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GAUDUCHON

*Ambitoric geometry II:  
Extremal toric surfaces and Einstein 4-orbifolds*

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# AMBITORIC GEOMETRY II: EXTREMAL TORIC SURFACES AND EINSTEIN 4-ORBIFOLDS

BY VESTISLAV APOSTOLOV, DAVID M. J. CALDERBANK  
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**ABSTRACT.** – We provide an explicit resolution of the existence problem for extremal Kähler metrics on toric 4-orbifolds  $M$  with second Betti number  $b_2(M) = 2$ . More precisely we show that  $M$  admits such a metric if and only if its rational Delzant polytope (which is a labelled quadrilateral) is  $K$ -polystable in the relative, toric sense (as studied by S. Donaldson, E. Legendre, G. Székelyhidi et al.). Furthermore, in this case, the extremal Kähler metric is *ambitoric*, i.e., compatible with a conformally equivalent, oppositely oriented toric Kähler metric, which turns out to be extremal as well. These results provide a computational test for the  $K$ -stability of labelled quadrilaterals.

Extremal ambitoric structures were classified locally in Part I of this work, but herein we only use the straightforward fact that explicit Kähler metrics obtained there are extremal, and the identification of Bach-flat (conformally Einstein) examples among them. Using our global results, the latter yield countably infinite families of compact toric Bach-flat Kähler orbifolds, including examples which are globally conformally Einstein, and examples which are conformal to complete smooth Einstein metrics on an open subset, thus extending the work of many authors.

**RÉSUMÉ.** – Nous donnons une solution complète et explicite du problème d'existence de métriques kählériennes extrémales sur un orbifold torique  $M$  de dimension réelle 4, dont le nombre de Betti  $b_2(M)$  est égal à 2. Nous montrons plus précisément que  $M$  admet de telles métriques si et seulement si son polytope de Delzant rationnel — qui est alors un quadrilatère étiqueté — est  $K$ -polystable, suivant la théorie générale développée dans le cas torique par S. K. Donaldson, E. Legendre, G. Székelyhidi et al., et que ces métriques sont alors ambitoriques, donc complètement explicites d'après la classification figurant dans la première partie de ce travail. Notre approche donne de surcroît une façon effective de tester la stabilité des quadrilatères étiquetés. Parmi les métriques kählériennes construites dans cet article figurent celles dont le tenseur de Bach est nul, qui sont à la fois extrémales et conformément Einstein. Nous obtenons ainsi, en dimension 4, de nouveaux exemples explicites d'orbifolds d'Einstein compacts ou de variétés d'Einstein non-compactes, complètes et lisses.

## Introduction

This paper concerns the explicit construction of extremal Kähler metrics on compact 4-orbifolds, including Kähler metrics which are conformally Einstein (either globally or on

the complement of real hypersurface). The examples we construct are toric with second Betti number two, i.e., their rational Delzant polytope (which is the image of the momentum map of the 2-torus action [22, 45]) is a quadrilateral. More precisely, we use extremal *ambitoric* metrics, which we classified locally in Part I of this work, to resolve completely the existence problem in the quadrilateral case.

There are several narratives to which this paper may be viewed as a contribution. A general theme is the interplay between the abstract existence theory for a geometric PDE, and the construction of explicit solutions associated to special geometric structures. Extremal Kähler metrics were introduced by E. Calabi [16, 17] to address the problem of finding canonical Kähler metrics with Kähler form in a given cohomology class  $\Omega$  on a compact complex manifold. The  $L_2$  norm of the scalar curvature yields a functional on  $\Omega$ , and its critical points are the extremal metrics. They are thus natural generalizations of constant curvature metrics on Riemann surfaces; in general, the Euler-Lagrange equation asserts that a Kähler metric is extremal if its scalar curvature is Hamiltonian for a Killing vector field. As a geometric PDE, this is quasilinear of fourth order, and no general methods are currently available.

Nevertheless, considerable progress on the existence theory has been made, following the seminal work of Calabi [15] on the non-positive Kähler-Einstein case and the resolution of his famous conjecture by T. Aubin [9] and S-T. Yau [59]. Conjectures going back to Yau [60], G. Tian [56] and S. Donaldson [26] state that the obstruction to the existence of an extremal Kähler metric in the class  $\Omega = 2\pi c_1(\mathcal{L})$  of a polarized complex manifold  $(M, \mathcal{L})$  should be a purely algebro-geometric “stability condition” on the pair  $(M, \mathcal{L})$ , and these conjectures may be extended to orbifolds [49]. Defining a precise notion of stability is part of the problem, one candidate being “K-(poly)stability” [56, 26]: the necessity of K-polystability has been proven for constant scalar curvature metrics [28, 20, 51, 46], and a version of K-polystability relative to a maximal torus of the automorphism group of  $(M, \mathcal{L})$ , developed by G. Székelyhidi [54, 53], is necessary for the existence of an extremal Kähler metric of non-constant scalar curvature [52].

A major difficulty with the theory is that in practice it is not only difficult to determine whether a given polarized variety admits an extremal Kähler metric—it is also difficult to verify a proposed stability condition. Consequently classes of complex manifolds or orbifolds for which extremality and stability are more tractable play an important role. These examples come in two main flavors: *ruled* and *toric*. Using a construction due to Calabi [16], ruled surfaces and other projective line bundles provide a setting for many explicit extremal Kähler metrics [57, 37, 54, 7]. We refer to these as metrics of *Calabi type*; they admit a Hamiltonian 2-form of order one [4, 5]. The extremality equations reduce to ODEs with explicit polynomial solutions, and stability amounts to a positivity condition on the solution [54, 8]. For toric varieties, in contrast, the extremality equations only reduce to a nonlinear fourth order PDE in the momenta; explicit solutions are hard to find, but the existence theory is well-developed [26, 27] and there is a well-understood notion of “relative K-polystability with respect to toric degenerations” which is widely believed to be equivalent to existence [26, 53, 62, 61]. Explicit examples are largely limited to orthotoric  $2m$ -orbifolds, which admit a Hamiltonian 2-form of order  $m$  and have a convex  $m$ -cube (or degeneration) for their rational Delzant polytope [4, 7, 42].

In dimension four, examples and theory come together to provide a fairly complete picture. Extremal Kähler surfaces of Calabi type are locally toric (as the base is a constant curvature Riemann surface) and there are specific results for toric surfaces. The equivalence of existence and (relative) K-polystability is established in [30, 29] in the constant scalar curvature case, while more recent work [18] relaxes this to the assumption that the zero locus of the scalar curvature does not contain a toric divisor.

Our paper is closely related to work of E. Legendre [42], who investigated systematically the extent to which explicit methods resolve the existence problem when the rational Delzant polytope is a convex quadrilateral. Her solution highlighted the role of the *extremal affine function*  $\zeta$  on the rational Delzant polytope, a combinatorial invariant which pulls back to the scalar curvature in the extremal case. Her main results show that Hamiltonian 2-form methods suffice only for “equiposed” quadrilaterals, for which  $\zeta$  has equal values at the midpoints of the diagonals. A key ingredient in Legendre’s work is the observation that  $\zeta$  is linear in the inverse lengths of the normals. Using this, she resolved the existence problem for the codimension one family of equiposed quadrilaterals using orthotoric, Calabi type or product metrics.

The theory of Hamiltonian 2-forms in four dimensions [4] implies that these toric metrics are in fact *ambitoric*, i.e., toric with respect to a pair of oppositely oriented but conformally equivalent Kähler metrics. The local classification of ambitoric structures [6] implies that the “regular” examples (i.e., neither a product nor of Calabi type) are determined by a quadratic polynomial  $q$  and two functions  $A, B$  of one variable. Regular ambitoric structures reduce to orthotoric metrics precisely when  $q$  has vanishing discriminant. The extremality conditions for regular ambitoric structures can be explicitly solved with  $A, B$  given by quartic polynomials [6], and this generalization suffices to remove the equiposedness constraint introduced by Legendre.

To prove this, we use, in addition to ambitoric geometry, two further ingredients. The first is an analysis of rational Delzant quadrilaterals building on [41]. We compute the extremal affine function  $\zeta$  and establish a notion of “temperateness” for polystable quadrilaterals which implies  $\zeta$  is positive at the midpoints of the diagonals.

The second is the concept of a “factorization structure”, which makes precise the separation of variables technique that underpins explicit solutions of geometric PDEs on toric 4-orbifolds. One can hope such an approach will work in  $2m$ -dimensions when the rational Delzant polytope is a convex  $m$ -cube (or degeneration), with the  $2m$  facets providing boundary conditions for the  $m$  functions of one variable determining the solution. In particular, by the uniqueness of toric extremal Kähler metrics [34], we might *expect* a rational Delzant polytope to select an essentially unique adapted factorization structure for the solution. This is indeed what happens for  $m = 2$ .

The fruit of this analysis is Theorem 1, which establishes in an explicit way, and for arbitrary quadrilaterals, the conjecture [26] that the existence of an extremal Kähler metric is equivalent to relative K-polystability with respect to toric degenerations. Indeed, we show that temperate quadrilaterals admit a factorization structure which relates the polystability condition directly to the positivity of the quartics  $A$  and  $B$  appearing in the expression for the extremal ambitoric metrics. This explicitly computable criterion yields new examples both of

extremal toric 4-orbifolds, and of (unstable) toric 4-orbifolds admitting no extremal Kähler metric.

In our discussion of examples, we return to another motivation for ambitoric geometry: Kähler metrics which are conformally Einstein. Since the work of D. Page [48] and E. Calabi [16], such metrics have been an important source of examples, with contributions by L. Bérard-Bergery [10], R. Bryant [14], A. Derdzinski [23], G. Maschler [24], and C. LeBrun [38, 39, 40] among others. In part I of this work [6], we classified locally 4-dimensional Einstein metrics with degenerate half-Weyl tensors using Bach-flat ambitoric structures (which are extremal and locally conformally Einstein). Here we show that Bach-flat ambitoric 4-orbifolds are abundant, and include examples which are globally conformally Einstein, as well as examples with an open set where the Kähler metric is conformal to a smooth, complete (conformally compact) Einstein metric on a covering. These extend in particular the examples of R. Bryant [14].

The organization is as follows. In Section 1 we review the theory of compact toric Kähler orbifolds [2, 22, 26, 35, 45], but adopting an affine invariant viewpoint. We begin our analysis of quadrilaterals in Section 2 where affine invariance provides an effective tool to compute, for example, the extremal vector field, without extensive calculus. By considering the affine structure as a variable, we similarly use projective invariance to simplify our later discussion of factorization structures. This approach is closely related to 5-dimensional contact, CR and Sasakian geometry, cf. [41], which we discuss in Appendix C. The main results are established in Sections 3–4 which concern the compactification of ambitoric metrics in general, in terms of *factorization structures*, and extremal ambitoric metrics in particular in terms of *adapted factorization structures*. Examples, including the new Einstein metrics, are given in Section 5. In Appendix B, we study the set of inward normals for which a given quadrilateral is K-semistable but not stable, and show that any quadrilateral which is not a parallelogram can be made K-unstable by suitably chosen normals.

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## 1. Toric orbifolds, Kähler metrics and polystability

We review the theory of toric Kähler  $2m$ -orbifolds  $M$ , primarily adopting the symplectic point of view, as in [2, 22, 35, 36, 45]. We denote the  $m$ -torus acting on  $M$  by  $\mathbb{T} = \mathfrak{t}/2\pi\Lambda$ , where  $\mathfrak{t}$  is its (Abelian) Lie algebra, and  $\Lambda$  its lattice of circle subgroups. Our applications have  $m = 2$  and a geometry which may not be compatible with the lattice or origin in  $\mathfrak{t}^*$ . Hence we use basis-independent and affine-invariant language.

**1.1. Toric symplectic orbifolds**

Let  $\mathfrak{h}$  be an  $(m+1)$ -dimensional real vector space, and  $\iota: \mathbb{R} \rightarrow \mathfrak{h}$  a 1-dimensional subspace with quotient  $\mathfrak{t}$ . Dually,  $\mathfrak{h}^*$  has  $\mathfrak{t}^*$  as a subspace with quotient  $\iota^\top: \mathfrak{h}^* \rightarrow \mathbb{R}$ . The inverse image of  $1 \in \mathbb{R}$  under  $\iota^\top$  is an affine subspace  $\Xi$  of  $\mathfrak{h}^*$ , modeled on  $\mathfrak{t}^*$ . The space of affine functions  $f$  on  $\Xi$  is canonically isomorphic to  $\mathfrak{h}$ , where the constant functions are  $\iota(c)$ ,  $c \in \mathbb{R}$ , and the projection of  $f$  to  $\mathfrak{t}$ , viewed as a linear form on  $\mathfrak{t}^*$ , is its derivative  $df$  (at every point of  $\Xi$ ).

DEFINITION 1. – Let  $L_1, \dots, L_n$  be affine functions on  $\Xi$  such that the convex polytope

$$\Delta := \{\xi \in \Xi : L_j(\xi) \geq 0, j = 1, \dots, n\}$$

is compact and nonempty.<sup>(1)</sup> Then  $(\Delta, L_1, \dots, L_n)$  is a *rational Delzant polytope* in  $\Xi$  iff

- (i)  $\forall j \in \{1, \dots, n\}$  the normals  $u_j := dL_j \in \mathfrak{t}$  belong to the lattice  $\Lambda \subset \mathfrak{t}$  and
- (ii)  $\forall \xi \in \Delta$ ,  $N_\xi := \{u_j \in \mathfrak{t} : L_j(\xi) = 0\}$  is linearly independent in  $\mathfrak{t}$ .

The term “rational” refers to the fact that the normals  $u_j$  span an  $m$ -dimensional vector space over  $\mathbb{Q}$ . If the *affine normals*  $L_j$  span an  $(m+1)$ -dimensional vector space over  $\mathbb{Q}$ , we say the polytope is *strongly rational*. The *faces*  $F$  of  $\Delta$  are intersections of the *facets* (codimension one faces)  $F_j = \Delta \cap \{\xi \in \Xi : L_j(\xi) = 0\}$  which have inward normals  $u_j$ . A rational Delzant polytope is *simple* or *m-valent*:  $m$  facets and  $m$  edges meet at each vertex. The primitive inward normals, which are uniquely determined by  $\Delta$  and  $\Lambda$ , have the form  $u_j/m_j$  for some positive integer labelling  $m_j$  of the facets  $F_j$ , so rational Delzant polytopes are also called *labelled polytopes* [45]. It is convenient to encode the labelling in the row vector  $\mathbf{L} = (L_1, \dots, L_n) \in \text{Hom}(\mathbb{R}^n, \mathfrak{h}) \cong \mathfrak{h} \otimes \mathbb{R}^{n*}$  of affine normals, and denote the normals by  $d\mathbf{L} = (u_1, \dots, u_n) \in \text{Hom}(\mathbb{R}^n, \mathfrak{t})$ .

Compact toric symplectic  $2m$ -orbifolds are classified (up to equivariant symplectomorphism) by rational Delzant polytopes (up to lattice preserving affine equivalences) [22, 45]. In one direction, if  $(M, \omega)$  is a toric symplectic orbifold under the action of  $\mathbb{T}$ , with Lie algebra  $\mathfrak{t} \subseteq C^\infty(M, TM)$ , we let  $\mathfrak{h} \subseteq C^\infty(M, \mathbb{R})$  be the space of Hamiltonian generators  $f$  (with  $\text{grad}_\omega f \in \mathfrak{t}$ ) and  $\mu: M \rightarrow \mathfrak{h}^*$  be the natural momentum map (with  $\langle \mu(x), f \rangle = f(x)$ ); then  $\Delta$  is the image of  $\mu$ , and  $(\text{span}_{\mathbb{R}} u_j \cap \Lambda) / \text{span}_{\mathbb{Z}} u_j \cong \mathbb{Z}/m_j \mathbb{Z}$  is the local uniformizing group of every point in  $\mu^{-1}(F_j^0)$ . (For any face  $F$ , we denote by  $F^0$  its interior.) Conversely,  $(\Delta, \mathbf{L})$  determines  $(M, \omega)$  as a symplectic quotient of  $\mathbb{C}^n$  by an  $(n - m)$ -dimensional subgroup  $G$  of the standard  $n$ -torus  $(S^1)^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$ :  $G$  is the kernel of the map  $(S^1)^n \rightarrow \mathbb{T} = \mathfrak{t} / 2\pi\Lambda$  induced by the natural map  $d\mathbf{L}: \mathbb{R}^n \rightarrow \mathfrak{t}$  (with kernel the Lie algebra  $\mathfrak{g}$  of  $G$ ); the composite of  $\mathbf{L}^\top: \mathfrak{h}^* \rightarrow \mathbb{R}^{n*}$  with the transpose  $\mathbb{R}^{n*} \rightarrow \mathfrak{g}^*$  of the inclusion therefore vanishes on  $\mathfrak{t}^*$  and hence induces a map  $\lambda: \mathbb{R} = \mathfrak{h}^* / \mathfrak{t}^* \rightarrow \mathfrak{g}^*$ —the momentum level for the symplectic quotient of  $\mathbb{C}^n$  by  $G$  is then  $\lambda(1)$ .

REMARK 1. – Affine functions  $L_j$  defining a rational Delzant polytope  $\Delta \subset \Xi$  do so with respect to *any* lattice containing the normals  $u_j$ . There is clearly a smallest such lattice  $\Lambda := \text{span}_{\mathbb{Z}}\{u_j : j = 1, \dots, n\}$ , and any other such lattice  $\Lambda'$  contains  $\Lambda$  as a sublattice (of finite index). The torus  $\mathbb{T}' = \mathfrak{t} / 2\pi\Lambda'$  is the quotient of  $\mathbb{T} = \mathfrak{t} / 2\pi\Lambda$  by a finite Abelian group  $\Gamma \cong \Lambda' / \Lambda$ , and the corresponding toric symplectic orbifolds  $M$  and  $M'$  (under the

<sup>(1)</sup> We implicitly assume each  $L_j$  vanishes somewhere on  $\Delta$ , otherwise it may be discarded.

tori  $\mathbb{T}$  and  $\mathbb{T}'$ ) are related by a regular orbifold covering [55]:  $M' = M/\Gamma$ . In fact  $M$  is a *simply connected orbifold* in the sense of W. Thurston [55] and is the universal orbifold cover of  $M'$  [45]. We therefore say that the rational Delzant polytope is *simply connected*. Simply connected rational Delzant polytopes are entirely determined by the affine normals  $\mathbf{L}$ .

## 1.2. Toric Kähler orbifolds

We next consider Kähler metrics compatible with a toric symplectic structure. On the union  $M^0 := \mu^{-1}(\Delta^0)$  of the generic orbits, such metrics have an explicit general expression due to V. Guillemin [35, 36]. In this description, the momentum map  $\mu: M^0 \rightarrow \Xi$  is supplemented by angular coordinates  $\mathbf{t}: M \rightarrow \mathfrak{t}/2\pi\Lambda$  such that the kernel of  $d\mathbf{t}$  is orthogonal to the torus orbits. These action-angle coordinates  $(\mu, \mathbf{t})$  identify each tangent space to  $M^0$  with  $\mathfrak{t} \oplus \mathfrak{t}^*$ , and the symplectic form is  $\omega = \langle d\mu \wedge d\mathbf{t} \rangle$ , where  $\langle \cdot \rangle$  denotes contraction of  $\mathfrak{t}$  and  $\mathfrak{t}^*$ . Hence invariant  $\omega$ -compatible Kähler metrics on  $M^0$  have the form

$$(1) \quad g = \langle d\mu, \mathbf{G}, d\mu \rangle + \langle d\mathbf{t}, \mathbf{H}, d\mathbf{t} \rangle,$$

where  $\mathbf{G}$  is a positive definite  $S^2\mathfrak{t}$ -valued function of  $\mu$ ,  $\mathbf{H}$  is its pointwise inverse in  $S^2\mathfrak{t}^*$  (at each point,  $\mathbf{G}$  and  $\mathbf{H}$  define mutually inverse linear maps  $\mathfrak{t}^* \rightarrow \mathfrak{t}$  and  $\mathfrak{t} \rightarrow \mathfrak{t}^*$ ) and  $\langle \cdot, \cdot, \cdot \rangle$  denotes the pointwise contraction  $\mathfrak{t}^* \times S^2\mathfrak{t} \times \mathfrak{t}^* \rightarrow \mathbb{R}$  or the dual contraction. The corresponding almost complex structure is defined by

$$(2) \quad Jd\mathbf{t} = -\langle \mathbf{G}, d\mu \rangle,$$

and  $J$  is integrable if and only if  $\mathbf{G}$  is the Hessian of a function [35].

