ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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Publiées avec le concours du Centre National de la Recherche Scientifique

NONLINEAR ANALYSIS
WITH RESURGENT FUNCTIONS

BY David SAUZIN

ABSTRACT. – We provide estimates for the convolution product of an arbitrary number of "resurgent functions", that is holomorphic germs at the origin of $\mathbb{C}$ that admit analytic continuation outside a closed discrete subset of $\mathbb{C}$ which is stable under addition. Such estimates are then used to perform nonlinear operations like substitution in a convergent series, composition or functional inversion with resurgent functions, and to justify the rules of "alien calculus"; they also yield implicitly defined resurgent functions. The same nonlinear operations can be performed in the framework of Borel-Laplace summability.

RÉSUMÉ. – Nous obtenons des estimations pour le produit de convolution d'un nombre arbitraire de « fonctions résurgentes », c'est-à-dire de fonctions holomorphes à l'origine du plan complexe qui se prolongent anaytiquement en dehors d'un sous-ensemble fermé discret stable par addition. Ces estimations sont ensuite utilisées pour effectuer des opérations non linéaires avec les fonctions résurgentes, comme la substitution dans une série convergente, la composition ou l'inversion fonctionnelle, et pour justifier les règles du « calcul étranger » ; elles permettent aussi d'obtenir un théorème des fonctions résurgentes implicites. Les mêmes opérations non linéaires peuvent être effectuées dans le cadre de la sommabilité de Borel-Laplace.

1. Introduction

In the 1980s, to deal with local analytic problems of classification of dynamical systems, J. Écalle started to develop his theory of resurgent functions and alien derivatives [10, 11], [13], which is an efficient tool for dealing with divergent series arising from complex dynamical systems or WKB expansions, analytic invariants of differential or difference equations, linear and nonlinear Stokes phenomena [25, 26], [12, 3], [6, 2], [7], [20, 29], [34, 4], [35, 22], [24, 17], [32, 21], [8, 9]; connections were also recently found with Painlevé asymptotics [19], Quantum Topology [18, 5] and Wall Crossing [23].

The starting point in Écalle's theory is the definition of certain subalgebras of the algebra of formal power series by means of the formal Borel transform

$$(1) \qquad \mathcal{B} \colon \tilde{\varphi}(z) = \sum_{n=0}^{\infty} a_n z^{-n-1} \in z^{-1}\mathbb{C}[[z^{-1}]] \mapsto \hat{\varphi}(\zeta) = \sum_{n=0}^{\infty} a_n \frac{\zeta^n}{n!} \in \mathbb{C}[[\zeta]]$$

(using negative power expansions in the left-hand side and changing the name of the indeterminate from z to ζ are just convenient conventions).

It turns out that, for a lot of interesting functional equations, one can find formal solutions which are divergent for all z and whose Borel transforms define holomorphic germs at 0 with particular properties of analytic continuation. The simplest examples are the Euler series [3, 32], which can be written $\tilde{\varphi}^{\mathrm{E}}(z) = \sum_{n=0}^{\infty}(-1)^n n! z^{-n-1}$ and solves a first-order linear non-homogeneous differential equation, and the Stirling series [10, Vol. 3]

$$\tilde{\varphi}^{\mathrm{S}}(z) = \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)} z^{-2k+1}$$

(here expressed in terms of the Bernoulli numbers), solution of a linear non-homogeneous difference equation derived from the functional equation for Euler's Gamma function by taking logarithms. In both examples the Borel transform gives rise to convergent series with a meromorphic extension to the ζ-plane, namely $(1 + \zeta)^{-1}$ for the Euler series and $\zeta^{-2}\left(\frac{\zeta}{2}\coth\frac{\zeta}{2} - 1\right)$ for the Stirling series (see [37]). In fact, holomorphic germs at 0 with meromorphic or algebraic analytic continuation are examples of "resurgent functions"; more generally, what is required for a resurgent function is the possibility of following the analytic continuation without encountering natural barriers.

One is thus led to distinguish certain subspaces $\hat{\mathscr{R}}$ of $\mathbb{C}\{\zeta\}$, characterized by properties of analytic continuation which ensure a locally discrete set of singularities for each of its members (and which do not preclude multiple-valuedness of the analytic continuation), and to consider

$$\tilde{\mathscr{R}} := \mathbb{C} \oplus \mathcal{B}^{-1}(\hat{\mathscr{R}}) \subset \mathbb{C}[[z^{-1}]].$$

Typically one has the strict inclusion $\mathbb{C}\{z^{-1}\} \subsetneq \tilde{\mathscr{R}}$ but the divergent series in $\tilde{\mathscr{R}}$ can be "summed" by means of Borel-Laplace summation. The formal series in $\tilde{\mathscr{R}}$ as well as the holomorphic functions whose germ at 0 belongs to $\hat{\mathscr{R}}$ are termed "resurgent". (One also defines, for each $\omega \in \mathbb{C}^*$, an "alien operator" which measures the singularities at ω of certain branches of the analytic continuation of $\hat{\varphi}$.)

Later we shall be more specific about the definition of $\hat{\mathscr{R}}$. This article is concerned with the convolution of resurgent functions: the convolution in $\mathbb{C}\{\zeta\}$ is the commutative associative product defined by

$$(2) \qquad \hat{\varphi}_1 * \hat{\varphi}_2(\zeta) = \int_0^{\zeta} \hat{\varphi}_1(\zeta_1)\hat{\varphi}_2(\zeta - \zeta_1)\,\mathrm{d}\zeta_1 \qquad \text{for } |\zeta| \text{ small enough,}$$

for any $\hat{\varphi}_1, \hat{\varphi}_2 \in \mathbb{C}\{\zeta\}$, which reflects the Cauchy product of formal series via the formal Borel transform:

$$\mathcal{B}\tilde{\varphi}_1 = \hat{\varphi}_1 \text{ and } \mathcal{B}\tilde{\varphi}_2 = \hat{\varphi}_2 \implies \mathcal{B}(\tilde{\varphi}_1\tilde{\varphi}_2) = \hat{\varphi}_1 * \hat{\varphi}_2.$$

Since the theory was designed to deal with nonlinear problems, it is of fundamental importance to control the convolution product of resurgent functions; however, this requires to

follow the analytic continuation of the function defined by (2), which turns out not to be an easy task. In fact, probably the greatest difficulties in understanding and applying resurgence theory are connected with the problem of controlling the analytic continuation of functions defined by such integrals or by analogous multiple integrals. Even the mere stability under convolution of the spaces $\hat{\mathscr{R}}$ is not obvious [10, 3], [30, 36].

We thus need to estimate the convolution product of two or more resurgent functions, both for concrete manipulations of resurgent functions in nonlinear contexts and for the foundations of the resurgence theory. For instance, such estimates will allow us to check that, when we come back to the resurgent series via $\mathscr{B}$, the exponential of a resurgent series is resurgent and that more generally one can substitute resurgent series in convergent power expansions, or define implicitly a resurgent series, or develop "alien calculus" when manipulating Écalle's alien derivatives. They will also show that the group of "formal tangent-to-identity diffeomorphisms at ∞", i.e., the group (for the composition law) $z + \mathbb{C}[[z^{-1}]]$, admits $z + \tilde{\mathscr{R}}$ as a subgroup, which is particularly useful for the study of holomorphic tangent-to-identity diffeomorphisms f (in this classical problem of local holomorphic dynamics [27], the Fatou coordinates have the same resurgent asymptotic expansion, the so-called direct iterator $f^* \in z + \tilde{\mathscr{R}}$ of [10]; thus its inverse, the inverse iterator, also belongs to $z + \tilde{\mathscr{R}}$, as well as its exponential, which appears in the Bridge equation connected with the "horn maps"—see § 3.3).

Such results of stability of the algebra of resurgent series under nonlinear operations are mentioned in Écalle's works, however the arguments there are sketchy and it was desirable to provide a proof.[1] Indeed, the subsequent authors dealing with resurgent series either took such results for granted or simply avoided resorting to them. The purpose of this article is to give clear statements with rigorous and complete proofs, so as to clarify the issue and contribute to make resurgence theory more accessible, hopefully opening the way for new applications of this powerful theory.

In this article, we shall deal with a particular case of resurgence called Ω-continuability or Ω-resurgence, which means that we fix in advance a discrete subset Ω of $\mathbb{C}$ and restrict ourselves to those resurgent functions whose analytic continuations have no singular point outside of Ω. Many interesting cases are already covered by this definition (one encounters Ω-continuable germs with $\Omega = \mathbb{Z}$ when dealing with differential equations formally conjugate to the Euler equation or in the study of the saddle-node singularities [11, 35], or with $\Omega = 2\pi i\mathbb{Z}$ when dealing with certain difference equations like Abel's equation for tangent-to-identity diffeomorphisms [10, 34], [8]). We preferred to restrict ourselves to this situation so as to make our method more transparent, even if more general definitions of resurgence can be handled—see Section 3.4. An outline of the article is as follows:

– In Section 2, we recall the precise definition of the corresponding algebras of resurgent functions, denoted by $\hat{\mathscr{R}}_\Omega$, and state Theorem 1, which is our main result on the control of the convolution product of an arbitrary number of Ω-continuable functions.

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[1] This was one of the tasks undertaken in the seminal book [3] but, despite its merits, one cannot say that this book clearly settled this particular issue: the proof of the estimates for the convolution is obscure and certainly contains at least one mistake (see Remark 7.3).

– In Section 3, we give applications to the construction of a Fréchet algebra structure on $\tilde{\hat{\mathscr{R}}}_\Omega$ (Theorem 2) and to the stability of Ω-resurgent series under substitution (Theorem 3), implicit function (Theorem 4) and composition (Theorem 5); we also mention other possible applications and similar results for 1-summable series.
– The proof of Theorem 1 is given in Sections 4–7.
– Finally, there is an appendix on a few facts of the theory of currents which are used in the proof of the main theorem.

Our method consists in representing the analytic continuation of a convolution product as the integral of a holomorphic n-form on a singular n-simplex obtained as a suitable deformation of the standard n-simplex; we explain in Sections 4–5 what kind of deformations ("adapted origin-fixing isotopies" of the identity) are licit in order to provide the analytic continuation and how to produce them. We found the theory of currents very convenient to deal with our integrals of holomorphic forms, because it allowed us to content ourselves with little regularity: the deformations we use are only Lipschitz continuous, because they are built from the flow of non-autonomous Lipschitz vector fields—see Section 6. Section 7 contains the last part of the proof, which consists in providing appropriate estimates.

2. The convolution of Ω-continuable germs

NOTATION 2.1. – *For any $R > 0$ and $\zeta_0 \in \mathbb{C}$ we use the notations $D(\zeta_0, R) := \{\, \zeta \in \mathbb{C} \mid |\zeta - \zeta_0| < R \,\}$, $\mathbb{D}_R := D(0, R)$, $\mathbb{D}_R^* := \mathbb{D}_R \setminus \{0\}$.*

Let Ω be a closed, discrete subset of $\mathbb{C}$ containing 0. We set

$$\rho(\Omega) := \min \{\, |\omega|, \ \omega \in \Omega \setminus \{0\} \,\}.$$

Recall [36] that the space $\hat{\mathscr{R}}_\Omega$ of all Ω-continuable germs is the subspace of $\mathbb{C}\{\zeta\}$ which can be defined by the fact that, for arbitrary $\zeta_0 \in \mathbb{D}_{\rho(\Omega)}$,

$$\hat{\varphi} \in \hat{\mathscr{R}}_\Omega \quad \Longleftrightarrow \quad \left|\begin{array}{l} \hat{\varphi} \text{ germ of holomorphic function of } \mathbb{D}_{\rho(\Omega)} \text{ admitting analytic continuation} \\ \text{along any path } \gamma\colon [0,1] \to \mathbb{C} \text{ such that } \gamma(0) = \zeta_0 \text{ and } \gamma\big((0,1]\big) \subset \mathbb{C} \setminus \Omega. \end{array}\right.$$

For example, for the Euler series, resp. the Stirling series, the Borel transform belongs to $\hat{\mathscr{R}}_\Omega$ as soon as $1 \in \Omega$, resp. $2\pi i \mathbb{Z}^* \subset \Omega$.

It is convenient to rephrase the property of Ω-continuability as holomorphy on a certain Riemann surface spread over the complex plane, $(\mathscr{S}_\Omega, \pi_\Omega)$.

DEFINITION 2.2. – *Let $I := [0,1]$ and consider the set $\mathscr{P}_\Omega$ of all paths $\gamma\colon I \to \mathbb{C}$ such that either $\gamma(I) = \{0\}$ or $\gamma(0) = 0$ and $\gamma\big((0,1]\big) \subset \mathbb{C} \setminus \Omega$. We denote by*

$$\mathscr{S}_\Omega := \mathscr{P}_\Omega / \sim$$

the quotient set of $\mathscr{P}_\Omega$ by the equivalence relation $\sim$ defined by

$$\gamma \sim \gamma' \quad \Longleftrightarrow \quad \exists (\gamma_s)_{s \in I} \text{ such that } \begin{cases} \text{for each } s \in I, \ \gamma_s \in \mathscr{P}_\Omega \text{ and } \gamma_s(1) = \gamma(1) \\ (s,t) \in I \times I \mapsto \gamma_s(t) \in \mathbb{C} \text{ is continuous, } \gamma_0 = \gamma, \ \gamma_1 = \gamma' \end{cases}$$

for $\gamma, \gamma' \in \mathscr{P}_\Omega$ (homotopy with fixed endpoints). The map $\gamma \in \mathscr{P}_\Omega \mapsto \gamma(1) \in \{0\} \cup \mathbb{C} \setminus \Omega$ passes to the quotient and defines the "projection"

$$(3) \qquad \pi_\Omega \colon \zeta \in \mathscr{S}_\Omega \to \overset{\bullet}{\zeta} \in \{0\} \cup \mathbb{C} \setminus \Omega.$$

We equip $\mathscr{S}_\Omega$ with the unique structure of Riemann surface which turns π_Ω into a local biholomorphism. The equivalence class of the trivial path $\gamma(t) \equiv 0$ is denoted by 0_Ω and called the origin of $\mathscr{S}_\Omega$.

We obtain a connected, simply connected Riemann surface $\mathscr{S}_\Omega$, which is somewhat analogous to the universal cover of $\mathbb{C} \setminus \Omega$ except for the special role played by 0 and 0_Ω: since we assumed $0 \in \Omega$, the equivalence class 0_Ω of the trivial path is reduced to the trivial path and is the only point of $\mathscr{S}_\Omega$ which projects onto 0. It belongs to the *principal sheet of $\mathscr{S}_\Omega$*, defined as the set of all $\zeta \in \mathscr{S}_\Omega$ which can be represented by a line segment (i.e., such that the path $t \in [0,1] \mapsto t\overset{\bullet}{\zeta}$ belongs to $\mathscr{P}_\Omega$ and represents ζ); observe that π_Ω induces a biholomorphism from the principal sheet of $\mathscr{S}_\Omega$ to the cut plane $U_\Omega := \mathbb{C} \setminus \bigcup_{\omega \in \Omega \setminus \{0\}} \omega[1, +\infty)$.

Any holomorphic function of $\mathscr{S}_\Omega$ identifies itself with a convergent germ at the origin of $\mathbb{C}$ which admits analytic continuation along all the paths of $\mathscr{P}_\Omega$, so that

$$\hat{\mathscr{R}}_\Omega \simeq \mathscr{O}(\mathscr{S}_\Omega)$$

(see [10, 34]). We usually use the same symbol $\hat{\varphi}$ for a function of $\mathscr{O}(\mathscr{S}_\Omega)$ or the corresponding germ of holomorphic function at 0 (i.e., its Taylor series). Each $\hat{\varphi} \in \hat{\mathscr{R}}_\Omega$ has a well-defined principal branch holomorphic in U_Ω, obtained (via π_Ω) by restriction to the principal sheet of $\mathscr{S}_\Omega$, for which 0 is a regular point, but the points of $\mathscr{S}_\Omega$ which lie outside of the principal sheet correspond to branches of the analytic continuation which might have a singularity at 0 (for instance, as soon as $\{0, 1\} \subset \Omega$, the Taylor series $\sum_{n \geq 0} \frac{\zeta^n}{n+1} = -\frac{1}{\zeta} \log(1 - \zeta)$ defines a member of $\hat{\mathscr{R}}_\Omega$ of which all branches except the principal one have a simple pole at 0).

From now on we assume that Ω is stable under addition. According to [36], this ensures that $\hat{\mathscr{R}}_\Omega$ is stable under convolution. Our aim is to provide explicit bounds for the analytic continuation of a convolution product of two or more factors belonging to $\hat{\mathscr{R}}_\Omega$.

It is well-known that, if $U \subset \{0\} \cup \mathbb{C} \setminus \Omega$ is open and star-shaped with respect to 0 (as is U_Ω) and two functions $\hat{\varphi}_1, \hat{\varphi}_2$ are holomorphic in U, then their convolution product has an analytic continuation to U which is given by the very same Formula (2); by induction, one gets a representation of a product of n factors $\hat{\varphi}_j \in \mathscr{O}(U)$ as an iterated integral,

$$(4) \quad \hat{\varphi}_1 * \cdots * \hat{\varphi}_n(\zeta)$$

$$= \int_0^\zeta \mathrm{d}\zeta_1 \int_0^{\zeta - \zeta_1} \mathrm{d}\zeta_2 \cdots \int_0^{\zeta - (\zeta_1 + \cdots + \zeta_{n-2})} \mathrm{d}\zeta_{n-1} \, \hat{\varphi}_1(\zeta_1) \cdots \hat{\varphi}_{n-1}(\zeta_{n-1}) \hat{\varphi}_n(\zeta - (\zeta_1 + \cdots + \zeta_{n-1}))$$

for any $\zeta \in U$, which leads to

$$(5) \qquad |\hat{\varphi}_1 * \cdots * \hat{\varphi}_n(\zeta)| \leq \frac{|\zeta|^{n-1}}{(n-1)!} \max_{[0,\zeta]} |\hat{\varphi}_1| \cdots \max_{[0,\zeta]} |\hat{\varphi}_n|, \qquad \zeta \in U.$$

This allows one to control convolution products in the principal sheet of $\mathscr{S}_\Omega$ (which is already sufficient to deal with 1-summability issues—see Section 3.5) but, to reach the other sheets, Formula (2) must be replaced by something else, as explained e.g., in [36]. What about the bounds for a product of n factors then? To state our main result, we introduce

NOTATION 2.3. – *The function $R_\Omega \colon \mathscr{S}_\Omega \to (0, +\infty)$ is defined by*

$$(6) \quad \zeta \in \mathscr{S}_\Omega \;\mapsto\; R_\Omega(\zeta) := \begin{cases} \operatorname{dist}\left(\overset{\bullet}{\zeta}, \Omega \setminus \{0\}\right) & \textit{if } \zeta \textit{ belongs to the principal sheet of } \mathscr{S}_\Omega \\[2mm] \operatorname{dist}\left(\overset{\bullet}{\zeta}, \Omega\right) & \textit{if not} \end{cases}$$

(where $\overset{\bullet}{\zeta}$ is the shorthand for $\pi_\Omega(\zeta)$ defined by (3)). For $\delta, L > 0$, we set

$$(7) \quad \mathscr{K}_{\delta,L}(\Omega) := \big\{\, \zeta \in \mathscr{S}_\Omega \mid \exists \gamma \text{ path of } \mathscr{S}_\Omega \text{ with endpoints } 0_\Omega \text{ and } \zeta, \text{ of length} \leq L,$$
$$\text{such that } R_\Omega(\gamma(t)) \geq \delta \text{ for all } t \,\big\}.$$

Informally, $\mathscr{K}_{\delta,L}(\Omega)$ consists of the points of $\mathscr{S}_\Omega$ which can be joined to 0_Ω by a path of length $\leq L$ "staying at distance $\geq \delta$ from the boundary".[2] Observe that $\big(\mathscr{K}_{\delta,L}(\Omega)\big)_{\delta,L>0}$ is an exhaustion of $\mathscr{S}_\Omega$ by compact subsets. If $L + \delta < \rho(\Omega)$, then $\mathscr{K}_{\delta,L}(\Omega)$ is just the lift of the closed disc $\overline{\mathbb{D}}_L$ in the principal sheet of $\mathscr{S}_\Omega$.

