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MONODROMY AND
THE LEFSCHETZ FIXED POINT FORMULA

BY Ehud HRUSHOVSKI AND François LOESER

To Jan Denef as a token of admiration and friendship
on the occasion of his 60th birthday

ABSTRACT. – We give a new proof—not using resolution of singularities—of a formula of Denef and the second author expressing the Lefschetz number of iterates of the monodromy of a function on a smooth complex algebraic variety in terms of the Euler characteristic of a space of truncated arcs. Our proof uses ℓ-adic cohomology of non-archimedean spaces, motivic integration and the Lefschetz fixed point formula for finite order automorphisms. We also consider a generalization due to Nicaise and Sebag and at the end of the paper we discuss connections with the motivic Serre invariant and the motivic Milnor fiber.

RÉSUMÉ. – Nous donnons une nouvelle preuve — n'utilisant pas la résolution des singularités — d'une formule de Denef et du second auteur exprimant le nombre de Lefschetz des itérés de la monodromie d'une fonction sur une variété algébrique complexe en fonction de la caractéristique d'Euler d'un espace d'arcs tronqués. Notre preuve utilise la cohomologie ℓ-adique des espaces non-archimédiens, l'intégration motivique, ainsi que la formule des points fixes de Lefschetz pour les automorphismes d'ordre fini. Nous considérons également une généralisation due à Nicaise et Sebag et la fin de l'article est consacrée aux relations avec l'invariant de Serre motivique et la fibre de Milnor motivique.

1. Introduction

1.1. – Let X be a smooth complex algebraic variety of dimension d and let $f \colon X \to \mathbb{A}^1_{\mathbb{C}}$ be a non-constant morphism to the affine line. Let x be a singular point of $f^{-1}(0)$, that is, such that $df(x) = 0$.

Fix a distance function δ on an open neighborhood of x induced from a local embedding of this neighborhood in some complex affine space. For $\varepsilon > 0$ small enough, one may consider the corresponding closed ball $B(x, \varepsilon)$ of radius ε around x. For $\eta > 0$ we denote by D_η the closed disk of radius η around the origin in $\mathbb{C}$.

By Milnor's local fibration Theorem (see [30], [14]), there exists $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$, there exists $0 < \eta < \varepsilon$ such that the morphism f restricts to a fibration, called the Milnor fibration,

$$(1.1.1) \qquad B(x, \varepsilon) \cap f^{-1}(D_\eta \setminus \{0\}) \longrightarrow D_\eta \setminus \{0\}.$$

The Milnor fiber at x,

$$(1.1.2) \qquad F_x = f^{-1}(\eta) \cap B(x, \varepsilon),$$

has a diffeomorphism type that does not depend on δ, η and ε. The characteristic mapping of the fibration induces on F_x an automorphism which is defined up to homotopy, the monodromy M_x. In particular the singular cohomology groups $H^i(F_x, \mathbb{Q})$ are endowed with an automorphism M_x, and for any integer m one can consider the Lefschetz numbers

$$(1.1.3) \qquad \Lambda(M_x^m) = \mathrm{tr}(M_x^m; H^\bullet(F_x, \mathbb{Q})) = \sum_{i \geq 0} (-1)^i \mathrm{tr}(M_x^m; H^i(F_x, \mathbb{Q})).$$

In [1], A'Campo proved that if x is a singular point of $f^{-1}(0)$, then $\Lambda(M_x^1) = 0$ and this was later generalized by Deligne to the statement that $\Lambda(M_x^m) = 0$ for $0 < m < \mu$, with μ the multiplicity of f at x, cf. [2].

In [13], Denef and Loeser proved that $\Lambda(M_x^m)$ can be expressed in terms of Euler characteristics of arc spaces as follows. For any integer $m \geq 0$, let $\mathscr{L}_m(X)$ denote the space of arcs modulo t^{m+1} on X: a $\mathbb{C}$-rational point of $\mathscr{L}_m(X)$ corresponds to a $\mathbb{C}[t]/t^{m+1}$-rational point of X, cf. [10]. Consider the locally closed subset $\mathscr{X}_{m,x}$ of $\mathscr{L}_m(X)$

$$(1.1.4) \qquad \mathscr{X}_{m,x} = \{\varphi \in \mathscr{L}_m(X); f(\varphi) = t^m \mod t^{m+1}, \varphi(0) = x\}.$$

Note that $\mathscr{X}_{m,x}$ can be viewed in a natural way as the set of closed points of a complex algebraic variety.

THEOREM 1.1.1 ([13]). – *For every $m \geq 1$,*

$$(1.1.5) \qquad \chi_c(\mathscr{X}_{m,x}) = \Lambda(M_x^m).$$

Here χ_c denotes the usual Euler characteristic with compact supports. Note that one recovers Deligne's statement as a corollary since $\mathscr{X}_{m,x}$ is empty for $0 < m < \mu$. The original proof in [13] proceeds as follows. One computes explicitly both sides of (1.1.5) on an embedded resolution of the hypersurface defined by $f = 0$ and checks that both quantities are equal. The computation of the left-hand side relies on the change of variable formula for motivic integration in [10] and the one on the right-hand side on A'Campo's formula in [2]. The problem of finding a geometric proof of Theorem 1.1.1 not using resolution of singularities is raised in [27]. The aim of this paper is to present such a proof.

1.2. – Our approach uses étale cohomology of non-archimedean spaces and motivic integration. Nicaise and Sebag introduced in [33] the analytic Milnor fiber $\mathscr{F}_x$ of the function f at a point x which is a rigid analytic space over $\mathbb{C}((t))$. Let $\mathscr{F}_x^{\mathrm{an}}$ denote its analytification in the sense of Berkovich. Using a comparison theorem of Berkovich, they show that, for every $i \geq 0$, the étale ℓ-adic cohomology group $H^i(\mathscr{F}_x^{\mathrm{an}} \widehat{\otimes \mathbb{C}((t))^{\mathrm{alg}}}, \mathbb{Q}_\ell)$ is isomorphic to $H^i(F_x, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Furthermore, these étale ℓ-adic cohomology groups are naturally endowed with an action of the Galois group $\mathrm{Gal}(\mathbb{C}((t))^{\mathrm{alg}}/\mathbb{C}((t)))$ of the algebraic closure of $\mathbb{C}((t))$, and under this isomorphism the action of the topological generator $(t^{1/n} \mapsto \exp(2i\pi/n)t^{1/n})_{n \geq 1}$ of $\hat{\mu}(\mathbb{C}) = \mathrm{Gal}(\mathbb{C}((t))^{\mathrm{alg}}/\mathbb{C}((t)))$ corresponds to the monodromy M_x.

Another fundamental tool in our approach is provided by the theory of motivic integration developed in [20] by Hrushovski and Kazhdan. Their logical setting is that of the theory $\mathrm{ACVF}(0,0)$ of algebraically closed valued fields of equal characteristic zero, with two sorts VF and RV. If L is a field endowed with a valuation $v : L \to \Gamma(L)$, with valuation ring $\mathcal{O}_L$ and maximal ideal $\mathscr{M}_L$, $\mathrm{VF}(L) = L$ and $\mathrm{RV}(L) = L^\times/(1+\mathscr{M}_L)$. Thus $\mathrm{RV}(L)$ can be inserted in an exact sequence

$$(1.2.1) \qquad\qquad 1 \to \mathbf{k}^\times(L) \to \mathrm{RV}(L) \to \Gamma(L) \to 0$$

with $\mathbf{k}(L)$ the residue field of L. Let us work with $\mathbb{C}((t))$ as a base field. One of the main result of [20] is the construction of an isomorphism

$$(1.2.2) \qquad\qquad \oint : K(\mathrm{VF}) \longrightarrow K(\mathrm{RV}[*])/I_{\mathrm{sp}}$$

between the Grothendieck ring $K(\mathrm{VF})$ of definable sets in the VF-sort and the quotient of a graded version $K(\mathrm{RV}[*])$ of the Grothendieck ring of definable sets in the RV-sort by an explicit ideal I_{sp}. At the Grothendieck rings level, the extension (1.2.1) is reflected by the fact that $K(\mathrm{RV}[*])$ may be expressed as a tensor product of the graded Grothendieck rings $K(\Gamma[*])$ and $K(\mathrm{RES}[*])$ for a certain sort RES. A precise definition of RES will be given in 2.2, but let us say that variables in the RES sort range not only over the residue field but also over certain torsors over the residue field so that definable sets in the RES sort are twisted versions of constructible sets over the residue field. This reflects the fact that the extension (1.2.1) has no canonical splitting. Furthermore, there is a canonical isomorphism between a quotient $!K(\mathrm{RES})$ of the Grothendieck ring $K(\mathrm{RES})$ and $K^{\hat{\mu}}(\mathrm{Var}_\mathbb{C})$, the Grothendieck ring of complex algebraic varieties with $\hat{\mu}$-action, as considered in [12] and [27]. Let $[\mathbb{A}^1]$ denote the class of the affine line. In [20] a canonical morphism

$$(1.2.3) \qquad\qquad \mathrm{EU}_\Gamma : K(\mathrm{VF}) \longrightarrow !K(\mathrm{RES})/([\mathbb{A}^1]-1)$$

is constructed. We shall make essential use of that construction, which is recalled in detail in 2.5. It roughly corresponds to applying the o-minimal Euler characteristic to the Γ-part of the product decomposition of the right-hand side of (1.2.2). Denote by $K(\hat{\mu}\text{-Mod})$ the Grothendieck ring of the category of finite dimensional $\mathbb{Q}_\ell$-vector spaces with $\hat{\mu}$-action. There is a canonical morphism $K^{\hat{\mu}}(\mathrm{Var}_\mathbb{C}) \to K(\hat{\mu}\text{-Mod})$ induced by taking the alternating sum of cohomology with compact supports from which one derives a morphism

$$(1.2.4) \qquad\qquad \mathrm{eu}_{\text{ét}} : !K(\mathrm{RES})/([\mathbb{A}^1]-1) \longrightarrow K(\hat{\mu}\text{-Mod}).$$

Our strategy is the following. Instead of trying to prove directly a Lefschetz fixed point formula for objects of VF, that are infinite dimensional in nature when considered as objects over $\mathbb{C}$, we take advantage of the morphism EU_Γ for reducing to finite dimensional spaces. To this aim, using étale cohomology of Berkovich spaces, developed by Berkovich in [3], we construct a natural ring morphism

$$(1.2.5) \qquad\qquad \mathrm{EU}_{\text{ét}} \colon K(\mathrm{VF}) \longrightarrow K(\hat{\mu}\text{-Mod})$$

and we prove a key result, Theorem 5.4.1, which states that the diagram

$$(1.2.6) \qquad \begin{array}{ccc} K(\mathrm{VF}) & \xrightarrow{\;\;\mathrm{EU}_\Gamma\;\;} & !K(\mathrm{RES})/([\mathbb{A}^1]-1) \\ & \searrow{\scriptstyle \mathrm{EU}_{\text{ét}}} \quad \swarrow{\scriptstyle \mathrm{eu}_{\text{ét}}} & \\ & K(\hat{\mu}\text{-Mod}) & \end{array}$$

is commutative. Using this result, we are able to reduce the proof of Theorem 1.1.1 to a classical statement, the Lefschetz fixed point theorem for finite order automorphisms acting on complex algebraic varieties (Proposition 5.5.1).

Since our approach makes no use of resolution of singularities, it would be tempting to try extending it to situations in positive residue characteristic. In order to do that, a necessary prerequisite would be to find the right extension of the results of [20] beyond equicharacteristic 0.

1.3. – Using the same circle of ideas, we also obtain several new results and constructions dealing with the motivic Serre invariant and the motivic Milnor fiber.

More precisely, in Section 7, we explain the connexion between the morphism EU_Γ and the motivic Serre invariant of [28]. We show in Proposition 7.2.1 that if X is a smooth proper algebraic variety over $F((t))$ with F a field of characteristic zero, with base change $X(m)$ over $F((t^{m^{-1}}))$, then the motivic Serre invariant $S(X(m))$ can be expressed in terms of the part of $\mathrm{EU}_\Gamma(X)$ fixed by the m-th power of a topological generator of $\hat{\mu}$. This allows in particular to provide a proof of a fixed point theorem originally proved by Nicaise and Sebag in [33] that circumvents the use of resolution of singularities.

In Section 8 we show how one can recover the motivic zeta function and the motivic Milnor fiber of [9] and [12], after inverting the elements $1 - [\mathbb{A}^1]^i$, $i \geq 1$, from a single class in the measured Grothendieck ring of definable objects over VF, namely the class of the set $\mathcal{X}_x$ of points y in $X(\mathbb{C}[[t]])$ such that $\mathrm{rv}f(y) = \mathrm{rv}(t)$ and $y(0) = x$. This provides a new construction of the motivic Milnor fiber that seems quite useful. It has already been used by Lê Quy Thuong [25] to prove an integral identity conjectured by Kontsevich and Soibelman in their work on motivic Donaldson-Thomas invariants [24].

2. Preliminaries on Grothendieck rings of definable sets, after [20]

2.1. – We shall consider the theory $\mathrm{ACVF}(0,0)$ of algebraically closed valued fields of equal characteristic zero, with two sorts VF and RV. This will be more suitable here than the more classical signature with three sorts $(\mathrm{VF}, \Gamma, \mathbf{k})$. The language on VF is the ring language, and the language on RV consists of abelian group operations $\cdot$ and $(\cdot)^{-1}$, a unary predicate $\mathbf{k}^{\times}$ for a subgroup, an operation $+\colon \mathbf{k}^2 \to \mathbf{k}$, where $\mathbf{k}$ is $\mathbf{k}^{\times}$ augmented by a symbol zero, and a function symbol rv for a function $\mathrm{VF}^{\times} \to \mathrm{RV}$. Here, $\mathrm{VF}^{\times}$ stands for $\mathrm{VF} \setminus \{0\}$.

Let L be a valued field, with valuation ring $\mathcal{O}_L$ and maximal ideal $\mathcal{M}_L$. We set $\mathrm{VF}(L) = L$, $\mathrm{RV}(L) = L^{\times}/(1 + \mathcal{M}_L)$, $\Gamma(L) = L^{\times}/\mathcal{O}_L^{\times}$ and $\mathbf{k}(L) = \mathcal{O}_L/\mathcal{M}_L$. We have an exact sequence

$$(2.1.1) \qquad\qquad 1 \to \mathbf{k}^{\times} \to \mathrm{RV} \to \Gamma \to 0,$$

where we view Γ as an imaginary sort. We denote by $\mathrm{rv}\colon \mathrm{VF}^{\times} \to \mathrm{RV}$, $\mathrm{val}\colon \mathrm{VF}^{\times} \to \Gamma$ and $\mathrm{val}_{\mathrm{rv}}\colon \mathrm{RV} \to \Gamma$ the natural maps.

2.2. – Fix a base structure L_0 which is a nontrivially valued field. We shall view L_0-definable sets as functors from the category of valued field extensions of L_0 with no morphisms except the identity to the category of sets. For each $\gamma \in \mathbb{Q} \otimes \Gamma(L_0)$, we consider the definable set V_{γ}

$$(2.2.1) \qquad\qquad L \longmapsto V_{\gamma}(L) = \{0\} \cup \{x \in L^{\times}; \mathrm{val}(x) = \gamma\}/(1 + \mathcal{M}_L)$$

on valued field extensions L of L_0. Note that when $\gamma - \gamma' \in \Gamma(L_0)$, $V_{\gamma}(L)$ and $V_{\gamma'}(L)$ are definably isomorphic. For $\bar{\gamma} = (\gamma_1, \ldots, \gamma_n) \in (\mathbb{Q} \otimes \Gamma(L_0))^n$ we set $V_{\bar{\gamma}} = \prod_i V_{\gamma_i}$. By a $\bar{\gamma}$-weighted monomial, we mean an expression $a_{\nu} X^{\nu} = a_{\nu} \prod_i X_i^{\nu_i}$ with $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$ a multi-index, such that a_{ν} is an L_0-definable element of RV with $\mathrm{val}_{\mathrm{rv}}(a_{\nu}) + \sum_i \nu_i \gamma_i = 0$. A $\bar{\gamma}$-polynomial is a finite sum of $\bar{\gamma}$-weighted monomials. Such a $\bar{\gamma}$-polynomial H gives rise to a function $H\colon V_{\bar{\gamma}} \to \mathbf{k}$ so we can consider its zero set $Z(H)$. The intersection of finitely many such sets is called a generalized algebraic variety over the residue field. The generalized residue structure RES consists of the residue field, together with the collection of the definable sets V_{γ}, for $\gamma \in \mathbb{Q} \otimes \Gamma(L_0)$, and the functions $H\colon V_{\bar{\gamma}} \to \mathbf{k}$ associated with each $\bar{\gamma}$-polynomial.

2.3. – If S is a sort, we write S^* to mean S^m, for some m. We shall view varieties over L_0 as definable sets over L_0. We denote by $\mathrm{VF}[n]$ the category of definable subsets of n-dimensional varieties over L_0. By Lemma 8.1 of [20] this category is equivalent to the category whose objects are the definable subsets X of $\mathrm{VF}^* \times \mathrm{RV}^*$ such that there exists a definable map $X \to \mathrm{VF}^n$ with finite fibers. By abuse of notation we shall sometimes also denote by $\mathrm{VF}[n]$ that category.

We denote by $\mathrm{RV}[n]$ the category of definable pairs (X, f) with $X \subset \mathrm{RV}^*$ and $f\colon X \to \mathrm{RV}^n$ a definable map with finite fibers and by $\mathrm{RES}[n]$ the full subcategory consisting of objects with X such that $\mathrm{val}_{\mathrm{rv}}(X)$ is finite (which is equivalent to the condition that X is isomorphic to a definable subset of RES^*). By Remark 3.67 of [20], the forgetful map $(X, f) \mapsto X$ induces an equivalence of categories between $\mathrm{RV}[n]$ and the category of all definable subsets of RV^* of RV-dimension $\leq n$, that is, such that there exists a definable map with finite fibers to RV^n. Nonetheless, the morphism f will be useful for defining $\mathbb{L}$ in 2.4.

