ISSN 0012-9593

quatrième série - tome 48 fascicule 4 juillet-août 2015

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Arend BAYER & Brendan HASSETT & Yuri TSCHINKEL

Mori cones of holomorphic symplectic varieties of K3 type

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition / *Publication*

Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : smf@smf.univ-mrs.fr

Tarifs

Europe : 515 €. Hors Europe : 545 €. Vente au numéro : 77 €.

ISSN 0012-9593

Directeur de la publication : Marc Peigné
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^{e} série, t. 48, 2015, p. 941 à 950

MORI CONES OF HOLOMORPHIC SYMPLECTIC VARIETIES OF K3 TYPE

BY AREND BAYER, BRENDAN HASSETT AND YURI TSCHINKEL

ABSTRACT. – We determine the Mori cone of holomorphic symplectic varieties deformation equivalent to the punctual Hilbert scheme on a K3 surface. Our description is given in terms of Markman's extended Hodge lattice.

RÉSUMÉ. – On détermine les cônes de Mori des variétés symplectiques holomorphes qui se déforment au schéma de Hilbert de points sur une surface K3. Notre description est donnée en termes de structure de Hodge élargie de Markman.

Introduction

Let X be an irreducible holomorphic symplectic manifold. Let $(,)$ denote the Beauville-Bogomolov form on $H^2(X,\mathbb{Z})$; we may embed $H^2(X,\mathbb{Z})$ in $H_2(X,\mathbb{Z})$ via this form. Fix a polarization h on X; by a fundamental result of Huybrechts [17], X is projective if it admits a divisor class H with $(H,H) > 0$. It is expected that finer birational properties of X are also encoded by the Beauville-Bogomolov form and the Hodge structure on $H^2(X)$, along with appropriate extension data. In particular, natural cones appearing in the minimal model program—the moving cone, the nef cone, the pseudo-effective cone—should have a description in terms of this form.

Now assume X is deformation equivalent to the punctual Hilbert scheme $S^{[n]}$ of a K3 surface S with $n > 1$. Recall that

$$H^2(S^{[n]},\mathbb{Z})_{(,)} = H^2(S,\mathbb{Z}) \oplus_{\perp} \mathbb{Z}\delta, \quad (\delta,\delta) = -2(n-1) \tag{1}$$

where the restriction of the Beauville-Bogomolov form to the first factor is just the intersection form on S, and 2δ is the class of the locus of non-reduced subschemes. Recall from [20] that for K3 surfaces S, the cone of (pseudo-)effective divisors is the closed cone generated by

$$\{D \in \operatorname{Pic}(S) : (D,D) \geq -2, (D,h) > 0\}.$$

The first attempt to extend this to higher dimensions was [13]. Further work on moving cones was presented in [14, 24], which built on Markman's analysis of monodromy groups. The characterization of extremal rays arising from Lagrangian projective spaces $\mathbb{P}^n \hookrightarrow X$ has been addressed in [14, 12] and [3]. The paper [15] proposed a general framework describing all types of extremal rays; however, Markman found counterexamples in dimensions ≥ 10, presented in [5].

The formalism of Bridgeland stability conditions [7, 8] has led to breakthroughs in the birational geometry of moduli spaces of sheaves on surfaces. The case of punctual Hilbert schemes of $\mathbb{P}^2$ and del Pezzo surfaces was investigated by Arcara, Bertram, Coskun, and Huizenga [2, 16, 6, 10]. The effective cone on $(\mathbb{P}^2)^{[n]}$ has a beautiful and complex structure as n increases, which only becomes transparent in the language of stability conditions. Bayer and Macrì resolved the case of punctual Hilbert schemes and more general moduli spaces of sheaves on K3 surfaces [5, 4]. Abelian surfaces, whose moduli spaces of sheaves include generalized Kummer varieties, have been studied as well [31, 32].

In this note, we extend the results obtained for moduli spaces of sheaves over K3 surfaces to all holomorphic symplectic manifolds arising as deformations of punctual Hilbert schemes of K3 surfaces. Our principal result is Theorem 1 below, providing a description of the Mori cone (and thus dually of the nef cone).