THEOREM 1. – *Let $\Omega \subset \mathbb{C}$ be closed, discrete, stable under addition, with $0 \in \Omega$. Let $\delta, L > 0$ with $\delta < \rho(\Omega)$ and*

$$(8) \qquad C := \rho(\Omega)\, e^{3 + 6L/\delta}, \qquad \delta' := \frac{1}{2}\rho(\Omega)\, e^{-2 - 4L/\delta}, \qquad L' := L + \frac{\delta}{2}.$$

Then, for any $n \geq 1$ and $\hat{\varphi}_1, \ldots, \hat{\varphi}_n \in \hat{\mathscr{R}}_\Omega$,

$$(9) \qquad \max_{\mathscr{K}_{\delta,L}(\Omega)} |\hat{\varphi}_1 * \cdots * \hat{\varphi}_n| \leq \frac{2}{\delta} \cdot \frac{C^n}{n!} \cdot \max_{\mathscr{K}_{\delta',L'}(\Omega)} |\hat{\varphi}_1| \cdots \max_{\mathscr{K}_{\delta',L'}(\Omega)} |\hat{\varphi}_n|.$$

The proof of Theorem 1 will start in Section 4. We emphasize that $\delta, \delta', L, L', C$ do not depend on n, which is important in applications.

REMARK 2.4. – In fact, *a posteriori*, one can remove the assumption $0 \in \Omega$. Suppose indeed that Ω is a non-empty closed discrete subset of $\mathbb{C}$ which does *not* contain 0. Defining the space $\hat{\mathscr{R}}_\Omega$ of Ω-continuable germs as above [36], we then get $\hat{\mathscr{R}}_\Omega \simeq \mathscr{O}(\mathscr{S}_\Omega)$, where $\mathscr{S}_\Omega$ is the universal cover of $\mathbb{C} \setminus \Omega$ with base point at the origin (the fibre of 0 is no longer exceptional). Clearly $\hat{\mathscr{R}}_\Omega \subset \hat{\mathscr{R}}_{\{0\} \cup \Omega}$, but the inclusion is strict, because Ω-continuable germs are required to extend analytically through 0 even when following a path which has turned around the points of Ω and e.g., $\sum_{n \geq 0} \frac{\zeta^n}{(n+1)\omega^{n+1}} = -\frac{1}{\zeta}\log(1 - \frac{\zeta}{\omega})$ is in $\hat{\mathscr{R}}_{\{0\} \cup \Omega}$ but not in $\hat{\mathscr{R}}_\Omega$ for any $\omega \in \Omega$. Suppose moreover that Ω is stable under addition. It is shown in [36] that also in this case $\hat{\mathscr{R}}_\Omega$ is stable under convolution. *One can adapt all the results of this article to this case.* It is sufficient to observe that any point ζ of $\mathscr{S}_\Omega$ can be defined by a path $\gamma \colon [0,1] \to \mathbb{C}$ such that $\gamma(0) \in \mathbb{D}_{\rho(\Omega)}$, $\gamma\big((0,1)\big) \cap (\Omega \cup \{0\}) = \varnothing$ and $\gamma(1) \notin \Omega$; if $\gamma(1) \neq 0$, then the situation is explicitly covered by this article; if $\gamma(1) = 0$, then we can still apply our results to the neighbourhing points and make use of the maximum principle.

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[2] Given $\zeta \in \mathscr{S}_\Omega$, observe that any $\hat{\varphi} \in \hat{\mathscr{R}}_\Omega$ induces a function holomorphic in $D\big(\overset{\bullet}{\zeta}, R_\Omega(\zeta)\big)$ and $R_\Omega(\zeta)$ is maximal for that property. The idea is that R_Ω measures the distance to the closest possibly singular point, i.e., the distance to Ω except that on the principal sheet 0 must not be considered as a possibly singular point.

3. Application to nonlinear operations with Ω-resurgent series

3.1. Fréchet algebra structure on $\tilde{\mathscr{R}}_\Omega$

Recall that Ω is a closed discrete subset of $\mathbb{C}$ which contains 0 and is stable under addition. The space of Ω-resurgent series is

$$\tilde{\mathscr{R}}_\Omega = \mathbb{C} \oplus \mathscr{B}^{-1}(\hat{\mathscr{R}}_\Omega).$$

As a vector space, it is isomorphic to $\mathbb{C} \times \mathscr{O}(\mathscr{S}_\Omega)$. We now define seminorms on $\tilde{\mathscr{R}}_\Omega$ which will ease the exposition.

DEFINITION 3.1. – *Let $K \subset \mathscr{S}_\Omega$ be compact. We define the seminorm $\|\cdot\|_K \colon \tilde{\mathscr{R}}_\Omega \to \mathbb{R}^+$ by*

$$\tilde{\phi} \in \tilde{\mathscr{R}}_\Omega \mapsto \|\tilde{\phi}\|_K := |c| + \max_K |\hat{\varphi}|,$$

where $\tilde{\phi} = c + \mathscr{B}^{-1}\hat{\varphi}$, $c \in \mathbb{C}$, $\hat{\varphi} \in \hat{\mathscr{R}}_\Omega$.

Choosing $K_N = \mathscr{K}_{\delta_N, L_N}(\Omega)$, $N \in \mathbb{N}^*$, with any pair of sequences $\delta_N \downarrow 0$ and $L_N \uparrow \infty$ (so that $\mathscr{S}_\Omega$ is the increasing union of the compact sets K_N), we get a countable family of seminorms which defines a structure of Fréchet space on $\tilde{\mathscr{R}}_\Omega$. A direct consequence of Theorem 1 is the continuity of the Cauchy product for this Fréchet structure. More precisely:

THEOREM 2. – *For any K there exist $K' \supset K$ and $C > 0$ such that, for any $n \geq r \geq 0$,*

$$\tag{10} \|\tilde{\phi}_1 \cdots \tilde{\phi}_n\|_K \leq \frac{C^n}{r!} \|\tilde{\phi}_1\|_{K'} \cdots \|\tilde{\phi}_n\|_{K'}$$

for every sequence $(\tilde{\phi}_1, \ldots, \tilde{\phi}_n)$ of Ω-resurgent series, r of which have no constant term.

In particular, $\tilde{\mathscr{R}}_\Omega$ is a Fréchet algebra.

Proof. – Let us fix K compact and choose $\delta, L > 0$ so that $K \subset \mathscr{K}_{\delta,L}(\Omega)$. Let δ', L' be as in (8) and $K' := \mathscr{K}_{\delta',L'}(\Omega)$. According to Theorem 1, we can choose $C \geq 1$ large enough so that for any $m \geq 1$ and $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_m \in \mathscr{B}^{-1}(\hat{\mathscr{R}}_\Omega)$,

$$\tag{11} \|\tilde{\varphi}_1 \cdots \tilde{\varphi}_m\|_K \leq \frac{C^m}{m!} \|\tilde{\varphi}_1\|_{K'} \cdots \|\tilde{\varphi}_m\|_{K'}.$$

Let $n \geq rs := n - r$. Given n resurgent series among which r have no constant term, we can label them so that

$$\tilde{\phi}_1 = c_1 + \tilde{\varphi}_1, \ \ldots, \ \tilde{\phi}_s = c_s + \tilde{\varphi}_s, \ \ \tilde{\phi}_{s+1} = \tilde{\varphi}_{s+1}, \ \ldots, \ \tilde{\phi}_n = \tilde{\varphi}_n,$$

with $c_1, \ldots, c_s \in \mathbb{C}$ and $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_n \in \mathscr{B}^{-1}(\hat{\mathscr{R}}_\Omega)$. Then $\tilde{\phi}_1 \cdots \tilde{\phi}_n = c + \tilde{\psi}$ with

$$\tilde{\psi} = \sum_I c_{i_1} \cdots c_{i_p} \tilde{\varphi}_{j_1} \cdots \tilde{\varphi}_{j_q} \tilde{\varphi}_{s+1} \cdots \tilde{\varphi}_n \in \mathscr{B}^{-1}(\hat{\mathscr{R}}_\Omega),$$

where either $r \geq 1$, $c = 0$ and the summation is over all subsets $I = \{i_1, \ldots, i_p\}$ of $\{1, \ldots, s\}$ (of any cardinality p), with $\{j_1, \ldots, j_q\} := \{1, \ldots, s\} \setminus I$, or $r = 0$, $c = c_1 \cdots c_n$ and the summation is restricted to the proper subsets of $\{1, \ldots, n\}$. Using inequality (11), we get

$$\|\tilde{\phi}_1 \cdots \tilde{\phi}_n\|_K \leq \sum_I \frac{C^{q+r}}{(q+r)!} |c_{i_1} \cdots c_{i_p}| \, \|\tilde{\varphi}_{j_1}\|_{K'} \cdots \|\tilde{\varphi}_{j_q}\|_{K'} \|\tilde{\varphi}_{s+1}\|_{K'} \cdots \|\tilde{\varphi}_n\|_{K'}$$

$$\leq \frac{C^n}{r!} \|\tilde{\phi}_1\|_{K'} \cdots \|\tilde{\phi}_n\|_{K'}$$

(even if $r = 0$, in which case we include $I = \{1, \ldots, n\}$ in the summation and use $C \geq 1$).

The continuity of the multiplication in $\tilde{\mathscr{R}}_\Omega$ follows, as a particular case when $n = 2$.   □

REMARK 3.2. – $\tilde{\mathscr{R}}_\Omega$ is even a differential Fréchet algebra since $\frac{d}{dz}$ induces a continuous derivation of $\tilde{\mathscr{R}}_\Omega$. Indeed, the very definition of $\mathscr{B}$ in (1) shows that

$$\tilde{\phi} = c + \mathscr{B}^{-1}\hat{\varphi} \quad \Longrightarrow \quad \frac{d\tilde{\phi}}{dz} = \mathscr{B}^{-1}\hat{\psi} \text{ with } \hat{\psi}(\zeta) = -\zeta\hat{\varphi}(\zeta),$$

whence $\|\frac{d\tilde{\phi}}{dz}\|_K \leq D(K)\|\tilde{\phi}\|_K$ with $D(K) = \max_{\zeta \in K}|\zeta|$.

3.2. Substitution and implicit resurgent functions

DEFINITION 3.3. – *For any* $r \in \mathbb{N}^*$, *we define* $\tilde{\mathscr{R}}_\Omega\{w_1, \ldots, w_r\}$ *as the subspace of* $\tilde{\mathscr{R}}_\Omega[[w_1, \ldots, w_r]]$ *consisting of all formal power series*

$$\tilde{H} = \sum_{\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r} \tilde{H}_{\boldsymbol{k}}(z)\, w_1^{k_1} \cdots w_r^{k_r}$$

with coefficients $\tilde{H}_{\boldsymbol{k}} = \tilde{H}_{\boldsymbol{k}}(z) \in \tilde{\mathscr{R}}_\Omega$ *such that, for every compact* $K \subset \mathscr{S}_\Omega$, *there exist positive numbers* A_K, B_K *such that*

(12)
$$\|\tilde{H}_{\boldsymbol{k}}\|_K \leq A_K\, B_K^{|\boldsymbol{k}|}$$

for all $\boldsymbol{k} \in \mathbb{N}^r$ *(with the notation* $|\boldsymbol{k}| = k_1 + \cdots + k_r$*).*

The idea is to consider formal series "resurgent in z and convergent in $w_1, \ldots, w_r$". We now show that one can substitute resurgent series in such a convergent series. Observe that $\tilde{\mathscr{R}}_\Omega\{w_1, \ldots, w_r\}$ can be considered as a subspace of $\mathbb{C}[[z^{-1}, w_1, \ldots, w_r]]$.

THEOREM 3. – (i) *The space* $\tilde{\mathscr{R}}_\Omega\{w_1, \ldots, w_r\}$ *is a subalgebra of* $\mathbb{C}[[z^{-1}, w_1, \ldots, w_r]]$.
(ii) *Suppose that* $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r \in \tilde{\mathscr{R}}_\Omega$ *have no constant term. Then for any*

$$\tilde{H} = \sum \tilde{H}_{\boldsymbol{k}}\, w_1^{k_1} \cdots w_r^{k_r} \in \tilde{\mathscr{R}}_\Omega\{w_1, \ldots, w_r\},$$

the series

$$\tilde{H}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r) := \sum_{\boldsymbol{k} \in \mathbb{N}^r} \tilde{H}_{\boldsymbol{k}}\, \tilde{\varphi}_1^{k_1} \cdots \tilde{\varphi}_r^{k_r} \in \mathbb{C}[[z^{-1}]]$$

is convergent in $\tilde{\mathscr{R}}_\Omega$ *and, for every compact* $K \subset \mathscr{S}_\Omega$, *there exist a compact* $K' \supset K$ *and a constant* $C > 0$ *so that*

$$\|\tilde{H}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r)\|_K \leq CA_{K'}\, e^{CB_{K'}\left(\|\tilde{\varphi}_1\|_{K'} + \cdots + \|\tilde{\varphi}_r\|_{K'}\right)}$$

(with notations similar to those of Definition 3.3 for $A_{K'}, B_{K'}$*).*
(iii) *The map* $\tilde{H} \in \tilde{\mathscr{R}}_\Omega\{w_1, \ldots, w_r\} \mapsto \tilde{H}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r) \in \tilde{\mathscr{R}}_\Omega$ *is an algebra homomorphism.*

Proof. – The proof of the first statement is left as an exercise. Observe that the series of formal series

$$\tilde{\chi} = \sum_{\boldsymbol{k} \in \mathbb{N}^r} \tilde{H}_{\boldsymbol{k}}\, \tilde{\varphi}_1^{k_1} \cdots \tilde{\varphi}_r^{k_r}$$

is formally convergent[3] in $\mathbb{C}[[z^{-1}]]$, because $\tilde{H}_{\boldsymbol{k}}\,\tilde{\varphi}_1^{k_1}\cdots\tilde{\varphi}_r^{k_r}$ has order $\geq |\boldsymbol{k}|$; this is in fact a particular case of composition of formal series and the fact that the map

$$\tilde{H} \in \tilde{\mathbb{C}}[[z^{-1}, w_1, \ldots, w_r]] \mapsto \tilde{H}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r) \in \mathbb{C}[[z^{-1}]]$$

is an algebra homomorphism is well-known. The last statement will thus follow from the second one.

Let us fix $K \subset \mathscr{S}_\Omega$ compact. We first choose K' and C as in Theorem 2, and then $A = A_{K'}$, $B = B_{K'}$ so that (12) holds relatively to K'. For each $\boldsymbol{k} \in \mathbb{N}^r$, inequality (10) yields

$$\|\tilde{H}_{\boldsymbol{k}}\,\tilde{\varphi}_1^{k_1}\cdots\tilde{\varphi}_r^{k_r}\|_K \leq \frac{C^{|\boldsymbol{k}|+1}}{|\boldsymbol{k}|!}\|\tilde{H}_{\boldsymbol{k}}\|_{K'}\|\tilde{\varphi}_1\|_{K'}^{k_1}\cdots\|\tilde{\varphi}_r\|_{K'}^{k_r} \leq CA\frac{(CB)^{|\boldsymbol{k}|}}{|\boldsymbol{k}|!}\|\tilde{\varphi}_1\|_{K'}^{k_1}\cdots\|\tilde{\varphi}_r\|_{K'}^{k_r},$$

and the conclusion follows easily. $\qquad\square$

As an illustration, for $\tilde{\phi} = c + \tilde{\varphi}$ with $c \in \mathbb{C}$ and $\tilde{\varphi} \in \mathscr{B}^{-1}(\hat{\mathscr{R}}_\Omega)$, we have

$$\exp(\tilde{\phi}) = \mathrm{e}^c \sum_{n \geq 0} \frac{1}{n!}\tilde{\varphi}^n \in \tilde{\mathscr{R}}_\Omega$$

and, if moreover $c \neq 0$,

$$1/\tilde{\phi} = \sum_{n \geq 0} (-1)^n c^{-n-1}\tilde{\varphi}^n \in \tilde{\mathscr{R}}_\Omega.$$

REMARK 3.4. – An example of application of Theorem 3 is provided by the exponential of the Stirling series $\tilde{\varphi}^{\mathrm{S}}$ mentioned in the introduction: we obtain the $2\pi\mathrm{i}\mathbb{Z}$-resurgence of the divergent series $\exp(\tilde{\varphi}^{\mathrm{S}})$ which, according to the refined Stirling formula, is the asymptotic expansion of $\frac{1}{\sqrt{2\pi}}z^{\frac{1}{2}-z}\mathrm{e}^z\,\Gamma(z)$ (in fact the formal series $\exp(\tilde{\varphi}^{\mathrm{S}})$ is 1-summable in the directions of $(-\frac{\pi}{2}, \frac{\pi}{2})$, and this function is its Borel-Laplace sum in the sector $-\pi < \arg z < \pi$; see Section 3.5).

We now show an implicit function theorem for resurgent series.

THEOREM 4. – *Let* $F(x, y) \in \mathbb{C}[[x, y]]$ *be such that* $F(0, 0) = 0$ *and* $\partial_y F(0, 0) \neq 0$, *and call* $\varphi(x)$ *the unique solution in* $x\mathbb{C}[[x]]$ *of the equation*

(13) $$F(x, \varphi(x)) = 0.$$

Let $\tilde{F}(z, y) := F(z^{-1}, y) \in \mathbb{C}[[z^{-1}, y]]$ *and* $\tilde{\varphi}(z) := \varphi(z^{-1}) \in z^{-1}\mathbb{C}[[z^{-1}]]$, *so that* $\tilde{\varphi}$ *is implicitly defined by the equation* $\tilde{F}(z, \tilde{\varphi}(z)) = 0$. *Then*

$$\tilde{F}(z, y) \in \tilde{\mathscr{R}}_\Omega\{y\} \quad \Longrightarrow \quad \tilde{\varphi}(z) \in \tilde{\mathscr{R}}_\Omega.$$

[3] A family of formal series in $\mathbb{C}[[z^{-1}]]$ is formally summable if it has only finitely many members of order $\leq N$ for every $N \in \mathbb{N}$. Notice that if a formally summable family is made up of Ω-resurgent series and is summable for the semi-norms $\|\cdot\|_K$, then the formal sum in $\mathbb{C}[[z^{-1}]]$ and the sum in $\tilde{\mathscr{R}}_\Omega$ coincide (because the Borel transform of the formal sum is nothing but the Taylor series at 0 of the Borel transform of the sum in $\tilde{\mathscr{R}}_\Omega$).

Proof. – Without loss of generality we can assume $\partial_y F(0,0) = -1$ and write

$$F(x,y) = -y + f(x) + R(x,y)$$

with $f(x) = F(x,0) \in x\mathbb{C}[[x]]$ and a quadratic remainder

$$R(x,y) = \sum_{n \geq 1} R_n(x)y^n, \qquad R_n(x) \in \mathbb{C}[[x]], \quad R_1(0) = 0.$$

When viewed as formal transformation in y, the formal series $\theta(x,y) := y - R(x,y)$ is invertible, with inverse given by the Lagrange reversion formula: the series

$$H(x,y) := y + \sum_{k \geq 1} \frac{1}{k!} \partial_y^{k-1}(R^k)(x,y)$$

is formally convergent (the order of $\partial_y^{k-1}(R^k)$ is at least $k+1$ because the order of R is at least 2) and satisfies $\theta\big(x, H(x,y)\big) = y$. Rewriting (13) as $\theta\big(x, \varphi(x)\big) = f(x)$, we get $\varphi(x) = H\big(x, f(x)\big)$.

Now, the y-expansion of H can be written $H(x,y) = \sum_{m \geq 1} H_m(x)y^m$ with

$$H_1 = (1 - R_1)^{-1} \quad \text{and} \quad H_m = \sum_{k \geq 1} \frac{(m+k-1)!}{m!\, k!} \sum_{\boldsymbol{n}} R_{n_1} \cdots R_{n_k} \text{ for } m \geq 2,$$

where the last summation is over all k-tuples of integers $\boldsymbol{n} = (n_1, \ldots, n_k)$ such that $n_1, \ldots, n_k \geq 1$ and $n_1 + \cdots + n_k = m + k - 1$. For $m \geq 2$, grouping together the indices i such that $n_i = 1$, we get an expression of H_m as a formally convergent series in $\mathbb{C}[[x]]$:

$$(14) \qquad H_m = \sum_{r \geq 0} \sum_{s \geq 1} \frac{(m+r+s-1)!}{m!\, r!\, s!} \sum_{\boldsymbol{j}} R_1^r R_{j_1} \cdots R_{j_s},$$

where the last summation is over all s-tuples of integers $\boldsymbol{j} = (j_1, \ldots, j_s)$ such that $j_1, \ldots, j_s \geq 2$ and $j_1 + \cdots + j_s = m + s - 1$. Observe that one must restrict oneself to $s \leq m-1$ and that there are $\binom{m-2}{s-1} \leq 2^{m-2}$ summands in the $\boldsymbol{j}$-summation.