Let $A = \Gamma(L_0)$ or more generally any ordered abelian group in Γ. One defines $\Gamma[n]$ as the category whose objects are finite disjoint union of subsets of Γ^n defined by linear equalities and inequalities with $\mathbb{Z}$-coefficients and parameters in A. Given objects X and Y in $\Gamma[n]$, a morphism f between X and Y is a bijection such that there exists a finite partition of X into objects X_i of $\Gamma[n]$, such that the restriction of f to X_i is of the form $x \mapsto M_i x + a_i$ with $M_i \in \mathrm{GL}_n(\mathbb{Z})$ and $a_i \in A^n$. We define $\Gamma^{\mathrm{fin}}[n]$ as the full subcategory of $\Gamma[n]$ consisting of finite sets.

We shall consider the categories

$$(2.3.1) \qquad\qquad \mathrm{RV}[\leq n] = \bigoplus_{0 \leq k \leq n} \mathrm{RV}[k],$$

$$(2.3.2) \qquad\qquad \mathrm{RV}[*] = \bigoplus_{n \geq 0} \mathrm{RV}[n],$$

$$(2.3.3) \qquad\qquad \mathrm{RES}[*] = \bigoplus_{n \geq 0} \mathrm{RES}[n],$$

$$(2.3.4) \qquad\qquad \Gamma[*] = \bigoplus_{n \geq 0} \Gamma[n]$$

and

$$(2.3.5) \qquad\qquad \Gamma^{\mathrm{fin}}[*] = \bigoplus_{n \geq 0} \Gamma^{\mathrm{fin}}[n].$$

Let C be any of the symbols RV, RES, Γ and Γ^{fin}. We shall denote by $K_+(C[n])$, $K_+(C[*])$, resp. $K(C[n])$, $K(C[*])$, the Grothendieck monoid, resp. the Grothendieck group, of the corresponding categories as defined in [20]. The Grothendieck monoid $K_+(C[*])$ decomposes as a direct sum $K_+(C[*]) = \bigoplus_{0 \leq n} K_+(C[n])$ and admits a natural structure of graded semi-ring with $K_+(C[n])$ as degree n part. Similarly, $K(C[*])$ admits a natural structure of graded ring with $K(C[n])$ as degree n part. The Grothendieck monoid $K_+(\mathrm{RV}[n])$ is isomorphic to the Grothendieck monoid of definable subsets X of RV^* of RV-dimension $\leq n$.

One also considers $K_+(\mathrm{VF})$, resp. $K(\mathrm{VF})$, the Grothendieck semi-ring, resp. the Grothendieck ring, of the category of definable subsets of L_0-varieties of any dimension. The product is induced by cartesian product and $K_+(\mathrm{VF})$ and $K(\mathrm{VF})$ are filtered by dimension. By Lemma 8.1 of [20], $K_+(\mathrm{VF})$, resp. $K(\mathrm{VF})$, can be identified with the Grothendieck semi-ring, resp. the Grothendieck ring, of the category of definable subsets X of $\mathrm{VF}^* \times \mathrm{RV}^*$ such that there exists, for some n, a definable map $X \to \mathrm{VF}^n$ with finite fibers. Similarly, one denotes by $K_+(\mathrm{RV})$, $K(\mathrm{RV})$, $K_+(\mathrm{RES})$, and $K(\mathrm{RES})$, the Grothendieck semi-rings and rings of the categories of definable subsets of RV^* and RES^*, respectively.

The mapping $X \mapsto \mathrm{val}_{\mathrm{rv}}^{-1}(X)$ induces a functor $\Gamma[n] \to \mathrm{RV}[n]$, hence a morphism $K_+(\Gamma[n]) \to K_+(\mathrm{RV}[n])$ which restricts to a morphism $K_+(\Gamma^{\mathrm{fin}}[n]) \to K_+(\mathrm{RES}[n])$. We also have a morphism $K_+(\mathrm{RES}[n]) \to K_+(\mathrm{RV}[n])$ induced by the inclusion functor $\mathrm{RES}[n] \to \mathrm{RV}[n]$. There is a unique morphism of graded semi-rings

$$(2.3.6) \qquad \Psi \colon K_+(\mathrm{RES}[*]) \otimes_{K_+(\Gamma^{\mathrm{fin}}[*])} K_+(\Gamma[*]) \longrightarrow K_+(\mathrm{RV}[*])$$

sending $[X] \times [Y]$ to $[X \times \mathrm{val}_{\mathrm{rv}}^{-1}(Y)]$, for X in $\mathrm{RES}[m]$ and Y in $\Gamma[n]$ and it is proved in Corollary 10.3 of [20] that Ψ is an isomorphism.

2.4. – One defines

$$(2.4.1) \qquad\qquad \mathbb{L} \colon \mathrm{ObRV}[n] \longrightarrow \mathrm{ObVF}[n]$$

by sending a definable pair (X, f) with $X \subset \mathrm{RV}^*$ and $f \colon X \to \mathrm{RV}^n$ a definable map with finite fibers to

$$(2.4.2) \qquad \mathbb{L}(X, f) = \{(y_1, \ldots, y_n, x) \in (\mathrm{VF}^\times)^n \times X; (\mathrm{rv}(y_i)) = f(x)\}.$$

Note that by Proposition 6.1 of [20], the isomorphism class of $\mathbb{L}(X, f)$ does not depend on f, so we shall sometimes write $\mathbb{L}(X)$ instead of $\mathbb{L}(X, f)$. This mapping induces a morphism of filtered semi-rings

$$(2.4.3) \qquad\qquad \mathbb{L} \colon K_+(\mathrm{RV}[*]) \longrightarrow K_+(\mathrm{VF})$$

sending the class of an object X of $\mathrm{RV}[n]$ to the class of $\mathbb{L}(X)$.

If X is a definable subset of RV^n, we denote by $[X]_n$ the class of (X, Id) in $K_+(\mathrm{RV}[n])$ or in $K(\mathrm{RV}[n])$. Similarly, if X is a definable subset of RES^n or Γ^n, we denote by $[X]_n$ the class of X in $K_+(\mathrm{RES}[n])$ and $K_+(\Gamma[n])$, respectively, or in the corresponding Grothendieck ring. In particular, we can assign to the point $1 \in k^* \subset \mathrm{RV}$ a class $[1]_1$ in $K_+(\mathrm{RV}[1])$, and the point of RV^0 a class $[1]_0$ in $K_+(\mathrm{RV}[0])$. Set $\mathrm{RV}^{>0} = \{x \in \mathrm{RV}; \mathrm{val}_{\mathrm{rv}}(x) > 0\}$. Observe the identity $\mathbb{L}([1]_1) = \mathbb{L}([1]_0) + \mathbb{L}([\mathrm{RV}^{>0}]_1)$ in $K_+(\mathrm{VF})$; the left-hand side is the open ball $1 + \mathscr{M}$, while the right-hand side is $(0) + (\mathscr{M} \setminus (0))$. Let I_{sp} be the semi-ring congruence generated by the relation $[1]_1 \sim [1]_0 + [\mathrm{RV}^{>0}]_1$. By Theorem 8.8 of [20], $\mathbb{L}$ is surjective with kernel I_{sp}. Thus, by inverting $\mathbb{L}$, one gets a canonical isomorphism of filtered semi-rings

$$(2.4.4) \qquad\qquad \oint \colon K_+(\mathrm{VF}) \longrightarrow K_+(\mathrm{RV}[*])/I_{\mathrm{sp}}.$$

2.5. – Let $I!$ be the ideal of $K(\mathrm{RES}[*])$ generated by the differences $[\mathrm{val}_{\mathrm{rv}}^{-1}(a)]_1 - [\mathrm{val}_{\mathrm{rv}}^{-1}(0)]_1$ where a runs over $\Gamma(L_0) \otimes \mathbb{Q}$. We denote by $!K(\mathrm{RES}[*])$ the quotient of $K(\mathrm{RES}[*])$ by $I!$ and by $!K(\mathrm{RES}[n])$ its graded piece of degree n (note that passing to the quotient by $I!$ preserves the graduation). One defines similarly $!K(\mathrm{RES})$.

Let us still denote by I_{sp} the ideal in $K(\mathrm{RV}[*])$ generated by the similar object of $K_+(\mathrm{RV}[*])$. We shall now recall the construction of group morphisms

$$(2.5.1) \qquad\qquad \mathscr{E}_n \colon K(\mathrm{RV}[\leq n])/I_{\mathrm{sp}} \longrightarrow !K(\mathrm{RES}[n])$$

and

$$(2.5.2) \qquad\qquad \mathscr{E}'_n \colon K(\mathrm{RV}[\leq n])/I_{\mathrm{sp}} \longrightarrow !K(\mathrm{RES}[n])$$

given in Theorem 10.5 of [20].

The morphism $\mathscr{E}_n$ is induced by the group morphism

$$(2.5.3) \qquad\qquad \gamma \colon \bigoplus_{m \leq n} K(\mathrm{RV}[m]) \longrightarrow !K(\mathrm{RES}[n])$$

given by

$$(2.5.4) \qquad\qquad \gamma = \sum_m \beta_m \circ \chi[m],$$

with $\beta_m \colon !K(\mathrm{RES}[m]) \to !K(\mathrm{RES}[n])$ given by $[X] \mapsto [X \times \mathbb{A}^{n-m}]$ and $\chi[m] \colon K(\mathrm{RV}[m]) \to !K(\mathrm{RES}[m])$ defined as follows. The isomorphism (2.3.6) induces an isomorphism

$$(2.5.5) \qquad K(\mathrm{RV}[m]) \simeq \bigoplus_{1 \leq \ell \leq m} K(\mathrm{RES}[m - \ell]) \otimes_{K(\Gamma^{\mathrm{fin}})} K(\Gamma[\ell]),$$

and $\chi[m]$ is defined as $\bigoplus_{1 \leq \ell \leq m} \chi_\ell$ with χ_ℓ sending $a \otimes b$ in $K(\mathrm{RES}[m - \ell]) \otimes_{K(\Gamma^{\mathrm{fin}})} K(\Gamma[\ell])$ to $\chi(b) \cdot [\mathbb{G}_m]^\ell \cdot a$, where $\chi \colon K(\Gamma[\ell]) \to \mathbb{Z}$ is the o-minimal Euler characteristic (cf. Lemma 9.5 of [20]). Here $\mathbb{G}_m$ denotes the multiplicative torus of the residue field, thus $[\mathbb{G}_m] = [\mathbb{A}^1] - 1$.

The definition of $\mathscr{E}'_n$ is similar, replacing β_m by the map $[X] \mapsto [X] \times [1]_1^{n-m}$ and χ by the "bounded" Euler characteristic $\chi' \colon K(\Gamma[\ell]) \to \mathbb{Z}$ (cf. Lemma 9.6 of [20]) given by $\chi'(Y) = \lim_{r \to \infty} \chi(Y \cap [-r, r]^n)$ for Y a definable subset of Γ^n.

We will now consider $!K(\mathrm{RES}[n])$ modulo the ideal of multiples of the class of $[\mathbb{G}_m]_1$, which we denote by $!K(\mathrm{RES}[n])/[\mathbb{G}_m]_1$. By formulas (1) and (3) in Theorem 10.5 of [20] the morphisms $\mathscr{E}_n$ and $\mathscr{E}'_n$ induce the same morphism

$$(2.5.6) \qquad E_n \colon K(\mathrm{RV}[\leq n])/I_{\mathrm{sp}} \longrightarrow !K(\mathrm{RES}[n])/[\mathbb{G}_m]_1.$$

These morphisms are compatible, thus passing to the limit one gets a morphism

$$(2.5.7) \qquad E \colon K(\mathrm{RV}[*])/I_{\mathrm{sp}} \longrightarrow !K(\mathrm{RES})/([\mathbb{A}^1] - 1).$$

In fact, the morphism E is induced from both the morphisms $\mathscr{E}$ and $\mathscr{E}'$ from (2) and (4) in Theorem 10.5 of [20].

The morphism E maps $[\mathrm{RV}^{>0}]_1$ to 0, and $[X]_k$ to $[X]_k$ for $X \in \mathrm{RES}[k]$. Composing E with the morphism $K(\mathrm{VF}) \to K(\mathrm{RV}[*])/I_{\mathrm{sp}}$ obtained by groupification of the morphism $\oint$ in (2.4.4) one gets a ring morphism

$$(2.5.8) \qquad \mathrm{EU}_\Gamma \colon K(\mathrm{VF}) \longrightarrow !K(\mathrm{RES})/([\mathbb{A}^1] - 1).$$

2.6. – The rest of this section is not really needed; it shows however that the introduction of Euler characteristics for Γ can be bypassed in the construction of EU_Γ.

Let $\mathrm{val} = \mathrm{val}_{\mathrm{rv}}$ denote the canonical map $\mathrm{RV} \to \Gamma$. Let I'_Γ be the ideal of $K(\mathrm{RV}[*])$ generated by all classes $[\mathrm{val}^{-1}(U)]_m$, for U a definable subset of Γ^m, $m \geq 1$, and let I_* be the ideal generated by I'_Γ along with I_{sp}. Since $[\mathrm{RV}^{>0}]_1 \in I'_\Gamma$, the canonical generator $[\mathrm{RV}^{>0}]_1 + [1]_0 - [1]_1$ reduces, modulo I'_Γ, to $[1]_0 - [1]_1$, i.e., the different dimensions are identified. Thus $K(\mathrm{RV}[*])/I_* = K(\mathrm{RV})/I_\Gamma$, where on the right we have the ideal of $K(\mathrm{RV})$ generated by all classes $[\mathrm{val}^{-1}(U)]$, for any definable $U \subset \Gamma^m$, $m \geq 1$, or equivalently just by $\mathrm{val}^{-1}(\{0\})$.

LEMMA 2.6.1. – *The inclusion functor* $\mathrm{RES} \to \mathrm{RV}$ *induces an isomorphism*

$$!K(\mathrm{RES})/([\mathbb{A}^1] - 1) \longrightarrow K(\mathrm{RV})/I_\Gamma.$$

Proof. – This is already true even at the semi-ring level, as follows from Proposition 10.2 of [20]. The elements $[\mathrm{val}^{-1}(U)]$ of $K_+(\mathrm{RV})$ are those of the form $1 \otimes b$ in the tensor product description, with $b \in K_+(\Gamma[n])$, $n \geq 1$. Moding out the tensor product $K_+(\mathrm{RES}) \otimes K_+(\Gamma[*])$ by these elements we obtain simply $K_+(\mathrm{RES}) \otimes K_+(\Gamma[0]) \simeq K_+(\mathrm{RES})$. Now taking into account the relations of the tensor product amalgamated over $K_+(\Gamma^{\mathrm{fin}})$, namely $1 \otimes [\gamma]_1 = [\mathrm{rv}^{-1}(\gamma)] \otimes [1]_0$, as the left-hand side vanishes, we obtain the relation $[\mathrm{rv}^{-1}(\gamma)] = 0$.

These are precisely the relations defining $!K(\text{RES})$ (namely $[\text{rv}^{-1}(\gamma)] = [\text{rv}^{-1}(\gamma')]$) along with the relation $\text{rv}^{-1}(0) = 0$ (i.e., $[\mathbb{A}^1] - 1 = 0$). $\qquad\qquad\square$

REMARK 2.6.2. – It is also easy to compute that the map

$$(2.6.1) \qquad\qquad \mathscr{E}: K(\text{RV}[*])/I_{\text{sp}} \longrightarrow !K(\text{RES})[[\mathbb{A}^1]^{-1}]$$

from [20], Theorem 10.5, composed with the natural map from $!K(\text{RES})[[\mathbb{A}^1]^{-1}]$ to $!K(\text{RES})/([\mathbb{A}^1] - 1)$, induces the retraction $K(\text{RV})/I_\Gamma \to !K(\text{RES})/([\mathbb{A}^1] - 1)$ above.

3. Invariant admissible transformations

In this section we introduce a notion of invariance for definable sets and functions. The main result is Proposition 3.2.2 which provides a refinement of Proposition 4.5 of [20] within the invariant context. This will be used in Proposition 4.2.2, in order to evaluate $\text{EU}_{\Gamma,m}$ on invariant sets in terms of reductions. We continue to work in $\text{ACVF}(0,0)$ over a base structure L_0 which is a nontrivially valued field.

3.1. – For $\alpha \in \Gamma(L_0)$, one sets $\mathcal{O}\alpha = \{x : \text{val}(x) \geq \alpha\}$, and $\mathcal{M}\alpha = \{x : \text{val}(x) > \alpha\}$. For $x = (x', x''), y = (y', y'') \in \text{VF}^n \times \text{RV}^m$, write $v(x - y) > \alpha$ if $x' - y' \in (\mathcal{M}\alpha)^n$. If f is a definable function on a definable subset X of $\text{VF}^n \times \text{RV}^m$, say f is α-invariant, resp. α^+-invariant, if $f(x + y) = f(x)$ whenever $x, x + y \in X$ and $y \in (\mathcal{O}\alpha)^n$, resp. $y \in (\mathcal{M}\alpha)^n$. Say a definable set Y is α-invariant, resp. α^+-invariant, if the characteristic function $1_Y: \text{VF}^n \times \text{RV}^m \to \{0, 1\}$ is α-invariant, resp. α^+-invariant.

Call a definable set of imaginaries *non-field* if it admits no definable map onto a non-empty open disk (over parameters). Any imaginary set of the form GL_n/H, where H is a definable subgroup of GL_n containing a valuative neighborhood of 1, has this property. By [18], ACVF admits elimination of imaginaries to the level of certain "geometric sorts"; these include the valued field K itself and certain other sorts of the form GL_n/H as above. We may thus restrict our attention to such sorts in the lemma below. Note that for a separable topological field L, $\text{GL}_n(L)$ is separable while $H(L)$ is an open subgroup, so $(\text{GL}_n/H)(L)$ is countable.