In any given situation, this also leads to an effective method to determine the list of marked minimal models (i.e., birational maps $f\colon X \dashrightarrow Y$ where Y is also a holomorphic symplectic manifold): the movable cone has been described by Markman [23, Lemma 6.22]; by [14], it admits a wall-and-chamber decomposition whose walls are the orthogonal complements of extremal curves on birational models, and whose closed chambers correspond one-to-one to marked minimal model, as the pull-backs of the corresponding nef cones.

Acknowledgments:

The first author was supported by NSF grant 1101377; the second author was supported by NSF grants 0901645, 0968349, and 1148609; the third author was supported by NSF grants 0968318 and 1160859. We are grateful to Emanuele Macrì for helpful conversations, to Eyal Markman for constructive criticism and correspondence, to Claire Voisin for helpful comments on deformation-theoretic arguments in a draft of this paper, and to Ekatarina Amerik for discussions on holomorphic symplectic contractions. We are indebted to the referees for their careful reading of our manuscript. The first author would also like to thank Giovanni Mongardi for discussions and a preliminary version of [25]. Related questions for general hyperkähler manifolds have been treated in [1].

1. Statement of results

Let X be deformation equivalent to the Hilbert scheme of length-n subschemes of a K3 surface. Markman, see [22, Theorem 1.10] and [23, Cor. 9.5], describes an extension of lattices

$$H^2(X,\mathbb{Z}) \subset \widetilde{\Lambda}$$

and weight-two Hodge filtrations

$$H^2(X,\mathbb{C}) \subset \widetilde{\Lambda}_{\mathbb{C}}$$

with the properties listed below. We will write

$$\theta_X \colon H^2(X) \subset \widetilde{\Lambda}_X$$

to denote the extension of Hodge structures with pairing; here θ_X is defined canonically up to a choice of sign.

- The orthogonal complement of $\theta_X\big(H^2(X,\mathbb{Z})\big)$ has rank one, and is generated by a primitive vector of square $2n-2$ and type $(1,1)$;
- as a lattice
$$\widetilde{\Lambda} \simeq U^4 \oplus (-E_8)^2$$
where U is the hyperbolic lattice and E_8 is the positive definite lattice associated with the corresponding Dynkin diagram;
- any parallel transport operator $\phi\colon H^2(X,\mathcal{Z}) \to H^2(X',\mathcal{Z})$ naturally lifts to an isometry of lattices $\tilde{\phi}\colon \widetilde{\Lambda}_X \to \widetilde{\Lambda}_{X'}$ such that
$$\tilde{\phi} \circ \theta_X = \theta_{X'} \circ \phi;$$
the induced action of the monodromy group on $\widetilde{\Lambda}/H^2(X,\mathbb{Z})$ is encoded by a character *cov* (see [21, Sec. 4.1]);
- we have the following Torelli-type statement: X_1 and X_2 are birational if and only if there is Hodge isometry
$$\widetilde{\Lambda}_{X_1} \simeq \widetilde{\Lambda}_{X_2}$$
taking $H^2(X_1,\mathbb{Z})$ isomorphically to $H^2(X_2,\mathbb{Z})$;
- if X is a moduli space $M_v(S)$ of sheaves (or of Bridgeland-stable complexes) over a K3 surface S with Mukai vector v then there is an isomorphism from $\widetilde{\Lambda}$ to the Mukai lattice of S taking $H^2(X,\mathbb{Z})$ to $v^\perp$.

Generally, we use v to denote a primitive generator for the orthogonal complement of $H^2(X,\mathbb{Z})$ in $\widetilde{\Lambda}$. Note that $v^2 = (v,v) = 2n-2$. When $X \simeq M_v(S)$ we may take the Mukai vector v as the generator.

As the dual of θ_X we obtain a homomorphism[(1)]

$$\theta_X^\vee \colon \widetilde{\Lambda}_X \twoheadrightarrow H_2(X,\mathbb{Z})$$

which restricts to an inclusion

$$H^2(X,\mathbb{Z}) \subset H_2(X,\mathbb{Z})$$

of finite index. By extension, it induces a $\mathbb{Q}$-valued Beauville-Bogomolov form on $H_2(X,\mathbb{Z})$.

