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THE GEOMETRIC SATAKE CORRESPONDENCE FOR RAMIFIED GROUPS

BY XINWEN ZHU

ABSTRACT. – We prove the geometric Satake isomorphism for a reductive group defined over $F = k(\!(\mathrm{t})\!)$, and split over a tamely ramified extension. As an application, we give a description of the nearby cycles on certain Shimura varieties via the Rapoport-Zink-Pappas local models.

RÉSUMÉ. – Nous démontrons l'isomorphisme de Satake géométrique pour un groupe réductif défini sur $F = k(\!(\mathrm{t})\!)$ et déployé sur une extension modérément ramifiée. Nous donnons comme application une description des cycles évanescents sur certaines variétés de Shimura via les modèles locaux de Rapoport-Zink-Pappas.

Introduction

The Satake isomorphism (for unramified groups) is the starting point of the Langlands duality. Let us first recall its statement. Let F be a non-Archimedean local field with ring of integers $\mathcal{O}$ and residue field k, and let G be a connected unramified reductive group over F (e.g., $G = \mathrm{GL}_n$). Let $A \subset G$ be a maximal split torus of G, and W_0 be the Weyl group of (G, A). Let K be a hyperspecial subgroup of $G(F)$ containing $A(\mathcal{O})$ (e.g., $K = \mathrm{GL}_n(\mathcal{O})$). Then the classical Satake isomorphism describes the spherical Hecke algebra $\mathrm{Sph} = C_c(K \backslash G(F)/K)$, the algebra of compactly supported bi-K-invariant functions on $G(F)$ under convolution. Namely, there is an isomorphism of algebras

$$\mathrm{Sph} \simeq \mathbb{C}[\mathbb{X}_\bullet(A)]^{W_0},$$

where $\mathbb{X}_\bullet(A)$ is the coweight lattice of A, and $\mathbb{C}[\mathbb{X}_\bullet(A)]^{W_0}$ denotes the W_0-invariants of the group algebra of $\mathbb{X}_\bullet(A)$.

If F has positive characteristic $p > 0$, then the classical Satake correspondence has a vast enhancement. For simplicity, let us assume that G is split over F (for the general case, see Theorem A.12). Let us write $G = H \otimes_k F$ for some split group H over k so that $K = H(\mathcal{O})$. Let $\mathrm{Gr}_H = H(F)/H(\mathcal{O})$ be the affine Grassmannian of H. Choose ℓ a prime different from p,

and let Sat_H be the category of $(K \otimes \bar{k})$-equivariant perverse sheaves with $\overline{\mathbb{Q}}_\ell$-coefficients on $\mathrm{Gr}_H \otimes \bar{k}$. Then this is a Tannakian category and there is an equivalence

$$\mathrm{Sat}_H \simeq \mathrm{Rep}(G^\vee_{\overline{\mathbb{Q}}_\ell}),$$

where $G^\vee_{\overline{\mathbb{Q}}_\ell}$ is the dual group of G and $\mathrm{Rep}(G^\vee_{\overline{\mathbb{Q}}_\ell})$ is the tensor category of algebraic representations of $G^\vee_{\overline{\mathbb{Q}}_\ell}$ (cf. [10, 19]).

There is also a version of Satake isomorphism for an arbitrary reductive group over F, as recently proved by Haines and Rostami (cf. [12])[(1)]. Namely, let $\mathscr{B}(G)$ be the Bruhat-Tits building of G and $v \in \mathscr{B}(G)$ be a special vertex. Let $K_v \subset G(F)$ be the special parahoric subgroup of $G(F)$ corresponding to v. Let A be a maximal split F-torus of G such that $K_v \supset A(\mathcal{O})$, let M be the centralizer of A in G and $W_0 = N_G(A)/M$ be the Weyl group as before. Let M_1 be the unique parahoric subgroup of $M(F)$, and $\Lambda_M = M(F)/M_1$, which is a finitely generated Abelian group. Then

$$C_c(K_v \backslash G(F)/K_v) \simeq \mathbb{C}[\Lambda_M]^{W_0}. \tag{0.1}$$

More explicitly, suppose that G is quasi-split so that $M = T$ is a maximal torus. Then

$$\Lambda_M = (\mathbb{X}_\bullet(T)_I)^\sigma,$$

where I is the inertial group and σ is the Frobenius, and $(\mathbb{X}_\bullet(T)_I)^\sigma$ denotes the σ-invariants of the I-coinvariants of the group $\mathbb{X}_\bullet(T)$.

The goal of this paper is to provide a geometric version of the above isomorphism when F has positive characteristic p and the group G is quasi-split and splits over a *tamely* ramified extension. More precisely, let k be an algebraically closed field and let $\ell \neq \mathrm{char}\, k$ be a prime. Let G be a group over the local field $F = k(\!(\mathrm{t})\!)$ (so that G is quasi-split automatically), which is split over a tamely ramified extension. That is, there is a finite extension $\tilde{F}/F$ such that $G_{\tilde{F}}$ is split and $\mathrm{char}\, k \nmid [\tilde{F} : F]$. Let $v \in \mathscr{B}(G)$ be a special vertex in the building of G and let $\underline{G}_v$ be the parahoric group scheme over $\mathcal{O} = k[\![t]\!]$ (in the sense of Bruhat-Tits), determined by v. We write LG for the loop space of G and $K_v = L^+\underline{G}_v$ for the jet space of $\underline{G}_v$. By definition, for any k-algebra R, $LG(R) = G(R\hat{\otimes}_k F)$ and $K_v(R) = \underline{G}_v(R\hat{\otimes}_k \mathcal{O})$. Let

$$\mathscr{F}\ell_v = LG/K_v$$

be the (twisted) affine flag variety[(2)], which is an ind-scheme over k. Let $\mathscr{P}_v = \mathscr{P}_{K_v}(\mathscr{F}\ell_v)$ be the category of K_v-equivariant perverse sheaf on $\mathscr{F}\ell_v$, with coefficients in $\overline{\mathbb{Q}}_\ell$. Let H be a split Chevalley group over $\mathbb{Z}$ such that $G \otimes_F F^s \simeq H \otimes F^s$, where F^s is a (fixed) separable closure of F. Then there is a natural action of $I = \mathrm{Gal}(F^s/F)$ on $H^\vee := H^\vee_{\overline{\mathbb{Q}}_\ell}$ (preserving a fixed pinning).

THEOREM 0.1. – *The category $\mathscr{P}_v$ has a natural tensor structure. In addition, as tensor categories, there is an equivalence*

$$\mathscr{RS} : \mathrm{Rep}((H^\vee)^I) \simeq \mathscr{P}_v,$$

(1) There is another version, known earlier, as in [6].

(2) One would call $\mathscr{F}\ell_v$ the affine Grassmannian of G. However, we reserve the name "affine Grassmannian" of G for another object, as defined in Definition A.2.

such that $\mathrm{H}^* \circ \mathcal{R}\mathcal{J}$ *is isomorphic to the forgetful functor, where* H^* *is the hypercohomology functor.*

This theorem can be regarded as a categorification of (0.1) in the case when k is algebraically closed and the group splits over a tamely ramified extension of $k((\mathrm{t}))$. For the description of $(H^\vee)^I$ when H is absolutely simple and simply-connected, see §4.

Let us point out the following remarkable facts when the group is ramified. First, the group $(H^\vee)^I$ is not necessarily connected as is shown in Remark (4.4). Second, it is well-known that if G is unramified over F, then all the hyperspecial subgroups of G are conjugate under $G_{\mathrm{ad}}(F)$ ([27, §2.5]), where G_{ad} is the adjoint group of G. However, this is no longer true for special parahoric of G if G is ramified. An example is given by the odd ramified unitary similitude group GU_{2m+1}. There are essentially two types of special parahorics of GU_{2m+1}, as given in (7.1). One of them has reductive quotient GO_{2m+1} (denoted by $\underline{G}_{v_0}$), and the other has reductive quotient GSp_{2m} (denoted $\underline{G}_{v_1}$). Accordingly, the geometry of the corresponding flag varieties $\mathcal{F}\ell_{v_0}$ and $\mathcal{F}\ell_{v_1}$ are very different, while $\mathcal{P}_{v_0} \simeq \mathcal{P}_{v_1}$. Indeed, their Schubert varieties (i.e., closures of K_{v_i}-orbits) are both parameterized by irreducible representations of GO_{2m+1}. Let $\mathcal{F}\ell_{v_0\bar{\mu}_{2m,1}}$ (resp. $\mathcal{F}\ell_{v_1\bar{\mu}_{2m,1}}$) be the Schubert variety in $\mathcal{F}\ell_{v_0}$ (resp. $\mathcal{F}\ell_{v_1}$) parameterized by the standard representation of GO_{2m+1}. Then it is shown in [31] that $\mathcal{F}\ell_{v_0\bar{\mu}_{2m,1}}$ is not Gorenstein, while in [25] that $\mathcal{F}\ell_{v_1\bar{\mu}_{2m,1}}$ is smooth. On the other hand, the intersection cohomology of both varieties gives the standard representation of GO_{2m+1}. In addition, the stalk cohomologies of both sheaves are the "same". See Theorem 0.3 below.

REMARK 0.1. – Instead of considering a special parahoric K_v of LG, one can begin with the special maximal "compact" K'_v, (i.e., $K'_v = L^+\underline{G}'_v$, where $\underline{G}'_v$ is the stabilizer group scheme of v as constructed by Bruhat-Tits), and consider the category of K'_v-equivariant perverse sheaves on LG/K'_v. However, from a geometric point of view, this is less natural since K'_v is not necessarily connected and the category of K'_v-equivariant perverse sheaves is complicated. In fact, we do not know how to relate this category to the Langlands dual group yet. In addition, when we discuss the Langlands parameters in Section 6, it is also more "correct" to consider K_v rather than K'_v.

The idea of the proof of the theorem is as follows. Using Gaitsgory's nearby cycle functor construction as in [8, 31], we construct a functor

$$\mathcal{Z} : \mathrm{Sat}_H \to \mathcal{P}_v,$$

which is a central functor in the sense of [2]. By standard arguments in the theory of Tannakian equivalence and the Mirkovic-Vilonen theorem, this already implies that $\mathcal{P}_v \simeq \mathrm{Rep}(\tilde{G}^\vee)$ for certain closed subgroup $\tilde{G}^\vee \subset H^\vee$. Then we identify $\tilde{G}^\vee$ with $(H^\vee)^I$ using the parametrization of the K_v-orbits on $\mathcal{F}\ell_v$.

REMARK 0.2. – (i) We believe that the same argument (maybe with small modifications) should work for groups split over wild ramified extensions. However, we have not checked this carefully.

(ii) Our approach is more inspired by [8] rather than [19]. However, it would be interesting to know whether there is the similar theory of MV-cycles in the ramified case. It seems that

the geometry of semi-infinite orbits on $\mathcal{F}\ell_v$ is similar to the unramified case, except when $\mathcal{F}\ell_v$ corresponds to one type of special parahorics for odd unitary groups (the one denoted by $\underline{G}_{v_1}$ as above). We do not know what happens in this last case.

(iii) Our theorem and the method also share the similar features with the results of Nadler on geometric Satake for real groups [20].

When the group G is quasi-split over the non-Archimedean local field $F = \mathbb{F}_q(\!(\mathrm{t})\!)$ and v is a special vertex of $\mathcal{B}(G, F)$, the affine flag variety $\mathcal{F}\ell_v$ is defined over $\mathbb{F}_q$. We assume that v is very special, i.e., it remains special when we base change G to $\overline{\mathbb{F}}_q(\!(\mathrm{t})\!)$ (see § 6 for more discussions of this notion). Then we can consider the category of K_v-equivariant semi-simple perverse sheaves on $\mathcal{F}\ell_v$, pure of weight zero, and denote it by $\mathscr{P}_v^0$. On the other hand, let I be the inertial group of F and σ be the Frobenius of $\mathrm{Gal}(k/\mathbb{F}_q)$, where $k = \overline{\mathbb{F}}_q$. Then the action of $\mathrm{Gal}(F^s/F)$ on $H^\vee$ (via the pinned automorphisms) induces a canonical action of σ on $(H^\vee)^I$, denoted by $\mathrm{act}^{\mathrm{alg}}$. One can form the semidirect product $(H^\vee)^I \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(k/\mathbb{F}_q)$, which can be regarded as a proalgebraic group over $\overline{\mathbb{Q}}_\ell$, and consider the category of algebraic representations of $(H^\vee)^I \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(k/\mathbb{F}_q)$, denoted by $\mathrm{Rep}((H^\vee)^I \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(k/\mathbb{F}_q))$.

THEOREM 0.2. – *In this case, the functor $\mathcal{RS}$ in Theorem 0.1 can be extended to an equivalence*

$$\mathcal{RS} : \mathrm{Rep}((H^\vee)^I \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(k/\mathbb{F}_q)) \simeq \mathscr{P}_v^0,$$

whose composition with H^* *is isomorphic to the forgetful functor.*

Let us mention that under this equivalence, the restriction to $\mathrm{Gal}(k/\mathbb{F}_q)$ of the representation $(H^\vee)^I \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(k/\mathbb{F}_q)$ on $\mathrm{H}^*(\mathcal{F})$ for $\mathcal{F} \in \mathscr{P}_v^0$ is NOT the natural Galois action of $\mathrm{Gal}(k/\mathbb{F}_q)$ on $\mathrm{H}^*(\mathcal{F})$. However, their difference can be described explicitly. See Section 4 and appendix for more details.

Our next result is to use the ramified geometric Satake isomorphism to obtain the stalk cohomology of sheaves on $\mathcal{F}\ell_v$ (i.e., the corresponding Lusztig-Kato polynomial in ramified case), following an idea of Ginzburg (cf. [10]). Let us state the result precisely. The centralizer of A in our case is a maximal torus of G, denoted by T. Then the K-orbits on $\mathcal{F}\ell_v$ are labeled by $\mathbb{X}_\bullet(T)_I/W_0$, W_0-orbits of the coinvariants of the cocharacter group of T. For $\bar{\mu} \in \mathbb{X}_\bullet(T)_I$, let $\mathring{\mathcal{F}}\ell_{v\bar{\mu}}$ be the corresponding orbit. For a representation V of $(H^\vee)^I$, let $V(\bar{\mu})$ be the weight space of V for $(T^\vee)^I$. Let $X^\vee \in \mathrm{Lie}(H^\vee)^I$ be a certain principal nilpotent element (see Section 5 for the details), which induces a filtration $F_iV(\bar{\mu}) = (\ker X^\vee)^{i+1} \cap V(\bar{\mu})$ on $V(\bar{\mu})$, called the Brylinski-Kostant filtration. Then we have

THEOREM 0.3. – *For $V \in \mathrm{Rep}((H^\vee)^I)$, let $\mathcal{RS}(V) \in \mathscr{P}_v$ be the corresponding sheaf. Then*

$$\dim \mathscr{H}^{2i-(2\rho,\bar{\mu})}\mathcal{RS}(V)|_{\mathcal{F}\ell_{v\bar{\mu}}} = \dim \mathrm{gr}_i^F V(\bar{\mu}).$$

Here $\mathscr{H}^*$ denotes the cohomology sheaves, and 2ρ is the sum of positive roots of H, see Section 1 for the meaning of $(2\rho, \bar{\mu})$.

One of our main motivations of this work is to apply these results to the calculation of the nearby cycles of certain ramified unitary Shimura varieties, via the Rapoport-Zink-Pappas local models. For example, we obtain the following theorem (see Section 7 for details).

THEOREM 0.4. – *Let $G = \mathrm{GU}(r,s)$ be a unitary similitude group associated to an imaginary quadratic extension $F/\mathbb{Q}$ and a Hermitian space (W,ϕ) over $F/\mathbb{Q}$. Let $p > 2$ be a prime where $F/\mathbb{Q}$ is ramified and the Hermitian form is split. Let K_p be a special parahoric subgroup of $G = G(\mathbb{Q}_p)$. Let $K = K_p K^p \subset G(\mathbb{Q}_p)G(\mathbb{A}_f^p)$ be a compact open subgroup with K^p small enough. Let Sh_K be the associated Shimura variety over the reflex field E and Sh_{K_p} be the integral model of Sh_K over $\mathcal{O}_{E_p}$ (as defined in* [23]*). Then for $\ell \neq p$, the action of the inertial subgroup I of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_p)$ on the nearby cycle $\Psi_{\mathrm{Sh}_{K_p}\otimes_{\mathcal{O}_{E_p}}\mathcal{O}_{F_p}}(\mathbb{Q}_\ell)$ is trivial.*

By applying Theorem 0.3, it will not be hard to determine the traces of Frobenius on these sheaves explicitly, which will be the input of the Langlands-Kottwitz method to determine the local Zeta factors of Sh_K. Instead, we characterize these traces of Frobenius in terms of Langlands parameters, which verifies a conjecture of Haines and Kottwitz in this case (see Proposition 7.4).

REMARK 0.3. – (i) While the definition of the integral model of a PEL-type Shimura variety at an "unramified" prime p (i.e., the group is unramified at p and K_p is hyperspecial) is well-known (cf. [15]), the definition of such a model at the ramified prime p (even for K_p special) is a subtle issue. In [21, 23], the integral models Sh_{K_p} are defined as certain closed subschemes of certain moduli problems of Abelian varieties. Except a few cases (e.g., $(r,s) = (n-1,1)$ and $n = r+s$ is small), there is no moduli description of Sh_{K_p} so far. In general, Sh_{K_p} are not smooth. Indeed, as shown in [21, 23], when $n = r+s$ is odd and $(r,s) = (n-1,1)$, for the special parahoric K_p of $G(\mathbb{Q}_p)$ with reductive quotient GO_n, Sh_{K_p} is not even semi-stable.

(ii) If $r \neq s$, then we know that $E = F$ and the above theorem gives a complete description of the monodromy on the nearby cycles of Sh_{K_p}. If $r = s$, then $E = \mathbb{Q}$, and the complete description of the monodromy is more complicated. See Section 7 for details. In any case, the action of inertia on the nearby cycle is semi-simple.

(iii) We hope that there will be a "good" compactification of such Shimura varieties Sh_{K_p}. Then the above theorem, together with the existence of such compactification, would imply that the monodromy of $H_c^*(\mathrm{Sh}_K \otimes_{E_p} F_p)$ is trivial.

(iv) The triviality of the monodromy as above would have the following surprising consequence for the Albanese of Picard modular surfaces. Namely, in the case when $(r,s) = (2,1)$, $F/\mathbb{Q}$ is ramified at $p > 2$ and $K_p = G(\mathbb{Q}_p)$ is a special parahoric, the Albanese $\mathrm{Alb}(\mathrm{Sh}_{K_p})$ of Sh_{K_p} is trivial. It will be interesting to find the "optimal" level structure at p so that $\mathrm{Alb}(\mathrm{Sh}_{K_p})$ can be possibly non-trivial. More detailed discussion will appear elsewhere.

Let us quickly describe the organization of the paper. We will prove Theorem 0.1 and Theorem 0.2 in §1-4. Then we prove Theorem 0.3 in §5.

In § 6, we briefly discuss the Langlands parameters associated to a smooth representation of a quasi-split p-adic group, which has a vector fixed by a special parahoric. We call them "spherical" representations, and we will see that their Langlands parameters can be described easily. Again, the correct point of view is to consider the special parahoric rather than the special maximal compact. Then in § 7, we apply the previous results to study the nearby cycles on certain unitary Shimura varieties.

The paper contains an appendix, joint with T. Richarz, where we recover the full Langlands dual group via the Tannakian formalism. In particular, we give the geometric Satake correspondence for unramified groups. We hope that this formulation will be of independent interest. In addition, we observe that for a reductive group defined over k, the Tannakian formalism provides a natural action $\mathrm{act}^{\mathrm{geom}}$ of $\mathrm{Gal}(k^s/k)$ on the dual group $G^\vee$, which differs from the usual pinned action $\mathrm{act}^{\mathrm{alg}}$ of $\mathrm{Gal}(k^s/k)$ on $G^\vee$ by the twist of "the half sum of the positive roots". This gives a geometric explanation of the two natural normalizations of the Satake parameters.

Notations. – Let k be a field. We denote by k^s a separable closure of k.

For a (not necessarily connected) diagonalizable algebraic group C defined over a field F, we denote by $\mathbb{X}^\bullet(C)$ its group of characters and by $\mathbb{X}_\bullet(C)$ its group of cocharacters over the separable closure F^s. The Galois group $I = \mathrm{Gal}(F^s/F)$ acts on $\mathbb{X}^\bullet(C)$ (resp. $\mathbb{X}_\bullet(C)$) and the invariants (resp. coinvariants) are denoted by $\mathbb{X}^\bullet(C)^I$ (resp. $\mathbb{X}^\bullet(C)_I, \mathbb{X}_\bullet(C)^I, \mathbb{X}_\bullet(C)_I$). We will always use $\lambda, \mu, \ldots$ to denote elements in $\mathbb{X}^\bullet(C)$ or $\mathbb{X}_\bullet(C)$ and $\bar{\lambda}, \bar{\mu}$ to denote elements in $\mathbb{X}^\bullet(C)_I$ or $\mathbb{X}_\bullet(C)_I$. In general, let I be a group acting on a set S. We denote S^I to be the subset of fixed points.

If G is an algebraic group defined over a field E, we denote by $\mathrm{Rep}(G)$ the category of finite dimensional representations of G over E. If G is connected reductive, we denote $G_{\mathrm{ad}}, G_{\mathrm{der}}, G_{\mathrm{sc}}$ to be its adjoint quotient, its derived group, and the simply-connected cover of its derived group.

Let k be a field and $\mathcal{O} = k[[t]], F = k(\!(\mathrm{t})\!) := k[[t]][t^{-1}]$. For an $\mathcal{O}$-scheme X, we denote L^+X to be the jet space over k so that for any k-algebra R, $L^+X(R) = X(R[[t]])$. For an F-scheme X, we denote LX to be its loop space so that $LX(R) = X(R(\!(\mathrm{t})\!))$. If X is defined over k, we write L^+X for $L^+(X \otimes \mathcal{O})$ and LX for $L(X \otimes F)$ if no confusion will arise.

For a variety X over k, we denote $D(X)$ the usual (bounded) derived category of ℓ-adic sheaves on X ($\ell \nmid \mathrm{char}\, k$). If $X = \lim X_i$ is an ind-scheme of ind-finite type, $D(X) = \lim D(X_i)$ as usual. If there is an action of an algebraic group G on X, the G-equivariant derived category is denoted by $D_G(X)$ (see [1] for the details). All the functors like $f_*, f_!, f^*, f^!$ are understood in the derived sense unless otherwise specified.

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1. Reminders on the affine flag variety associated to a special parahoric

In this section, we collect basic facts about the affine flag varieties associated to a special parahoric of G. Another purpose of this section is to fix notations that are used in the later sections.

Let k be an algebraically closed field and let G be a group over the local field $F = k(\!(\mathrm{t})\!)$, which is split over a tamely ramified extension. Let us choose A to be a maximal F-split torus

of G and T be its centralizer. Then T is a maximal torus of G since G is quasi-split. Let us choose a rational Borel subgroup $B \supset T$.

Let H be a split Chevalley group over $\mathbb{Z}$ such that $H \otimes F^s \simeq G \otimes F^s$. We need to choose this isomorphism carefully. Let us fix a pinning (H, B_H, T_H, X) of H over $\mathbb{Z}$. Let us recall that this means that B_H is a Borel subgroup of H, T_H is a split maximal torus contained in B_H, and $X = \Sigma_{\tilde{a} \in \Delta_H} X_{\tilde{a}} \in \mathrm{Lie} B$, where Δ_H is the set of simple roots of H, $\tilde{U}_{\tilde{a}}$ is the root subgroup corresponding to $\tilde{a}$ and $X_{\tilde{a}}$ is a generator in the rank one free $\mathbb{Z}$-module $\mathrm{Lie}\, \tilde{U}_{\tilde{a}}$. Let Ξ be the group of pinned automorphisms of (H, B_H, T_H, X), which is canonically isomorphic to the group of the automorphisms of the root datum $(\mathbb{X}^\bullet(T_H), \Delta_H, \mathbb{X}_\bullet(T_H), \Delta_H^\vee)$.

Let us choose an isomorphism $(G, B, T) \otimes_F \tilde{F} \simeq (H, B_H, T_H) \otimes_{\mathbb{Z}} \tilde{F}$, where $\tilde{F}/F$ is a cyclic extension such that $G \otimes \tilde{F}$ splits. This induces an isomorphism of the root data $(\mathbb{X}^\bullet(T_H), \Delta_H, \mathbb{X}_\bullet(T_H), \Delta_H^\vee) \simeq (\mathbb{X}^\bullet(T), \Delta, \mathbb{X}_\bullet(T), \Delta^\vee)$. Now the action of $I = \mathrm{Gal}(\tilde{F}/F)$ on $G \otimes_F \tilde{F}$ induces a homomorphism $\psi : I \to \Xi$. Then we can always choose an isomorphism

$$(G, B, T) \otimes_F \tilde{F} \simeq (H, B_H, T_H) \otimes_{\mathbb{Z}} \tilde{F} \tag{1.1}$$

such that the action of $\gamma \in I$ on the left hand side corresponds to $\psi(\gamma) \otimes \gamma$. In the rest of the paper, we fix such an isomorphism.