LEMMA 3.1.1. – *Let A be a set of imaginaries. Let $X \subset \text{VF}^n$ be an A-definable subset bounded and closed in the valuation topology. Let $f: X \to W$ be A-definable, where W is a non-field set of imaginaries. Fix α in $\Gamma(L_0)$. Then there exists a $\beta \geq \alpha$, a β^+-invariant A-definable map $g: X \to W$ such that for any $x \in X$, for some $y \in X$, $v(x - y) > \alpha$ and $g(x) = f(y)$.*

Proof. – We use induction on $\dim(X)$. If $\dim(X) = 0$, X is finite so we can take $f = g$, and β the maximum of α and the maximal valuative distance between two distinct points of X. So assume $\dim(X) > 0$.

Let us start by proving that there exists a relatively Zariski closed definable subset $Y \subset X$ such that $\dim(Y) < \dim(X)$ and such that f is locally constant on $X \setminus Y$. To do this, we work within the Zariski closure $\bar{X}$ of X which has same dimension as X. We use both the Zariski topology and the valuation topology on $\bar{X}$; when referring to the latter we use the prefix v. It follows from quantifier-elimination that any definable subset differs from a v-open set by a set contained in a subvariety of $\bar{X}$ of dimension $< \dim(X)$. In particular, a definable subset of $\bar{X}$ of dimension $\dim(X)$ must contain a non-empty v-open set. Now the locus Z

where f is locally constant is definable. Set $Y = X \setminus Z$. Assume by contradiction that Z does not contain a Zariski dense open subset of $\bar{X}$. Then its complement contains a non-empty v-open set e. Note that on every non-empty v-open definable subset of e, f is non-constant, since otherwise it would intersect Z. It follows that the following property holds:

($*$) the Zariski closure of $e \cap f^{-1}(w)$ is of dimension $< n$ for every w in W.

Thus, for any model of ACVF$(0,0)$, there exists $X' \subset \mathrm{VF}^n$ definable bounded and closed in the valuation topology, $f' \colon X' \to W'$ definable with W' a non-field set of imaginaries and a non-empty v-open definable subset e' such that ($*$) holds for X', W', f' and e'. By compactness, such X', W', f' and e' can each be defined uniformly, i.e., they each belong to a single definable family of definable sets. By compactness again, it follows that for p large enough there exist such X', W', f' and e' defined over the algebraic closure of $\mathbb{Q}_p$ such that ($*$) holds. Take a finite extension L of $\mathbb{Q}_p$ over which X', W', f' and e' are defined. As was observed above Lemma 3.1.1, $W'(L)$ is then a countable set. By ($*$), $f'^{-1}(w') \cap e'(L)$ is of measure zero, for each $w' \in W'(L)$. It follows that $e'(L)$ is of measure zero, a contradiction.

By the inductive hypothesis, there exist $\beta' \geq \alpha$, a β'^+-invariant function $g_Y \colon Y \to W$ such that for any $y \in Y$, for some $z \in Y$, $v(y - z) > \alpha$ and $g_Y(y) = f(z)$.

Let $Y' = \{ x \in X : (\exists y \in Y)(v(x - y) > \beta') \}$. One extends g_Y to a function g' on Y' by defining $g'(x) = g_Y(y)$ where y is an element of Y such that $v(x - y) > \beta'$. By the β'^+-invariance of g_Y, this is well-defined. Moreover, for any $y \in Y'$, there exists $z \in Y$ such that $v(y - z) > \alpha$ and $g'(y) = f(z)$.

For each x in $X \setminus Y$, we denote by $\delta(x)$ the valuative radius of the maximal open ball around x contained in $X \setminus Y$ on which f is constant. Since $X \setminus Y'$ is closed and bounded, δ is bounded on $X \setminus Y'$ by Lemma 11.6 of [20]. Thus, there exists $\beta \geq \beta'$ such that if $x, x' \in X \setminus Y'$ and $x - x' \in \mathcal{M}\beta$, then $f(x) = f(x')$. We now define g on X by $g(x) = g'(x)$ for $x \in Y'$, and $g(x) = f(x)$ for $x \in X \setminus Y'$. Note that if $x, x' \in X$ and $v(x - x') > \beta (\geq \beta')$, then either $x, x' \in Y'$ or $x, x' \in X \setminus Y'$; in both cases, $g(x) = g(x')$. We have already seen that the last condition in the statement of the lemma holds on Y'; it clearly holds for $x \in X \setminus Y'$, with $y = x$.   □

We repeat here Corollary 2.29 of [19].

LEMMA 3.1.2. – *Let D be a C-definable set in* ACVF *that may contain imaginary elements. Then the following are equivalent:*

(1) *There exists a definable surjective map $g \colon (\mathcal{O}/\mathcal{O}\beta)^n \to D$.*
(2) *There is no definable function $f \colon D \to \Gamma$ with unbounded image.*
(3) *For some $\beta_0 \leq 0 \leq \beta_1 \in \Gamma(C)$, for any $e \in D$, $e \in \mathrm{dcl}(C, \mathcal{O}\beta_0/\mathcal{M}\beta_1)$.*

A definable set D (of imaginary elements) satisfying (1-3) will be called *boundedly imaginary*. An infinite subset of the valued field can never be boundedly imaginary; a subset of the value group, or of Γ^n, is boundedly imaginary iff it is bounded; a subset of RV^n is boundedly imaginary iff its image in Γ^n under the valuation map is bounded (i.e., contained in a box $[-\gamma, \gamma]^n$). We shall say a subset of RV^n is bounded below if its image in Γ^n under the valuation map is bounded below (i.e., contained in a box $[\gamma, \infty)^n$).

LEMMA 3.1.3. – *Let T be a boundedly imaginary definable set. Let $X \subset \mathrm{VF}^n \times T$, and, for $t \in T$, set $X_t = \{x : (x,t) \in X\}$. Assume each X_t is bounded and closed in the valuation topology. Let W be a non-field set of imaginaries and let $f : X \to W$ be a definable map. Fix α in $\Gamma(L_0)$. Then there exist $\beta \geq \alpha$, a β^+-invariant definable function $g : X \to W$ such that for any $t \in T$ and $x \in X_t$, there exists $y \in X_t$, $v(x - y) > \alpha$ and $g(x,t) = f(y,t)$.*

Proof. – For each t we obtain, from Lemma 3.1.1, an $A(t)$-definable element $\beta(t) \geq \alpha$, and a $\beta(t)^+$-invariant $g_t : X_t \to W$, with the stated property. As T is boundedly imaginary, $\beta(t)$ is bounded on T and $\beta = \sup_t \beta(t) \in \Gamma$. For each t, the statement remains true with $\beta(t)$ replaced by β. By the usual compactness / glueing argument, as explained for instance in Section 2.1.1 of [20], we may take g_t to be uniformly definable, i.e., $g_t(x) = g(x,t)$. $\square$

3.2. – We now define an invariant analogue of the admissible transformations of [20], Definition 4.1.

Let $n \geq 1$ an integer and let $\beta = (\beta_1, \ldots, \beta_n) \in \Gamma^n$. Let $\mathrm{VF}^n / \mathcal{O}\beta = \prod_{1 \leq i \leq n}(\mathrm{VF}/\mathcal{O}\beta_i)$, and let $\pi = \pi_\beta : \mathrm{VF}^n \to \mathrm{VF}^n / \mathcal{O}\beta$ be the natural map. Also write $\pi(x, y) = (\pi(x), y)$ if $x \in \mathrm{VF}^n$ and $y \in \mathrm{RV}^m$. Say $X \subseteq \mathrm{VF}^n \times \mathrm{RV}^m$ is β-invariant if it is a pullback via π_β; and that $f : \mathrm{VF}^n \times \mathrm{RV}^* \to \mathrm{VF}$ is (β, α)-covariant if it induces a map $\mathrm{VF}^n / \mathcal{O}\beta \times \mathrm{RV}^* \to \mathrm{VF}/\mathcal{O}\alpha$, via (π_β, π_α).

DEFINITION 3.2.1. – *Let A be a base structure. Let $n \geq 1$ be an integer and let $\beta = (\beta_1, \ldots, \beta_n) \in \Gamma^n$.*

(1) *An elementary β-invariant admissible transformation over A is a function of one of the following types:*
   (i) *a function $\mathrm{VF}^n \times \mathrm{RV}^m \to \mathrm{VF}^n \times \mathrm{RV}^m$ of the form*

$$(x_1, \ldots, x_n, y_1, \ldots, y_m) \longmapsto (x_1, \ldots, x_{i-1}, x_i + a, x_{i+1}, \ldots, x_n, y_1, \ldots, y_m)$$

   with $a = a(x_1, \ldots, x_{i-1}, y_1, \ldots, y_l)$ a (β, β_i)-covariant A-definable function and $m \geq 0$ an integer.
   (ii) *a function $\mathrm{VF}^n \times \mathrm{RV}^m \to \mathrm{VF}^n \times \mathrm{RV}^{m+1}$ of the form*

$$(x_1, \ldots, x_n, y_1, \ldots, y_l) \longmapsto (x_1, \ldots, x_n, y_1, \ldots, y_l, h(x_i))$$

   with h an A-definable β_i-invariant function $\mathrm{VF} \to \mathrm{RV}$ and $m \geq 0$ an integer.
(2) *Let m and m' be non negative integers. A function $\mathrm{VF}^n \times \mathrm{RV}^m \to \mathrm{VF}^n \times \mathrm{RV}^{m'}$ is called β-invariant admissible transformation over A if it is the composition of elementary β-invariant admissible transformations over A.*
(3) *Let $\mathscr{C}'_A(\beta)$ be the category whose objects are triples (m, W, X) with $m \geq 0$ an integer, W a boundedly imaginary definable set contained in RV^m and X a definable subset of $\mathrm{VF}^n \times W$ such that X_w is a bounded, β-invariant subset of VF^n, for every $w \in W$. We shall sometimes write X instead of (m, W, X). A morphism $(m, W, X) \to (m', W', X')$ in $\mathscr{C}'_A(\beta)$ is a definable map $X \to X'$ which is the restriction of some β-invariant admissible transformation $\mathrm{VF}^n \times \mathrm{RV}^m \to \mathrm{VF}^n \times \mathrm{RV}^{m'}$. We consider the full subcategory $\mathscr{C}_A(\beta)$ whose objects X satisfy the additional condition that the projection $X \to \mathrm{VF}^n$ has finite fibers.*

(4) *Let (m, W, X) be in $\mathscr{C}'_A(\beta)$. We say X is elementary if there exist an integer $m' \geq 0$, a β-invariant admissible transformation $T : \mathrm{VF}^n \times \mathrm{RV}^m \to \mathrm{VF}^n \times \mathrm{RV}^{m'}$, a definable subset H of $\mathrm{RV}^{m'}$, and a map $h \colon \{1, \ldots, n\} \to \{1, \ldots, m'\}$ such that*

$$T(X) = \{(a, b) \in \mathrm{VF}^n \times H; \mathrm{rv}(a_i) = b_{h(i)}, \text{for } 1 \leq i \leq n\}.$$

If $\beta = (\beta_1, \ldots, \beta_n)$ and $\beta' = (\beta'_1, \ldots, \beta'_n)$ are in Γ^n, we write $\beta \geq \beta'$ if $\beta_i \geq \beta'_i$ for every $1 \leq i \leq n$. If $\beta \geq \beta'$, we have a natural embedding of $\mathscr{C}'_A(\beta)$ as a (non-full) subcategory of $\mathscr{C}'_A(\beta')$. We denote by $\mathscr{C}'_A$, resp. $\mathscr{C}_A$, the direct limit over all β of the categories $\mathscr{C}'_A(\beta)$, resp. $\mathscr{C}_A(\beta)$.

The following proposition is an analogue of Proposition 4.5 of [20] in the category $\mathscr{C}_A$.

PROPOSITION 3.2.2. – *Let F be a subset of a model of $\mathrm{ACVF}(0, 0)$, in any finite product of sorts, such that for each $\gamma \in \Gamma(F)$, there exists $f \in \mathrm{VF}(F)$ such that $\mathrm{val}(f) > \gamma$. We work in ACVF_F. Let $\alpha \in \Gamma^n$ and let (ℓ, W, X) be an object in $\mathscr{C}_F(\alpha)$. There exists $\beta \geq \alpha$ such that X is a Boolean combination of finitely many β-invariant definable subsets Z which are elementary in the sense of Definition 3.2.1 (4). Furthermore, if the projection $X \to \mathrm{VF}^n$ has finite fibers, one may assume that for each such Z, the projection $H \to \mathrm{RV}^n$ given by $b \mapsto (b_{h(1)}, \ldots, b_{h(n)})$ has finite fibers.*

Proof. – Note that the hypothesis on the base set F is preserved if we move from F to $F(w)$, where w lies in a boundedly imaginary definable set. This permits the inductive argument below to work.

We now explain how to adapt the proof of Proposition 4.5 of [20] to the present setting. We add the hypothesis that X is invariant and want to obtain the conclusion that Z is invariant. The proof will be essentially the same except that we have to pay attention that certain sets are boundedly imaginary. We first adapt Lemma 4.2 of [20]. In that lemma, if X is α^+-invariant, the proof gives α^+-invariant sets Z_i and transformations T_i. As stated there, the RV sets $H_i \subset \mathrm{RV}^{\ell_i}_\infty$ are bounded below, since the assumption made on X implies that $X \times W \subset B \times W$, for some bounded $B \subset \mathrm{VF}^n$. However we need to modify the proof there in order to obtain boundedly imaginary sets. This occurs where X is a ball around 0, namely in cases 1 and 2 in the proof of Lemma 4.2 of [20]. In these cases choose a definable $f \in \mathrm{VF}$ such that $\mathrm{val}(f)$ is bigger than the radius of X. Let Y be an open ball around 0 of radius $\mathrm{val}(f)$. Then $X \setminus Y$ is the pullback from RV of a boundedly imaginary set. As for Y we may move it to $f + Y$, which is the pullback from RV of a single element. It is at this point that we require Boolean combinations instead of unions.

Next, let us adapt the argument in the proof of Proposition 4.5 of [20]. Given a definable map $\pi \colon X \to U$, with U a definable subset of $\mathrm{VF}^{n-1} \times V$ with V a boundedly imaginary definable set contained in RV^ℓ, such that U_v is a bounded subset of VF^{n-1}, for every $v \in V$, such that X, U and π are all α^+-invariant, we obtain a partition and transformations of X over U, such that each fiber becomes an RV-pullback, and each piece of each fiber is α^+-invariant. Note that the fiber above u depends only on $u + (\mathscr{M}\alpha)^{n-1}$. Note also that U, being α^+-invariant, is clopen in the valuation topology. Using Lemma 3.1.3, we may modify the partition and the admissible transformations so as to be β^+-invariant, for some $\beta \geq \alpha$. With this, the inductive proof of [20], Proposition 4.5 goes through to give the invariant result. $\qquad\square$

4. Working over $F((t))$

4.1. – We now work over the base field $L_0 = F((t))$, with F a trivially valued algebraically closed field of characteristic zero and $\mathrm{val}(t)$ positive and denoted by 1. Then the sorts of RES are the **k**-vector spaces $V_\gamma = \{x \in \mathrm{RV} : \mathrm{val}_{\mathrm{rv}}(x) = \gamma\} \cup \{0\}$, for $\gamma \in \mathbb{Q}$. Let $k \in \mathbb{Z}$ and let m be a positive integer. Since we have a definable bijection $V_{k/m} \to V_{(k+m)/m}$ given by multiplication by $\mathrm{rv}(t)$, it suffices to consider $V_{k/m}$ with $0 \leq k < m$ and m a positive integer.

The Galois group of $F((t))^{\mathrm{alg}}/F((t))$ may be identified with the group $\hat{\mu} = \varprojlim \mu_n$ of roots of unity and it acts on RES by automorphisms. On $V_{k/n}$, a primitive n-th root of 1, say ζ, acts by multiplication by ζ^k. We have an induced action on $K(\mathrm{RES})$. The classes $[V_{k/n}]$ are fixed by this action; and so an action is induced on $!K(\mathrm{RES})$.

Given a positive integer m, let $\mathrm{RES}_{m^{-1}\mathbb{Z}}$ denote the sorts of RES fixed by $\hat{\mu}^m$, the kernel of $\hat{\mu} \to \mu_m$, namely, $V_{k/m}$ for $k \in \mathbb{Z}$.

Projection on $\mathrm{RES}_{m^{-1}\mathbb{Z}}$ provides a canonical morphism

$$(4.1.1) \qquad \Delta_m : K_+(\mathrm{RES}) \longrightarrow K_+(\mathrm{RES}_{m^{-1}\mathbb{Z}})$$

inducing

$$(4.1.2) \qquad \Delta_m : !K(\mathrm{RES})/([\mathbb{A}^1]-1) \longrightarrow !K(\mathrm{RES}_{m^{-1}\mathbb{Z}})/([\mathbb{A}^1]-1),$$

where $!K(\mathrm{RES}_{m^{-1}\mathbb{Z}})$ is defined similarly as was $!K(\mathrm{RES})$ in 2.5. One denotes by $\mathrm{EU}_{\Gamma,m}$ the morphism

$$(4.1.3) \qquad \mathrm{EU}_{\Gamma,m} : K(\mathrm{VF}) \longrightarrow !K(\mathrm{RES}_{m^{-1}\mathbb{Z}})/([\mathbb{A}^1]-1)$$

obtained by composing EU_Γ in (2.5.8) and Δ_m in (4.1.2).