Assume X is projective. Let $H^2(X)_{\mathrm{alg}} \subset H^2(X,\mathbb{Z})$ and $\widetilde{\Lambda}_{\mathrm{alg}} \subset \widetilde{\Lambda}_X$ denote the algebraic classes, i.e., the integral classes of type $(1,1)$. Since the orthogonal complement of $i_X\big(H^2(X)\big)$ is generated by an algebraic class, it follows dually that $a \in \widetilde{\Lambda}_X$ is of type $(1,1)$ if and only if $\theta^\vee(a)$ is. The Beauville-Bogomolov form on $H^2(X)_{\mathrm{alg}}$ has signature $(1,\rho(X)-1)$, where $\rho(X) = \dim(H^2_{\mathrm{alg}}(X))$. The *Mori cone* of X is defined as the closed cone in $H_2(X,\mathbb{R})_{\mathrm{alg}}$ containing the classes of algebraic curves in X. The *positive cone* (or more accurately, non-negative cone) in $H^2(X,\mathbb{R})_{\mathrm{alg}}$ is the closure of the connected component of the cone

$$\{D \in H^2(X,\mathbb{R})_{\mathrm{alg}} : D^2 > 0\}$$

(1) We will often drop the subscript X from the notation when the context is clear.

containing an ample class. The dual of the positive cone in $H^2(X,\mathbb{R})_{\mathrm{alg}}$ is the positive cone.

THEOREM 1. – *Let (X,h) be a polarized holomorphic symplectic manifold as above. The Mori cone in $H_2(X,\mathbb{R})_{\mathrm{alg}}$ is generated by classes in the positive cone and the images under $\theta^\vee$ of the following:*

$$\{a \in \widetilde{\Lambda}_{\mathrm{alg}} : a^2 \geq -2, |(a,v)| \leq v^2/2, (h, \theta^\vee(a)) > 0\}.$$

This generalizes [4, Theorem 12.2], which treated the case of moduli spaces of sheaves on K3 surfaces. This allows us to compute the full nef cone of X from its Hodge structure once a single ample divisor is given. As another application of our methods, we can bound the length of extremal rays of the Mori cone with respect to the Beauville-Bogomolov pairing:

PROPOSITION 2. – *Let X be a projective holomorphic symplectic manifold as above. Then any extremal ray of its Mori cone contains an effective curve class R with*

$$(R,R) \geq -\frac{n+3}{2}.$$

The value $-\frac{n+3}{2}$ had been conjectured in [15]. Proposition 2 has been obtained independently by Mongardi [25]. His proof is based on twistor deformations, and also applies to non-projective manifolds.

2. Deforming extremal rational curves

In this section, we consider general irreducible holomorphic symplectic manifolds, not necessarily of K3 type. Our arguments are based on the deformation theory of rational curves on holomorphic symplectic manifolds, as first studied in [27]. Recall the definition of a *parallel transport operator* $\phi : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ between manifolds of a fixed deformation type: there is a smooth proper family $\pi : \mathcal{X} \to B$ over a connected analytic space, points $b, b' \in B$ with $\mathcal{X}_b := \pi^{-1}(b) \simeq X$ and $\mathcal{X}_{b'} \simeq X'$, and a continuous path $\gamma : [0,1] \to B, \gamma(0) = b, \gamma(1) = b'$, such that parallel transport along γ induces ϕ.

PROPOSITION 3. – *Let X be a projective holomorphic symplectic variety and R the class of an extremal rational curve $\mathbb{P}^1 \subset X$ with $(R,R) < 0$. Suppose that X' is deformation equivalent to X and $\phi : H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z})$ is a parallel transport operator associated with some family. If $R' := \phi(R)$ is a Hodge class, and if there exists a Kähler class κ on X' with $\kappa.R' > 0$, then a multiple of R' is effective and represented by a cycle of rational curves.*

Note that X' need not be projective here.

Proof. – Fix a proper holomorphic family $\pi : \mathcal{X} \to B$ over an irreducible analytic space B with $X = \mathcal{X}_b$. We claim there exists a rational curve $\xi : \mathbb{P}^1 \to X$ with class $[\xi(\mathbb{P}^1)] \in \mathbb{Q}_{\geq 0}R$ satisfying the following property: for each b'' near b such that R remains algebraic there exists a deformation $\xi_{b''} : \mathbb{P}^1 \to \mathcal{X}_{b''}$ of ξ.