Necessary and sufficient conditions for  $\mathbf{H}$  to come from a globally defined metric on  $M$  are obtained in [2, 7, 27]. Here we use the first order boundary conditions given in [7, §1]. In order to state them, we denote by  $\mathfrak{t}_F \subseteq \mathfrak{t}$  (for any face  $F \subseteq \Delta$ ) the vector subspace spanned by the inward normals  $u_j \in \mathfrak{t}$  to facets containing  $F$ . Thus the tangent plane to points in  $F^0$  is the annihilator  $\mathfrak{t}_F^0 \cong (\mathfrak{t}/\mathfrak{t}_F)^*$  of  $\mathfrak{t}_F$  in  $\mathfrak{t}^*$ .

**PROPOSITION 1.** – *Let  $(M, \omega)$  be a compact toric symplectic  $2m$ -manifold or orbifold with natural momentum map  $\mu: M \rightarrow \Delta \subset \Xi \subset \mathfrak{h}^*$ , and  $\mathbf{H}$  be a positive definite  $S^2\mathfrak{t}^*$ -valued function on  $\Delta^0$ . Then  $\mathbf{H}$  defines a  $\mathbb{T}$ -invariant,  $\omega$ -compatible almost Kähler metric  $g$  via (1) if and only if it satisfies the following conditions:*

- [smoothness]  $\mathbf{H}$  is the restriction to  $\Delta^0$  of a smooth  $S^2\mathfrak{t}^*$ -valued function on  $\Delta$ ;
- [boundary values] for any point  $\xi$  on the facet  $F_j \subset \Delta$  with inward normal  $u_j$ ,

$$(3) \quad \mathbf{H}_\xi(u_j, \cdot) = 0 \quad \text{and} \quad (d\mathbf{H})_\xi(u_j, u_j) = 2u_j,$$

where the differential  $d\mathbf{H}$  is viewed as a smooth  $S^2\mathfrak{t}^* \otimes \mathfrak{t}$ -valued function on  $\Delta$ ;

- [positivity] for any point  $\xi$  in interior of a face  $F \subseteq \Delta$ ,  $\mathbf{H}_\xi(\cdot, \cdot)$  is positive definite when viewed as a smooth function with values in  $S^2(\mathfrak{t}/\mathfrak{t}_F)^*$ .



**1.3. The extremal affine function and K-polystability**

Let  $(M, J, g, \omega)$  be a compact Kähler orbifold invariant under the action of a maximal torus  $G$  in the reduced automorphism group  $H_0(M, J)$  of  $(M, J)$ . (By a result of Calabi [16], any extremal Kähler metric is invariant under such a  $G$ .) Following [32], the *extremal potential* is the  $L_2$ -projection of the scalar curvature  $s_g$  onto the space of Killing potentials (with respect to  $\omega$ ) of elements of the Lie algebra  $\mathfrak{g}$  and the *extremal vector field* is its symplectic gradient. A. Futaki and T. Mabuchi [32] show that the extremal vector field is independent of the choice of a  $G$ -invariant Kähler metric within the given Kähler class  $[\omega]$  on  $(M, J)$ . Since, the extremal vector field is central,  $G$  can also be taken to be a maximal compact subgroup. Furthermore, by adopting the symplectic viewpoint [31, 25, 43], the extremal potential becomes a natural deformation invariant of the complex structure, for fixed  $(M, \omega, G)$ .

For toric symplectic orbifolds  $(M, \omega, \mathbb{T})$ , the extremal potential is an element  $\zeta$  of  $\mathfrak{h}$ , called the *extremal affine function* [8, 34, 43], and is defined (in the notation of Section 1.1) by the following vector equation in  $\mathfrak{h}^*$ :

$$\int_{\xi \in \Delta} \langle \xi, \zeta \rangle \xi \, d\lambda = \int_{\xi \in \Delta} s_g(\xi) \xi \, d\lambda = 2 \int_{\xi \in \partial\Delta} \xi \, d\nu$$

where  $d\lambda$  is a (constant) volume form on  $\Xi$ , the  $(m - 1)$ -form  $d\nu$  satisfies  $u_j \wedge d\nu = -d\lambda$  on the facet  $F_j$  of  $\partial\Delta$  with normal  $u_j$ , and  $s_g$  is the scalar curvature of a compatible Kähler metric viewed as a function on  $\Delta$ . The combinatorial boundary integral for the first moment of  $s_g$  is an application of the Abreu formula [1]

$$(4) \quad s_g = -\operatorname{div} \delta \mathbf{H} := - \sum_{r,s} \frac{\partial^2 H_{rs}}{\partial \xi_r \partial \xi_s}$$

for the scalar curvature of the compatible metric defined by  $\mathbf{H}$ , together with the divergence theorem; the latter calculation uses only the boundary conditions of Proposition 1, and not the positive definiteness of  $\mathbf{H}$  on the faces of  $\Delta$ , nor the fact that  $\mathbf{H}$  is the inverse hessian of a symplectic potential. We deduce that if  $\mathbf{H}$  satisfies the boundary conditions and  $-\operatorname{div} \delta \mathbf{H}$  is an affine function, then this is the extremal affine function; such an  $\mathbf{H}$  is called a *formal extremal solution*.

The extremal affine function is important not only as the scalar curvature of a compatible extremal Kähler metric, but also because it may be used to define a *relative Futaki invariant* and hence a combinatorial K-polystability criterion [26, 30, 53, 62].

DEFINITION 2. – The *relative Futaki invariant*  $\mathcal{F}_{\Delta, \mathbf{L}}$  of a compact toric symplectic  $2m$ -orbifold  $(M, \omega, \mathbb{T})$  with rational Delzant polytope  $(\Delta, \mathbf{L})$  is defined by

$$(5) \quad \mathcal{F}_{\Delta, \mathbf{L}}(f) := \int_{\xi \in \partial\Delta} f(\xi) \, d\nu - \frac{1}{2} \int_{\xi \in \Delta} \langle \xi, \zeta \rangle f(\xi) \, d\lambda$$

for any continuous function  $f$  on  $\Delta$ . Note that  $\mathcal{F}_{\Delta, \mathbf{L}}$  vanishes on affine functions  $f$ .

Note that if  $\mathbf{H}$  is a formal extremal solution, we may substitute  $\zeta = -\operatorname{div} \delta \mathbf{H}$  in this formula and integrate by parts to obtain

$$(6) \quad \mathcal{F}_{\Delta, \mathbf{L}}(f) = \frac{1}{2} \int_{\xi \in \Delta} \operatorname{tr}(\mathbf{H} \operatorname{Hess} f) \, d\lambda.$$

Let  $\mathcal{P}\mathcal{L}(\Delta)$  be the space of continuous piecewise-linear (PL) convex functions  $f$  on  $\Delta$  (thus  $f$  is the maximum of a finite collection of affine linear functions). Although (6) involves two derivatives of  $f$ , it may be used in a distributional sense to compute  $\mathcal{F}_{\Delta, \mathbf{L}}(f)$  for  $f \in \mathcal{P}\mathcal{L}(\Delta)$ . In particular (cf. [42]) let  $f$  be a simple convex PL function with crease on the line  $\{\xi \in \Xi : \langle \xi, u_f \rangle = 0\}$  (with  $u_f$  normalized to be the change in  $df$  along the line) and let  $S_f$  be the intersection of this line with  $\Delta$ . Then

$$(7) \quad \mathcal{F}_{\Delta, \mathbf{L}}(f) = \int_{S_f} \mathbf{H}(u_f, u_f) d\nu_f,$$

where  $\nu_f$  is the positive measure on  $S_f$  such that  $u_f \wedge d\nu_f = d\lambda$ .

**DEFINITION 3.** –  $(M, \omega, \mathbb{T})$  is said to be (*analytically, relatively*) *K-polystable (with respect to toric degenerations)* provided that  $\mathcal{F}_{\Delta, \mathbf{L}}(f) \geq 0$  for all  $f \in \mathcal{P}\mathcal{L}(\Delta)$ , with equality iff  $f$  is an affine function.

The main conjecture of [26] is that a compact toric orbifold  $(M, \omega, \mathbb{T})$  admits a compatible extremal Kähler metric if and only if it is K-polystable in this toric sense. The forward implication has been established by Zhou and Zhu [61]. Conversely, in [30], Donaldson shows that for polygons with zero extremal vector field, this toric K-polystability criterion implies existence of a CSC metric. The general extremal case remains open, which motivates its study in the ambitoric context.

## 2. Simplices and quadrilaterals

Rational Delzant polytopes may be considered from a projective viewpoint, not just an affine one. To fix notation, for a real vector space  $V$ , we denote by  $P(V) = V^\times / \mathbb{R}^\times$  the nonzero vectors in  $V$  up to scale:  $P(V)$  is isomorphic to the set of 1-dimensional subspaces of  $V$ , where the equivalence class  $[v] \in P(V)$  of a nonzero vector  $v \in V$  is mapped to  $\langle v \rangle \subseteq V$ , its span.

Given a rational Delzant polytope  $(\Delta, \mathbf{L})$ , where  $\Delta \subset \Xi \subset \mathfrak{h}^*$  and  $\Xi$  is an affine subspace of  $\mathfrak{h}^*$  modeled on  $\mathfrak{t}^* \subseteq \mathfrak{h}^*$ , observe that  $\Xi$  is an affine chart for  $P(\mathfrak{h}^*)$ . Hence it is natural to identify  $\Delta$  with its image in  $P(\mathfrak{h}^*)$ , and also with the convex cone in  $\mathfrak{h}^*$  of its nonnegative multiples. Dually, the space  $\Delta^* \subseteq \mathfrak{h}$  of affine functions which are nonnegative on  $\Delta$  is a convex cone,<sup>(2)</sup> given by the nonnegative linear combinations of the affine normals  $L_1, \dots, L_n$ , which we may identify with its image in  $P(\mathfrak{h})$ .

The incarnations of  $\Delta$  determine one another uniquely, but depend upon the choice of  $\Xi$ , or equivalently, the inclusion  $\iota: \mathbb{R} \rightarrow \mathfrak{h}$  or the *affine structure*  $\iota(1) \in \mathfrak{h}$ ; note that  $\iota(1)$  is in the interior of  $\Delta^*$ .

<sup>(2)</sup> In fact  $\Delta^*$  is a *strictly* convex cone: it contains no nontrivial linear subspace.

**2.1. Rational Delzant simplices**

The case of  $m$ -simplices is well understood, but we summarize it briefly, both as a warm-up, and because we shall use the case of triangles ( $m = 2$ ) as a limiting case of quadrilaterals. All simplices are affine equivalent, so simply connected rational Delzant simplices are parametrized by the choices of scale for the normals. Concretely let  $\Delta \subset \Xi$  be the  $m$ -simplex on which  $\ell_j(\xi) \geq 0$  for affine functions  $\ell_0, \ell_1, \dots, \ell_m$  with  $\ell_0 + \ell_1 + \dots + \ell_m = 1$  on  $\Xi$ , so that each  $\ell_j = 1$  at the vertex  $v_j$  opposite to the facet  $F_j$  on which it vanishes. Then for any  $r_0, r_1, \dots, r_m \in \mathbb{R}^+$  with rational ratios, affine normals  $L_j := \ell_j/r_j$  define a rational Delzant simplex  $(\Delta, \mathbf{L})$  with  $\mathbf{L} = (L_0, L_1, \dots, L_m)$  and  $\sum_{j=0}^m r_j L_j = 1$ .

The corresponding symplectic orbifolds are *weighted projective spaces*: the vector  $(r_0, r_1, \dots, r_m)$  spans the kernel of the map  $d\mathbf{L}: \mathbb{R}^{m+1} \rightarrow \mathfrak{t}$  sending  $(x_0, x_1, \dots, x_m)$  to  $\sum_{j=0}^m x_j u_j$  (where  $u_j = dL_j$ ), and some multiple  $(w_0, w_1, \dots, w_m)$  of  $(r_0, r_1, \dots, r_m)$  is a list of positive integers with no common multiple; the rational Delzant construction therefore yields the weighted projective space  $\mathbb{C}P_{w_0, \dots, w_m}^m$  as the symplectic quotient of  $\mathbb{C}^{m+1}$  by the diagonal action of  $S^1$  with weights  $w_0, \dots, w_m$ . Each weighted projective space has a unique Kähler class (up to scale), and this class contains a unique extremal Kähler metric (up to homothety and biholomorphism); moreover, this extremal Kähler metric is Bochner-flat [14]. In four dimensions, a Kähler metric is Bochner-flat iff it is selfdual ( $W_- = 0$ ) which is equivalent to the existence of many (local) opposite complex structures, although none of these are globally defined on a weighted projective plane.

The toric geometry of weighted projective spaces (and their quotients) has been worked out in detail by M. Abreu [2] in specific coordinates. Here we give an affine invariant derivation, as we shall use similar ideas to simplify computation in the more complicated case of quadrilaterals.

LEMMA 1. – *Let  $\Delta$  be an  $m$ -simplex in  $\Xi$  as above, let  $\lambda$  be a translation invariant measure on  $\Xi$ , and  $A_1, A_2: \Xi \rightarrow \mathbb{R}$  be affine functions whose values on the vertices  $v_0, v_1, \dots, v_m$  of  $\Delta$  are given by  $a_1, a_2 \in \mathbb{R}^{m+1}$ . Then*

$$\int_{\Delta} A_1 A_2 d\lambda = B(a_1, a_2)\lambda(\Delta)$$

where  $B$  is the symmetric bilinear form on  $\mathbb{R}^{m+1}$  with  $B_{jk} = \frac{1+\delta_{jk}}{(m+1)(m+2)}$ .

*Proof.* – Since all  $m$ -simplices are affine equivalent, and any affine function is uniquely determined by its values on the  $m + 1$  vertices of a simplex, the integral must have the given form for some symmetric bilinear form  $B$ . The entries of  $B$  must be permutation invariant, so  $B_{jk} = a + b\delta_{jk}$ . Substituting  $A_1 = A_2 = 1$ , we obtain  $(m + 1)^2 a + (m + 1)b = 1$ . If  $A_1 = A_2 = \ell_0$  (i.e., equal to 1 at  $v_0$  and 0 at  $v_1, \dots, v_m$ ), then we observe that for  $0 \leq x \leq 1$ ,  $\lambda(\{p \in \Delta : \ell_0(p)^2 \geq x\}) = \lambda(\{p \in \Delta : 1 - \ell_0(p) \leq 1 - \sqrt{x}\}) = \lambda((1 - \sqrt{x})\Delta) = (1 - \sqrt{x})^m \lambda(\Delta)$ , and integrating over  $x$ , the integral evaluates to  $2\lambda(\Delta)/(m + 1)(m + 2)$ . Thus  $(m + 1)(m + 2)(a + b) = 2$  and  $a = b = 1/(m + 1)(m + 2)$ .  $\square$

The measure  $\nu$ , with  $u_j \wedge d\nu = -d\lambda$  on the facet  $F_j$  where  $\ell_j = 0$ , satisfies  $\nu(F_j) = m r_j \lambda(\Delta)$ , since  $L_j = 1/r_j$  at the opposite vertex  $v_j$ . Consequently, for any

affine function  $A$ ,

$$\int_{F_j} A \, d\nu = r_j \lambda(\Delta) \sum_{k \neq j} A(v_k).$$

Since  $(B^{-1})_{jk} = (m+1)((m+2)\delta_{jk} - 1)$ , the extremal affine function  $\zeta$  of  $(\Delta, \mathbf{L})$  is  $\sum_{j=0}^m \zeta_j r_j$  where

$$\zeta_j = \sum_{k=0}^m \frac{1}{2}(m+1)(2 - (m+2)\delta_{jk})\ell_k.$$

Note that  $\zeta$  is linear in the parameters  $r_j$ : this is the reason for using such an inverse scale to parametrize the normals. For  $m=2$ ,  $\zeta/3 = (-\ell_0 + \ell_1 + \ell_2)r_0 + (\ell_0 - \ell_1 + \ell_2)r_1 + (\ell_0 + \ell_1 - \ell_2)r_2$ , which is positive on the interior of the medial triangle (with vertices at the midpoints of the edges of  $\Delta$ ). This positivity has an analogue for convex quadrilaterals, to which we now turn.

## 2.2. Rational Delzant quadrilaterals

Quadrilaterals are not all affine equivalent, but they are projectively equivalent since the vertices (or the projective normals) give four points in general position in  $\mathbb{P}(\mathfrak{h}^*)$  (or  $\mathbb{P}(\mathfrak{h})$ ). Consequently, quadrilaterals can be parametrized conveniently by varying the affine structure, an approach adopted by E. Legendre in [42, 41] and closely related to 5-dimensional toric Sasakian geometry (see Appendix C). Following Legendre, let  $\Delta = \{[w, x, y] \in \mathbb{P}(\mathbb{R}^{3*}) : w \geq |x|, w \geq |y|\}$  be the quadrilateral with vertices  $[1, \pm 1, \pm 1]$ . In the affine subspace  $\{(w, x, y) \in \mathbb{R}^{3*} : w = 1\}$  defined by  $(1, 0, 0) \in \mathbb{R}^3$ ,  $\Delta$  is a square. More generally, affine subspaces meeting  $\Delta$  in a compact convex quadrilateral are parametrized by vectors in the interior of the dual cone  $\Delta^*$ , spanned by  $(1, \pm 1, 0)$  and  $(1, 0, \pm 1)$ . Any such vector is a positive multiple of  $(1, \frac{1}{2}(\varepsilon + \eta), \frac{1}{2}(\varepsilon - \eta))$  for some  $\varepsilon, \eta \in \mathbb{R}$  with  $|\varepsilon| < 1$  and  $|\eta| < 1$ . The corresponding affine subspace is  $\{(w, x, y) : 2w + (\varepsilon + \eta)x + (\varepsilon - \eta)y = 2\}$  and the vertices of  $\Delta$  in this subspace are

$$v_{00} = \frac{(1, -1, -1)}{1 - \varepsilon}, \quad v_{0\infty} = \frac{(1, -1, 1)}{1 - \eta}, \quad v_{\infty 0} = \frac{(1, 1, -1)}{1 + \eta}, \quad v_{\infty\infty} = \frac{(1, 1, 1)}{1 + \varepsilon}$$

with  $v_{00}$  opposite to  $v_{\infty\infty}$  and  $v_{0\infty}$  opposite to  $v_{\infty 0}$ . An affine function  $A$  is uniquely determined by its values at the vertices, but these values are constrained by the equality of two expressions for (twice) the value of  $A$  at the intersection of the diagonals:

$$(8) \quad (1 - \varepsilon)A(v_{00}) + (1 + \varepsilon)A(v_{\infty\infty}) = (1 - \eta)A(v_{0\infty}) + (1 + \eta)A(v_{\infty 0}).$$

The affine functions obtained by restricting  $w + x$ ,  $w - x$ ,  $w + y$  and  $w - y$  to this affine subspace will be denoted  $\ell'_{\alpha,0}$ ,  $\ell'_{\alpha,\infty}$ ,  $\ell'_{\beta,0}$  and  $\ell'_{\beta,\infty}$  respectively. They clearly satisfy  $\ell'_{\alpha,0} + \ell'_{\alpha,\infty} = \ell'_{\beta,0} + \ell'_{\beta,\infty}$ . We also set

$$\begin{aligned} \ell_{\alpha,0} &= \frac{1}{4}(1 + \varepsilon)(1 + \eta)\ell'_{\alpha,0}, & \ell_{\alpha,\infty} &= \frac{1}{4}(1 - \varepsilon)(1 - \eta)\ell'_{\alpha,\infty}, \\ \ell_{\beta,0} &= \frac{1}{4}(1 + \varepsilon)(1 - \eta)\ell'_{\beta,0}, & \ell_{\beta,\infty} &= \frac{1}{4}(1 - \varepsilon)(1 + \eta)\ell'_{\beta,\infty}, \end{aligned}$$

which satisfy  $\ell_{\alpha,0} + \ell_{\alpha,\infty} + \ell_{\beta,0} + \ell_{\beta,\infty} = 1$ , and whose nonzero values on vertices are

$$\begin{aligned} \ell_{\alpha,0}(v_{\infty 0}) &= \ell_{\beta,0}(v_{0\infty}) = \frac{1}{2}(1 + \varepsilon), & \ell_{\alpha,0}(v_{\infty\infty}) &= \ell_{\beta,\infty}(v_{00}) = \frac{1}{2}(1 + \eta) \\ \ell_{\alpha,\infty}(v_{00}) &= \ell_{\beta,0}(v_{\infty\infty}) = \frac{1}{2}(1 - \eta), & \ell_{\alpha,\infty}(v_{0\infty}) &= \ell_{\beta,\infty}(v_{\infty 0}) = \frac{1}{2}(1 - \varepsilon). \end{aligned}$$

These affine functions provide an affine invariant description of a family of quadrilaterals  $\Delta_{\varepsilon,\eta}$  with  $(\varepsilon, \eta) \in (-1, 1) \times (-1, 1)$ , and we can drop the  $(w, x, y)$  coordinate system. The parameters  $\varepsilon, \eta$  can be interpreted geometrically in terms of the diagonals, which bisect each other in the ratios  $1 - \varepsilon : 1 + \varepsilon$  and  $1 - \eta : 1 + \eta$ . The rescaled affine normals  $L_{\alpha,k} := \ell_{\alpha,k}/r_{\alpha,k}$  and  $L_{\beta,k} := \ell_{\beta,k}/r_{\beta,k}$ , where  $r_{\alpha,k}, r_{\beta,k}$  ( $k \in \{0, \infty\}$ ) are positive real numbers, define a rational Delzant quadrilateral  $(\Delta, \mathbf{L})$  in  $P(\mathfrak{h}^*)$  (with an ordering of its vertices and the affine normals  $\mathbf{L}$  indexed  $L_{\beta,0}, L_{\alpha,0}, L_{\beta,\infty}, L_{\alpha,\infty}$ ) provided that the normals  $u_{\alpha,k} = dL_{\alpha,k}$  and  $u_{\beta,k} = dL_{\beta,k}$  span a lattice in  $\mathfrak{t}$ .