The following statement is straightforward:

LEMMA 4.1.1. – *Let r and n be integers, let X be a definable subset of VF^r, let Y be a definable subset of RES^n. Assume that $\mathrm{EU}_\Gamma([X]) = [Y]$. Then, for any positive integer m, $\mathrm{EU}_{\Gamma,m}([X])$ is the class of the subset of Y fixed by $\hat{\mu}^m$.* $\qquad\square$

4.2. – Inside a given algebraic closure of $F((t))$, the field $K_m = F((t^{1/m}))$ does not depend on a particular choice of $t^{1/m}$, and μ_m acts on it. Let $\beta \in \frac{1}{m}\mathbb{Z}^n \subset \Gamma^n$, and let $X \subset \mathrm{VF}^n \times \mathrm{RV}^\ell$ be a β-invariant K-definable set such that the projection $X \to \mathrm{VF}^n$ has finite fibers. We assume X is contained in $\mathrm{VF}^n \times W$ with W a boundedly imaginary definable subset of in RV^ℓ, and that, for every $w \in W$, X_w is bounded. Thus, X_w is β-invariant for each w in RV^ℓ, the projection of X to Γ^ℓ is contained in a cube $[-\alpha, \alpha]^\ell$, and the projection of X to VF^n is contained in $c\mathscr{O}^n$ for some c. For notational simplicity, and since this is what we will use, we shall assume $X \subset \mathscr{O}^n \times \mathrm{RV}^\ell$.

Then $X(K_m)$ is the pullback of some subset $X[m; \beta] \subseteq \Pi_{i=1}^n F[t^{1/m}]/t^{\beta_i} \times \mathrm{RV}^\ell$; and the projection $X[m; \beta] \to \mathscr{O}^n$ has finite fibers.

We can identify $F[t^{1/m}]/t^N$ with $\bigoplus_{0 \leq k < mN} V_{k/m} \cong \bigoplus_{0 \leq k < m} V_{k/m}^N$. Also, if Y is definable in RV and $\mathrm{val}_{\mathrm{rv}}(Y) \subset [-\alpha, \alpha]$, then

$$(4.2.1) \qquad Y(F((t^{1/m}))) \subset \cup\{V_\gamma : \gamma \in m^{-1}\mathbb{Z} \cap [-\alpha, \alpha]\}.$$

Thus $X[m; \beta]$ can be viewed as a subset of the structure $\mathrm{RES}_{m^{-1}\mathbb{Z}}$ (over F). Here are three ways to see it is definable. The first one is to say it is definable in $(F((t^{1/m})), t)$; the induced

structure on the sorts $V_{k/m}$ is the same as the structure induced from ACVF. The second one is to remark that after finitely many invariant admissible transformations, X becomes a set in standard form, a pullback from RV. These operations induce quantifier-free definable maps on the sets $X[m; \beta]$; so it suffices to take X in standard form, and then the statement is clear. Thirdly, in the structure $F((t))^{\text{alg}}$ with a distinguished predicate for F, it is clear that $F((t^{1/m}))$ is definable and so $X(F((t^{1/m})))$ is definable; and here too (cf. [22], Lemma 6.3) the induced structure on F is just the field structure, and the induced structure on the sorts $V_{k/m}$ is the same as the structure induced from ACVF.

LEMMA 4.2.1. – *Let X be as above and let β' in $\frac{1}{m}\mathbb{Z}^n$, with $\beta_i \leq \beta'_i$, for every $1 \leq i \leq n$. Then $[X[m; \beta']] = [X[m; \beta]] \times [\mathbb{A}^{m \sum_i (\beta'_i - \beta_i)}]$ in $K(\text{RES}_{m^{-1}\mathbb{Z}})$.*

Proof. – We shall assume β differs from β' in one coordinate, say the first, and that $\beta'_1 = \beta_1 + \frac{1}{m}$. Consider the projection $X[m, \beta'] \to X[m, \beta]$. Working over a parameter $t^{1/m}$, this is a morphism of ACF-constructible sets, whose fibers are $\mathbb{A}^1(\mathbf{k})$-torsors; so by Hilbert 90, there exists a constructible section. Now this section may not be μ_m-invariant, but after averaging the μ_m-conjugates one finds a μ_m-invariant section, which is $F((t))$-definable. It follows that $X[m; \beta'] = X[m; \beta] \times \mathbb{A}^1$, as required. □

Thus the class of $[X[m; \beta]]/[\mathbb{A}^{m(\sum_i \beta_i) - n}]$ in the localization $K(\text{RES}_{m^{-1}\mathbb{Z}})[[\mathbb{A}^1]^{-1}]$ does not depend on β; let us denote it by $\widetilde{X}[m]$.

We also denote by $X[m]$ the image of $[X[m; \beta]]$ in $!K(\text{RES}_{m^{-1}\mathbb{Z}})/([\mathbb{A}^1] - 1)$, or in $!K(\text{RES})/([\mathbb{A}^1] - 1)$, which does not depend on β.

Let X be as before and let $f: X \to Y$ be a β-invariant admissible bijection in $\mathscr{C}(\beta)$. Since f induces a bijection between $X[m; \beta]$ and $Y[m; \beta]$, it follows that $\widetilde{X}[m] = \widetilde{Y}[m]$ and $X[m] = Y[m]$.

PROPOSITION 4.2.2. – *Let X be a β-invariant $F((t))$-definable subset of $\mathcal{O}^n \times \text{RV}^\ell$, for some β. Assume the projection $X \to \text{VF}^n$ has finite fibers. Then, for every $m \geq 1$, $\text{EU}_{\Gamma,m}(X) = X[m]$ as classes in $!K(\text{RES}_{m^{-1}\mathbb{Z}})/([\mathbb{A}^1] - 1)$.*

Proof. – Both sides being invariant under the transformations of Definition 3.2.1 and Boolean combinations, we may assume by Proposition 3.2.2 that there exist a definable boundedly imaginary subset H of $\text{RV}^{\ell'}$ and a map $h: \{1, \ldots, n\} \to \{1, \ldots, \ell'\}$ such that

(4.2.2) $$X = \{(a, b); b \in H, \text{rv}(a_i) = b_{h(i)}, 1 \leq i \leq n\}$$

and the map $r: H \to \text{RV}^n$ given by $b \mapsto (b_{h(1)}, \ldots, b_{h(n)})$ is finite to one. According to (2.3.6) we may assume that the class $[(H, r)]$ is equal to $\Psi([W] \otimes [\Delta])$ with W in $\text{RES}[\ell]$ and Δ bounded in $\Gamma[n - \ell]$. By induction on dimension and considering products, it is enough to prove the result when X is the lifting of an object of Γ or RES. Let us prove that the image of the canonical lift from Γ vanishes for both invariants. In the case of $\text{EU}_{\Gamma,m}$, the lift of any $Z \subset \Gamma^q$, $q \geq 1$, to $K(\text{RV})$ vanishes modulo $[\mathbb{A}^1] - 1$. In the case of $X[m]$, finitely many points of the value group of K_m in the cube $[0, N]^n$ lie in Z; again for each such point, the class of $!K(\text{RES})$ lying above it is divisible by $[\mathbb{A}^1] - 1$. On the other hand on RES, both $\text{EU}_{\Gamma,m}$ and $X[m]$ correspond to intersection with $\text{RES}_{m^{-1}\mathbb{Z}}$. □

COROLLARY 4.2.3. – *Let X be a smooth variety over F, f a regular function on X and x a closed point of $f^{-1}(0)$. Set $\mathcal{O} = F[\![t]\!]$ and let π denote the reduction map $X(\mathcal{O}) \to X(F)$. Let*

$$X_{t,x} = \{y \in X(\mathcal{O}); f(y) = t \text{ and } \pi(y) = x\}$$

and let

$$\mathcal{X}_x = \{y \in X(\mathcal{O}); \mathrm{rv}f(y) = \mathrm{rv}(t) \text{ and } \pi(y) = x\}.$$

Then $\mathcal{X}_x$ is β-invariant for every $\beta > 0$, and, for every $m \geq 1$, $\mathrm{EU}_{\Gamma,m}(X_{t,x}) = \mathcal{X}_x[m]$ as classes in $!K(\mathrm{RES})/([\mathbb{A}^1] - 1)$.

Proof. – The β-invariance of $\mathcal{X}_x$ is clear. Consider the canonical morphism

$$\oint : K_+(\mathrm{VF}) \longrightarrow K_+(\mathrm{RV}[*])/I_{\mathrm{sp}}$$

of (2.4.4). For any t' with $\mathrm{rv}(t') = \mathrm{rv}(t)$, there is an automorphism over F fixing RV that sends t to t', thus

$$\oint [X_{t',x}] = \oint [X_{t,x}].$$

It follows that

$$(4.2.3) \qquad \oint([\mathcal{X}_x]) = \oint \left[\bigcup_{t'} \{X_{t',x} : \mathrm{rv}(t') = \mathrm{rv}(t)\} \right] = \left(\oint [X_{t,x}] \right) \cdot e,$$

where e is the class of an open ball, i.e., $e = [1]_1$. Applying EU_Γ we find that $\mathrm{EU}_\Gamma(\mathcal{X}_x) = \mathrm{EU}_\Gamma(X_{t,x})$, and the statement follows from Proposition 4.2.2. □

4.3. – We say a $\hat{\mu}$-action is *good* if it factorizes through some μ_n-action, for some $n \geq 1$. We denote by $K_+^{\flat,\hat{\mu}}(\mathrm{Var}_F)$ the quotient of the abelian monoid generated by isomorphism classes of quasi-projective varieties over F with a good $\hat{\mu}$-action by the standard cut and paste relations. We denote by $K_+^{\hat{\mu}}(\mathrm{Var}_F)$ the Grothendieck semi-ring of F-varieties with $\hat{\mu}$-action as considered in [12] and [27]. It is the quotient of $K_+^{\flat,\hat{\mu}}(\mathrm{Var}_F)$ by the following additional relations: for every quasi-projective F-variety X with good $\hat{\mu}$-action, for every finite dimensional F-vector space V endowed with two good linear actions ϱ and ϱ', the class of $X \times (V, \varrho)$ is equal to the class of $X \times (V, \varrho')$. We denote by $K^{\hat{\mu}}(\mathrm{Var}_F)$ the corresponding Grothendieck ring.

For any $s \in \mathbb{Q}_{>0}$, let $t_s \in F(\!(t)\!)^{\mathrm{alg}}$ such that $t_1 = t$ and $t_{as} = t_s^a$ for any s and any $a \geq 1$. Set $\mathbf{t}_{k/m} = \mathrm{rv}(t_{k/m}) \in V_{k/m}$.

Let X be an $F(\!(t)\!)$-definable set in the generalized residue structure RES. Thus, for some $n \geq 0$, X is an $F(\!(t)\!)$-definable subset of RV^n whose image in Γ^n under $\mathrm{val}_{\mathrm{rv}}$ is finite. When the image is a single point, there exist a positive integer m and integers k_i, $1 \leq i \leq n$, such that X is an $F(\!(t)\!)$-definable subset of $\prod_{1 \leq i \leq n} V_{k_i/m}$. The $\hat{\mu}$-action on X factors through a μ_m-action. The image $\Theta(X)$ of the set X by the $F(\!(t^{1/m})\!)$-definable function

$$(4.3.1) \qquad g(x_1, \ldots, x_n) = (x_1/\mathbf{t}_{k_1/m}, \ldots, x_n/\mathbf{t}_{k_n/m})$$

is an F-definable subset of $\mathbf{k}^n$ which is endowed with a μ_m-action coming from the one on X. In general, the set X is a disjoint union of definable subsets X_j of the previous type. Since

an F-definable subset of $\mathbf{k}^n$ is nothing but a constructible subset of $\mathbb{A}_F^n$, there is a unique morphism of semi-rings

$$(4.3.2) \qquad \Theta \colon K_+(\mathrm{RES}) \longrightarrow K_+^{\flat,\hat{\mu}}(\mathrm{Var}_F)$$

such that, for every $F((t))$-definable set X in the structure RES of the form $X = \cup X_j$ with $X_j \subset \prod_{1 \le i \le n} V_{k_i/m}$,

$$(4.3.3) \qquad \Theta([X]) = \sum_j [\Theta(X_j)].$$

One derives from (4.3.2) a ring morphism

$$(4.3.4) \qquad \Theta \colon !K(\mathrm{RES}) \longrightarrow K^{\hat{\mu}}(\mathrm{Var}_F).$$

We shall also consider the morphism

$$(4.3.5) \qquad \Theta_0 \colon !K(\mathrm{RES}) \longrightarrow K(\mathrm{Var}_F)$$

obtained by composing the morphism (4.3.4) with the morphism $K^{\hat{\mu}}(\mathrm{Var}_F) \to K(\mathrm{Var}_F)$ induced by forgetting the $\hat{\mu}$-action.

PROPOSITION 4.3.1. – *The morphisms (4.3.2) and (4.3.4) are isomorphisms.*

Proof. – Let us prove that (4.3.2) is injective. Let X and X' be respectively $F((t))$-definable subsets of RV^n and $\mathrm{RV}^{n'}$ whose respective images in Γ^n and $\Gamma^{n'}$ under $\mathrm{val}_{\mathrm{rv}}$ are a single point and choose a positive integer m such that the $\hat{\mu}$-action on X and X' factors through a μ_m-action. Consider the $F((t^{1/m}))$-definable functions g and g' associated respectively to X and X' as in (4.3.1). Let f be a $\hat{\mu}_m$-invariant isomorphism between $g'(X')$ and $g(X)$. Then $g^{-1} \circ f \circ g' \colon X' \to X$ is an $F((t^{1/m}))$-definable bijection $X' \to X$, which moreover is invariant under the Galois group of $F((t^{1/m}))/F((t))$ hence is an $F((t))$-definable bijection. In general, when the images of X and X' under $\mathrm{val}_{\mathrm{rv}}$ are only supposed to be finite, if $\Theta([X]) = \Theta([X'])$, one can write X and X' as a disjoint union of definable subsets X_j and X'_j of the previous type, $1 \le j \le r$, such that for all j, $\Theta(X_j) = \Theta(X'_j)$, and injectivity of (4.3.2) follows.

For surjectivity, by induction on dimension, it is enough to prove that, for $m \ge 1$, if V is an irreducible quasi-projective variety over F endowed with a μ_m-action, then there exists an $F((t))$-definable set W over RES such that $\Theta(W)$ is a dense subset of V. We may assume, by partitioning, that the kernel of the action is constant, so that the action is equivalent to an effective $\mu_{m'}$-action for some $m' \mid m$, and for notational simplicity we take $m = m'$. Set $U = V/\mu_m$. By Kummer theory there exists $f \in F(U)$ such that $F(V) = F(U)(f^{1/m})$. Up to shrinking V, we may assume f is regular and does not vanish on U. It follows that V is isomorphic to the closed set $V^* = \{(u,z) \in U \times \mathbb{G}_m; f(u) = z^m\}$, with μ_m-action the trivial action on the U-factor and the standard one on the $\mathbb{G}_m$-factor. If one sets $W = \{(u,z) \in U \times V_{1/m}; f(u) = tz^m\}$, one gets that $\Theta(W) = V$.

Since any linear μ_m-action on $\mathbb{A}_F^n$ is diagonalizable, the relations involved in dropping the "flat" from (4.3.2) to (4.3.4) are just those implicit in adding the ! on the left-hand side. So the bijectivity of (4.3.4) follows from the one of (4.3.2). $\qquad\qquad \square$

5. Étale Euler characteristics with compact supports

5.1. Étale cohomology with compact supports of semi-algebraic sets

Let K be a complete non-archimedean normed field. Let X be an algebraic variety over K and write X^{an} for its analytification in the sense of Berkovich. Assume now X is affine. A semi-algebraic subset of X^{an}, in the sense of [16], is a subset of X^{an} defined by a finite Boolean combination of inequalities $|f| \leq \lambda|g|$ with f and g regular functions on X and $\lambda \in \mathbb{R}$.

We denote by $\overline{K}$ the completion of a separable closure of K and by G the Galois group $\mathrm{Gal}(\overline{K}/K)$. We set $\overline{X^{\mathrm{an}}} = X^{\mathrm{an}}\widehat{\otimes}\overline{K}$ and for U a semi-algebraic subset of X^{an} we denote by $\overline{U}$ the preimage of U in $\overline{X^{\mathrm{an}}}$ under the canonical morphism $\overline{X^{\mathrm{an}}} \to X^{\mathrm{an}}$. Let ℓ be a prime number different from the residue characteristic of K.

Let U be a locally closed semi-algebraic subset of X^{an}. For any finite torsion ring R, the theory of germs in [3] provides étale cohomology groups with compact supports $H^i_c(\overline{U}, R)$ which coincide with the ones defined there when U is an affinoid domain of X^{an}. These groups are also endowed with an action of the Galois group G. We shall set $H^i_c(\overline{U}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim H^i_c(\overline{U}, \mathbb{Z}/\ell^n)$.

We shall use the following properties of the functor $U \mapsto H^i_c(\overline{U}, \mathbb{Q}_\ell)$ which are proved by F. Martin in [29]:

THEOREM 5.1.1. – *Let X be an affine algebraic variety over K of dimension d. Let U be a locally closed semi-algebraic subset of X^{an}.*

(1) *The groups $H^i_c(\overline{U}, \mathbb{Q}_\ell)$ are finite dimension $\mathbb{Q}_\ell$-vector spaces, endowed with a G-action, and $H^i_c(\overline{U}, \mathbb{Q}_\ell) = 0$ for $i > 2d$.*

(2) *If V is a semi-algebraic subset of U which is open in U with complement $F = U \setminus V$, there is a long exact sequence*

$$(5.1.1) \qquad \longrightarrow H^{i-1}_c(\overline{F}, \mathbb{Q}_\ell) \longrightarrow H^i_c(\overline{V}, \mathbb{Q}_\ell) \longrightarrow H^i_c(\overline{U}, \mathbb{Q}_\ell) \longrightarrow H^i_c(\overline{F}, \mathbb{Q}_\ell) \longrightarrow .$$

(3) *Let Y be an affine algebraic variety over K and let V be a locally closed semi-algebraic subset of Y^{an}. There are canonical Künneth isomorphisms*

$$(5.1.2) \qquad \bigoplus_{i+j=n} H^i_c(\overline{U}, \mathbb{Q}_\ell) \otimes H^j_c(\overline{V}, \mathbb{Q}_\ell) \simeq H^n_c(\overline{U \times V}, \mathbb{Q}_\ell).$$

REMARK 5.1.2. – We shall only make use of Theorem 5.1.1 when $X = \mathbb{A}^n$ and $K = F((t))$ with F a field of characteristic zero. Note also that, though in subsequent arXiv versions of [29] the proof of Theorem 5.1.1 relies on Theorem 1.1 of [6] which uses de Jong's results on alterations and Gabber's weak uniformization theorem, the first version is based on Corollary 5.5 of [4], which does not use any of these results.