Replacing x by z^{-1}, we get

$$\tilde{\varphi}(z) = \tilde{H}\big(z, \tilde{f}(z)\big)$$

with $\tilde{f}(z) := f(z^{-1}) \in \tilde{\mathscr{R}}_\Omega$ without constant term and

$$\tilde{H}(z,y) := \sum_{m \geq 1} \tilde{H}_m(z)y^n, \qquad \tilde{H}_m(z) := H_m(z^{-1}) \text{ for } m \geq 1.$$

In view of Theorem 3 it is thus sufficient to check that $\tilde{H} \in \tilde{\mathscr{R}}_\Omega\{y\}$.

Let $K \subset \mathscr{S}_\Omega$ be compact. Setting $\tilde{R}_n(z) := R_n(z^{-1})$ for all $n \geq 1$, by Theorem 2 we can find $K' \supset K$ compact and $C > 0$ such that

$$\left\| \tilde{R}_1^r \tilde{R}_{j_1} \cdots \tilde{R}_{j_s} \right\|_K \leq \frac{C^{r+s}}{r!} \left\| \tilde{R}_1 \right\|_{K'}^r \left\| \tilde{R}_{j_1} \right\|_{K'} \cdots \left\| \tilde{R}_{j_s} \right\|_{K'}.$$

Assuming $\tilde{F}(z,y) \in \tilde{\mathscr{R}}_\Omega\{y\}$, we can find $A, B > 0$ such that $\|\tilde{R}_n\|_{K'} \leq AB^n$ for all $n \geq 1$. Enlarging A if necessary, we can assume $3ABC \geq 1$. We then see that the series (14) is

convergent in $\tilde{\mathscr{R}}_\Omega$: for $m \geq 2$,

$$\|\tilde{H}_m\|_K \leq \sum_{r \geq 0} \sum_{s=1}^{m-1} \frac{(m+r+s-1)!}{m!\,r!\,s!} \frac{2^{m-2}C^{r+s}}{r!} A^{r+s} B^{m+r+s-1}$$

$$\leq \frac{2^{m-2}}{m} \sum_{r \geq 0} \frac{1}{r!} \sum_{s=1}^{m-1} 3^{m+r+s-1}(CA)^{r+s} B^{m+r+s-1} \leq \tfrac{1}{2}(6B)^{m-1} \sum_{r \geq 0} \frac{1}{r!}(3ABC)^{r+m-1},$$

which is $\leq \alpha\beta^{m-1}$ with $\alpha = \tfrac{1}{2}\exp(3ABC)$ and $\beta = 18AB^2C$. On the other hand, $\tilde{H}_1 \in \tilde{\mathscr{R}}_\Omega$ by Theorem 3. $\qquad\square$

3.3. The group of resurgent tangent-to-identity diffeomorphisms

One of the first applications by J. Écalle of his resurgence theory was the iteration theory for tangent-to-identity local analytic diffeomorphisms [10, Vol. 2]. In the language of holomorphic dynamics, this corresponds to a parabolic fixed point in one complex variable, for which, classically, one introduces the Fatou coordinates to describe the dynamics and to define the "horn map" [27]. In the resurgent approach, one places the variable at infinity and deals with formal diffeomorphisms: starting from $F(w) = w + O(w^2) \in \mathbb{C}\{w\}$ or $\mathbb{C}[[w]]$, one gets $f(z) := 1/F(1/z) = z + \sum_{m=0}^{\infty} a_m z^{-m} \in z + \mathbb{C}\{z^{-1}\}$ or $z + \mathbb{C}[[z^{-1}]]$. The set

$$\tilde{\mathscr{G}} := z + \mathbb{C}[[z^{-1}]]$$

is a group for the composition law: this is the group of formal tangent-to-identity diffeomorphisms.

Convergent diffeomorphisms form a subgroup $z + \mathbb{C}\{z^{-1}\}$. In the simplest case, one is given a specific dynamical system $z \mapsto f(z) = z + \alpha + O(z^{-1}) \in z + \mathbb{C}\{z^{-1}\}$ with $\alpha \in \mathbb{C}^*$ and there is a formal conjugacy between f and the trivial dynamics $z \mapsto z + \alpha$, i.e., the equation $\tilde{v} \circ f = \tilde{v} + \alpha$ admits a solution $\tilde{v} \in \tilde{\mathscr{G}}$ (strictly speaking, an assumption is needed for this to be true, without which one must enlarge slightly the theory to accept a logarithmic term in $\tilde{v}(z)$; we omit the details here—see [10, 34]). One can give a direct proof [8] that $\tilde{v}(z) - z$ is Ω-resurgent with $\Omega = 2\pi i\alpha^{-1}\mathbb{Z}$. The inverse of $\tilde{v}$ is a solution $\tilde{u}$ of the difference equation $\tilde{u}(z + \alpha) = f(\tilde{u}(z))$ and the exponential of $\tilde{v}$ plays a role in Écalle's "bridge equation" [9], which is related to the Écalle-Voronin classification theorem and to the horn map (again, we refrain from giving more details here).

This may serve as a motivation for the following

THEOREM 5. – *Assume that Ω is a closed discrete subset of $\mathbb{C}$ which contains 0 and is stable under addition. Then the Ω-resurgent tangent-to-identity diffeomorphisms make up a subgroup*

$$\tilde{\mathscr{G}}_\Omega := z + \tilde{\mathscr{R}}_\Omega \subset \tilde{\mathscr{G}},$$

which contains $z + \mathbb{C}\{z^{-1}\}$.

Proof. – We must prove that, for arbitrary $\tilde{f}(z) = z + \tilde{\phi}(z), \tilde{g}(z) = z + \tilde{\psi}(z) \in \tilde{\mathscr{G}}_\Omega$, both $\tilde{f} \circ \tilde{g}$ and $\tilde{h} := \tilde{f}^{\circ(-1)}$ belong to $\tilde{\mathscr{R}}_\Omega$.

We have $\tilde{f} \circ \tilde{g} = \tilde{g} + \tilde{\phi} \circ \tilde{g}$, where the last term can be defined by the formally convergent series

$$(15) \qquad \tilde{\phi} \circ \tilde{g} = \tilde{\phi} + \sum_{n \geq 1} \frac{1}{n!} \tilde{\psi}^n \Big(\frac{\mathrm{d}}{\mathrm{d}z} \Big)^n \tilde{\phi}.$$

Let $K \subset \mathscr{S}_\Omega$ be compact, and let $K' \supset K$ and $C > 0$ be as in Theorem 2. We have

$$\Big\| \tilde{\psi}^n \Big(\frac{\mathrm{d}}{\mathrm{d}z} \Big)^n \tilde{\phi} \Big\|_K \leq C^{n+1} \big\| \tilde{\psi} \big\|_{K'}^n \Big\| \Big(\frac{\mathrm{d}}{\mathrm{d}z} \Big)^n \tilde{\phi} \Big\|_{K'} \leq C^{n+1} D(K')^n \big\| \tilde{\psi} \big\|_{K'}^n \big\| \tilde{\phi} \big\|_{K'},$$

where $D(K') := \max_{\zeta \in K'} |\zeta|$ (by Remark 3.2), hence the series (15) is convergent in $\tilde{\mathscr{R}}_\Omega$, and $\| \tilde{\phi} \circ \tilde{g} \|_K \leq C \| \tilde{\phi} \|_{K'} \exp \big(C D(K') \| \tilde{\psi} \|_{K'} \big)$.

As for $\tilde{h}$, the Lagrange reversion formula yields it in the form of a formally convergent series

$$(16) \qquad \tilde{h} = z + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \Big(\frac{\mathrm{d}}{\mathrm{d}z} \Big)^{k-1} (\tilde{\phi}^k).$$

We have

$$\Big\| \Big(\frac{\mathrm{d}}{\mathrm{d}z} \Big)^{k-1} (\tilde{\phi}^k) \Big\|_K \leq D(K)^{k-1} \big\| \tilde{\phi}^k \big\|_K \leq D(K)^{k-1} C^k \big\| \tilde{\phi} \big\|_{K'}^k$$

(again by Remark 3.2 and Theorem 2), hence the series (16) is convergent in $\tilde{\mathscr{R}}_\Omega$, and $\| \tilde{h} - z \|_K \leq C \| \tilde{\phi} \|_{K'} \exp \big(C D(K) \| \tilde{\phi} \|_{K'} \big)$. $\qquad \square$

REMARK 3.5. – One can easily deduce from the estimates obtained in the above proof that $\tilde{\mathscr{G}}_\Omega$ is a topological group: composition and inversion are continuous if we transport the topology of $\tilde{\mathscr{R}}_\Omega$ onto $\tilde{\mathscr{G}}_\Omega$ by the bijection $\tilde{\phi} \mapsto z + \tilde{\phi}$.

3.4. Other applications

In this article, we stick to the simplest case which presents itself in resurgence theory: formal expansions in negative integer powers of z, whose Borel transforms converge and extend analytically outside a set Ω fixed in advance, but

– the condition of Ω-continuability can be substituted with "continuability without a cut" or "endless continuability" which allow for Riemann surfaces much more general than $\mathscr{S}_\Omega$ [10, Vol. 3], [3];
– the theory of "resurgent singularities" was developed by J. Écalle to deal with much more general formal objects than power series.

The extension to more general Rieman surfaces is necessary in certain problems, particularly those involving parametric resurgence or quantum resurgence (in relation with semi-classical asymptotics). To make our method accomodate the notion of continuability without a cut, one could for instance imitate the way [31] deals with "discrete filtered sets". The point is that, when convolving germs in the ζ-plane, the singular points of the analytic continuation of each factor may produce a singularity located at the sum of these singular points, but being continuable without a cut means that the set of singular points is locally finite, thus one can explore sequentially the Riemann surface of the convolution product, considering longer and longer paths of analytic continuation and saturating the corresponding Riemann surface by removing at each step the (finitely many) sums of singular points already encountered.

The formalism of general resurgent singularities also can be accomodated. The reader is referred to [10] and [34] for the corresponding extension of the definition of convolution (see also [9] and [37]). In short, the formal Borel transform (1), which must be considered as a termwise inverse Laplace transform, can be generalized by considering the action of the Laplace transform on monomials like $\zeta^\alpha (\log \zeta)^m$ with $m \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ for instance. One is thus led to deal with holomorphic functions of ζ defined for arbitrarily small nonzero $|\zeta|$ but not holomorphic at the origin: one must rather work in subsets of the Riemann surface of the logarithm (without even assuming the existence of any kind of expansion for small $|\zeta|$) before considering their analytic continuation for large values of $|\zeta|$. If one restricts oneself to functions which are integrable at 0, like the convergent expansions involving monomials $\zeta^\alpha (\log \zeta)^m$ with $\Re e\, \alpha > -1$, then Formula (2) may still be used to define the convolution. To deal with general resurgent singularities, one must replace it with the so-called convolution of majors. This should be the subject of another article, but we can already mention that it is in the context of resurgent singularities that the alien operators Δ_ω associated with nonzero complex numbers ω are defined in the most efficient way.

These operators can be proved to be derivations (they satisfy the Leibniz rule with respect to the convolution law) independent between them and independent of the natural derivation $\frac{\mathrm{d}}{\mathrm{d}z}$ except for the relations $\left[\Delta_\omega, \frac{\mathrm{d}}{\mathrm{d}z}\right] = -\omega \Delta_\omega$ (this is why they were called "alien derivatives" by Écalle). They annihilate the convergent series (because Δ_ω measures the singularity at ω of a combination of branches of the Borel transform and the Borel transform of a convergent series has no singularity at all) and a suitable adaptation of Theorem 1 allows one to check the rules of "alien calculus", e.g.,

$$\Delta_\omega\big(\tilde{H}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r)\big) = (\Delta_\omega \tilde{H})(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r) + \sum_{j=1}^r (\Delta_\omega \tilde{\varphi}_j) \cdot \frac{\partial \tilde{H}}{\partial w_j}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r)$$

$$\Delta_\omega(\tilde{f} \circ \tilde{g}) = \mathrm{e}^{-\omega(\tilde{g}-z)} \cdot (\Delta_\omega \tilde{f}) \circ \tilde{g} + \left(\frac{\mathrm{d}\tilde{f}}{\mathrm{d}z} \circ \tilde{g}\right) \cdot \Delta_\omega \tilde{g}$$

in the situations of Theorems 3 and 5 (where $\Delta_\omega \tilde{H}$ is defined, with the notation of Theorem 3, as the formal series $\sum(\Delta_\omega \tilde{H}_{\boldsymbol{k}})(z)\, w_1^{k_1} \cdots w_r^{k_r}$, and $(\Delta_\omega \tilde{H})(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r)$ and $(\Delta_\omega \tilde{f}) \circ \tilde{g}$ must be defined properly; see Theorem 30.9 of [37] for an example).

As another possible application, it would be worth trying to adapt our method to the weighted convolution products which appear in [14]. Their definition is as follows: given a sequence of pairs $B_1 = (\omega_1, b_1)$, $B_2 = (\omega_2, b_2)$, etc. with $\omega_n \in \mathbb{C}$ and $b_n \in \mathbb{C}\{\zeta\}$ and assuming that

$$\breve{\omega}_n = \omega_1 + \cdots + \omega_n \neq 0, \qquad n \in \mathbb{N}^*,$$

one defines a sequence $\hat{S}^{B_1}, \hat{S}^{B_1, B_2}, \ldots \in \mathbb{C}\{\zeta\}$ by the formulas

$$\hat{S}^{B_1}(\zeta) := \frac{1}{\omega_1} b_1\left(\frac{\zeta}{\omega_1}\right), \qquad \hat{S}^{B_1, B_2}(\zeta) := \frac{1}{\omega_1} \int_0^{\zeta/\breve{\omega}_2} b_1\left(\frac{\zeta - \omega_2 \xi_2}{\omega_1}\right) b_2(\xi_2)\, \mathrm{d}\xi_2,$$

$$\hat{S}^{B_1, B_2, B_3}(\zeta) := \frac{1}{\omega_1} \int_0^{\zeta/\breve{\omega}_3} \mathrm{d}\xi_3 \int_{\xi_3}^{(\zeta - \omega_3 \xi_3)/\breve{\omega}_2} \mathrm{d}\xi_2\, b_1\left(\frac{\zeta - \omega_2 \xi_2 - \omega_3 \xi_3}{\omega_1}\right) b_2(\xi_2) b_3(\xi_3), \quad \text{etc.}$$

The general formula is $\hat{S}^{B_1,\ldots,B_n}(\zeta) := \frac{1}{\omega_1} \int d\xi_n \cdots d\xi_2\, b_1(\xi_1) b_2(\xi_2) \cdots b_n(\xi_n)$, where the integral is taken over

$$\xi_n \in \left[0, \frac{\zeta}{\breve{\omega}_n}\right], \qquad \xi_i \in \left[\xi_{i+1}, \frac{\zeta - (\omega_{i+1}\xi_{i+1} + \cdots + \omega_n\xi_n)}{\breve{\omega}_i}\right] \text{ for } i = n-1, n-2, \ldots, 2$$

and $\xi_1 := \frac{\zeta - (\omega_2\xi_2 + \cdots + \omega_n\xi_n)}{\breve{\omega}_1}$. There is a relation with the ordinary convolution called symmetrality: if $\boldsymbol{B'} = B^{i_1} \cdots B^{i_n}$ and $\boldsymbol{B''} = B^{j_1} \cdots B^{j_m}$, then $\hat{S}^{\boldsymbol{B'}} * \hat{S}^{\boldsymbol{B''}}$ is the sum $\sum \hat{S}^{\boldsymbol{B}}$ over all words $\boldsymbol{B}$ belonging to the shuffle of $\boldsymbol{B'}$ and $\boldsymbol{B''}$, e.g.,

$$\hat{S}^{B_1} * \hat{S}^{B_2} = \hat{S}^{B_1,B_2} + \hat{S}^{B_2,B_1}, \qquad \hat{S}^{B_1,B_2} * \hat{S}^{B_3} = \hat{S}^{B_1,B_2,B_3} + \hat{S}^{B_1,B_3,B_2} + \hat{S}^{B_3,B_1,B_2}, \quad \text{etc.}$$

It is argued in [14] that the weighted convolutions $\hat{S}^{B_1,\ldots,B_n}$ associated with endlessly continuable germs $b_1, b_2, \ldots$ are themselves endlessly continuable and constitute the "building blocks" of the resurgent functions which appear in parametric resurgence or quantum resurgence problems (see [33] for an example with $\omega_i = 1$ for all i). It would thus be interesting and natural (because the weighted convolution products present themselves as multiple integrals not so different from the n-fold integrals (20) below) to try to deform the integration simplex, in a manner similar to the one that will be employed for convolution products in Sections 4–7, in order to control the analytic continuation of $\hat{S}^{B_1,\ldots,B_r}$.

3.5. Nonlinear analysis with 1-summable series

For the resurgent series encountered in practice, one is often interested in applying Borel-Laplace summation. It is thus important to notice that the property of 1-summability too is compatible with the nonlinear operations described in the previous sections.

We recall that, given a non-trivial interval $\mathcal{C}$ of $\mathbb{R}$, a formal series $\tilde{\phi}(z) \in \mathbb{C}[[z^{-1}]]$ is said to be 1-*summable in the directions of* $\mathcal{C}$ if it can be written $\tilde{\phi} = c + \mathscr{B}^{-1}\hat{\varphi}$ with $c \in \mathbb{C}$ and $\hat{\varphi}(\zeta) \in \mathbb{C}\{\zeta\}$, there exists $\rho > 0$ such that $\hat{\varphi}$ extends analytically to

$$S(\rho, \mathcal{C}) := \mathbb{D}_\rho \cup \left\{ r\,e^{i\theta} \mid r > 0,\ \theta \in \mathcal{C} \right\}$$

and there exist $\tau \in \mathbb{R}$ and $C > 0$ such that $|\hat{\varphi}(\zeta)| \le C\,e^{\tau|\zeta|}$ for all $\zeta \in S(\rho, \mathcal{C})$. In such a case, the *Borel sum of* $\tilde{\phi}$ is the function obtained by glueing the Laplace transforms of $\hat{\varphi}$ associated with the directions of $\mathcal{C}$ (the Cauchy Theorem entails that they match) and adding the constant term c, i.e., the function $\mathrm{Sum}^{\mathcal{C}}\tilde{\phi}$ holomorphic in the union of half-planes[4] $\Sigma_\tau^{\mathcal{C}} := \bigcup_{\theta \in \mathcal{C}}\{z \mid \Re e(z\,e^{i\theta}) > \tau\}$ defined by $\left(\mathrm{Sum}^{\mathcal{C}}\tilde{\phi}\right)(z) := c + \int_0^{e^{i\theta}\infty} \hat{\varphi}(\zeta)\,e^{-z\zeta}\,d\zeta$ with any $\theta \in \mathcal{C}$ such that $\Re e(z\,e^{i\theta}) > \tau$. This function admits $\tilde{\phi}(z)$ as Gevrey asymptotic expansion and is the only one with this property—see e.g., [2, 32].