Let ω denote the holomorphic symplectic form on X, $f : X \to Y$ the birational contraction associated with R, E an irreducible component of the exceptional locus of f, Z its image in Y, and F a generic fiber of $E \to Z$. We recall structural results about the contraction f:

– ω restricts to zero on F [18, Lemma 2.7];
– the smooth locus of Z is symplectic with two-form pulling back to $\omega|E$ [18, Thm. 2.5] [26, Prop. 1.6];
– the dimension r of F equals the codimension of E [30, Thm. 1.2].

Second, we review general results about rational curves $\xi : \mathbb{P}^1 \to X$:

– a non-constant morphism $\xi : \mathbb{P}^1 \to X$ deforms in at least a $(2n+1)$-dimensional family [27, Cor. 5.1];
– the fibers of $E \to Z$ are rationally chain connected [11, Cor. 1.6];
– a non-constant morphism $\xi : \mathbb{P}^1 \to F$ deforms in at least a $(2r+1)$-dimensional family [30, Thm. 1.2].

Let $\xi : \mathbb{P}^1 \to F \subset X$ be a rational curve of minimal degree passing through the generic point of F. We do not assume *a priori* that F is smooth. The normal bundle N_ξ was determined completely in [9, §9], which gives a precise classification of F. The fact that rational curves in F deform in $(2r-2)$-dimensional families implies that every rational curve through the generic point of F is doubly dominant, i.e., it passes through *two* generic points of F. Using a bend-and-break argument [9, Thm. 2.8 and 4.2], we may conclude that the normalization of F is isomorphic to $\mathbb{P}^r$. Note that the generic $\xi : \mathbb{P}^1 \to F$ is immersed in X by [19, §3].

Using standard exact sequences for normal bundles and the fact that $\xi : \mathbb{P}^1 \to F$ is immersed in X, one sees that (cf. [9, Lemma 9.4])

$$N_\xi \simeq \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{r-1} \oplus \mathcal{O}_{\mathbb{P}^1}^{2(n-r)} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{r-1}.$$

The crucial point is that $h^1(N_\xi) = 1$. Thus we may apply [27, Cor. 3.2] to deduce that the deformation space of $\xi(\mathbb{P}^1) \subset X$ has dimension $2n-2$; [27, Cor. 3.3] then implies that $\xi(\mathbb{P}^1)$ persists in deformations of X for which R remains a Hodge class. This proves our claim.

EXAMPLE. – The extremality assumption is essential, as shown by an example suggested by Voisin: Let S be a K3 surface arising as a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over a curve of bidegree $(4,4)$ and $X = S^{[2]}$. We may regard $\mathbb{P}^1 \times \mathbb{P}^1 \subset X$ as a Lagrangian surface. Consider a smooth curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1 \subset X$ of bidegree $(1,1)$. The curve C persists only in the codimension-*two* subspace of the deformation space of X where $\mathbb{P}^1 \times \mathbb{P}^1$ deforms (see [29]); note that $N_{C/X} \simeq \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)^2$.

We return to the proof of Proposition 3. Consider the relative Douady space parametrizing rational curves of class $[\xi(\mathbb{P}^1)]$ in fibers of $\mathcal{X} \to B$ and their specializations. Remmert's Proper Mapping theorem [28, Satz 23] implies that its image $B_R \subset B$ is proper and that over each $b' \in B_R$ there exists a cycle of rational curves in $\mathcal{X}'_b$ that is a specialization of $\xi_{b''}(\mathbb{P}^1)$.

To prove the Proposition 3, we need to produce a family $\varpi : \mathcal{X}^+ \to B^+$ over an irreducible base, with both X and X' as fibers, such that X' lies over a point of B^+_R and $R' = \phi(R)$ coincides with $\phi^+(R)$. Here ϕ^+ is the parallel transport mapping associated with ϖ. Then the Proper Mapping theorem would guarantee that R' is in the Mori cone of X'.