5.2. Definition of $\mathrm{EU}_{\acute{e}t}$

We denote by $G\text{-Mod}$ the category of $\mathbb{Q}_\ell[G]$-modules that are finite dimensional as $\mathbb{Q}_\ell$-vector spaces and by $K(G\text{-Mod})$ the corresponding Grothendieck ring. Let K be a valued field endowed with a rank one valuation, that is, with $\Gamma(K) \subset \mathbb{R}$. We can consider the norm $\exp(-\mathrm{val})$ on K. Let U be an ACVF_K-definable subset of VF^n. By quantifier elimination it is defined by a finite Boolean combination of inequalities $\mathrm{val}(f) \geq \mathrm{val}(g) + \alpha$ where f and g are polynomials and α in $\Gamma(K) \otimes \mathbb{Q}$. Thus, after exponentiating, one attaches canonically to U the semi-algebraic subset U^{an} of $(\mathbb{A}^n_{\widehat{K}})^{\mathrm{an}}$ defined by the corresponding inequalities, with $\widehat{K}$ the completion of K, and also a semi-algebraic subset $\overline{U^{\mathrm{an}}}$ of $\mathbb{A}^n_{\overline{K}}$. When U^{an} is locally closed, we define $\mathrm{EU}_{\acute{e}t}(U)$ as the class of

$$(5.2.1) \qquad \sum_i (-1)^i [H^i_c(\overline{U^{\mathrm{an}}}, \mathbb{Q}_\ell)]$$

in $K(G\text{-Mod})$. It follows from (1) of Theorem 5.1.1 that this is well-defined.

LEMMA 5.2.1. – *Let U be an ACVF_K-definable subset of VF^n. Then there exists a finite partition of U into ACVF_K-definable subsets U_i such that each U_i^{an} is locally closed.*

Proof. – The set U is the union of sets U_i defined by conjunctions of formulas of the form $\mathrm{val}(f) < \mathrm{val}(g)$, $f = 0$, or $\mathrm{val}(f) = \mathrm{val}(g)$, with f and g polynomials. Since the intersection and intersection of two locally closed sets are locally closed, it suffices to show that each of these basic subsets are locally closed. Since $|f|$ and $|g|$ are continuous functions for the Berkovich topology with values in $\mathbb{R}_{\geq 0}$, the sets defined by $f = 0$ and $\mathrm{val}(f) = \mathrm{val}(g)$ are closed, as well as $\mathrm{val}(f) \leq \mathrm{val}(g)$. The remaining kind of set, $\mathrm{val}(f) < \mathrm{val}(g)$, is the difference between $\mathrm{val}(f) \leq \mathrm{val}(g)$ and $\mathrm{val}(f) = \mathrm{val}(g)$, hence is locally closed. $\square$

PROPOSITION 5.2.2. – *There exists a unique ring morphism*

$$(5.2.2) \qquad \mathrm{EU}_{\acute{e}t} \colon K(\mathrm{VF}) \longrightarrow K(G\text{-Mod})$$

such that $\mathrm{EU}_{\acute{e}t}([U]) = \mathrm{EU}_{\acute{e}t}(U)$ when U is an ACVF_K-definable subset of VF^n such that U^{an} is locally closed.

Proof. – Let U be an ACVF_K-definable subset of VF^n. Choose a partition of U into ACVF_K-definable subsets U_i, $1 \leq i \leq r$, such that each U_i^{an} is locally closed. If U^{an} is locally closed, it follows from (2) in Theorem 5.1.1, using induction on r, that $\mathrm{EU}_{\acute{e}t}(U) = \sum_i \mathrm{EU}_{\acute{e}t}(U_i)$. For general U, set $\mathrm{EU}_{\acute{e}t}(U) = \sum_i \mathrm{EU}_{\acute{e}t}(U_i)$. This is independent of the choice of the partition U_i. Indeed, if (U'_j) is a finer such partition with $(U'_j)^{\mathrm{an}}$ locally closed, then $\sum_i \mathrm{EU}_{\acute{e}t}(U_i) = \sum_j \mathrm{EU}_{\acute{e}t}(U'_j)$ by the previous remark, and two such partitions always have a common refinement. Note that $\mathrm{EU}_{\acute{e}t}(U)$ depends only on the isomorphism class of U as a definable set. Indeed, when f is a polynomial isomorphism $f \colon U \to U'$ (with inverse given by a polynomial function), and U^{an} and $(U')^{\mathrm{an}}$ are locally closed, this is clear by functoriality of $H^\bullet_c$, and in general one can reduce to this case by taking suitable partitions U_i and U'_i of U and U'. Thus, if now U is an ACVF_K-definable subset of X, with X an affine variety over K, if i is some embedding of X in an affine space $\mathbb{A}^n$, $\mathrm{EU}_{\acute{e}t}(i(U))$ will not depend on i, so one may set $\mathrm{EU}_{\acute{e}t}(U) = \mathrm{EU}_{\acute{e}t}(i(U))$. Note that, by definition, $K(\mathrm{VF})$ is generated by classes of ACVF_K-definable subsets of affine algebraic

varieties over K. Furthermore, by (2) in Theorem 5.1.1, $\mathrm{EU}_{\text{ét}}$ satisfies the additivity relation, thus existence and uniqueness of an additive map $\mathrm{EU}_{\text{ét}}\colon K(\mathrm{VF}) \to K(G\text{-Mod})$ with the required property follows. Its multiplicativity is a consequence from Property (3) in Theorem 5.1.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

5.3. Definition of $\mathrm{eu}_{\text{ét}}$

We now assume for the rest of this section that $K = F((t))$ with F algebraically closed of characteristic zero. Thus the Galois group G may be identified with $\hat{\mu}$ in the standard way, namely to an element $\sigma \in G$ corresponds the unique element $\zeta = (\zeta_n)_{n\geq 1} \in \hat{\mu}$ such that, for any $n \geq 1$, $\sigma(x) = \zeta_n x$ if $x^n = t$.

Let X be an F-variety endowed with a $\hat{\mu}$-action factoring for some n through a μ_n-action. The ℓ-adic étale cohomology groups $H_c^i(X, \mathbb{Q}_\ell)$ are endowed with a $\hat{\mu}$-action, and we may consider the element

$$(5.3.1) \qquad\qquad \mathrm{eu}_{\text{ét}}(X) := \sum_i (-1)^i [H_c^i(X, \mathbb{Q}_\ell)]$$

in $K(\hat{\mu}\text{-Mod})$. Note that $\mathrm{eu}_{\text{ét}}([V, \varrho]) = 1$ for any finite dimensional F-vector space V endowed with a $\hat{\mu}$-action factoring for some n through a linear μ_n-action. Thus, $\mathrm{eu}_{\text{ét}}$ factors to give rise to a morphism

$$(5.3.2) \qquad\qquad \mathrm{eu}_{\text{ét}}\colon K^{\hat{\mu}}(\mathrm{Var}_F) \longrightarrow K(\hat{\mu}\text{-Mod}).$$

Furthermore, the morphism $\mathrm{eu}_{\text{ét}} \circ \Theta$, with Θ as in (4.3.4), factors through $!K(\mathrm{RES})/([\mathbb{A}^1]-1)$ and gives rise to a morphism

$$(5.3.3) \qquad\qquad \mathrm{eu}_{\text{ét}}\colon !K(\mathrm{RES})/([\mathbb{A}^1]-1) \longrightarrow K(\hat{\mu}\text{-Mod}).$$

5.4. Compatibility

We have the following fundamental compatibility property between $\mathrm{EU}_{\text{ét}}$ and $\mathrm{eu}_{\text{ét}}$.

THEOREM 5.4.1. – *The diagram*

$$(5.4.1)$$

is commutative.

Proof. – It is enough to prove that if X is a definable subset of VF^n, then $\mathrm{EU}_{\text{ét}}(X) = \mathrm{eu}_{\text{ét}}(\mathrm{EU}_\Gamma([X]))$. Using the notation of (2.4.1) and the isomorphism (2.3.6), we may assume the class of X in $K_+(\mathrm{VF})$ is of the form $\mathbb{L}(\Psi(a \otimes b))$ with a in $K_+(\mathrm{RES}[m])$ and b in $K_+(\Gamma[r])$.

If $r \geq 1$, $\mathrm{EU}_\Gamma([X]) = 0$ by construction of EU_Γ. Indeed, with the notations from 2.5, $\chi_r([a \otimes b]) = \chi(b) \cdot [\mathbb{G}_m]^r \cdot a$, which implies that $E_n(a \otimes b) = 0$ for $n \geq r$, and $\mathrm{EU}_\Gamma([X]) = 0$ follows. To prove that $\mathrm{EU}_{\text{ét}}(X) = 0$, it is enough by multiplicativity of $\mathrm{EU}_{\text{ét}}$, to prove that $\mathrm{EU}_{\text{ét}}(\mathbb{L}(\Psi(1 \otimes b))) = 0$, which follows from Lemma 5.4.2.

Thus, we may assume $r = 0$ and $[X] = \mathbb{L}(\Psi([Z] \otimes 1))$, with Z a definable subset in $\mathrm{RES}[n]$. Since $\mathrm{EU}_\Gamma(\mathbb{L}(Z))$ is equal to the class of Z in $!K(\mathrm{RES})/([\mathbb{A}^1] - 1)$, it is enough to prove that $\mathrm{EU}_{\text{ét}}([\mathbb{L}(Z)]) = \mathrm{eu}_{\text{ét}}([Z])$.

Let Z be a definable subset of $\mathrm{RES}[n]$. For some integer d, Z is a definable subset of RES^d and, after partitioning Z into a finite number of definable sets, we may assume that, with the notation of §4.3, there exist a positive integer m and integers k_i, $1 \leq i \leq d$, such that Z is a definable subset of $\prod_{1 \leq i \leq d} V_{k_i/m}$. Consider the $F((t^{1/m}))$-definable isomorphism $g \colon \prod_{1 \leq i \leq d} V_{k_i/m} \to \mathbb{A}_F^d$ given by $g(x_1, \ldots, x_d) = (x_1/\mathbf{t}_{k_1/m}, \ldots, x_d/\mathbf{t}_{k_d/m})$. The Galois action on the space $\prod_{1 \leq i \leq d} V_{k_i/m}$ factorizes through a μ_m-action and g becomes μ_m-equivariant if one endows $\mathbb{A}_F^d$ with the action ϱ of μ_m given by $\zeta \cdot (y_1, \ldots, y_d) = (\zeta^{k_1} y_1, \ldots, \zeta^{k_d} y_d)$. The set $Y = g(Z)$ is an F-definable subset of $\mathbb{A}_F^d$. We shall still denote by g the induced $F((t^{1/m}))$-definable isomorphism $g \colon Z \to Y$ and by ϱ the induced action of μ_m on Y. We may assume Z is of RV-dimension n. Indeed, if Z is of RV-dimension $< n$, there exists a definable morphism $h \colon Z \to \mathrm{RV}^{n-1}$ with finite fibers. Let $i \colon \mathrm{RV}^{n-1} \to \mathrm{RV}^n$ denote the inclusion $(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1}, 0)$ and set $f = i \circ h$. Since $\mathbb{L}((Z, f)) = \mathbb{L}((Z, h)) \times \mathcal{M}$, with $\mathcal{M}$ the maximal ideal and $\mathrm{EU}_{\text{ét}}([\mathcal{M}]) = 1$, it follows that $\mathrm{EU}_{\text{ét}}([\mathbb{L}((Z, f))]) = \mathrm{EU}_{\text{ét}}([\mathbb{L}((Z, h))])$ by multiplicativity and we may conclude by induction on n in this case. Thus, by additivity, we may assume Y is a smooth variety over F of pure dimension n and that the morphism $f_Y \colon Y \to \mathbb{A}_F^n$ given by projection to the first n factors has finite fibers. It follows that the morphism $f_Z \colon Z \to \prod_{1 \leq i \leq n} V_{k_i/m}$ given by projecting to the first n factors has finite fibers too. The definable subset $\mathbb{L}((Y, f_Y))$ of $(\mathrm{VF}^\times)^n \times \mathbb{A}_F^d$ is the isomorphic image of the subset $\mathbb{L}((Z, f_Z))$ of $(\mathrm{VF}^\times)^n \times \prod_{1 \leq i \leq d} V_{k_i/m}$ under the mapping $\tilde{g} \colon (z_1, \ldots, z_n, x_1, \ldots, x_d) \mapsto (z_1/t_{k_1/m}, \ldots, z_n/t_{k_n/m}, x_1/\mathbf{t}_{k_1/m}, \ldots, x_d/\mathbf{t}_{k_d/m})$. It is endowed with a μ_m-action $\tilde{\varrho}$ given by $\zeta \cdot (z_1, \ldots, z_n, y_1, \ldots, y_d) = (\zeta^{k_1} z_1, \ldots, \zeta^{k_n} z_n, \zeta^{k_1} y_1, \ldots, \zeta^{k_d} y_d)$.

Let us consider the formal completion $\mathcal{Y}$ of $Y \otimes F[[t]]$. Denote by $\mathcal{Y}_\eta$ the analytic generic fiber of $\mathcal{Y}$ and by π the reduction map $\pi \colon \mathcal{Y}_\eta \to Y$. The μ_m-action ϱ induces an action on $\mathcal{Y}$ and $\mathcal{Y}_\eta$ which we still denote by ϱ. By Lemma 13.2 in [20] and its proof, $\mathbb{L}(Y)^{\mathrm{an}}$ is isomorphic to $\pi^{-1}(Y)$. Furthermore, under this isomorphism the μ_m-action on $\mathbb{L}(Y)^{\mathrm{an}}$ induced from $\tilde{\varrho}$ corresponds to the action ϱ on $\pi^{-1}(Y)$. Denote by π_m the projection $\hat{\mu} \to \mu_m$. The mapping $\tilde{g}$ induces an isomorphism between the spaces $\overline{\mathbb{L}(Z)^{\mathrm{an}}}$ and $\overline{\mathbb{L}(Y)^{\mathrm{an}}}$ under which the Galois action on $\overline{\mathbb{L}(Z)^{\mathrm{an}}}$ corresponds to the Galois action twisted by $\tilde{\varrho}$ on $\overline{\mathbb{L}(Y)^{\mathrm{an}}}$, namely the action for which an element σ of $\hat{\mu}$ acts on $\overline{\mathbb{L}(Y)^{\mathrm{an}}}$ by $y \mapsto \sigma \cdot \tilde{\varrho}(\pi_m(\sigma)) \cdot y = \varrho(\pi_m(\sigma)) \cdot \sigma \cdot y$. It follows that, for $i \geq 0$, $H_c^i(\overline{\mathbb{L}(Z)^{\mathrm{an}}}, \mathbb{Q}_\ell)$ is isomorphic to $H_c^i(\overline{\pi^{-1}(Y)}, \mathbb{Q}_\ell)$ and that, since the Galois action on $H_c^i(\overline{\pi^{-1}(Y)}, \mathbb{Q}_\ell)$ is trivial, cf. Lemma 5.4.3, that the Galois action on $H_c^i(\overline{\mathbb{L}(Z)^{\mathrm{an}}}, \mathbb{Q}_\ell)$ factorizes through μ_m and corresponds to the action induced by ϱ on $H_c^i(\overline{\pi^{-1}(Y)}, \mathbb{Q}_\ell)$. By Lemma 5.4.3 there is a canonical isomorphism, equivariant for the action ϱ,

$$(5.4.2) \qquad H_c^i(\overline{\pi^{-1}(Y)}, \mathbb{Q}_\ell) \simeq H_c^{2n-i}(Y, \mathbb{Q}_\ell(n))^\vee,$$

with the superscript $^\vee$ standing for the dual of a $\mathbb{Q}_\ell$-vector space. Since $\mathrm{EU}_{\text{ét}}([\mathbb{L}(Z)])$ is equal to $\sum_i (-1)^i [H_c^i(\overline{\mathbb{L}(Z)^{\mathrm{an}}}, \mathbb{Q}_\ell)]$, it follows from (5.4.2) that it is equal to $\sum_i (-1)^i [H_c^{2n-i}(Y, \mathbb{Q}_\ell(n))^\vee] = \sum_i (-1)^i [H_c^i(Y, \mathbb{Q}_\ell(n))^\vee]$, with the μ_m-action induced from ϱ. Let us note that a finite dimensional vector space $\mathbb{Q}_\ell$-vector space V with

μ_m-action has the same class in $K(\hat{\mu}\text{-Mod})$ as its dual $V^\vee$ endowed with the dual action and that, for any integer n, V and the Tate twist $V(n)$ have the same class in $K(\hat{\mu}\text{-Mod})$. It follows that $\mathrm{EU}_{\text{ét}}([\mathbb{L}(Z)]) = \sum_i (-1)^i [H^i_c(Y, \mathbb{Q}_\ell)]$, with action on the right-hand side induced from ϱ, hence $\mathrm{EU}_{\text{ét}}([\mathbb{L}(Z)]) = \mathrm{eu}_{\text{ét}}([Z])$. $\qquad\square$

LEMMA 5.4.2. – *Let $m \geq 1$ be an integer and let Z be a definable subset of Γ^m. Then we have*

$$(5.4.3) \qquad\qquad \mathrm{EU}_{\text{ét}}(\mathrm{val}^{-1}(Z)) = 0.$$

Proof. – Denote by 1 the class of the rank one $\mathbb{Q}_\ell$-vector space with trivial $\hat{\mu}$-action in $K(\hat{\mu}\text{-Mod})$. Let $D \subset \mathbb{A}^1_K$ be an open ball or $\mathbb{A}^1_K$. Recall that $H^i_c(\overline{D^{\text{an}}}, \mathbb{Q}_\ell)$ is zero if $i \neq 2$ and is isomorphic to $\mathbb{Q}_\ell(-1)$ for $i = 2$. In particular, $\mathrm{EU}_{\text{ét}}([D]) = 1$. By additivity of $\mathrm{EU}_{\text{ét}}$, it follows that, for any rational number c, if E is a subset of VF defined by one of the conditions $\mathrm{val}(x) \leq c$, $\mathrm{val}(x) < c$, $\mathrm{val}(x) = c$, or $\mathrm{val}(x) < \infty$, $\mathrm{EU}_{\text{ét}}([E]) = 0$.