Let us denote by $\tilde{\mathscr{S}}^{\mathcal{C}}$ the subspace of all 1-summable series, so that

$$\mathbb{C}\{z^{-1}\} \subset \tilde{\mathscr{S}}^{\mathcal{C}} \subset \mathbb{C}[[z^{-1}]]$$

($\mathrm{Sum}^{\mathcal{C}}$ coincides with ordinary summation in restriction to $\mathbb{C}\{z^{-1}\}$). The open sets $S(\rho, \mathcal{C})$ being star-shaped with respect to 0, we can use (4) and check that the properties imposed

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[4] Viewed as a subset of the Riemann surface of the logarithm if $\tau \ge 0$ and $\mathcal{C}$ has length $\ge \pi$.

to the $\hat{\varphi}$'s to define 1-summability (analytic continuation to $S(\rho, \mathscr{C})$ and exponential bound) are stable under convolution. More precisely, we get

$$(17) \quad |\hat{\varphi}_j(\zeta)| \leq C_j \, \mathrm{e}^{\tau|\zeta|} \text{ for } \zeta \in S(\rho, \mathscr{C}) \text{ and } j = 1, \ldots, n$$

$$\implies \quad |\hat{\varphi}_1 * \cdots * \hat{\varphi}_n(\zeta)| \leq \frac{|\zeta|^{n-1}}{(n-1)!} C_1 \cdots C_n \, \mathrm{e}^{\tau|\zeta|} \text{ for } \zeta \in S(\rho, \mathscr{C}).$$

It follows that $\tilde{\mathscr{S}}^{\mathscr{C}}$ is a subalgebra of $\mathbb{C}[[z^{-1}]]$ and, since the Laplace transform maps the convolution product onto the multiplication of functions, $\mathrm{Sum}^{\mathscr{C}}(\tilde{\phi}_1 \tilde{\phi}_2) = (\mathrm{Sum}^{\mathscr{C}} \tilde{\phi}_1)(\mathrm{Sum}^{\mathscr{C}} \tilde{\phi}_2)$. In view of Remark 3.2, it is even a differential subalgebra and $\mathrm{Sum}^{\mathscr{C}} \frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}z} \mathrm{Sum}^{\mathscr{C}} \tilde{\phi}$.

To go farther, we fix a non-trivial interval $\mathscr{C}$ of $\mathbb{R}$ and set

$$\|\tilde{\phi}\|_{\rho,\tau} := |c| + \sup_{S(\rho, \mathscr{C})} \, \mathrm{e}^{-\tau|\zeta|}|\hat{\varphi}(\zeta)|$$

for any $\rho > 0$, $\tau \in \mathbb{R}$ and $\tilde{\phi} = c + \mathscr{B}^{-1}\hat{\varphi} \in \mathbb{C} \oplus \mathscr{B}^{-1}\big(\mathscr{O}(S(\rho, \mathscr{C}))\big)$, so that a formal series $\tilde{\phi}(z)$ belongs to $\tilde{\mathscr{S}}^{\mathscr{C}}$ if and only if there exist ρ and τ such that $\|\tilde{\phi}\|_{\rho,\tau} < \infty$, and $\mathrm{Sum}^{\mathscr{C}} \tilde{\phi}$ is then holomorphic at least in $\Sigma_\tau^{\mathscr{C}}$. The results of the previous sections can be complemented with the following four theorems, the proof of which will be outlined at the end of this section:

THEOREM 2'. – *Suppose $n \geq 1$, $\tilde{\phi}_1, \ldots, \tilde{\phi}_n \in \tilde{\mathscr{S}}^{\mathscr{C}}$ and $N \in \mathbb{N}$. Then*

$$\|\tilde{\phi}_1 \cdots \tilde{\phi}_n\|_{\rho,\tau+\varepsilon} \leq \max(1, \tfrac{1}{\varepsilon^{n-1}})\|\tilde{\phi}_1\|_{\rho,\tau} \cdots \|\tilde{\phi}_n\|_{\rho,\tau}, \qquad \left\|\frac{\mathrm{d}^N \tilde{\phi}_1}{\mathrm{d}z^N}\right\|_{\rho,\tau+\varepsilon} \leq \frac{N!}{\varepsilon^N}\|\tilde{\phi}_1\|_{\rho,\tau}$$

for every $\rho > 0$, $\tau \in \mathbb{R}$ and $\varepsilon > 0$.

THEOREM 3'. – *Suppose that*

$$\tilde{H} = \sum_{\boldsymbol{k}=(k_1,\ldots,k_r)\in\mathbb{N}^r} \tilde{H}_{\boldsymbol{k}}(z)\, w_1^{k_1} \cdots w_r^{k_r} \in \mathbb{C}[[z^{-1}, w_1, \ldots, w_r]]$$

has its coefficients 1-summable in the directions of $\mathscr{C}$ and $\|\tilde{H}_{\boldsymbol{k}}\|_{\rho,\tau} \leq A\, B^{|\boldsymbol{k}|}$ for all $\boldsymbol{k} \in \mathbb{N}^r$, with some $\rho, A, B > 0$ and $\tau \in \mathbb{R}$ independent of $\boldsymbol{k}$. Then

$$\mathscr{H}^{\mathscr{C}}(z, w_1, \ldots, w_r) := \sum_{\boldsymbol{k}=(k_1,\ldots,k_r)\in\mathbb{N}^r} (\mathrm{Sum}^{\mathscr{C}} \tilde{H}_{\boldsymbol{k}})(z)\, w_1^{k_1} \cdots w_r^{k_r}$$

is holomorphic in $\Sigma_\tau^{\mathscr{C}} \times \mathbb{D}_{1/B} \cdots \times \mathbb{D}_{1/B}$ and, for all $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r \in \tilde{\mathscr{S}}^{\mathscr{C}}$ without constant term,

$$\tilde{H}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r) \in \tilde{\mathscr{S}}^{\mathscr{C}} \quad \text{and} \quad \mathrm{Sum}^{\mathscr{C}}\big(\tilde{H}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r)\big)(z) = \mathscr{H}^{\mathscr{C}}\big(z, \mathrm{Sum}^{\mathscr{C}} \tilde{\varphi}_1(z), \ldots, \mathrm{Sum}^{\mathscr{C}} \tilde{\varphi}_r(z)\big)$$

for $z \in \Sigma_{\tau'}^{\mathscr{C}}$ as soon as τ' is large enough. One can take $\tau' = \tau + B\big(\|\tilde{\varphi}_1\|_{\rho,\tau} + \cdots + \|\tilde{\varphi}_r\|_{\rho,\tau}\big)$ if this number is finite (if not, take τ larger and ρ smaller), in which case

$$\|\tilde{H}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r)\|_{\rho,\tau'} \leq A\Big(2 + B\big(\|\tilde{\varphi}_1\|_{\rho,\tau} + \cdots + \|\tilde{\varphi}_r\|_{\rho,\tau}\big)\Big).$$

THEOREM 4'. – *Suppose that $\tilde{F} = \sum_{k\geq 0} \tilde{F}_k(z)y^k \in \mathbb{C}[[z^{-1}, y]]$ and $\|\tilde{F}_k\|_{\rho,\tau} \leq AB^k$ for all $k \in \mathbb{N}$, with some $\rho, A, B > 0$ and $\tau \in \mathbb{R}$ independent of k, so that*

$$\mathscr{F}^{\mathscr{C}}(z, y) := \sum_{k\geq 0} (\mathrm{Sum}^{\mathscr{C}} \tilde{F}_k)(z)\, y^k$$

is holomorphic in $\Sigma_\tau^{\mathcal{A}} \times \mathbb{D}_{1/B}$. *Suppose moreover that* $\tilde{F}_0(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ *and that the constant term of* $\tilde{F}_1$ *is nonzero, so that the equation* $\tilde{F}(z, \tilde{\varphi}(z)) = 0$ *implicitly defines a formal series* $\tilde{\varphi}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$. *Then this unique formal solution is 1-summable in the directions of* $\mathcal{A}$, *and* $\mathrm{Sum}^{\mathcal{A}} \tilde{\varphi}$ *is a solution of the corresponding functional equation* $\mathcal{F}^{\mathcal{A}}(z, (\mathrm{Sum}^{\mathcal{A}} \tilde{\varphi})(z)) = 0$ *which is holomorphic in* $\Sigma_{\tau'}^{\mathcal{A}}$, *for* τ' *large enough.*

We also define the set of "1-summable tangent-to-identity diffeomorphisms"

$$\tilde{\mathscr{G}}^{\mathcal{A}} := z + \tilde{\mathscr{S}}^{\mathcal{A}} \subset \tilde{\mathscr{G}} = z + \mathbb{C}[[z^{-1}]]$$

and use the notations $\|\tilde{f}\|_{\rho,\tau} := \|\tilde{\phi}\|_{\rho,\tau}$ and $\mathrm{Sum}^{\mathcal{A}} \tilde{f} := \mathrm{Id} + \mathrm{Sum}^{\mathcal{A}} \tilde{\phi}$ for any $\tilde{f}(z) = z + \tilde{\phi}(z) \in \tilde{\mathscr{G}}^{\mathcal{A}}$.

THEOREM 5′. – *The set* $\tilde{\mathscr{G}}^{\mathcal{A}}$ *is a subgroup of* $\tilde{\mathscr{G}}$ *and contains* $z + \mathbb{C}\{z^{-1}\}$. *For* $\tilde{f}, \tilde{g} \in \tilde{\mathscr{G}}^{\mathcal{A}}$ *with* $\|\tilde{f}\|_{\rho,\tau}, \|\tilde{g}\|_{\rho,\tau} < \infty$, *one has*

$$\|\tilde{g} \circ \tilde{f}\|_{\rho,\tau'} \leq \|\tilde{f}\|_{\rho,\tau} + \|\tilde{g}\|_{\rho,\tau}, \qquad \|\tilde{f}^{\circ(-1)}\|_{\rho,\tau'} \leq \|\tilde{f}\|_{\rho,\tau},$$

with $\tau' := \tau + 1 + \|\tilde{f}\|_{\rho,\tau}$. *Moreover, the composition* $(\mathrm{Sum}^{\mathcal{A}} \tilde{g}) \circ (\mathrm{Sum}^{\mathcal{A}} \tilde{f})$ *is well-defined and coincides with* $\mathrm{Sum}^{\mathcal{A}}(\tilde{g} \circ \tilde{f})$ *on* $\Sigma_{\tau'}^{\mathcal{A}}$ *and, for* τ'' *large enough,* $\mathrm{Sum}^{\mathcal{A}} \tilde{f}$ *is injective on* $\Sigma_{\tau''}^{\mathcal{A}}$, *and the composition* $(\mathrm{Sum}^{\mathcal{A}}(\tilde{f}^{\circ(-1)})) \circ (\mathrm{Sum}^{\mathcal{A}} \tilde{f})$ *is well-defined and coincides with* Id *on* $\Sigma_{\tau''}^{\mathcal{A}}$.

Before proceeding with the proof of these statements, we first mention

LEMMA 3.6. – *Let* $\tilde{\phi} \in \tilde{\mathscr{S}}^{\mathcal{A}}$ *with* $\|\tilde{\phi}\|_{\rho,\tau} < \infty$. *Then*

$$|\mathrm{Sum}^{\mathcal{A}} \tilde{\phi}(z)| \leq D(z)\|\tilde{\phi}\|_{\rho,\tau} \quad \text{for } z \in \Sigma_\tau^{\mathcal{A}}, \quad \text{with } D(z) = \max\left(1, \frac{1}{\sup_{\theta \in \mathcal{A}} \Re e(z\, e^{i\theta} - \tau)}\right).$$

If $\tilde{\phi}$ *has no constant term and* $z \in \Sigma_{\tau'}^{\mathcal{A}}$, *with* $\tau' > \tau$, *then one can take* $D(z) = \frac{1}{\tau'-\tau}$.

Proof. – Write $\tilde{\phi} = c + \tilde{\varphi}$ with $\tilde{\varphi}$ without constant term and take $z \in \Sigma_\tau^{\mathcal{A}}$. For any $\theta \in \mathcal{A}$ such that $\delta_\theta(z) := \Re e(z\, e^{i\theta} - \tau) > 0$, we have $|\mathrm{Sum}^{\mathcal{A}} \tilde{\varphi}(z)| \leq \frac{1}{\delta_\theta(z)}\|\tilde{\varphi}\|_{\rho,\tau}$, whence the conclusion follows. □

Outline of the proof of Theorem 2′. – For $j = 1, \ldots, n$, we write $\tilde{\phi}_j = c_j + \tilde{\varphi}_j$ with $c_j \in \mathbb{C}$ and $\tilde{\varphi}_j$ without constant term. For $N \geq 1$, $\frac{d^N \tilde{\phi}_1}{dz^N}$ is the inverse Borel transform of $(-\zeta)^N \hat{\varphi}_1(\zeta)$, whose modulus is bounded on $S(\rho, \mathcal{A})$ by $\frac{N!}{\varepsilon^N} \frac{(\varepsilon|\zeta|)^N}{N!} \|\tilde{\varphi}_1\|_{\rho,\tau}\, e^{\tau|\zeta|} \leq \frac{N!}{\varepsilon^N} \|\tilde{\phi}_1\|_{\rho,\tau}\, e^{(\tau+\varepsilon)|\zeta|}$.

The first statement results from the identity $\tilde{\phi}_1 \cdots \tilde{\phi}_n = c + \tilde{\psi}$ with $c = c_1 \cdots c_n$ and $\hat{\psi} = \sum c_{i_1} \cdots c_{i_p} \hat{\varphi}_{j_1} * \cdots * \hat{\varphi}_{j_q}$, with summation over all proper subsets $I = \{i_1, \ldots, i_p\}$ of $\{1, \ldots, n\}$ and $\{j_1, \ldots, j_q\} := \{1, \ldots, s\} \setminus I$: (17) yields

$$\left|\hat{\psi}(\zeta)\right| \leq \sum |c_{i_1} \cdots c_{i_p}| \|\tilde{\varphi}_{j_1}\|_{\rho,\tau} \cdots \|\tilde{\varphi}_{j_q}\|_{\rho,\tau} \frac{|\zeta|^{q-1}}{(q-1)!} e^{\tau|\zeta|}$$

on $S(\rho, \mathcal{A})$ and $\frac{|\zeta|^{q-1}}{(q-1)!} \leq \max(1, \frac{1}{\varepsilon^{n-1}}) e^{\varepsilon|\zeta|}$, while $|c| \leq \max(1, \frac{1}{\varepsilon^{n-1}})|c_{i_1} \cdots c_{i_p}|$. □

REMARK 3.7. – A simple modification of the previous argument, in the spirit of the proof of Theorem 2, shows that, if among $\tilde{\phi}_1, \ldots, \tilde{\phi}_n$ at least $r \geq 1$ formal series have no constant term, then $\tilde{\phi}_1 \cdots \tilde{\phi}_n = \tilde{\psi}$ is the inverse Borel transform of a function which satisfies

$$|\hat{\psi}(\zeta)| \leq \max(1, \tfrac{1}{\varepsilon^{n-r}}) \|\tilde{\phi}_1\|_{\rho,\tau} \cdots \|\tilde{\phi}_n\|_{\rho,\tau} \frac{|\zeta|^{r-1}}{(r-1)!} e^{(\tau+\varepsilon)|\zeta|} \quad \text{for } \zeta \in S(\rho, \mathscr{C}).$$

Outline of the proof of Theorem 3′. – The first statement follows from Lemma 3.6. Suppose $C_j := \|\tilde{\varphi}_j\|_{\rho,\tau} < \infty$ for each j, then Lemma 3.6 also shows that

$$\mathscr{H}^{\mathscr{C}}\big(z, \operatorname{Sum}^{\mathscr{C}} \tilde{\varphi}_1(z), \ldots, \operatorname{Sum}^{\mathscr{C}} \tilde{\varphi}_r(z)\big)$$

is well-defined for $z \in \Sigma^{\mathscr{C}}_{\tau'}$ as soon as $\tau' - \tau > B \max(C_1, \ldots, C_n)$.

Let us write $\tilde{H}_k = c_k + \tilde{G}_k$ with $\tilde{G}_k$ without constant term. Then $\tilde{H}(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r) = c_0 + \tilde{\psi}$,

$$\tilde{\psi} = \mathscr{B}^{-1} \hat{\psi}, \qquad \hat{\psi} = \sum_{\boldsymbol{k} \in \mathbb{N}^r \setminus \{0\}} c_{\boldsymbol{k}} \hat{\varphi}_1^{k_1} * \cdots * \hat{\varphi}_r^{k_r} + \sum_{\boldsymbol{k} \in \mathbb{N}^r} \hat{G}_{\boldsymbol{k}} * \hat{\varphi}_1^{k_1} * \cdots * \hat{\varphi}_r^{k_r}.$$

By inequality (17), representing by $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r)$ the canonical basis of $\mathbb{R}^r$, we get

$$\left|\hat{\psi}(\zeta)\right| \leq \sum_{\boldsymbol{k} \in \mathbb{N}^r} \Big(\sum_{j=1}^r |c_{\boldsymbol{k}+\boldsymbol{e}_j}| C_j + \left\|\tilde{G}_{\boldsymbol{k}}\right\|_{\rho,\tau}\Big) C_1^{k_1} \cdots C_r^{k_r} \frac{|\zeta|^{|\boldsymbol{k}|}}{|\boldsymbol{k}|!} e^{\tau|\zeta|}$$

$$\leq A\big(B(C_1 + \cdots + C_r) + 1\big) e^{\tau'|\zeta|}. \qquad\qquad \square$$

Outline of the proof of Theorem 4′. – Dividing $\tilde{F}$ by the appropriate factor, we can suppose that $\tilde{F}(z,y) = -y + \tilde{f}(z) + \sum_{n \geq 1} \tilde{R}_n(z) y^n$, where $\tilde{f}$ and $\tilde{R}_1$ have no constant term, $\|\tilde{f}\|_{\rho,\tau} \leq A$ and $\|\tilde{R}_n\|_{\rho,\tau} \leq AB^n$ for every $n \geq 1$. Arguing as in the proof of Theorem 4, we have $\tilde{\varphi} = \sum_{m \geq 1} \tilde{H}_m \tilde{f}^m$ with $\tilde{H}_1 := (1 - \tilde{R}_1)^{-1}$ and $\tilde{H}_m$ given by (14) for $m \geq 2$. The main task consists in showing that there exist $\alpha, \beta > 0$ and τ_1 such that $\|\tilde{H}_m\|_{\rho,\tau_1} \leq \alpha \beta^m$ for all m, so that Theorem 3′ can be applied.

In the r-summation defining $\tilde{H}_m$, we separate $r = 0$ and $r \geq 1$, so that $\tilde{H}_m = \tilde{H}'_m + \tilde{H}''_m$ with $\tilde{H}'_1 := 1$, $\tilde{H}''_1 := \sum_{r \geq 1} \tilde{R}_1^r$, and, for $m \geq 2$,

$$\tilde{H}'_m := \sum_{s=1}^{m-1} \frac{(m+s-1)!}{m!\,s!} \sum_{\boldsymbol{j}} \tilde{R}_{j_1} \cdots \tilde{R}_{j_s}, \quad \tilde{H}''_m := \sum_{r \geq 1} \sum_{s=1}^{m-1} \frac{(m+r+s-1)!}{m!\,r!\,s!} \sum_{\boldsymbol{j}} \tilde{R}_1^r \tilde{R}_{j_1} \cdots \tilde{R}_{j_s}$$

with summation over all $\boldsymbol{j} = (j_1, \ldots, j_s)$ such that $j_1, \ldots, j_s \geq 2$ and $j_1 + \cdots + j_s = m+s-1$. Enlarging A if necessary, we suppose $2AB \geq 1$. Theorem 3′ yields, for $m \geq 2$,

$$\left\|\tilde{H}'_m\right\|_{\rho,\tau+1} \leq \sum_{s=1}^{m-1} \frac{(m+s-1)!}{m!\,s!} \binom{m-2}{s-1} A^s B^{m+s-1}$$

$$\leq \sum_{s=1}^{m-1} \frac{2^{m+s-1}}{m} 2^{m-2}(AB)^s B^{m-1} \leq \frac{(4B)^{m-1}}{2m} \sum_{s=1}^{m-1} (2AB)^s,$$

whence $\|\tilde{H}'_m\|_{\rho,\tau+1} \leq (8AB^2)^{m-1}$ for all $m \geq 1$. On the other hand, $\|\tilde{H}''_1\|_{\rho,\tau+AB} < \infty$ by Theorem 3' and, by Remark 3.7, for $m \geq 2$ and $\zeta \in S(\rho, \mathcal{C})$,

$$\left|\hat{H}''_m(\zeta)\right|\mathrm{e}^{-(\tau+1)|\zeta|} \leq \sum_{r \geq 1}\sum_{s=1}^{m-1}\frac{(m+r+s-1)!}{m!\,r!\,s!}\binom{m-2}{s-1}A^{r+s}B^{m+r+s-1}\frac{|\zeta|^{r-1}}{(r-1)!}$$

$$\leq \sum_{r \geq 1}\sum_{s=1}^{m-1}\frac{3^{m+r+s-1}}{m}2^{m-2}(AB)^{r+s}B^{m-1}\frac{|\zeta|^{r-1}}{(r-1)!}$$

$$\leq \tfrac{1}{2}(6B)^{m-1}\sum_{r \geq 1}(3AB)^{r+m-1}\frac{|\zeta|^{r-1}}{(r-1)!}$$

(because $3AB \geq 1$), whence $\|\tilde{H}''_m\|_{\rho,\tau+1+3AB} \leq \frac{3AB}{2}(18AB^2)^{m-1}$. $\square$

Outline of the proof of Theorem 5'. – Write $\tilde{f} = z + c + \tilde{\varphi}(z)$, $\tilde{g} = z + c' + \tilde{\psi}(z)$, with $c, c' \in \mathbb{C}$, $\tilde{\varphi}$ and $\tilde{\psi}$ without constant term, and let $A := \|\tilde{\varphi}\|_{\rho,\tau}$ and $B := \|\tilde{\psi}\|_{\rho,\tau}$. The function $\left(\mathrm{Sum}^{\mathcal{C}}\tilde{g}\right) \circ \left(\mathrm{Sum}^{\mathcal{C}}\tilde{f}\right)$ is well-defined on $\Sigma^{\mathcal{C}}_{\tau'}$ because, by Lemma 3.6, $|\mathrm{Sum}^{\mathcal{C}}\tilde{f}(z) - z| \leq |c| + \frac{A}{\tau'-\tau} \leq \tau' - \tau$.