The following diagram shows the projection of the quadrilateral onto the  $(x, y)$  plane. This normal form is orthodiagonal and so its Varignon parallelogram (whose vertices are the midpoints of the sides of the quadrilateral) is a rectangle. Also shown are the midpoints  $v_\varepsilon, v_\eta$  of the diagonals and the centroid  $v_0$ .

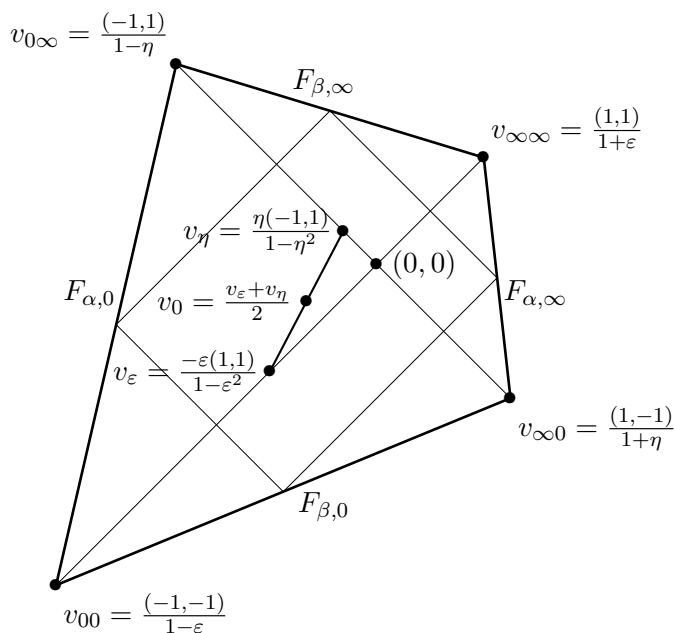


FIGURE 1. A rational Delzant quadrilateral with its diagonals and Newton line.

When  $\varepsilon = \eta = 0$ ,  $\Delta$  is a parallelogram (these are all affine equivalent); the associated simply connected symplectic 4-orbifolds are products of weighted projective lines (including  $\mathbb{C}P^1 \times \mathbb{C}P^1$  when  $r_{\alpha,0} = r_{\alpha,\infty}$  and  $r_{\beta,0} = r_{\beta,\infty}$ ). If  $\eta = \pm\varepsilon$ ,  $\Delta$  is a trapezium<sup>(3)</sup>; the associated simply connected symplectic 4-orbifolds are orbifold weighted projective line bundles over a weighted projective line (which include the smooth Hirzebruch surfaces  $P(\theta \oplus \theta(k)) \rightarrow \mathbb{C}P^1$ ).

The extremal affine function  $\zeta$  may be written  $\zeta = c(\varepsilon, \eta) \sum_{k=0,\infty} (\zeta_{\alpha,k} r_{\alpha,k} + \zeta_{\beta,k} r_{\beta,k})$  where the normalization constant  $c(\varepsilon, \eta)$  will be chosen shortly. By symmetry, it suffices to

<sup>(3)</sup> I.e., a quadrilateral with at least two parallel sides, also known as a trapezoid.

compute  $\zeta_{\alpha,0}$ , noting that its integral over  $\Delta$  against any affine function  $A$  depends only on the value of  $A$  at the midpoint of the edge  $v_{00}v_{0\infty}$ .

LEMMA 2. –  $\zeta_{\alpha,0}$  satisfies the following equations:

$$(9) \quad \zeta_{\alpha,0}(v_{0\infty}) + \zeta_{\alpha,0}(v_{\infty 0}) + (1 - \varepsilon)\zeta_{\alpha,0}(v_{\infty\infty}) = 0$$

$$(10) \quad \zeta_{\alpha,0}(v_{00}) + \zeta_{\alpha,0}(v_{\infty\infty}) + (1 - \eta)\zeta_{\alpha,0}(v_{\infty 0}) = 0.$$

*Proof.* – Let  $A_\varepsilon$  be an affine function which is constant on the diagonal  $v_{\infty 0}v_{0\infty}$  and vanishes at the midpoint of  $v_{00}v_{0\infty}$ . Up to scale, we may take  $A_\varepsilon(v_{\infty 0}) = A_\varepsilon(v_{0\infty}) = 1 + \varepsilon$ ,  $A_\varepsilon(v_{00}) = -(1 + \varepsilon)$ , and  $A_\varepsilon(v_{\infty\infty}) = 3 - \varepsilon$  (which verifies (8)). The integral of  $\zeta_{\alpha,0}A_\varepsilon$  over  $\Delta$  (which vanishes) may be computed using Lemma 1 by splitting  $\Delta$  into two triangles along the diagonal  $v_{\infty 0}v_{0\infty}$ . Up to a positive constant, the result is

$$(1 - \varepsilon)(3 + \varepsilon)(\zeta_{\alpha,0}(v_{0\infty}) + \zeta_{\alpha,0}(v_{\infty 0})) + 4(1 - \varepsilon)\zeta_{\alpha,0}(v_{\infty\infty}) + (1 + \varepsilon)^2\zeta_{\alpha,0}(v_{00})$$

which readily yields (9); (10) follows similarly, using the diagonal  $v_{00}v_{\infty\infty}$ .  $\square$

It follows that  $\zeta_{\alpha,0}$  is a constant multiple of the affine function

$$\begin{aligned} (2 + (1 - \varepsilon)(1 - \eta))(\ell_{\alpha,\infty} - \ell_{\alpha,0}) + (2 - (1 - \varepsilon)(1 - \eta))(\ell_{\beta,0} + \ell_{\beta,\infty}) \\ = 2(1 - 2\ell_{\alpha,0}) + (1 - \varepsilon)(1 - \eta)(2\ell_{\alpha,\infty} - 1). \end{aligned}$$

With some further work we can compute the constant: if we set  $c(\varepsilon, \eta) = 1$ , it equals  $24/(4 - (1 - \varepsilon^2)(1 - \eta^2))$ . We can instead take this constant as the definition of  $c(\varepsilon, \eta)$  (by symmetry, it is the same for all four components of  $\zeta$ ). For  $\zeta_{\alpha,0}$ , we then have

$$\begin{aligned} \zeta_{\alpha,0}(v_{00}) &= 2 - \eta(1 - \varepsilon)(1 - \eta) & \zeta_{\alpha,0}(v_{\infty\infty}) &= -2 + (1 + \varepsilon)(1 - \eta) \\ \zeta_{\alpha,0}(v_{0\infty}) &= 2 - \varepsilon(1 - \varepsilon)(1 - \eta) & \zeta_{\alpha,0}(v_{\infty 0}) &= -2 + (1 - \varepsilon)(1 + \eta) \\ 2\zeta_{\alpha,0}(v_\varepsilon) &= (1 - \eta)(2 - (1 - \varepsilon)(1 + \eta)) & 2\zeta_{\alpha,0}(v_\eta) &= (1 - \varepsilon)(2 - (1 + \varepsilon)(1 - \eta)) \end{aligned}$$

$$(11) \quad 4\zeta_{\alpha,0}(v_0) = (1 + \varepsilon^2)(1 - \eta) + (1 + \eta^2)(1 - \varepsilon)$$

where  $v_\varepsilon = \frac{1}{2}(v_{00} + v_{\infty\infty})$  and  $v_\eta = \frac{1}{2}(v_{0\infty} + v_{\infty 0})$  are the midpoints of the diagonals, and  $v_0 = \frac{1}{2}(v_\varepsilon + v_\eta)$  is the centroid of  $\Delta$ .

LEMMA 3. – *The extremal affine function  $\zeta$  is positive at the centroid of  $\Delta$ , and is also positive at the midpoints of the diagonals if  $(1 + |\varepsilon|)(1 + |\eta|) < 2$ . In general, the value of  $\zeta$  at the midpoint of a diagonal is a positive multiple of the Futaki invariant of a simple PL convex function with crease along that diagonal.*

*Proof.* – By (11) and analogous formulae for the other components,  $\zeta$  is positive at the centroid  $v_0$ , since  $|\varepsilon|, |\eta| < 1$ . Similarly, it is positive at the midpoints of the diagonals for  $(1 + |\varepsilon|)(1 + |\eta|) < 2$ . We now compute the Futaki invariant of a simple PL function  $H$  with a crease along the diagonal  $v_{00}v_{\infty\infty}$  through  $v_\varepsilon$ . By symmetry, it suffices to compute the  $r_{\alpha,0}$  component  $\zeta_{\alpha,0}$ . Modulo an affine function (on which the Futaki invariant vanishes), we may assume  $f$  vanishes on  $F_{\alpha,0}$ , so that we only need to compute the integral of  $-f\zeta_{\alpha,0}$  over the triangle  $T = v_{00}v_{\infty\infty}v_{\infty 0}$ , on which we may suppose  $f = \ell'_{\alpha,0} - \ell'_{\beta,0}$ , i.e.,  $f(v_{\infty 0}) = 2/(1 + \eta)$ . By Lemma 1, the integral evaluates to a universal constant positive

multiple of  $-\lambda(T)f(v_{\infty 0})(\zeta_{\alpha,0}(v_{00}) + \zeta_{\alpha,0}(v_{\infty\infty}) + 2\zeta_{\alpha,0}(v_{\infty 0}))$ . By Lemma 2, this is a universal constant positive multiple of

$$-\frac{(\zeta_{\alpha,0}(v_{00}) + \zeta_{\alpha,0}(v_{\infty\infty}))(1 - 2/(1 - \eta))}{(1 + \varepsilon)(1 - \varepsilon)(1 + \eta)^2} = \frac{2\zeta_{\alpha,0}(v_\varepsilon)}{(1 + \varepsilon)(1 - \varepsilon)(1 + \eta)(1 - \eta)}.$$

Hence the Futaki invariant is a positive multiple of  $\zeta(v_\varepsilon)$ , as required. The argument for the other diagonal/midpoint is similar.  $\square$

We conclude that if a rational Delzant quadrilateral is K-polystable, then the extremal affine function must be positive at the diagonal midpoints.

**DEFINITION 4.** – Let  $(\Delta, \mathbf{L})$  be a rational Delzant quadrilateral with vertices  $v_{00}, v_{\infty\infty}, v_{0\infty}, v_{\infty 0}$  so that  $v_{00}$  and  $v_{\infty\infty}$  are opposite. Then  $\Delta$  is said to be *equipoised* if  $\zeta(v_{00}) + \zeta(v_{\infty\infty}) = \zeta(v_{0\infty}) + \zeta(v_{\infty 0})$ , and *temperate* if  $\zeta(v_{00}) + \zeta(v_{\infty\infty})$  and  $\zeta(v_{0\infty}) + \zeta(v_{\infty 0})$  are both positive.

The condition to be equipoised was introduced in [42], and means equivalently that the extremal affine function has equal values on the midpoints  $v_\varepsilon$  and  $v_\eta$  of the diagonals of  $\Delta$ . This is automatic if  $\Delta$  is a parallelogram (when  $v_\varepsilon = v_\eta$ )—otherwise it means that the extremal affine function is constant (hence positive by Lemma 3) on the *Newton line*  $v_\varepsilon v_\eta$ . The condition to be temperate is the much weaker condition that  $\zeta$  is positive on the line segment between  $v_\varepsilon$  and  $v_\eta$ .

### 3. Orbifold compactifications of ambitoric structures

#### 3.1. Ambitoric structures and their local classification

In [6], we studied the following 4-dimensional geometric structure.

**DEFINITION 5.** – An *ambikähler structure* on a real 4-manifold or orbifold  $M$  consists of a pair of Kähler metrics  $(g_-, J_-, \omega_-)$  and  $(g_+, J_+, \omega_+)$  such that

- $g_-$  and  $g_+$  induce the same conformal structure (i.e.,  $g_- = f^2 g_+$  for a positive function  $f$  on  $M$ );
- $J_-$  and  $J_+$  have opposite orientations (equivalently the volume elements  $\frac{1}{2}\omega_- \wedge \omega_-$  and  $\frac{1}{2}\omega_+ \wedge \omega_+$  on  $M$  have opposite signs).

The structure is said to be *ambitoric* if in addition

- there is a 2-dimensional subspace  $\mathfrak{t}$  of vector fields on  $M$ , linearly independent on a dense open set, whose elements are Hamiltonian and Poisson-commuting Killing vector fields with respect to both  $(g_-, \omega_-)$  and  $(g_+, \omega_+)$ .

Thus  $M$  has a pair of conformally equivalent but oppositely oriented Kähler metrics, invariant under a local 2-torus action, and both locally toric with respect to that action.

**EXAMPLES.** – There are three classes of examples of ambitoric structures.

(i) *Toric Kähler products.* Let  $(\Sigma_1, g_1, J_1, \omega_1)$  and  $(\Sigma_2, g_2, J_2, \omega_2)$  be (locally) toric 2-dimensional Kähler manifolds or orbifolds, with Hamiltonian Killing vector fields  $K_1$  and  $K_2$ . Then  $M = \Sigma_1 \times \Sigma_2$  is ambitoric, with  $g_\pm = g_1 \oplus g_2$ ,  $J_\pm = J_1 \oplus (\pm J_2)$ ,  $\omega_\pm = \omega_1 \oplus (\pm \omega_2)$  and  $\mathfrak{t}$  spanned by  $K_1$  and  $K_2$ .

(ii) *Toric Calabi geometries.* Let  $(\Sigma, g, J, \omega)$  be a 2-dimensional Kähler manifold or orbifold with Hamiltonian Killing vector field  $K$ , let  $\pi: P \rightarrow \Sigma$  be a circle bundle with connection  $\theta$  and curvature  $d\theta = \pi^*\omega$ , and let  $A$  be a positive function on an open subset  $U$  of  $\mathbb{R}^+$ . Then  $M = P \times U$  is ambitoric, with

$$g_{\pm} = z^{\pm 1} \left( g + \frac{(z^{-1} dz)^2}{A(z)} + A(z)\theta^2 \right),$$

$$\omega_{\pm} = z^{\pm 1} (\omega \pm z^{-1} dz \wedge \theta), \quad J_{\pm}(z^{-1} dz) = \pm A(z)\theta,$$

and the local torus action spanned by  $K$  and the generator of the circle action. Here  $z: M \rightarrow \mathbb{R}^+$  is the projection onto  $U \subseteq \mathbb{R}^+$ .

(iii) *Regular ambitoric structures.* Let  $q(z) = q_0 z^2 + 2q_1 z + q_2$  be a quadratic polynomial and let  $M$  be a 4-dimensional manifold or orbifold with real-valued functions  $(x, y, \tau_0, \tau_1, \tau_2)$  such that  $x > y$ ,  $2q_1 \tau_1 = q_0 \tau_2 + q_2 \tau_0$ , and their exterior derivatives span each cotangent space. Let  $\mathfrak{t}$  be the 2-dimensional space of vector fields  $K$  on  $M$  with  $dx(K) = 0 = dy(K)$  and  $d\tau_j(K)$  constant, and let  $A$  and  $B$  be positive functions on open neighborhoods of the images of  $x$  and  $y$  in  $\mathbb{R}$ , on whose product  $q(x, y) := q_0 xy + q_1(x + y) + q_2$  is positive. Then  $M$  is ambitoric with

$$(12) \quad g_{\pm} = \left( \frac{x-y}{q(x,y)} \right)^{\pm 1} \left( \frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + A(x) \left( \frac{y^2 d\tau_0 + 2y d\tau_1 + d\tau_2}{(x-y)q(x,y)} \right)^2 + B(y) \left( \frac{x^2 d\tau_0 + 2x d\tau_1 + d\tau_2}{(x-y)q(x,y)} \right)^2 \right),$$

$$(13) \quad \omega_{\pm} = \left( \frac{x-y}{q(x,y)} \right)^{\pm 1} \frac{dx \wedge (y^2 d\tau_0 + 2y d\tau_1 + d\tau_2) \pm dy \wedge (x^2 d\tau_0 + 2x d\tau_1 + d\tau_2)}{(x-y)q(x,y)},$$

$$(14) \quad J_{\pm} dx = A(x) \frac{y^2 d\tau_0 + 2y d\tau_1 + d\tau_2}{(x-y)q(x,y)}, \quad J_{\pm} dy = \pm B(y) \frac{x^2 d\tau_0 + 2x d\tau_1 + d\tau_2}{(x-y)q(x,y)}.$$

If  $(g_{\pm}, J_{\pm}, \omega_{\pm}, \mathfrak{t})$  is a regular ambitoric structure for which the quadratic form  $q$  has vanishing discriminant, then  $(g_+, J_+, \omega_+)$  is *orthotoric* in the sense of [4]: see [6].

It is easy to check that these explicit geometries are ambitoric (cf. [4, 6]).

**THEOREM ([6]).** – *Let  $(M, g_{\pm}, J_{\pm}, \omega_{\pm}, \mathfrak{t})$  be an ambitoric 4-manifold or orbifold. Then any point in an open dense subset of  $M$  has a neighborhood on which  $(g_{\pm}, J_{\pm}, \omega_{\pm}, \mathfrak{t})$  is either a toric Kähler product, a toric Calabi geometry, or a regular ambitoric structure.*

### 3.2. Invariant geometry in momentum coordinates

In order to compactify ambitoric structures using the approach of Section 1.2, we need to describe the metrics in momentum coordinates. For toric Kähler products, toric Calabi geometries and orthotoric metrics, this has been done systematically by E. Legendre [42] in her resolution of existence problem for extremal metrics over *equipoised* rational Delzant quadrilaterals. Hence we concentrate on the general regular ambitoric case.