Recall that the Kottwitz homomorphism $\kappa : T(F) \to \mathbb{X}_\bullet(T)_I$ (cf. [16, 11]) induces an isomorphism

$$\mathbb{X}_\bullet(T)_I \simeq T(F)/\underline{T}^{\flat,0}(\mathcal{O}),$$

where $\underline{T}^{\flat,0}$ is the unique parahoric group scheme of T over $\mathcal{O}$ (the connected Néron model). Our convention of Kottwitz homomorphism is that the action of $t \in T(F)$ on $A(G, A)$ (the apartment associated to (G, A)) is given by $v \mapsto v - \kappa(\mathrm{t})$. Let $W_0 = W(G, A)$ be the relative Weyl group of G. It acts on T and therefore on $\mathbb{X}_\bullet(T)_I$. In addition, its action on the torsion subgroup $\mathbb{X}_\bullet(T)_{I,\mathrm{tor}} \subset \mathbb{X}_\bullet(T)_I$ is trivial.

The Borel subgroup B determines a set of positive roots $\Phi^+ = \Phi(G, A)^+$ for G. There is a natural map $\mathbb{X}_\bullet(T)_I \to \mathbb{X}_\bullet(T)_I \otimes \mathbb{R} \simeq \mathbb{X}_\bullet(A)_{\mathbb{R}}$. We define the set of dominant elements in $\mathbb{X}_\bullet(T)_I$ to be

$$\mathbb{X}_\bullet(T)_I^+ = \{\bar{\mu} | (\bar{\mu}, a) \geq 0 \text{ for } a \in \Phi^+\}. \tag{1.2}$$

Then the natural map $\mathbb{X}_\bullet(T)_I^+ \subset \mathbb{X}_\bullet(T)_I \to \mathbb{X}_\bullet(T)_I/W_0$ is bijective. Let us define an order $\preceq$ on $\mathbb{X}_\bullet(T)_I$ as follows. Let Q_H be the coroot lattice for H. The action of I on Q_H will send the positive coroots of H (determined by the chosen Borel) to positive coroots. Therefore, it makes sense to talk about positive elements in $(Q_H)_I$. Namely, an element in $(Q_H)_I$ is positive if its preimage in Q_H is a sum of positive coroots (of H). Since $(Q_H)_I \subset \mathbb{X}_\bullet(T)_I$, we can define for $\lambda, \mu \in \mathbb{X}_\bullet(T)_I$,

$$\bar{\lambda} \preceq \bar{\mu} \text{ if } \bar{\mu} - \bar{\lambda} \text{ is positive in } (Q_H)_I. \tag{1.3}$$

Let $\underline{G}_v$ be a special parahoric group scheme of G over $\mathcal{O} = k[[t]]$ in the sense of Bruhat-Tits (see [27] for a summary of the theory), such that the natural inclusion $A \subset G$ extends to $A_{\mathcal{O}} \subset \underline{G}_v$ (i.e., the vertex v in the (reduced) building of G corresponding to $\underline{G}_v$ is contained in the apartment $A(G, A)$. For examples of such group schemes, we refer to Section 7. We write $K_v = L^+\underline{G}_v$, and consider the (twisted) affine flag variety

$$\mathcal{F}\ell_v = LG/K_v.$$

This is an ind-projective scheme (cf. [22]). As is shown in loc. cit., when G is semi-simple and simply-connected, $\mathscr{F}\ell_v$ is just a partial flag variety of certain (twisted) affine Kac-Moody group. The K_v-orbits on $\mathscr{F}\ell_v$ are parameterized by $\mathbb{X}_\bullet(T)_I^+$. For $\bar{\mu} \in \mathbb{X}_\bullet(T)_I^+$, let $s_{\bar{\mu}}$ denote the point in $\mathscr{F}\ell_v$ corresponding to $\bar{\mu}$. More precisely, this point is the image of $\bar{\mu}$ under the map $\mathbb{X}_\bullet(T)_I \simeq T(F)/\underline{T}^{\flat,0}(\mathcal{O}) \to G(F)/K_v = \mathscr{F}\ell_v(\mathrm{k})$. Let $\mathscr{F}\ell_{v\bar{\mu}}$ be the corresponding Schubert variety, i.e., the closure of K-orbit through $s_{\bar{\mu}}$. Then

$$\dim \mathscr{F}\ell_{v\bar{\mu}} = (2\rho, \bar{\mu}),$$

where 2ρ is the sum of all positive roots (for H), and by definition $(2\rho, \bar{\mu}) = (2\rho, \mu)$ for any lift μ of $\bar{\mu}$ to $\mathbb{X}_\bullet(T)$. In addition, $\mathscr{F}\ell_{v\bar{\lambda}} \subset \mathscr{F}\ell_{v\bar{\mu}}$ if and only if $\bar{\lambda} \preceq \bar{\mu}$. In this case, $\dim \mathscr{F}\ell_{v\bar{\mu}} - \dim \mathscr{F}\ell_{v\bar{\lambda}}$ is an even integer. For the proof of these facts, see [25, 31].

Let $\mathscr{P}_v = \mathscr{P}_{K_v}(\mathscr{F}\ell_v)$ be the category of K_v-equivariant perverse sheaves on $\mathscr{F}\ell_v$, with coefficients in $\overline{\mathbb{Q}}_\ell$. By the above facts, in each connected component of $\mathscr{F}\ell_v$, the dimensions of K_v-orbits have the constant parity. Therefore, we have

LEMMA 1.1. – *$\mathscr{P}_v$ is a semi-simple Abelian category.*

Proof. – By the argument as in [8, Proposition 1], it is enough to show that the stalks of the intersection cohomology sheaves have the parity vanishing property. But this follows from the existence of Demazure resolutions of Schubert varieties in $\mathscr{F}\ell_v$ whose fibers have pavings by affine spaces (for example see [8, A.7]). More precisely, the existence of such resolutions were constructed in [22, Section 8] for twisted affine flag varieties, and the arguments as in [13] apply in this situation to show that the fibers have pavings by affine spaces. □

In [31, 24], a natural $\mathbb{G}_m$-action on $\mathscr{F}\ell_v$ is constructed. In the Kac-Moody setting, it is just the action of the "rotation torus" on $\mathscr{F}\ell_v$. Each Schubert cell is invariant under this action.

COROLLARY 1.2. – *Any K_v-equivariant perverse sheaf on $\mathscr{F}\ell_v$ is automatically $\mathbb{G}_m$-equivariant.*

Proof. – Clearly, the intersection complex is $\mathbb{G}_m$-equivariant. Then the assertion follows from the semisimplicity of $\mathscr{P}_v$. □

EXAMPLE 1.3. – In the special case when $G = H \otimes_k F$ and $\underline{G}_v = H \otimes_k \mathcal{O}$ is hyperspecial, then $\mathscr{F}\ell_v$ is just the usual affine Grassmannian Gr_H of H, and $\mathscr{P}_v$ is the Satake category of H, i.e., the category of L^+H-equivariant perverse sheaves on Gr_H. All the above facts are well-known.

2. Construction of the functor $\mathcal{Z}$

We continue the notations as in the previous sections and let $\underline{G}_v$ be a special parahoric group scheme of G over $\mathcal{O}$. In [31], a group scheme $\mathscr{G}$ over $\mathbb{A}_k^1$ is constructed such that

1. $\mathscr{G}_\eta$ is connected reductive, splits over a (tamely) ramified extension, where η is the generic point of $\mathbb{A}_k^1$;
2. For some choice of isomorphism $F_0 \simeq F$, $\mathscr{G}_{F_0} \simeq G$, where for a point $x \in \mathbb{A}_k^1$, $\mathcal{O}_x$ denotes the completed local ring at x and F_x denotes the fractional field of $\mathcal{O}_x$;
3. For any $y \neq 0$, $\mathscr{G}_{\mathcal{O}_y}$ is hyperspecial, (non-canonically) isomorphic to $H \otimes \mathcal{O}_y$;

4. $\mathscr{G}_{\mathscr{O}_0} = \underline{G}_v$ under the isomorphism $\mathscr{G}_{F_0} \simeq G$.

The construction is as follows. Regard I as the Galois group of the cyclic cover $[e] : \mathbb{G}_m \to \mathbb{G}_m$ of degree e. Then the group I acts $H \times \mathbb{G}_m$. Namely, it acts on H via the pinned automorphism $\psi : I \to \Xi$, and on $\mathbb{G}_m$ via transport of structures. Then $\mathscr{G}|_{\mathbb{G}_m} = (\mathrm{Res}_{\mathbb{G}_m/\mathbb{G}_m}(H \times \mathbb{G}_m))^I$, and $\mathscr{G}$ is the extension of it to $\mathbb{A}^1$ so that $\mathscr{G}_{\mathscr{O}_0} = \underline{G}_v$.

Let $\mathrm{Gr}_{\mathscr{G}}$ be the global affine Grassmannian of $\mathscr{G}$, which is an ind-scheme over $\mathbb{A}^1$ (see, for example [24, §5] for the ind-representability of $\mathrm{Gr}_{\mathscr{G}}$). Recall that it classifies triples $(y, \mathscr{E}, \beta)$ where y is a point on $\mathbb{A}^1_k$, $\mathscr{E}$ is a $\mathscr{G}$-torsor on $\mathbb{A}^1$ and β is a trivialization of this $\mathscr{G}$-torsor away from y. Let $[e] : \mathbb{A}^1 \to \mathbb{A}^1$ be the natural extension of the cyclic cover of $[e] : \mathbb{G}_m \to \mathbb{G}_m$. Let $\widetilde{\mathrm{Gr}}_{\mathscr{G}} := \mathrm{Gr}_{\mathscr{G}} \times_{\mathbb{A}^1,[e]} \mathbb{A}^1$ be the base change. Then

$$(\widetilde{\mathrm{Gr}}_{\mathscr{G}})_0 \simeq \mathscr{F}\ell_v, \qquad \widetilde{\mathrm{Gr}}_{\mathscr{G}}|_{\mathbb{G}_m} \simeq \mathrm{Gr}_H \times \mathbb{G}_m.$$

Since I acts on H via pinned automorphisms, it acts on Gr_H, still denoted by ψ. The following lemma is clear from the above construction.

LEMMA 2.1. – *Under the isomorphism* $\mathrm{Gr}_{\mathscr{G}} \times_{\mathbb{G}_m} \mathbb{G}_m \simeq \mathrm{Gr}_H \times \mathbb{G}_m$, *the action of* $\gamma \in I$ *on the left hand side (via the Galois action on the second factor) corresponds to the action of* $\psi(\gamma) \times \gamma$ *on the right hand side.*

REMARK 2.2. – One should be warned that $\widetilde{\mathrm{Gr}}_{\mathscr{G}} \neq \mathrm{Gr}_{\widetilde{\mathscr{G}}}$, where $\widetilde{\mathscr{G}}$ is the base change of $\mathscr{G}$ along $[e] : \mathbb{A}^1 \to \mathbb{A}^1$.

Recall that we denote Sat_H to be the Satake category for H, i.e., the category of L^+H-equivariant perverse sheaves on Gr_H, which is equivalent to $\mathrm{Rep}(H^\vee)$ via the geometric Satake correspondence. Let

$$\mathscr{J} : \mathrm{Rep}(H^\vee) \to \mathrm{Sat}_H \tag{2.1}$$

be this equivalence. We define a functor

$$\mathcal{Z} : \mathrm{Sat}_H \to \mathscr{P}_v$$

by taking the nearby cycles. More precisely, let

$$\mathcal{Z}(\mathcal{F}) = \Psi_{\widetilde{\mathrm{Gr}}_{\mathscr{G}}}(\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell[1]), \tag{2.2}$$

where for an (ind)-scheme $\mathcal{X}$ of (ind)-finite type over $\mathbb{A}^1$, $\Psi_{\mathcal{X}}$ denotes the usual nearby cycle functor (see SGA 7, XIII for the definition of nearby cycles, and [8, A.2] for the explanation why the nearby cycles functors extend to ind-schemes of ind-finite type). Recall that the theory of nearby cycles provides an action of $\mathrm{Gal}(F^s/F)$ on the functor $\mathcal{Z}$ via automorphisms, usually called the monodromy action.

LEMMA 2.3. – *The monodromy of* $\mathcal{Z}(\mathcal{F})$ *is trivial.*

Proof. – This follows from the fact that there is a $\mathbb{G}_m$-action on $\widetilde{\mathrm{Gr}}_{\mathscr{G}}$ making the natural map $\widetilde{\mathrm{Gr}}_{\mathscr{G}} \to \mathbb{A}^1$ a $\mathbb{G}_m$-equivariant morphism, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ via natural dilatations (cf. [31, 24]). In addition, the restriction of this $\mathbb{G}_m$-action on $(\widetilde{\mathrm{Gr}}_{\mathscr{G}})_0 = \mathscr{F}\ell_v$ coincides with the action of the "rotation torus" on $\mathscr{F}\ell_v$ as mentioned in Lemma 1.2.

Then for any $\mathcal{F} \in \mathrm{Sat}_H$, the sheaf $\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell[1]$ is $\mathbb{G}_m$-equivariant so that the monodromy of the nearby cycle $\mathcal{Z}(\mathcal{F}) = \Psi_{\widetilde{\mathrm{Gr}}_{\mathscr{G}}}(\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell)$ is opposite to the $\mathbb{G}_m$-monodromic action on $\mathcal{Z}(\mathcal{F})$

(see [28] for the definition of $\mathbb{G}_m$-monodromic sheaves and $\mathbb{G}_m$-monodromic actions, and in particular Proposition 7.1 of loc. cit. for this statement). By Corollary 1.2, $\mathcal{Z}(\mathcal{F})$ is $\mathbb{G}_m$-equivariant which exactly means that the monodromy is trivial. □

REMARK 2.4. – A mixed characteristic analogue of this lemma also holds (Theorem 7.2).

Let $i_0 : (\widetilde{\mathrm{Gr}}_{\mathscr{G}})_0 \to \widetilde{\mathrm{Gr}}_{\mathscr{G}}$ be the closed embedding of the special fiber and $j : \widetilde{\mathrm{Gr}}_{\mathscr{G}} \setminus (\widetilde{\mathrm{Gr}}_{\mathscr{G}})_0 \to \widetilde{\mathrm{Gr}}_{\mathscr{G}}$ be the open complement.

COROLLARY 2.5. – *There is a canonical isomorphism* $\mathcal{Z}(\mathcal{F}) \simeq i_0^* j_{!*}(\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell)$.

Proof. – This is standard. Since the monodromy is trivial, from the distinguished triangle

$$i_0^* j_*(\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell) \to \mathcal{Z}(\mathcal{F}) \xrightarrow{0} \mathcal{Z}(\mathcal{F}) \to$$

we obtain that $i_0^* j_*(\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell)$ lives in perverse cohomological degree 0 and 1, and both cohomology sheaves are isomorphic to $\mathcal{Z}(\mathcal{F})$. But $i_0^* j_{!*}(\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell) = {}^p\mathrm{H}^0 i_0^* j_*(\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell)$, where ${}^p\mathrm{H}^*$ stands for the perverse cohomology. □

In what follows, for $\mathcal{F} \in \mathrm{Sat}_H$, we denote $\mathcal{F}_{\mathbb{G}_m} = \mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell[1]$ over $\widetilde{\mathrm{Gr}}_{\mathscr{G}}|_{\mathbb{G}_m}$, and $\mathcal{F}_{\mathbb{A}^1} = j_{!*}\mathcal{F}_{\mathbb{G}_m}$.

Recall that the irreducible objects in Sat_H are the intersection cohomology sheaves $\mathscr{IC}(V_\mu)$ on Gr_H where $\mu \in \mathbb{X}_\bullet(T_H)^+ = \mathbb{X}_\bullet(T)^+$. On the other hand, the irreducible objects in $\mathscr{P}_v$ are the intersection cohomology sheaves $\mathrm{IC}_{\bar{\mu}}$ on $\mathcal{G}\ell_v$, where $\bar{\mu} \in \mathbb{X}_\bullet(T)^+_I$. For $\bar{\mu} \in \mathbb{X}_\bullet(T)^+_I$, let $j_{\bar{\mu}} : \mathring{\mathcal{G}}\ell_{v\bar{\mu}} \to \mathcal{G}\ell_v$ be the corresponding locally closed embedding of K_v-orbits.

LEMMA 2.6. – *For any* $\mu \in \mathbb{X}_\bullet(T)^+$, *let* $\bar{\mu}$ *be its image in* $\mathbb{X}_\bullet(T)^+_I$. *Then* $j_{\bar{\mu}}^* \mathcal{Z}(\mathscr{IC}(V_\mu)) \simeq \overline{\mathbb{Q}}_\ell[(2\rho, \mu)]$.

Proof. – Consider $s_\mu \times \mathbb{G}_m \subset \mathrm{Gr}_H \times \mathbb{G}_m \subset \widetilde{\mathrm{Gr}}_{\mathscr{G}}|_{\mathbb{G}_m}$. Since $\widetilde{\mathrm{Gr}}_{\mathscr{G}}$ is ind-proper over $\mathbb{A}^1$, it extends to a section $\mathbb{A}^1 \to \widetilde{\mathrm{Gr}}_{\mathscr{G}}$, still denoted by s_μ. By [31, Proposition 3.5], $s_\mu(0) \in \mathcal{G}\ell_v$ is just the point $s_{\bar{\mu}}$, where $\bar{\mu}$ is the image of μ under $\mathbb{X}_\bullet(T_H) \to \mathbb{X}_\bullet(T_H)_I$.

Recall that $\mathscr{IC}(V_\mu)$ is supported on $\overline{\mathrm{Gr}}_\mu$, the Schubert variety in Gr_H corresponding to s_μ. Let $\widetilde{\mathrm{Gr}}_\mu \subset \widetilde{\mathrm{Gr}}_{\mathscr{G}}$ be the closure of $\overline{\mathrm{Gr}}_\mu \times \mathbb{G}_m \subset \widetilde{\mathrm{Gr}}_{\mathscr{G}}|_{\mathbb{G}_m}$. Then by the above fact, $\mathcal{G}\ell_{v\bar{\mu}}$ is contained in the special fiber of $\widetilde{\mathrm{Gr}}_\mu$. In fact, it is proved in [31] that the special fiber of $\widetilde{\mathrm{Gr}}_\mu$ is $\mathcal{G}\ell_{v\bar{\mu}}$. In addition, it is shown in loc. cit. that the point $s_{\bar{\mu}}$ is smooth in $\widetilde{\mathrm{Gr}}_\mu$. The lemma then is clear. □

The following key result is established in [8] for the split case and in [31, Theorem 7.3] in general. Let $D_{K_v}(\mathcal{G}\ell_v)$ be the K_v-equivariant derived category on $\mathcal{G}\ell_v$, and $\star : \mathscr{P}_v \times \mathscr{P}_v \to D_{K_v}(\mathcal{G}\ell_v)$ be the convolution product functor. For a precise definition of the convolution product, see for example [19, 31].

PROPOSITION 2.7. – *For any* $\mathcal{F}_1 \in \mathrm{Sat}_H$ *and* $\mathcal{F}_2 \in \mathscr{P}_v$, *there is a canonical isomorphism* $\mathcal{Z}(\mathcal{F}_1) \star \mathcal{F}_2 \simeq \mathcal{F}_2 \star \mathcal{Z}(\mathcal{F}_1)$ *and both are objects in* $\mathscr{P}_v$.

A brief review of the construction of this canonical isomorphism is given in § 3.

COROLLARY 2.8. – *The convolution of* $\mathscr{P}_v$ *is bi-exact. Therefore,* $\mathscr{P}_v$ *is a monoidal category.*

Proof. – Observe that $\mathscr{P}_v$ is semi-simple (Lemma 1.1) and every irreducible object in $\mathscr{P}_v$ is a direct summand of some $\mathcal{Z}(\mathscr{F})$. Indeed, for $\bar{\mu} \in \mathbb{X}_\bullet(T)^+_I$, let μ be a lift of it in $\mathbb{X}_\bullet(T)^+$. Then by Lemma 2.6, IC_μ appears as a direct summand of $\mathcal{Z}(\mathscr{J}(V_\mu))$ with multiplicity one. As the convolution (for left and right) with $\mathcal{Z}(\mathscr{J}(V_\mu))$ is exact, the convolution with its direct summand is also exact. The first claim follows.

It is well-known that the convolution functor $D_{K_v}(\mathscr{F}\ell_v) \times D_{K_v}(\mathscr{F}\ell_v) \to D_{K_v}(\mathscr{F}\ell_v)$ is monoidal. Its restriction to $\mathscr{P}_v \times \mathscr{P}_v$ takes value in $\mathscr{P}_v$. As $\mathscr{P}_v$ is a full subcategory of $D_{K_v}(\mathscr{F}\ell_v)$, the associativity constraints are morphisms in $\mathscr{P}_v$. Therefore, $\mathscr{P}_v$ is a monoidal subcategory of $D_{K_v}(\mathscr{F}\ell_v)$. □

Remark 2.9. – (i) According to [19, Remark 4.5], the exactness of the convolution product probably would imply that $LG \times^K \mathscr{F}\ell_v \to \mathscr{F}\ell_v$ is (stratified) semi-small.

(ii) By the same argument, the convolution bi-functor $\star : \mathscr{P}(\mathscr{F}\ell_v) \times \mathscr{P}_v \to \mathscr{P}(\mathscr{F}\ell_v)$ is also exact, where $\mathscr{P}(\mathscr{F}\ell_v)$ is the category of perverse sheaves on $\mathscr{F}\ell_v$.

3. $\mathcal{Z}$ is a central functor

In this section, we show that $\mathcal{Z}$ is a central functor in the sense of [2]. By Lemma 2.6, together with some general nonsense, this already implies that $\mathscr{P}_v$ is equivalent to $\mathrm{Rep}(\tilde{G}^\vee)$ for some closed subgroup $\tilde{G}^\vee \subset H^\vee$. In the next section, we will identify $\tilde{G}^\vee$ explicitly. We will also determine a fiber functor of $\mathscr{P}_v$.

Theorem-Definition 3.1. – *The functor* $\mathcal{Z} : \mathrm{Sat}_H \to \mathscr{P}_v$ *is naturally a monoidal functor.*

Proof. – The proof is literally the same as the proof in [8, Theorem 1(c)]. We repeat the argument here in order to make the definition of this monoidal structure explicit.

Let $\mathrm{Gr}_{\mathcal{G}} \tilde{\times} \mathrm{Gr}_{\mathcal{G}}$ be the ind-scheme over $\mathbb{A}^2$ classifying

$$\mathrm{Gr}_{\mathcal{G}} \tilde{\times} \mathrm{Gr}_{\mathcal{G}}(R) = \left\{ (x, y, \mathcal{E}, \mathcal{E}', \beta, \beta') \;\middle|\; \begin{array}{l} x, y \in \mathbb{A}^1(R), \mathcal{E}, \mathcal{E}' \text{ are two } \mathcal{G}\text{-torsors} \\ \text{on } \mathbb{A}^1_R, \beta : \mathcal{E}|_{\mathbb{A}^1_R - \Gamma_x} \simeq \mathcal{E}^0|_{\mathbb{A}^1_R - \Gamma_x} \text{ is a} \\ \text{trivialization}, \beta' : \mathcal{E}'|_{\mathbb{A}^1_R - \Gamma_y} \simeq \mathcal{E}|_{\mathbb{A}^1_R - \Gamma_y} \end{array} \right\}. \tag{3.1}$$

Let $\mathrm{Gr}_{\mathcal{G}} \underset{\mathbb{A}^1}{\tilde{\times}} \mathrm{Gr}_{\mathcal{G}}$ denote the restriction of $\mathrm{Gr}_{\mathcal{G}} \tilde{\times} \mathrm{Gr}_{\mathcal{G}}$ along the diagonal $\Delta \to \mathbb{A}^2$. Then $\mathrm{Gr}_{\mathcal{G}} \underset{\mathbb{A}^1}{\tilde{\times}} \mathrm{Gr}_{\mathcal{G}}$ is a kind of twisted product. Indeed, let $\mathscr{L}^+\mathcal{G}$ be the global jet group of $\mathcal{G}$, which classifies a point on $\mathbb{A}^1$ and a trivialization of the trivial $\mathcal{G}$-torsor over the formal neighborhood of this point (cf. [31, §3.1]). Then $\mathscr{L}^+\mathcal{G}$ naturally acts on $\mathrm{Gr}_{\mathcal{G}}$. In addition, there is a $\mathscr{L}^+\mathcal{G}$-torsor over $\mathrm{Gr}_{\mathcal{G}}$ classifying quadruples $(y, \mathcal{E}, \beta, \gamma)$, where the triple $(y, \mathcal{E}, \beta)$ is as in the definition of $\mathrm{Gr}_{\mathcal{G}}$ and γ is a trivialization of $\mathcal{E}$ over the formal neighborhood of y (this is indeed the global loop group $\mathscr{L}\mathcal{G}$ of $\mathcal{G}$ introduced in [31, §3.1]). Then $\mathrm{Gr}_{\mathcal{G}} \underset{\mathbb{A}^1}{\tilde{\times}} \mathrm{Gr}_{\mathcal{G}} \simeq \mathscr{L}\mathcal{G} \overset{\mathscr{L}^+\mathcal{G}}{\times} \mathrm{Gr}_{\mathcal{G}}$. Let us denote the base change of this isomorphism along $[e] : \mathbb{A}^1 \to \mathbb{A}^1$ by $\widetilde{\mathrm{Gr}}_{\mathcal{G}} \underset{\mathbb{A}^1}{\tilde{\times}} \widetilde{\mathrm{Gr}}_{\mathcal{G}} \simeq \widetilde{\mathscr{L}\mathcal{G}} \overset{\widetilde{\mathscr{L}^+\mathcal{G}}}{\times} \widetilde{\mathrm{Gr}}_{\mathcal{G}}$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two objects in Sat_H. We can form the twisted product $(\mathcal{F}_1)_{\mathbb{G}_m} \tilde{\times} (\mathcal{F}_2)_{\mathbb{G}_m}$ over $\widetilde{\mathrm{Gr}}_{\mathcal{G}}|_{\mathbb{G}_m} \underset{\mathbb{G}_m}{\tilde{\times}} \widetilde{\mathrm{Gr}}_{\mathcal{G}}|_{\mathbb{G}_m}$. We claim that there is a canonical isomorphism

$$\Psi_{\widetilde{\mathrm{Gr}}_{\mathcal{G}} \underset{\mathbb{A}^1}{\tilde{\times}} \widetilde{\mathrm{Gr}}_{\mathcal{G}}}((\mathcal{F}_1)_{\mathbb{G}_m} \tilde{\times} (\mathcal{F}_2)_{\mathbb{G}_m}) \simeq \mathcal{Z}(\mathcal{F}_1) \tilde{\times} \mathcal{Z}(\mathcal{F}_2).$$

Indeed, let $V_i \subset \widetilde{\mathrm{Gr}}_{\mathcal{G}}$ be the closure of the support of $\mathcal{F}_i \boxtimes \overline{\mathbb{Q}}_\ell[1]$ in $\widetilde{\mathrm{Gr}}_{\mathcal{G}}|_{\mathbb{G}_m}$. Let $\mathcal{L}_n^+\mathcal{G}$ be the nth jet group such that the action of $\mathcal{L}^+\mathcal{G}$ on V_i factors through $\mathcal{L}_n^+\mathcal{G}$. The corresponding $\mathcal{L}_n^+\mathcal{G}$ torsor over $\mathrm{Gr}_{\mathcal{G}}$ is denoted by $\mathcal{L}_n\mathcal{G}$. Let us denote by $\widetilde{\mathcal{L}_n^+\mathcal{G}}$ and $\widetilde{\mathcal{L}_n\mathcal{G}}$ their base changes along $[e]$. Then one can check the isomorphism after pullback along $\widetilde{\mathcal{L}_n\mathcal{G}} \times_{\mathbb{A}^1} V_2 \to \widetilde{\mathcal{L}_n\mathcal{G}} \overset{\widetilde{\mathcal{L}_n^+\mathcal{G}}}{\times} V_2$, and the isomorphism follows from [8, Theorem 5.2.1].