Let $\tilde{\varphi}_0(z) := \tilde{\varphi}(z - c)$: we have $\tilde{g} \circ \tilde{f} = z + c + c' + \tilde{\varphi} + \tilde{\chi}$ with

$$\tilde{\chi} := \tilde{\chi}_0 \circ (\mathrm{Id} + c), \quad \tilde{\chi}_0 := \tilde{\psi} \circ (\mathrm{Id} + \tilde{\varphi}_0), \quad \text{hence } \hat{\chi}(\zeta) = \hat{\chi}_0(\zeta)\,\mathrm{e}^{-c\zeta}, \quad \hat{\chi}_0(\zeta) = \sum_{k \geq 0}\left(\frac{(-\zeta)^k}{k!}\hat{\psi}\right) * \hat{\varphi}_0^{*k}.$$

Given $\theta \in \mathcal{C}$ and $\tau_+ := \tau + \Re e(c\,\mathrm{e}^{i\theta})$, since $\hat{\varphi}_0(\zeta) = \mathrm{e}^{c\zeta}\varphi(\zeta)$, we have $|\hat{\varphi}_0(\zeta)| \leq A\,\mathrm{e}^{\tau_+|\zeta|}$ on $\mathbb{R}^+\,\mathrm{e}^{i\theta}$, thus $|\left(\frac{(-\zeta)^k}{k!}\hat{\psi}\right) * \hat{\varphi}_0^{*k}(\zeta)| \leq BA^k\frac{|\zeta|^{2k}}{(2k)!}\mathrm{e}^{\max\{\tau,\tau_+\}|\zeta|}$ and $|\hat{\chi}_0(\zeta)| \leq B\,\mathrm{e}^{(\sqrt{A}+\max\{\tau,\tau_+\})|\zeta|}$, whence $|\hat{\chi}(\zeta)| \leq B\,\mathrm{e}^{(\sqrt{A}+\tau+|c|)|\zeta|} \leq B\,\mathrm{e}^{\tau'|\zeta|}$ and $\|\tilde{g} \circ \tilde{f}\|_{\rho,\tau'} \leq |c| + |c'| + A + B$.

Using the Lagrange reversion formula to compute $\tilde{f}^{\circ(-1)} = (\mathrm{Id} + \tilde{\varphi}_0)^{-1} - c$, we get

$$\tilde{f}^{\circ(-1)} = \mathrm{Id} - c + \tilde{\varphi}_-, \quad \tilde{\varphi}_- := \sum_{k \geq 1}\frac{(-1)^k}{k!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{k-1}\tilde{\varphi}_0^k, \quad \text{hence } \hat{\varphi}_-(\zeta) = -\sum_{k \geq 1}\frac{\zeta^{k-1}}{k!}\hat{\varphi}_0^{*k}.$$

On $S(\rho, \mathcal{C})$, $|\hat{\varphi}_0^{*k}(\zeta)| \leq A^k\frac{|\zeta|^{k-1}}{(k-1)!}\mathrm{e}^{(\tau+|c|)|\zeta|}$, thus $|\hat{\varphi}_-(\zeta)| \leq A\,\mathrm{e}^{(\tau+2\sqrt{A})|\zeta|} \leq A\,\mathrm{e}^{\tau'|\zeta|}$. $\square$

4. The initial n-dimensional integration current

We now begin the proof of Theorem 1. Notice that convolution with the constant germ 1 amounts to integration from 0, according to (2), thus $\frac{\mathrm{d}}{\mathrm{d}\zeta}(1 * \hat{\varphi}) = \hat{\varphi}$ and, by associativity of the convolution,

$$\text{(18)} \qquad\qquad \hat{\varphi}_1 * \cdots * \hat{\varphi}_n = \frac{\mathrm{d}}{\mathrm{d}\zeta}\big(1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n\big)$$

for any $\hat{\varphi}_1, \ldots, \hat{\varphi}_n \in \mathbb{C}\{\zeta\}$.

We shall now dedicate ourselves to the proof of a statement similar to Theorem 1 for convolution products of the form $1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n$, with $\hat{\varphi}_1, \ldots, \hat{\varphi}_n \in \mathcal{O}(\mathscr{S}_\Omega)$; this will be Theorem 1' of Section 7. The proof of Theorem 1 itself will then follow by the Cauchy inequalities.

It turns out that, for $\zeta \in \mathscr{S}_\Omega$ close to 0_Ω, there is a natural way of representing $1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n(\zeta)$ as the integral of a holomorphic n-form over an n-dimensional chain of

the complex manifold $\mathscr{S}_\Omega^n$; this is Formula (19) of Proposition 4.3, which will be our starting point for the proof of Theorem $1'$.

NOTATION 4.1. – *Given $\zeta \in \mathscr{S}_\Omega$, we denote by $\mathscr{L}_\zeta \colon \mathbb{D}_{R_\Omega(\zeta)} \to \mathscr{S}_\Omega$ the holomorphic map defined by*

$$\mathscr{L}_\zeta(\xi) := \text{endpoint of the lift which starts at } \zeta \text{ of the path } t \in [0,1] \mapsto \overset{\bullet}{\zeta} + t\xi$$

(so that $\mathscr{L}_\zeta(\xi)$ can be thought of as "the lift of $\overset{\bullet}{\zeta} + \xi$ wich sits on the same sheet of $\mathscr{S}_\Omega$ as ζ"). We shall often use the shorthand

$$\zeta + \xi := \mathscr{L}_\zeta(\xi)$$

(beware that, in the latter formula, $\xi \in \mathbb{D}_{R_\Omega(\zeta)}$ is a complex number but not ζ nor $\zeta + \xi$, which are points of $\mathscr{S}_\Omega$). If $n \geq 1$ and $\underline{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathscr{S}_\Omega^n$, we also set

$$S_n(\underline{\zeta}) := \overset{\bullet}{\zeta}_1 + \cdots + \overset{\bullet}{\zeta}_n \in \mathbb{C},$$

$$\underline{\mathscr{L}}_{\underline{\zeta}}(\underline{\xi}) := \underline{\zeta} + \underline{\xi} := \big(\mathscr{L}_{\zeta_1}(\xi_1), \ldots, \mathscr{L}_{\zeta_n}(\xi_n) \big) \in \mathscr{S}_\Omega^n$$

for $\underline{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ close enough to 0 (it suffices that $|\xi_j| < R_\Omega(\zeta_j)$; observe that $S_n(\underline{\zeta} + \underline{\xi}) = S_n(\underline{\zeta}) + \xi_1 + \cdots + \xi_n$).

NOTATION 4.2. – *For any $n \geq 1$, we denote by Δ_n the n-dimensional simplex[5]*

$$\Delta_n := \{ (s_1, \ldots, s_n) \in \mathbb{R}^n \mid s_1, \ldots, s_n \geq 0 \text{ and } s_1 + \cdots + s_n \leq 1 \}$$

with the standard orientation, and by $[\Delta_n] \in \mathscr{E}_n(\mathbb{R}^n)$ the corresponding integration current:

$$[\Delta_n] \colon \alpha \text{ complex-valued smooth } n\text{-form on } \mathbb{R}^n \mapsto \int_{\Delta_n} \alpha \in \mathbb{C}.$$

For every $\zeta \in \mathbb{D}_{\rho(\Omega)}$, we consider the map

$$\underline{\mathscr{D}}(\zeta) \colon \underline{s} = (s_1, \ldots, s_n) \mapsto \underline{\mathscr{D}}(\zeta, \underline{s}) := 0_\Omega + (s_1\zeta, \ldots, s_n\zeta) \in \mathscr{S}_\Omega^n,$$

defined in a neighbourhood of Δ_n in $\mathbb{R}^n$, and denote by $\underline{\mathscr{D}}(\zeta)_\#[\Delta_n] \in \mathscr{E}_n(\mathscr{S}_\Omega^n)$ the push-forward of $[\Delta_n]$ by $\underline{\mathscr{D}}(\zeta)$:

$$\underline{\mathscr{D}}(\zeta)_\#[\Delta_n] \colon \beta \text{ smooth } n\text{-form on } \mathscr{S}_\Omega^n \mapsto [\Delta_n]\big(\underline{\mathscr{D}}(\zeta)^\# \beta\big).$$

See Appendix for our notations in relation with currents. Notice that the last formula makes sense because $\underline{\mathscr{D}}(\zeta)$ is a smooth map, thus the pullback form $\underline{\mathscr{D}}(\zeta)^\# \beta$ is well-defined in a neighbourhood of Δ_n. The reason for using the language of currents and Geometric Measure Theory is that later we shall require the push-forward of integration currents by Lipschitz maps which are not smooth everywhere. The reader is referred to Appendix for a survey of a few facts of the theory which will be useful for us.

PROPOSITION 4.3. – *For $\hat{\varphi}_1, \ldots, \hat{\varphi}_n \in \hat{\mathscr{R}}_\Omega$ and $\zeta \in \mathbb{D}_{\rho(\Omega)}$, one has*

$$(19) \quad 1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n(\zeta) = \underline{\mathscr{D}}(\zeta)_\#[\Delta_n](\beta) \quad \text{with } \beta = \hat{\varphi}_1(\zeta_1) \cdots \hat{\varphi}_n(\zeta_n) \, \mathrm{d}\zeta_1 \wedge \cdots \wedge \mathrm{d}\zeta_n,$$

where we denote by $\mathrm{d}\zeta_1 \wedge \cdots \wedge \mathrm{d}\zeta_n$ the pullback in $\mathscr{S}_\Omega^n$ by $\pi_\Omega^{\otimes n} \colon \underline{\zeta} \in \mathscr{S}_\Omega^n \mapsto \underline{\xi} = \big(\overset{\bullet}{\zeta}_1, \ldots, \overset{\bullet}{\zeta}_n\big)$ of the n-form $\mathrm{d}\xi_1 \wedge \cdots \wedge \mathrm{d}\xi_n$ of $\mathbb{C}^n$.

[5] This Δ_n has nothing to do with Écalle's alien derivatives Δ_ω mentioned earlier.

Proof. – Since $(\zeta_1, \ldots, \zeta_n) \in \mathbb{D}^n_{\rho(\Omega)} \mapsto 0_\Omega + (\zeta_1, \ldots, \zeta_n) \in \mathscr{S}^n_\Omega$ is an analytic chart which covers a neighbourhood of $\underline{\mathscr{D}}(\zeta)(\Delta_n)$, we can write

$$\underline{\mathscr{D}}(\zeta)^\# \beta = \hat{\varphi}_1(s_1\zeta) \cdots \hat{\varphi}_n(s_n\zeta) \mathrm{d}s_1 \wedge \cdots \wedge \mathrm{d}s_n.$$

Since

$$\Delta_n = \{\, (s_1, \ldots, s_n) \in \mathbb{R}^n \mid s_1 \in [0,1], s_2 \in [0, 1-s_1], \ldots, s_n \in [0, 1-(s_1 + \cdots + s_{n-1})] \,\}$$

with the standard orientation, the right-hand side of the identity stated in (19) can be rewritten

or

$$\zeta^n \int_0^1 \mathrm{d}s_1 \int_0^{1-s_1} \mathrm{d}s_2 \cdots \int_0^{1-(s_1+\cdots+s_{n-1})} \mathrm{d}s_n \, \hat{\varphi}_1(s_1\zeta) \cdots \hat{\varphi}_n(s_n\zeta)$$

$$(20) \qquad \int_0^\zeta \mathrm{d}\zeta_1 \int_0^{\zeta-\zeta_1} \mathrm{d}\zeta_2 \cdots \int_0^{\zeta-(\zeta_1+\cdots+\zeta_{n-1})} \mathrm{d}\zeta_n \, \hat{\varphi}_1(\zeta_1) \cdots \hat{\varphi}_n(\zeta_n).$$

When $n = 1$, Formula (19) is thus the very definition of $1 * \hat{\varphi}_1(\zeta)$. Writing

$$1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n(\zeta) = \int_0^\zeta \mathrm{d}\zeta_1 \, \hat{\varphi}_1(\zeta_1) \big(1 * \hat{\varphi}_2 * \cdots * \hat{\varphi}_n \big)(\zeta - \zeta_1),$$

we get the general case by induction. $\qquad \square$

5. Deformation of the n-dimensional integration current in $\mathscr{S}^n_\Omega$

In this section, we fix an interval $J = [a,b]$ and a path $\gamma \colon J \mapsto \mathbb{C} \setminus \Omega$ such that $\gamma(a) \in \mathbb{D}^*_{\rho(\Omega)}$; we denote by $\tilde{\gamma}$ the lift of γ which starts in the principal sheet of $\mathscr{S}_\Omega$. In order to obtain the analytic continuation of Formula (19), we shall deform the n-dimensional integration current $\underline{\mathscr{D}}(\zeta)_\#[\Delta_n]$ as indicated in Proposition 5.2 below.

DEFINITION 5.1. – *Given $n \geq 1$, for $\zeta \in \mathbb{C}$ and $j = 1, \ldots, n$, we set*

$$\mathscr{N}(\zeta) := \{\, \underline{\zeta} \in \mathscr{S}^n_\Omega \mid S_n(\underline{\zeta}) = \zeta \,\}, \qquad \mathscr{N}_j := \{\, \underline{\zeta} = (\zeta_1, \ldots, \zeta_n) \in \mathscr{S}^n_\Omega \mid \zeta_j = 0_\Omega \,\}.$$

We call γ-adapted origin-fixing isotopy in $\mathscr{S}^n_\Omega$ any family $(\Psi_t)_{t \in J}$ of homeomorphisms of $\mathscr{S}^n_\Omega$ such that $\Psi_a = \mathrm{Id}$, the map $(t, \underline{\zeta}) \in J \times \mathscr{S}^n_\Omega \mapsto \Psi_t(\underline{\zeta}) \in \mathscr{S}^n_\Omega$ is locally Lipschitz,[6] and for any $t \in J$ and $j = 1, \ldots, n$,

$$\underline{\zeta} \in \mathscr{N}\big(\gamma(a)\big) \quad \Rightarrow \quad \Psi_t(\underline{\zeta}) \in \mathscr{N}\big(\gamma(t)\big),$$
$$\underline{\zeta} \in \mathscr{N}_j \quad \Rightarrow \quad \Psi_t(\underline{\zeta}) \in \mathscr{N}_j.$$

PROPOSITION 5.2. – *Suppose that $(\Psi_t)_{t \in J}$ is a γ-adapted origin-fixing isotopy in $\mathscr{S}^n_\Omega$. Then, for any $\hat{\varphi}_1, \ldots, \hat{\varphi}_n \in \hat{\mathscr{R}}_\Omega$, the analytic continuation of $1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n$ along γ is given by*

$$(21) \qquad \big(1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n \big)\big(\tilde{\gamma}(t)\big) = \big(\Psi_t \circ \underline{\mathscr{D}}(\gamma(a)) \big)_\#[\Delta_n](\beta), \qquad t \in J,$$

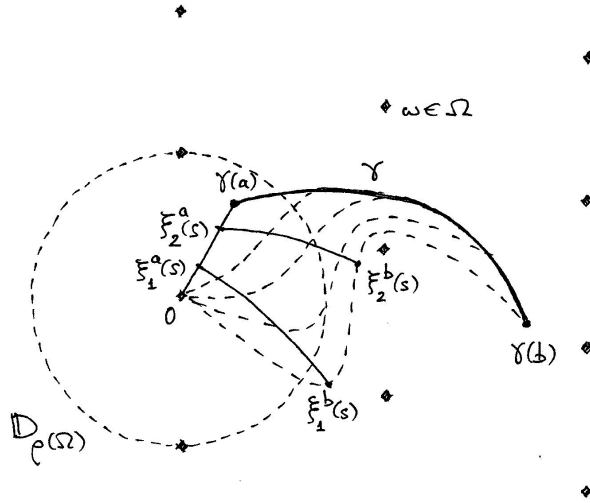*with $\beta = \hat{\varphi}_1(\zeta_1) \cdots \hat{\varphi}_n(\zeta_n) \, \mathrm{d}\zeta_1 \wedge \cdots \wedge \mathrm{d}\zeta_n$.*

FIGURE 1. Projections of $\left(\xi_1^t(\underline{s}), \ldots, \xi_n^t(\underline{s})\right) := \Psi_t\left(s_1\gamma(a), \ldots, s_n\gamma(a)\right) = \Psi_t \circ \mathscr{D}\left(\gamma(a)\right)(\underline{s})$.

See Figure 1. Observe that, for each $t \in J$, the map $\Psi_t \circ \mathscr{D}\left(\gamma(a)\right) \colon \Delta_n \to \mathscr{S}_\Omega^n$ is Lipschitz, so that the push-forward $\left(\Psi_t \circ \mathscr{D}(\gamma(a))\right)_\#[\Delta_n]$ is a well-defined n-dimensional current of $\mathscr{S}_\Omega^n$ (see the appendix). The proof of Proposition 5.2 relies on the following more general statement:

NOTATION 5.3. – *Given a map* $\underline{C} = (C_1, \ldots, C_n) \colon J \times \Delta_n \to \mathscr{S}_\Omega^n$, *for each* $t \in J$ *we denote by* $\underline{C}_t \colon \Delta_n \to \mathscr{S}_\Omega^n$ *the partial map defined by*

$$\underline{s} \in \Delta_n \mapsto \underline{C}_t(\underline{s}) := \underline{C}(t, \underline{s})$$

(not to be confused with the components $C_j \colon J \times \Delta_n \to \mathscr{S}_\Omega$, $j = 1, \ldots, n$*).*

PROPOSITION 5.4. – *Let* β *be a holomorphic* n*-form on* $\mathscr{S}_\Omega^n$ *and*

$$F \colon \zeta \in \mathbb{D}_{\rho(\Omega)} \mapsto F(\zeta) := \mathscr{D}(\zeta)_\#[\Delta_n](\beta).$$

Then F *is a holomorphic function in* $\mathbb{D}_{\rho(\Omega)}$.

Let $\underline{C} \colon J \times \Delta_n \to \mathscr{S}_\Omega^n$ *be a Lipschitz map*[7] *such that the partial map corresponding to* $t = a$ *satisfies*

$$\underline{C}_a = \mathscr{D}\left(\gamma(a)\right)$$

and that, for every $t \in J$, $\underline{s} = (s_1, \ldots, s_n) \in \Delta_n$ *and* $j = 1, \ldots, n$,

$$s_1 + \cdots + s_n = 1 \quad \Rightarrow \quad \underline{C}(t, \underline{s}) \in \mathscr{N}\left(\gamma(t)\right)$$

$$s_j = 0 \qquad \Rightarrow \qquad \underline{C}(t, \underline{s}) \in \mathscr{N}_j.$$

[6] By that, we mean that each point of $\mathscr{S}_\Omega^n$ admits an open neighbourhood $\mathscr{U}$ on which $\pi_\Omega^{\otimes n} \colon \mathscr{S}_\Omega^n \to \mathbb{C}^n$ induces a biholomorphism and such that the map $(t, \underline{\xi}) \in J \times \pi_\Omega^{\otimes n}(\mathscr{U}) \mapsto \pi_\Omega^{\otimes n} \circ \Psi_t \circ \left((\pi_\Omega^{\otimes n})_{|\mathscr{U}}\right)^{-1}(\underline{\xi}) \in \mathbb{C}^n$ is Lipschitz.
[7] In the sense that $\pi_\Omega^{\otimes n} \circ \underline{C} \colon J \times \Delta_n \to \mathbb{C}^n$ is Lipschitz.