We shall make essential use of the underlying geometry of regular ambitoric structures. For this we recall (from [6]) that there is a natural  $\mathrm{PSL}(2, \mathbb{R})$  gauge freedom in regular ambitoric structures, which may be described in an invariant geometric way by viewing the codomain of the  $x$  and  $y$  coordinates as a projective line  $P(W)$ , where  $W$  is a 2-dimensional



real vector space on which we fix a (nonzero) area form  $\kappa$ . Note that the tensor product  $\mathcal{O}(1) \otimes W$  of the dual tautological line bundle  $\mathcal{O}(1)$  and the trivial bundle  $P(W) \times W$  has a tautological section which we use to embed  $P(W)$  into the total space. We use bold font for the maps  $\mathbf{x}, \mathbf{y}: M \rightarrow P(W) \subseteq \mathcal{O}(1) \otimes W$  and  $x, y: M \rightarrow \mathbb{R}$  for their expression in an affine coordinate on  $P(W)$ ; in the affine trivialization of  $\mathcal{O}(1)$ ,  $\mathbf{x}, \mathbf{y}$  are homogeneous coordinates corresponding to the inhomogeneous coordinates  $x, y$ .

The quadratic form  $q$  in (12)–(14) is naturally an element of  $S^2W^*$  (i.e., an algebraic section of  $\mathcal{O}(2) \rightarrow P(W)$ ) and the Lie algebra  $\mathfrak{t}$  of the torus is the subspace  $S^2_{0,q}W^*$  orthogonal to  $q$  with respect to the inner product  $\kappa \otimes \kappa$  on  $W^* \otimes W^*$  (which restricts to the polarization of the discriminant on  $S^2W^*$ ). Note that  $S^2W^* \cong \mathfrak{sl}(W)$  has a Lie algebra structure (the Poisson bracket, or Wronskian) and the Poisson bracket with  $q$  induces an isomorphism  $\text{ad}_q: S^2W^*/\langle q \rangle \rightarrow S^2_{0,q}W^*$ . Following [6], we distinguish these isomorphisms of  $\mathfrak{t}$  by writing  $\boldsymbol{\tau} = \text{ad}_q(\mathbf{t})$  where  $\boldsymbol{\tau}$  and  $\mathbf{t}$  take values in  $S^2_{0,q}W^*$  and  $S^2W^*/\langle q \rangle$  respectively (modulo corresponding lattices). For  $z \in W$ , we denote by  $\mathbf{z}^\flat = \kappa(z, \cdot)$  the corresponding element of  $W^*$ , using  $\sharp$  for the inverse isomorphism.

DEFINITION 6. – A regular ambitoric structure is said to be of *elliptic, parabolic* or *hyperbolic* type if  $q$  has (respectively) zero, one or two distinct real roots (on  $P(W)$ ).

The spaces  $\mathfrak{h}_\pm$  of Hamiltonian generators of the torus action with respect to  $\omega_\pm$  are readily computed using (13) (see [6, (24)–(25)]); these yield the affine structures  $\iota_\pm(1) \in \mathfrak{h}_\pm$  and the natural momentum maps  $\mu^\pm: M \rightarrow \mathfrak{h}_\pm^*$  as functions of  $\mathbf{x}$  and  $\mathbf{y}$ .

Negative structures. – We identify  $\mathfrak{h}_-$  with  $S^2_{0,q}W^* \oplus \wedge^2W^* \subseteq W^* \otimes W^*$  and

$$(15) \quad \mu^-(\mathbf{x}, \mathbf{y}) = -\frac{\mathbf{x} \otimes \mathbf{y} \pmod{q^\sharp}}{\kappa(\mathbf{x}, \mathbf{y})} \quad \text{in} \quad \mathfrak{h}_-^* \cong W \otimes W / \langle q^\sharp \rangle,$$

where  $q^\sharp \in S^2W$  is dual to  $q$  using  $\kappa$  (i.e.,  $q^\sharp(\mathbf{z}^\flat) = q(\mathbf{z})$ ). Modulo a sign convention, the affine structure is  $\kappa$ , with  $\langle \mu^-, \kappa \rangle = 1$ . For  $\gamma \in W$  (or in  $P(W) \subseteq \mathcal{O}(1) \otimes W$ ), we define  $\lambda^{(\gamma)} = \gamma^\flat \otimes q(\gamma, \cdot)$  and  $\rho^{(\gamma)} = q(\gamma, \cdot) \otimes \gamma^\flat$ , which are decomposables in  $W^* \otimes W^*$  orthogonal to  $q$ , and hence in  $\mathfrak{h}_-$ . The contractions of  $\lambda^{(\gamma)}$  and  $\rho^{(\gamma)}$  with  $\mu^-(\mathbf{x}, \mathbf{y})$  vanish when  $\mathbf{x} = \gamma$  or  $\mathbf{y} = \gamma$  respectively. In affine coordinates on  $P(W)$ ,

$$\begin{aligned} \lambda^{(\gamma)}(x, y) &= (x - \gamma)q(\gamma, y), & \rho^{(\gamma)}(x, y) &= q(x, \gamma)(y - \gamma), \\ \langle \mu^-(x, y), \lambda^{(\gamma)} \rangle &= -\frac{(x - \gamma)q(y, \gamma)}{x - y}, & \langle \mu^-(x, y), \rho^{(\gamma)} \rangle &= -\frac{(y - \gamma)q(x, \gamma)}{x - y}. \end{aligned}$$

Hence  $\lambda^{(\gamma)}$  and  $\rho^{(\gamma)}$  are dual (i.e., normal) to level surfaces of  $x$  and  $y$  respectively.

Positive structures. – Here we have  $\mathfrak{h}_+ \cong S^2W^*$  and

$$(16) \quad \mu^+(\mathbf{x}, \mathbf{y}) = -\frac{\mathbf{x} \odot \mathbf{y}}{q(\mathbf{x}, \mathbf{y})} \quad \text{in} \quad \mathfrak{h}_+^* \cong W \otimes W / \wedge^2W \cong S^2W.$$

The affine structure is  $q$ , with  $\langle \mu^+, q \rangle = 1$ . Decomposables now all have the form  $\sigma^{(\gamma)} = \gamma^\flat \otimes \gamma^\flat \in \mathfrak{h}_+ = S^2W^* \subseteq W^* \otimes W^*$ , for some  $\gamma \in W$  (or  $P(W) \subseteq \mathcal{O}(1) \otimes W$ ). The

contraction of  $\sigma^{(\gamma)}$  with  $\mu^+(x, y)$  vanishes when  $x = \gamma$  or  $y = \gamma$ . In affine coordinates on  $P(W)$ ,  $\sigma^{(\gamma)}$  polarizes the quadratic  $\sigma^{(\gamma)}(z) = (z - \gamma)^2$ , with

$$\sigma^{(\gamma)}(x, y) = (x - \gamma)(y - \gamma), \quad \langle \mu^+(x, y), \sigma^{(\gamma)} \rangle = -\frac{(x - \gamma)(y - \gamma)}{q(x, y)}.$$

Hence  $\sigma^{(\gamma)}$  are dual (i.e., normal) to level surfaces of  $x$  and  $y$ .

The constants in  $\mathfrak{h}_\pm$  are elements of  $\langle \kappa \rangle = \wedge^2 W^*$  and  $\langle q \rangle \subseteq S^2 W^*$  respectively, and the map  $w \mapsto p = \text{ad}_q(w): S^2 W^* \rightarrow S_{0,q}^2 W^*$  sends a Killing potential with respect to  $\omega_+$  to a Killing potential for the same vector field with respect to  $\omega_-$ . We denote by  $K^{(p)}$  the corresponding vector field on  $M$ .

It is now straightforward to compute the torus metrics  $\mathbf{H}^\pm$  of  $g_\pm$ :

$$(17) \quad \mathbf{H}_{\mu^-(x,y)}^-(p, \tilde{p}) = g_-(K^{(p)}, K^{(\tilde{p})}) = \frac{A(x)p(y)\tilde{p}(y) + B(y)p(x)\tilde{p}(x)}{(x-y)^3 q(x,y)},$$

$$(18) \quad \mathbf{H}_{\mu^+(x,y)}^+(p, \tilde{p}) = g_+(K^{(p)}, K^{(\tilde{p})}) = \frac{A(x)p(y)\tilde{p}(y) + B(y)p(x)\tilde{p}(x)}{(x-y)q(x,y)^3},$$

where  $p, \tilde{p}$  in  $S_{0,q}^2 W^* \cong \mathfrak{t}$ . Up to a constant multiple (depending on a choice of basis for  $\mathfrak{t}^*$ ), we have

$$(19) \quad \det \mathbf{H}_{\mu^-(x,y)}^- = \frac{A(x)B(y)}{(x-y)^4}, \quad \det \mathbf{H}_{\mu^+(x,y)}^+ = \frac{A(x)B(y)}{q(x,y)^4}.$$

### 3.3. Orbifold compactifications of ambitoric Kähler surfaces

The existence of an ambitoric structure on a compact 4-orbifold  $M$  places strong (and well-known) constraints on the topology of  $M$ .

**PROPOSITION 2.** – *Let  $M$  be a compact connected 4-orbifold with an effective action of a 2-torus  $\mathbb{T}$ , and suppose that  $(g_\pm, J_\pm, \omega_\pm)$  is an ambitoric structure on  $M$  with respect to the derivative  $\mathfrak{t} \hookrightarrow C^\infty(M, TM)$  of the  $\mathbb{T}$  action. Then the images of the momentum maps of the  $\mathbb{T}$  action (with respect to  $\omega_+$  and  $\omega_-$ ) are quadrilaterals (i.e.,  $b_2(M) = 2$ ). In particular, if  $M$  is smooth, then for some  $k \in \mathbb{N}$ ,  $(M, J_+)$  and  $(M, J_-)$  are biholomorphic to a Hirzebruch surface  $P(\mathcal{O} \oplus \mathcal{O}(k)) \rightarrow \mathbb{C}P^1$ .*

*Proof.* – If  $M$  is a compact Kähler surface admitting a holomorphic Hamiltonian action of a 2-torus  $\mathbb{T}$  and  $M^0$  is the union of the generic  $\mathbb{T}$ -orbits, then the anticanonical bundle has a holomorphic section with zeroset  $M \setminus M^0$ . The canonical bundle therefore has no holomorphic sections, and so  $h^{2,0} = h^{0,2} = 0$ . In the ambitoric case, the deRham cohomology group  $H^2(M) = H^{1,1}(M)$  with respect to both  $J_+$  and  $J_-$ , hence is represented by a constant linear combination of the harmonic forms  $\omega_+$  and  $\omega_-$  (since  $g_+$  and  $g_-$  are conformally equivalent). It follows that  $M$  has second Betti number  $b_2(M) = 2$  ( $b_+(M) = b_-(M) = 1$ ). Standard results about compact simply connected 4-orbifolds with 2-torus actions (e.g., [12]) then imply that the rational Delzant polygons have  $b_2(M) + 2 = 4$  sides.  $\square$

REMARK 2. – Conversely, it is well-known that any Hirzebruch surface  $M$  admits compatible ambitoric structures: indeed it admits toric *extremal* metrics of Calabi type in each Kähler class [16], where the base metric  $g_\Sigma$  is the Fubini-Study metric on  $\mathbb{C}P^1$ —such metrics are ambitoric by [6, Proposition 9]. Furthermore, any ambikähler structure  $(g_\pm, \omega_\pm)$  on  $M$  with  $g_+$  or  $g_-$  extremal is of this type: if  $g_+$  is extremal, then by uniqueness for extremal Kähler metrics in their Kähler class [20], we may assume  $g_+$  is of Calabi type, hence ambitoric with respect to a negative complex structure  $\tilde{J}_-$ . However,  $g_+$  cannot have selfdual Weyl tensor, so  $J_-$  must equal  $\pm\tilde{J}_-$ .

Note that we have assumed above that both Kähler metrics  $(g_\pm, J_\pm, \omega_\pm)$  are globally defined on  $M$ , not just at points in generic  $\mathbb{T}$ -orbits. In the following we shall weaken this assumption slightly.

DEFINITION 7. – An *ambitoric compactification* is a compact connected oriented 4-orbifold  $M$  with an effective action of a 2-torus  $\mathbb{T}$  such that on the (dense) union  $M^0$  of the free  $\mathbb{T}$ -orbits, there is an ambitoric structure  $(g_\pm, J_\pm, \omega_\pm, \mathfrak{t})$  (with  $\mathfrak{t}$  the Lie algebra of  $\mathbb{T}$ ) for which at least one of the Kähler metrics extends smoothly to a toric Kähler metric on  $(M, \mathbb{T})$ . An ambitoric compactification is *regular* if the ambitoric structure on  $M^0$  is regular with  $(x, y)$ -coordinates that are globally defined on  $M$ .

Henceforth, we consider only regular ambitoric compactifications (without loss of generality if we are interested in extremal ambitoric metrics, as Theorem 1 below will show). We say the ambitoric compactification  $M$  is positive and/or negative if  $g_+$  and/or  $g_-$  extends smoothly to  $M$ . ( $M$  can be both positive and negative.)

By Proposition 1, if  $g_\pm$  compactifies, the determinant (19) of  $\mathbf{H}^\pm$  must be smooth on  $M$ , positive on  $M^0$ , and vanish on (the pre-image of) the boundaries of  $\Delta_\pm$ . Hence the image of  $M^0$  under  $(x, y)$  must be a domain  $D^0 := (\alpha_0, \alpha_\infty) \times (\beta_0, \beta_\infty)$  where  $A(z)$  and  $B(z)$  are positive on  $(\alpha_0, \alpha_\infty)$  and  $(\beta_0, \beta_\infty)$  respectively, with zeros at the endpoints; furthermore, if  $g_+$  and/or  $g_-$  are globally defined, then  $q(x, y) \neq 0$  and/or  $x - y \neq 0$  on the closure  $D = [\alpha_0, \alpha_\infty] \times [\beta_0, \beta_\infty]$  of  $D^0$ . If both  $g_+$  and  $g_-$  are globally defined,  $\alpha_0 > \beta_\infty$  and  $\Delta_\pm$  are both quadrilaterals. However, in order to apply limiting arguments, we also need to allow  $\beta_\infty = \alpha_0$  when  $\Delta_+$  is a simplex and  $M$  is a weighted projective plane. In this case  $g_-$  does not compactify.

The polytopes  $\Delta_\pm \subseteq \mathfrak{h}_\pm^*$  are the images of  $D \subseteq P(W) \times P(W)$  (using the chosen affine chart on  $P(W)$ ) under the formulae (15)–(16) for the momenta  $\mu^\pm(x, y)$ . Since the level surfaces  $x = \gamma, y = \gamma$  have normals  $\lambda^{(\gamma)}$  and  $\rho^{(\gamma)}$  (respectively) in the negative case, and  $\sigma^{(\gamma)}$  in the positive case, we can take  $\gamma = \alpha_0, \alpha_\infty, \beta_0$  and  $\beta_\infty$  to determine straightforwardly that

$$\Delta_- = \left\{ \xi \in \mathfrak{h}_-^* : \langle \xi, \kappa \rangle = 1, \langle \xi, \lambda^{(\alpha_0)} \rangle \leq 0, \langle \xi, \lambda^{(\alpha_\infty)} \rangle \geq 0, \langle \xi, \rho^{(\beta_0)} \rangle \leq 0, \langle \xi, \rho^{(\beta_\infty)} \rangle \geq 0 \right\};$$

$$\Delta_+ = \left\{ \xi \in \mathfrak{h}_+^* : \langle \xi, q \rangle = 1, \langle \xi, \sigma^{(\alpha_0)} \rangle \leq 0, \langle \xi, \sigma^{(\alpha_\infty)} \rangle \geq 0, \langle \xi, \sigma^{(\beta_0)} \rangle \geq 0, \langle \xi, \sigma^{(\beta_\infty)} \rangle \leq 0 \right\}.$$

Normals to  $\Delta_\pm$  may be written  $u_{\alpha,k}^\pm = p^{(\alpha_k)}/r_{\alpha,k}^\pm$  and  $u_{\beta,k}^\pm = p^{(\beta_k)}/r_{\beta,k}^\pm$  for  $k = 0, \infty$  and constants  $r_{\alpha,k}^\pm$  and  $r_{\beta,k}^\pm$ , where  $p^{(\gamma)} = \gamma^b \odot q(\gamma, \cdot) \in \mathfrak{t}$  has the affine expression

$$p^{(\gamma)}(x, y) = \frac{1}{2} (q(x, \gamma)(y - \gamma) + (x - \gamma)q(y, \gamma)).$$

The boundary conditions  $\mathbf{H}_{\mu^\pm(\alpha_k, y)}^\pm(u_{\alpha, k}^\pm, \cdot) = 0 = \mathbf{H}_{\mu^\pm(x, \beta_k)}^\pm(u_{\beta, k}^\pm, \cdot)$  (see (3)) are equivalent to  $A(\alpha_k) = 0 = B(\beta_k)$ , and the remaining boundary conditions simplify to  $A'(\alpha_k) = 2r_{\alpha, k}^\pm$  and  $B'(\beta_k) = \mp 2r_{\beta, k}^\pm$ , using e.g.,

$$d\mathbf{H}_{\mu^+(\alpha_0, y)}^+(p^{(\alpha_0)}, p^{(\alpha_0)}) = \frac{A'(\alpha_0)(p^{(\alpha_0)}(y))^2}{(\alpha_0 - y)q(\alpha_0, y)^3} dx = \frac{A'(\alpha_0)(\alpha_0 - y)}{q(\alpha_0, y)} dx,$$

which is equal to  $-\frac{1}{2}A'(\alpha_0)p^{(\alpha_0)}$ . We deduce that  $r_{\alpha, k} := r_{\alpha, k}^+ = r_{\alpha, k}^-$  and  $r_{\beta, k} := r_{\beta, k}^+ = -r_{\beta, k}^-$ . The construction of (simply connected) regular ambitoric compactifications is now completed by ensuring the normals are inward, and satisfy the integrality condition that they span a lattice.

**PROPOSITION 3.** – *Any compact, simply connected regular ambitoric compactification is determined by the following data:*

- real numbers  $\alpha_k, \beta_k, r_{\alpha, k}, r_{\beta, k}$  ( $k = 0, \infty$ ), subject to the inequalities

$$\beta_0 < \beta_\infty \leq \alpha_0 < \alpha_\infty, \quad r_{\alpha, 0} < 0 < r_{\alpha, \infty}, \quad r_{\beta, 0} > 0 > r_{\beta, \infty},$$

and the integrality condition that, with  $u_{\alpha, k} = p^{(\alpha_k)}/r_{\alpha, k}$  and  $u_{\beta, k} = p^{(\beta_k)}/r_{\beta, k}$ ,

$$(20) \quad \text{span}_{\mathbb{Z}}\{u_{\alpha, 0}, u_{\alpha, \infty}, u_{\beta, 0}, u_{\beta, \infty}\} \cong \mathbb{Z}^2;$$

- a quadratic  $q(z)$  and two smooth functions of one variable,  $A(z)$  and  $B(z)$ , satisfying the positivity conditions that  $q(x, y) > 0$  on  $D^0 = (\alpha_0, \alpha_\infty) \times (\beta_0, \beta_\infty)$ ,  $A(z) > 0$  on  $(\alpha_0, \alpha_\infty)$  and  $B(z) > 0$  on  $(\beta_0, \beta_\infty)$ , and the boundary conditions that

$$(21) \quad A(\alpha_k) = 0 = B(\beta_k), \quad A'(\alpha_k) = -2r_{\alpha, k}, \quad B'(\beta_k) = 2r_{\beta, k} \quad (k = 0, \infty).$$

It is positive if  $q(x, y) > 0$  on the closure  $D$  of  $D^0$ , and negative if  $\beta_\infty < \alpha_0$ .

**REMARK 3.** – A particular case where (20) holds automatically is when  $q$  has rational coefficients and  $\alpha_k, \beta_k$  and  $r_{\alpha, k}, r_{\beta, k}$  are all rational: since the condition (20) is clearly invariant under an overall multiplication of  $r_{\alpha, k}$  and  $r_{\beta, k}$  by a nonzero real constant, we can choose this constant such that  $u_{\alpha, k}$  and  $u_{\beta, k}$  have integer coordinates.