Now $\mathrm{Gr}_{\mathcal{G}} \underset{\mathbb{A}^1}{\tilde{\times}} \mathrm{Gr}_{\mathcal{G}} \to \mathrm{Gr}_{\mathcal{G}}, (y, \mathcal{E}, \mathcal{E}', \beta, \beta) \mapsto (y, \mathcal{E}', \beta\beta')$ is ind-proper and taking nearby cycles commutes with proper push-forward. Therefore we obtain the canonical isomorphism

$$\mathcal{Z}(\mathcal{F}_1 \star \mathcal{F}_2) \simeq \mathcal{Z}(\mathcal{F}_1) \star \mathcal{Z}(\mathcal{F}_2).$$

In addition, working over $\mathrm{Gr}_{\mathcal{G}} \underset{\mathbb{A}^1}{\tilde{\times}} \mathrm{Gr}_{\mathcal{G}} \underset{\mathbb{A}^1}{\tilde{\times}} \mathrm{Gr}_{\mathcal{G}}$, one can see that this isomorphism makes $\mathcal{Z}$ a monoidal functor. □

Let us recall the definition of central functors as in [2]. Namely, if $F : \mathcal{C} \to \mathcal{D}$ is a monoidal functor between two monoidal categories and assume that $\mathcal{C}$ is a symmetric monoidal category, then F (together with the following data) is called central if

1. there is an isomorphism c of the bi-functors $\mathcal{C} \times \mathcal{D} \to \mathcal{D}, (X, Y) \mapsto F(X) \otimes Y$ and $(X, Y) \mapsto Y \otimes F(X)$, i.e., an isomorphism $c_{X,Y} : F(X) \otimes Y \simeq Y \otimes F(X)$ functorial in X, Y;
2. for $X, X' \in \mathcal{C}$, the following diagram is commutative
$$\begin{array}{ccc} F(X) \otimes F(X') & \xrightarrow{c_{X,F(X')}} & F(X') \otimes F(X) \\ \downarrow & & \downarrow \\ F(X \otimes X') & \xrightarrow{F(\sigma_{X,X'})} & F(X' \otimes X), \end{array}$$
where σ is the commutativity constraint of $\mathcal{C}$;
3. for $X \in \mathcal{C}$ and $Y, Y' \in \mathcal{D}$, the following diagram is commutative
$$\begin{array}{ccc} F(X) \otimes Y \otimes Y' & \xrightarrow{c_{X,Y} \otimes \mathrm{id}} & Y \otimes F(X) \otimes Y' \\ {\scriptstyle c_{X,Y \otimes Y'}}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes c_{X,Y'}} \\ Y \otimes Y' \otimes F(X) & = & Y \otimes Y' \otimes F(X); \end{array}$$
4. for $X, X' \in \mathcal{C}, Y \in \mathcal{D}$, the following diagram is commutative
$$\begin{array}{ccccc} F(X) \otimes F(X') \otimes Y & \xrightarrow{\mathrm{id} \otimes c_{X',Y}} & F(X) \otimes Y \otimes F(X') & \xrightarrow{c_{X,Y} \otimes \mathrm{id}} & Y \otimes F(X) \otimes F(X') \\ \downarrow & & & & \downarrow \\ F(X \otimes X') \otimes Y & \xrightarrow{c_{X \otimes X',Y}} & Y \otimes F(X \otimes X') & = & Y \otimes F(X \otimes X'). \end{array}$$

PROPOSITION 3.2. – *The functor $\mathcal{Z}$, together with the canonical isomorphisms provided in Proposition 2.7, is a central functor. There is an algebraic group $\tilde{G}^\vee \subset H^\vee$ together with an equivalence $S : \mathcal{P}_v \simeq \mathrm{Rep}(\tilde{G}^\vee)$ such that $S \circ \mathcal{Z} \simeq \mathrm{Res}^{\tilde{G}^\vee}_{H^\vee}$ as tensor functors, where $\mathrm{Res}^{\tilde{G}^\vee}_{H^\vee}$ is the restriction functor from $\mathrm{Rep}(H^\vee)$ to $\mathrm{Rep}(\tilde{G}^\vee)$.*

Proof. – Since every object in $\mathcal{P}_v$ appears as a direct summand of some object in the essential image of $\mathcal{Z}$, the second statement of the proposition is a direct consequence of the first statement and Proposition 1 of [2].

The first statement can be checked literally the same as in [9]. For this, let us briefly review the construction of the canonical isomorphisms in Proposition 2.7. Let us define the Beilinson-Drinfeld Grassmannian

$$\mathrm{Gr}^{BD}_{\mathscr{G}}(R) = \left\{ (y, \mathcal{E}, \beta) \;\middle|\; \begin{array}{l} y \in \mathbb{A}^1(R), \mathcal{E} \text{ is a } \mathscr{G}\text{-torsor on } \mathbb{A}^1_R, \text{ and} \\ \beta : \mathcal{E}|_{(\mathbb{G}_m)_R - \Gamma_y} \simeq \mathcal{E}^0|_{(\mathbb{G}_m)_R - \Gamma_y} \text{ is a trivialization} \end{array} \right\}, \tag{3.2}$$

where $\mathcal{E}^0$ denotes the trivial torsor. This is the restriction to $\{0\} \times \mathbb{A}^1$ (or to $\mathbb{A}^1 \times \{0\}$) of the usual Beilinson-Drinfeld Grassmannian over $\mathbb{A}^2$ (e.g., the one considered in [19]). In particular, we have

$$\mathrm{Gr}^{BD}_{\mathscr{G}}|_{\mathbb{G}_m} \simeq \mathrm{Gr}_{\mathscr{G}}|_{\mathbb{G}_m} \times (\mathrm{Gr}_{\mathscr{G}})_0, \quad (\mathrm{Gr}^{BD}_{\mathscr{G}})_0 \simeq (\mathrm{Gr}_{\mathscr{G}})_0 \simeq \mathscr{F}\ell_v.$$

As before, we denote by $\widetilde{\mathrm{Gr}}^{BD}_{\mathscr{G}}$ the base change along $[e] : \mathbb{A}^1 \to \mathbb{A}^1$. Over $\widetilde{\mathrm{Gr}}^{BD}_{\mathscr{G}}|_{\mathbb{G}_m} \simeq \widetilde{\mathrm{Gr}}_{\mathscr{G}}|_{\mathbb{G}_m} \times (\mathrm{Gr}_{\mathscr{G}})_0$, we can form the external product $(\mathcal{T}_1)_{\mathbb{G}_m} \boxtimes \mathcal{T}_2$. Then the isomorphism in the proposition is induced from the canonical isomorphisms

$$\mathcal{Z}(\mathcal{T}_1) \star \mathcal{T}_2 \simeq \Psi_{\widetilde{\mathrm{Gr}}^{BD}_{\mathscr{G}}}((\mathcal{T}_1)_{\mathbb{G}_m} \boxtimes \mathcal{T}_2) \simeq \mathcal{T}_2 \star \mathcal{Z}(\mathcal{T}_1),$$

where the two isomorphisms are induced from the natural maps

$$\mathrm{Gr}_{\mathscr{G}} \tilde{\times} \mathrm{Gr}_{\mathscr{G}}|_{\mathbb{A}^1 \times \{0\}} \to \mathrm{Gr}^{BD}_{\mathscr{G}} \leftarrow \mathrm{Gr}_{\mathscr{G}} \tilde{\times} \mathrm{Gr}_{\mathscr{G}}|_{\{0\} \times \mathbb{A}^1}.$$

See [31, Proposition 7.4] for more details.

Again, since the monodromy of $\Psi_{\widetilde{\mathrm{Gr}}^{BD}_{\mathscr{G}}}((\mathcal{T}_1)_{\mathbb{G}_m} \boxtimes \mathcal{T}_2)$ is trivial, we have

$$\mathcal{Z}(\mathcal{T}_1) \star \mathcal{T}_2 \simeq i_0^* j_{!*}((\mathcal{T}_1)_{\mathbb{G}_m} \boxtimes \mathcal{T}_2) \simeq \mathcal{T}_2 \star \mathcal{Z}(\mathcal{T}_1), \tag{3.3}$$

where i_0, j are corresponding closed and open embedding.

Back to the proof of the proposition. The conditions (3) and (4) are checked the same way as in [9]. To check condition (2) is even simpler than loc. cit. Observe that the monodromy of all the nearby cycles involved is trivial. Then the nearby cycles can be expressed via intermediate extensions as in Corollary 2.5 and (3.3), rather than via the homotopy (co)limits of certain ind-pro system of sheaves as in loc. cit. □

Now we would like to endow $\mathcal{P}_v$ with a fiber functor. We begin with the following general lemma.

LEMMA 3.3. – *Let $G_1 \subset G_2$ be a closed embedding of affine algebraic groups over a field E (of characteristic zero). Let $F : \mathrm{Rep}(G_1) \to \mathrm{Vect}_E$ be an E-linear exact and faithful functor. Assume that:* (i) *$F(X \otimes Y)$ and $F(X) \otimes F(Y)$ are (non-canonically) isomorphic;* (ii) *$F \circ \mathrm{Res}^{G_1}_{G_2}$ is a fiber functor of $\mathrm{Rep}(G_2)$. Then F has a unique fiber functor structure which induces the fiber functor structure of $F \circ \mathrm{Res}^{G_1}_{G_2}$ as in* (ii).

REMARK 3.4. – We are not sure whether the first assumption is necessary.

Proof. – The uniqueness is clear. We write $R = \mathrm{Res}^{G_1}_{G_2}$ for simplicity. For any $X \in \mathrm{Rep}(G_2)$, let $\langle X\rangle$ denote the full subcategory of $\mathrm{Rep}(G_2)$ consisting of the objects that are isomorphic to subquotients of $X^n, n \in \mathbb{N}$, and $\langle R(X)\rangle$ denote the full subcategory of $\mathrm{Rep}(G_1)$ consisting of the objects that are isomorphic to subquotients of $R(X)^n, n \in \mathbb{N}$. Let us denote $\mathrm{End}(FR|_{\langle X\rangle})$ (resp. $\mathrm{End}(F|_{\langle R(X)\rangle})$) the endomorphism algebra of the restriction of the functor FR (resp. F) to $\langle X\rangle$ (resp. $\langle R(X)\rangle$). They are finite dimensional E-algebras and clearly, the E-algebra homomorphism $\mathrm{End}(F|_{\langle R(X)\rangle}) \to \mathrm{End}(FR|_{\langle X\rangle})$ is injective. According to [7, Lemma 2.13], there are canonical equivalences and the following commutative diagram

$$\begin{array}{ccccc}
\langle X\rangle & \xrightarrow[\simeq]{a_X} & \mathrm{End}(FR|_{\langle X\rangle})\text{-Mod} & \xrightarrow{\omega} & \mathrm{Vect}_E \\
{\scriptstyle R}\downarrow & & \downarrow & & \| \\
\langle R(X)\rangle & \xrightarrow[\simeq]{b_{R(X)}} & \mathrm{End}(F|_{\langle R(X)\rangle})\text{-Mod} & \xrightarrow{\omega} & \mathrm{Vect}_E\,.
\end{array}$$

In addition, $\omega a_X \simeq FR$ and $\omega b_{R(X)} \simeq F$. Observe that if $\langle X\rangle$ is a subcategory of $\langle Y\rangle$. Then we have a natural algebra homomorphism $\mathrm{End}(FR|_{\langle Y\rangle}) \to \mathrm{End}(FR|_{\langle X\rangle})$. Then $A = \varinjlim_{X\in \mathrm{Rep}G_2} \mathrm{End}(FR|_{\langle X\rangle})^\vee$ is a coalgebra. Similarly, we can define $B = \varinjlim_{X\in \mathrm{Rep}G_2} \mathrm{End}(F|_{\langle R(X)\rangle})^\vee$. We have the surjective map of coalgebras $A \to B$, and

$$\begin{array}{ccccc}
\mathrm{Rep}G_2 & \xrightarrow[\simeq]{a} & A\text{-Comod} & \xrightarrow{\omega} & \mathrm{Vect}_E \\
{\scriptstyle R}\downarrow & & \downarrow & & \| \\
\mathrm{Rep}G_1 & \xrightarrow[\simeq]{b} & B\text{-Comod} & \xrightarrow{\omega} & \mathrm{Vect}_E\,.
\end{array}$$

By the assumption (i) and [7, Proposition 2.16], the tensor structures on $\mathrm{Rep}G_1$ and $\mathrm{Rep}G_2$ induce $B \otimes B \to B$ and $A \otimes A \to A$ respectively. Since the restriction functor R is a tensor functor, we have the commutative diagram

$$\begin{array}{ccc}
A \otimes A & \longrightarrow & A \\
\downarrow & & \downarrow \\
B \otimes B & \longrightarrow & B.
\end{array}$$

By assumption (ii), $\omega a \simeq FR$ is a fiber functor of $\mathrm{Rep}G_2$, and therefore we know that $A = \mathcal{O}_{G_2}$ and the map $A \otimes A \to A$ is the usual multiplication. Since the map $A \to B$ is surjective, this implies that the map $B \otimes B \to B$ is also associative and commutative. By [7, Proposition 2.16] again, this implies that the functor F is compatible with the associativity and the commutativity constraints. The lemma follows. □

COROLLARY 3.5. – *The functor given by taking the cohomology* $\mathrm{H}^* : \mathscr{P}_v \to \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$ *has a natural structure as a fiber functor.*

Proof. – It is well-known (e.g., from the decomposition theorem) that there exists an isomorphism $\mathrm{H}^*(\mathcal{F}\ell_v, \mathcal{F}_1 \star \mathcal{F}_2) \simeq \mathrm{H}^*(\mathcal{F}\ell_v, \mathcal{F}_1) \otimes \mathrm{H}^*(\mathcal{F}\ell_v, \mathcal{F}_2)$ (non-canonically). Since taking nearby cycles commutes with proper push-forward, we have a canonical isomorphism

$\mathrm{H}^* \circ \mathcal{Z} \simeq \mathrm{H}^*$. Since $\mathrm{H}^* : \mathrm{Sat}_H \to \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$ is a fiber functor, the assertion follows from the above lemma. □

4. Identification of the group $\tilde{G}^\vee$ with $(H^\vee)^I$

We need to describe the group $\tilde{G}^\vee$ from the last section. To begin with, let us review how the geometric Satake correspondence (together with a choice of an ample line bundle on Gr_H) gives rise to a pinned group $(H^\vee, B_H^\vee, T_H^\vee, X^\vee)$. First, once we choose $T_H \subset B_H \subset H$, the construction of [19] gives us $T_H^\vee \subset B_H^\vee \subset H^\vee$. Namely, let $U_H \subset B_H$ be its unipotent radical. For $\mu \in \mathbb{X}_\bullet(T_H)$, let S_μ be the semi-infinite orbit on Gr_H passing through s_μ as introduced in [19] (i.e., the LU_H-orbit passing through s_μ). Let $S_{\leq\mu} = \cup_{\lambda \preceq \mu} S_\lambda$ and $S_{<\mu} = \cup_{\lambda \prec \mu} S_\lambda$. Then the fiber functor $\mathrm{H}^* : \mathrm{Sat}_H \to \mathrm{Vect}_{\mathbb{Q}_\ell}$ has a canonical filtration (called the *MV filtration*) given by $\ker(\mathrm{H}^*(\mathrm{Gr}_H, -) \to \mathrm{H}^*(S_{<\mu}, -))$. This defines a Borel $B_H^\vee \subset H^\vee$. In addition, it is proved that the filtration admits a canonical splitting, i.e., a canonical isomorphism $\mathrm{H}^*(\mathrm{Gr}_H, -) \simeq \bigoplus_\mu \mathrm{H}_c^*(S_\mu, -)$. This provides a maximal torus $T_H^\vee \subset B_H^\vee$. Let $\mathcal{L}$ be an ample line bundle on Gr_H, and let $c(\mathcal{L}) \in \mathrm{H}^2(\mathrm{Gr}_H, \overline{\mathbb{Q}}_\ell)$ be its Chern class. Then it is shown in [10, 29] that the cup product with this class realizes $c(\mathcal{L})$ as a principal nilpotent element in $X^\vee \in \mathfrak{h}^\vee = \mathrm{Lie} H^\vee$. In addition, by [29, Proposition 5.6], the quadruple $(H^\vee, B_H^\vee, T_H^\vee, X^\vee)$ is indeed a pinned reductive group.

REMARK 4.1. – One remark is in order. In [29], all the assertions are proved for the affine Grassmannian defined over $\mathbb{C}$. The only place where the complex topology is used, besides the issue of dealing with $\mathbb{Z}$-coefficients as in [19], is to define the coproduct on $\mathrm{H}^*(\mathrm{Gr}_H, \mathbb{Z})$ by realizing Gr_H as being homotopic to the based loop space of a maximal compact subgroup of $H_{\mathbb{C}}$. However, one can provide a commutative and cocommutative Hopf algebra structure on $\mathrm{H}^*(\mathrm{Gr}_H, \mathbb{Z})$ using the Beilinson-Drinfeld Grassmannian. More precisely, one can use the isomorphism (2.11) in loc. cit. to define the comultiplication map by the formula (2.12) in loc. cit. This map on the other hand can be realized as follows. There is the Beilinson-Drinfeld Grassmannian $\pi : \mathrm{Gr}_2 \to \mathbb{A}^2$ whose fiber over a point in the diagonal $\Delta \subset \mathbb{A}^2$ is Gr_H and whose fiber over a point off the diagonal is $\mathrm{Gr}_H \times \mathrm{Gr}_H$ (cf. [19, Section 5]). Then $R^i\pi_*\mathbb{Q}_\ell$ is a constructible sheaf on $\mathbb{A}^2$, constant along the stratification $\mathbb{A}^2 = \Delta \cup (\mathbb{A}^2 - \Delta)$. Now the usual cospecialization map of constructible sheaves gives rise to the comultiplication. From this latter definition, the usual arguments for the commutativity constraints as in [19] show that this defined comultiplication is indeed cocommutative. The proof of [29, Lemma 5.1] that $c^{T_H}(\mathcal{L})$ is primitive under this Hopf algebra structure can be replaced by the following argument: as is well-known (e.g., see [30, 1.1.9]), if $\mathcal{L}$ is ample on Gr_H, then there is an ample line bundle on Gr_2, which away from the diagonal is $\mathcal{L} \boxtimes \mathcal{L}$ and on the diagonal is $\mathcal{L}$. Now the above arguments and all the remaining arguments of [29] apply to the situation when Gr_H is defined over arbitrary field k and sheaves have $\overline{\mathbb{Q}}_\ell$-coefficients.

REMARK 4.2. – As explained in [19, Theorem 3.6], the above pinning $(H^\vee, B_H^\vee, T_H^\vee, X^\vee)$ is in fact independent of the choice of $T \subset B$. Another way to deduce this fact is as follows. The natural grading on the cohomological functor H^* defines a one-parameter subgroup $\mathbb{G}_m \to H^\vee$ and $T_H^\vee$ is just the centralizer of this subgroup, which is independent of the choice of $T \subset B$. On the other hand, $B_H^\vee$ is completely determined by $X^\vee$, which is also

independent of the choice of $T \subset B$. In other words, there is a canonical morphism from the Lefschetz SL_2 to $H^\vee$, which gives a principal SL_2 in $H^\vee$, and the pinning is determined by this principal SL_2.

Recall that we denote by ψ the action of I on Gr_H. The action of $\gamma \in I$ will map $\overline{\mathrm{Gr}}_\lambda$ isomorphically to $\overline{\mathrm{Gr}}_{\gamma(\lambda)}$. Therefore the pushforward functor $\gamma_* : D(\mathrm{Gr}_H) \to D(\mathrm{Gr}_H)$ for $\gamma \in I$ naturally gives rise to $\gamma_* : \mathrm{Sat}_H \to \mathrm{Sat}_H$. In this way, I acts on Sat_H via tensor automorphisms. Under the geometric Satake correspondence, I acts on $H^\vee$ clearly as pinned automorphisms with respect to the pinning we mentioned above.

THEOREM 4.3. – $\tilde{G}^\vee \simeq (H^\vee)^I$.

REMARK 4.4. – Observe that $(H^\vee)^I$ is not necessarily a connected reductive group. For example: let $H^\vee = \mathrm{GL}_{2n+1}$, let J be the matrix with $1s$ on the anti-diagonal and $0s$ elsewhere. Let $\Gamma = \{1, \gamma\}$ and γ act on $H^\vee$ via $g \mapsto J(g^t)^{-1}J$. Then $(H^\vee)^I = \mathrm{O}_{2n+1}$.

Proof. – Since $\mathscr{P}_v$ is semi-simple, $\tilde{G}^\vee$ is a reductive subgroup of $H^\vee$. We first see that $\tilde{G}^\vee \subset (H^\vee)^I$. The following lemma is a direct consequence of [7, Corollary 2.9].

LEMMA 4.5. – *Let $f : H_2 \to H_1$ be a homomorphism of algebraic groups and let ω^f denote the induced tensor functor* $\mathrm{Rep}(H_1) \to \mathrm{Rep}(H_2)$ *(if f is a closed embedding then ω^f is the restriction functor* $\mathrm{Res}^{H_2}_{H_1}$*). Let $I \subset \mathrm{Aut}(H_1)$ so that it acts on* $\mathrm{Rep}(H_1)$ *via tensor automorphisms. If for any $\gamma \in I$, $\omega^f \circ \omega^\gamma \simeq \omega^f$, then f factors through $f : H_2 \to H_1^I \subset H_1$.*

Now, I acts on $\widetilde{\mathrm{Gr}}_{\mathscr{G}} = \mathrm{Gr}_{\mathscr{G}} \times_{\mathbb{A}^1} \mathbb{A}^1$ via the action on the second factor $\mathbb{A}^1$ by deck transformations. By Lemma 2.1, we have

$$\mathcal{Z}(\gamma_*\mathcal{F}) = \Psi_{\widetilde{\mathrm{Gr}}_{\mathscr{G}}}(\gamma_*\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell[1]) \simeq \Psi_{\widetilde{\mathrm{Gr}}_{\mathscr{G}}}((\psi(\gamma) \times \gamma)_*(\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell[1])) \simeq \Psi_{\widetilde{\mathrm{Gr}}_{\mathscr{G}}}(\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell[1]) = \mathcal{Z}(\mathcal{F}).$$

In other words, we have the tensor isomorphism between $\mathcal{Z} \circ \gamma_*$ and $\mathcal{Z}$ for all $\gamma \in I$. From the above lemma, $\tilde{G}^\vee \subset (H^\vee)^I$.

Therefore, we have successive restriction functors

$$\mathrm{Rep}(H^\vee) \to \mathrm{Rep}((H^\vee)^I) \to \mathrm{Rep}(\tilde{G}^\vee).$$

To prove that $\tilde{G}^\vee = (H^\vee)^I$, it is enough to show that the above restriction induces an isomorphism of K-groups $K(\mathrm{Rep}(H^\vee)^I) \simeq K(\mathrm{Rep}(\tilde{G}^\vee))$.