LEMMA 4. – *Let X, X', R be as in Proposition 3. There exists a smooth proper family $\varpi\colon \mathcal{X}^+ \to B^+$ over an irreducible analytic space, points $b, b' \in B^+$ with $\mathcal{X}^+_b \simeq X$ and $\mathcal{X}^+_{b'} \simeq X'$, and a section*

$$\rho\colon B^+ \to \mathbb{R}^2\varpi_*\mathbb{Z}$$

of type $(1,1)$, such that $\rho(b) = R$ and $\rho(b') = R'$.

Proof. – This proof is essentially the same as the argument for Proposition 5.12 of [24]. We summarize the key points.

Let $\mathfrak{M}$ denote the moduli space of marked holomorphic symplectic manifolds of K3 type [17, Sec. 1]. Essentially, this is obtained by gluing together all the local Kuranishi spaces of the relevant manifolds. It is non-Hausdorff. Let $\mathfrak{M}^\circ$ denote a connected component of $\mathfrak{M}$ containing X equipped with a suitable marking.

Consider the subspace $\mathfrak{M}^\circ_R$ such that R is of type $(1,1)$ and $\kappa.R > 0$ for *some* Kähler class, which may vary from point to point of the moduli space. This coincides with an open subset of the preimage of the hyperplane $R^\perp$ under the period map P [24, Claim 5.9]. Furthermore, for general periods τ—those for which R is the unique integral class of type $(1,1)$—the preimage $P^{-1}(\tau)$ consists of a single marked manifold [24, Cor. 5.10]. The proof of this in [24] only requires that $(R, R) < 0$. (The Torelli Theorem implies two manifolds share the same period point only if they are bimeromorphic [23, Th. 1.2], but if R is the only algebraic class, the only other bimeromorphic model would not admit a Kähler class κ' with $\kappa'.R > 0$.) Finally, $\mathfrak{M}^\circ_R$ is path-connected by [24, Cor. 5.11].

Choose a path $\gamma : [0,1] \to \mathfrak{M}^\circ_R$ joining X and X' equipped with suitable markings, taking R and R' to the distinguished element R in the reference lattice. Cover the image with a finite number of small connected neighborhoods U_i admitting Kuranishi families. We claim there exists an analytic space B^+

$$\gamma([0,1]) \subset B^+ \subset \cup_{i=1}^m U_i$$

with a universal family. Indeed, we choose B^+ to be an open neighborhood of $\gamma([0,1])$ admitting a deformation retract onto the path, but small enough so it is contained in the union of the U_i's. The topological triviality of B^+ means there is no obstruction to gluing local families. □

This completes the proof of Proposition 3. □

3. Proof of Theorem 1

In the case where $X = M_v(S)$ is a smooth moduli space of Gieseker-stable sheaves (or, indeed, of Bridgeland-stable objects) on a K3 surface S, the statement is proven in [4, Theorem 12.2]. We will prove Theorem 1 by reduction to this case.

The key argument is based on important results of Markman on the cone of movable divisors and its relation to the monodromy group. Let $\mathcal{C}^o_{\mathrm{mov}}$ be the intersection of the movable cone with the positive cone in $H^2(X, \mathbb{R})_{\mathrm{alg}}$. Each wall of the movable cone corresponds to a divisorial contraction of an irreducible exceptional divisor E on some birational model of X; the wall is contained in the orthogonal complement $E^\perp$ of E with respect to the Beauville-Bogomolov form.

THEOREM 5 (Markman). – 1. *Let X be an irreducible holomorphic symplectic manifold. Consider the reflection $\rho_E\colon H^2(X,\mathbb{R}) \to H^2(X,\mathbb{R})$ that leaves $E^\perp$ fixed and sends E to $-E$. Then ρ_E is defined over $\mathbb{Z}$, acts by a monodromy transformation, and extends to a Hodge isometry of the extended lattice $H^2(X) \subset \widetilde{\Lambda}$.*

2. *Let W_{Exc} be the Weyl group generated by reflections ρ_E for all irreducible exceptional divisors E on all marked birational models of X. Then $\mathscr{C}^o_{\mathrm{mov}}$ is a fundamental domain of the action of W_{Exc} on the positive cone.*