**REMARK 4.** – One can allow some (but not all) of  $r_{\alpha, k}$  and  $r_{\beta, k}$  in Proposition 3 to be zero. In terms of the theory reviewed in Section 1, this is a limiting case in which some of the normals  $u_j$  are infinite, and hence the measure  $d\nu$  on the corresponding facet  $F_j$  is zero. On such an “omitted” facet  $F_j$ , the first order boundary conditions of Proposition 1 become

$$(22) \quad \mathbf{H}_\xi(\tilde{u}_j, \cdot) = 0 \quad \text{and} \quad (d\mathbf{H})_\xi(\tilde{u}_j, \tilde{u}_j) = 0, \quad \forall \xi \in F_j,$$

where  $\tilde{u}_j$  is any nonzero normal vector to  $F_j$ . This is the setting of [26, Conjecture 7.2.3] and [53, §3.1] and yields complete Kähler metrics on the complement of a toric divisor (the inverse image of the omitted facets) in a compact toric orbifold  $M$ .

Proposition 3 extends the characterization of regular ambitoric compactifications by allowing complete ends of this form. When  $r_{\alpha, k}$  or  $r_{\beta, k}$  is zero, the boundary conditions (21) apply without change, but in (20), we replace  $u_{\alpha, k} = p^{(\alpha_k)}/r_{\alpha, k}$  or  $u_{\beta, k} = p^{(\beta_k)}/r_{\beta, k}$  by some other multiples of  $p^{(\alpha_k)}$  or  $p^{(\beta_k)}$ .

### 3.4. Factorization structures for triangles and quadrilaterals

In the previous two subsections, we found that in an ambitoric compactification, the coordinate lines (in particular, the facets) in  $\Delta \subseteq P(\mathfrak{h}^*)$  were dual to (projective) normals in  $P(\mathfrak{h})$  which were decomposable with respect to an inclusion of  $\mathfrak{h}$  into a tensor product of 2-dimensional vector spaces  $W^* \otimes W^*$ . In order to obtain a converse, and determine when a rational Delzant quadrilateral arises from an ambitoric compactification, we formalize this phenomenon by introducing (2-dimensional) factorization structures; for a more general context, see Appendix A.

Throughout this section, we adopt the notation of §2.2 for  $(\Delta, \mathbf{L})$  in  $P(\mathfrak{h}^*)$ : the affine normals will be indexed  $L_{\beta,0}, L_{\alpha,0}, L_{\beta,\infty}, L_{\alpha,\infty}$ . We also allow the quadrilateral to degenerate to a triangle with  $L_{\alpha,0} = L_{\beta,\infty}$ .

DEFINITION 8. – Let  $(\Delta, \mathbf{L})$  be a rational Delzant quadrilateral or a triangle in  $P(\mathfrak{h}^*)$ , let  $W_1, W_2$  be 2-dimensional vector spaces, and let  $S: P(W_1) \times P(W_2) \rightarrow P(W_1 \otimes W_2)$  be the Segre embedding, sending  $([w_1], [w_2])$  to  $[w_1 \otimes w_2]$ . A *factorization structure* is a rational map  $S_\varphi: P(W_1) \times P(W_2) \dashrightarrow P(\mathfrak{h}^*)$  obtained by composing  $S$  with a projection  $P(W_1 \otimes W_2) \dashrightarrow P(\mathfrak{h}^*)$  dual to a linear injection  $\varphi: \mathfrak{h} \rightarrow W_1^* \otimes W_2^*$ .

(i)  $S_\varphi$  is a *Segre factorization structure* if the image of  $\varphi$  is  $\gamma_1^0 \otimes W_2^* + W_1^* \otimes \gamma_2^0 \subset W_1^* \otimes W_2^*$ , where  $\gamma_j^0 \subseteq W_j^*$  is the annihilator of some  $\gamma_j \in W_j$  (for  $j = 1, 2$ ).

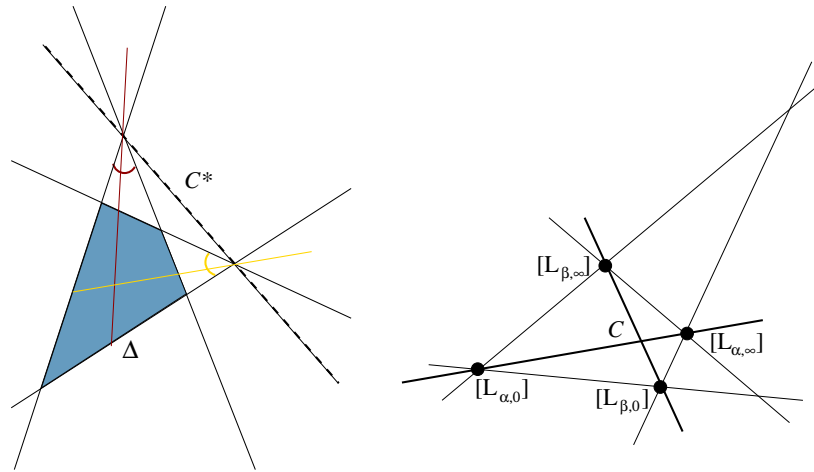


FIGURE 2. Segre factorization compatible with  $\Delta \subseteq P(\mathfrak{h}^*)$ : the induced conic is a line-pair (singular conic)  $\mathcal{C}$  passing through the projective normals in  $P(\mathfrak{h})$ ; points in  $\Delta$  are parametrized by lines through the two points on the dual conic  $\mathcal{C}^* \subseteq P(\mathfrak{h}^*)$ .

(ii)  $S_\varphi$  is a *Veronese factorization structure* if there is an isomorphism  $W_1 \cong W_2$  (so we drop the index) such that the image of  $\varphi$  is  $S^2W^* \subset W^* \otimes W^*$ .

We say  $S_\varphi$  is *compatible* with  $(\Delta, \mathbf{L})$  if  $S_\varphi$  maps a product  $I_1 \times I_2$  of closed intervals in  $P(W_1) \times P(W_2)$  bijectively onto  $\Delta$ .

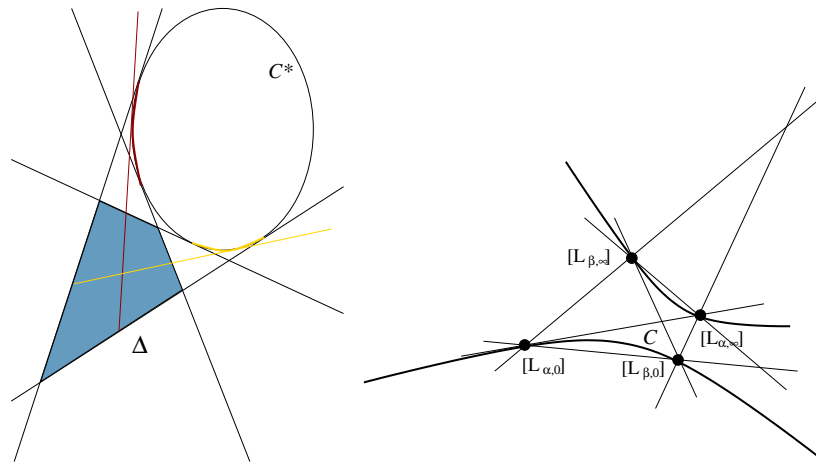


FIGURE 3. Veronese factorization compatible with  $\Delta \subseteq P(\mathfrak{h}^*)$ : the induced conic  $\mathcal{C}$  is a nonsingular conic passing through the projective normals in  $P(\mathfrak{h})$ ; points in  $\Delta$  are parameterized by pairs of tangent lines to the dual conic  $\mathcal{C}^* \subseteq P(\mathfrak{h}^*)$ .

The image of the Segre embedding  $S$  is a nonsingular ruled quadric surface, and the pullback by  $\varphi$  is a hyperplane section. This is a conic  $\mathcal{C}$  in  $P(\mathfrak{h})$  which we call the *induced conic* of the factorization structure; it is a line-pair in the Segre case, and nonsingular (and nonempty) in the Veronese case. Conversely, any two nonsingular ruled quadric surfaces are projectively equivalent, as are any two line-pairs, or any two nonsingular nonempty conics, in a projective plane. Hence a Segre or Veronese factorization structure is determined up to isomorphism by a line-pair or nonsingular nonempty conic  $\mathcal{C}$  in  $P(\mathfrak{h})$ .

Such a conic  $\mathcal{C}$  in  $P(\mathfrak{h})$  has a dual  $\mathcal{C}^*$  in  $P(\mathfrak{h}^*)$ : in the Segre case (Figure 2),  $\mathcal{C}^*$  is a “double” line (dual to the vertex of the line-pair  $\mathcal{C}$ ) with two marked points (dual to the two lines), while in the Veronese case (Figure 3),  $\mathcal{C}^*$  is the conic of tangent lines to  $\mathcal{C}$ . The coordinate lines of the factorization structure (whose duals, i.e., projective normals, are the points of  $\mathcal{C}$ ) are

- the lines through the two marked points on  $\mathcal{C}^*$  in the Segre case;
- the tangent lines to  $\mathcal{C}^*$  in the Veronese case.

For compatibility with  $(\Delta, \mathbf{L})$ , the projective normals  $[L_{\beta,0}]$ ,  $[L_{\alpha,0}]$ ,  $[L_{\beta,\infty}]$ , and  $[L_{\alpha,\infty}]$  must be on  $\mathcal{C}$  (so that the facets of  $\Delta$  are on coordinate lines). This is not quite sufficient: in the Segre case,  $\mathcal{C}^*$  must not meet the interior of  $\Delta$ , while in the Veronese case,  $\Delta$  must be entirely in the “exterior” (union of tangent lines) of  $\mathcal{C}^*$ . Using the projectivized dual cone  $\Delta^* \subseteq P(\mathfrak{h})$ , the following ensures both requirements.

CONDITION 1. –  $\mathcal{C}$  meets the interior of  $\Delta^*$ .

When  $\Delta$  is a quadrilateral (bounded by four lines in general position), there is a pencil of conics through the four projective normals, and  $\mathcal{C}$  can be any conic in the pencil satisfying Condition 1.

To complete our analysis, we need to discuss what happens to the affine structure  $\iota(1) \in \mathfrak{h}$  under the factorization structure. In the Segre case, there are three possibilities for  $\varphi(\iota(1))$ :

if  $\varphi(\iota(1)) \in \gamma_1^0 \otimes \gamma_2^0$ , then  $\mathcal{C}^*$  is the line at infinity and  $\Delta$  is a parallelogram; otherwise  $\varphi(\iota(1))$  is either decomposable, in which case  $\Delta$  is a trapezium (two parallel sides), or indecomposable, in which case  $\Delta$  has no parallel sides. In the Veronese case,  $\varphi(\iota(1)) = q \in S^2W^*$  and there are also three possibilities:  $q$  may have positive, zero or negative discriminant.

**PROPOSITION 4.** – *Let  $(\Delta, \mathbf{L})$  be a rational Delzant quadrilateral, and  $\mathcal{C}$  a conic through the projective normals of  $\Delta$  which satisfies Condition 1. Then if  $\mathcal{C}$  is nonsingular (respectively  $[\iota(1)]$  is not on  $\mathcal{C}$ ), there is a positive (respectively negative) ambitoric compactification with rational Delzant polytope  $\Delta$  and induced conic  $\mathcal{C}$ .*

*Proof.* – For  $\mathcal{C}$  nonsingular, Condition 1 implies there is a Veronese factorization  $\varphi$  compatible with  $(\Delta, \mathbf{L})$ . We identify  $\mathfrak{h}$  with  $S^2W^*$  using  $\varphi$ , and fix an area form  $\kappa$  on  $W$ , hence an isomorphism  $\gamma \mapsto \gamma^b \in \gamma^0$  from  $W$  to  $W^*$ . Up to an overall sign, the affine normals therefore have the form  $L_{\beta,0} = -\beta_0^b \otimes \beta_0^b$ ,  $L_{\alpha,0} = \alpha_0^b \otimes \alpha_0^b$ ,  $L_{\beta,\infty} = \beta_\infty^b \otimes \beta_\infty^b$  and  $L_{\alpha,\infty} = \alpha_\infty^b \otimes \alpha_\infty^b$ ; it is straightforward to check that  $[\beta_0], [\beta_\infty], [\alpha_0], [\alpha_\infty]$  are in cyclic order on  $P(W)$  and hence choose an affine chart in which they are represented by  $\beta_0 < \beta_\infty \leq \alpha_0 < \alpha_\infty$ . Proposition 3 now implies that there is a positive ambitoric compactification with rational Delzant polytope isomorphic to  $\Delta$  and induced conic  $\mathcal{C}$ .

For negative compactifications, we identify  $\mathcal{C} \subseteq P(\mathfrak{h})$  with the conic of decomposables in  $\psi^0 \subseteq W_1^* \otimes W_2^*$ , where  $\psi \in W_1 \otimes W_2$  is decomposable if  $\mathcal{C}$  is singular, and indecomposable otherwise. (By Appendix A.2, the factorization is Veronese if  $\mathcal{C}$  is nonsingular and Segre otherwise; in the singular case,  $\mathcal{C}$  must be a line-pair, since the projective normals are not collinear.) Now if  $\iota(1) \in \mathfrak{h}$  is not on  $\mathcal{C}$ , its image in  $W_1^* \otimes W_2^*$  is not decomposable, and may be used to identify  $W_1$  and  $W_2^*$ ; we may identify  $W_1$  with  $W_2$  and drop subscripts by fixing also an area form  $\kappa$ . Then  $\psi$  is dual to a quadratic form  $q \in S^2W^*$ , i.e.,  $\mathfrak{h} = q^\perp$ . We conclude, similarly to the positive case, that Condition 1 implies that the normals have the form  $L_{\beta,0} = q(\beta_0, \cdot) \otimes \beta_0^b$ ,  $L_{\alpha,0} = \alpha_0^b \otimes q(\alpha_0, \cdot)$ ,  $L_{\beta,\infty} = -q(\beta_\infty, \cdot) \otimes \beta_\infty^b$  and  $L_{\alpha,\infty} = -\alpha_\infty^b \otimes q(\alpha_\infty, \cdot)$ ; the rest of the construction, using Proposition 3, follows the positive case. □

**REMARK 5.** – Both types of compactification exist if  $\mathcal{C}$  is nonsingular and does not pass through  $[\iota(1)]$ . They are then related by interchanging the roles of  $q$  (nonsingular) and  $\kappa$ . If neither exists,  $\mathcal{C}$  is the line-pair joining opposite normals, and  $[\iota(1)]$  lies on one of its lines. In particular,  $\Delta$  is a trapezium or parallelogram.

Condition 1 means that the conic  $\mathcal{C}^*$  tangent to the four lines of  $\Delta$  (the dual conic if  $\mathcal{C}$  is nonsingular, or the double line dual to the vertex of  $\mathcal{C}$  if it is a line-pair) does not meet  $\Delta$ , i.e.,  $\mathcal{C}^*$  is not an inscribed ellipse, or a degeneration of such an ellipse to a double line through opposite points of  $\Delta$ . We conclude this section by using the affine structure  $\iota(1) \in \Delta^*$  to provide a sufficient criterion for Condition 1.

**PROPOSITION 5.** – *Let  $(\Delta, \mathbf{L})$  be a rational Delzant quadrilateral with affine structure  $\iota(1) \in \mathfrak{h}$  and let  $\mathcal{C}$  be a conic in the pencil through the four projective normals such that  $[\iota(1)]$  is not a singular point on  $\mathcal{C}$ . Suppose there is an affine function orthogonal to  $\iota(1)$  with no zero on the segment of the Newton line between the midpoints of the diagonals of  $\Delta$ . Then  $\mathcal{C}$  satisfies Condition 1.*

*Proof.* – The centre of  $\mathcal{C}^*$  is the point dual to the line orthogonal to  $[\iota(1)]$  with respect to  $\mathcal{C}$  (or the midpoint of the vertices of  $\Delta$  on  $\mathcal{C}^*$  if  $\mathcal{C}$  is singular). Thus all affine functions orthogonal to  $\iota(1)$  with respect to  $\mathcal{C}$  vanish there. Newton's theorem for convex quadrilaterals implies that inscribed ellipses (or their degenerations) have centers (or midpoints) on the segment of the Newton line between the midpoints of the diagonals of  $\Delta$ . Hence  $\mathcal{C}^*$  cannot be among these, so Condition 1 holds.  $\square$

#### 4. Extremal ambitoric 4-orbifolds and convex quadrilaterals

##### 4.1. Extremal ambitoric metrics and adapted factorizations

A toric Kähler metric is extremal if and only if its scalar curvature is equal to the extremal affine function. For regular ambitoric structures, a straightforward computation of the scalar curvatures of the two Kähler metrics yields the following result [6].

**THEOREM.** – *Let  $(J_+, J_-, g_+, g_-, \iota)$  be a regular ambitoric structure given by a quadratic  $q$  and functions of one variable  $A, B$ . Then  $(g_+, J_+)$  is an extremal Kähler metric if and only if  $(g_-, J_-)$  is an extremal Kähler metric if and only if*

$$(23) \quad \begin{aligned} A(z) &= q(z)\pi(z) + P(z), \\ B(z) &= q(z)\pi(z) - P(z), \end{aligned}$$

where  $\pi$  is a polynomial of degree at most two orthogonal to  $q$  and  $P$  is polynomial of degree at most four. In this case

$$(24) \quad s_- = -\frac{24\pi(x, y)}{x - y},$$

$$(25) \quad s_+ = -\frac{w(x, y)}{q(x, y)},$$

where the quadratic  $w$  (defining  $w(x, y) = w_0xy + w_1(x + y) + w_2$ ) is equal to  $C_q(P)$ , where  $C_q$  is a surjective linear map from quartics to quadratics orthogonal to  $q$ .

In [6], we also proved that the ambitoric structure is locally conformally Einstein if in addition the quadratics  $\pi$  and  $w$  in this theorem are linearly dependent. We shall use this to construct examples later. We shall also need the explicit formula for  $C_q(P)(z)$ , which is the Poisson bracket (with the normalization convention  $\{f_1, f_2\} := f_1'f_2 - f_1f_2'$ ) of  $q(z)$  with  $q(z)P''(z) - 3q'(z)P'(z) + 6q''(z)P(z)$ .

For ambitoric compactifications, we deduce the following from the above theorem.

**COROLLARY 1.** – *For an extremal regular ambitoric compactification with induced conic  $\mathcal{C}$ , the extremal affine function  $\zeta_{\pm} \in \mathfrak{h}_{\pm}$  of  $\omega_{\pm}$  is orthogonal to the affine structure  $\iota(1) \in \mathfrak{h}_{\pm}$  with respect to  $\mathcal{C}$ .*

Given a rational Delzant quadrilateral  $(\Delta, \mathbf{L})$  in  $\Xi \subset \mathbb{P}(\mathfrak{h}^*)$ , there is a unique conic  $\mathcal{C}(\Delta, \mathbf{L}) \subset \mathbb{P}(\mathfrak{h})$  in the pencil through the normals such that  $[\iota(1)]$  is orthogonal to  $[\zeta]$ . Now  $\mathcal{C}(\Delta, \mathbf{L})$  corresponds to a conic in  $\mathbb{P}(\mathfrak{h}^*)$  such that  $\zeta$  vanishes at its centre (if nonsingular) or midpoint (if a double line). We now apply Proposition 5.



LEMMA 4. – *A rational Delzant quadrilateral  $(\Delta, \mathbf{L})$  is equiposed iff  $[\iota(1)]$  lies on the conic  $\mathcal{C}(\Delta, \mathbf{L})$ ; if  $(\Delta, \mathbf{L})$  is temperate, then the conic  $\mathcal{C}(\Delta, \mathbf{L})$  satisfies Condition 1.*

Consequently, by Proposition 4, for temperate rational Delzant quadrilaterals, there is an ambitoric compactification unless  $\mathcal{C}(\Delta, \mathbf{L})$  is the diagonal line pair, and the affine structure  $[\iota(1)]$  lies on one of the diagonals, i.e.,  $\Delta$  is an equiposed trapezium.

**4.2. Extremal ambitoric orbifolds and K-polystability**

In order to construct extremal ambitoric orbifolds, we specialize the discussion of Section 3.3 to the case that  $A(z)$  and  $B(z)$  in Proposition 3 are polynomials of degree  $\leq 4$ . Our approach follows [7, 8, 42, 43] to which we refer the reader for further details.