As the group $(H^\vee)^I$ may not be connected, we need to be careful to describe its representation ring.

Let $(H^\vee)^{I,0}$ denote the neutral connected component of $(H^\vee)^I$. This is a connected reductive group with maximal torus $(T^\vee)^{I,0}$, the neutral connected component of $(T^\vee)^I$. The key fact is the following lemma.

LEMMA 4.6. – *The natural map*

$$(T^\vee)^I/(T^\vee)^{I,0} \to (H^\vee)^I/(H^\vee)^{I,0}$$

is an isomorphism.

Proof. – We will not distinguish a group from its E-points. As we purely work with dual groups, we switch the notation $H^\vee$ to H, etc. in the proof. Let us choose γ to be a generator of I. We need to show that $\pi_0(T^I) = \pi_0(H^I)$. Let N be the normalizer of T in H and let $W = N/T$ be the Weyl group. Then I acts on W naturally. Let us define a right action of W^I on $\mathrm{H}^1(I,T)$ as follows. Suppose w is in W^I and c be a cohomology class; lift w to $n \in N$ and lift c to a cocycle $\varphi : I \to T$. Then we set $(c \cdot w)(\gamma) = [n^{-1}\varphi(\gamma)\gamma(\mathrm{n})]$. It is clear that this is independent of all choices. We will deduce Lemma 4.6 from the following fact.

LEMMA 4.7. – *Under the above definition, every element $w \in W^I$ acts on $\mathrm{H}^1(I,T)$ via a group automorphism. In addition, $\mathrm{H}^1(I,H)$ is the quotient of $\mathrm{H}^1(I,T)$ via the above action.*

Proof. – Observe that the map $N^I \to W^I$ is surjective. Indeed, let H_{der} be the derived group of H and H_{sc} be the simply-connected cover of H_{der}. We have corresponding groups $N_{\mathrm{der}}, N_{\mathrm{sc}}, T_{\mathrm{der}}, T_{\mathrm{sc}}$. We now apply the argument of [26] p. 55 (5) to H_{sc} and γ. Our assumption that γ is pinned allows us to take $t = 1$ in loc. cit. It follows that the natural map $N^I_{\mathrm{sc}} \to W^I$ is surjective; therefore the same is true for $N^I \to W^I$ (another argument of this surjectivity can be found in [3, Lemma 6.2]). By taking the lift w to $n \in N^I$, it is clear that w acts on $\mathrm{H}^1(I,T)$ via group automorphisms. The second statement was proved in [24]. □

COROLLARY 4.8. – *The preimage of $1 \in \mathrm{H}^1(I,H)$ under $\mathrm{H}^1(I,T) \to \mathrm{H}^1(I,H)$ is* 1.

We continue to prove Lemma 4.6. First, if H is simply-connected, then H^I is connected as is shown in [26, Theorem 8.2]. On the other hand, I acts on T via permuting a basis of $\mathbb{X}^\bullet(T)$. Therefore, T^I is also connected. The lemma holds in this case. For general H, let H_{der} be the derived group of H and H_{sc} be the simply-connected cover of H_{der}. Let T_{der} and T_{sc} be the corresponding preimages of T. Write

$$1 \to Z \to H_{\mathrm{sc}} \to H_{\mathrm{der}} \to 1$$

which then gives

$$1 \to \pi_0(H^I_{\mathrm{der}}) \to \mathrm{H}^1(I,Z) \to \mathrm{H}^1(I,H_{\mathrm{sc}}). \tag{4.1}$$

Similarly, the sequence of maximal tori

$$1 \to Z \to T_{\mathrm{sc}} \to T_{\mathrm{der}} \to 1$$

gives

$$1 \to \pi_0(T^I_{\mathrm{der}}) \to \mathrm{H}^1(I,Z) \to \mathrm{H}^1(I,T_{\mathrm{sc}}). \tag{4.2}$$

Comparing (4.1) and (4.2) and applying Corollary 4.8, we obtain that the natural map

$$\pi_0(T^I_{\mathrm{der}}) \xrightarrow{\sim} \pi_0(H^I_{\mathrm{der}}) \tag{4.3}$$

is an isomorphism. Now consider

$$1 \to H_{\mathrm{der}} \to H \to D \to 1$$

which gives

$$1 \to \pi_0(H^I_{\mathrm{der}}) \to \pi_0(H^I) \to \pi_0(D^I) \to \mathrm{H}^1(I,H_{\mathrm{der}}). \tag{4.4}$$

Similarly, the sequence of maximal tori

$$1 \to T_{\mathrm{der}} \to T \to D \to 1$$

give

$$1 \to \pi_0(T^I_{\mathrm{der}}) \to \pi_0(T^I) \to \pi_0(D^I) \to \mathrm{H}^1(I, T_{\mathrm{der}}). \tag{4.5}$$

Comparing (4.4) and (4.5) and using Corollary 4.8 again, we obtain that the natural map

$$\pi_0(T^I) \xrightarrow{\sim} \pi_0(H^I) \tag{4.6}$$

is an isomorphism. □

REMARK 4.9. – Note that the above proof of Lemma 4.6 in fact applies to any cyclic subgroup I of the group of pinned automorphisms of H. With a little more effort, one can prove the same statement for any solvable subgroup. Also note that Lemma 4.7 implies that if I acts on H via pinned automorphisms, then $\mathrm{H}^1(I, H)$ has a canonical Abelian group structure. This does not necessarily hold for arbitrary action of I on H.

Recall that there is a natural partial order "$\preceq$" on $\mathbb{X}_\bullet(T)_I$ given by (1.3). Let $\bar{\mu} \in \mathbb{X}_\bullet(T)^+_I$. We say an irreducible representation W of $(H^\vee)^I$ is of highest weight $\bar{\mu}$ if $\bar{\mu}$ appears as a weight under the weight decomposition of W with respect to $(T^\vee)^I$ and any other weight appearing in this weight decomposition is $\preceq \bar{\mu}$. We claim that

LEMMA 4.10. – (i) *For $\bar{\mu} \in \mathbb{X}_\bullet(T)^+_I$, there is a unique (up to isomorphism) irreducible representation $W_{\bar{\mu}}$ of $(H^\vee)^I$ of highest weight $\bar{\mu}$. In addition, any irreducible representation of $(H^\vee)^I$ is of this form.*

(ii) *The multiplicity of the $\bar{\mu}$-weight in $W_{\bar{\mu}}$ is one.*

Proof. – Indeed, let $\bar{\bar{\mu}}$ be the image of $\bar{\mu}$ in $\mathbb{X}_\bullet(T)_I/\mathbb{X}_\bullet(T)_{I,\mathrm{tor}}$, and $W_{\bar{\bar{\mu}}}$ be the unique irreducible representation of $(H^\vee)^{I,0}$ of highest weight $\bar{\bar{\mu}}$. Let W be an irreducible representation of $(H^\vee)^I$ appearing as a direct summand of $\mathrm{ind}_{(H^\vee)^{I,0}}^{(H^\vee)^I} W_{\bar{\bar{\mu}}}$. Then by the Frobenius reciprocity, $W \otimes \chi$ also appears as a direct summand of $\mathrm{ind}_{(H^\vee)^{I,0}}^{(H^\vee)^I} W_{\bar{\bar{\mu}}}$ for every character χ of $(H^\vee)^I/(H^\vee)^{I,0}$, regarded as a representation of $(H^\vee)^I$. Then by Lemma 4.6 and counting the dimensions,

$$\mathrm{ind}_{(H^\vee)^{I,0}}^{(H^\vee)^I} W_{\bar{\bar{\mu}}} \simeq \bigoplus_{\chi \in \mathrm{Rep}((T^\vee)^I/(T^\vee)^{I,0})} W \otimes \chi,$$

and the restriction W to $(H^\vee)^{I,0}$ is irreducible. Therefore the restriction W to $(H^\vee)^{I,0}$ is isomorphic to $W_{\bar{\bar{\mu}}}$. It is then clear that exactly one $(W \otimes \chi)$ appearing in the direct sum is an irreducible representation of $(H^\vee)^I$ of highest weight $\bar{\mu}$. This proves the existence. Then uniqueness is also clear because by the Frobenius reciprocity, every irreducible representation of $(H^\vee)^I$ appears as a direct summand in $\mathrm{ind}_{(H^\vee)^{I,0}}^{(H^\vee)^I} W_{\bar{\bar{\mu}}}$ for some $\bar{\bar{\mu}} \in \mathbb{X}_\bullet(T)_I/\mathbb{X}_\bullet(T)_{I,\mathrm{tor}}$. □

Now we finish the proof of the theorem. Let $\mu \in \mathbb{X}_\bullet(T)^+$ be a lift of $\bar{\mu}$. Then, $\mathrm{Res}^{(H^\vee)^I}_{H^\vee}[V_\mu] = [W_{\bar{\mu}}] + \sum_{\bar{\lambda}\prec\bar{\mu}} c_{\bar{\lambda}\bar{\mu}}[W_{\bar{\lambda}}]$, where $[X]$ stands for the element in the K-group corresponding to X. Therefore,

$$\mathrm{Res}^{\tilde{G}^\vee}_{H^\vee}[V_\mu] = \mathrm{Res}^{\tilde{G}^\vee}_{(H^\vee)^I}[W_{\bar{\mu}}] + \sum_{\bar{\lambda}\prec\bar{\mu}} c_{\bar{\lambda}\bar{\mu}}\mathrm{Res}^{\tilde{G}^\vee}_{(H^\vee)^I}[W_{\bar{\lambda}}].$$

On the other hand, for $\bar{\mu} \in \mathbb{X}_\bullet(T)^+_I$, the intersection cohomology sheaf $\mathrm{IC}_{\bar{\mu}} \in \mathscr{P}_v$ gives rise to an irreducible object $U_{\bar{\mu}}$ in $\mathrm{Rep}(\tilde{G}^\vee)$. By Lemma 2.6, we have

$$\mathrm{Res}^{\tilde{G}^\vee}_{H^\vee}[V_\mu] = [U_{\bar{\mu}}] + \sum_{\bar{\lambda}\prec\bar{\mu}} d_{\bar{\lambda}\bar{\mu}}[U_{\bar{\lambda}}].$$

By induction on $\bar{\mu}$, one immediately obtains that

$$\mathrm{Res}^{\tilde{G}^\vee}_{(H^\vee)^I}[W_{\bar{\mu}}] = U_{\bar{\mu}} + \sum_{\bar{\lambda}\prec\bar{\mu}} e_{\bar{\lambda}\bar{\mu}}[U_{\bar{\lambda}}].$$

Since $[W_{\bar{\mu}}]$ (resp. $[U_{\bar{\mu}}]$) forms a $\mathbb{Z}$-basis of $K(\mathrm{Rep}((H^\vee)^I))$ (resp. $K(\mathrm{Rep}(\tilde{G}^\vee))$), this implies that $\mathrm{Res}^{\tilde{G}^\vee}_{(H^\vee)^I}$ is an isomorphism and therefore $\tilde{G}^\vee = (H^\vee)^I$. □

The following table is a list of the group $(H^\vee)^I$ when G over $k(\!(\mathrm{t})\!)$ is absolutely simple and simply-connected, and *non-split* . Note that in this case, I acts on the coweight lattice of T_H by permutations of a basis and therefore $\mathbb{X}_\bullet(T)_I$ is torsion free. Therefore, by Lemma 4.6, $(H^\vee)^I$ is connected.

G	H	I	$(H^\vee)^I$
SU_{2n-1}	$\mathrm{SL}_{2n-1}, n \geq 2$	$\mathbb{Z}/2$	SO_{2n-1}
SU_{2n}	$\mathrm{SL}_{2n}, n \geq 2$	$\mathbb{Z}/2$	$\mathbf{P}\mathrm{Sp}_{2n}$
${}^2D_n$	$\mathrm{Spin}_{2n}, n \geq 4$	$\mathbb{Z}/2$	SO_{2n-1}
${}^3D_4$	Spin_8	$\mathbb{Z}/3$	G_2
${}^2E_6$	E_6	$\mathbb{Z}/2$	F_4

Here $\mathbf{P}\mathrm{Sp}_{2n}$ denotes the adjoint form of Sp_{2n}, and ${}^2D_n, {}^3D_4, {}^2E_6$ are the quasi-split but non-split forms of D_n and E_6.

Now, we switch to Theorem 0.2. Therefore, we will assume that G is a quasi-split reductive group defined over the non-Archimedean local field $F = \mathbb{F}_q(\!(\mathrm{t})\!)$ (we can in fact replace $\mathbb{F}_q$ by any other perfect field). Let $k = \overline{\mathbb{F}}_q$, and σ be the Frobenius element in $\mathrm{Gal}(k/\mathbb{F}_q)$. Let $v \in \mathscr{B}(G, F)$ be a special vertex in the building which remains to be special when base change to $k(\!(\mathrm{t})\!)$ (such a vertex is called very special, see § 6 for more discussions). Let $\underline{G}_v$ be the special parahoric group scheme over $\mathbb{F}_q[[t]]$ corresponding to v and $K_v = L^+\underline{G}_v$. Then the affine flag variety $\mathscr{F}\ell_v = LG/K_v$ is defined over $\mathbb{F}_q$ and when base change to k, $\mathscr{F}\ell_v \otimes k$ is the affine flag variety considered in the previous sections, and we have the Tannakian category $\mathscr{P}_v = \mathscr{P}_{K_v\otimes k}(\mathscr{F}\ell_v \otimes k)$ with a fiber functor H^*.

As in Lemma A.5, there is an action of σ on $\mathscr{P}_v$, and therefore an action of σ on $(H^\vee)^I$. Following the notation as in the appendix, we denote this action by $\mathrm{act}^{\mathrm{geom}}$. On the other hand, since there is a canonical pinning $(H^\vee, B^\vee_H, T^\vee_H, X^\vee)$, there is a canonical action of $\mathrm{Gal}(F^s/F)$ on $H^\vee$ by pinned automorphisms and therefore an action of σ on $(H^\vee)^I$ by

pinned automorphisms. We denote this action by $\mathrm{act}^{\mathrm{alg}}$. As in the appendix, we denote cycl to be the cyclotomic character of $\mathrm{Gal}(k/\mathbb{F}_q)$, so that $\mathrm{cycl}(\sigma) = q$. Let

$$\chi = \rho \circ \mathrm{cycl} : \mathrm{Gal}(k/\mathbb{F}_q) \to (H^\vee_{\mathrm{ad}})^I.$$

As in Proposition A.6, since the action of $\mathrm{act}^{\mathrm{geom}}$ on $(H^\vee)^I$ fixes the cohomological grading and acts on $X^\vee$ via the cyclotomic character, we know that

$$\mathrm{act}^{\mathrm{geom}} = \mathrm{act}^{\mathrm{alg}} \circ \mathrm{Ad}_\chi,$$

and there is an isomorphism

$$(H^\vee)^I \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(k/\mathbb{F}_q) \to (H^\vee)^I \rtimes_{\mathrm{act}^{\mathrm{geom}}} \mathrm{Gal}(k/\mathbb{F}_q), \quad (g,\sigma) \mapsto (\mathrm{Ad}_{\chi(\sigma)^{-1}} g, \sigma). \tag{4.7}$$

Now regarding $(H^\vee)^I \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(k/\mathbb{F}_q)$ as a pro-algebraic group over $\overline{\mathbb{Q}}_\ell$, as in the appendix, we have the category $\mathrm{Rep}((H^\vee)^I \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(k/\mathbb{F}_q))$ of algebraic representations of $(H^\vee)^I \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(k/\mathbb{F}_q)$. Now Theorem 0.2 follows from the same line as in the appendix.

5. IC-stalks, q-analogy of the weight multiplicity, and the Lusztig-Kato polynomial

Let $\bar{\mu} \in \mathbb{X}_\bullet(T)_I$ and $\mathcal{T} \in \mathscr{P}_v$. We determine the stalk cohomology $\mathcal{T}$ at the point $s_{\bar{\mu}}$. By abuse of notation, the inclusion map $s_{\bar{\mu}} \in \mathcal{T}\ell_v$ is still denoted by $s_{\bar{\mu}}$. It will be convenient to define

$$\mathrm{Stalk}_{\bar{\mu}}(\mathcal{T}) = s^*_{\bar{\mu}}\mathcal{T}[-(2\rho,\bar{\mu})], \quad \mathrm{Costalk}_{\bar{\mu}}(\mathcal{T}) = s^!_{\bar{\mu}}\mathcal{T}[(2\rho,\bar{\mu})].$$

Let $X^\vee$ be the regular nilpotent element of $\mathrm{Lie}(H^\vee)^I$ given by the pinning. It defines an increasing filtration (the Brylinski-Kostant filtration) on any representation V of $H^\vee$ or $(H^\vee)^I$,

$$F_i V = (\ker X^\vee)^{i+1}. \tag{5.1}$$

For $\bar{\mu} \in \mathbb{X}_\bullet(T)_I$, denote by $V(\bar{\mu})$ the $\bar{\mu}$-weight subspace of V, under the action of $(T^\vee_H)^I$. Then filtration (5.1) induces

$$F_i V(\bar{\mu}) = V(\bar{\mu}) \cap F_i V. \tag{5.2}$$

Let

$$P_{\bar{\mu}}(V,q) = \sum \mathrm{gr}^F_i V(\bar{\mu}) q^i$$

be the q-analogue weight multiplicity polynomial.

THEOREM 5.1. – *Let $\mathcal{T} \in \mathscr{P}_v$ and let $V = \mathrm{H}^*(\mathcal{T}\ell_v, \mathcal{T})$ be the corresponding representation of $(H^\vee)^I$. Then*

$$P_{\bar{\mu}}(V,q) = \sum \dim \mathrm{H}^{-2i}(\mathrm{Stalk}_{\bar{\mu}}(\mathcal{T}))q^i = \sum \dim \mathrm{H}^{2i}(\mathrm{Costalk}_{\bar{\mu}}(\mathcal{T}))q^i.$$

Observe that by the parity vanishing property of $\mathcal{T}$, $\mathrm{Stalk}_{\bar{\mu}}\,\mathcal{T}$ and $\mathrm{Costalk}_{\bar{\mu}}(\mathcal{T})$ only concentrate on even degrees.

In the split case, this is proved in [18, 4]. A more geometric proof is given by Ginzburg [10], which relies on the geometric Satake isomorphism and certain techniques of equivariant cohomology. We will follow Ginzburg's idea.

Let us give a quick review of equivariant cohomology (see [1] and [10, §8] for more details). Let M be a variety with an action of a torus A. Let $\mathbb{B}A$ be the classifying space (stack) of A. Let $R_A = \mathrm{H}^*(\mathbb{B}A)$ and recall that $\mathrm{Spec}R_A \simeq \mathfrak{a} = \mathrm{Lie}A$. Let $t \in \mathfrak{a}$ be an element. We denote

by $\kappa(\mathrm{t})$ the residue field of t and let $\mathrm{H}_t := \mathrm{H}^*_A \otimes_{R_A} \kappa(\mathrm{t})$. If $t = 0$, this functor H_0 inherits a canonical grading. For general t, this functor equips with a canonical filtration by

$$\mathrm{H}_t^{\leq i} := \mathrm{Im}(\sum_{j \leq i} \mathrm{H}^j_A \to \mathrm{H}_t),$$

and there is a canonical isomorphism $\mathrm{gr}^* \mathrm{H}_t \simeq \mathrm{H}^*_0$. For every $\mathscr{F} \in D_A(M)$, there is a spectral sequence $E_2^{p,q} = \mathrm{H}^p(\mathbb{B}A, H^q(M, \mathscr{F})) \Rightarrow \mathrm{H}^{p+q}_A(M, \mathscr{F})$. If this spectral sequence degenerates at the E_2-term (which is always the case in the following discussion), then $\mathrm{H}^*_0(M, \mathscr{F}) \simeq \mathrm{H}^*(M, \mathscr{F})$ and therefore we have a canonical isomorphism $\mathrm{gr}\, \mathrm{H}_t(M, \mathscr{F}) \simeq \mathrm{H}^*(M, \mathscr{F})$.

Now assume that the action of A on M has only isolated fixed points M^A, and let $\eta \in \mathfrak{a}$ be the generic point. Then the localization theorem claims that there is an isomorphism

$$\bigoplus_{x \in M^A} \mathrm{H}_\eta(i_x^! \mathscr{F}) \simeq \mathrm{H}_\eta(M, \mathscr{F}) \simeq \bigoplus_{x \in M^A} \mathrm{H}_\eta(i_x^* \mathscr{F}), \tag{5.3}$$

where i_x is the inclusion of the point x.

Next we review some results for split groups, which are essentially contained in [10]. However, our presentation here follows [29] closely. First, it is proved in [29, Lemma 2.2] that there is a canonical grading preserving isomorphism

$$\mathrm{H}^*_{T_H} \simeq \mathrm{H}^* \otimes R_{T_H} : \mathrm{Sat}_H \to R_{T_H}\text{-Mod}, \tag{5.4}$$

which endows $\mathrm{H}^*_{T_H}$ with a structure of tensor functors and defines a canonically trivialized $H^\vee$-torsor $\mathscr{E} \simeq H^\vee \times \mathfrak{t}_H$ on $\mathrm{Spec} R_{T_H} = \mathfrak{t}_H =: \mathrm{Lie} T_H$. In other words, the group scheme $\mathrm{Aut}^\otimes \mathrm{H}^*_{T_H}$ over $\mathfrak{t}_H$ of the tensor automorphism of this fiber functor, which a priori is an inner form of $H^\vee$, is canonically isomorphic to $H^\vee \times \mathfrak{t}_H$. In addition, the MV filtration and its canonical splitting extend in the equivariant setting [29, Lemma 2.2] and provide $T_H^\vee \times \mathfrak{t}_H \subset B_H^\vee \times \mathfrak{t}_H \subset H^\vee \times \mathfrak{t}_H$. Now, let $c^{T_H}(\mathscr{L}) \in \mathrm{H}^2_{T_H}(\mathrm{Gr}_H)$ denote the equivariant Chern class of $\mathscr{L}$[3]. Then the action of $c^{T_H}(\mathscr{L})$ on $\mathrm{H}^*_{T_H}(\mathrm{Gr}_H, \mathscr{F})$ for $\mathscr{F} \in \mathrm{Sat}_H$ can be identified with the action of an element

$$e^{T_H} \in \Gamma(\mathfrak{t}_H, \mathrm{Lie}(\mathrm{ad}\mathscr{E})).$$

Since $\mathscr{E}$ is canonically trivialized, e^{T_H} can be regarded as a map $\mathfrak{t}_H \to \mathfrak{h}^\vee$. Observe that e^{T_H} is *not* the constant map $X^\vee$. In fact,

$$e^{T_H} = X^\vee + h,$$

where $h : \mathfrak{t}_H \to \mathfrak{t}_H^\vee \simeq (\mathfrak{t}_H)^*$ is given by a nondegenerate invariant bilinear form (cf. [29, Proposition 5.7]). In particular, $(H^\vee, B_H^\vee, T_H^\vee, e^{T_H})$ is not a pinning over $\mathfrak{t}_H$.

The equivariant homology $\mathrm{H}_*^{T_H}(\mathrm{Gr}_H)$ is a commutative and cocommutative Hopf algebra and $J^\vee = \mathrm{Spec} \mathrm{H}_*^{T_H}(\mathrm{Gr}_H)$ is a flat group scheme over $\mathfrak{t}_H$, acting on every $\mathrm{H}^*_{T_H}(\mathrm{Gr}_H, \mathscr{F}), \mathscr{F} \in \mathrm{Sat}_H$. By Tannakian formulism, this induces a map $\iota : J^\vee \to H^\vee \times \mathfrak{t}_H$. In [29], it is shown that this is a closed embedding, which identifies $J^\vee$ with the $(H^\vee \times \mathfrak{t}_H)^{e^{T_H}}$, the centralizer of e^{T_H} in $H^\vee \times \mathfrak{t}_H$.

[3] By replacing $\mathscr{L}$ by a power of it, we can assume that $\mathscr{L}$ is T_H-equivariant.

Let η be the generic point of $\mathfrak{t}_H$. Then $J_\eta^\vee$ is indeed a torus in $H_\eta^\vee$ since $e^{T_H}(\eta) \in \mathfrak{h}_\eta^\vee$ is regular semisimple. Then localization theorem gives rise to an isomorphism of $J_\eta^\vee$-modules

$$\bigoplus_{\mu \in \mathbb{X}_\bullet(T_H)} \mathrm{H}_\eta(s_\mu^! \mathscr{F}) \simeq \mathrm{H}_\eta(\mathrm{Gr}_H, \mathscr{F}). \tag{5.5}$$

Following the idea of Ginzburg, we claim that this decomposition corresponds to the weight decomposition under $J_\eta^\vee \subset H_\eta^\vee$. First, let $B_H^\vee \to T_H^\vee$ be the natural projection. As is shown in [29], $J^\vee \subset B_H^\vee \times \mathfrak{t}_H$ and the composition $J^\vee \to B_H^\vee \times \mathfrak{t}_H \to T_H^\vee \times \mathfrak{t}_H$ is identified with the map (cf. Remark 3.4 of loc. cit.)

$$R_{T_H}[\mathbb{X}_\bullet(T_H)] \simeq \bigoplus_{\mu \in \mathbb{X}_\bullet(T_H)} \mathrm{H}_*^{T_H}(s_\mu) \to \mathrm{H}_*^{T_H}(\mathrm{Gr}_H).$$

Over η, this is an isomorphism and therefore we obtain a canonical isomorphism $J_\eta^\vee \simeq (T_H^\vee)_\eta$. In addition, the action of $J_\eta^\vee$ on $\mathrm{H}_{\eta,c}(S_\mu, \mathscr{F})$ via $J_\eta^\vee \to (T_H^\vee)_\eta$ is identified with the natural action of $J_\eta^\vee$ on $\mathrm{H}_\eta(s_\mu^! \mathscr{F}) \simeq \mathrm{H}_{\eta,c}(S_\mu, \mathscr{F})$. Therefore, we obtain the following proposition, originally proved by Ginzburg by another method.