Then F admits analytic continuation along γ and, for each $t \in J$,

$$(22) \qquad\qquad F\big(\tilde{\gamma}(t)\big) = (\underline{C}_t)_{\#}[\Delta_n](\beta).$$

The proof of Proposition 5.4 requires the following consequence of the Cauchy-Poincaré Theorem [38]:

LEMMA 5.5. – *Let M be a complex analytic manifold of dimension n and let $N_0, N_1, \ldots, N_n$ be complex analytic hypersurfaces of M. Let $\underline{H} \colon [0,1] \times \Delta_n \to M$ be a Lipschitz map such that, for every $\tau \in [0,1]$, $\underline{s} = (s_1, \ldots, s_n) \in \Delta_n$ and $j = 1, \ldots, n$,*

$$s_1 + \cdots + s_n = 1 \quad \Rightarrow \quad \underline{H}(\tau, s) \in N_0$$

$$s_j = 0 \qquad\qquad \Rightarrow \quad \underline{H}(\tau, s) \in N_j.$$

Then the partial maps $\underline{H}_0$ and $\underline{H}_1$ corresponding to $\tau = 0$ and $\tau = 1$ satisfy

$$(23) \qquad\qquad (\underline{H}_0)_{\#}[\Delta_n](\beta) = (\underline{H}_1)_{\#}[\Delta_n](\beta)$$

for any holomorphic n-form β on M.

Proof of Lemma 5.5. – Let β be a holomorphic n-form on M. Let us consider $P := [0,1] \times \Delta_n$ and the corresponding $(n+1)$-dimensional integration current $[P] \in \mathscr{E}_{n+1}(\mathbb{R}^{n+1})$. Its boundary can be written

$$\partial[P] = Q_1 - Q_0 + B_0 + \cdots + B_n,$$

where each summand is an n-dimensional current with compact support:

$$\operatorname{spt} Q_i = \{i\} \times \Delta_n, \qquad \operatorname{spt} B_j = [0,1] \times F_j$$

with $F_j :=$ the face of $\partial\Delta_n$ defined by $s_j = 0$ if $j \geq 1$ or $s_1 + \cdots + s_n = 1$ if $j = 0$. This is a simple adaptation of Formula (47) of the appendix; in fact, $Q_i = [A_i(\Delta_n)]$ with an affine map $A_i \colon \underline{x} \in \mathbb{R}^n \mapsto (i, \underline{x}) \in \mathbb{R}^{n+1}$ and $B_j = \pm[A_j^*(\Delta_n)]$ with some other injective affine maps $A_j^* \colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ mapping Δ_n to $[0,1] \times F_j$. In this situation, according to Lemma A.3 and Formula (44), we have

$$\partial \underline{H}_{\#}[P] = \underline{H}_{\#}\partial[P], \qquad \underline{H}_{\#}Q_i = (\underline{H} \circ A_i)_{\#}[\Delta_n], \qquad \underline{H}_{\#}B_j = (\underline{H} \circ A_j^*)_{\#}[\Delta_n].$$

On the one hand, the Cauchy-Poincaré Theorem tells us that $\partial \underline{H}_{\#}[P](\beta) = 0$ (because $d\beta = 0$), and $H \circ A_i = H_i$, thus

$$(\underline{H}_0)_{\#}[\Delta_n](\beta) - (\underline{H}_1)_{\#}[\Delta_n](\beta) = \underline{H}_{\#}B_0(\beta) + \cdots + \underline{H}_{\#}B_n(\beta).$$

On the other hand $\operatorname{spt} \underline{H}_{\#}B_j \subset N_j$ and the restriction of β to any complex hypersurface vanishes identically (because it is a holomorphic form of maximal degree), thus $\underline{H}_{\#}B_j(\beta) = 0$, and (23) is proved. $\qquad\square$

Proof of Proposition 5.4. – Observe that the function R_Ω defined by (6) is continuous, thus we can define a positive number

$$R^* := \min\big\{ R_\Omega\big(C_j(t, \underline{s})\big) \mid t \in J,\ \underline{s} \in \Delta_n,\ j = 1, \ldots, n \big\}$$

and, for each $t \in J$ and $\zeta \in D\big(\gamma(t), R^*\big)$, a Lipschitz map and a complex number

$$\underline{\mathscr{D}}_t(\zeta) \colon \underline{s} \in \Delta_n \mapsto \underline{C}(t, \underline{s}) + \big(\zeta - \gamma(t)\big)\underline{s} \in \mathscr{S}_\Omega^n, \qquad G_t(\zeta) := \underline{\mathscr{D}}_t(\zeta)_{\#}[\Delta_n](\beta).$$

For $\zeta \in D\big(\gamma(a), R^*\big)$, we have $\underline{\mathscr{D}}_a(\zeta) = \underline{\mathscr{D}}(\zeta)$, hence $G_a(\zeta) = F(\zeta)$. For $t \in J$, we have $\underline{\mathscr{D}}_t\big(\gamma(t)\big) = \underline{C}_t$, hence

$$G_t\big(\gamma(t)\big) = (\underline{C}_t)_{\#}[\Delta_n](\beta).$$

Therefore it suffices to show that, for each $t \in J$,

- i) the function G_t is holomorphic in $D\big(\gamma(t), R^*\big)$ (and $G_a = F$ is even holomorphic in $\mathbb{D}_{\rho(\Omega)}$);
- ii) for any $t' \in J$ close enough to t, the functions G_t and $G_{t'}$ coincide in a neighbourhood of $\gamma(t)$.

i) The case of $G_a = F$ is easier because, for $\zeta \in \mathbb{D}_{\rho(\Omega)} \cup D\big(\gamma(a), R^*\big)$, the range of $\underline{\mathscr{D}}_a(\zeta) = \underline{\mathscr{D}}(\zeta)$ entirely lies in a domain $\mathscr{U} = \mathscr{U}_1 \times \cdots \mathscr{U}_n$, where each $\mathscr{U}_j$ is an open subset of $\mathscr{S}_\Omega$ in restriction to which π_Ω is injective, so that

$$(24) \qquad \chi = \pi_\Omega^{\otimes n} \colon (\zeta_1, \dots, \zeta_n) \in \mathscr{U} \mapsto (\xi_1, \dots, \xi_n) = (\overset{\bullet}{\zeta}_1, \dots, \overset{\bullet}{\zeta}_n)$$

is an analytic chart of $\mathscr{S}_\Omega^n$; we can write $\chi^{\#}\beta = f(\xi_1, \dots, \xi_n)\, \mathrm{d}\xi_1 \wedge \cdots \wedge \mathrm{d}\xi_n$ with a holomorphic function f and $\chi \circ \underline{\mathscr{D}}_a(\zeta)(\underline{s}) = (s_1\zeta, \dots, s_n\zeta)$, therefore

$$G_a(\zeta) = F(\zeta) = \zeta^n \int_{\Delta_n} f(s_1\zeta, \dots, s_n\zeta)\, \mathrm{d}s_1 \cdots \mathrm{d}s_n$$

is holomorphic.

Given $t \in J$, by compactness, we can cover Δ_n by simplices $Q^{[m]}$, $1 \le m \le M$, so that any intersection $Q^{[m]} \cap Q^{[m']}$ is contained in an affine hyperplane of $\mathbb{R}^n$ and each $Q^{[m]}$ is small enough for $\bigcup_{\zeta \in D(\gamma(t), R^*)} \underline{\mathscr{D}}_t(\zeta)\big(Q^{[m]}\big)$ to be contained in the domain $\mathscr{U}^{[m]}$ of an analytic chart $\chi^{[m]}$ similar to (24) (i.e., $\mathscr{U}^{[m]}$ is a product of factors on which π_Ω is injective and $\chi^{[m]}$ is defined by the same formula as χ but on $\mathscr{U}^{[m]}$). For each m, we can write $\big(\chi^{[m]}\big)^{\#}\beta = f^{[m]}(\xi_1, \dots, \xi_n)\, \mathrm{d}\xi_1 \wedge \cdots \wedge \mathrm{d}\xi_n$ with a holomorphic function $f^{[m]}$ and $\chi^{[m]} \circ \underline{\mathscr{D}}_t(\zeta) = \big(\xi_1^{[m]}(\zeta, \cdot), \dots, \xi_n^{[m]}(\zeta, \cdot)\big)$ with, for each $j = 1, \dots, n$,

$$(\zeta, \underline{s}) \in D\big(\gamma(t), R^*\big) \times Q^{[m]} \mapsto \xi_j^{[m]}(\zeta, \underline{s}) = \pi_\Omega \circ C_j(t, \underline{s}) + s_j\big(\zeta - \gamma(t)\big).$$

These functions $\xi_j^{[m]}$ are holomorphic in ζ; applying Rademacher's theorem to $\underline{s} \mapsto \pi_\Omega \circ C_j(t, \underline{s})$ (recall that t is fixed), we see that, for almost every $\underline{s}$, the partial derivatives of $\xi_j^{[m]}$ exist and are holomorphic in ζ, therefore

$$G_t(\zeta) = \sum_{m=1}^{M} \int_{Q^{[m]}} f^{[m]}\big(\xi_1^{[m]}(\zeta, \underline{s}), \dots, \xi_n^{[m]}(\zeta, \underline{s})\big) \det\left[\frac{\partial \xi_i^{[m]}}{\partial s_j}(\zeta, \underline{s})\right]_{1 \le i, j \le n} \mathrm{d}s_1 \cdots \mathrm{d}s_n$$

is holomorphic for $\zeta \in D\big(\gamma(t), R^*\big)$.

ii) We now fix $t \in J$. By compactness, for $t' \in J$ close enough to t, we can write

$$\underline{C}(t', \underline{s}) = \underline{C}(t, \underline{s}) + \underline{\delta}(\underline{s})$$

for all $\underline{s} \in \Delta_n$, with

$$\delta_j(\underline{s}) := \pi_\Omega\big(C_j(t', \underline{s}) - C_j(t, \underline{s})\big) \in \mathbb{D}_{\frac{R^*}{2n}}, \qquad j = 1, \dots, n.$$

Then $\gamma(t') \in D\big(\gamma(t), R^*/2\big)$ (because $s_1 + \cdots + s_n = 1$ implies $S_n \circ \delta(\underline{s}) = \gamma(t') - \gamma(t)$) and, for $\zeta \in D\big(\gamma(t'), R^*/2\big)$, we have

$$G_t(\zeta) := \underline{\mathscr{D}}_t(\zeta)_{\#}[\Delta_n](\beta), \qquad G_{t'}(\zeta) := \underline{\mathscr{D}}_{t'}(\zeta)_{\#}[\Delta_n](\beta)$$

with

$$\underline{\mathscr{D}}_t(\zeta)(\underline{s}) = \underline{C}(t, \underline{s}) + \big(\zeta - \gamma(t)\big)\underline{s}, \qquad \underline{\mathscr{D}}_{t'}(\zeta)(\underline{s}) = \underline{C}(t, \underline{s}) + \underline{\delta}(s) + \big(\zeta - \gamma(t')\big)\underline{s}.$$

Let us define a Lipschitz map $\underline{H} \colon [0, 1] \times \Delta_n \to \mathscr{S}_\Omega^n$ by

$$\underline{H}(\tau, \underline{s}) := \underline{C}(t, \underline{s}) + (1 - \tau)\big(\zeta - \gamma(t)\big)\underline{s} + \tau\big(\underline{\delta}(s) + \big(\zeta - \gamma(t')\big)\underline{s}\big).$$

An easy computation yields

$$s_1 + \cdots + s_n = 1 \quad \Rightarrow \quad S_n \circ \underline{H}(\tau, \underline{s}) = \zeta$$
$$s_j = 0 \qquad \Rightarrow \quad H_j(\tau, \underline{s}) = 0_\Omega.$$

We can thus apply Lemma 5.5 with $N_0 = \mathscr{N}(\zeta)$ and $N_j = \mathscr{N}_j$, and conclude that $G_t \equiv G_{t'}$ on $D\big(\gamma(t'), R^*/2\big)$. $\qquad\square$

Proof of Proposition 5.4. – In view of Proposition 4.3, we can apply Proposition 5.4 with $\beta = \hat{\varphi}_1(\zeta_1) \cdots \hat{\varphi}_n(\zeta_n) \,\mathrm{d}\zeta_1 \wedge \cdots \wedge \mathrm{d}\zeta_n$ and $\underline{C}_t = \Psi_t \circ \underline{\mathscr{D}}\big(\gamma(a)\big)$. $\qquad\square$

6. Construction of an adapted origin-fixing isotopy in $\mathscr{S}_\Omega^n$

To prove Theorem 1, Formula (18) tells us that it is sufficient to deal with the analytic continuation of products of the form $1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n$ instead of $\hat{\varphi}_1 * \cdots * \hat{\varphi}_n$ itself, and Proposition 5.2 tells us that, to do so, we only need to construct explicit γ-adapted origin-fixing isotopies (Ψ_t) and to provide estimates.

This section aims at constructing (Ψ_t) for any given C^1 path γ (estimates are postponed to Section 7). Our method is inspired by an appendix of [3] and is a generalization of Section 6.2 of [36].

PROPOSITION 6.1. – *Let* $\gamma \colon J = [a, b] \to \mathbb{C} \setminus \Omega$ *be a* C^1 *path such that* $\gamma(a) \in \mathbb{D}_{\rho(\Omega)}^*$, *and let* $\eta \colon \mathbb{C} \to [0, +\infty)$ *be a locally Lipschitz function such that*

$$\{ \xi \in \mathbb{C} \mid \eta(\xi) = 0 \} = \Omega.$$

Then the function

$$(25) \qquad (t, \underline{\zeta}) \in J \times \mathscr{S}_\Omega^n \mapsto D(t, \underline{\zeta}) := \eta(\overset{\bullet}{\zeta}_1) + \cdots + \eta(\overset{\bullet}{\zeta}_n) + \eta\big(\gamma(t) - S_n(\underline{\zeta})\big)$$

is everywhere positive and the formula

$$(26) \qquad X(t, \underline{\zeta}) = \left| \begin{array}{l} X_1 := \dfrac{\eta(\overset{\bullet}{\zeta}_1)}{D(t, \underline{\zeta})}\gamma'(t) \\[2ex] \quad\vdots \\[2ex] X_n := \dfrac{\eta(\overset{\bullet}{\zeta}_n)}{D(t, \underline{\zeta})}\gamma'(t) \end{array} \right.$$

defines a non-autonomous vector field $X(t, \underline{\zeta}) \in T_{\underline{\zeta}}(\mathscr{S}_{\Omega}^n) \simeq \mathbb{C}^n$ *(using the canonical identification between the tangent space of* $\mathscr{S}_{\Omega}$ *at any point and* $\mathbb{C}$ *provided by the tangent map of the local biholomorphism* π_{Ω} *) which admits a flow map* Ψ_t *between time* a *and time* t *for every* $t \in J$ *and induces a* γ*-adapted origin-fixing isotopy* $(\Psi_t)_{t \in J}$ *in* $\mathscr{S}_{\Omega}^n$.

An example of function which satisfies the assumptions of Proposition 6.1 is

$$\eta(\xi) := \mathrm{dist}(\xi, \Omega), \qquad \xi \in \mathbb{C}.$$

Proof of Proposition 6.1. – (a) Observe that $D\big(t, (\zeta_1, \ldots, \zeta_n)\big) = \tilde{D}\big(t, (\overset{\bullet}{\zeta}_1, \ldots, \overset{\bullet}{\zeta}_n)\big)$ with

$$(27) \qquad (t, \underline{\xi}) \in J \times \mathbb{C}^n \mapsto \tilde{D}(t, \underline{\xi}) := \eta(\xi_1) + \cdots + \eta(\xi_n) + \eta\big(\gamma(t) - \tilde{S}_n(\underline{\xi})\big)$$

and $\tilde{S}_n(\underline{\xi}) := \xi_1 + \cdots + \xi_n$ for any $\underline{\xi} \in \mathbb{C}^n$. The function $\tilde{D}$ is everywhere positive: suppose indeed $\tilde{D}(t, \underline{\xi}) = 0$ with $t \in J$ and $\underline{\xi} \in \mathbb{C}^n$, we would have

$$\xi_1, \ldots, \xi_n, \gamma(t) - \tilde{S}(\underline{\xi}) \in \Omega,$$

whence $\gamma(t) \in \Omega$ by the stability under addition of Ω, but this is contrary to the hypothesis on γ.

Therefore $D > 0$, the vector field X is well-defined and in fact

$$X\big(t, (\zeta_1, \ldots, \zeta_n)\big) = \tilde{X}\big(t, (\overset{\bullet}{\zeta}_1, \ldots, \overset{\bullet}{\zeta}_n)\big)$$

with a non-autonomous vector field $\tilde{X}$ defined in $J \times \mathbb{C}^n$, the components of which are

$$(28) \qquad \tilde{X}_j(t, \underline{\xi}) := \frac{\eta(\xi_j)}{\tilde{D}(t, \underline{\xi})} \gamma'(t), \qquad j = 1, \ldots, n.$$

These functions are locally Lipschitz on $J \times \mathbb{C}^n$, thus we can apply the Cauchy-Lipschitz theorem on the existence and uniqueness of solutions to differential equations to $d\underline{\xi}/dt = \tilde{X}(t, \underline{\xi})$: for every $t^* \in J_k$ and $\underline{\xi} \in \mathbb{C}^n$, there is a unique maximal solution $t \mapsto \tilde{\Phi}^{t^*, t}(\underline{\xi})$ such that $\tilde{\Phi}^{t^*, t^*}(\underline{\xi}) = \underline{\xi}$. The fact that the vector field $\tilde{X}$ is bounded implies that $\tilde{\Phi}^{t^*, t}(\underline{\xi})$ is defined for all $t \in J$ and the classical theory guarantees that $(t^*, t, \underline{\xi}) \mapsto \tilde{\Phi}^{t^*, t}(\underline{\xi})$ is locally Lipschitz on $J \times J \times \mathbb{C}^n$.

(b) For each $\omega \in \Omega$ and $j = 1, \ldots, n$, we set

$$\tilde{\mathscr{N}}_j(\omega) := \{ \underline{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n \mid \xi_j = \omega \}.$$

We have $\tilde{X}_j \equiv 0$ on $J \times \tilde{\mathscr{N}}_j(\omega)$, thus $\tilde{\Phi}^{t^*, t}$ leaves $\tilde{\mathscr{N}}_j(\omega)$ invariant for every $(t^*, t) \in J \times J$. In particular, since $0 \in \Omega$,

$$(29) \qquad \underline{\xi} \in \tilde{\mathscr{N}}_j(0) \quad \Rightarrow \quad \tilde{\Phi}^{t^*, t}(\underline{\xi}) \in \tilde{\mathscr{N}}_j(0).$$

The non-autonomous flow property $\tilde{\Phi}^{t, t^*} \circ \tilde{\Phi}^{t^*, t} = \tilde{\Phi}^{t^*, t} \circ \tilde{\Phi}^{t, t^*} = \mathrm{Id}$ implies that, for each $(t^*, t) \in J \times J$, $\tilde{\Phi}^{t^*, t}$ is a homeomorphism the inverse of which is $\tilde{\Phi}^{t, t^*}$, which leaves $\tilde{\mathscr{N}}_j(\omega)$ invariant, hence also

$$(30) \qquad \underline{\xi} \in \mathbb{C}^n \setminus \tilde{\mathscr{N}}_j(\omega) \quad \Rightarrow \quad \tilde{\Phi}^{t^*, t}(\underline{\xi}) \in \mathbb{C}^n \setminus \tilde{\mathscr{N}}_j(\omega).$$

Properties (29) and (30) show that the flow map between times t^* and t for X is well-defined in $\mathscr{S}_{\Omega}^n$: for $\underline{\zeta} \in \mathscr{S}_{\Omega}^n$, the solution $t \mapsto \Phi^{t^*, t}(\underline{\zeta})$ can be obtained as the lift starting at $\underline{\zeta}$ of the

path $t \mapsto \tilde{\Phi}^{t^*,t}\big(\overset{\bullet}{\zeta}_1, \ldots, \overset{\bullet}{\zeta}_n\big)$ (indeed, each component of this path has its range either reduced to $\{0\}$ or contained in $\mathbb{C} \setminus \Omega$).