The boundary conditions (21) have the general solution

$$A(z) = (z - \alpha_0)(z - \alpha_\infty)((c + d)(z - \alpha_0)(z - \alpha_\infty) + N_{\alpha,0}(z - \alpha_\infty) + N_{\alpha,\infty}(z - \alpha_0))$$

$$B(z) = (z - \beta_0)(z - \beta_\infty)((c - d)(z - \beta_0)(z - \beta_\infty) + N_{\beta,0}(z - \beta_\infty) + N_{\beta,\infty}(z - \beta_0))$$

for  $A(z)$  and  $B(z)$  in terms of  $(\alpha_k, r_{\alpha,k})$  and  $(\beta_k, r_{\beta,k})$  (for  $k = 0, \infty$ ), where  $N_{\alpha,k} = 2r_{\alpha,k}/(\alpha_\infty - \alpha_0)^2$  and  $N_{\beta,k} = 2r_{\beta,k}/(\beta_\infty - \beta_0)^2$  (for  $k = 0, \infty$ ), and  $c, d$  are two free parameters. For fixed  $q(z)$ , the extremality conditions of Section 4.1 state that  $A(z) + B(z) = q(z)\pi(z)$  with  $\pi$  orthogonal to  $q$ . These impose three further linear conditions on  $A$  and  $B$ , which we may solve for  $c$  and  $d$ , leaving one linear condition on  $(r_{\alpha,k}, r_{\beta,k})$  whose coefficients depend rationally on  $\alpha_k$  and  $\beta_k$  ( $k = 0, \infty$ ).

EXAMPLE 1. – When  $q(z) = 1$  (the orthotoric case), we have  $c = 0$  and two formulae for  $d$  whose equality yields the equation

$$(N_{\alpha,0} \ N_{\alpha,\infty} \ N_{\beta,0} \ N_{\beta,\infty}) \begin{pmatrix} (\alpha_0 + \alpha_\infty - \beta_0 - \beta_\infty)^2 + 2(\alpha_\infty - \beta_0)(\alpha_\infty - \beta_\infty) \\ (\alpha_0 + \alpha_\infty - \beta_0 - \beta_\infty)^2 + 2(\alpha_0 - \beta_0)(\alpha_0 - \beta_\infty) \\ (\alpha_0 + \alpha_\infty - \beta_0 - \beta_\infty)^2 + 2(\beta_\infty - \alpha_0)(\beta_\infty - \alpha_\infty) \\ (\alpha_0 + \alpha_\infty - \beta_0 - \beta_\infty)^2 + 2(\beta_0 - \alpha_0)(\beta_0 - \alpha_\infty) \end{pmatrix} = 0$$

found by E. Legendre [42].<sup>(4)</sup> She proved that this condition on  $(r_{\alpha,k}, r_{\beta,k})$  is equivalent to  $\Delta_+$  being equiposed (relative to the corresponding normals) and thus showed that the existence of extremal Kähler metrics is equivalent to (toric) K-polystability in this case. However, it turns out that when  $\Delta_+$  is equiposed, it is automatically K-polystable: for  $q(z) = 1$ ,  $\deg(A + B) \leq 1$ , and so between any maximum of  $A$  on  $(\alpha_0, \alpha_\infty)$  and  $B$  on  $(\beta_0, \beta_\infty)$ , the quadratic  $A'' = -B''$  has a unique root; the boundary conditions thus force  $A$  and  $B$  to be positive on  $(\alpha_0, \alpha_\infty)$  and  $(\beta_0, \beta_\infty)$  respectively.

For equiposed trapezia (which cannot be orthotoric), Legendre [42] established similar existence and K-polystability results using ambitoric metrics of Calabi type.

We now generalize these results to arbitrary quadrilaterals, on which we relate the existence of ambitoric extremal Kähler metrics to the toric K-polystability criteria.

THEOREM 1. – *Let  $(M, \omega, \mathbb{T})$  be a toric symplectic orbifold whose rational Delzant polytope  $\Delta$  is a quadrilateral (i.e.,  $b_2(M) = 2$ ). Then the following are equivalent:*

<sup>(4)</sup> Hence the linear system always determines  $c$  and  $d$ , since this condition is open and natural in  $q$ .

- (i)  $(M, \omega)$  admits a  $\mathbb{T}$ -invariant extremal Kähler metric;
- (ii)  $(M, \omega, \mathbb{T})$  is analytically relatively  $K$ -polystable wrt. toric degenerations;
- (iii)  $(M, \omega)$  admits a  $\mathbb{T}$ -invariant ambitoric extremal Kähler metric  $g$  which is regular on generic orbits, unless  $\Delta$  is an equiposed trapezium, in which case,  $g$  has Calabi type or is a Kähler product.

In particular, if  $(M, \omega, \mathbb{T})$  admits an extremal Kähler metric, it must be ambitoric.

*Proof.* – We use the notation of Sections 1.1 and 2.2, so that  $\Delta \subset \Xi \subset \mathfrak{h}^*$ , where  $\iota^\top(\Xi) = \{1\}$  for an affine structure  $\iota: \mathbb{R} \rightarrow \mathfrak{h}$  on  $P(\mathfrak{h}^*)$  which we identify with  $\iota(1) \in \mathfrak{h}$ . Since  $\Delta$  is convex,  $\iota(1)$  is interior to the strictly convex cone spanned by the normal rays of  $\Delta$ ; thus  $[\iota(1)]$  is interior to the dual polytope  $\Delta^* \subset P(\mathfrak{h})$  which is the projectivization of this cone. Let  $\zeta \in \mathfrak{h}$  be the extremal affine function and  $\mathcal{C}(\Delta, \mathbf{L})$  the unique conic in  $P(\mathfrak{h})$  passing through the normals, and such that  $[\iota(1)]$  and  $[\zeta]$  are orthogonal.

*Case 1.* – Suppose  $\Delta$  is temperate and  $\mathcal{C}(\Delta, \mathbf{L})$  is nonsingular. Then by Lemma 4,  $\mathcal{C}(\Delta, \mathbf{L})$  satisfies Condition 1 and so Proposition 4 implies that there are positive ambitoric compactifications with rational Delzant polytope  $\Delta$  and induced conic  $\mathcal{C}(\Delta, \mathbf{L})$ .

Fixing  $\mathcal{C} = \mathcal{C}(\Delta, \mathbf{L})$  and the associated factorization structure, such compactifications also exist for arbitrary positive rational rescalings of the normals of  $\Delta$ . Hence we are in a position to apply an argument pioneered by E. Legendre [42] in the parabolic case. The positive ambitoric ansatz, with fixed  $\alpha_k, \beta_k$  and  $q$  yields a linear condition on the normal parameters  $r_{\alpha,k}, r_{\beta,k}$  for the existence of quartics  $A, B$  satisfying the boundary conditions (21) such that  $A(z) + B(z) = q(z)\pi(z)$  with  $\pi$  orthogonal to  $q$ . If such  $A$  and  $B$  exist, then even if they do not satisfy the positivity requirement to define an extremal Kähler metric, we can use them to compute that the extremal affine function is orthogonal to  $q$ . However, this latter condition is also a linear condition on the normal parameters  $r_{\alpha,k}, r_{\beta,k}$ . We conclude that the two linear conditions agree. For the normals  $\mathbf{L}$  of  $\Delta$ ,  $\zeta$  is orthogonal to  $q$ ; hence there do exist quartics  $A, B$  defining a formal extremal solution  $\mathbf{H} = \mathbf{H}^+$  on  $\Delta = \Delta_+$ .

*Case 2.* – Suppose  $\Delta$  is temperate, but  $\mathcal{C}(\Delta, \mathbf{L})$  is singular so that it is the line-pair in the pencil of conics through the four normals which meets the interior of  $\Delta^*$ . If  $[\iota(1)]$  is on  $\mathcal{C}(\Delta, \mathbf{L})$ , then  $\Delta$  is an equiposed trapezium, hence admits formal extremal solution of Calabi type [42]. Otherwise, Proposition 4 implies that there are negative ambitoric compactifications with rational Delzant polytope  $\Delta$ . As in case 1, we conclude that there are quartics  $A, B$  defining a formal extremal solution  $\mathbf{H} = \mathbf{H}^-$  on  $\Delta = \Delta_-$ .

If  $\Delta$  is intemperate, it is not  $K$ -polystable by Lemma 3. Otherwise, either case 1 or case 2 provides a formal extremal solution. This may not yield a positive definite metric, but it can be used to compute the toric  $K$ -polystability criterion. In the Calabi or product case, this has been done in [42]; it remains to consider the regular ambitoric case.

Let  $\mathbf{H}$  be the formal extremal solution given by the quartics  $A, B$  as above. If  $A$  is positive on  $(\alpha_0, \alpha_\infty)$  and  $B$  is positive on  $(\beta_0, \beta_\infty)$ , then  $\mathbf{H}$  is positive definite. Hence (6) implies that  $\mathcal{F}_{\Delta, \mathbf{L}}(f) > 0$  for  $f \in \mathcal{P}\mathcal{L}(\Delta)$ , unless  $\text{Hess } f = 0$  on  $\Delta$ , i.e.,  $f$  is affine. The rational Delzant polytope  $(\Delta, \mathbf{L})$  is therefore  $K$ -polystable.

To establish a converse, we consider special families of simple convex PL functions determined by the ambitoric factorization. For any  $x_0 \in (\alpha_0, \alpha_\infty)$  consider the line segment

$\{(x_0, y) : y \in (\beta_0, \beta_\infty)\}$ . Under  $\mu^\pm$  it transforms to a line segment  $S_{x_0}$  in the interior of  $\Delta_\pm$ . Let  $u_{x_0}$  be a normal of  $S_{x_0}$ . It is straightforward to check that  $\mathbf{H}_{(x_0, y)}^\pm(u_{x_0}, u_{x_0})$  is positive multiple of  $A(x_0)$ . Thus, if  $(M, \omega)$  is analytically relatively  $\mathbf{K}$ -polystable with respect to toric degenerations, then (7) implies  $A(x_0)$  must be positive for any  $x_0 \in (\alpha_0, \alpha_\infty)$ ; the argument for  $B$  is similar.

*Conclusion.* – We conclude that (ii) and (iii) are equivalent, and evidently (iii) implies (i). The implication (i)  $\Rightarrow$  (ii) follows from [61, Theorem 1.3], and the final assertion follows from the uniqueness of the extremal toric Kähler metrics, modulo automorphisms, established in [34].  $\square$

REMARK 6. – The general theory from [26] and [62] implies that in order to check the  $\mathbf{K}$ -polystability of a rational Delzant polygon  $(\Delta, \mathbf{L})$ , it is only necessary to consider a particular kind of PL convex function: the simple PL convex functions whose crease meets the interior of the polytope  $\Delta$ . Theorem 1 shows that in the case of a quadrilateral, it suffices to consider the cases that the crease is either one of the diagonals or meets the polytope in a segment corresponding to  $\{(x_0, y) : y \in (\beta_0, \beta_\infty)\}$  or  $\{(x, y_0) : x \in (\alpha_0, \alpha_\infty)\}$  under the unique ambitoric compactification given by the conic  $\mathcal{C}(\Delta, \mathbf{L})$ , which may be found by solving linear equations.

In the light of [26] and its extension to orbifolds in [49], when the rational Delzant polytope  $\Delta$  has rational vertices with respect to the dual lattice, one can also consider a weaker version of algebraic relative (toric)  $\mathbf{K}$ -polystability by requiring that  $\mathcal{F}_{\Delta, \mathbf{L}}(f) \geq 0$  for any rational PL continuous convex function  $f$  with equality if and only if  $f$  is an affine function. Presumably, this condition corresponds to an algebro-geometric notion of stability for the corresponding (log) variety. A key observation in [26] is that in the case of a rational polygon with vanishing extremal vector field, the algebraic relative  $\mathbf{K}$ -polystability with respect to toric degenerations is equivalent to the analytic one. This phenomenon is well demonstrated on our classification: if  $\alpha_k, \beta_k, r_{\alpha, k}, r_{\beta, k}$  are all rational numbers as in Remark 3 (so that the vertices of  $\Delta$  are rational with respect to the dual lattice) and if  $\mathcal{F}_{\Delta, \mathbf{L}} > 0$  on rational PL convex functions which are not affine on  $\Delta$ , we then conclude as in the proof of Theorem 1 that  $A(z)$  must be positive at any rational point in  $(\alpha_0, \alpha_\infty)$ . It follows that  $A(z) \geq 0$  on  $(\alpha_0, \alpha_\infty)$  with (possibly) a repeated irrational root in this interval. As the  $\alpha_k$ 's and  $r_{\alpha, k}$ 's are rational, by the first order boundary conditions  $A(z)$  is a (multiple of) degree 4 polynomial with rational coefficients with two simple (rational) roots  $\alpha_0$  and  $\alpha_\infty$ . In particular, any double root of  $A$  (if any) must be rational too, showing that  $A(z)$  must be strictly positive on  $(\alpha_0, \alpha_\infty)$ . Similarly,  $B(z) > 0$  on  $(\beta_0, \beta_\infty)$ .

This provides a computational test for  $\mathbf{K}$ -polystability of quadrilaterals which is guaranteed to terminate in the unstable case. We will further use these observations in Appendix B to show that any compact convex quadrilateral which is not a parallelogram can be made  $\mathbf{K}$ -unstable by a suitable choice of the affine normals  $\mathbf{L}$ .

REMARK 7. – In view of Remark 4, the equivalence (ii)  $\Leftrightarrow$  (iii) of Theorem 1 extends to complements of toric divisors in compact toric orbifolds (with  $b_2 = 2$ ), for ambitoric extremal Kähler metrics satisfying (21) with  $r_{\alpha, k}$  or  $r_{\beta, k}$  zero on omitted facets.

### 5. Examples

Our results show, as in [42], that for any convex quadrilateral, there is a nonempty open subset of scales for the normals such that the corresponding toric 4-orbifold has an extremal Kähler metric. By considering data close to well-known Bochner-flat Kähler metrics, we shall demonstrate this explicitly. We shall restrict attention to rational data in the sense of Remark 3. More precisely, if  $\alpha_k, \beta_k$  and the coefficients of  $q$  are rational, then the parameters  $\varepsilon$  and  $\eta$  defining the quadrilateral are rational, and the normal scales  $r_{\alpha,k}, r_{\beta,k}$  are constrained by a single rational linear relation.

A 4-dimensional extremal Kähler metric with nonzero scalar curvature is locally conformally Einstein iff it is Bach-flat, and globally so if the scalar curvature is nonvanishing [23]; the compact smooth examples have been classified [19, 38, 39], so we seek complete or compact orbifold examples.

A 4-dimensional Kähler metric is Bochner-flat iff it is selfdual ( $W_- = 0$ ); hence it is Bach-flat and locally conformally Einstein. According to R. Bryant [14] such metrics exist on weighted projective planes  $\mathbb{C}P^2_{w_1, w_2, w_3}$  (where  $w_1, w_2, w_3$  are positive integers with no common factor), cf. Section 2.1 and [2, 7]. Since  $W^- = 0$ , there is some freedom in the choice of negative complex structure  $J_-$ , and hence a family of ambitoric structures compatible with a given Bochner-flat Kähler metric (note however, that  $J_-$  is not globally defined).

#### 5.1. Bochner-flat ambitoric structures on weighted projective planes

The Kähler metric  $(g_+, J_+, \omega_+)$  of an ambitoric structure is Bochner-flat if  $A(z) = P(z)$ ,  $B(z) = -P(z)$  for an arbitrary polynomial  $P$  of degree  $\leq 4$ . The parabolic case (with  $q(z) = 1$ ) has been studied in [7]; we now consider arbitrary  $q$ .

We set  $P(z) = -(z - z_0)(z - z_1)(z - z_2)(z - z_3)$ , where  $z_0 < z_1 < z_2 < z_3$  and  $q(x, y)$  is positive on  $[z_2, z_3] \times [z_1, z_2]$  (e.g.,  $q(z)$  positive on  $[z_1, z_3]$ ). Since  $\Delta_+$  is a simplex, we are in the degenerate case of Section 3.3 where  $\alpha_0 = \beta_\infty$ , and we set  $\beta_0 = z_1, \beta_\infty = z_2 = \alpha_0$  and  $\alpha_\infty = z_3$ . The boundary conditions give

$$r_{\beta,0} = -2P'(z_1), \quad r_{\beta,\infty} = r_{\alpha,0} = -2P'(z_2), \quad r_{\alpha,\infty} = -2P'(z_3),$$

which we can ensure are rational by taking  $z_0, \dots, z_3$  rational. By Remark 3, taking  $q$  also rational gives condition (20) for the normals  $u_1 := p^{(\beta_0)}/r_{\beta,1}, u_2 := p^{(\alpha_0)}/r_{\alpha,1}$  and  $u_3 := p^{(\alpha_\infty)}/r_{\alpha,2}$ . These normals are  $u_j = -p^{(z_j)}/2P'(z_j)$ , which satisfy

$$(z_1 - z_0)q(z_2, z_3)u_1 + (z_2 - z_0)q(z_1, z_3)u_2 + (z_3 - z_0)q(z_1, z_2)u_3 = 0$$

and so the weights  $w_1, w_2, w_3$  are a multiple of  $(z_1 - z_0)q(z_2, z_3), (z_2 - z_0)q(z_1, z_3), (z_3 - z_0)q(z_1, z_2)$ . Any weighted projective plane  $\mathbb{C}P^2_{w_1, w_2, w_3}$  with distinct weights has a Bochner-flat ambitoric structure of any type (parabolic, hyperbolic or elliptic).

Since the scalar curvature  $s_+$  of  $g_+$  is an affine function of the momenta, it attains its maximum and minimum values at the vertices of the momentum simplex, which are the images of  $(x, y) = (z_2, z_1), (z_3, z_1)$  and  $(z_3, z_2)$ . If we write, for  $0 < i < j \leq 3$ ,  $P(z) = -(z - z_i)(z - z_j)p_{ij}(z)$ , then we compute from Section 4.1 that

$$s_+(z_j, z_i) = 3 \frac{q(z_i)p_{ij}(z_j) - q(z_j)p_{ij}(z_i)}{z_j - z_i}.$$

We deduce (assuming  $q(z) > 0$  on  $[z_1, z_3]$ ) that  $s_+$  is positive at  $(z_3, z_1)$ ; it is also positive at  $(z_3, z_2)$  for  $z_2 - z_1$  sufficiently small. On the other hand, at  $(z_2, z_1)$ , for  $z_3 - z_2$  small,  $s_+$  changes sign as a function of  $z_0 \in (-\infty, z_1)$ , being negative at  $z_0 = z_1$ , but positive once  $z_1 - z_0$  is sufficiently large.

Under these conditions,  $s_+$  is everywhere positive for  $z_0 \ll z_1$ , and hence  $g_+$  is globally conformal to an Einstein hermitian metric  $s_+^{-2}g$  of positive scalar curvature. On the other hand, as  $z_0$  increases,  $s_+$  becomes nonpositive on the preimage  $(\mu^+)^{-1}(T)$  of a triangle  $T \subset \Delta_+$  containing the vertex  $(z_2, z_1)$ . This preimage  $N$  is a compact orbifold with boundary (the latter being the zero locus of  $s_+$ ), but it is straightforward to see that  $N$  is covered by a compact manifold  $\tilde{N}$  with boundary, cf. [14]. Indeed let  $C$  be the 2-dimensional cone defined by the two facets of  $\Delta_+$  which bound  $T$  and let  $\Lambda$  be the lattice generated by the normals to these facets; Delzant theory identifies  $(C, \Lambda)$  as the image by the momentum map of a (standard) toric  $\mathbb{C}^2$ ; the preimage  $\tilde{N}$  of  $T \subset C$  is the closure of a bounded domain biholomorphic to the unit ball in  $\mathbb{C}^2$ . The lift of  $g = s_+^{-2}g_+$  to  $\tilde{N} \setminus \partial\tilde{N}$  is a conformally compact, Einstein hermitian metric of negative scalar curvature, which is complete since  $ds_+ \neq 0$  on  $\partial\tilde{N}$  (cf. [3]).