Proof. – These results are reviewed in [23, Section 6]. The first statement was originally proved in [24, Corollary 3.6]. The second statement is [23, Lemma 6.22]. (Note that the definition of W_{Exc} in [23, Definition 6.8] is slightly different to the one given above; by [23, Theorem 6.18, part (3)] they are equivalent.) □

COROLLARY 6. – *Let $R \in H_2(X)$ be an algebraic class with $(R,R) < 0$. Then there exists a birational model X' of X, and a parallel transport operator $\psi\colon H^2(X) \to H^2(X')$ such that one of the two following conditions holds:*

1. *$\psi(R)$ generates an extremal ray of the Mori cone.*
2. *Neither $\psi(R)$ nor $-\psi(R)$ is in the Mori cone.*

In either case X' admits a Kähler class κ with $\kappa.\psi(R) > 0$.

Note that ψ may be non-trivial even when $X = X'$.

Proof. – The statement immediately follows from the following claim: *There exists X',ψ such that the orthogonal complement $\psi(R)^\perp$ intersects the nef cone in full dimension, and such that there exists an ample class h with $h.\psi(R) > 0$.* Case (1) corresponds to the case that $\psi(R)^\perp$ contains a wall of the nef cone, and case (2) to the case that $\psi(R)^\perp$ intersects the interior. Either way, we have a Kähler class κ meeting $\psi(R)$ positively.

We first prove the claim with "nef cone" replaced by "movable cone" and "ample class" by "movable class". Since $(R,R) < 0$, the orthogonal complement $R^\perp$ intersects the positive cone; therefore, we can use the Weyl group action of W_{Exc} to force the intersection of $\psi(R)^\perp$ and the movable cone to be full-dimensional. In case $\psi(R)^\perp$ contains a wall of the movable cone, R is proportional to an irreducible exceptional divisor $E^\perp$, and the reflection ρ_E at E can be used to ensure the second condition.

Now we use the chamber decomposition of the movable cone, whose chambers are given by pull-backs of nef cones of marked birational models (see [14]): at least one of the closed chambers intersects $\psi(R)^\perp$ in full dimension, such that part or all of the interior lies on the side with positive intersection with $\psi(R)$. The identification of H^2 of different birational models is induced by a parallel transport operator. □

To prove Theorem 1, we will use the following facts:

– By assumption, there exists a deformation of X to a Hilbert scheme $S^{[n]}$ of a projective K3 surface S; by the surjectivity of the Torelli map for K3 surface, we may further deform S such that a given class in $\widetilde{\Lambda}_X$ becomes algebraic in $H^*(S) \cong \widetilde{\Lambda}_{S^{[n]}}$.
– By [4, Theorem 12.2], the main theorem holds for any moduli space $M_\sigma(v)$ of σ-stable objects of given primitive Mukai vector v on any projective K3 surface (in particular, for any Hilbert scheme).

– By [4, Theorem 1.2], any birational model of $M_\sigma(v)$ is also a moduli space of stable objects (with respect to a different stability condition), and in particular the main Theorem holds.

We will prove Theorem 1 by deformation to the Hilbert scheme X', followed by a second deformation to a birational model X'' of X' using Corollary 6. By abuse of notation, we will use the same letters ϕ, ψ to denote the parallel transport operators on H^2, H_2 and $\widetilde{\Lambda}$ for the deformations from X to X', and from X' to X'', respectively.

We first prove that the Mori cone of (X, h) is contained in the cone described in Theorem 1. Let R be a generator of one of its extremal rays. Let X' be a deformation-equivalent Hilbert scheme with parallel transport operator ϕ such that $\phi(R)$ is algebraic. We apply Corollary 6 to $\phi(R)$; thus there exists a birational model X'' of X' such that $\psi \circ \phi(R)$ satisfies property (1) or (2) as stated in the corollary. By Proposition 3, $\psi \circ \phi(R)$ is effective, excluding case (2); thus $\psi \circ \phi(R)$ is extremal on X''. Since X'' is a moduli space of stable objects on a K3 surface, it is of the form $\theta^\vee(a)$ with a as stated in the theorem. Since the Mori cone is generated by the positive cone and its extremal rays, this proves the claim.