We thus define, for each $t \in J$, a homeomorphism of $\mathscr{S}_\Omega^n$ by $\Psi_t := \Phi^{a,t}$ and observe that $\Psi_t(\mathscr{N}_j) \subset \mathscr{N}_j$, $\Psi_a = \mathrm{Id}$ and $(t, \underline{\varsigma}) \mapsto \Psi_t(\underline{\varsigma})$ is locally Lipschitz on $J \times \mathscr{S}_\Omega^n$.

(c) It only remains to be proved that

$$(31) \qquad\qquad \Psi_t\big(\mathscr{N}(\gamma(a))\big) \subset \Psi_t\big(\mathscr{N}(\gamma(t))\big)$$

for every $t \in J$.

Given $\underline{\varsigma} \in \mathscr{S}_\Omega^n$, the function defined by

$$\xi_0 \colon t \in J \mapsto \gamma(t) - S_n \circ \Psi_t(\underline{\varsigma})$$

is C^1 on J and an easy computation yields its derivative in the form $\xi_0'(t) = h(t)\gamma'(t)/d(t)$, with Lipschitz functions

$$h(t) := \eta\big(\xi_0(t)\big), \qquad d(t) := D\big(t, \Psi_t(\underline{\varsigma})\big).$$

Since η is Lipschitz on the range of ξ_0, say with Lipschitz constant K, the function $h = \eta \circ \xi_0$ is Lipschitz on J, hence its derivative h' exists almost everywhere on J; writing $|h(t') - h(t)| \le K|\xi_0(t') - \xi_0(t)|$, we see that $|h'(t)| \le K|\xi_0'(t)| \le Kh(t)\max_J|\frac{\gamma'}{d}|$ a.e., hence

$$g(t) := \frac{h'(t)}{h(t)} \text{ exists a.e. and defines } g \in L^\infty(J).$$

By the fundamental theorem of Lebesgue integral calculus, $t \mapsto \int_a^t g(\tau)\,\mathrm{d}\tau$ is differentiable a.e. and

$$h(t) = h(a) \exp\Big(\int_a^t g(\tau)\,\mathrm{d}\tau\Big), \qquad t \in J.$$

Now, if $\underline{\varsigma} \in \mathscr{N}\big(\gamma(a)\big)$, then $\xi_0(a) = 0$, thus $h(a) = 0$, thus $h \equiv 0$ on J, thus $\xi_0(t)$ stays in Ω for all $t \in J$, thus $\xi_0 \equiv 0$ on J, i.e., $\Psi_t(\underline{\varsigma}) \in \mathscr{N}\big(\gamma(t)\big)$. $\qquad\square$

7. Estimates

We are now ready to prove

THEOREM 1′. – *Let $\delta, L > 0$ with $\delta < \rho(\Omega)/2$ and*

$$(32) \qquad\qquad \delta' := \frac{1}{2}\rho(\Omega)\,\mathrm{e}^{-2L/\delta}.$$

Then, for any $n \ge 1$ and $\hat{\varphi}_1, \ldots, \hat{\varphi}_n \in \hat{\mathscr{R}}_\Omega$,

$$(33) \qquad \max_{\mathscr{K}_{\delta,L}(\Omega)} |1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n| \le \frac{1}{n!}\big(\rho(\Omega)\,\mathrm{e}^{3L/\delta}\big)^n \max_{\mathscr{K}_{\delta',L}(\Omega)} |\hat{\varphi}_1| \cdots \max_{\mathscr{K}_{\delta',L}(\Omega)} |\hat{\varphi}_n|.$$

The proof of Theorem 1′ will follow from

PROPOSITION 7.1. – *Let $\delta, L > 0$. Let $\gamma\colon J = [a,b] \to \mathbb{C} \setminus \Omega$ be a C^1 path such that $\gamma(a) \in \mathbb{D}^*_{\rho(\Omega)/2}$, $|\gamma(a)| + b - a \le L$ and*

$$|\gamma'(t)| = 1 \ \text{ and } \ \text{dist}\left(\gamma(t), \Omega\right) \ge \delta, \qquad t \in J.$$

Consider the γ-adapted origin-fixing isotopy $(\Psi_t)_{t\in J}$ defined as in Proposition 6.1 by the flow of the vector field (26) with the choice $\eta(\xi) = \text{dist}(\xi, \Omega)$. Then, for each $t \in J$,

- *the Lipschitz map $\Psi_t \circ \mathscr{D}\left(\gamma(a)\right) = (\xi_1^t, \ldots, \xi_n^t)$ maps Δ_n in $\left(\mathscr{K}_{\delta',L}(\Omega)\right)^n$, with δ' as in (32),*

- *the almost everywhere defined partial derivatives $\frac{\partial \dot{\xi}_i^t}{\partial s_j}\colon \Delta_n \to \mathbb{C}$ satisfy*

$$(34) \qquad \left| \det \left[\frac{\partial \dot{\xi}_i^t}{\partial s_j}(\underline{s}) \right]_{1 \le i,j \le n} \right| \le \left(\rho(\Omega)\, \mathrm{e}^{3L/\delta} \right)^n \quad \text{for a.e. } \underline{s} \in \Delta_n.$$

Proof of Proposition 7.1. – We first fix $\underline{s} \in \Delta_n$, omitting it in the notations, and study the solution

$$t \in J \mapsto \underline{\xi}^t := (\xi_1^t, \ldots, \xi_n^t) := \Psi_t\left(\mathscr{D}\left(\gamma(a)\right)(\underline{s})\right)$$

of the vector field X defined by (26), the components of the initial condition being $\xi_i^0 = 0_\Omega + s_i \gamma(a)$.

(a) We observe that $\mathrm{d}\dot{\xi}_i^t/\mathrm{d}t = X_i(t, \underline{\xi}^t)$ has modulus ≤ 1 for each $i = 1, \ldots, n$, thus the path $t \in J \mapsto \dot{\xi}_i^t$ has length $\le b - a$ and stays in $\mathbb{D}_L$.

(b) The denominator (25) is

$$d(t) := D(t, \underline{\xi}^t) \ge \delta, \qquad t \in J.$$

Indeed, we can write $d(t) = \eta\left(\dot{\xi}_0^t\right) + \eta\left(\dot{\xi}_1^t\right) + \cdots + \eta\left(\dot{\xi}_n^t\right)$ with $\dot{\xi}_0^t := \gamma(t) - S_n(\underline{\xi}^t)$, and, since Ω is stable under addition and $\dot{\xi}_0^t + \dot{\xi}_1^t + \cdots + \dot{\xi}_n^t = \gamma(t)$, the triangle inequality yields

$$d(t) = \sum_{i=0}^n \text{dist}\left(\dot{\xi}_i^t, \Omega\right) \ge \text{dist}\left(\gamma(t), \Omega\right),$$

which is $\ge \delta$ by assumption.

(c) We now check that for $t \in J$ and $i = 1, \ldots, n$,

$$(35) \qquad \mathrm{e}^{-L/\delta}\, \eta\left(\dot{\xi}_i^a\right) \le \eta\left(\dot{\xi}_i^t\right) \le \mathrm{e}^{L/\delta}\, \eta\left(\dot{\xi}_i^a\right).$$

Since η is 1-Lipschitz, the function $h_i := \eta \circ \dot{\xi}_i^t$ is Lipschitz on J and its derivative exists a.e.; writing $|h_i(t') - h_i(t)| \le |\dot{\xi}_i(t') - \dot{\xi}_i(t)|$, we see that a.e. $|h_i'(t)| \le |\dot{\xi}_i'(t)| = h_i(t)/d(t)$ hence $g_i(t) := \frac{h_i'(t)}{h_i(t)}$ exists a.e. and defines $g_i \in L^\infty(J)$ with

$$(36) \qquad |g_i(t)| \le 1/\delta \quad \text{for a.e. } t \in J.$$

By the fundamental theorem of Lebesgue integral calculus, $t \mapsto \int_a^t g_i(\tau)\,\mathrm{d}\tau$ is differentiable a.e. and

$$h_i(t) = h_i(a) \exp\left(\int_a^t g_i(\tau)\,\mathrm{d}\tau \right), \qquad t \in J,$$

whence (35) follows in view of (36).

(d) Now, the fact that $\overset{\bullet}{\xi}{}_i^a = s_i\gamma(a) \in \mathbb{D}_{\rho(\Omega)/2}$ implies that dist $\big(\overset{\bullet}{\xi}{}_i^a, \Omega\setminus\{0\}\big) \geq \rho(\Omega)/2$, whence

$$\eta\big(\overset{\bullet}{\xi}{}_i^a\big) = \text{dist}\,\big(\overset{\bullet}{\xi}{}_i^a, \Omega\big) = |\overset{\bullet}{\xi}{}_i^a| \leq \rho(\Omega)/2.$$

If $|\overset{\bullet}{\xi}{}_i^a| < \frac{1}{2}\rho(\Omega)\,\mathrm{e}^{-L/\delta}$, then the second inequality in (35) shows that $\eta\big(\overset{\bullet}{\xi}{}_i^t\big)$ stays $< \frac{1}{2}\rho(\Omega)$, hence ξ_i^t stays in the lift of $\mathbb{D}_{\rho(\Omega)/2}$ in the principal sheet and $R_\Omega(\xi_i^t)$ stays $\geq \frac{1}{2}\rho(\Omega) > \delta'$.

If $|\overset{\bullet}{\xi}{}_i^a| \geq \frac{1}{2}\rho(\Omega)\,\mathrm{e}^{-L/\delta}$, then the first inequality in (35) shows that $\eta\big(\overset{\bullet}{\xi}{}_i^t\big) \geq \frac{1}{2}\rho(\Omega)\,\mathrm{e}^{-2L/\delta}$ which equals δ', hence $R_\Omega(\xi_i^t)$ stays $\geq \delta'$.

We infer that $\xi_i^t \in \mathscr{K}_{\delta',L}(\Omega)$ for all $t \in J$ in both cases (in view of point (a), since $\xi_i^t \in \mathscr{S}_\Omega$ can be represented by the path $\Gamma_s|t \in \mathscr{P}_\Omega$ which is obtained by concatenation of $[0, s_i\gamma(a)]$ and $\tau \in [a,t] \mapsto \overset{\bullet}{\xi}{}_i^\tau$ and has length $\leq |\gamma(a)| + b - a \leq L$).

(e) It only remains to study the partial derivatives $\frac{\partial \xi_i^t}{\partial s_j}(\underline{s})$ which, given $t \in J$, exist for almost every $\underline{s} \in \Delta_n$ by virtue of Rademacher's theorem. We first prove that for every $t \in J$, $\underline{s}, \underline{s}' \in \Delta_n$,

$$(37) \qquad \sum_{i=1}^n \Big|\overset{\bullet}{\xi}{}_i^t(\underline{s}') - \overset{\bullet}{\xi}{}_i^t(\underline{s})\Big| \leq \mathrm{e}^{3L/\delta}|\gamma(a)| \sum_{i=1}^n |s_i' - s_i|.$$

LEMMA 7.2. – *Whenever the function η is 1-Lipschitz on $\mathbb{C}$ and $|\gamma'(\tau)| \leq 1$ for all $\tau \in J$, the vector field X defined by (25)–(26) satisfies*

$$(38) \qquad \sum_{i=1}^n |X_i(\tau,\underline{\varsigma}') - X_i(\tau,\underline{\varsigma})| \leq \frac{3}{D(\tau,\underline{\varsigma}')} \sum_{i=1}^n \Big|\overset{\bullet}{\varsigma}{}_i' - \overset{\bullet}{\varsigma}{}_i\Big|$$

for any $\tau \in J$ and $\underline{\varsigma}, \underline{\varsigma}' \in \mathscr{S}_\Omega^n$.

Proof of Lemma 7.2. – Let $\tau \in J$ and $\underline{\varsigma}, \underline{\varsigma}' \in \mathscr{S}_\Omega^n$. For $i = 1, \ldots, n$, we can write

$$X_i(\tau,\underline{\varsigma}') - X_i(\tau,\underline{\varsigma}) = \left(\eta\big(\overset{\bullet}{\varsigma}{}_i'\big) - \eta\big(\overset{\bullet}{\varsigma}{}_i\big) - \big(D(\tau,\underline{\varsigma}') - D(\tau,\underline{\varsigma})\big)\frac{\eta\big(\overset{\bullet}{\varsigma}{}_i\big)}{D(\tau,\underline{\varsigma})}\right)\frac{\gamma'(\tau)}{D(\tau,\underline{\varsigma}')},$$

with $D(\tau,\underline{\varsigma}') - D(\tau,\underline{\varsigma}) = \sum_{j=0}^n \big(\eta\big(\overset{\bullet}{\varsigma}{}_j'\big) - \eta\big(\overset{\bullet}{\varsigma}{}_j\big)\big)$, using the notations $\overset{\bullet}{\varsigma}{}_0 = \gamma(\tau) - S_n(\underline{\varsigma})$, $\overset{\bullet}{\varsigma}{}_0' = \gamma(\tau) - S_n(\underline{\varsigma}')$. Since η is 1-Lipschitz, we have $|\eta\big(\overset{\bullet}{\varsigma}{}_j'\big) - \eta\big(\overset{\bullet}{\varsigma}{}_j\big)| \leq |\overset{\bullet}{\varsigma}{}_j' - \overset{\bullet}{\varsigma}{}_j|$ for $j = 0, \ldots, n$ and $|\overset{\bullet}{\varsigma}{}_0' - \overset{\bullet}{\varsigma}{}_0| \leq \sum_{j=1}^n |\overset{\bullet}{\varsigma}{}_j' - \overset{\bullet}{\varsigma}{}_j|$, whence $|D(\tau,\underline{\varsigma}') - D(\tau,\underline{\varsigma})| \leq \sum_{j=0}^n |\overset{\bullet}{\varsigma}{}_j' - \overset{\bullet}{\varsigma}{}_j| \leq 2\sum_{j=1}^n |\overset{\bullet}{\varsigma}{}_j' - \overset{\bullet}{\varsigma}{}_j|$. The result follows because $\sum_{i=1}^n \eta\big(\overset{\bullet}{\varsigma}{}_i\big) \leq D(\tau,\underline{\varsigma})$. $\qquad\square$

Proof of inequality (37). – Let us fix $\underline{s}, \underline{s}' \in \Delta_n$ and denote by $\Delta(t)$ the left-hand side of (37), i.e.,

$$\Delta(t) = \sum_{i=1}^n |\Delta_i(t)|, \qquad \Delta_i(t) := \overset{\bullet}{\xi}{}_i^t(\underline{s}') - \overset{\bullet}{\xi}{}_i^t(\underline{s}).$$

For every $t \in J$, we have

$$\Delta_i(t) = \Delta_i(a) + \int_a^t \Big(X_i\big(\tau, \underline{\xi}^\tau(\underline{s}')\big) - X_i\big(\tau, \underline{\xi}^\tau(\underline{s})\big)\Big)\,\mathrm{d}\tau, \qquad i = 1, \ldots, n.$$

By Lemma (7.2), we get

$$|\Delta(t) - \Delta(a)| \leq \sum_{i=1}^n |\Delta_i(t) - \Delta_i(a)| \leq \int_a^t \frac{3}{D\big(\tau, \underline{\xi}^\tau(\underline{s}')\big)}\Delta(\tau)\,\mathrm{d}\tau.$$

We have seen that $D\big(\tau, \underline{\xi}^{\tau}(\underline{s}')\big)$ stays $\geq \delta$ (this was point (b)), thus $|\Delta(t) - \Delta(a)| \leq \frac{3}{\delta}\int_a^t \Delta(\tau)\,\mathrm{d}\tau$ for all $t \in J$. Gronwall's lemma yields

$$|\Delta(t)| \leq \Delta(a)\,\mathrm{e}^{3(t-a)/\delta}, \qquad t \in J,$$

and, in view of the initial conditions $\Delta_i(a) = (s_i' - s_i)\gamma(a)$, (37) is proved. $\qquad\square$

(f) Let us fix $t \in J$. For any $\underline{s} \in \Delta_n$ at which $(\overset{\bullet}{\xi}_1^t, \ldots, \overset{\bullet}{\xi}_n^t)$ is differentiable, because of (37), the entries of the matrix $\mathscr{J} := \left[\frac{\partial \overset{\bullet}{\xi}_i^t}{\partial s_j}(\underline{s})\right]_{1 \leq i,j \leq n}$ satisfy

$$\sum_{i=1}^n \left|\frac{\partial \overset{\bullet}{\xi}_i^t}{\partial s_j}(\underline{s})\right| \leq \mathrm{e}^{3L/\delta}|\gamma(a)|, \qquad j = 1, \ldots, n.$$

We conclude by observing that

$$|\det(\mathscr{J})| \leq \Big(\sum_{i=1}^n |\mathscr{J}_{i,1}|\Big) \cdots \Big(\sum_{i=1}^n |\mathscr{J}_{i,n}|\Big) \leq \big(\mathrm{e}^{3L/\delta}|\gamma(a)|\big)^n$$

(because the left-hand side is bounded by the sum of the products $\big|\mathscr{J}_{\sigma(1),1} \cdots \mathscr{J}_{\sigma(n),n}\big|$ over all bijective maps $\sigma\colon [1,n] \to [1,n]$, while the middle expression is equal to the sum of the same products over *all* maps $\sigma\colon [1,n] \to [1,n]$). $\qquad\square$

Proof of Theorem 1'. – Let $\delta, L > 0$ with $\delta < \rho(\Omega)/2$ and $\zeta \in \mathscr{K}_{\delta,L}(\Omega)$. We want to prove

$$|1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n(\zeta)| \leq \frac{1}{n!}\big(\rho(\Omega)\,\mathrm{e}^{3L/\delta}\big)^n \max_{\mathscr{K}_{\delta',L}(\Omega)} |\hat{\varphi}_1| \cdots \max_{\mathscr{K}_{\delta',L}(\Omega)} |\hat{\varphi}_n|$$

for any $n \geq 1$ and $\hat{\varphi}_1, \ldots, \hat{\varphi}_n \in \hat{\mathscr{R}}_\Omega$.

We may assume $\zeta \notin \mathscr{L}_{0_\Omega}(\mathbb{D}_{\rho(\Omega)})$ (since the behaviour of convolution products on the principal sheet is already settled by (5) and $\zeta \in \mathscr{L}_{0_\Omega}(\mathbb{D}_{\rho(\Omega)})$ would imply $\frac{|\zeta|}{n+1} < \rho(\Omega)\,\mathrm{e}^{3L/\delta}$). We can then choose a representative of ζ in $\mathscr{P}_\Omega$ which is a C^1 path, the initial part of which is a line segment ending in $\mathbb{D}_{\rho(\Omega)/2} \setminus \mathbb{D}_\delta$; since we prefer to parametrize our paths by arc-length, we take $\tilde{\gamma}\colon [\tilde{a}, b] \to \mathbb{C}$ with $\tilde{\gamma}'(t) \equiv 1$ and $\mathrm{length}(\tilde{\gamma}) = b - \tilde{a} \leq L$, and $a \in (\tilde{a}, b)$ such that

- $\tilde{\gamma}(a) \in \mathbb{D}_{\rho(\Omega)/2}$,
- $\tilde{\gamma}(t) = \frac{t-\tilde{a}}{a-\tilde{a}}\tilde{\gamma}(a)$ for all $t \in [\tilde{a}, a]$,
- $\mathrm{dist}\big(\tilde{\gamma}(t), \Omega\big) \geq \delta$ for all $t \in [a, b]$.