PROPOSITION 5.2. – *Let $V \in \mathrm{Rep}(H^\vee)$ and $\mathscr{S}(V) \in \mathrm{Sat}_H$ be the corresponding sheaf (see* (2.1)*). Under the identification of the weight lattice of $J_\eta^\vee$ with $\mathbb{X}_\bullet(T_H^\vee)$ via the canonical isomorphism $J_\eta^\vee \to (T_H^\vee)_\eta$, the direct summand $\mathrm{H}_\eta(s_\mu^! \mathscr{S}(V)) \subset \mathrm{H}_\eta(\mathrm{Gr}_H, \mathscr{S}(V))$ corresponds to the weight subspace $V(\mu) \subset V$ for $J_\eta^\vee$.*

REMARK 5.3. – Let us observe that the localization isomorphism (5.5) holds over $\mathfrak{t}_H$ after we remove all root hyperplanes. This is because for every T_H-invariant finite dimensional closed subvariety $Z \subset \mathrm{Gr}_H$, there are only finitely many 1-dimensional T_H-orbits in Z, and T_H acts on these orbits via rotations determined by roots. Therefore, in all the discussions above, we can replace the generic point η by any (closed) point in $\mathfrak{t}_H \setminus \{\tilde{a} = 0, \tilde{a} \in \Phi_H\}$, where Φ_H is the set of roots of H.

Now let t be a closed point on $\mathfrak{t}_H$ such that

$$h(\mathrm{t}) = 2\rho \tag{5.6}$$

so that $e^{T_H}(\mathrm{t}) = X^\vee + h(\mathrm{t}) = X^\vee + 2\rho$. According to [29, Proposition 5.7], such point exists (unique up to adding an element in the center $\mathfrak{z}(\mathfrak{h})$ of $\mathfrak{h}$) and does not belong to any root hyperplanes. Therefore, the localization isomorphism (5.5) holds for H_t by the above remark. From now on, we will always choose the point t satisfying (5.6).

Recall that under the geometric Satake isomorphism $\mathrm{H}^* : \mathrm{Sat}_H \simeq \mathrm{Rep}(H^\vee)$, the natural grading on the cohomology functor corresponds to the principal grading on representations of $H^\vee$. More precisely, consider the cocharacter $2\rho : \mathbb{G}_m \to T_H^\vee \subset H^\vee$. Then the grading on the cohomology functor corresponds to the grading given by 2ρ on the representations. This follows from the fact that $\mathrm{H}_c^*(S_\mu, \mathscr{F})$ is nonzero only in degree $(2\rho, \mu)$. Now it is clear from (5.4) that for the closed point $t \in \mathfrak{t}_H$, the filtration $\mathrm{H}_t^{\leq i}$ corresponds to the increasing filtration on the representations associated to the gradings given by 2ρ. For $i \in \mathbb{Z}$, let

$$\mathbb{X}_\bullet(T_H)_i = \{\mu \in \mathbb{X}_\bullet(T_H) \mid (2\rho, \mu) = i\}.$$

Let V be a representation of $H^\vee$. Denote

$$V(\mathrm{i}) = \sum_{\mu \in \mathbb{X}_\bullet(T_H)_i} V(\mu),$$

where $V(\mu)$ is the μ-weight space of $J_t^\vee$. Let us identify $\mathrm{H}_t(\mathrm{Gr}_H, \mathcal{J}(V))$ with V canonically, so that $\mathrm{H}_t(s_\mu^* \mathcal{J}(V))$ is identified with $V(\mu)$ by Proposition 5.2. Then we have the following proposition.

PROPOSITION 5.4. – *Let t be as in* (5.6). *Write $\mathrm{H}_t = \mathrm{H}_t(\mathrm{Gr}_H, \mathcal{J}(V))$ for simplicity. Then for any $m \in \mathbb{Z}$,*

$$\mathrm{H}_t^{\leq 2i+m} \cap \bigoplus_{\mu \in \mathbb{X}_\bullet(T)_m} \mathrm{H}_t(s_\mu^! \mathcal{J}(V)) = \mathrm{H}_t^{\leq 2i+1+m} \cap \bigoplus_{\mu \in \mathbb{X}_\bullet(T)_m} \mathrm{H}_t(s_\mu^! \mathcal{J}(V)) = F_i V \cap V(\mathrm{m}),$$

where $F_i V$ is defined as in (5.1).

Proof. – Let n be the unique element in $U_H^\vee$ such that $\mathrm{Ad}_n(X^\vee + 2\rho) = 2\rho$. Then the canonical isomorphism $J_t^\vee \simeq T_H^\vee$ is given by $\mathrm{Ad}_n : J_t^\vee \to T_H^\vee$. The proposition clearly follows from the following purely representation theoretical lemma. □

LEMMA 5.5. – *Let V be a representation of $H^\vee$. Let*

$$V = \sum V^1(\mathrm{i}), \quad V = \sum V^2(\mathrm{i})$$

be two gradings on V, given by the cocharacters $2\rho : \mathbb{G}_m \to H^\vee$ and $\mathrm{Ad}_{n^{-1}} 2\rho : \mathbb{G}_m \to H^\vee$ respectively. Let $F_\bullet^1 V$ and $F_\bullet^2 V$ be two filtrations on V given by

$$F_i^1 V = \sum_{j \leq i} V^1(\mathrm{j}), \quad F_i^2 V = (\ker X^\vee)^{i+1}.$$

Then for any $m \in \mathbb{Z}$, $V^2(\mathrm{m}) \cap F_{2i+m}^1 V = V^2(\mathrm{m}) \cap F_{2i+1+m}^1 V = V^2(\mathrm{m}) \cap F_i^2 V$.

Proof. – Let $Y^\vee \in \mathfrak{h}^\vee$ so that $\{X^\vee, 2\rho, Y^\vee\}$ forms an $\mathfrak{sl}_2$-triple. Then the lemma is purely a statement about this $\mathfrak{sl}_2$ and can be checked easily by direct calculation. □

This finishes the discussion for split groups. Now let G be as before and A be its maximal split torus. By the isomorphism (1.1), we can regard A as a subtorus of T_H. Note that we can restrict everything discussed above to $A \subset T_H$. In particular, we can choose the point $t \in \mathfrak{a}$ such that $h(\mathrm{t}) = 2\rho$. This is because $h : \mathfrak{t}_H \to \mathfrak{t}_H^\vee$ is equivariant under the automorphisms of the based root datum and 2ρ is a fixed point under these automorphisms.

We begin to prove the theorem. Observe that it is enough to prove the theorem for objects in $\mathscr{P}_v$ of the form $\mathcal{Z}(\mathcal{F})$, where $\mathcal{F} \in \mathrm{Sat}_H$. Indeed, both maps $\mathscr{P}_v \to \mathbb{Z}[q]$ given by $\mathcal{F} \mapsto P_{\bar{\mu}}(\mathrm{H}^*(\mathcal{F}), q)$ and $\mathcal{F} \mapsto \sum \dim \mathrm{H}^{2i}(\mathrm{Costalk}_{\bar{\mu}}(\mathcal{F}))q^i$ factor through the Grothendieck group, and as observed in the proof of Theorem 0.1 in Section 4, the objects of the form $\mathcal{Z}(\mathcal{F}), \mathcal{F} \in \mathrm{Sat}_H$ generate the Grothendieck group of $\mathscr{P}_v$.

Recall that the maximal torus $A \subset G$ extends naturally to a split torus over $\mathcal{O}$ and $A_\mathcal{O} \subset \underline{G}_v$. Therefore, we can regard A as a subtorus of K_v, as A is a natural subgroup of $L^+ A_\mathcal{O}$ consisting of "constant" elements. The set of fixed points of the action of A on $\mathcal{F}\ell_v$ are exactly $\{s_{\bar{\mu}} | \bar{\mu} \in \mathbb{X}_\bullet(T)_I\}$. This will be clear if we regard LG as a Kac-Moody group and

A as its maximal torus. We consider the A-equivariant cohomology $\mathrm{H}^*_A : \mathscr{P}_v \to R_A$-Mod. Since the nearby cycles commute with proper base change, we have a canonical isomorphism

$$\mathrm{H}^*_A \circ \mathcal{Z} \simeq \mathrm{H}^*_A : \mathrm{Sat}_H \to R_A\text{-Mod}.$$

Indeed, when fixing a cohomological degree, we can replace $\mathbb{B}A$ by $(\mathbb{P}^n)^{\mathrm{rk}\,A}$ for n large enough, and consider the nearby cycle functors for the family $\widetilde{\mathrm{Gr}}_{\mathscr{G}} \times^A (\mathbb{P}^n)^{\mathrm{rk}\,A}$. The claim then is clear.

REMARK 5.6. – One should be able to argue as defining the tensor structure of H^*, that there is a canonical isomorphism

$$\mathrm{H}^*_A \simeq \mathrm{H}^* \otimes R_A : \mathscr{P}_v \to R_A\text{-Mod},$$

which endows H^*_A with a fiber functor structure, and the corresponding $(H^\vee)^I$ torsor on $\mathrm{Spec}R_A = \mathfrak{a} =: \mathrm{Lie}A$ is canonically trivialized. However, we did not investigate this.

Let $p : \mathbb{X}_\bullet(T) \to \mathbb{X}_\bullet(T)_I$ be the projection. Let η be the generic point of $\mathfrak{a}$. Now, the key observation is

LEMMA 5.7. – *Under the canonical isomorphism* $\mathrm{H}_\eta(\mathcal{F}\ell_v, \mathcal{Z}(\mathcal{F})) \simeq \mathrm{H}_\eta(\mathrm{Gr}_H, \mathcal{F})$*, the direct summand* $\mathrm{H}^*_\eta(s^!_{\bar{\mu}}\mathcal{Z}(\mathcal{F}))$ *corresponds to* $\bigoplus_{\mu \in p^{-1}(\bar{\mu})} \mathrm{H}^*_\eta(s^!_\mu \mathcal{F})$.

Proof. – Recall that if $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of varieties over $\mathbb{A}^1$, then there are always the natural maps $f^*\Psi_{\mathcal{Y}} \to \Psi_{\mathcal{X}} f^*$ and $\Psi_{\mathcal{Y}} f_* \to f_* \Psi_{\mathcal{X}}$ (see SGA 7 Exposé XIII, (2.1.7.1) (2.1.7.2)). In addition, these two maps fit into the following commutative diagram

$$\begin{array}{ccc} \Psi_{\mathcal{Y}}(\mathcal{F}) & \longrightarrow & f_* f^* \Psi_{\mathcal{Y}}(\mathcal{F}) \\ \downarrow & & \downarrow \\ \Psi_{\mathcal{Y}}(f_* f^* \mathcal{F}) & \longrightarrow & f_* \Psi_{\mathcal{X}}(f^* \mathcal{F}). \end{array}$$

Now let $\mu \in \mathbb{X}_\bullet(T_H)$ and apply this remark to $s_\mu : \mathbb{A}^1 \to \widetilde{\mathrm{Gr}}_{\mathscr{G}}$ as defined in the proof of Lemma 2.6. By taking the cohomology H^*_A, we obtain, for any $\mathcal{F} \in \mathrm{Sat}_H$, the following commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^*_A(\mathrm{Gr}_H, \mathcal{F}) & \xrightarrow{\simeq} & \mathrm{H}^*_A(\mathcal{F}\ell_v, \mathcal{Z}(\mathcal{F})) & \longrightarrow & \mathrm{H}^*_A(s^*_{\bar{\mu}}\mathcal{Z}(\mathcal{F})) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}^*_A(s^*_\mu \mathcal{F}) & \xrightarrow{\simeq} & \mathrm{H}^*_A(\mathcal{F}\ell_v, \Psi_{\widetilde{\mathrm{Gr}}_{\mathscr{G}}}(s_{\mu *} s^*_\mu \mathcal{F})) & \xrightarrow{\simeq} & \mathrm{H}^*_A(\Psi_{\mathbb{A}^1}(s^*_\mu \mathcal{F})). \end{array} \tag{5.7}$$

In other words, the composition $\mathrm{H}^*_A(\mathcal{F}\ell_v, \mathcal{Z}(\mathcal{F})) \to \mathrm{H}^*_A(\mathrm{Gr}_H, \mathcal{F}) \to \mathrm{H}^*_A(s^*_\mu \mathcal{F})$ factors as $\mathrm{H}^*_A(\mathcal{F}\ell_v, \mathcal{Z}(\mathcal{F})) \to \mathrm{H}^*_A(s^*_{\bar{\mu}}\mathcal{Z}(\mathcal{F})) \to \mathrm{H}^*_A(s^*_\mu \mathcal{F})$. On the other hand, by the localization theorem, over the generic η of $\mathfrak{a}$, we have

$$\begin{array}{ccc} \mathrm{H}^*_\eta(\mathcal{F}\ell_v, \mathcal{Z}(\mathcal{F}))) & \xrightarrow{\simeq} & \bigoplus_{\bar{\mu} \in \mathbb{X}_\bullet(T)_I} \mathrm{H}^*_\eta(s^*_{\bar{\mu}}\mathcal{Z}(\mathcal{F})) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathrm{H}^*_\eta(\mathrm{Gr}_H, \mathcal{F}) & \xrightarrow{\simeq} & \bigoplus_{\mu \in \mathbb{X}_\bullet(T)} \mathrm{H}^*_\eta(s^*_\mu \mathcal{F}). \end{array} \tag{5.8}$$

Observe that in the localization Theorem (5.3), for $x, y \in M^A$ and $x \neq y$, the composition $\mathrm{H}_\eta(i^!_x \mathcal{F}) \to \mathrm{H}_\eta(i^*_y \mathcal{F})$ is zero. Therefore, the lemma follows from (5.7) and (5.8). □

As a result, by Proposition 5.2, if $\mathcal{T} = \mathcal{J}(V)$, this direct summand can be further identified with $\bigoplus_{\mu \in p^{-1}(\bar{\mu})} V(\mu)$, the weight subspaces under $J_\eta^\vee$. In addition, by Proposition 5.4,

$$\mathrm{H}_t^{\leq 2i+(2\rho,\bar{\mu}))} \cap \mathrm{H}_t(s_{\bar{\mu}}^! Z(\mathcal{J}(V))) = \mathrm{H}_t^{\leq 2i+1+(2\rho,\bar{\mu}))} \cap \mathrm{H}_t(s_{\bar{\mu}}^! Z(\mathcal{J}(V))) = F_i V \cap \bigoplus_{\mu \in p^{-1}(\bar{\mu})} V(\mu),$$

where we write $\mathrm{H}_t = \mathrm{H}_t(\mathcal{T}\ell_v, Z(\mathcal{J}(V)))$ for brevity. Therefore, to finish the proof of the theorem, it remains to show that

LEMMA 5.8. – *Let $\mathcal{T} \in \mathcal{P}_v$. Then the canonical map*

$$\mathrm{H}_A^*(s_{\bar{\mu}}^! \mathcal{T}) \to \mathrm{H}_A^*(\mathcal{T}) \tag{5.9}$$

is a splitting injective map of free R_A-modules. Therefore,

$$\mathrm{H}_t^{\leq i}(\mathcal{T}\ell_v, Z(\mathcal{J}(V))) \cap \mathrm{H}_t(s_{\bar{\mu}}^! Z(\mathcal{J}(V))) = \mathrm{H}_t^{\leq i}(s_{\bar{\mu}}^! Z(\mathcal{J}(V))).$$

These are general facts about flag varieties for Kac-Moody groups. The basic geometric fact behind this proposition is that the "big open cell" of the flag variety contracts to a point under certain $\mathbb{G}_m$-action. Then the statement follows using an argument with weights (cf. [10]). We here reproduce the proof for completeness.

Proof. – Without loss of generality, we can assume that $\mathcal{T}$ is an intersection cohomology complex. First, we claim that it is enough to prove a dual statement: the map

$$\mathrm{H}_A^*(\mathcal{T}) \to \mathrm{H}_A^*(s_{\bar{\mu}}^* \mathcal{T}) \tag{5.10}$$

is surjective. To see this, recall that since each of $\mathrm{H}^*(\mathcal{T})$, $s_{\bar{\mu}}^* \mathcal{T}$, $s_{\bar{\mu}}^! \mathcal{T}$ concentrates in cohomological degrees of the same parity, the spectral sequence calculating the A-equivariant cohomology degenerates at the E_2-term, which implies that all $\mathrm{H}_A^*(\mathcal{T})$, $\mathrm{H}_A^*(s_{\bar{\mu}}^* \mathcal{T})$, $\mathrm{H}_A^*(s_{\bar{\mu}}^! \mathcal{T})$ are finite free R_A-modules. Then taking $\mathrm{Hom}(-, R_A)$ interchanges (5.9) and (5.10).

Since $\mathcal{T}\ell_v$ is the flag variety of certain Kac-Moody group (cf. [22, Section 9.h]), for every $s_{\bar{\mu}}$, there is a $\mathbb{G}_m$-action on $\mathcal{T}\ell_v$, contracting an open neighborhood of $s_{\bar{\mu}}$ in $\mathcal{T}\ell_v$ to $s_{\bar{\mu}}$. In addition, this $\mathbb{G}_m$-action stabilizes every Schubert cell $\mathring{\mathcal{T}}\ell_{\bar{\lambda}}^s$, and commutes with the action of A on $\mathcal{T}\ell_v$. Denote this open neighborhood by $j : U_{\bar{\mu}} \hookrightarrow \mathcal{T}\ell_v$. Then $U_{\bar{\mu}}$ is an inductive limit of affine spaces. Indeed, $U_{\bar{0}}$ is just the big open cell in the flag variety, and $U_{\bar{\mu}}$ is the translate $U_{\bar{0}}$ via $s_{\bar{\mu}}$ (lifted to an element in $T(F)$).

Now we can assume that our group is defined over $\mathbb{F}_q(\!(\mathrm{t})\!)$ and splits over a totally ramified extension. All the discussion above remains unchanged in this setting. Recall that we denote $\mathcal{P}_v^0$ to be the semisimple K_v-equivariant perverse sheaves on $\mathcal{T}\ell_v$, pure of weight zero. It is well-known that for every object $\mathcal{T}$ in $\mathcal{P}_v$, $\mathrm{H}_A^*(s_{\bar{\mu}}^* \mathcal{T})$ is pure of weight zero (i.e., $\mathrm{H}_A^i(s_{\bar{\mu}}^! \mathcal{T})$ is pure of weight i), essentially due to the existence of Demazure resolutions. To show (5.10) is surjective, we decompose this map into

$$\mathrm{H}_A^*(\mathcal{T}) \to \mathrm{H}_A^*(U_{\bar{\mu}}, j^* \mathcal{T}) \to \mathrm{H}_A^*(s_{\bar{\mu}}^* \mathcal{T}).$$

It is well-known that the second map is an isomorphism since $j^* \mathcal{T}$ is equivariant under this $\mathbb{G}_m$-action, which contracts $(U_{\bar{\mu}} \cap \mathrm{Supp}\, \mathcal{T})$ to $s_{\bar{\mu}}$. In particular, $\mathrm{H}_A^*(U_{\bar{\mu}}, j^* \mathcal{T})$ is pure of weight zero. Therefore, it is enough to show that the first map is surjective. Denote $i : Z = \mathcal{T}\ell_v \setminus U_{\bar{\mu}} \hookrightarrow \mathcal{T}\ell_v$ to be the complement. Then we have the distinguished triangle

$$i_* i^! \mathcal{T} \to \mathcal{T} \to j_* j^* \mathcal{T} \to$$

and therefore

$$0 \to \mathrm{H}^*_A(i^!\mathscr{F}) \to \mathrm{H}^*_A(\mathscr{F}) \to \mathrm{H}^*_A(U_{\bar{\mu}}, j^*\mathscr{F}) \to 0.$$

The last map is surjective because the weights of $\mathrm{H}^*_A(i^!\mathscr{F})$ are ≥ 0. □

6. The Langlands parameter

In this section, we briefly discuss the Langlands parameters for smooth "spherical" representations of a quasi-split p-adic group. The parameters themselves can be described easily, and they will be used when we discuss the Frobenius trace of nearby cycles for certain unitary Shimura varieties.

We will assume that F is a non-Archimedean local field with finite residue field and that G is a connected reductive group over F. First, we generalize the hyperspecial vertex of an unramified group as follows. Recall that by [27], the building of $G(F)$ can be embedded into the building of $G(L)$, where L is the completion of a maximal unramified extension of F.

DEFINITION 6.1. – A special vertex v of G is called geometrically special (or very special) if it remains special in G_L. The parahoric subgroup of G corresponding to a geometrically special vertex is called a geometrically special (or very special) parahoric subgroup of G.

Clearly, if G is an unramified group, then very special vertices of G are the same as hyperspecial vertices of G.

LEMMA 6.1. – *A very special vertex of G exists if and only if G is quasi-split over F.*

Proof. – Assume that G is quasi-split. Then the existence of such points follows exactly by the same argument as in [27, 1.10.2]. We prove the converse. Let v be a very special point. Choose a maximal F-split torus A of G such that the corresponding apartment $A(G, A, F)$ containing v. Let S be a maximal L-split torus defined over F and containing A. We identify the apartment $A(G_L, S_L, L)$ with $\mathbb{X}_\bullet(S) \otimes \mathbb{R}$ by v. As v is special, there is a bijection between the finite Weyl chambers for (G_L, S_L) and the affine Weyl chambers (or called alcove) with v as a vertex, and this bijection is compatible with the action of $\mathrm{Gal}(L/F)$. To show that G is quasi-split, it is enough to find an L-rational Borel containing S stable under $\mathrm{Gal}(L/F)$, which is equivalent to finding a finite Weyl chamber in $\mathbb{X}_\bullet(S) \otimes \mathbb{R}$, stable under $\mathrm{Gal}(L/F)$. Therefore, it enough to show that among all alcoves with v as a vertex, there is one stable under $\mathrm{Gal}(L/F)$. But as it is known, one of such alcoves intersects with $A(G, A, F)$ (since every reductive group over F is residually quasi-split, see [27, §1.10]), which is stable under $\mathrm{Gal}(L/F)$. □

In fact, by checking the classification of central isogeny classes of quasi-simple, absolutely simple reductive group over F as in [27, §4], we find that if G is quasi-split, then every special vertex of G is very special except the following case: up to central isogeny, G is an unramified odd unitary group. Then there are two special vertices in its relative local Dynkin diagram, only one of which is hyperspecial. To prove this assertion, one uses the following observation: Using the notation as in loc. cit., a vertex v in Δ (the relative local Dynkin diagram of G) is very special if and only if the corresponding $\mathrm{Gal}(L/F)$-orbit $O(\mathrm{v}) \subset \Delta_1$ (the absolute local Dynkin diagram of G) consists of one point, which is special in Δ_1.

Next we turn to representations.

DEFINITION 6.2. – We call an irreducible smooth representation V of G spherical, if there is some $v \in V, v \neq 0$, which is fixed by some very special parahoric subgroup of G.

REMARK 6.2. – Again, one could try to define spherical representations of G as those with a vector fixed by some special maximal compact subgroup of G. However, from the point of view of Langlands parameters discussed below, this is not correct.

Clearly, if G is unramified, spherical representations are those usually called unramified representations. For the unramified representations, the description the associated Langlands parameters is well-known (for example, see [3, Chapter II]). Let us explain the Langlands parameters of spherical representations for quasi-split ramified groups.

Following the notation in the previous sections, we denote by $H^\vee$ its dual group in the sense of Langlands defined over $\mathbb{C}$, i.e., the root datum is dual to the root datum of G. Let us equip $H^\vee$ with a pinning $(H^\vee, B^\vee, T^\vee, X^\vee)$[4]. Then $\mathrm{Gal}(F^s/F)$ acts on $H^\vee$ via pinned automorphisms, which we denote by $\mathrm{act}^{\mathrm{alg}}$, and we can form the Langlands dual group of G as

$$^L G^{\mathrm{alg}} = H^\vee \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(F^s/F).$$

Let W_F be the Weil group of F. A Langlands parameter is a continuous homomorphism (up to conjugation by $H^\vee$) $\rho : W_F \to {}^L G^{\mathrm{alg}}$, such that its composition with the canonical projection ${}^L G^{\mathrm{alg}} \to \mathrm{Gal}(F^s/F)$ is the natural inclusion $W_F \to \mathrm{Gal}(F^s/F)$ and $\rho(W_F)$ consists of semisimple elements of ${}^L G^{\mathrm{alg}}$ (see [3, 8.2] for the unexplained terminology).

We write $\rho(\gamma) = (\rho_1(\gamma), \gamma)$ for $\gamma \in W_F$, where ρ_1 is a map from W_F to $H^\vee$.

DEFINITION 6.3. – A "spherical" parameter (or Langlands-Satake parameter) is a Langlands parameter ρ which can be conjugated to the form $\rho(\gamma) = (1, \gamma)$ for γ in the inertial group I.