By quantifier elimination and cell decomposition in o-minimal structures, cf. [15], using additivity of $\mathrm{EU}_{\text{ét}}$, we may assume there exists a definable subset Z' of Γ^{m-1}, affine linear forms with rational coefficients L_1 and L_2 in variables $u_1, \ldots, u_{m-1}$ such that Z is defined by the conditions $(u_1, \ldots, u_{m-1}) \in Z'$ and

$$(5.4.4) \qquad L_1(u_1, \ldots, u_{m-1}) \ \square_1 \ u_m \ \square_2 \ L_2(u_1, \ldots, u_{m-1}),$$

where $\square_1$ and $\square_2$ are of one of the following four types:

(1) no condition,
(2) = and no condition,
(3) < and <,
(4) < and no condition,
(5) no condition and <.

Thus $\mathrm{val}^{-1}(Z)$ is the set defined by the conditions $(x_1, \ldots, x_{m-1}) \in \mathrm{val}^{-1}(Z')$ and

$$(5.4.5) \qquad L_1(\mathrm{val}(x_1), \ldots, \mathrm{val}(x_{m-1})) \ \square_1 \ \mathrm{val}(x_m) \ \square_2 \ L_2(\mathrm{val}(x_1), \ldots, \mathrm{val}(x_{m-1})).$$

In case (1), $\mathrm{val}^{-1}(Z)$ is equal to the product of $\mathrm{val}^{-1}(Z')$ by the open annulus $C = \mathrm{VF} \setminus \{0\}$ and we deduce $\mathrm{EU}_{\text{ét}}(\mathrm{val}^{-1}(Z)) = 0$ from the fact that $\mathrm{EU}_{\text{ét}}(C) = 0$. In case (2), Z is defined by the conditions $(u_1, \ldots, u_{m-1}) \in Z'$ and $u_m = \sum_{1 \leq i < m} a_i u_i + b$ with $a_i \in \mathbb{Z}$, $1 \leq i < m$, and b in $\mathbb{Q}$. We may rewrite the last condition in the form $\sum_{1 \leq i \leq m} b_i u_i = c$ with $(b_1, \ldots, b_m)$ a primitive vector in Γ^m and $c \in \mathbb{Q}$. Thus, up to changing the coordinates in Γ^m, we may assume Z is defined by $(u_1, \ldots, u_{m-1}) \in Z'$ and $u_m = c$, so that $\mathrm{val}^{-1}(Z)$ is equal to the product of $\mathrm{val}^{-1}(Z')$ by the closed annulus D defined by $\mathrm{val}(x_m) = c$. Since $\mathrm{EU}_{\text{ét}}(D) = 0$, we get that $\mathrm{EU}_{\text{ét}}(\mathrm{val}^{-1}(Z)) = 0$ in this case. To deal with the remaining cases, consider the sets Z_1 and Z_2 defined respectively by $(u_1, \ldots, u_{m-1}) \in Z'$ and $u_m \leq L_1(u_1, \ldots, u_{m-1})$, resp. $u_m \geq L_2(u_1, \ldots, u_{m-1})$. It is enough to prove that $\mathrm{EU}_{\text{ét}}(\mathrm{val}^{-1}(Z_1)) = 0$ and $\mathrm{EU}_{\text{ét}}(\mathrm{val}^{-1}(Z_2)) = 0$, since then, by additivity, the result will follow from case (1). Let us prove $\mathrm{EU}_{\text{ét}}(\mathrm{val}^{-1}(Z_2)) = 0$. Similarly as in case (2), after a change of variable one may assume Z_2 is defined by $(u_1, \ldots, u_{m-1}) \in Z'$ and $u_m \geq c$, for some rational number c, so that $\mathrm{val}^{-1}(Z_2)$ is equal to the product of $\mathrm{val}^{-1}(Z')$ by the set E defined by $\mathrm{val}(x_m) \geq c$. Since $\mathrm{EU}_{\text{ét}}(E) = 0$, we deduce that $\mathrm{EU}_{\text{ét}}(\mathrm{val}^{-1}(Z_2)) = 0$. The proof that $\mathrm{EU}_{\text{ét}}(\mathrm{val}^{-1}(Z_1)) = 0$ is similar. $\qquad\square$

LEMMA 5.4.3. – *Let $\mathcal{X}$ be a smooth formal scheme of finite type over the valuation ring of K with special fiber X of pure dimension n and analytic generic fiber $\mathcal{X}_\eta$. Let $\pi \colon \mathcal{X}_\eta \to X$ be the reduction map. Let S be a smooth closed subscheme of X. Then there exist canonical isomorphisms*

$$(5.4.6) \qquad H^i_c(\overline{\pi^{-1}(S)}, \mathbb{Q}_\ell) \simeq H^{2n-i}_c(S, \mathbb{Q}_\ell(n))^\vee,$$

with $^\vee$ standing for the dual vector space. In particular, the Galois action on $H^i_c(\overline{\pi^{-1}(S)}, \mathbb{Q}_\ell)$ is trivial for $i \geq 0$. Assume furthermore that a finite group H acts on $\mathcal{X}$ inducing an action on X such that S is globally invariant by H. Then the isomorphism (5.4.6) *is equivariant for the H-action induced on both sides.*

Proof. – By Corollary 2.5 of [5], for any finite torsion group Λ, we have a canonical isomorphism

$$(5.4.7) \qquad R\Gamma_c(\overline{\pi^{-1}(S)}, \Lambda_{\mathcal{X}_\eta}) \simeq R\Gamma_S(X, R\psi_\eta(\Lambda_{\mathcal{X}_\eta})).$$

One checks by inspection of the proof in [5] that this isomorphism is H-equivariant. By triviality of vanishing cycles for smooth analytic spaces, cf. Corollary 5.7 of [4], $R^q\psi_\eta(\Lambda_{\mathcal{X}_\eta}) = 0$ for $q > 0$ and $R^0\psi_\eta(\Lambda_{\mathcal{X}_\eta}) = \Lambda_X$, hence it follows that there exist canonical H-equivariant isomorphisms

$$(5.4.8) \qquad H^i_c(\overline{\pi^{-1}(S)}, \Lambda) \simeq H^i_S(X, \Lambda).$$

We may assume S is of pure codimension r, hence, by purity, we have canonical H-equivariant isomorphisms $H^i_S(X, \Lambda) \simeq H^{i-2r}(S, \Lambda(-r))$, so we get canonical H-equivariant isomorphisms

$$(5.4.9) \qquad H^i_c(\overline{\pi^{-1}(S)}, \Lambda) \simeq H^{i-2r}(S, \Lambda(-r)).$$

Note that S is smooth of dimension $d = n - r$. Thus, for $j = 2d - (i - 2r) = 2n - i$, the canonical morphism

$$(5.4.10) \qquad H^j_c(S, \Lambda(d+r)) \times H^{i-2r}(S, \Lambda(-r)) \longrightarrow H^{2d}_c(S, \Lambda(d)) \simeq \Lambda$$

is a perfect pairing of finite groups by Poincaré Duality. The statement follows by passing to the limit over torsion coefficients $\mathbb{Z}/\ell^m\mathbb{Z}$ and tensoring with $\mathbb{Q}_\ell$. $\qquad\square$

5.5. A fixed point formula

The following version of the Lefschetz fixed point Theorem is classical and follows in particular from Theorem 3.2 of [8]:

PROPOSITION 5.5.1. – *Let Y be a quasi-projective variety over an algebraically closed field of characteristic zero. Let T be a finite order automorphism of Y. Let Y^T be the fixed point set of T and denote by $\chi_c(Y^T, \mathbb{Q}_\ell)$ its ℓ-adic Euler characteristic with compact supports. Then*

$$(5.5.1) \qquad \chi_c(Y^T, \mathbb{Q}_\ell) = \operatorname{tr}(T; H^\bullet_c(Y, \mathbb{Q}_\ell)).$$

Let us denote by Θ_0 the morphism

$$(5.5.2) \qquad \Theta_0 \colon {}!K(\mathrm{RES})/([\mathbb{A}^1] - 1) \longrightarrow K(\mathrm{Var}_F)/([\mathbb{A}^1] - 1)$$

induced by the morphism Θ_0 of (4.3.5). Denote by χ_c the morphism

$$\chi_c \colon K(\mathrm{Var}_F)/([\mathbb{A}^1] - 1) \longrightarrow \mathbb{Z}$$

induced by the ℓ-adic Euler characteristic with compact supports.

Combining Theorem 5.4.1 with Proposition 5.5.1 we obtain the following fixed point formula:

THEOREM 5.5.2. – *Let X be an ACVF_K-definable subset of VF^n. Let φ be a topological generator of $G = \mathrm{Gal}(\overline{K}/K)$. Then, for every $m \geq 1$,*

$$(5.5.3) \qquad \mathrm{tr}(\varphi^m; \mathrm{EU}_{\acute{e}t}([X])) = \chi_c(\Theta_0 \circ \mathrm{EU}_{\Gamma,m}([X])).$$

Proof. – Let $m \geq 1$. By Theorem 5.4.1,

$$(5.5.4) \qquad \mathrm{tr}(\varphi^m; \mathrm{EU}_{\acute{e}t}([X])) = \mathrm{tr}(\varphi^m; \mathrm{eu}_{\acute{e}t}(\mathrm{EU}_\Gamma([X]))).$$

On the other hand, it follows from Lemma 4.1.1 and Proposition 5.5.1 that

$$(5.5.5) \qquad \mathrm{tr}(\varphi^m; \mathrm{eu}_{\acute{e}t}(\mathrm{EU}_\Gamma([X]))) = \chi_c(\Theta_0 \circ \mathrm{EU}_{\Gamma,m}([X])).$$

The result follows. $\qquad\square$

6. Proof of Theorem 1.1.1

In this section we are working over $F((t))$, with $F = \mathbb{C}$. Our aim is to prove Theorem 1.1.1, namely that, for every $m \geq 1$, with the notation from the introduction,

$$\chi_c(\mathscr{X}_{m,x}) = \Lambda(M_x^m).$$

6.1. Using comparison results

Let X be a smooth complex variety and f be a regular function on X. Let x be a closed point of the fiber $f^{-1}(0)$. We shall use the notation introduced in Corollary 4.2.3. Thus π denotes the reduction map $X(\mathcal{O}) \to X(\mathbf{k})$, and we consider the $\mathrm{ACVF}_{F((t))}$-definable sets

$$(6.1.1) \qquad X_{t,x} = \{y \in X(\mathcal{O}); f(y) = t \text{ and } \pi(y) = x\}$$

and

$$(6.1.2) \qquad \mathscr{X}_x = \{y \in X(\mathcal{O}); \mathrm{rv}\, f(y) = \mathrm{rv}(t) \text{ and } \pi(y) = x\}.$$

The definable set $X_{t,x}$ is closely related to the analytic Milnor fiber $\mathscr{F}_x$ introduced in §9.1 of [33] whose definition we now recall. Let X_∞ be the t-adic completion of $f\colon X \to \mathrm{Spec}\,\mathbb{C}[t]$ and let X_η be its generic fiber (in the category of rigid $F((t))$-varieties). There is a canonical specialization morphism $\mathrm{sp}\colon X_\eta \to X_\infty$ (cf. §2.2 of [33]) and $\mathscr{F}_x$ is defined as $\mathrm{sp}^{-1}(x)$. It is an open rigid subspace of X_η. It follows directly from the definitions that $X_{t,x}^{\mathrm{an}}$ and $\mathscr{F}_x^{\mathrm{an}}$ may be canonically identified.

Fix a prime number ℓ and denote by φ the topological generator of $\hat{\mu}(\mathbb{C}) = \mathrm{Gal}(\mathbb{C}((t))^{\mathrm{alg}}/\mathbb{C}((t)))$ given by the family $(\zeta_n)_{n \geq 1}$ with $\zeta_n = \exp(2i\pi/n)$. It follows from Theorem 9.2 from [33] (more precisely, from its proof ; note that in the notation of loc. cit.

the exponent an is omitted), which is a consequence of the second isomorphism proved in [5] Corollary 3.5, that there exist isomorphisms

$$(6.1.3) \qquad H^i(F_x, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq H^i(\widehat{\mathscr{S}_x^{\mathrm{an}} \widehat{\otimes} \mathbb{C}((t))}^{\mathrm{alg}}, \mathbb{Q}_\ell)$$

compatible with the action of M_x and φ. Here F_x is the topological Milnor fiber defined in (1.1.2), $H^i(F_x, \mathbb{Q})$ is a singular cohomology group, while the cohomology group appearing on the right-hand side is an étale cohomology group. It follows that, for every $m \geq 0$,

$$(6.1.4) \qquad \Lambda(M_x^m) = \mathrm{tr}(\varphi^m; H^\bullet(\widehat{\mathscr{S}_x^{\mathrm{an}} \widehat{\otimes} \mathbb{C}((t))}^{\mathrm{alg}}, \mathbb{Q}_\ell)).$$

By Poincaré Duality as established in §7.3 of [3], there is a perfect duality

$$H^i(\widehat{\mathscr{S}_x^{\mathrm{an}} \widehat{\otimes} \mathbb{C}((t))}^{\mathrm{alg}}, \mathbb{Z}/\ell^n\mathbb{Z})) \times H_c^{2d-i}(\widehat{\mathscr{S}_x^{\mathrm{an}} \widehat{\otimes} \mathbb{C}((t))}^{\mathrm{alg}}, \mathbb{Z}/\ell^n\mathbb{Z}(d)) \to \mathbb{Z}/\ell^n\mathbb{Z},$$

with d the dimension of X, which is compatible with the φ-action. Hence, after taking the limit over n and tensoring with $\mathbb{Q}_\ell$, one deduces that, for every $m \geq 0$,

$$(6.1.5) \qquad \Lambda(M_x^m) = \mathrm{tr}(\varphi^m; H_c^\bullet(\widehat{\mathscr{S}_x^{\mathrm{an}} \widehat{\otimes} \mathbb{C}((t))}^{\mathrm{alg}}, \mathbb{Q}_\ell)),$$

which may be rewritten as

$$(6.1.6) \qquad \Lambda(M_x^m) = \mathrm{tr}(\varphi^m; H_c^\bullet(\widehat{X_{t,x}^{\mathrm{an}} \widehat{\otimes} \mathbb{C}((t))}^{\mathrm{alg}}, \mathbb{Q}_\ell)).$$

REMARK 6.1.1. – With the notations of Corollary 3.5 of [5], when $\mathscr{Y}$ is proper, it is explained in Remark 3.8 (i) of [5] how to deduce the first isomorphism of Corollary 3.5 of [5] directly from Theorem 5.1 in [4] in the way indicated in [17]. When furthermore $\mathscr{X}_\eta$ is smooth (keeping the notations of loc. cit.), the second isomorphism of Corollary 3.5 of [5] follows from the first by Poincaré Duality and Corollary 5.3.7 of [3]. In particular, for the use which is made of Corollary 3.5 of [5] in this paper, one may completely avoid using de Jong's results on stable reduction and one may rely only on results from [3] and [4].

6.2. Proof of Theorem 1.1.1

Let $m \geq 1$. With the previous notations, one may rewrite (6.1.6) as

$$(6.2.1) \qquad \Lambda(M_x^m) = \mathrm{tr}(\varphi^m; \mathrm{EU}_{\text{ét}}([X_{t,x}])).$$

On the other hand, by Theorem 5.5.2 we have

$$(6.2.2) \qquad \mathrm{tr}(\varphi^m; \mathrm{EU}_{\text{ét}}([X_{t,x}])) = \chi_c(\Theta_0(\mathrm{EU}_{\Gamma,m}([X_{t,x}]))).$$

In Corollary 4.2.3, it is proven that $\mathrm{EU}_{\Gamma,m}(X_{t,x}) = \mathscr{X}_x[m]$ as classes in $!K(\mathrm{RES})/([\mathbb{A}^1]-1)$. In particular,

$$(6.2.3) \qquad \chi_c(\Theta_0(\mathrm{EU}_{\Gamma,m}([X_{t,x}]))) = \chi_c(\mathscr{X}_x[m]).$$

To conclude the proof it is thus enough to check that

$$(6.2.4) \qquad \chi_c(\mathscr{X}_x[m]) = \chi_c(\mathscr{X}_{m,x}).$$

This may be seen as follows. For $m \geq 1$,

$$(6.2.5) \qquad \mathscr{X}_{m,x} = \{\varphi \in X(\mathbb{C}[t]/t^{m+1}); f(\varphi) = t^m \mod t^{m+1}, \varphi(0) = x\}$$

may be rewritten as

(6.2.6) $$\{\varphi \in X(\mathbb{C}[t^{1/m}]/t^{(m+1)/m}); f(\varphi) = t \mod t^{(m+1)/m}, \varphi(0) = x\}$$

or as

(6.2.7) $$\{\varphi \in X(\mathbb{C}[t^{1/m}]/t^{(m+1)/m}); \mathrm{rv}(f(\varphi)) = \mathrm{rv}(t), \varphi(0) = x\}.$$

Thus $\Theta(\mathscr{X}_x[m])$ and $\mathscr{X}_{m,x}$ have the same class in $K^{\hat{\mu}}(\mathrm{Var}_F)/([\mathbb{A}^1] - 1)$. The equality $\chi_c(\mathscr{X}_x[m]) = \chi_c(\mathscr{X}_{m,x})$ follows. $\qquad\square$

7. Trace formulas and the motivic Serre invariant

7.1. – In this section F denotes a field of characteristic zero, $K = F((t))$, $K_m = F((t^{1/m}))$ and $\bar{K} = \bigcup_{m \geq 1} K_m$. If X is an ACVF_K-definable set or an algebraic variety over K, we write $X(m)$ and $\bar{X}$ for the objects obtained by extension of scalars to K_m and $\bar{K}$, respectively. As in (5.5.2) we denote by Θ_0 the morphism

(7.1.1) $$\Theta_0 \colon !K(\mathrm{RES})/([\mathbb{A}^1] - 1) \longrightarrow K(\mathrm{Var}_F)/([\mathbb{A}^1] - 1)$$

induced by the morphism Θ_0 of (4.3.5).