### 5.2. Extremal ambitoric compactifications

In order to obtain new examples, which have those of the previous subsection as limiting cases, we let  $P(z), q(z)$  and  $z_0 < z_1 < z_2 < z_3$  be as before, and consider rational  $\alpha_k$  and  $\beta_k$  satisfying  $z_1 \approx \beta_0 < \beta_\infty \lesssim z_2 \lesssim \alpha_0 < \alpha_\infty \approx z_3$ . We now set  $A(z) = q(z)\pi_A(z) + P(z)$  and  $B(z) = q(z)\pi_B(z) - P(z)$  where  $\pi_A$  and  $\pi_B$  are quadratic polynomials uniquely determined by three rational (affine) linear conditions: each is orthogonal to  $q$ ,  $A(\alpha_0) = 0 = A(\alpha_\infty)$  and  $B(\beta_0) = 0 = B(\beta_\infty)$ . Note that  $A(z) + B(z) = q(z)(\pi_A + \pi_B)(z)$  and that  $A(z) - B(z) = q(z)(\pi_A - \pi_B)(z) + 2P(z)$ .

For  $\beta_0 = z_1, \beta_\infty = z_2 = \alpha_0, \alpha_\infty = z_3$ , the unique solution is  $\pi_A = \pi_B = 0$  and the quartics  $A$  on  $(z_3, z_2)$  and  $B$  on  $(z_2, z_1)$  are positive and define a Bochner-flat extremal metric. Hence for a small perturbation of the endpoints,  $A$  and  $B$  remain positive on  $(\alpha_0, \alpha_\infty)$  and  $(\beta_0, \beta_\infty)$  respectively (having roots close to  $z_1, z_2, z_3$  and  $z_0 < z_1$ ). The boundary conditions  $r_{\alpha,k} = -2A'(\alpha_k), r_{\alpha,k} = 2B'(\beta_k)$  give rational scales for the normals with the right signs to obtain an extremal Kähler metrics over a rational Delzant quadrilateral  $\Delta_+$ . Since  $\beta_\infty$  and  $\alpha_0$  are very close, the sides  $F_{\beta,\infty}$  and  $F_{\alpha,0}$  are almost parallel, meaning that the quadrilateral  $\Delta = \Delta_+$  has parameter  $\eta$  close to  $-1$ , but  $\varepsilon \in (-1, 1) \cap \mathbb{Q}$  is unconstrained.

The parametrization of these solutions by  $P, \alpha_k, \beta_k$  is not effective, because  $P(z)$  is only determined up to the addition of  $q(z)\pi(z)$  with  $\pi$  orthogonal to  $q$ . This overcounting matches precisely with the dimension of the space of rational Delzant quadrilaterals. By symmetry, we see that for quadrilaterals with rational parameters  $\varepsilon$  and  $\eta$ , one of these being sufficiently close to  $\pm 1$ , there is a nonempty open subset of rational normal scales—belonging, up to homothety, a nonempty open subset of  $\mathbb{Q}P^3$ —for which the corresponding toric 4-orbifold has an extremal Kähler metric. There are thus infinitely many ambitoric extremal compact 4-orbifolds with  $b_2 = 2$ , depending on 5 rational parameters.

### 5.3. Conformally Einstein Kähler orbifolds and complete Einstein metrics

A regular extremal ambitoric structure, given by quartic polynomials  $A = q\pi + P$ ,  $B = q\pi - P$  is Bach-flat iff the quadratics  $\pi$  and  $C_q(P)$  (which are both orthogonal to  $q$ ) are linearly dependent. For fixed  $q$ , this is a singular quadric hypersurface in the  $\mathbb{Q}P^6$  of coefficients of  $(\pi, P)$  up to homothety. Bochner-flat metrics are Bach-flat with  $\pi = 0$ , and so the quadric meets any open neighborhood of  $\pi = 0$ . Hence, as in the Bochner-flat case, we obtain locally or globally conformally Einstein metrics according to whether the scalar curvature  $s_+$  of  $g_+$  changes sign or is positive.

We can make this more explicit using the approach developed in the previous two subsections, where  $A = P$ ,  $B = -P$  gives a known Bochner-flat Kähler metric with nonzero scalar curvature. Fix  $\pi, \tilde{\pi} := C_q(P)$  as a basis for the quadratic polynomials orthogonal to  $q$ , and consider the equations  $A+B = \delta(\lambda\tilde{\pi} + \mu\pi)q$ ,  $C_q(A-B) = \gamma(\lambda\tilde{\pi} + \mu\pi)$ ,  $A(\alpha_k) = 0 = B(\beta_k)$ . For fixed  $(\delta, \gamma) \approx (0, 1)$  and  $z_0 \approx \beta_0 < \beta_\infty \lesssim z_1 \lesssim \alpha_0 < \alpha_\infty \approx z_2$ , this has a unique (and appropriately positive) solution up to scale (with  $\lambda \approx 2$  and  $\mu \approx 0$ ). The solution depends rationally on  $[\delta : \gamma]$  up to scale, hence for given  $\alpha_k, \beta_k$ , we have a one parameter family of Bach-flat ambitoric orbifolds.

Positivity of  $s_+$  can be obtained by a limiting argument, provided  $P$  is chosen so that the corresponding Bochner-flat metric (in 5.1) has positive scalar curvature. We thus obtain infinitely many new examples of compact ambihermitian Einstein 4-orbifolds of positive scalar curvature. If instead we choose  $P$  so that the corresponding Bochner-flat metric has scalar curvature positive for  $(x, y) = (z_3, z_1), (z_3, z_2), (z_2, z_2)$ , and negative for  $(x, y) = (z_2, z_1)$ , the analysis in 5.1 generalizes to yield new complete ambihermitian Einstein 4-manifolds of negative scalar curvature.

We do not attempt to classify explicitly the data yielding Bach-flat (or extremal) ambitoric compactifications, but examples are not confined to limiting cases. For instance, let  $q(z) = z$  and consider the quartics  $A, B$  with parameters  $(s, t)$  given by

$$\begin{aligned} A(z) &= tz^4 + (s-1)(t+1)z^3 - (st+4s+2t-2)z^2 - 2s(t-2)z \\ &= z(z-2)(tz^2 + (st+s+t-1)z + s(t-2)), \\ B(z) &= -tz^4 + (s-1)(t-2)z^3 + (st+4s+2t-2)z^2 - 2s(t+1)z \\ &= -z(z-1)(tz^2 - (st-2s-2t+2)z - 2s(t+1)). \end{aligned}$$

In this family, the roots  $z = 0, 1, 2$  are fixed, which is a slightly special choice because  $q(z)$  vanishes at  $z = 0$ , and so  $(A+B)(0) = 0$  is a consequence of the extremality condition. The latter equation is satisfied by the family, since

$$(A+B)(z) = z(2t-1)((s-1)z^2 - 2s)$$

and the Bochner-flat case is  $t = 1/2$ , with  $A(z) = -B(z) = \frac{1}{2}z(z-1)(z-2)(z+3s)$ . With three roots fixed, the extremal family is parametrized by an open subset of  $\mathbb{Q}P^3$ , and  $s, t$  are affine coordinates on the quadric surface given by the Bach-flatness condition  $a_1a_3 = b_1b_3$  on the coefficients of  $A$  and  $B$ .

For  $s > 0$ , after negating  $A, B$ , we are in the situation considered before, with  $z_0 = -3s$ ,  $z_1 = 0$ ,  $z_2 = 1$  and  $z_3 = 2$ : the Bochner-flat metric with  $\beta_0 = 0, \beta_\infty = 1$  and  $\alpha_\infty = 2$  has positive scalar curvature. Varying  $t$  in  $[2/3(s+2), 1/2]$ ,  $A$  has a root  $1 < \alpha_0 < 2$ ,

yielding Bach-flat examples over  $[\alpha_0, 2] \times [0, 1]$ . We also get such examples for  $s < -2/3$  using a slight variant of the same approach in which  $z_0 = -3s > 2$ . Here  $A, B$  (unnegated) satisfy positivity on  $[\alpha_0, 2] \times [0, 1]$  with  $1 < \alpha_0 < 2$  provided  $-1 < s < -2/3$  and  $1/2 < t < 2/3(s+2)$ , or  $s < -1$  and  $1/2 > t > 2/3(1-s)$ . The quadrilaterals corresponding to these examples have moduli  $\varepsilon = 1/2$  and  $\eta = -1/\alpha_0$ .

Similar examples to these can be found by considering the Bochner-flat metrics on  $[2, -3s] \times [1, 2]$  for  $s < -2/3$ . However, there are plenty of examples which are not deformations of the Bochner-flat family in this way. Other convenient families are given by  $(s, t)$  coordinate lines tangent to the discriminant of  $A$  or  $B$ , so that one of the quartics splits over  $\mathbb{Q}(s)$  or  $\mathbb{Q}(t)$ . These lines are  $s = 0, -1, 1/3, 2/3, 2, \infty$  and  $t = 0, -1, 1/2, 2, \infty$ . Many of these only yield singular or indefinite examples. However, after multiplying  $A$  and  $B$  by  $-1/t$ , we have, for  $s = 1/3, t = 2/(1 + 3u)$ ,

$$A(z) = -z^4 + (u + 1)z^3 - (u - 2)z^2 - 2uz = -(z + 1)z(z - 2)(z - u)$$

$$B(z) = z^4 - 2uz^3 + (u - 2)z^2 + (u + 1)z = z(z - 1)(z^2 - (2u - 1)z - (u + 1)),$$

and for  $u > 2$ ,  $A(z)$  is positive on  $(2, u)$  while the nontrivial roots of  $B(z)$  have opposite sign and sum at least 3, so that  $B(z)$  is positive on  $(0, 1)$ . Hence after rescaling, we obtain Bach-flat examples on  $[2, \alpha_\infty] \times [0, 1]$  with  $\alpha_\infty = u > 2$ . The quadrilaterals corresponding to these examples have moduli  $\varepsilon = 1/\alpha_\infty$  and  $\eta = -1/2$ .

For a final example, let  $t = 0$ , scale by  $-1/(s - 1)$  and set  $s = u/(u - 1)$  so that

$$A(z) = -z^3 + 2(u + 1)z^2 - 4uz = -z(z - 2)(z - 2u)$$

$$B(z) = 2z^3 - 2(u + 1)z^2 + 2uz = 2z(z - 1)(z - u).$$

For  $1/2 < u < 1$  this yields Bach-flat examples on  $[\alpha_0, 2] \times [0, \beta_\infty]$  with  $\beta_\infty = u$  and  $\alpha_0 = 2u$ , while for  $u > 1$  we obtain instead examples on  $[2, \alpha_\infty] \times [0, 1]$  with  $\alpha_\infty = 2u$ .

**5.4. Hirzebruch orbifold surfaces**

Another interesting class of examples are the toric orbifolds for which the rational Delzant polytope is a trapezium but not a parallelogram. It is shown in [42] that these are precisely the toric orbifolds which admit toric Kähler metrics of Calabi type. Up to an orbifold covering, these orbifolds are fibre bundles of the form  $M = P \times_{S^1} \mathbb{C}P_{w_1, w_2} \rightarrow \mathbb{C}P_{v_1, v_2}$ , where the fibre and the base are weighted projective lines  $\mathbb{C}P_{w_1, w_2}$  and  $\mathbb{C}P_{v_1, v_2}$  respectively, and  $P$  is a principal  $S^1$ -orbibundle over  $\mathbb{C}P_{v_1, v_2}$ . It follows from [42] that such a Hirzebruch orbifold surface admits an extremal Kähler metric of Calabi type (in some and hence any Kähler class) if and only if the base admits a CSC Kähler metric, i.e.,  $v_1 = v_2 = 1$ . In our formalism, this corresponds to the case when  $\Delta$  is an equiposed trapezium, and  $(M, \omega, \mathbb{T})$  is automatically K-polystable with respect to toric degenerations [42].

When  $v_1 \neq v_2$ , the corresponding trapezia are not equiposed and extremal Kähler metrics must be obtained from the hyperbolic ambitoric ansatz. Rational Delzant trapezia which are close to but different from equiposed ones provide such examples. On the other hand, one can readily find K-unstable trapezia by violating the condition  $(1 + |\varepsilon|)(1 + |\eta|) < 2$  in Lemma 3; then there exist affine normals such that the trapezium is intemperate. More generally, Proposition 6 shows that any quadrilateral which is not a parallelogram is K-unstable for some choice of affine normals.

## Appendix A

### Factorization structures

The idea behind factorization structures is to separate variables using a rational map from a product of projective lines to a projective space. In order to explain our terminology, and place our constructions in a natural context, we discuss this idea in greater generality than we need in the body of the paper.

#### A.1. Factorization for rational Delzant polytopes

Let  $\mathfrak{h}$  be a real vector space of dimension  $m + 1$  and  $\Delta \subseteq \mathbb{P}(\mathfrak{h}^*)$  the image of a strictly convex cone in  $\mathfrak{h}^*$ .

**DEFINITION 9.** – A *factorization structure* over  $\mathbb{P}(\mathfrak{h}^*)$  is an injective linear map  $\varphi: \mathfrak{h} \rightarrow V_1^* \otimes V_2^* \otimes \cdots \otimes V_m^*$ , where  $V_1, \dots, V_m$  are 2-dimensional real vector spaces, such that composite  $S_\varphi$  of the *Segre embedding*

$$\begin{aligned} \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_m) &\rightarrow \mathbb{P}(V_1 \otimes \cdots \otimes V_m) \\ ([v_1], \dots, [v_m]) &\mapsto [v_1 \otimes \cdots \otimes v_m] \end{aligned}$$

with the dual projection  $\mathbb{P}(V_1 \otimes \cdots \otimes V_m) \dashrightarrow \mathbb{P}(\mathfrak{h}^*)$  maps any coordinate hyperplane ( $[v_j]$  constant for some  $j$ ) into a hyperplane in  $\mathbb{P}(\mathfrak{h}^*)$ . We say  $\varphi$  is *compatible* with  $\Delta$  (or a *factorization structure for  $\Delta$* ) if  $S_\varphi$  maps a product  $I_1 \times \cdots \times I_m$  of intervals  $I_j \subseteq \mathbb{P}(V_j)$  bijectively onto  $\Delta \subseteq \mathbb{P}(\mathfrak{h}^*)$ .

Note that the coordinate hyperplane condition is automatic for  $m \leq 2$  (since  $S_\varphi$  maps coordinate lines to lines). Also  $S_\varphi$  maps the boundary of  $I_1 \times \cdots \times I_m$  to the boundary of  $\Delta$ , so  $\Delta$  has at most  $2m$  facets. In our application,  $\Delta$  and the intervals  $I_j$  will be closed, so  $S_\varphi$  is also a bijection between boundaries.

If  $\varphi$  is understood, we typically regard it as an inclusion and identify  $\mathfrak{h}$  with its image  $\varphi(\mathfrak{h})$  in  $V_1^* \otimes \cdots \otimes V_m^*$ . The examples we consider are all of the following form.

**EXAMPLES.** – Let  $(m_1, \dots, m_k)$  be a partition of  $m$  and let  $W_1, \dots, W_k$  be 2-dimensional vector spaces. Then  $\varphi: \mathfrak{h} \rightarrow \bigotimes_{i=1}^k (\bigotimes^{m_i} W_i^*)$  is a *Segre-Veronese factorization structure* of *type  $(m_1, \dots, m_k)$*  iff  $\varphi(\mathfrak{h}) = \sum_{i=1}^k \langle \pi_i \rangle \otimes \cdots \otimes S^{m_i} W_i^* \otimes \cdots \otimes \langle \pi_k \rangle \subseteq \bigotimes_{i=1}^k S^{m_i} W_i^*$  for some decomposable  $\pi_i = \gamma_i^{\odot m_i} \in S^{m_i} W_i^*$  (for  $j = 1 \dots k$ ,  $\gamma_j \in W_j^*$ ).

The map  $S_\varphi: \mathbb{P}(W_1)^{m_1} \times \cdots \times \mathbb{P}(W_k)^{m_k} \rightarrow \mathbb{P}(\mathfrak{h}^*)$  sends a coordinate hyperplane with one component equal to  $[\alpha_j] \in \mathbb{P}(W_j)$  to the hyperplane in  $\mathbb{P}(\mathfrak{h}^*)$  dual to  $[\pi_1 \otimes \cdots \otimes \theta_j^{\odot m_j} \otimes \cdots \otimes \pi_k] \in \mathbb{P}(\mathfrak{h})$ , where  $\ker \theta_j = \langle \alpha_j \rangle$ . This is an element in the image of the (dual) mixed *Segre-Veronese embedding*  $\mathbb{P}(W_1^*) \times \cdots \times \mathbb{P}(W_k^*) \rightarrow \mathbb{P}(\bigotimes_{i=1}^k S^{m_i} W_i^*)$ .

The extreme partitions  $(1, 1, \dots, 1)$  and  $(m)$  correspond to pure Segre and Veronese embeddings respectively. For toric 4-orbifolds ( $m = 2$ ), these are the only cases.



**A.2. Factorizations on toric 4-orbifolds**

When  $m := \dim \mathfrak{t} = 2$ , the image of any factorization structure  $\mathfrak{h} \rightarrow W_1^* \otimes W_2^*$  ( $\dim W_i = 2$ ) is the annihilator of an element  $\chi$  of  $W_1 \otimes W_2$ . If  $\chi = \gamma_1 \otimes \gamma_2$  is decomposable, the image of  $\mathfrak{h}$  is  $\gamma_1^0 \otimes W_2^* + W_1^* \otimes \gamma_2^0$ , where  $\gamma_j^0 \subseteq W_j^*$  is the annihilator of  $\gamma_j \in W_j$ . If not,  $\chi$  defines an isomorphism  $W_1^* \rightarrow W_2$ , and hence, fixing a nonzero area form on  $W_1$ , an isomorphism  $W_1 \rightarrow W_2$ . Using this to identify  $W_1$  with  $W_2$  and dropping subscripts, the factorization structure  $\mathfrak{h} \rightarrow W^* \otimes W^*$  has image annihilating  $\wedge^2 W \subset W \otimes W$ , i.e., equal to  $S^2 W^*$ .

Thus, up to isomorphism, any factorization structure is Segre-Veronese of type (1, 1) or (2); these are the Segre and Veronese factorizations used in the paper. In the Segre case,  $P(W_1) \times P(W_2) \dashrightarrow P(\mathfrak{h}^*)$  is projection away from the point  $[\gamma_1 \otimes \gamma_2]$  on the quadric surface in  $P(W_1 \otimes W_2)$ ; this is the famous birational map identifying the blow-up of  $P(W_1) \times P(W_2)$  at  $([\gamma_1], [\gamma_2])$  with the blow-up of  $P(\mathfrak{h}^*)$  at two points. In the Veronese case, the map  $P(W) \times P(W) \dashrightarrow P(\mathfrak{h}^*)$  is projection away from a point *off* the quadric surface, which is a branched double cover over a conic.

**Appendix B**

**The semistability surface**

In this appendix we consider the dependence of the toric K-polystability condition (and hence the existence of extremal metrics) on the rational Delzant quadrilateral  $(\Delta, \mathbf{L})$ , which is determined by a positivity property of its Futaki functional  $\mathcal{F}_{\Delta, \mathbf{L}}$  on the space  $\mathcal{PL}(\Delta)$  of PL convex functions. For a fixed quadrilateral  $\Delta$ ,  $\mathcal{F}_{\Delta, \mathbf{L}}$  depends *linearly* on inverse scales  $r_{\alpha, k}$  and  $r_{\beta, k}$  ( $k = 0, \infty$ ) for the normals  $\mathbf{L}$ . We can thus parameterize a choice of normals, up to overall scale, by a point  $[r_{\alpha, 0}, r_{\alpha, \infty}, r_{\beta, 0}, r_{\beta, \infty}]$  in the positive quadrant of  $\mathbb{Q}P^3$ , and a given choice will be K-polystable provided this point lies in the open subset  $R_\Delta$  of  $\mathbb{R}P^3$  on which the Futaki functional has constant sign (with only trivial zeros). It follows from [42] that  $R_\Delta \subseteq \mathbb{R}P^3$  has nonempty intersection with the positive rational quadrant.

We refer to the boundary  $S_\Delta$  of  $R_\Delta$  as the *semistability surface* of  $\Delta$ . At any point in  $S_\Delta$  there must be a nontrivial Futaki invariant which is zero, and since it is linear on  $\mathbb{R}P^3$ , this Futaki invariant defines a supporting hyperplane for  $R_\Delta$ . Consequently, we can hope to describe the *dual surface* of  $S_\Delta$  explicitly in terms of Futaki invariants, and then consider its dependence on  $\Delta$ .