Let $(H^\vee)^I$ be the I-fixed point subgroup of $H^\vee$ (which could be non-connected according to Remark 4.4). Then $\mathrm{Gal}(F^s/F)/I$ acts on $(H^\vee)^I$ through a finite cyclic group $\langle\sigma\rangle$, where $\sigma \in \mathrm{Gal}(F^s/F)/I$ is the Frobenius element.

LEMMA 6.3. – *"Spherical" Langlands parameters $\rho : W_F \to {}^L G^{\mathrm{alg}}$ are in one-to-one correspondence to semi-simple elements in $(H^\vee)^I \times \sigma \subset (H^\vee)^I \rtimes_{\mathrm{act}^{\mathrm{alg}}} \langle\sigma\rangle$ up to conjugacy by $(H^\vee)^I$.*

We denote the set of semi-simple elements in $(H^\vee)^I \times \sigma$ by $((H^\vee)^I \times \sigma)_{ss}$.

Proof. – First, observe that elements $h \in H^\vee$ satisfy that $(h, 1)(1, \gamma)(h^{-1}, 1) = (1, \gamma)$ for all $\gamma \in I$ if and only if $h \in (H^\vee)^I$.

Let Φ be a lift of the Frobenius element to W_F. Then ρ is uniquely determined by $\rho_1(\Phi)$, which is a semi-simple element in $H^\vee$. Let $\gamma \in I$. Then $(1, \Phi\gamma\Phi^{-1}) = \rho(\Phi\gamma\Phi^{-1}) = (\rho_1(\Phi), \Phi)(1, \gamma)(\rho_1(\Phi^{-1}), \Phi^{-1})$ implies that $\rho_1(\Phi)$ is invariant under I. Conversely, if $g \in H^\vee \rtimes_{\mathrm{act}^{\mathrm{alg}}} I$, then the formulas $\rho_1(I) = 1, \rho_1(\Phi) = g$ define ρ. □

[4] In fact, by the construction of the appendix, there is a canonical pinned of $H^\vee$ provided by the geometric Satake correspondence.

Let A be a maximal F-split torus and let T be the centralizer of A, which is a maximal torus of G. Let $W_0 = W(G,A)$ be the Weyl group. As explained in [11, Remark 9], we can identify W_0 with the σ invariants of the Weyl group $W((H^\vee)^I,(T^\vee)^I)$ (observe that the latter group was denoted by W_0 in Section 4), and let N_0 be the inverse image of W_0 inside the normalizer of $(T^\vee)^I$ in $(H^\vee)^I$. Let $\mathrm{Rep}((H^\vee)^I \rtimes_{\mathrm{act}^{\mathrm{alg}}} \langle\sigma\rangle)$ be a category of algebraic representations of $(H^\vee)^I \rtimes_{\mathrm{act}^{\mathrm{alg}}} \langle\sigma\rangle$.

For every $W \in \mathrm{Rep}((H^\vee)^I \rtimes_{\mathrm{act}^{\mathrm{alg}}} \langle\sigma\rangle)$, by restriction of its character to $(H^\vee)^I \times \sigma$, we obtain a function ch_W on $(H^\vee)^I \times \sigma$. We denote by R the algebra of functions on $(H^\vee)^I \times \sigma$, generated by all ch_W. We can adapt the proofs in [3, 6.4-6.7] to the group $(H^\vee)^I \rtimes \langle\sigma\rangle$, and obtain

PROPOSITION 6.4. – (i) *The natural map* $\mathbb{X}_\bullet(T)_I^\sigma \subset \mathbb{X}_\bullet(T)_I$ *induces an isomorphism*

$$\alpha : (T^\vee)^I \rtimes \sigma / \operatorname{Int} N_0 \simeq \mathrm{Spec}\mathbb{C}[\mathbb{X}_\bullet(T)_I^\sigma]^{W_0}.$$

(ii) *The natural map* $(T^\vee)^I \rtimes \sigma \to (H^\vee)^I \rtimes \sigma$ *induces an isomorphism*

$$\beta : (T^\vee)^I \rtimes \sigma / \operatorname{Int} N_0 \simeq ((H^\vee)^I \rtimes \sigma)_{ss} / \operatorname{Int}((H^\vee)^I).$$

(iii) *The composition* $\beta\alpha^{-1} : \mathrm{Spec}\mathbb{C}[\mathbb{X}_\bullet(T)_I^\sigma]^{W_0} \to ((H^\vee)^I \times \sigma)_{ss} / \operatorname{Int}((H^\vee)^I)$ *induces an isomorphism*

$$\mathbb{C}[\mathbb{X}_\bullet(T)_I^\sigma]^{W_0} \simeq \mathrm{R}$$

as functions on $((H^\vee)^I \times \sigma)_{ss} / \operatorname{Int}(H^\vee)^I$.

Therefore, the set of spherical Langlands parameters can be identified with the set of all characters of R. Namely if ρ is a spherical parameter, then the corresponding character $\chi_\rho : R \to \mathbb{C}$ is given by

$$\chi_\rho(\mathrm{ch}_W) = \mathrm{tr}(\rho(\Phi), W), \tag{6.1}$$

where we assume (after conjugation) that $\rho(\Phi) \in ((H^\vee)^I \times \sigma)_{ss}$, and $\in (H^\vee)^I \rtimes \langle\sigma\rangle$.

Now let us explain how to attach to a spherical representation its spherical parameter. Let π be a spherical representation of G, such that $\pi^{K_v} \neq 0$ for a very special parahoric subgroup $K_v \subset G(F)$. Therefore, π determines a character χ_π of $C_c(K_v \backslash G(F)/K_v)$ by

$$\chi_\pi(\mathrm{f}) = \mathrm{tr}(\pi(\mathrm{f})), \tag{6.2}$$

where we fix a measure on $G(F)$ so that the volume of K_v is one.

DEFINITION 6.4. – We define the spherical parameter associated to π to be the unique Langlands parameter

$$\mathrm{Sat}(\pi) : W_F \to {}^L G$$

such that $\chi_{\mathrm{Sat}(\pi)} = \chi_\pi$ under the Satake isomorphism

$$C_c(K_v \backslash G(F)/K_v) \simeq \mathrm{R}\,. \tag{6.3}$$

As explained in Lemma A.13, in the case $F = \mathbb{F}_q(\!(\mathrm{t})\!)$, the isomorphism (6.3) can be deduced from 0.2 under the sheaf-function dictionary. We come back to the notation as in §4, in particular $k = \overline{\mathbb{F}}_q$. Let $\bar{\mu} \in \mathbb{X}_\bullet(T)_I$. By abuse of notation, we denote the corresponding Schubert variety in $\mathscr{F}\ell_v \otimes k$ by $\mathscr{F}\ell_{v\bar{\mu}} \otimes k$. If $\bar{\mu}$ is defined $\mathbb{F}_q$, i.e., $\bar{\mu} \in (\mathbb{X}_\bullet(T)_I)^\sigma$, then $\mathscr{F}\ell_{v\bar{\mu}} \otimes k$ is also defined over $\mathbb{F}_q$ and we denote the corresponding Schubert variety in $\mathscr{F}\ell_v$ by $\mathscr{F}\ell_{v\bar{\mu}}$. In this case the intersection cohomology sheaf $\mathrm{IC}_{\bar{\mu}}$ is naturally in $\mathscr{P}_v^0$, and $\mathrm{H}^*(\mathrm{IC}_{\bar{\mu}})$ is a representation of $(H^\vee)^I \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(k/\mathbb{F}_q)$. When restricted to $(H^\vee)^I$, it is the highest representation $W_{\bar{\mu}}$. By abuse of notation, this algebraic representation of $(H^\vee)^I \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(k/\mathbb{F}_q)$ is still denoted by $W_{\bar{\mu}}$. Let $A_{\bar{\mu}} \in C_c(K_v \backslash G(F)/K_v)$ be the associated function under Grothendieck's sheaf-function dictionary. Combining (6.1) and (6.2), we have

$$\mathrm{tr}(\pi(A_\mu)) = \mathrm{tr}(\mathrm{Sat}(\pi)(\Phi), W_{\bar{\mu}}). \tag{6.4}$$

7. Applications to the nearby cycles on certain Shimura varieties

One of the main motivations of this work is to calculate the nearby cycles for certain unitary Shimura varieties. This is achieved by the so-called Rapoport-Zink-Pappas local models.

Let $F/\mathbb{Q}$ be a quadratic imaginary field; we fix an embedding $F \subset \mathbb{C}$. Let (W, ϕ) be a Hermitian space over $F/\mathbb{Q}$, of dimension $n = \dim W \geq 3$. Let $G = \mathrm{GU}(W, \phi)$ be the group of unitary similitudes defined by

$$G(R) = \{g \in \mathrm{GL}_F(W \otimes_{\mathbb{Q}} R) \mid \phi(gv, gw) = c(\mathrm{g})\phi(v, w), c(\mathrm{g}) \in R^\times\}.$$

Assume that $(W_{\mathbb{R}}, \phi_{\mathbb{R}}) \simeq (\mathbb{C}^n, H)$, where H is the standard Hermitian matrix on $\mathbb{C}^n$ of signature (r, s), i.e., $H = \mathrm{diag}\{(-1)^{(\mathrm{s})}, 1^{(\mathrm{r})}\}$ is the diagonal matrix with -1 repeated at the first s places and 1 repeated at the remaining r places[5]. Without loss of generality, we can assume that $s \leq r$. Let $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$ be the homomorphism given by $h(\mathrm{z}) = \mathrm{diag}\{z^{(\mathrm{s})}, \bar{z}^{(\mathrm{r})}\}$. Let $K \subset G(\mathbb{A}_f)$ be an open compact subgroup, small enough (i.e., K is contained in some principal congruence subgroup for some $N \geq 3$). Then associated to the data $(G, \{h\}, K)$, one can define a Shimura variety $\mathrm{Sh}(G, K)$ over a number field E, where $E = \mathbb{Q}$ if $r = s$, and $E = F$ if $r \neq s$. Let us recall that h also determines a conjugacy class of one parameter subgroups of $G_{\mathbb{C}}$ (the Shimura cocharacter), defined over E. In our case, $G_{\mathbb{C}} \simeq \mathrm{GL}_n \times \mathbb{G}_m$ and the one parameter subgroups are conjugate to $\mu_{r,s}(\mathrm{z}) = (\mathrm{diag}\{z^{(\mathrm{s})}, 1^{(\mathrm{r})}\}, z)$.

Let us fix a prime $p > 2$ and assume that $F/\mathbb{Q}$ is ramified at p. We denote F_p (resp. E_p) the completion of F (resp. E) at the unique place over p. In addition, we assume that (W, ϕ) is a split Hermitian form at p. In other words, $(W, \phi)_{\mathbb{Q}_p} \simeq (F_p^n, J)$, where J is the split Hermitian matrix on F_p^n with all its anti-diagonal entries 1, and 0 elsewhere. Observe that this assumption automatically holds if n is odd. Then $G_{\mathbb{Q}_p}$ is quasi-split. We will assume that $K = K_p K^p \subset G(\mathbb{Q}_p)G(\mathbb{A}_f^p)$ and K_p is a special parahoric of $G_{\mathbb{Q}_p}$, which is automatically very special in the sense of (6.1). Let us make this more concretely.

[5] The corresponding Hermitian form is $H(z, w) = \underline{\bar{z}}^t H \underline{w}$ for $\underline{z}, \underline{w} \in \mathbb{C}^n$.

Let π be a uniformizer of F_p so that $\pi^2 = ap$ with a a Teichmüller lifting of $\mathbb{F}_p^\times$. Let $\{e_1, \dots, e_n\}$ be a basis of F_p^n so that $\phi(e_i, e_j)$ is given by J. Let

$$\Lambda_i = \mathrm{Span}_{\mathcal{O}_{F_p}}\{\pi^{-1}e_1, \dots, \pi^{-1}e_i, e_{i+1}, \dots, e_n\}.$$

If $n = 2m+1$, we consider two integral models for $G_{\mathbb{Q}_p}$,

$$\underline{G}_{v_0} = \{g \in G, g\Lambda_0 = \Lambda_0\}, \quad \underline{G}_{v_1} = \{g \in G, g\Lambda_m = \Lambda_m\}. \tag{7.1}$$

If $n = 2m$, we consider the integral model

$$\underline{G}_v = \{g \in G, g\Lambda_m = \Lambda_m\}. \tag{7.2}$$

As explained in [23, Section 1.2], these $\underline{G}_v$ are special parahoric group schemes and essentially all special parahoric group schemes of G are conjugate to these ones.

Let $K_p = \underline{G}_v(\mathbb{Z}_p)$. In this case, the Shimura variety $\mathrm{Sh}(G, K)$ has a well-defined model over $\mathcal{O}_{E_p}$, as in [23]. Let us denote the integral model by Sh_{K_p}. In addition, there is the so-called local model diagram

$$\begin{array}{ccc} & \widetilde{\mathrm{Sh}}_{K_p} & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\varphi} \\ \mathrm{Sh}_{K_p} & & \mathrm{M}^{\mathrm{loc}}_{K_p}, \end{array}$$

where the scheme $\mathrm{M}^{\mathrm{loc}}_{K_p}$, which is called the local model of Sh_{K_p}, is projective over $\mathcal{O}_{E_p}$ with an action of $\underline{G}_v \otimes_{\mathbb{Z}_p} \mathcal{O}_{E_p}$, and is étale locally isomorphic to Sh_{K_p}. In addition, $\pi : \widetilde{\mathrm{Sh}}_{K_p} \to \mathrm{Sh}_{K_p}$ is a $\underline{G}_v \otimes_{\mathbb{Z}_p} \mathcal{O}_{E_p}$-torsor, and $\varphi : \widetilde{\mathrm{Sh}}_{K_p} \to \mathrm{M}^{\mathrm{loc}}_{K_p}$ is $\underline{G}_v \otimes_{\mathbb{Z}_p} \mathcal{O}_{E_p}$-equivariant and is formally smooth (cf. [23] for details).

We are interested in the nearby cycle $\Psi_{\mathrm{Sh}_{K_p}}(\mathbb{Q}_\ell)$, which is an ℓ-adic complex on $\mathrm{Sh}_{K_p} \otimes_{\mathcal{O}_{E_p}} \overline{\mathbb{F}}_p$, on which $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}_p/E)$ acts continuously, compatibly with the action of Γ on $\mathrm{M}^{\mathrm{loc}}_{K_p} \otimes \overline{\mathbb{F}}_p$ through $\Gamma \to \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. From the local model diagram, we have

$$\pi^*\Psi_{\mathrm{Sh}_{K_p}}(\mathbb{Q}_\ell) \simeq \varphi^*\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}(\mathbb{Q}_\ell).$$

Therefore, it is essentially enough to determine $\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}(\mathbb{Q}_\ell)$. For this purpose, we need to recall the geometry of $\mathrm{M}^{\mathrm{loc}}_{K_p}$.

First, let $F' = \mathbb{F}_p(\!(\mathrm{u})\!)$ be a ramified quadratic extension of $\mathbb{F}_p(\!(\mathrm{t})\!)$ with $u^2 = at$, where $a \in \mathbb{F}_p^\times$ as before. Let $W' = F'e_1 + \cdots + F'e_n$ and ϕ' be a split Hermitian form on W' given by $\phi'(e_i, e_{n+1-j}) = \delta_{ij}$. Let G' be the corresponding unitary similitude group over $\mathbb{F}_p(\!(\mathrm{t})\!)$. The parahoric group scheme $\underline{G}_v$ of G over $\mathbb{Z}_p$ has an obvious counterpart $\underline{G}'_v$ over $\mathbb{F}_p[\![t]\!]$. Namely, consider

$$\Lambda'_i = \mathrm{Span}_{\mathcal{O}_{F'}}\{\pi^{-1}e_1, \dots, \pi^{-1}e_i, e_{i+1}, \dots, e_n\}.$$

If $\underline{G}_v = G_{\mathbb{Q}_p} \cap \mathrm{Aut}(\Lambda_i)$, then $\underline{G}'_v = G' \cap \mathrm{Aut}(\Lambda'_i)$. Observe that there is an isomorphism $\underline{G}_v \otimes \mathbb{F}_p \simeq \underline{G}'_v \otimes \mathbb{F}_p$ (given by the obvious identification of $(W, \phi)_{\mathbb{F}_p} \simeq (W', \phi')_{\mathbb{F}_p}$). In addition, the Shimura cocharacter $\mu_{r,s}$ makes sense as a cocharacter of $G' \otimes (\mathbb{F}_p(\!(\mathrm{t})\!))^s$. Let $\mathscr{F}\ell_v = LG'/L^+\underline{G}'_v$ be the associated affine flag variety considered before.

Now, we give the description of $\mathrm{M}^{\mathrm{loc}}_{K_p}$. The following statements can be extracted from [23, 25, 24].

PROPOSITION 7.1. – (i) *The generic fiber of* $\mathrm{M}^{\mathrm{loc}}_{K_p}$ *is isomorphic to* $\mathscr{P}_{\mu_{r,s}}$, *where* $\mathscr{P}_{\mu_{r,s}}$ *is the variety of maximal parabolic subgroups of* G_{E_p} *of the type given by* $\mu_{r,s}$.

(ii) *The special fiber is isomorphic to* $\mathscr{F}\ell_{v\bar{\mu}_{r,s}}$ *in an equivariant way. More precisely, the* $L^+\underline{G}'_v$*-action on* $\mathscr{F}\ell_{v\bar{\mu}_{r,s}}$ *factors through an action of* $\underline{G}'_v \otimes \mathbb{F}_p$, *and there is an isomorphism*

$$\mathrm{M}^{\mathrm{loc}}_{K_p} \simeq \mathscr{F}\ell_{v\bar{\mu}_{r,s}},$$

intertwining the $\underline{G}_v \otimes \mathbb{F}_p$ *action on the left and this* $\underline{G}'_v \otimes \mathbb{F}_p$*-action on the right.*

(iii) *The generic point of the special fiber* $\mathrm{M}^{\mathrm{loc}}_{K_p} \otimes \mathbb{F}_p$ *is smooth in* $\mathrm{M}^{\mathrm{loc}}_{K_p}$.

Having described the geometry of $\mathrm{M}^{\mathrm{loc}}_{K_p}$, let us state the main theorem of this section. First, let $\mathscr{P}_v$ be either

1. the category of $L^+\underline{G}'_v$-equivariant Weil perverse sheaves on $\mathscr{F}\ell_v$, constant along each $L^+\underline{G}'_v$-orbit; or
2. the category of $L^+\underline{G}'_v \otimes \overline{\mathbb{F}}_p$-equivariant perverse sheaves on $\mathscr{F}\ell_v \otimes \overline{\mathbb{F}}_p$.

Likewise, we understand $\mathrm{IC}_{\bar{\mu}_{r,s}}$ either as a pure perverse sheaf of weight zero, or just a geometric perverse sheaf. By Theorem 0.1, we have:

1. if $n = 2m+1$ is odd,
$$\mathscr{RS} : \mathrm{Rep}(\mathrm{GO}_{2m+1}) \simeq \mathscr{P}_v;$$
2. if $n = 2m$ is even,
$$\mathscr{RS} : \mathrm{Rep}(\mathrm{GSp}_{2m}) \simeq \mathscr{P}_v.$$

Next, let V be the standard representation of GL_n and $V_{r,s} = \wedge^s V$ to be its sth wedge power. We extend $V_{r,s}$ to a representation of $\mathrm{GL}_n \times \mathbb{G}_m$, on which $\mathbb{G}_m$ acts via the homotheties.

THEOREM 7.2. – *Regard* $V_{r,s}$ *as a representation of* $\mathrm{GO}_n \subset \mathrm{GL}_n \times \mathbb{G}_m$ *if* n *is odd, or of* $\mathrm{GSp}_n \subset \mathrm{GL}_n \times \mathbb{G}_m$ *if* n *is even by restriction. Then:*

1. *Denote by* Ψ^{geom} *the underlying complex of sheaves of* $\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}[rs](\frac{rs}{2})$ *on* $\mathrm{M}^{\mathrm{loc}}_{K_p} \otimes \overline{\mathbb{F}}_p$. *Then*
$$\Psi^{\mathrm{geom}} \simeq \mathscr{RS}(V_{r,s}) \simeq \begin{cases} \mathrm{IC}_{\bar{\mu}_{r,s}} & n \text{ odd,} \\ \sum_{s' \geq 0, s-s' \in 2\mathbb{Z}_{\geq 0}} \mathrm{IC}_{\bar{\mu}_{n-s',s'}} & n \text{ even.} \end{cases}$$
2. *Let* $r \neq s$. *Then the action of the inertial subgroup* $I \subset \Gamma$ *on* $\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}$ *is trivial so that* $\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}$ *admits a structure as a Weil sheaf on* $\mathscr{F}\ell_v$. *In addition, as Weil sheaves,*
$$\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}[rs](\frac{rs}{2}) \simeq \mathscr{RS}(V_{r,s}).$$
3. *Let* $r = s = m$ *where* $n = 2m$. *By (1)*
$$\Psi^{\mathrm{geom}} \simeq \sum_{m' \geq 0, m-m' \in 2\mathbb{Z}_{\geq 0}} \mathrm{IC}_{\bar{\mu}_{n-m',m'}}.$$
The action of the inertial subgroup I *on* $\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}$ *factors through* $I \to \mathrm{Gal}(F_p/\mathbb{Q}_p) \simeq \mathbb{Z}/2$. *In addition, the action of* $\mathbb{Z}/2$ *on* $\mathrm{IC}_{\bar{\mu}_{n-m',m'}}$ *is trivial if* $4 \mid m - m'$ *and is through the non-trivial character if* $4 \nmid m - m'$. *As Weil sheaves,*
$$(\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}})^I[rs](\frac{rs}{2}) \simeq \sum_{m' \geq 0, m-m' \in 4\mathbb{Z}_{\geq 0}} \mathrm{IC}_{\bar{\mu}_{n-m',m'}},$$

where $(\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}})^I$ *denotes the inertial invariants of* $\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}$.

Proof. – Observe that Ψ^{geom} is an object in $\mathscr{P}_v$. This follows from Proposition 7.1 (ii). Observe that $\mathrm{IC}_{\bar{\mu}_{r,s}}$ is a direct summand of Ψ^{geom}. This follows from the fact that $\mathrm{M}^{\mathrm{loc}}_{K_p}$ is flat over $\mathcal{O}_{E_p}$ with special fiber isomorphic to $\mathscr{F}\ell_{v\bar{\mu}_{r,s}}$.

The partially flag variety $\mathscr{P}_{\mu_{r,s}}$ is a Schubert variety in the affine Grassmanian of G_{E_p} over E_p, (see (A.2)). Therefore, $\mathrm{H}^*(\mathscr{P}_{\mu_{r,s}})$ is a natural representation of $\mathrm{GL}_n \times \mathbb{G}_m$ (the dual group of G_{E_p}), which is indeed just $V_{r,s}$. Since nearby cycles commute with proper push-forward, we have

$$V_{r,s} \simeq \mathrm{H}^*(\mathscr{P}_{\mu_{r,s}}) \simeq \mathrm{H}^*(\Psi^{\mathrm{geom}}). \tag{7.3}$$

Part (1) of the theorem would follow if we can show that this isomorphism is an isomorphism of GO_n or GSp_n-modules. This is indeed the case, and can be shown using the constructions in [24]. In fact, as a further application of the main results of this paper, we will prove the corresponding Part (1) for all ramified groups in loc. cit. Here for the ramified unitary groups, we give a more direct (and easier) argument, without showing that (7.3) is an isomorphism of GO_n or GSp_n-modules. (However, we do need the existence of this natural isomorphism as graded vector spaces.) In fact, we will deduce the theorem from the following description of the restriction of $V_{r,s}$ as a representation of $(\mathrm{GL}_n \times \mathbb{G}_m)^I$. Note that if n is odd, then $V_{r,s}$ remains irreducible as a representation of $\mathrm{GO}_n = (\mathrm{GL}_n \times \mathbb{G}_m)^I$, denoted by $W_{\bar{\mu}_{r,s}}$. If n is even, the symplectic form induces a surjective map $V_{r,s} \to V_{r+2,s-2}$ and the kernel is the irreducible representation of $\mathrm{GSp}_n = (\mathrm{GL}_n \times \mathbb{G}_m)^I$ of highest weight $\bar{\mu}_{r,s}$, denoted by $W_{\bar{\mu}_{r,s}}$. Finally, if $(r,s) = (m,m)$, $V_{m,m}$ canonically extends to a representation of $(\mathrm{GL}_n \times \mathbb{G}_m) \rtimes I$ (e.g., see Corollary A.11).

LEMMA 7.3. – (i) *As* $(\mathrm{GL}_n \times \mathbb{G}_m)^I$*-modules,*

$$V_{r,s} \simeq \begin{cases} W_{\bar{\mu}_{r,s}} & n \text{ odd,} \\ \sum_{s' \geq 0, s-s' \in 2\mathbb{Z}_{\geq 0}} W_{\bar{\mu}_{n-s',s'}} & n \text{ even.} \end{cases}$$

(ii) *The representation* $V_{m,m}$ *of* $(\mathrm{GL}_n \times \mathbb{G}_m) \rtimes I$*, when restricted to* $\mathrm{GSp}_n \rtimes I = \mathrm{GSp}_n \times I$*, decomposes as*

$$V_{m,m} = \sum_{m' \geq 0, m-m' \in 2\mathbb{Z}_{\geq 0}} W_{\bar{\mu}_{n-m',m'}} \otimes \chi_{m'}, \tag{7.4}$$

where $\chi_{m'}$ *is the trivial character of* $\mathrm{Gal}(F_p/\mathbb{Q}_p)$ *if* $4 \mid m - m'$*, and is the non-trivial character if* $4 \nmid m - m'$.