7.2. The motivic Serre invariant

Let R be a complete discrete valuation ring, with perfect residue field F and field of fractions K. We denote by R^{sh} a strict Henselization of R and by K^{sh} its field of fractions. Let X be a smooth quasi-compact rigid K-variety. In [28], using motivic integration on formal schemes, for any such X a canonical class $S(X) \in K(\mathrm{Var}_F)/([\mathbb{A}^1]-1)$ is constructed, called the motivic Serre invariant of X. If X is a smooth proper algebraic variety over K, one sets $S(X) = S(X^{rig})$, with X^{rig} the rigid analytification of X.

We have the following comparison between the morphism EU_Γ and the motivic Serre invariant in residue characteristic zero via the morphism Θ_0:

PROPOSITION 7.2.1. – *Let $K = F((t))$ with F a field of characteristic zero. Let X be a smooth proper algebraic variety over K. Then, for every $m \geq 1$,*

(7.2.1) $$\Theta_0(\mathrm{EU}_{\Gamma,m}([X])) = S(X(m)).$$

Proof. – After replacing $F((t))$ by $F((t^{1/m}))$ we may assume $m = 1$. Let $\mathscr{X}$ be a weak Néron model of X, cf. Section 2.7 of [28]. This means that $\mathscr{X}$ is a smooth R-variety endowed with an isomorphism $\mathscr{X}_K \to X$ such that the natural map $\mathscr{X}(R^{sh}) \to X(K^{sh})$ is a bijection. Consider the unique definable subset X_1 of X such that for any valued field extension K' of K, with valuation ring R', $X_1(K')$ is the image of $\mathscr{X}(R')$ under the canonical mapping $\mathscr{X}(R') \to X(K')$ (in fact $\mathscr{X}$ gives rise to a definable set and X_1 is its image through the natural map $\mathscr{X} \to X$). Let $X_{\neq 1}$ be the complement of X_1 in X. By the very construction of $\mathrm{EU}_{\Gamma,1}$ and $S(X)$, $\Theta_0(\mathrm{EU}_{\Gamma,1}([X_1])) = S(X)$. Thus it is enough to prove that $\mathrm{EU}_{\Gamma,1}([X_{\neq 1}]) = 0$. Since $X_{\neq 1}(F'((t))) = \varnothing$ for every field extension F' of F by the Néron property of $\mathscr{X}$, this follows from Lemma 7.2.2. $\qquad\square$

LEMMA 7.2.2. – *Let X be an $F((t))$-definable subset of VF^n. Assume that $X(F'((t))) = \varnothing$ for every field extension F' of F. Then $\mathrm{EU}_{\Gamma,1}([X]) = 0$.*

Proof. – Using the notation of (2.4.1) and the isomorphism (2.3.6), we may assume X is of the form $[X] = \mathbb{L}(\Psi(a \otimes b))$ with a in $K_+(\mathrm{RES}[m])$ and b in $K_+(\Gamma[r])$. If $r \geq 1$, $\mathrm{EU}_\Gamma([X]) = 0$ by construction of EU_Γ. Thus, we may assume $r = 0$ and $b = 1$. Let ℓ be an integer and Z a definable subset in RES^ℓ such that $a = [Z]$. By construction, Z and $\mathrm{EU}_\Gamma(X)$ have the same class in $!K(\mathrm{RES})/([\mathbb{A}^1]-1)$. In particular $Z \cap \mathbf{k}^\ell$ and $\mathrm{EU}_{\Gamma,1}(X)$ have the same class in $!K(\mathrm{RES}_\mathbb{Z})/([\mathbb{A}^1]-1)$. The statement follows, since if $X(F'((t))) = \varnothing$ for every field extension F' of F, then $Z \cap \mathbf{k}^\ell = \varnothing$. □

In particular, we obtain the following:

COROLLARY 7.2.3 ([33]). – *Let $K = F((t))$ with F an algebraically closed field of characteristic zero. Let X be a smooth proper algebraic variety over K. Then, for every $m \geq 1$,*

$$(7.2.2) \qquad \mathrm{tr}(\varphi^m; H^\bullet(\bar{X}, \mathbb{Q}_\ell)) = \chi_c(S(X(m))).$$

Proof. – By Corollary 7.5.4 of [3], for every $q \geq 0$ there are canonical isomorphisms $H^q(\bar{X}, \mathbb{Q}_\ell) \simeq H^q(\overline{X^{\mathrm{an}}}, \mathbb{Q}_\ell)$. On the other hand, X being proper, $H^q(\overline{X^{\mathrm{an}}}, \mathbb{Q}_\ell)$ is canonically isomorphic to $H^q_c(\overline{X^{\mathrm{an}}}, \mathbb{Q}_\ell)$. Let $m \geq 1$. Using Proposition 5.5.2 one deduces that $\mathrm{tr}(\varphi^m; H^\bullet(\bar{X}, \mathbb{Q}_\ell)) = \chi_c(\Theta_0(\mathrm{EU}_{\Gamma,m}([X])))$ and the result follows from Proposition 7.2.1. □

The original proof in Corollary 5.5 [33] of Corollary 7.2.3 uses resolution of singularities, which is not the case of the proof given here.

REMARK 7.2.4. – Our results also provide a new construction, not using resolution of singularities, of the motivic Serre invariant of arbitrary algebraic varieties in equal characteristic zero. This motivic Serre invariant was constructed in equal characteristic zero and mixed characteristic in Theorem 5.4 of [32], using resolution of singularities, weak factorization and a refinement of the Néron smoothening process to pairs of varieties. In equal characteristic zero, the trace formula extends to arbitrary varieties by a formal additivity argument, see Theorem 6.4 and Corollary 6.5 of [32].

7.3. Analytic variants

Assume again R is a complete discrete valuation ring, with perfect residue field F and field of fractions K. In [31], the construction of the motivic Serre invariant was extended to the class of generic fibers of generically smooth special formal R-schemes. Special formal R-schemes are obtained by gluing formal spectra of quotient of R-algebras of the form $R\{T_1, \ldots, T_r\}[\![S_1, \ldots, S_s]\!]$, cf. [31]. In particular, if $\mathscr{X}_\eta$ is such a generic fiber and $K = F((t))$ with F an algebraically closed field of characteristic zero, then it follows from Theorem 6.4 of [31], generalizing Theorem 5.4 of [33], that, with the obvious notations, for every $m \geq 1$,

$$(7.3.1) \qquad \mathrm{tr}(\varphi^m; H^\bullet_c(\bar{\mathscr{X}}_\eta, \mathbb{Q}_\ell)) = \chi_c(S(\mathscr{X}_\eta(m))).$$

In this setting it is natural to replace the theory $\mathrm{ACVF}(0,0)$ considered in the present paper by its rigid analytic variant $\mathrm{ACVF}^R(0,0)$ introduced by Lipshitz in [26] and one may expect that the results from this section still hold for $\mathrm{ACVF}^R(0,0)$-definable sets. It is quite likely that it should be possible to prove such extensions using arguments similar to ours once some appropriate extension of Theorem 5.1.1 to this analytic setting is established. In

particular, one should be able to extend this way Proposition 7.2.1 and Corollary 7.2.3 to generic fibers of generically smooth special formal R-schemes. This would provide a proof of (7.3.1) which would not use resolution of singularities, unlike the original proof in [31].

8. Recovering the motivic zeta function and the motivic Milnor fiber

In this section, we shall work within the framework of 4.1. In particular the base structure is the field $L_0 = F((t))$, with F a trivially valued algebraically closed field of characteristic zero and $\mathrm{val}(t)$ positive and denoted by 1.

8.1. Some notations and constructions from [21]

Let A be an ordered abelian group and n a non-negative integer. An A-definable subset of Γ^n will be called bounded if it is contained in $[-\gamma, \gamma]^n$ for some A-definable $\gamma \in \Gamma$. An A-definable subset of Γ^n will be called bounded below if it is contained in $[\gamma, \infty)^n$ for some A-definable $\gamma \in \Gamma$. We recall from [21], Definition 2.4, the definition of various categories $\Gamma_A[n]$, $\Gamma_A^{\mathrm{bdd}}[n]$, $\mathrm{vol}\Gamma_A[n]$ and $\mathrm{vol}\Gamma_A^{\mathrm{bdd}}[n]$. Thus, $\Gamma_A[n]$ is the category already defined in §2.3, $\Gamma_A^{\mathrm{bdd}}[n]$ is the subcategory of bounded subsets while $\mathrm{vol}\Gamma_A[n]$ has the same objects as $\Gamma_A[n]$ with morphisms $f\colon X \to Y$, those morphisms in $\Gamma_A[n]$ such that $\sum_i x_i = \sum_i y_i$ whenever $(y_1, \ldots, y_n) = f(x_1, \ldots, x_n)$, $\mathrm{vol}\Gamma_A^{\mathrm{bdd}}[n]$ is the subcategory of $\mathrm{vol}\Gamma_A[n]$ whose objects are bounded below. Finally, we denote by $\mathrm{vol}\Gamma_A^{\mathrm{2bdd}}[n]$ the subcategory of $\mathrm{vol}\Gamma_A[n]$ whose objects are bounded.

We shall also consider the corresponding Grothendieck monoids $K_+(\Gamma_A[n])$, $K_+(\Gamma_A^{\mathrm{bdd}}[n])$, $K_+(\mathrm{vol}\Gamma_A[n])$, $K_+(\mathrm{vol}\Gamma_A^{\mathrm{bdd}}[n])$, and $K_+(\mathrm{vol}\Gamma_A^{\mathrm{2bdd}}[n])$. We also set $K_+(\Gamma_A^{\mathrm{bdd}}[*]) = \bigoplus_n K_+(\Gamma_A^{\mathrm{bdd}}[n])$ with the associated ring $K(\Gamma_A^{\mathrm{bdd}})$, and similar notation for the other categories.

Let $[0]_1$ denote the class of $\{0\}$ in $K_+(\Gamma_A^{\mathrm{bdd}}[1])$. We set

$$(8.1.1) \qquad K_+^{df}(\Gamma_A^{\mathrm{bdd}}) = (K_+(\Gamma_A^{\mathrm{bdd}}[*])[[0]_1^{-1}])_0,$$

where $(K_+(\Gamma_A^{\mathrm{bdd}}[*])[[0]_1^{-1}])_0$ is the sub-semi-ring of the graded semi-ring $K_+(\Gamma_A^{\mathrm{bdd}}[*])[[0]_1^{-1}]$ consisting of elements of degree 0. One defines similarly $K_+^{df}(\mathrm{vol}\Gamma_A^{\mathrm{bdd}})$, $K_+^{df}(\mathrm{vol}\Gamma_A^{\mathrm{2bdd}})$ and denote by $K^{df}(\Gamma_A^{\mathrm{bdd}})$, $K^{df}(\mathrm{vol}\Gamma_A^{\mathrm{bdd}})$ and $K^{df}(\mathrm{vol}\Gamma_A^{\mathrm{2bdd}})$ the corresponding rings.

For $x = (x_1, \ldots, x_n) \in \mathrm{RV}^n$, set $\mathrm{w}(x) = \sum_{1 \le i \le n} \mathrm{val}_{\mathrm{rv}}(x_i)$. We recall from [21], Definition 3.14, the definition of the categories $\mathrm{volRV}[n]$, $\mathrm{volRES}[n]$ and $\mathrm{volRV}^{\mathrm{bdd}}[n]$, given a base structure A. The category $\mathrm{volRV}[n]$ has the same objects of the category $\mathrm{RV}[n]$, namely pairs (X, f) with $X \subset \mathrm{RV}^*$ and $f\colon X \to \mathrm{RV}^n$ a morphism with finite fibers, and a morphism $h\colon (X, f) \to (X', f')$ in $\mathrm{volRV}[n]$ is a definable bijection $h\colon X \to X'$ such that $\mathrm{w}(f(x)) = \mathrm{w}((f' \circ h)(x))$ for every $x \in X$. The category $\mathrm{volRES}[n]$ is the full subcategory of $\mathrm{volRV}[n]$ consisting of objects in $\mathrm{RES}[n]$ and $\mathrm{volRV}^{\mathrm{bdd}}[n]$ is the full subcategory of $\mathrm{volRV}[n]$ consisting of objects whose Γ-image is bounded below. One defines $\mathrm{volRV}^{\mathrm{2bdd}}[n]$ as the subcategory of $\mathrm{volRV}^{\mathrm{bdd}}[n]$ whose Γ-image is bounded. Similar notation as above for the various semi-rings and rings.

We have a map

$$(8.1.2) \qquad K_+(\mathrm{volRES}[n]) \longrightarrow K_+(\mathrm{volRV}^{\mathrm{bdd}}[n])$$

induced by inclusion and a map

$$(8.1.3) \qquad K_+(\mathrm{vol}\Gamma^{\mathrm{bdd}}[n]) \longrightarrow K_+(\mathrm{vol}\mathrm{RV}^{\mathrm{bdd}}[n])$$

induced by $X \mapsto \mathrm{rv}^{-1}(X)$. By §3.4 in [21], taking the tensor product, one gets a canonical morphism

$$(8.1.4) \qquad \Psi \colon K_+(\mathrm{vol}\mathrm{RES}[*]) \otimes K_+(\mathrm{vol}\Gamma^{\mathrm{bdd}}[*]) \longrightarrow K_+(\mathrm{vol}\mathrm{RV}^{\mathrm{bdd}}[*])$$

whose kernel is the congruence relation generated by pairs

$$(8.1.5) \qquad ([\mathrm{val}_{\mathrm{rv}}^{-1}(\gamma)]_1 \otimes 1, 1 \otimes [\gamma]_1),$$

with γ in Γ definable. Here the subscript 1 refers to the fact that the classes are considered in degree 1. Note that (8.1.4) restricts to a morphism

$$(8.1.6) \qquad \Psi \colon K_+(\mathrm{vol}\mathrm{RES}[*]) \otimes K_+(\mathrm{vol}\Gamma^{2\mathrm{bdd}}[*]) \longrightarrow K_+(\mathrm{vol}\mathrm{RV}^{2\mathrm{bdd}}[*]).$$

Similarly, cf. Proposition 10.10 of [20], there is a canonical morphism

$$(8.1.7) \qquad \Psi \colon K_+(\mathrm{vol}\mathrm{RES}[*]) \otimes K_+(\mathrm{vol}\Gamma[*]) \longrightarrow K_+(\mathrm{vol}\mathrm{RV}[*])$$

whose kernel is generated by the elements (8.1.5).

Consider the category $\mathrm{vol}\mathrm{VF}[n]$ of Definition 3.20 in [21] and its bounded version $\mathrm{vol}\mathrm{VF}^{\mathrm{bdd}}[n]$. There is a lift of the mapping $\mathbb{L}$ to a mapping

$$(8.1.8) \qquad \mathbb{L} \colon \mathrm{Ob}\, \mathrm{vol}\mathrm{RV}[n] \longrightarrow \mathrm{Ob}\, \mathrm{vol}\mathrm{VF}[n].$$

We will denote by I'_{sp} the congruence generated by $[1]_1 = [\mathrm{RV}^{>0}]_1$ in either $K_+(\mathrm{vol}\mathrm{RV}[*])$ or $K_+(\mathrm{vol}\mathrm{RV}^{\mathrm{bdd}}[*])$, or in one of the monoids $K_+(\mathrm{vol}\mathrm{RV}[n])$ or $K_+(\mathrm{vol}\mathrm{RV}^{\mathrm{bdd}}[n])$; the context will determine the ambient monoid or semi-ring. By Lemma 3.21 of [21] and Theorems 8.28 and 8.29 of [20], there are canonical isomorphisms

$$(8.1.9) \qquad \int \colon K_+(\mathrm{vol}\mathrm{VF}[n]) \longrightarrow K_+(\mathrm{vol}\mathrm{RV}[n])/I'_{\mathrm{sp}}$$

and

$$(8.1.10) \qquad \int \colon K_+(\mathrm{vol}\mathrm{VF}^{\mathrm{bdd}}[n]) \longrightarrow K_+(\mathrm{vol}\mathrm{RV}^{\mathrm{bdd}}[n])/I'_{\mathrm{sp}}$$

which are characterized by the prescription that, for X in $\mathrm{vol}\mathrm{VF}[n]$ and V in $\mathrm{vol}\mathrm{RV}[n]$ (resp. $\mathrm{vol}\mathrm{VF}^{\mathrm{bdd}}[n]$ and $\mathrm{vol}\mathrm{RV}^{\mathrm{bdd}}[n]$), $\int([X])$ is equal to the class of $[V]$ in $K_+(\mathrm{vol}\mathrm{RV}[n])/I'_{\mathrm{sp}}$ (resp. $K_+(\mathrm{vol}\mathrm{RV}^{\mathrm{bdd}}[n])/I'_{\mathrm{sp}}$) if and only if $[X] = [\mathbb{L}(V)]$. We denote similarly the corresponding isomorphisms between Grothendieck rings.

8.2. The morphisms h_m and $\tilde{h}_m$

For $\gamma \in \Gamma^n$, let $\mathrm{w}(\gamma) = \sum_{1 \leq i \leq n} \gamma_i$.

Let $\mathbb{Z}[T, T^{-1}]_{\mathrm{loc}}$ denote the localisation of the ring of Laurent polynomials $\mathbb{Z}[T, T^{-1}]$ with respect to the multiplicative family generated by the polynomials $1 - T^{-i}$, $i \geq 1$.

