\tau)} \|(j+1)^{3/4}\|\omega\|_{\mathscr{H}_\gamma^{j-1}}\|_{l^2(\tau)}$$

$$\lesssim \||\omega\|_{\mathscr{H}_\gamma^j}\|_{l^2(\tau)} \|(j+1)^{-1}\|\omega\|_{\mathscr{H}_\gamma^j}\|_{l^2(\tau)} \lesssim \sqrt{E_\omega(t,\tau)}.$$

Combining the bounds on A_j and B_j yields the result.

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David Gérard-Varet
Institut de Mathématiques de Jussieu
et
Université Paris 7
175 rue du Chevaleret
75013 Paris, France
E-mail: `david.gerard-varet@imj-prg.fr`

Nader Masmoudi
Courant Institute, NYU
251 Mercer Street
New York, NY 10012, USA
E-mail: `masmoudi@courant.nyu.edu`