Now the restriction γ of $\tilde{\gamma}$ to $[a, b]$ satisfies all the assumptions of Proposition 7.1, while Formula (21) of Proposition 5.2 for $t = b$ can be interpreted as

$$(39) \quad 1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n(\zeta) = \int_{\Delta_n} \hat{\varphi}_1\big(\xi_1^b(\underline{s})\big) \cdots \hat{\varphi}_n\big(\xi_n^b(\underline{s})\big) \det\left[\frac{\partial \overset{\bullet}{\xi}_i^b}{\partial s_j}(\underline{s})\right]_{1 \leq i,j \leq n} \mathrm{d}s_1 \cdots \mathrm{d}s_n.$$

The conclusion follows immediately, since the Lebesgue measure of Δ_n is $1/n!$. $\qquad\square$

We can now prove the main result which was announced in Section 2.

Proof of Theorem 1. – Let $\delta, L > 0$ with $\delta < \rho(\Omega)$, $n \geq 1$ and $\hat{\varphi}_1, \ldots, \hat{\varphi}_n \in \hat{\mathscr{R}}_\Omega$. Let $\zeta \in \mathscr{K}_{\delta,L}(\Omega)$. We must prove

$$|\hat{\varphi}_1 * \cdots * \hat{\varphi}_n(\zeta)| \leq \frac{2}{\delta} \cdot \frac{C^n}{n!} \cdot \max_{\mathscr{K}_{\delta',L'}(\Omega)} |\hat{\varphi}_1| \cdots \max_{\mathscr{K}_{\delta',L'}(\Omega)} |\hat{\varphi}_n|.$$

One can check that any $\zeta' \in \mathscr{L}_\zeta(\mathbb{D}_{\delta/2}) = \{\, \zeta + w \mid |w| < \delta/2 \,\}$ satisfies

(40) $$\zeta' \in \mathscr{K}_{\delta/2,L'}(\Omega), \qquad \text{where } L' := L + \delta/2.$$

Indeed, ζ is the endpoint of a path γ starting from 0_Ω, of length $\leq L$, which has $R_\Omega(\gamma(t)) \geq \delta$. In particular $R_\Omega(\zeta) \geq \delta$ thus the path $t \in [0,1] \mapsto \sigma(t) := \zeta + t(\overset{\bullet}{\zeta'} - \overset{\bullet}{\zeta})$ is well-defined. Either ζ does not lie in the principal sheet of $\mathscr{S}_\Omega$, then $\mathrm{dist}(\overset{\bullet}{\zeta}, \Omega) \geq \delta$ implies $\mathrm{dist}(\sigma(t), \Omega) \geq \delta/2$ and, by concatenating γ and σ, we see that (40) holds; or ζ is in the principal sheet and then we can choose γ contained in the principal sheet and we have at least $\mathrm{dist}(\sigma(t), \Omega \setminus \{0\}) \geq \delta/2$; if moreover $\overset{\bullet}{\zeta} \in \mathbb{D}_{\rho(\Omega)}$ then also σ is contained in the principal sheet, with $R_\Omega(\sigma(t)) \geq \delta/2$, whereas if $\overset{\bullet}{\zeta} \notin \mathbb{D}_{\rho(\Omega)}$ then $\mathrm{dist}(\sigma(t), \{0\}) \geq \rho(\Omega) - \delta/2 \geq \delta/2$, hence again $R_\Omega(\sigma(t)) \geq \delta/2$, thus (40) holds in all cases.

Thus, by Theorem 1′,

$$\max_{\mathscr{L}_\zeta(\mathbb{D}_{\delta/2})} |1 * \hat{\varphi}_1 * \cdots * \hat{\varphi}_n| \leq \frac{C^n}{n!} \max_{\mathscr{K}_{\delta',L'}(\Omega)} |\hat{\varphi}_1| \cdots \max_{\mathscr{K}_{\delta',L'}(\Omega)} |\hat{\varphi}_n|$$

with $\delta' := \frac{1}{2}\rho(\Omega)\,\mathrm{e}^{-4L'/\delta}$ and $C := \rho(\Omega)\,\mathrm{e}^{6L'/\delta}$, which are precisely the values indicated in (8). The conclusion follows from the Cauchy inequalities. $\qquad\square$

REMARK 7.3. – As far as we understand, there is a mistake in [3], in the final argument given to bound a determinant analogous to our Formula (34): roughly speaking, these authors produce a deformation of the standard n-simplex through the flow of an autonomous vector field in $\mathbb{C}^n$ (the definition of which is not clear to us) and then use the linear differential equation satisfied by the Jacobian determinant of the flow; however, they overlook the fact that, since their vector field is not holomorphic, the Jacobian determinant which can be controlled this way is the real one, corresponding to the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, whereas the determinant which appears when computing the integral and that one needs to bound is a complex linear combination of the $n \times n$ minors of the $2n \times 2n$ real Jacobian matrix.

Appendix
Appendix: a class of rectifiable currents and their Lipschitz push-forwards

In this appendix, we single out a few facts from Geometric Measure Theory which are useful in the proof of our main result. Among the standard references on the subject one can quote [16, 39], [1, 28].

For a differentiable manifold M and an integer $m \geq 0$, we denote by $\mathscr{E}_m(M)$ the space of all m-dimensional currents with compact support, viewed as linear functionals on the space of all C^∞ differential m-forms (with complex-valued coefficients) which are continuous for the usual family of seminorms (defined by considering the partial derivatives of the coefficients of forms in compact subsets of charts). In fact, by taking real and imaginary parts, the situation is reduced to that of real-valued forms and real-valued currents. For us,

$M = \mathbb{R}^N$ or $M = \mathscr{S}_\Omega^n$, but in the latter case, as far as currents are concerned, the local biholomorphism $\pi_\Omega^{\otimes n}$ makes the difference between $\mathscr{S}_\Omega^n$ and $\mathbb{C}^n$ immaterial, and the complex structure plays no role, so that one loses nothing when replacing M with $\mathbb{R}^{2n}$.

Integration currents associated with oriented compact rectifiable sets

Let $m, N \in \mathbb{N}^*$. We denote by $\mathscr{H}^m$ the m-dimensional Hausdorff measure in $\mathbb{R}^N$. A basic example of m-dimensional current in $\mathbb{R}^N$ is obtained as follows:

DEFINITION A.1. – *Let S be an oriented compact m-dimensional rectifiable subset of $\mathbb{R}^N$ (i.e., S is compact, $\mathscr{H}^m$-almost all of S is contained in the union of the images of countably many Lipschitz maps from $\mathbb{R}^m$ to $\mathbb{R}^N$ and we are given a measurable orientation of the approximate tangent m-planes[8] to S) and, for $\mathscr{H}^m$-a.e. $x \in S$, let $\tau(x)$ be a unit m-vector orienting the tangent m-plane at x; then the formula*

$$(41) \qquad [S]\colon \alpha\ m\text{-form on } \mathbb{R}^N \mapsto \int_S \langle \tau(x), \alpha(x) \rangle \, \mathrm{d}\mathscr{H}^m(x)$$

defines a current $[S] \in \mathscr{E}_m(\mathbb{R}^N)$, the support of which is S.

This example belongs to the class of *integer rectifiable currents*, for which the right-hand side of (41) more generally assumes the form

$$\int_S \langle \tau(x), \alpha(x) \rangle \mu(x) \, \mathrm{d}\mathscr{H}^m(x),$$

where μ is a *multiplicity function*, i.e., an $\mathscr{H}^m$-integrable function $\mu\colon S \to \mathbb{N}^*$.

One must keep in mind that a rectifiable current is determined by a triple (S, τ, μ) where the orienting m-vector τ is tangent to the support S (at $\mathscr{H}^m$-almost every point); this is of fundamental importance in what follows (taking an m-vector field τ which is not tangent to S almost everywhere would lead to very different behaviours when applying the boundary operator). In this appendix we shall content ourselves with the case $\mu \equiv 1$.

An elementary example is $[\Delta_N] \in \mathscr{E}_N(\mathbb{R}^N)$, with the standard N-dimensional simplex $\Delta_N \subset \mathbb{R}^N$ of Notation 4.2 oriented by $\tau = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_N}$.

Push-forward by smooth and Lipschitz maps

The push-forward of a current $T \in \mathscr{E}_m(\mathbb{R}^N)$ by a smooth map $\Phi\colon \mathbb{R}^N \to \mathbb{R}^{N'}$ is classically defined by dualizing the pullback of differential forms:

$$\Phi_\# T(\beta) := T(\Phi^\# \beta), \qquad \beta \text{ any } m\text{-form on } \mathbb{R}^{N'},$$

which yields $\Phi_\# T \in \mathscr{E}_m(\mathbb{R}^{N'})$.

For an integration current $T = [S]$ as in (41), we observe that the smoothness of α is not necessary for the definition of $[S](\alpha)$ to make sense: it suffices that α be defined $\mathscr{H}^m$-almost everywhere on S, bounded and $\mathscr{H}^m$-measurable. Therefore, *in the top-dimensional case $m = N$, we can associate with the current $[S] \in \mathscr{E}_N(\mathbb{R}^N)$ a push-forward $\phi_\#[S] \in \mathscr{E}_N(\mathbb{R}^{N'})$ by any Lipschitz map $\phi\colon S \to \mathbb{R}^{N'}$, by means of the formula*

$$(42) \qquad \phi_\#[S](\beta) := [S](\phi^\# \beta), \qquad \beta \text{ any } N\text{-form on } \mathbb{R}^{N'}.$$

---

[8] Recall that, at $\mathscr{H}^m$-almost every point of S, the cone of approximate tangent vectors is an m-plane [16, 3.2.19], [28, 3.12]).

Indeed, Rademacher's theorem ensures that ϕ is differentiable $\mathscr{H}^N$-almost everywhere ($\mathscr{H}^N$ is the Lebesgue measure), with bounded partial derivatives, hence the pullback form $\phi^{\#}\beta$ is defined almost everywhere as

$$\beta = \sum_I g_I \, \mathrm{d}y^{I_1} \wedge \cdots \wedge \mathrm{d}y^{I_N} \quad \Longrightarrow$$

$$\phi^{\#}\beta = \sum_I (g_I \circ \phi) \, \mathrm{d}\phi^{I_1} \wedge \cdots \wedge \mathrm{d}\phi^{I_N} = \sum_I (g_I \circ \phi) \det\left[\frac{\partial \phi^{I_i}}{\partial x^j}\right]_{1 \leq i,j \leq N} \mathrm{d}x^1 \wedge \cdots \wedge x^N,$$

where the sums are over all $I = \{1 \leq I_1 < \cdots < I_N \leq N'\}$, the coordinates in $\mathbb{R}^{N'}$ are denoted by $(y^1, \ldots, y^{N'})$ and those in $\mathbb{R}^N$ by $(x^1, \ldots, x^N)$. The pullback form $\alpha = \phi^{\#}\beta$ has its coefficients in $L^\infty(\mathbb{R}^N)$, hence we can define $\phi_{\#}[S](\beta) = [S](\alpha)$ by (41).

Having defined $\phi_{\#}[S] \in \mathscr{E}_N(\mathbb{R}^{N'})$ by Formula (42), it is worth noticing that $\phi_{\#}[S]$ can also be obtained by a regularization process:

LEMMA A.2. – *Let S be an oriented compact N-dimensional rectifiable subset of $\mathbb{R}^N$ and let $\phi: S \to \mathbb{R}^{N'}$ be a Lipschitz map. Consider smooth Lipschitz maps $\Phi_\ell: \mathbb{R}^N \to \mathbb{R}^{N'}$, $\ell \in \mathbb{N}$, which have uniformly bounded Lipschitz constants and converge uniformly to ϕ on S as $\ell \to \infty$. Then*

$$(43) \qquad (\Phi_\ell)_{\#}[S](\beta) \xrightarrow[\ell\to\infty]{} \phi_{\#}[S](\beta), \qquad \beta \text{ any } N\text{-form on } \mathbb{R}^{N'}.$$

The proof relies on equicontinuity estimates derived from Reshetnyak's theorem[9] which guarantees that in this situation, not only do we have the weak-$*$ convergence in $L^\infty(\mathbb{R}^N)$ for the partial derivatives $\frac{\partial \Phi_\ell^{I_k}}{\partial x_j} \xrightarrow{*} \frac{\partial \phi^{I_k}}{\partial x_j}$, but also for the minors of the Jacobian matrix: $\det\left[\frac{\partial \phi_\ell^{I_i}}{\partial x^j}\right] \xrightarrow{*} \det\left[\frac{\partial \phi^{I_i}}{\partial x^j}\right]$, whence $\Phi_\ell^{\#}\beta \xrightarrow{*} \phi^{\#}\beta$ componentwise in $L^\infty(\mathbb{R}^N)$ and (43) follows.

Another case of interest is $T = [A(\Delta)] \in \mathscr{E}_m(\mathbb{R}^N)$ with $m \leq N$, Δ an oriented compact m-dimensional rectifiable subset of $\mathbb{R}^m$ and $A: \mathbb{R}^m \to \mathbb{R}^N$ an injective affine map (the unit m-vector field orienting $A(\Delta)$ is chosen to be a positive multiple of the image of the unit m-vector field orienting Δ by the m-linear extension of the linear part of A to $\Lambda_m\mathbb{R}^m$). We have $[A(\Delta)] = A_{\#}[\Delta]$, thus the natural definition of the push-forward of $[A(\Delta)]$ by a Lipschitz map $\phi: A(\Delta) \to \mathbb{R}^{N'}$ is clearly

$$(44) \qquad \phi_{\#}[A(\Delta)] := (\phi_A)_{\#}[\Delta] \in \mathscr{E}_{N-1}(\mathbb{R}^{N'}), \qquad \text{with } \phi_A := \phi \circ A: \Delta \to \mathbb{R}^{N'}.$$

Indeed, one easily checks that when ϕ is the restriction to $A(\Delta)$ of a smooth map $\Phi: \mathbb{R}^N \to \mathbb{R}^{N'}$, the above-defined push-forward $\phi_{\#}[A(\Delta)]$ coincides with the classical push-forward $\Phi_{\#}[A(\Delta)]$. Moreover, also in this case is the regularization process possible: for any sequence of smooth Lipschitz maps $\Phi_\ell: \mathbb{R}^N \to \mathbb{R}^{N'}$, $\ell \in \mathbb{N}$, which have uniformly bounded Lipschitz constants and converge uniformly to ϕ on $A(\Delta)$ as $\ell \to \infty$, we have

$$(45) \qquad (\Phi_\ell)_{\#}[A(\Delta)](\beta) \xrightarrow[\ell\to\infty]{} \phi_{\#}[A(\Delta)](\beta), \qquad \beta \text{ any } N\text{-form on } \mathbb{R}^{N'}$$

(simply because the left-hand side is $(\Phi_\ell \circ A)_{\#}[\Delta](\beta)$ and we can apply (43) to the sequence $\Phi_\ell \circ A$ uniformly converging to $\phi \circ A$ on Δ).

[9] See [15], § 8.2.4, lemma on the weak continuity of determinants.

The boundary operator and Stokes's theorem

The boundary operator is defined by duality on all currents $T \in \mathscr{E}_m(\mathbb{R}^N)$:

$$(46) \qquad \partial T(\alpha) := T(\mathrm{d}\alpha), \qquad \alpha \ m\text{-form on } \mathbb{R}^N.$$

The boundary of an integer rectifiable current T is not necessarily an integer rectifiable current; if it happens to be, then T is called an *integral current*. An example is provided by oriented smooth submanifolds M with boundary; Stokes's theorem then relates the action of the boundary operator ∂ on the corresponding integration currents with the action of the boundary operator ∂ of homology: $\partial[M] = [\partial M] \in \mathscr{E}_{m-1}(\mathbb{R}^N)$.

Another example is provided by the standard simplex $\Delta_N \subset \mathbb{R}^N$ of Notation 4.2; recall that the orienting unit n-vector field is $\tau := \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_N}$. Stokes's Theorem yields

$$\partial[\Delta_N] = [\Gamma_0] + \cdots + [\Gamma_N] \in \mathscr{E}_{N-1}(\mathbb{R}^N),$$

where

$$\Gamma_j = \begin{vmatrix} \Delta_N \cap \{x_1 + \cdots + x_N = 1\} & \text{if } j = 0, \\ \Delta_N \cap \{x_j = 0\} & \text{if } 1 \leq j \leq n, \end{vmatrix}$$

with orienting $(N-1)$-vectors τ_j defined by $\nu_j \wedge \tau_j = \tau$, where ν_j is the outward-pointing unit normal vector field for the piece Γ_j of $\partial \Delta_N$; with the notation $e_j = \frac{\partial}{\partial x_j}$, the result is $\tau_0 = \frac{1}{\sqrt{N}}(e_2 - e_1) \wedge (e_3 - e_1) \wedge \cdots \wedge (e_N - e_1)$ (because $\nu_0 = (e_1 + \cdots + e_N)/\sqrt{N}$) and $\tau_j = (-1)^j e_1 \wedge \cdots \wedge \overset{\wedge}{e_j} \wedge \cdots \wedge e_N$ for $j \geq 1$ (because $\nu_j = -e_j$).

Observe that one can write

$$(47) \qquad \partial[\Delta_N] = [A_0(\Delta_{N-1})] - [A_1(\Delta_{N-1})] + \cdots + (-1)^N[A_N(\Delta_{N-1})]$$

with an injective affine map $A_j \colon \mathbb{R}^{N-1} \to \mathbb{R}^N$ for each $j = 0, \ldots, N$ (taking $A_0(x_1, \ldots, x_{N-1}) = (1 - x_1 - \cdots - x_{N-1}, x_1, \ldots, x_{N-1})$ and $A_j(x_1, \ldots, x_{N-1}) = (x_1, \ldots, x_{j-1}, 0, x_j, \ldots, x_{N-1})$ for $j \geq 1$).

The commutation formula $\phi_\# \partial[P] = \partial \phi_\#[P]$

For any $T \in \mathscr{E}_m(\mathbb{R}^N)$ and any smooth map $\Phi \colon \mathbb{R}^N \to \mathbb{R}^{N'}$, the formula

$$(48) \qquad \Phi_\# \partial T = \partial \Phi_\# T \in \mathscr{E}_{m-1}(\mathbb{R}^{N'})$$

is a simple consequence of the identity $\mathrm{d} \circ \Phi^\# = \Phi^\# \circ \mathrm{d}$ on differential forms. We can also try to deal with a Lipschitz map ϕ when restricting ourselves to integral currents. The following is used in the proof of the main result of this article:

LEMMA A.3. – *Let $N \geq 1$ and let $\phi \colon \Delta_N \to \mathbb{R}^{N'}$ be Lipschitz; define $\phi_\#[\Delta_N]$ by means of (42) and $\phi_\# \partial[\Delta_N]$ by means of (47) and (44). Then*

$$\partial \phi_\#[\Delta_N] = \phi_\# \partial[\Delta_N].$$

In fact, it is with $P = [0,1] \times \Delta_n$ instead of Δ_N that this commutation formula is used in Section 5; moreover, the target space is $\mathscr{S}_\Omega^n$ instead of $\mathbb{R}^{N'}$ but, as mentioned above, this makes no difference (just take $N' = 2n$). We leave it to the reader to adapt the proof.

Proof of Lemma A.3. – We shall use the notation $T = [\Delta_N] \in \mathscr{E}_N(\mathbb{R}^N)$. Let β be a smooth $(N-1)$-form on $\mathbb{R}^{N'}$ and let $(\Phi_\ell)_{\ell \in \mathbb{N}}$ be any sequence of smooth maps from $\mathbb{R}^N$ to $\mathbb{R}^{N'}$ with uniformly bounded Lipschitz constants which converges uniformly to ϕ on Δ_N as $\ell \to \infty$. Then the sequence

$$\partial(\Phi_\ell)_\# T(\beta) = (\Phi_\ell)_\# T(\mathrm{d}\beta) \xrightarrow[\ell\to\infty]{} \phi_\# T(\mathrm{d}\beta) = \partial\phi_\# T(\beta)$$

by (46) and (43). But, by (48), this sequence coincides with

$$(\Phi_\ell)_\# \partial T(\beta) = \sum_{j=0}^{N}(-1)^j(\Phi_\ell)_\#[A_j(\Delta_{N-1})](\beta) \xrightarrow[\ell\to\infty]{} \sum_{j=0}^{N}(-1)^j\phi_\#[A_j(\Delta_{N-1})](\beta) = \phi_\# \partial T(\beta)$$

by (47) and (45). $\square$

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