It suffices to consider Futaki invariants defined by simple PL convex functions with a crease meeting opposite sides of  $\Delta$  (including the diagonals of  $\Delta$  as extreme cases): our main results show that the positivity of these invariants is not only necessary, but sufficient, for toric K-polystability. These invariants are still quite formidable in complexity, but are amenable to computation.

In our computations, we drop overall positive constants, such as the constant  $c(\varepsilon, \eta) = 24/(4 - (1 - \varepsilon^2)(1 - \eta^2))$  appearing in the extremal affine function, and employ the dihedral symmetry (which acts projectively on  $\Delta$ ) to minimize duplication of effort. This symmetry group, determined by its action on vertices, is generated by a “vertical” reflection

(cf. Figure 1)  $\sigma_\alpha: v_{00} \mapsto v_{\infty 0}, v_{0\infty} \mapsto v_{\infty\infty}$  and a diagonal reflection  $\sigma_\varepsilon: v_{00} \mapsto v_{\infty\infty}$  fixing  $v_{0\infty}$  and  $v_{\infty 0}$ , so that  $\rho := \sigma_\varepsilon \circ \sigma_\alpha$  is a  $\frac{\pi}{2}$  rotation, which acts on vertices and edges by

$$\begin{aligned} v_{00} &\mapsto v_{\infty 0} \mapsto v_{\infty\infty} \mapsto v_{0\infty} \mapsto v_{00}, \\ F_{\alpha,0} &\mapsto F_{\beta,0} \mapsto F_{\alpha,\infty} \mapsto F_{\beta,\infty} \mapsto F_{\alpha,0}. \end{aligned}$$

The remaining nonidentity elements consist of the other diagonal reflection  $\sigma_\eta = \sigma_\alpha \rho = \sigma_\alpha \sigma_\varepsilon \sigma_\alpha$ , the ‘‘horizontal’’ reflection  $\sigma_\beta = \rho \sigma_\varepsilon = \sigma_\varepsilon \sigma_\alpha \sigma_\varepsilon, \rho^2 = \sigma_\eta \sigma_\varepsilon$  and  $\rho^3 = \rho^{-1}$ . The dihedral action is only affine after permuting the labelling, so there is an induced action on the parameters  $(\varepsilon, \eta)$  which determine the affine class of  $\Delta$  as a labelled quadrilateral. Explicitly, we have  $\sigma_\alpha^*(\varepsilon, \eta) = (\eta, \varepsilon)$  and  $\sigma_\varepsilon^*(\varepsilon, \eta) = (-\varepsilon, \eta)$ , and hence  $\rho^*(\varepsilon, \eta) = (\eta, -\varepsilon), \sigma_\beta^*(\varepsilon, \eta) = (-\eta, -\varepsilon), \sigma_\eta^*(\varepsilon, \eta) = (\varepsilon, -\eta)$ .

The two families of simple PL convex functions whose Futaki invariants we need are  $f_{\mathbf{s},\mathbf{t}}^\alpha$ , with a crease joining  $\mathbf{s} \in F_{\alpha,0}$  to  $\mathbf{t} \in F_{\alpha,\infty}$ , and  $f_{\mathbf{s},\mathbf{t}}^\beta$ , with a crease joining  $\mathbf{s} \in F_{\beta,0}$  to  $\mathbf{t} \in F_{\beta,\infty}$ . We write

$$\mathcal{F}_{\Delta,\mathbf{L}}(f_{\mathbf{s},\mathbf{t}}^\alpha) = \sum_{j \in \{\alpha,\beta\} \times \{0,\infty\}} A_j(\varepsilon, \eta, \mathbf{s}, \mathbf{t}) r_j, \quad \mathcal{F}_{\Delta,\mathbf{L}}(f_{\mathbf{s},\mathbf{t}}^\beta) = \sum_{j \in \{\alpha,\beta\} \times \{0,\infty\}} B_j(\varepsilon, \eta, \mathbf{s}, \mathbf{t}) r_j$$

for functions  $A_j, B_j$  related by the following symmetries:

$$\begin{aligned} A_{\alpha,0}(\varepsilon, \eta, \mathbf{s}, \mathbf{t}) &= A_{\alpha,\infty}(-\varepsilon, -\eta, \mathbf{t}, \mathbf{s}) = B_{\beta,0}(\varepsilon, -\eta, \mathbf{s}, \mathbf{t}) = B_{\beta,\infty}(-\varepsilon, \eta, \mathbf{t}, \mathbf{s}) \\ B_{\alpha,0}(\varepsilon, \eta, \mathbf{s}, \mathbf{t}) &= B_{\alpha,\infty}(-\varepsilon, -\eta, \mathbf{t}, \mathbf{s}) = A_{\beta,0}(\varepsilon, -\eta, \mathbf{s}, \mathbf{t}) = A_{\beta,\infty}(-\varepsilon, \eta, \mathbf{t}, \mathbf{s}) \\ A_{\alpha,0}(\varepsilon, \eta, \mathbf{s}, \mathbf{t}) &= A_{\alpha,0}(-\eta, -\varepsilon, \mathbf{s}^*, \mathbf{t}^*), \quad B_{\alpha,0}(\varepsilon, \eta, \mathbf{s}, \mathbf{t}) = B_{\alpha,0}(-\eta, -\varepsilon, \mathbf{t}^*, \mathbf{s}^*), \end{aligned}$$

where the star denotes the (harmonic) inversion interchanging the diagonals  $l_\varepsilon$  and  $l_\eta$  and fixing the midpoints of the sides. It thus suffices to compute  $A_{\alpha,0}(\varepsilon, \eta, \mathbf{s}, \mathbf{t})$  and  $B_{\alpha,0}(\varepsilon, \eta, \mathbf{s}, \mathbf{t})$ . To parameterize the points  $\mathbf{s}, \mathbf{t}$  on the edges, a convenient reference space is the pencil of lines through the intersection  $O$  of the diagonals; this is a projective line with four harmonically separated marked points (the two diagonals  $l_\varepsilon$  and  $l_\eta$  and the two lines  $l_\alpha$  and  $l_\beta$  joining  $O$  to intersection points of opposite sides). In the concrete description of §2.2, the diagonals are  $x = \pm y$  and the other lines are  $x = 0$  and  $y = 0$ . We set  $l_\varepsilon = 0 = [1 : 0], l_\eta = \infty = [0 : 1]$ , so that we can use positive homogeneous coordinates  $\mathbf{s} = [s_0 : s_1], \mathbf{t} = [t_0 : t_1]$  on the edges; this fixes  $s_j$  and  $t_j$  up to independent scales. Each  $A, B$  is a polynomial of bidegree  $(3, 3)$  in  $\mathbf{s}, \mathbf{t}$ .

We compute that  $A_{\alpha,0}(\varepsilon, \eta, \mathbf{s}, \mathbf{t})$  is given (up to normalization) by

$$\begin{aligned} &2 (s_0 + s_1) \left( (1 + \eta)t_0 + (1 + \varepsilon)t_1 \right) \left( (1 - \eta)s_0 t_0 - (1 - \varepsilon)s_1 t_1 \right)^2 \\ &- (1 - \varepsilon)(1 - \eta) \left( (1 - \eta)s_0 + (1 - \varepsilon)s_1 \right) (t_0 + t_1) \left( (1 + \eta)s_0 t_0 - (1 + \varepsilon)s_1 t_1 \right)^2, \end{aligned}$$

whereas  $B_{\alpha,0}(\varepsilon, \eta, \mathbf{s}, \mathbf{t})$  is given (up to normalization) by

$$\begin{aligned} & (1 - \varepsilon)(1 + \eta)(1 + \varepsilon - \eta + \varepsilon\eta) s_0^3 t_0^3 \\ & + 2(1 + \varepsilon\eta + 1 + \varepsilon - \eta + \varepsilon\eta) s_0^2 s_1 t_0^3 \\ & + 2(1 - \varepsilon)(1 + \eta)(4 - (1 - \varepsilon^2)(1 - \eta^2)) s_0^3 t_0 t_1^2 \\ & + (1 - \varepsilon)(1 + \eta)((1 + \varepsilon)^2 + (1 - \eta)^2 + (1 + \varepsilon\eta)(2 + \varepsilon - \eta)) s_0^3 t_0^2 t_1 \\ & + ((1 + \varepsilon\eta)(10 + \varepsilon - \eta + (1 + \varepsilon)(1 - \eta)) + (1 + \varepsilon)^2 + (1 - \eta)^2) s_0^2 s_1 t_0^2 t_1 \\ & + 4(1 + \varepsilon\eta) s_0 s_1^2 t_0^2 t_1 + (40 - 12(1 - \varepsilon^2)(1 - \eta^2) - 8\varepsilon\eta) s_0^2 s_1 t_0 t_1^2 \\ & + ((1 + \varepsilon\eta)(10 - \varepsilon + \eta + (1 - \varepsilon)(1 + \eta)) + (1 - \varepsilon)^2 + (1 + \eta)^2) s_0 s_1^2 t_0 t_1^2 \\ & + (1 + \varepsilon)(1 - \eta)((1 - \varepsilon)^2 + (1 + \eta)^2 + (1 + \varepsilon\eta)(2 - \varepsilon + \eta)) s_0 s_1^2 t_1^3 \\ & + 2(1 + \varepsilon)(1 - \eta)(4 - (1 - \varepsilon^2)(1 - \eta^2)) s_0^2 s_1 t_1^3 \\ & + 2(1 + \varepsilon\eta + 1 - \varepsilon + \eta + \varepsilon\eta) s_1^3 t_0 t_1^2 \\ & + (1 + \varepsilon)(1 - \eta)(1 - \varepsilon + \eta + \varepsilon\eta) s_1^3 t_1^3. \end{aligned}$$

The latter expression typifies the contribution to the Futaki invariant from a side which does *not* meet the crease. Only the first and last two coefficients can be negative, and this can happen if and only if  $B_{\alpha,0}(\varepsilon, \eta, 0, 0)$  (i.e.,  $1 + \varepsilon - \eta + \varepsilon\eta$ ) or  $B_{\alpha,0}(\varepsilon, \eta, 1, 1)$  (i.e.,  $1 - \varepsilon + \eta + \varepsilon\eta$ ) is negative. This means that the expression already contributes negatively to the Futaki invariant of one of the diagonals, in which case the normals can be scaled to make the quadrilateral intemperate.

In contrast, the expression for  $A_{\alpha,0}$  typifies the contribution to the Futaki invariant from a side which *does* meet the crease. Here we have found a surprising factorization which shows that the contribution can be negative even when  $(1 + |\varepsilon|)(1 + |\eta|) < 2$  (so the quadrilateral is temperate for any choice of normals). We deduce the following.

**PROPOSITION 6.** – *Let  $\Delta$  be a compact convex quadrilateral.*

- *If  $\Delta$  is a parallelogram, then for any affine normals  $\mathbf{L}$ ,  $(\Delta, \mathbf{L})$  is  $K$ -polystable.*
- *If  $\Delta$  is not a parallelogram, then there exist choices for the affine normals  $\mathbf{L}$  such that  $(\Delta, \mathbf{L})$  is  $K$ -polystable as well as choices such that  $(\Delta, \mathbf{L})$  is  $K$ -unstable.*

*Proof.* – The stability results are straightforward [42], but the instability results stated in [42] are incorrect: by Example 1, equiposed rational Delzant quadrilaterals are  $K$ -polystable. However, if  $\Delta$  is not a parallelogram, then either  $\varepsilon \neq \eta$  or  $\varepsilon \neq -\eta$ . In the former case, put  $s_0 = (1 - \eta)t_1$  and  $s_1 = (1 - \varepsilon)t_0$  in  $f_{\mathbf{s},\mathbf{t}}^\alpha$  so that  $A_{\alpha,0}$  is negative. Then  $\mathcal{F}_{\Delta,\mathbf{L}}(f_{\mathbf{s},\mathbf{t}}^\alpha)$  can be made negative by taking  $r_{\alpha,0}$  large relative to the other inverse normals. When  $\varepsilon \neq -\eta$  a similar argument applies to  $\mathcal{F}_{\Delta,\mathbf{L}}(f_{\mathbf{s},\mathbf{t}}^\beta)$  and  $B_{\beta,0}$ . □

### Appendix C

#### Link with CR and Sasakian 5-manifolds

There are well known connections between symplectic and Kähler geometry in dimensions four and contact, CR and Sasakian geometry in dimension five [11, 44, 50]. In particular, quasiregular Sasaki-Einstein 5-manifolds have Kähler-Einstein orbifolds as quotients

by the Reeb vector field, and this provides one way of constructing them. As observed by D. Martelli and J. Sparks [47, 50], the Sasaki-Einstein manifolds of J. Gauntlett, D. Martelli, J. Sparks, D. Waldram [33] and M. Cvetič, H. Lü, D. Page, and C. Pope [21] have quotients which are of Calabi type and orthotoric respectively.

The general ambitoric context does not provide further Kähler-Einstein examples, but the extremal metrics may be used to construct continuous families of extremal Sasakian 5-manifolds, as well as Reeb directions which do not admit transversal extremal metrics (cf. [13, 42, 41]).

### C.1. Contact, CR and Sasakian structures

Recall that a contact manifold is an odd dimensional manifold  $N$  with a maximally non-integrable codimension one distribution  $\mathcal{H} \subset TN$ , i.e., the Lie bracket  $(X, Y) \mapsto [X, Y] \bmod \mathcal{H}$  defines a nondegenerate  $TN/\mathcal{H}$ -valued 2-form  $\Omega$  on  $\mathcal{H}$  called the *Levi form*. We assume that the line bundle  $TN/\mathcal{H}$  is oriented; positive sections  $\eta$  of the *contact line bundle*  $(TN/\mathcal{H})^* \subset T^*N$  are called *contact forms*. Such a contact form has pointwise kernel  $\mathcal{H}$  and induces a unique vector field  $K$  with  $\eta(K) = 1$  and  $\mathcal{L}_K \eta = 0$ , called the *Reeb vector field* of  $\eta$ .

An *almost CR structure* is a complex structure  $J$  on  $\mathcal{H}$  such that the Levi form is  $J$ -invariant, and  $(N, \mathcal{H}, J)$  is said to be a *CR manifold of Sasaki type* if there is a contact form  $\eta$  such that

$$g = dr^2 + r^2(d\eta(\cdot, J\cdot) + \eta^2), \quad \omega = d(r^2\eta) = 2r dr \wedge \eta + r^2 d\eta$$

is a Kähler metric on the cone  $N \times \mathbb{R}^+$ . The corresponding metric  $g_\eta = d\eta(\cdot, J\cdot) + \eta^2$  on  $M$  is called a compatible Sasakian metric. On a CR manifold of Sasaki type the contact forms  $\eta$  giving rise to compatible Sasakian metrics are those for which  $(\mathcal{H}, J, d\eta|_{\mathcal{H}})$  is invariant under the Reeb vector field and descends to a Kähler structure on local quotients by  $K$ ; it is called the *transverse Kähler geometry*. The Sasakian structure is said to be *quasiregular* if the quotient by  $K$  is an orbifold.

### C.2. CR structure associated to positive ambitoric metrics

We observe here that for fixed  $A(z)$  and  $B(z)$ , the ambitoric Kähler metrics  $(g_+, \omega_+)$  we obtain form a family of Sasakian metrics compatible with a fixed 5-dimensional CR-structure: this is similar to the well-known identification of Bochner-flat Kähler metrics with Sasakian structures compatible with the standard CR structure on an odd-dimensional sphere [58].

Suppose that  $(M, g_\pm, J_\pm, \omega_\pm, \mathfrak{t})$  be a regular ambitoric 4-orbifold. Then on the union  $M^0$  of the generic orbits the coordinates  $(\mathbf{x}, \mathbf{y}, \mathfrak{t})$  provide a diffeomorphism of  $M^0$  with  $D^0 \times \mathfrak{t}/2\pi\Lambda$  for a domain  $D^0$  in (an affine patch of)  $\mathbb{P}(W) \times \mathbb{P}(W)$ . Recall that  $\mathfrak{t} \cong S^2W^*/\langle q \rangle$  and the space of Hamiltonians  $\mathfrak{h}_+$  is isomorphic to  $S^2W^*$ . By passing to the universal cover of  $\mathfrak{t}/2\pi\Lambda$ , or introducing a lattice  $\tilde{\Lambda} \subset \mathfrak{h}_+$  covering  $\Lambda$ , we can pull back the Kähler structure along  $\pi_q: N^0 = D^0 \times \mathfrak{h}_+ \rightarrow D^0 \times \mathfrak{t}$ . Then

$$\pi_q^* \omega_+ = d\eta_q, \quad \text{where} \quad \eta_q = -\frac{\langle d\mathfrak{t}, \mathbf{x} \otimes \mathbf{y} \rangle}{q(\mathbf{x}, \mathbf{y})},$$

where  $dt$  is the tautological  $\mathfrak{h}_+ = S^2W^*$  valued 1-form on the 5-manifold. The kernel  $\mathcal{H} \cong \pi_q^*TM^0$  of  $\eta_q$ , together with  $J_+$  defines a CR structure of Sasakian type on  $N^0$ . The Reeb field of the contact form  $\eta_q$  is

$$K_q = -\langle q, X \rangle = -(q_0\partial_{t_0} + q_1\partial_{t_1} + q_2\partial_{t_2}),$$

where  $X \in \mathfrak{h}_+^* \otimes C^\infty(N^0, TN^0)$  is dual to  $dt$ , and the corresponding Sasakian metric is  $g_+ + \eta_q^2$ . Keeping the CR structure fixed, we now rescale the contact form and define

$$\eta = \frac{q(\mathbf{x}, \mathbf{y})}{\kappa(\mathbf{x}, \mathbf{y})} \eta_q = -\frac{\langle dt, \mathbf{x} \otimes \mathbf{y} \rangle}{\kappa(\mathbf{x}, \mathbf{y})} \quad \text{with}$$

$$d\eta = \frac{-d\mathbf{x} \wedge \langle dt, \mathbf{y} \otimes \mathbf{y} \rangle + d\mathbf{y} \wedge \langle dt, \mathbf{x} \otimes \mathbf{x} \rangle}{\kappa(\mathbf{x}, \mathbf{y})}.$$

The corresponding Reeb vector field is

$$K = -\langle x \odot y, X \rangle = -\partial_{t_0} + (1/2)(x + y)\partial_{t_1} - xy\partial_{t_2}.$$

This does not preserve the CR structure and hence only defines a transversal Hermitian metric, not a Sasakian metric. To compute this, we need to find the horizontal lift of  $g_0 = q(x, y)g_+/(x - y)$ . For this we observe that  $\langle dt, \{q, \mathbf{y} \otimes \mathbf{y}\} \rangle$  agrees with  $q(\mathbf{x}, \mathbf{y})\langle dt, \mathbf{y} \otimes \mathbf{y} \rangle/\kappa(\mathbf{x}, \mathbf{y})$  on  $\mathcal{H}$  (i.e., modulo  $\eta$ ) and the latter vanishes on  $K$ . Similarly, we replace  $\langle dt, \{q, \mathbf{x} \otimes \mathbf{x}\} \rangle$  by  $q(\mathbf{x}, \mathbf{y})\langle dt, \mathbf{x} \otimes \mathbf{x} \rangle/\kappa(\mathbf{x}, \mathbf{y})$ . Introducing affine coordinates, we conclude that the contact metric is

$$\frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + A(x)\left(\frac{y^2 dt_0 + 2y dt_1 + dt_2}{(x - y)^2}\right)^2 + B(y)\left(\frac{x^2 dt_0 + 2x dt_1 + dt_2}{(x - y)^2}\right)^2$$

$$+ \left(\frac{xy dt_0 + (x + y) dt_1 + dt_2}{x - y}\right)^2,$$

which is manifestly independent of  $q$ . Consequently this CR structure has a family of compatible Sasakian structures  $\eta_q$  (with Reeb vector fields  $K_q$ ) for  $q \in S^2W^*$ . If  $A(z)$  and  $B(z)$  are quartics such that  $A(z) + B(z) = q_1(z)q_2(z)$  for orthogonal quadratic forms  $q_1$  and  $q_2$ , then both Sasakian structures ( $q = q_1$  and  $q = q_2$ ) will be extremal.

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