Proof. – (i) is clear and we prove (ii). Clearly, there are some characters $\chi_{m'}$ of $\mathrm{Gal}(F_p/\mathbb{Q}_p)$ such that the decomposition (7.4) holds. We need to identify these characters.

First, it is clear that $\chi_m = 1$. This is because the lowest weight space of $V_{\bar{\mu}_{m,m}}$ is the same as the lowest weight space of $V_{m,m}$, which in turn is the same as $\mathrm{H}^0(\mathscr{P}_{\mu_{m,m}})$ as I-modules. But the action of I on $\mathrm{H}^0(\mathscr{P}_{\mu_{m,m}})$ is trivial. This shows that $\chi_m = 1$.

Now pick up $g \in I$ whose projection to $\mathrm{Gal}(F_p/\mathbb{Q}_p) \simeq \mathbb{Z}/2$ is non-trivial. To identify other $\chi_{m'}$, let us write the weight lattice of $\mathrm{GL}_n \times \mathbb{G}_m$ in a standard way to be $\mathbb{X}^\bullet = \bigoplus \mathbb{Z}\varepsilon_i \bigoplus \mathbb{Z}\varepsilon$ and the set of simple roots to be $\{\varepsilon_i - \varepsilon_{i+1}, 1 \leq i \leq n-1\}$. Then

the action of g on $\mathbb{X}^\bullet$ will send ε_i to $-\varepsilon_{n+1-i}$ and ε to $\varepsilon+\varepsilon_1+\cdots+\varepsilon_n$. The weight lattice of GSp_n is $\mathbb{X}^\bullet/\{\varepsilon_i+\varepsilon_{n+1-i}=0, i=1,\ldots,m\}$.

Let $\{v_1,\ldots,v_n\}$ be a standard basis of V so that v_i is a weight vector of GL_n of weight ε_i as usual. Then a basis of $V_{m,m}$ is given by $\{v_{i_1}\wedge\cdots\wedge v_{i_m} \mid 1\le i_1<\cdots<i_m\le n\}$. We divide this set of basis into two subsets A and B. A base vector $v_{i_1}\wedge\cdots\wedge v_{i_m}$ belongs to the subset A if $\{i,n+1-i\}\not\subseteq\{i_1,\ldots,i_m\}$ for any $1\le i\le m$. All remaining base vectors belong to B. It is clear that $\mathrm{Span}\{v \mid v\in A\}\subset V_{\bar\mu_{m,m}}$ and therefore, the action of g fixes each $v\in A$ since $\chi_m=1$. On the other hand, it is easy to see from the description of the action of g on $\mathbb{X}^\bullet$, that for $v\in B$, gv will be a multiple of some $w\in B, w\neq v$. From this, we deduce that for any t in the maximal torus of GSp_n,

$$\mathrm{tr}(gt,V_{m,m})=\varepsilon(\mathrm{t})\sum\varepsilon_1(\mathrm{t})^{\pm1}\cdots\varepsilon_m(\mathrm{t})^{\pm1}. \tag{7.5}$$

On the other hand, according to (7.4), we have

$$\mathrm{tr}(gt,V_{m,m})=\sum_{m'\ge0,m-m'\in2\mathbb{Z}_{\ge0}}\chi_{m'}(\mathrm{g})\,\mathrm{ch}(W_{\bar\mu_{n-m',m'}})(\mathrm{t}),$$

where $\mathrm{ch}(W_{\bar\mu_{r,s}})$ denotes the character of $W_{\bar\mu_{r,s}}$ as a GSp_n-module, and $\chi_{m'}(\mathrm{g})=\pm1$ according to whether $\chi_{m'}$ is trivial or not. Now it is easy to see that the above two identities force $\chi_{m'}=1$ if $4\mid m-m'$ and $\chi_{m'}\neq1$ if $4\nmid m-m'$. Indeed, let T be an indeterminant and write

$$\varepsilon(1+\varepsilon_1T)\cdots(1+\varepsilon_mT)(1+\varepsilon_1^{-1}T)\cdots(1+\varepsilon_m^{-1}T)=\sum a_kT^k,$$

then $a_{m+i}=a_{m-i}$ and $\mathrm{ch}(W_{\bar\mu_{r,s}})=a_s-a_{s-2}=a_r-a_{r+2}$. Put $T=\sqrt{-1}$, the left hand side becomes $(\sqrt{-1})^m\varepsilon(\varepsilon_1+\varepsilon_1^{-1})\cdots(\varepsilon_m+\varepsilon_m^{-1})$, which is exactly (7.5), and the right hand side is $(\sqrt{-1})^m(a_m-2a_{m-2}+2a_{m-4}-\cdots)$. The lemma is proved. □

Now we prove the theorem. We first assume that n is odd. As $V_{r,s}$ remains irreducible as a representation of GO_n. Therefore, $\mathscr{R}\mathscr{\ell}(V_{r,s})\simeq\mathrm{IC}_{\bar\mu_{r,s}}$ and we have

$$\dim V_{r,s}=\dim\mathrm{H}^*(\mathrm{IC}_{\bar\mu_{r,s}})\le\dim\mathrm{H}^*(\Psi^{\mathrm{geom}})=\dim\mathrm{H}^*(\mathscr{P}_{\mu_{r,s}})=\dim V_{r,s}.$$

Therefore, $\Psi^{\mathrm{geom}}=\mathrm{IC}_{\bar\mu_{r,s}}$. As $\mathrm{IC}_{\bar\mu_{r,s}}$ is irreducible, the inertial group I acts on $\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}$ via some character. Observe that the action of I on $\mathrm{H}^0(\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}})\simeq\mathrm{H}^0(\mathscr{P}_{\mu_{r,s}})\simeq\mathbb{Q}_\ell$ is via the same character, again due to the fact that nearby cycles commute with proper push-forward. Therefore the action of I on $\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}$ is trivial. Therefore, $\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}[rs](\frac{rs}{2})\simeq\mathrm{IC}_{\bar\mu_{r,s}}\otimes\mathscr{L}$ for some rank one local system $\mathscr{L}$ on $\mathrm{Spec}\mathbb{F}_p$. By comparing the action of Frobenius on $\mathrm{H}^*(\mathscr{P}_{\mu_{r,s}})$ and on $\mathrm{H}^*(\mathrm{IC}_{\bar\mu_{r,s}})$, we obtain the theorem in this case.

Now we assume that n is even. Recall that under the (ramified) geometric Satake isomorphism, the cohomological grading corresponds to the grading by $2\rho:\mathbb{G}_m\to\mathrm{GSp}_n\subset\mathrm{GL}_n\times\mathbb{G}_m$. Therefore, (7.3) is an isomorphism of the representations of $2\rho(\mathbb{G}_m)\subset\mathrm{GSp}_n$. We claim that this already implies Part (1) of the theorem. Indeed, for a representation V of GSp_n, we denote $V(\mathrm{i})$ to be the eigenspace of 2ρ of eigenvalue i. Let us write

$$\Psi^{\mathrm{geom}}=\sum_{m'\ge0,m-m'\in2\mathbb{Z}_{\le0}}c_{m'}\mathrm{IC}_{\bar\mu_{n-m',m'}},$$

we need to show that $c_{m'} = 1$. First as in Lemma 2.6, $c_s = 1$ by Proposition 7.1 (iii). Next, we show that $c_{s-2} = 1$. Observe that the gradings on $\mathrm{H}^*(\mathrm{IC}_{\bar{\mu}_{r,s}})$ range from rs to $-rs$. From

$$\dim \mathrm{H}^{(s-2)(r+2)}(\Psi^{\mathrm{geom}}) = c_{s-2} + \dim \mathrm{H}^{(s-2)(r+2)}(\mathrm{IC}_{\bar{\mu}_{r,s}}),$$

$$\dim V_{r,s}((s-2)(r+2)) = 1 + \dim(W_{\bar{\mu}_{r,s}}((s-2)(r+2))),$$

we conclude that $c_{s-2} = 1$. Now by induction, $c_{m'} = 1$ for all $m' \geq 0, m - m' \in 2\mathbb{Z}_{\geq 0}$. This shows that

$$\Psi^{\mathrm{geom}} \simeq \mathscr{R}\mathscr{J}(V_{r,s}).$$

Next, we determine the action of the inertial group I on $\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}$. First, assume that $r \neq s$. To show that the action of I on $\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}$ is trivial, we again observe that I acts on each irreducible direct summand of $\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}$ via a certain character. On the other hand, the group G_{E_p} is split. Therefore, the action of I on $\mathrm{H}^*(\mathscr{P}_{\mu_{r,s}})$ is trivial, and therefore is trivial on $\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}$. Again, comparing the action of the Frobenius, we conclude the result in this case.

Finally, let us assume that $r = s = m$, where $n = 2m$. Then $E_p = \mathbb{Q}_p$ and $G_{\mathbb{Q}_p}$ is not split. In addition the action of I on $\mathrm{H}^*(\mathscr{P}_{\mu_{m,m}})$ is not trivial. Indeed, as $\mathscr{P}_{\mu_{m,m}}$ is defined over $\mathbb{Q}_p$, according to the appendix, $V_{\mu_{m,m}} = \mathrm{H}^*(\mathscr{P}_{\mu_{m,m}})$ is a natural representation of ${}^L G^{\mathrm{geom}} = (\mathrm{GL}_n \times \mathbb{G}_m) \rtimes_{\mathrm{act}^{\mathrm{geom}}} \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, so that the natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $\mathrm{H}^*(\mathscr{P}_{\mu_{m,m}})$ is given by the restriction of this representation to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. This semidirect product $(\mathrm{GL}_n \times \mathbb{G}_m) \rtimes_{\mathrm{act}^{\mathrm{geom}}} \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is not the Langlands dual group ${}^L G^{\mathrm{alg}}_{\mathbb{Q}_p}$ of $G_{\mathbb{Q}_p}$. But if we form both semi-product using $I \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, they become the same because the cyclotomic character is trivial on I. On the other hand, since G is split over F_p, the action of I factors through $I \to \mathrm{Gal}(F_p/\mathbb{Q}_p) \simeq \mathbb{Z}/2$. We know that

$$\Psi^{\mathrm{geom}} = \sum_{m' \geq 0, m-m' \in 2\mathbb{Z}_{\geq 0}} \mathrm{IC}_{\bar{\mu}_{n-m',m'}}.$$

As argued in the case $r \neq s$, the action of I on $\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}$ also factors through $I \to \mathrm{Gal}(F_p/\mathbb{Q}_p)$. Assume that the action of I on $\mathrm{IC}_{\bar{\mu}_{n-m',m'}}$ is through the character $\chi'_{m'}$. We need to show that $\chi'_{m'} = \chi_{m'}$, where $\chi_{m'}$ is as in Lemma 7.3. Since $\mathrm{H}^*(\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}) \simeq \mathrm{H}^*(\mathscr{P}_{\mu_{m,m}}) \simeq V_{m,m}$ as $(2\rho(\mathbb{G}_m) \times I)$-modules, by taking the I-invariants, we obtain that

$$\sum_{\chi'_{m'}=1} \mathrm{H}^*(\mathrm{IC}_{\bar{\mu}_{n-m',m'}}) = \sum_{\chi_{m'}=1} W_{\bar{\mu}_{n-m',m'}}.$$

Again, as argued before by considering the gradings, it is easy to see that this forces $\chi'_{m'} = \chi_{m'}$. Finally, by comparing the action of Frobenius on $\mathrm{H}^*(\Psi^I_{\mathrm{M}^{\mathrm{loc}}_{K_p}})$ and on $\mathrm{H}^*(\mathscr{P}_{\mu_{m,m}})^I$, we conclude the theorem. □

Combining Theorem 7.2 and Theorem 5.1, it is not hard to obtain the explicit formula of the trace of Frobenius of $\Psi_{\mathrm{M}^{\mathrm{loc}}_{K_p}}$, which will be the input of the Langlands-Kottwitz method of calculating the local Zeta function of the Shimura varieties. Instead of writing down the explicit formula, let us characterize this function in terms of its trace on "unramified" representations of $G(F)$ (which clearly determines this function uniquely). The characterization verifies a conjecture of Haines and Kottwitz in this case.

PROPOSITION 7.4. – *Let $z_{r,s}$ be the function on $G'(F)$ associated to $\Psi^I_{\mathrm{M}^{\mathrm{loc}}_{K_p}}$ under the Grothendieck sheaf-function dictionary, and let $V_{r,s}$ be the representation of ${}^L G^{\mathrm{alg}}_{E_p}$ attached to $\mu_{r,s} \in \mathbb{X}_\bullet(T)$ as above (or in Corollary A.11). For π an "unramified" representation of $G'(F)$, with the Langlands parameter* $\mathrm{Sat}(\pi)$ *as defined in* (6.4)*, we have*

$$\mathrm{tr}(\pi(z_{r,s})) = \mathrm{tr}(\mathrm{Sat}(\pi)(\Phi), V^I_{r,s}).$$

The proof is a direct consequence of Theorem 7.2 and (6.4).

REMARK 7.5. – In the Langlands-Kottwitz methods of calculating the Zeta factors of Shimura varieties, one needs some mysterious test functions z_μ to be put into the trace formula. Assuming the Local Langlands, Haines and Kottwitz give a conjectural characterization of this test function z_μ in the general setting (i.e., arbitrary group and arbitrary level structure). In the special case when the group is quasi-split and the level structure is special parahoric, in which case the Langlands parameters is clear (as in Section 6), their characterization is reduced to the above proposition. Therefore, this proposition is the first example of their conjecture in the case when the group is ramified at p. In [24], we will show that the same characterization holds for arbitrary (tamely ramified) quasi-split groups with special parahoric level structure.

Finally, let us make Theorem 7.2 more explicit for some special cases.

COROLLARY 7.6. – *Let $(r,s) = (n-1,1)$. Then the inertial group acts on $\Psi_{\mathrm{Sh}_{K_p}}$ trivially. In addition, as Weil sheaves, $\Psi_{\mathrm{Sh}_{K_p}} \simeq \overline{\mathbb{Q}}_\ell$. In particular, for every $x \in \mathrm{Sh}_{K_p}(\mathbb{F}_{p^n})$, $\mathrm{tr}(\mathrm{Frob}_x, \Psi_{K_p}) = 1$.*

Proof. – The first statement follows from Theorem 7.2 and the local model diagram. We need to show that $\mathrm{IC}_{\bar{\mu}_{n-1,1}} \simeq \overline{\mathbb{Q}}_\ell[n-1](\frac{n-1}{2})$. However, according to Theorem 5.1, we know that $\mathrm{IC}_{\bar{\mu}_{n-1,1}}[1-n](\frac{1-n}{2})$ is a sheaf (for the standard t-structure) rather than a complex, with each stalk isomorphic to $\overline{\mathbb{Q}}_\ell$. As $\mathrm{IC}_{\bar{\mu}_{n-1,1}}[1-n](\frac{1-n}{2})$ is indecomposable as object in $D(\mathscr{G}\ell_v)$, this forces $\mathrm{IC}_{\bar{\mu}_{n-1,1}}[1-n](\frac{1-n}{2}) \simeq \overline{\mathbb{Q}}_\ell$. Now, since $\pi^*\Psi_{K_p} \simeq \varphi^*\mathrm{IC}_{\bar{\mu}_{n-1,1}}[1-n](\frac{1-n}{2})$ and π has geometrically connected fibers, the corollary follows. □

REMARK 7.7. – Concerning the part of the Frobenius trace, this corollary has been proven in [17, 23, 25]. Indeed, for the case n is odd, and the special parahoric is $\underline{G}_{v_0}$, this is a main result of [17]. In this case, Sh_{K_p} is not semi-stable. For the case n is odd and the parahoric is $\underline{G}_{v_1}$, it is shown in [25] that Sh_{K_p} is smooth. For the case n is even, it is shown in [23] that Sh_{K_p} is smooth.

Next, we consider the case $(r,s) = (2,2)$. Recall that the local model diagram can be written as a morphism

$$\mathrm{Sh}_{K_p} \to [\underline{G}_v \setminus \mathrm{M}^{\mathrm{loc}}_{K_p}],$$

where $[\underline{G}_v \setminus \mathrm{M}^{\mathrm{loc}}_{K_p}]$ denotes the stack quotient. Therefore, the Schubert stratification on $\mathrm{M}^{\mathrm{loc}}_{K_p} \otimes \overline{\mathbb{F}}_p$ induces a stratification on $\mathrm{Sh}_{K_p} \otimes \overline{\mathbb{F}}_p$, called the Kottwitz-Rapoport (KR) stratification. In the case $(r,s) = (2,2)$, the stratification has two strata $\mathrm{Sh}_{K_p,b}$ and $\mathrm{Sh}_{K_p,s}$. The smaller one $\mathrm{Sh}_{K_p,s}$ is zero-dimensional. Similarly to the previous case, we have

COROLLARY 7.8. – *Let* $(r,s)=(2,2)$. *Then* $\Psi_{\mathrm{Sh}_{K_p}}=\Psi^1_{\mathrm{Sh}_{K_p}}+\Psi^2_{\mathrm{Sh}_{K_p}}$. *The inertial action on* $\Psi^1_{\mathrm{Sh}_{K_p}}$ *is trivial and as Weil sheaves,* $\Psi^1_{\mathrm{Sh}_{K_p}}\simeq\mathbb{Q}_\ell$. *The vanishing cycle* $\Phi_{\mathrm{Sh}_{K_p}}(\mathbb{Q}_\ell)\simeq\Psi^2_{K_p}$. *When we forget the action of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $\Psi^2_{\mathrm{Sh}_{K_p}}=\sum_{x\in\mathrm{Sh}_{K_p,s}}\delta_x[-4]$, *where* δ_x *is the delta sheaf supported at* x. *In addition, the inertial action on* δ_x *factors through a non-trivial quadratic character. In particular, for every* $x\in\mathrm{Sh}_{K_p}(\mathbb{F}_{p^n})$, $\mathrm{tr}(\mathrm{Frob}_x,\Psi^I_{\mathrm{Sh}_{K_p}})=1$.

Appendix

Construction of the full Langlands dual group via the geometric Satake correspondence

by Timo Richarz & Xinwen Zhu

In the main body of the paper, we considered a reductive group G over $F=k(\!(\mathrm{t})\!)$ (k algebraically closed), split over a tamely ramified extension, and recovered $(H^\vee)^{\mathrm{Gal}(F^s/F)}$ by the Tannakian formalism from a certain category of perverse sheaves associated to G, where $H^\vee$ is the dual group of G, on which $\mathrm{Gal}(F^s/F)$ acts via pinned automorphisms. In this appendix, we take a different point of view to recover the full Langlands dual group ${}^LG=H^\vee\rtimes\mathrm{Gal}(F^s/F)$ of G by the Tannakian formalism. The construction is easy but we cannot find it in the literature. Most proofs will be omitted or rather sketched since they are very simple.

Let us begin with a review of certain general nonsense of Tannakian formalism. A similar discussion appears in [14, Appendix 2]. Let $(\mathscr{C},\omega)$ be a neutralized Tannakian category over a field E of characteristic zero with fiber functor ω. We define a monoidal category $\mathrm{Aut}^\otimes(\mathscr{C},\omega)$ as follows: objects are pairs (σ,α), where $\sigma:\mathscr{C}\to\mathscr{C}$ is a tensor automorphism and $\alpha:\omega\circ\sigma\simeq\omega$ is a natural isomorphism of tensor functors; morphisms between (σ,α) and (σ',α') are natural tensor isomorphisms between σ and σ' that are compatible with α,α' in an obvious way. The monoidal structure is given by compositions. Since ω is faithful, $\mathrm{Aut}^\otimes(\mathscr{C},\omega)$ is (equivalent to) a set, and in fact is a group. For example, $(\sigma,\alpha)=\mathrm{id}$ means that there is an isomorphism $\varepsilon:\sigma\simeq\mathrm{id}$ of tensor functors such that $\omega\varepsilon=\alpha$ (such ε will be unique).

Let $H=\mathrm{Aut}^\otimes_{\mathscr{C}}\omega$, the Tannakian group defined by $(\mathscr{C},\omega)$. Let $\mathrm{Aut}(H)$ be the group of automorphisms of H and $\mathrm{Out}(H)$ be the group of outer automorphisms of H.

LEMMA A.1. – *There is a canonical action of* $\mathrm{Aut}^\otimes(\mathscr{C},\omega)$ *on* H *by automorphisms. In addition, the map* $\mathrm{Aut}^\otimes(\mathscr{C},\omega)\to\mathrm{Aut}(H)$ *induces* $[\mathrm{Aut}^\otimes(\mathscr{C})]\to\mathrm{Out}(H)$, *where* $[\mathrm{Aut}^\otimes(\mathscr{C})]$ *is the group of isomorphism classes of tensor automorphisms of* $\mathscr{C}$.

The action of (σ,α) on H is given as follows. Let R be an E-algebra and $h:\omega_R\simeq\omega_R$ be an R-point of H. Then $(\sigma,\alpha)h$ is the following composition

$$\omega_R\overset{\alpha}{\leftarrow}\omega_R\circ\sigma\overset{h\circ\mathrm{id}}{\to}\omega_R\circ\sigma\overset{\alpha}{\to}\omega_R.$$

REMARK A.2. – As is shown [14], $\mathrm{Aut}^\otimes(\mathscr{C},\omega)$ can be upgraded into a *fppf* sheaf on the category of affine schemes over E, and as *fppf* sheaves $\mathrm{Aut}^\otimes(\mathscr{C},\omega)\simeq\mathrm{Aut}(H)$. We do not need this fact.

Let Γ be an abstract group. We define an action of Γ on $(\mathscr{C},\omega)$ to be a group homomorphism $\mathrm{act} : \Gamma \to \mathrm{Aut}^{\otimes}(\mathscr{C},\omega)$. Assume that Γ acts on $(\mathscr{C},\omega)$. We can then define $\mathscr{C}^{\Gamma}$, the category of Γ-equivariant objects in $\mathscr{C}$ as follows: objects are $(X,\{c_\gamma\}_{\gamma\in\Gamma})$, where X is an object in $\mathscr{C}$ and $c_\gamma : \mathrm{act}_\gamma(X) \simeq X$ is an isomorphism, satisfying the natural cocycle condition, i.e., $c_{\gamma'} \circ \mathrm{act}_{\gamma'}(c_\gamma) = c_{\gamma'\gamma}$; the morphisms between $(X,\{c_\gamma\}_{\gamma\in\Gamma})$ and $(X',\{c'_\gamma\}_{\gamma\in\Gamma})$ are morphisms between X and X', compatible with c_γ, c'_γ in an obvious way.

LEMMA A.3. – *Let Γ be a group acting on $(\mathscr{C},\omega)$. Then the category $\mathscr{C}^{\Gamma}$ is a neutral Tannakian category, with fiber functor ω. In addition, if Γ is a finite group, regarded as an algebraic E-group, then the Tannakian group $\mathrm{Aut}^{\otimes}_{\mathscr{C}^{\Gamma}}\omega$ is canonically isomorphic to $H \rtimes \Gamma$.*

Proof. – The monoidal structure on $\mathscr{C}^{\Gamma}$ is defined as

$$(X,\{c_\gamma\}_{\gamma\in\Gamma}) \otimes (X',\{c'_\gamma\}_{\gamma\in\Gamma}) = (X'',\{c''_\gamma\}_{\gamma\in\Gamma}),$$

where $X'' = X \otimes X'$ and $c''_\gamma : \mathrm{act}_\gamma(X'') \to X''$ is the composition

$$\mathrm{act}_\gamma(X\otimes X') \simeq \mathrm{act}_\gamma(X) \otimes \mathrm{act}_\gamma(X') \xrightarrow{c_\gamma \otimes c'_\gamma} X \otimes X'.$$

This gives $\mathscr{C}^{\Gamma}$ the structure of a Tannakian category. Now assume that Γ is finite, and hence $H \rtimes \Gamma$ is an affine group scheme. By [7, Prop. 2.8], it is enough to show that,

$$\mathrm{Rep}(H)^{\Gamma} \xrightarrow{\simeq} \mathrm{Rep}(H \rtimes \Gamma)$$

as tensor categories compatible with the forgetful functors. Let $((V,\rho),\{c_\gamma\}_{\gamma\in\Gamma}) \in \mathrm{Rep}(H)^{\Gamma}$. Then we define $(V,\tilde{\rho}) \in \mathrm{Rep}(H \rtimes \Gamma)$, for any k-algebra R $(h,\gamma) \in (H \rtimes \Gamma)(R)$, by

$$(h,\gamma) \longmapsto \rho(\mathrm{h}) \circ \alpha_{g,R}(V) \circ \omega_R \circ c_\gamma^{-1} \in \mathrm{GL}(V \otimes R),$$

where $\alpha_{g,R} : \omega_R \circ \sigma_g \simeq \omega_R$ is induced by the action of Γ as above. Using the cocycle relation one checks that this is indeed a representation, and that the map defines the desired equivalence. □

REMARK A.4. – If Γ is not finite, then the category $(\mathscr{C}^{\Gamma},\omega)$ is still Tannakian, but $\tilde{H} = \mathrm{Aut}^{\otimes}_{\mathscr{C}^{\Gamma}}\omega$ is no longer $H \rtimes \Gamma$ since the latter cannot be regarded as an affine group scheme (sometimes $\tilde{H}$ is called the algebraic envelop of $H \rtimes \Gamma$). However, there is always a group homomorphism

$$H(E) \rtimes \Gamma \to \tilde{H}(E).$$

Although this is not an isomorphism in general, we may still regard $\omega(X)$ for $X \in \mathscr{C}^{\Gamma}$ as a representation of $H(E) \rtimes \Gamma$.