Conversely, consider a class $R = \theta_X^\vee(a)$ where $a \in \widetilde{\Lambda}_{X,\mathrm{alg}}$ satisfies the assumptions in the theorem. We may assume $(R, R) < 0$. Again we deform to a Hilbert scheme X' such that $\phi(R)$ is algebraic, and apply Corollary 6 to $\phi(R)$. Let $R'' := \psi \circ \phi(R) \in H_2(X'')$ and $a'' := \psi \circ \phi(a) \in \widetilde{\Lambda}_{X''}$ be the corresponding classes; since R'' is algebraic, the same holds for a''. By Theorem [4, Theorem 12.2], the class R'' is effective; by the conclusion of the corollary, it has to be extremal. Thus we can apply Proposition 3 to R'', and conclude that R is effective.

This finishes the proof of Theorem 1.

Proof of Proposition 2. – In the case of moduli spaces of sheaves or Bridgeland-stable objects on a projective K3 surfaces, the statement is proved in [4, Proposition 12.6]. By the previous argument, there is a family $\pi : \mathcal{X} \to B$ such that $\mathcal{X}_{b_1} \cong X$ and $\mathcal{X}_{b_0}$ is a moduli space of sheaves on a K3 surface, and such that the parallel transport of R is extremal on $\mathcal{X}_{b_1}$. By [4, Theorem 1.2], there exists a wall in the space of Bridgeland stability conditions contracting R. Let R_0 be the rational curve on $\mathcal{X}_{b_0}$ in the ray $\mathbb{R}_{\geq 0}[R]$ with $(R_0, R_0) \geq -\frac{n+3}{2}$ given by [4, Proposition 12.6]. The curve R_0 is a minimal free curve in a generic fibre of the exceptional locus over B (see [4, Section 14]); therefore, the deformation argument in Proposition 3 applies directly to R_0 (rather than a multiple) and implies the conclusion. □

BIBLIOGRAPHY

[1] E. AMERIK, M. VERBITSKY, Rational curves on hyperkähler manifolds, preprint arXiv:1401.0479.

[2] D. ARCARA, A. BERTRAM, I. COSKUN, J. W. HUIZENGA, The minimal model program for the Hilbert scheme of points on $\mathbb{P}^2$ and Bridgeland stability, *Adv. Math.* **235** (2013), 580–626.

[3] B. BAKKER, A. JORZA, Lagrangian 4-planes in holomorphic symplectic varieties of $\mathrm{K3}^{[4]}$-type, *Cent. Eur. J. Math.* **12** (2014), 952–975.

[4] A. Bayer, E. Macrì, MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations, *Invent. Math.* **198** (2014), 505–590.

[5] A. Bayer, E. Macrì, Y. Toda, Bridgeland stability conditions on threefolds I: Bogomolov-Gieseker type inequalities, *J. Algebraic Geom.* **23** (2014), 117–163.

[6] A. Bertram, I. Coskun, The birational geometry of the Hilbert scheme of points on surfaces, in *Birational geometry, rational curves, and arithmetic*, Springer, New York, 2013, 15–55.

[7] T. Bridgeland, Stability conditions on triangulated categories, *Ann. of Math.* **166** (2007), 317–345.

[8] T. Bridgeland, Stability conditions on $K3$ surfaces, *Duke Math. J.* **141** (2008), 241–291.

[9] K. Cho, Y. Miyaoka, N. I. Shepherd-Barron, Characterizations of projective space and applications to complex symplectic manifolds, in *Higher dimensional birational geometry (Kyoto, 1997)*, Adv. Stud. Pure Math. **35**, Math. Soc. Japan, Tokyo, 2002, 1–88.

[10] I. Coskun, J. W. Huizenga, Interpolation, Bridgeland stability and monomial schemes in the plane, *J. Math. Pures Appl.* **102** (2014), 930–971.

[11] C. D. Hacon, J. Mckernan, On Shokurov's rational connectedness conjecture, *Duke Math. J.* **138** (2007), 119–136.