Now, we assume that G is a connected reductive group over *any* field k. We switch the notation to use $G^\vee$ to denote the reductive group over $E = \overline{\mathbb{Q}}_\ell$ dual to G in the sense of Langlands, i.e., the root datum of $G^\vee$ is dual to the root datum of G_{k^s}. Up to the choice of a pinning $(G^\vee,B^\vee,T^\vee,X^\vee)$ of $G^\vee$, we have an action of $\Gamma_k = \mathrm{Gal}(k^s/k)$ on $G^\vee$ via

$$\Gamma_k \to \mathrm{Out}(G_{k^s}) \simeq \mathrm{Out}(G^\vee) \simeq \mathrm{Aut}(G^\vee,B^\vee,T^\vee,X^\vee) \subset \mathrm{Aut}(G^\vee). \tag{A.1}$$

Then the Langlands dual group ${}^LG$ is defined to be $G^\vee \rtimes \Gamma_k$. Our goal is to recover this group via the above Tannakian formalism.

Let L^+G be the jet group of $G\otimes k[[t]]$ and LG be the loop group of $G\otimes k(\!(\mathrm{t})\!)$. Recall that by definition, for every k-algebra R, $L^+G(R)=G(R[[t]])$ and $LG(R)=G(R(\!(\mathrm{t})\!))$. Let

$$\mathrm{Gr}_G = LG/L^+G \tag{A.2}$$

be the affine Grassmannian of G over k. Let $\mathrm{Gr}_G\otimes k^s$ be its base change to the separable closure of k. From [31, Lemma 3.3], formation of the affine Grassmannian commutes with étale base change, we have $\mathrm{Gr}_G\otimes k^s\simeq \mathrm{Gr}_{G_{k^s}}$. Since G_{k^s} is split, we can consider the usual Satake category Sat on $\mathrm{Gr}_{G_{k^s}}$, i.e., the category of $(L^+G\otimes k^s)$-equivariant perverse sheaves on $\mathrm{Gr}_G\otimes k^s$, which is equivalent to $\mathrm{Rep}(G^\vee_{\overline{\mathbb{Q}}_\ell})$ via the geometric Satake correspondence of [19]. Note that the Galois group Γ_k acts on $\mathrm{Gr}_G\otimes k^s$. For $\gamma\in\Gamma_k$, the pullback functor $\gamma^*: D(\mathrm{Gr}_G\otimes k^s)\to D(\mathrm{Gr}_G\otimes k^s)$ clearly restricts to a functor $\gamma^*:\mathrm{Sat}\to\mathrm{Sat}$. In addition, there is a canonical isomorphism $\alpha_\gamma:\mathrm{H}^*(\gamma^*\mathscr{T})\simeq\mathrm{H}^*(\mathscr{T})$.

LEMMA A.5. – *The assignment $\gamma\mapsto(\gamma^*,\alpha_\gamma)$ defines an action of Γ_k on $(\mathrm{Sat},\mathrm{H}^*)$.*

According to Lemma A.1, there is a canonical action of Γ_k on $G^\vee$, denoted by $\mathrm{act}^{\mathrm{geom}}$. And we can form

$${}^LG^{\mathrm{geom}} := G^\vee\rtimes_{\mathrm{act}^{\mathrm{geom}}}\Gamma_k,$$

which we will call the geometric Langlands dual group.

Now, our goal is to understand the relation between ${}^LG^{\mathrm{geom}}$ and the usual Langlands dual group ${}^LG$. Recall that in Section 4, we explained that once we choose an ample line bundle on Gr_G, the geometric Satake isomorphism provides $G^\vee$ with a canonical pinning $(G^\vee,B^\vee,T^\vee,X^\vee)$. Therefore, there is an action of Γ_k on $G^\vee$ via (A.1), denoted by $\mathrm{act}^{\mathrm{alg}}$. Then we can form the usual Langlands dual group by ${}^LG^{\mathrm{alg}}=G^\vee\rtimes_{\mathrm{act}^{\mathrm{alg}}}\Gamma_k$. It turns out that the difference between $\mathrm{act}^{\mathrm{geom}}$ and $\mathrm{act}^{\mathrm{alg}}$ can be described explicitly.

Let

$$\mathrm{cycl}:\Gamma_k\to\mathbb{Z}_\ell^\times$$

be the cyclotomic character of Γ_k defined by the action of Γ_k on the ℓ^∞-roots of unity of k^s. Let $G^\vee_{\mathrm{ad}}$ be the adjoint group of $G^\vee$. Let ρ be the half sum of positive coroots of $G^\vee$, which gives rise to a one-parameter group $\rho:\mathbb{G}_m\to G^\vee_{\mathrm{ad}}$. We define a map

$$\chi:\Gamma_k\xrightarrow{\mathrm{cycl}}\mathbb{Z}_\ell^\times\xrightarrow{\rho}G^\vee_{\mathrm{ad}}(\overline{\mathbb{Q}}_\ell),$$

which gives a map $\mathrm{Ad}_\chi:\Gamma_k\to\mathrm{Aut}(G^\vee)$ to the inner automorphism of $G^\vee$.

PROPOSITION A.6. – *Let $(G^\vee,B^\vee,T^\vee,X^\vee)$ be the pinning defined by the geometric Satake isomorphism as in §4. We have $\mathrm{act}^{\mathrm{geom}}=\mathrm{act}^{\mathrm{alg}}\circ\mathrm{Ad}_\chi$.*

Proof. – Observe that the action of Γ_k preserves the cohomological grading. In addition, Γ_k acts on $X^\vee$ through cycl since $X^\vee$ is the Chern class of the chosen ample line bundle on Gr_G. Therefore, $\mathrm{Ad}_{\chi^{-1}}\circ\mathrm{act}^{\mathrm{geom}}$ preserves $(G^\vee,B^\vee,T^\vee,X^\vee)$. But since $\mathrm{Ad}_{\chi^{-1}}\circ\mathrm{act}^{\mathrm{geom}}$ and $\mathrm{act}^{\mathrm{alg}}$ act on the based root datum $(\mathbb{X}^\bullet(T),\Delta)$ by the same way, these two actions must coincide. □

REMARK A.7. – (i) An interesting corollary of the above proposition is that the $\mathrm{act}^{\mathrm{geom}}$ of Γ_k on $G^\vee$ only depends on the quasi-split form of G, since the same is true for $\mathrm{act}^{\mathrm{alg}}$.

(ii) The existence of these two actions $\mathrm{act}^{\mathrm{geom}}$ and $\mathrm{act}^{\mathrm{alg}}$ is a geometric source of the two natural normalizations for the Satake parameters. In addition, these two actions are also parallel to the notions of C-algebraic and L-algebraic, as recently introduced by Buzzard and Gee [5].

COROLLARY A.8. – *We have* ${}^LG^{\mathrm{alg}} \xrightarrow{\simeq} {}^LG^{\mathrm{geom}}$ *given by*

$$(g, \gamma) \mapsto (\mathrm{Ad}_{\chi(\gamma^{-1})}(\mathrm{g}), \gamma).$$

Following [3, 2.6], we define the notation of the algebraic representations of ${}^LG^{\mathrm{alg}}$ as follows. For every $k' \subset k^s$ finite over k such that $G_{k'}$ is split, one can form the "finite form" of Langlands dual group $G^\vee \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(k'/k)$, which can be regarded as an algebraic group over $\overline{\mathbb{Q}}_\ell$. Therefore, we may consider ${}^LG^{\mathrm{alg}} = \varprojlim G^\vee \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(k'/k)$ as a pro-algebraic group over $\overline{\mathbb{Q}}_\ell$, Then it makes sense to talk about the category of algebraic representations $\mathrm{Rep}({}^LG^{\mathrm{alg}})$ of ${}^LG^{\mathrm{alg}}$, which is the inductive limit of $\mathrm{Rep}(G^\vee \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(k'/k))$.

Now, we consider certain categories of perverse sheaves. First, according to the above discussions, we call the action of Γ_k on Sat via $\gamma \mapsto (\gamma^*, \alpha_\gamma)$ as in Lemma A.5 the geometric action. We can also define an algebraic action of Γ_k on Sat as $\gamma \mapsto (\gamma^*, \chi(\gamma)^{-1}\alpha_\gamma\chi(\gamma))$. Clearly, there is a canonical isomorphism between $\mathrm{Sat}^{\Gamma_k,\mathrm{geom}}$ and $\mathrm{Sat}^{\Gamma_k,\mathrm{alg}}$ sending $(\mathscr{T}, \{c_\gamma\}_{\gamma\in\Gamma_k})$ to $(\mathscr{T}, \{c_\gamma\}_{\gamma\in\Gamma_k})$. The reason we distinguish them is due to the following observation: since the algebraic action of Γ_k on $G^\vee$ factors through $\mathrm{Gal}(k'/k)$ if $G_{k'}$ is split, the algebraic action of Γ_k on Sat also factors through $\mathrm{Gal}(k'/k)$, and therefore it makes sense to talk about $\mathrm{Sat}^{\mathrm{Gal}(k'/k),\mathrm{alg}}$, which is naturally a full category of $\mathrm{Sat}^{\Gamma_k,\mathrm{alg}}$. In addition, according to Lemma A.3, $\mathrm{Sat}^{\mathrm{Gal}(k'/k),\mathrm{alg}}$ is equivalent to $\mathrm{Rep}(G^\vee \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(k'/k))$. Now, let $\mathrm{Sat}^{\Gamma_k,\mathrm{alg},f} \subset \mathrm{Sat}^{\Gamma_k,\mathrm{alg}}$ be the full subcategory, which is the union of all $\mathrm{Sat}^{\mathrm{Gal}(k'/k),\mathrm{alg}}$. We obtain that

$$\mathrm{H}^* : \mathrm{Sat}^{\Gamma_k,\mathrm{alg},f} \simeq \mathrm{Rep}({}^LG^{\mathrm{alg}}).$$

Our next goal then is to identify $\mathrm{Sat}^{\Gamma_k,\mathrm{alg},f}$ as a subcategory of $\mathrm{Sat}^{\Gamma_k,\mathrm{geom}}$.

As Γ_k is a topological group, a natural guess would be the full subcategory of $\mathrm{Sat}^{\Gamma_k,\mathrm{geom,ct}}$, consisting of objects on which Γ_k acts continuously. Equivalent, let $\mathscr{P}_{L^+G}(\mathrm{Gr}_G)$ be the category of perverse sheaves on Gr_G. Then the pullback functor to $\mathrm{Gr}_G \otimes k^s$ induces $\mathscr{P}_{L^+G}(\mathrm{Gr}_G) \simeq \mathrm{Sat}^{\Gamma_k,\mathrm{geom,ct}}$. However, it is not the case that $\mathrm{Sat}^{\Gamma_k,\mathrm{geom,ct}} = \mathrm{Sat}^{\Gamma_k,\mathrm{alg},f}$.

DEFINITION A.1. – (i) Let X be a smooth variety over k, we define a constant sheaf to be a direct sum of $(\overline{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\otimes \dim X}$.

(ii) We define $\mathscr{P}^f_{L^+G}(\mathrm{Gr}_G)$ to be the full subcategory of $\mathscr{P}_{L^+G}(\mathrm{Gr}_G)$, consisting of those sheaves $\mathscr{T}$, such that there exists some $k' \supset k$ such that $\mathscr{T} \otimes k'$ is constant along each $(L^+G \otimes k')$-orbit.

REMARK A.9. – (i) The toy model is when $G = \{e\}$ is the trivial group. Then $\mathscr{P}_{L^+G}(\mathrm{Gr}_G)$ is the category Γ_k-Mod of continuous representations of Γ_k while $\mathscr{P}^f_{L^+G}(\mathrm{Gr}_G)$ is the subcategory Γ_k-Mod^f consisting of representations of finite quotients of Γ_k. In particular, if k is a finite field, we can identify the latter as the category of semi-simple Γ_k-modules, pure of weight zero.

(ii) Observe that every object in $\mathscr{P}_{L^+G}(\mathrm{Gr}_G)$ is of the form $\oplus_i \mathrm{IC}_i \otimes \mathscr{L}_i$, where IC_i is an intersection cohomology sheaf on Gr_G, and $\mathscr{L}_i$ is a representation of Γ_k. Therefore, $\mathcal{T} \in \mathscr{P}^f_{L^+G}(\mathrm{Gr}_G)$ if and only if all $\mathscr{L}_i \in \Gamma_k\text{-Mod}^f$. In particular, if $k = \mathbb{F}_q$ is a finite field, the category $\mathscr{P}^f_{L^+G}(\mathrm{Gr}_G)$ is equivalent to the category of semi-simple L^+G-equivariant perverse sheaves on Gr_G, pure of weight zero.

PROPOSITION A.10. – *Under the canonical isomorphism* $\mathrm{Sat}^{\Gamma_k,\mathrm{geom}} = \mathrm{Sat}^{\Gamma_k,\mathrm{alg}}$*, we have the identification*

$$\mathscr{P}^f_{L^+G}(\mathrm{Gr}_G) = \mathrm{Sat}^{\Gamma_k,\mathrm{alg},f}.$$

Therefore, we obtain an equivalence of tensor categories

$$\mathrm{H}^* : \mathscr{P}^f_{L^+G}(\mathrm{Gr}_G) \to \mathrm{Rep}({}^L G^{\mathrm{alg}}).$$

Proof. – Let $k' \supset k$ such that $G_{k'}$ splits. We denote $\mathscr{P}_{L^+G,k'}(\mathrm{Gr}_G)$ the full subcategory of $\mathscr{P}^f_{L^+G}(\mathrm{Gr}_G)$ consisting of those $\mathcal{T}$ such that $\mathcal{T} \otimes k'$ is constant along each $(L^+G \otimes k')$-orbit. Then under

$$\mathscr{P}_{L^+G}(\mathrm{Gr}_G) \to \mathrm{Sat}^{\Gamma_k,\mathrm{geom}} \simeq \mathrm{Sat}^{\Gamma_k,\mathrm{alg}},$$

$\mathscr{P}_{L^+G,k'}(\mathrm{Gr}_G)$ maps to $\mathrm{Sat}^{\mathrm{Gal}(k'/k),\mathrm{alg}}$. To see this, one reduces to the case when G is split over k and $k' = k$. In this case $\mathscr{P}_{L^+G,k}(\mathrm{Gr}_G) \simeq \mathrm{Sat}$ and this statement is clear. □

For every $\mathcal{T} \in \mathscr{P}^f_{L^+G}(\mathrm{Gr}_G)$, let us describe $\mathrm{H}^*(\mathcal{T})$ as a representation of ${}^L G^{\mathrm{alg}}$ more explicitly. First, as an object in $\mathrm{Sat}^{\Gamma_k,\mathrm{geom}}$, it is a natural representation of ${}^L G^{\mathrm{geom}}$, on which $G^\vee$ acts via the usual geometric Satake isomorphism, and Γ_k acts via the natural Galois action. Then the action of ${}^L G^{\mathrm{alg}}$ is via (A.8).

In particular, if $\overline{\mathrm{Gr}}_\mu$ is a Schubert variety in Gr_G defined over k (i.e., the conjugacy class of the one-parameter subgroup determined by μ is defined over k), then $\mathrm{IC}_{\overline{\mathrm{Gr}}_\mu}$ is an object in $\mathscr{P}^f_{L^+G}(\mathrm{Gr}_G)$. Therefore, $\mathrm{H}^*(\mathrm{IC}_{\overline{\mathrm{Gr}}_\mu})$ is a representation of $G^\vee \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(k'/k)$, where k' is the splitting field of G. We thus obtain

COROLLARY A.11. – *Let V_μ be a representation of $G^\vee$ of highest weight μ. If the conjugacy class of the one parameter subgroup $\mu : \mathbb{G}_m \to G_{k^s}$ is defined over k, then V_μ can be extended canonically to a representation of $G^\vee \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(k'/k)$, where k' is the splitting field of G.*

Now we specialize to the case that $k = \mathbb{F}_q$ is a finite field, so that $G \otimes k[[t]]$ is a hyperspecial group scheme for the unramified group $G_{k((\mathrm{t}))}$. Note that ${}^L G^{\mathrm{alg}}_{k((\mathrm{t}))} = {}^L G^{\mathrm{alg}}$. We therefore obtain the geometric Satake isomorphism for unramified groups.

THEOREM A.12. – *Let $\mathscr{P}^f_{L^+G}(Gr_G)$ be the category of semi-simple, L^+G-equivariant perverse sheaves on Gr_G, pure of weight zero. Then we have an equivalence of tensor categories*

$$\mathrm{H}^* : \mathscr{P}^f_{L^+G}(\mathrm{Gr}_G) \simeq \mathrm{Rep}({}^L G^{\mathrm{alg}}).$$

Let σ be the Frobenius element in Γ_k. Denote by $\mathscr{H}_G$ the Grothendieck ring of $\mathscr{P}^f_{L^+G}(\mathrm{Gr}_G)$, tensored with $\overline{\mathbb{Q}}_\ell$, and by $\mathscr{R}_{^LG}$ the algebra associated to $\mathrm{Rep}({}^LG^{\mathrm{alg}})$. Theorem A.12 gives an isomorphism of algebras

$$\Phi : \mathscr{H}_G \to \mathscr{R}_{^LG}. \tag{A.3}$$

Let H_G be the spherical Hecke algebra of compactly supported bi-$G(\mathbb{F}_q[[t]])$-invariant functions. Let $\mathrm{R}_{^LG}$ be the algebra of $\overline{\mathbb{Q}}_\ell$-valued functions on $(G^\vee \times \sigma)_{ss}$, generated by the characters of elements in $\mathrm{Rep}({}^LG^{\mathrm{alg}})$. Here, $(G^\vee \times \sigma)_{ss}$ is the set of semi-simple elements in $G^\vee \times \sigma$, as defined in [3, Section 6]. We have a surjective map of algebras $\mathrm{Tr} : \mathscr{H}_G \to \mathrm{H}_G$ (resp. $\mathrm{Ch} : \mathscr{R}_{^LG} \to \mathrm{R}_{^LG}$) given by the trace of Frobenius (resp. by sending a representation to its character).

LEMMA A.13. – *The isomorphism* (A.3) *induces a unique isomorphism*

$$\phi : \mathrm{H}_G \to \mathrm{R}_{^LG}$$

such that $\mathrm{Ch} \circ \Phi = \phi \circ \mathrm{Tr}$.

Proof. – Uniqueness is clear and we show the existence. For an object X in either category, we denote by $[X]$ its class in the Grothendieck ring. Then it is easy to see that the kernel of the map $\mathrm{Tr} : \mathscr{R}_{^LG} \to \mathrm{R}_{^LG}$ is the ideal generated by elements of the form $[V \otimes \psi] - \psi(\sigma)[V]$, where $V \in \mathrm{Rep}({}^LG^{\mathrm{alg}})$ and $\psi : \Gamma_k \to \overline{\mathbb{Q}}_\ell^\times$ is a character of Γ_k factoring through a finite quotient. On the other hand, the kernel of the map $\mathrm{Ch} : \mathscr{H}_G \to \mathrm{H}_G$ is the ideal generated by elements of the form $[\mathscr{T} \otimes \mathscr{L}] - \mathrm{tr}(\sigma, \mathscr{L})[\mathscr{T}]$, where $\mathscr{T} \in \mathscr{P}^f_{L^+G}(\mathrm{Gr}_G)$, and $\mathscr{L}$ is a rank one local system on $\mathrm{Spec}\mathbb{F}_q$, pure of weight zero. But it is clear that these two ideals match under Φ. The lemma follows. □

Recall that by [3, 6.7] the classical Satake isomorphism also gives an isomorphism of algebras $\tilde{\phi} : \mathrm{H}_G \simeq \mathrm{R}_{^LG}$. By tracking back the construction of geometric Satake correspondence and the classical Satake isomorphism, one can show that $\phi = \tilde{\phi}$. Indeed, if $G = T$ is a toruse, this is clear. For general G, one observes that the fiber functor decomposes as a direct sum of weight functors [19, §3]: $\mathscr{P}^f_{L^+G}(\mathrm{Gr}_G) \to \mathscr{P}^f_{L^+T}(\mathrm{Gr}_T)$, and under the sheaf-function dictionary this corresponds to the constant term map $\mathrm{CT} : \mathrm{H}_G \to \mathrm{H}_T$. Therefore, either ϕ or $\tilde{\phi}$ is uniquely determined by the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}_G & \xrightarrow{\sim} & \mathrm{R}_{^LG} \\ {\scriptstyle \mathrm{CT}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathrm{H}_T & \xrightarrow{\sim} & \mathrm{R}_{^LT}\,. \end{array}$$

The general case follows.

BIBLIOGRAPHY

[1] J. BERNSTEIN, V. LUNTS, *Equivariant sheaves and functors*, Lecture Notes in Math. **1578**, Springer, Berlin, 1994.

[2] R. Bezrukavnikov, On tensor categories attached to cells in affine Weyl groups, in *Representation theory of algebraic groups and quantum groups*, Adv. Stud. Pure Math. **40**, Math. Soc. Japan, Tokyo, 2004, 69–90.

[3] A. Borel, Automorphic L-functions, in *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, 27–61.

[4] R. K. Brylinski, Limits of weight spaces, Lusztig's q-analogs, and fiberings of adjoint orbits, *J. Amer. Math. Soc.* **2** (1989), 517–533.

[5] K. Buzzard, T. Gee, The conjectural connections between automorphic representations and Galois representations, preprint arXiv:1009.0785.

[6] P. Cartier, Representations of p-adic groups: a survey, in *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, 111–155.

[7] P. Deligne, J. Milne, Tannakian categories, in *Hodge Cycles and Motives*, Lecture Notes in Math. **900**, 1982, 101–228.

[8] D. Gaitsgory, Construction of central elements in the affine Hecke algebra via nearby cycles, *Invent. Math.* **144** (2001), 253–280.

[9] D. Gaitsgory, Braiding compatibility, appendix to [2].

[10] V. A. Ginzburg, Perverse sheaves on a loop group, and Langlands duality, preprint arXiv:alg-geom/9511007.

[11] T. Haines, M. Rapoport, On parahoric subgroups, appendix to [22].

[12] T. J. Haines, S. Rostami, The Satake isomorphism for special maximal parahoric Hecke algebras, *Represent. Theory* **14** (2010), 264–284.

[13] T. Heines, A proof of the Kazhdan-Lusztig purity theorem via the decomposition theorem of BBD, preprint `www2.math.umd.edu/~tjh/KL_purity1.pdf`.

[14] J. Heinloth, B.-C. Ngô, Z. Yun, Kloosterman sheaves for reductive groups, *Ann. of Math.* **177** (2013), 241–310.

[15] R. E. Kottwitz, Points on some Shimura varieties over finite fields, *J. Amer. Math. Soc.* **5** (1992), 373–444.

[16] R. E. Kottwitz, Isocrystals with additional structure. II, *Compositio Math.* **109** (1997), 255–339.

[17] N. Krämer, Local models for ramified unitary groups, *Abh. Math. Sem. Univ. Hamburg* **73** (2003), 67–80.

[18] G. Lusztig, Singularities, character formulas, and a q-analog of weight multiplicities, in *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, Astérisque **101**, Soc. Math. France, Paris, 1983, 208–229.

[19] I. Mirković, K. Vilonen, Geometric Langlands duality and representations of algebraic groups over commutative rings, *Ann. of Math.* **166** (2007), 95–143.

[20] D. Nadler, Perverse sheaves on real loop Grassmannians, *Invent. Math.* **159** (2005), 1–73.

[21] G. Pappas, On the arithmetic moduli schemes of PEL Shimura varieties, *J. Algebraic Geom.* **9** (2000), 577–605.

[22] G. Pappas, M. Rapoport, Twisted loop groups and their affine flag varieties, *Adv. Math.* **219** (2008), 118–198.

[23] G. Pappas, M. Rapoport, Local models in the ramified case. III. Unitary groups, *J. Inst. Math. Jussieu* **8** (2009), 507–564.

[24] G. Pappas, X. Zhu, Local models of Shimura varieties and a conjecture of Kottwitz, *Invent. Math.* **194** (2013), 147–254.

[25] T. Richarz, Local models and Schubert varieties in twisted affine Grassmannians, preprint arXiv:1011.5416.

[26] R. Steinberg, *Endomorphisms of linear algebraic groups*, Memoirs of the American Mathematical Society **80**, Amer. Math. Soc., Providence, R.I., 1968.

[27] J. Tits, Reductive groups over local fields, in *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, 29–69.

[28] J.-L. Verdier, Spécialisation de faisceaux et monodromie modérée, in *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, Astérisque **101**, Soc. Math. France, Paris, 1983, 332–364.

[29] Z. Yun, X. Zhu, Integral homology of loop groups via Langlands dual groups, *Represent. Theory* **15** (2011), 347–369.

[30] X. Zhu, Affine Demazure modules and T-fixed point subschemes in the affine Grassmannian, *Adv. Math.* **221** (2009), 570–600.

[31] X. Zhu, On the coherence conjecture of Pappas and Rapoport, *Ann. of Math.* **180** (2014), 1–85.

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