[12] D. Harvey, B. Hassett, Y. Tschinkel, Characterizing projective spaces on deformations of Hilbert schemes of K3 surfaces, *Comm. Pure Appl. Math.* **65** (2012), 264–286.

[13] B. Hassett, Y. Tschinkel, Rational curves on holomorphic symplectic fourfolds, *Geom. Funct. Anal.* **11** (2001), 1201–1228.

[14] B. Hassett, Y. Tschinkel, Moving and ample cones of holomorphic symplectic fourfolds, *Geom. Funct. Anal.* **19** (2009), 1065–1080.

[15] B. Hassett, Y. Tschinkel, Intersection numbers of extremal rays on holomorphic symplectic varieties, *Asian J. Math.* **14** (2010), 303–322.

[16] J. W. Huizenga, Effective divisors on the Hilbert scheme of points in the plane and interpolation for stable bundles, preprint arXiv:1210.6576, to appear in *J. of Algebraic Geom*.

[17] D. Huybrechts, Compact hyper-Kähler manifolds: basic results, *Invent. Math.* **135** (1999), 63–113.

[18] D. Kaledin, Symplectic singularities from the Poisson point of view, *J. reine angew. Math.* **600** (2006), 135–156.

[19] S. Kebekus, Families of singular rational curves, *J. Algebraic Geom.* **11** (2002), 245–256.

[20] S. J. Kovács, The cone of curves of a $K3$ surface, *Math. Ann.* **300** (1994), 681–691.

[21] E. Markman, On the monodromy of moduli spaces of sheaves on $K3$ surfaces, *J. Algebraic Geom.* **17** (2008), 29–99.

[22] E. Markman, Integral constraints on the monodromy group of the hyperKähler resolution of a symmetric product of a $K3$ surface, *Internat. J. Math.* **21** (2010), 169–223.

[23] E. Markman, A survey of Torelli and monodromy results for holomorphic-symplectic varieties, in *Complex and differential geometry*, Springer Proc. Math. **8**, Springer, Heidelberg, 2011, 257–322.

[24] E. Markman, Prime exceptional divisors on holomorphic symplectic varieties and monodromy reflections, *Kyoto J. Math.* **53** (2013), 345–403.

[25] G. Mongardi, A note on the Kähler and Mori cones of manifolds of $K3^{[n]}$ type, preprint arXiv:1307.0393.

[26] Y. Namikawa, Deformation theory of singular symplectic n-folds, *Math. Ann.* **319** (2001), 597–623.

[27] Z. Ran, Hodge theory and deformations of maps, *Compositio Math.* **97** (1995), 309–328.

[28] R. Remmert, Holomorphe und meromorphe Abbildungen komplexer Räume, *Math. Ann.* **133** (1957), 328–370.

[29] C. Voisin, Sur la stabilité des sous-variétés lagrangiennes des variétés symplectiques holomorphes, in *Complex projective geometry (Trieste, 1989/Bergen, 1989)*, London Math. Soc. Lecture Note Ser. **179**, Cambridge Univ. Press, Cambridge, 1992, 294–303.

[30] J. Wierzba, Contractions of symplectic varieties, *J. Algebraic Geom.* **12** (2003), 507–534.

[31] S. Yanagida, K. Yoshioka, Bridgeland's stabilities on abelian surfaces, *Math. Z.* **276** (2014), 571–610.

[32] K. Yoshioka, Bridgeland's stability and the positive cone of the moduli spaces of stable objects on an Abelian surface, preprint arXiv:1206.4838.

(Manuscrit reçu le 15 juillet 2013 ;
accepté, après révision, le 22 juillet 2014.)

Arend Bayer
University of Edinburgh
School of Mathematics and Maxwell Institute
James Clerk Maxwell Building
The King's Buildings, Mayfield Road
Edinburgh EH9 3JZ, Scotland
E-mail: arend.bayer@ed.ac.uk

Brendan Hassett
Department of Mathematics
Rice University
MS 136, Houston, Texas 77251-1892, USA
E-mail: hassett@rice.edu

Yuri Tschinkel
Courant Institute
New York University
New York, NY 11012, USA

Simons Foundation
160 Fifth Avenue
New York, NY 10010, USA
E-mail: tschinkel@cims.nyu.edu