Let Δ be a bounded definable subset of Γ^n. For every integer $m \geq 1$, we set

$$(8.2.1) \qquad \alpha_m(\Delta) = (T - 1)^n \sum_{(\gamma_1, \ldots, \gamma_n) \in \Delta \cap (m^{-1}\mathbb{Z})^n} T^{-m\mathrm{w}(\gamma)}$$

in $\mathbb{Z}[T, T^{-1}]$.

Assume now Δ is a bounded below definable subset of Γ^n. The sum (8.2.1) is no longer finite, but it still makes sense as a Laurent series in T^{-1}, since in (8.2.1) only a finite number of terms have a given weight since Δ is bounded below (note that the weights are bounded below).

LEMMA 8.2.1. – *Let Δ be a bounded below definable subset of Γ^n. For every integer $m \geq 1$, the Laurent series*

$$(8.2.2) \qquad \tilde{\alpha}_m(\Delta) = (T-1)^n \sum_{(\gamma_1,\ldots,\gamma_n)\in\Delta\cap(m^{-1}\mathbb{Z})^n} T^{-m\mathrm{w}(\gamma)}$$

belongs to $\mathbb{Z}[T,T^{-1}]_{\mathrm{loc}}$.

Proof. – It is enough to prove the result for $m = 1$. We may assume Δ is convex and closed. Thus, it is the convex hull of a finite family of rational half-lines and points in $\mathbb{Q}^n$, i.e., a rational polytope according to the terminology of [7]. Consider the formal series

$$\Phi_\Delta(T_1,\ldots,T_n) := \sum_{(\gamma_1,\ldots,\gamma_n)\in\Delta\cap\mathbb{Z}^n} \prod T_i^{\gamma_i}.$$

It follows from [7] and [23] that $\Phi_\Delta(T_1,\ldots,T_n)$ belongs to the localisation of $\mathbb{Z}[T_1, T_1^{-1}, \ldots, T_n, T_n^{-1}]$ with respect to the multiplicative family generated by $1 - \prod T_i^{\gamma_i}$, $(\gamma_1,\ldots,\gamma_n) \in \mathbb{Z}^n \setminus \{0\}$. Indeed, the core of the paper [7] deals with integral polytopes, but in its §3.3 it is explained how to deduce the statement for rational polytopes. Since Δ is bounded below, $\Phi_\Delta(T_1,\ldots,T_n)$ belongs in fact to the localisation of $\mathbb{Z}[T_1, T_1^{-1}, \ldots, T_n, T_n^{-1}]$ with respect to the multiplicative family generated by $1 - \prod T_i^{\gamma_i}$, $(\gamma_1,\ldots,\gamma_n) \in \mathbb{N}^n \setminus \{0\}$. Then $\Phi_\Delta(T^{-1}, T^{-1}, \ldots, T^{-1})$ belongs to $\mathbb{Z}[T,T^{-1}]_{\mathrm{loc}}$ and one has $\tilde{\alpha}_m(\Delta) = (T-1)^n\Phi_\Delta(T^{-1}, T^{-1}, \ldots, T^{-1})$. $\qquad\square$

Let $!K(\mathrm{RES})([\mathbb{A}^1]^{-1})_{\mathrm{loc}}$ denote the localisation of $!K(\mathrm{RES})([\mathbb{A}^1]^{-1})$ with respect to the multiplicative family generated by the elements $1 - [\mathbb{A}^1]^{-i}$, $i \geq 1$. There are unique morphisms $\theta : \mathbb{Z}[T,T^{-1}] \to !K(\mathrm{RES})([\mathbb{A}^1]^{-1})$ and $\tilde{\theta} : \mathbb{Z}[T,T^{-1}]_{\mathrm{loc}} \to !K(\mathrm{RES})([\mathbb{A}^1]^{-1})_{\mathrm{loc}}$ sending T to $[\mathbb{A}^1]$.

If Δ is a bounded, resp. bounded below, definable subset of Γ^n, we set $a_m(\Delta) = \theta(\alpha_m(\Delta))$, resp. $\tilde{a}_m(\Delta) = \tilde{\theta}(\tilde{\alpha}_m(\Delta))$. By additivity, this gives rise to morphisms

$$(8.2.3) \qquad a_m \colon K(\mathrm{vol}\Gamma^{2\mathrm{bdd}}[*]) \longrightarrow !K(\mathrm{RES})([\mathbb{A}^1]^{-1})$$

and

$$(8.2.4) \qquad \tilde{a}_m \colon K(\mathrm{vol}\Gamma^{\mathrm{bdd}}[*]) \longrightarrow !K(\mathrm{RES})([\mathbb{A}^1]^{-1})_{\mathrm{loc}}.$$

Now consider $X = (X, f)$ in $\mathrm{RES}[n]$. Let $\gamma = (\gamma_1,\ldots,\gamma_n)$ and assume $f(X) \subset V_{\gamma_1} \times \cdots \times V_{\gamma_n}$. If $m(\gamma_1,\ldots,\gamma_n) \in \mathbb{Z}^n$, then set

$$(8.2.5) \qquad b_m^0(X) = [X]\left(\frac{[1]_1}{[\mathbb{A}^1]}\right)^{m\mathrm{w}(\gamma)}$$

in $!K(\mathrm{RES}[*])([\mathbb{A}^1]^{-1})$; note that $f(X) = f(X) \cap \mathrm{RES}_m$ in this case. Otherwise set $b_m^0(X) = 0$. This construction extends uniquely to a morphism

$$(8.2.6) \qquad b_m^0 \colon K(\mathrm{vol}\mathrm{RES}[*]) \longrightarrow !K(\mathrm{vol}\mathrm{RES}[*])([\mathbb{A}^1]^{-1}).$$

By composing b_m^0 with the forgetful morphism $!K(\text{volRES}[*])([\mathbb{A}^1]^{-1})$ to $!K(\text{RES})([\mathbb{A}^1]^{-1})$, one gets a morphism

(8.2.7) $b_m \colon K(\text{volRES}[*]) \longrightarrow !K(\text{RES})([\mathbb{A}^1]^{-1}).$

One denotes by $\tilde{b}_m$ the morphism

(8.2.8) $\tilde{b}_m \colon K(\text{volRES}[*]) \longrightarrow !K(\text{RES})([\mathbb{A}^1]^{-1})_{\text{loc}}$

obtained by composing b_m with the localisation morphism from $!K(\text{RES})([\mathbb{A}^1]^{-1})$ to $!K(\text{RES})([\mathbb{A}^1]^{-1})_{\text{loc}}$.

The morphism

(8.2.9) $b_m \otimes a_m \colon K(\text{volRES}[*]) \otimes K(\text{vol}\Gamma^{2\text{bdd}}[*]) \longrightarrow !K(\text{RES})([\mathbb{A}^1]^{-1})$

factors through the relations (8.1.5) and gives rise to a morphism

(8.2.10) $h_m \colon K(\text{volRV}^{2\text{bdd}}[*]) \longrightarrow !K(\text{RES})([\mathbb{A}^1]^{-1}).$

Indeed, if $\gamma = i/m$, then $a_m([\gamma]_1) = (\frac{1}{[\mathbb{A}^1]})^i([\mathbb{A}^1] - 1)$ and $[\text{val}_{\text{rv}}^{-1}(\gamma)]_1 = [\mathbb{A}^1] - [1]_1$ in $!K_+(\text{volRES}[1])$, thus $a_m([\gamma]_1) = b_m([\text{val}_{\text{rv}}^{-1}(\gamma)]_1)$. Similarly, the morphism

(8.2.11) $\tilde{b}_m \otimes \tilde{a}_m \colon K(\text{volRES}[*]) \otimes K(\text{vol}\Gamma^{\text{bdd}}[*]) \longrightarrow !K(\text{RES})([\mathbb{A}^1]^{-1})_{\text{loc}}$

gives rise to a morphism

(8.2.12) $\tilde{h}_m \colon K(\text{volRV}^{\text{bdd}}[*]) \longrightarrow !K(\text{RES})([\mathbb{A}^1]^{-1})_{\text{loc}}$

and the diagram

(8.2.13)
$$
\begin{array}{ccc}
K(\text{volRV}^{2\text{bdd}}[*]) & \xrightarrow{\ h_m\ } & !K(\text{RES})([\mathbb{A}^1]^{-1}) \\
\downarrow & & \downarrow \\
K(\text{volRV}^{\text{bdd}}[*]) & \xrightarrow{\ \tilde{h}_m\ } & !K(\text{RES})([\mathbb{A}^1]^{-1})_{\text{loc}}
\end{array}
$$

is commutative.

LEMMA 8.2.2. – *For every $m \geq 1$, the morphism $\tilde{h}_m$ vanishes on I'_{sp}.*

Proof. – Indeed, if ℓ denotes the open half-line $(0, \infty)$ in Γ,

$$\tilde{\alpha}_m(\ell) = (T - 1)\sum_{i>0} T^{-i} = 1,$$

therefore $\tilde{h}_m([\text{RV}^{>0}]_1) = 1$. On the other hand, $h_m([1]_1) = 1$ by definition. $\square$

It follows that the morphism $\tilde{h}_m$ factors through a morphism

(8.2.14) $\tilde{h}_m \colon K(\text{volRV}^{\text{bdd}}[*])/I'_{\text{sp}} \longrightarrow !K(\text{RES})([\mathbb{A}^1]^{-1})_{\text{loc}}.$

In particular, if α and α' are two elements in $K(\text{volRV}^{2\text{bdd}}[*])$ with same image in $K(\text{volRV}[*])/I'_{\text{sp}}$, then $h_m(\alpha)$ and $h_m(\alpha')$ have the same image in $!K(\text{RES})([\mathbb{A}^1]^{-1})_{\text{loc}}$.

Let us now state the analogue of Proposition 4.2.2 in this context.

Proposition 8.2.3. – *Let m be a positive integer. Let n and r be integers, let $\beta \in \Gamma^n$ and let X be a β-invariant $F((t))$-definable subset of $\mathcal{O}^n \times \mathrm{RV}^r$. We assume that X is contained in $\mathrm{VF}^n \times W$ with W a boundedly imaginary definable subset of RV^r, and that X_w is bounded, for every $w \in W$. We also assume that the projection $X \to \mathrm{VF}^n$ has finite fibers. Then $\tilde{h}_m(\int([X]))$ is equal to the image of the class $\tilde{X}[m]$ as defined in §4.2 in $!K(\mathrm{RES})([\mathbb{A}^1]^{-1})_{\mathrm{loc}}$.*

Proof. – Since both sides are invariant under the transformations of Proposition 3.2.2, we may assume by Proposition 3.2.2 that there exists a definable boundedly imaginary subset H of $\mathrm{RV}^{r'}$ and a map $h\colon \{1,\dots,n\} \to \{1,\dots,r'\}$ such that

$$(8.2.15) \qquad X = \{(a,b); b \in H, \mathrm{rv}(a_i) = b_{h(i)}, 1 \le i \le n\}$$

and the map $r\colon H \to \mathrm{RV}^n$ given by $b \mapsto (b_{h(1)},\dots,b_{h(n)})$ is finite to one. According to (8.1.6) we may assume $[H] = \Psi([W] \otimes [\Delta])$ with W in $\mathrm{RES}[r]$ and Δ bounded in $\Gamma[n-r]$. By induction on dimension and considering products, it is enough to prove the result when X is the lifting of W or Δ. In both cases, this is clear by construction. $\qquad\square$

Remark 8.2.4. – The definition of the morphisms of $\mathrm{volVF}[n]$ refers implicitly to the standard volume form on K^n, restricted to $\mathcal{O}^n$. When an n-dimensional variety is given without a specific embedding, we must specify a volume form since, in principle, integrals depend on the form, up to multiplication by a definable function into $\mathbb{G}_m(\mathcal{O})$. However, when V is a smooth variety over F, with a volume form ω (a nowhere vanishing section of $\bigwedge^{\mathrm{top}} TV$) defined over F, and X is a bounded, β-invariant $F((t))$-definable subset of $V(\mathcal{O})$, then $\int([X])$ does not depend on the choice of ω, as long as ω is chosen over F. The reason is that given another such form ω', we have $\omega' = g\omega$ for some non-vanishing regular functions g on V, defined over F. Thus, denoting by red the reduction mapping $V(\mathcal{O}) \to V$, for $u \in V$ we have $\mathrm{red}(g(u)) = g(\mathrm{red}(u)) \ne 0$ so $\mathrm{val}(g(u)) = 0$. In particular, we shall refer to $\int([X]) \in K(\mathrm{volRV}[n])$ in this setting without further mention of the volume form.

8.3. Expressing the motivic zeta function

Let $K^{\hat{\mu}}(\mathrm{Var}_F)_{\mathrm{loc}}$ denote the localisation of $K^{\hat{\mu}}(\mathrm{Var}_F)$ with respect to the multiplicative family generated by $[\mathbb{A}^1]$ and the elements $1 - [\mathbb{A}^1]^i$, $i \ge 1$. Note that this is equivalent to localising first with respect to the multiplicative family generated by $[\mathbb{A}^1]$ and then with respect to the multiplicative family generated by the elements $1 - [\mathbb{A}^1]^{-i}$, $i \ge 1$. One defines similarly $K(\mathrm{Var}_F)_{\mathrm{loc}}$. The isomorphism Θ of (4.3.4) induces isomorphisms

$$(8.3.1) \qquad \Theta\colon !K(\mathrm{RES})[[\mathbb{A}^1]^{-1}] \longrightarrow K^{\hat{\mu}}(\mathrm{Var}_F)[[\mathbb{A}^1]^{-1}]$$

and

$$(8.3.2) \qquad \Theta\colon !K(\mathrm{RES})[[\mathbb{A}^1]^{-1}]_{\mathrm{loc}} \longrightarrow K^{\hat{\mu}}(\mathrm{Var}_F)_{\mathrm{loc}}.$$

Let X be a smooth connected algebraic variety of dimension d over F and f a non-constant regular function $f\colon X \to \mathbb{A}^1_F$.

For any $m \ge 1$, we consider $\mathscr{X}_{m,x}$ as defined in (1.1.4)

$$(8.3.3) \qquad \mathscr{X}_{m,x} = \{\varphi \in X(\mathbb{C}[t]/t^{m+1}); f(\varphi) = t^m \mod t^{m+1}, \varphi(0) = x\}$$

and $\mathscr{X}_x$ from Corollary 4.2.3

$$(8.3.4) \qquad \mathscr{X}_x = \{y \in X(\mathcal{O}); \mathrm{rv} f(y) = \mathrm{rv}(t) \text{ and } \pi(y) = x\}.$$

Recall that $\mathscr{X}_x$ is β-invariant for every $\beta > 0$. After replacing X by an affine open containing x, we may assume the existence of a volume form on X defined over F. Thus, using the convention in Remark 8.2.4, we may consider $\tilde{h}_m(\int([\mathscr{X}_x]))$ in $!K(\mathrm{RES})[[\mathbb{A}^1]^{-1}]_{\mathrm{loc}}$.

We have the following interpretation for the class of $\mathscr{X}_{m,x}$.

PROPOSITION 8.3.1. – *Let X be a smooth connected variety over F of dimension d, f be a regular function on X which is non identically zero, and x be a closed point of $f^{-1}(0)$. Then, for every integer $m \geq 1$,*

$$\Theta\left(\tilde{h}_m\left(\int([\mathscr{X}_x])\right)\right) = [\mathscr{X}_{m,x}][\mathbb{A}^{md}]^{-1}$$

in $K^{\hat{\mu}}(\mathrm{Var}_F)_{\mathrm{loc}}$.

Proof. – By definition, using notation from 4.2,

$$\widetilde{\mathscr{X}_x}[m] = [\mathscr{X}_x[m; 1 + 1/m]][\mathbb{A}^{md}]^{-1}.$$

It follows from Proposition 8.2.3, by a similar argument as the one in the proof of Corollary 4.2.3, that $\tilde{h}_m(\int([\mathscr{X}_x]))$ and $\widetilde{\mathscr{X}_x}[m]$ have the same image in $!K(\mathrm{RES})([\mathbb{A}^1]^{-1})_{\mathrm{loc}}$. On the other hand, since, as already observed in 6.2, $\mathscr{X}_{m,x}$ is isomorphic to $\{\varphi \in X(\mathbb{C}[t^{1/m}]/t^{(m+1)/m}); \mathrm{rv}(f(\varphi)) = \mathrm{rv}(t), \varphi(0) = x\}$, $\mathscr{X}_{m,x}$ and $\Theta([\mathscr{X}_x[m; 1 + 1/m]])$ have the same class in in $K^{\hat{\mu}}(\mathrm{Var}_F)$. The result follows. □

The motivic zeta function $Z_{f,x}(T)$ attached to (f, x) is the following generating function, cf. [9], [12],

(8.3.5) $$Z_{f,x}(T) = \sum_{m \geq 1} [\mathscr{X}_{m,x}][\mathbb{A}^{md}]^{-1} T^m$$

in $K^{\hat{\mu}}(\mathrm{Var}_F)[[\mathbb{A}^1]^{-1}][\![T]\!]$.

Let $\iota\colon K^{\hat{\mu}}(\mathrm{Var}_F)[[\mathbb{A}^1]^{-1}] \to K^{\hat{\mu}}(\mathrm{Var}_F)_{\mathrm{loc}}$ denote the localisation morphism. Applying ι termwise to $Z_{f,x}(T)$ we obtain a series $\tilde{Z}_{f,x}$ in $K^{\hat{\mu}}(\mathrm{Var}_F)_{\mathrm{loc}}[\![T]\!]$.

Thus, by Proposition 8.3.1, $\tilde{Z}_{f,x}(T)$ may be expressed directly in terms of $\mathscr{X}_x$:

COROLLARY 8.3.2. – *Let X be a smooth connected variety over F of dimension d, f be a regular function on X which is non identically zero, and x be a closed point of $f^{-1}(0)$. Then,*

$$\tilde{Z}_{f,x}(T) = \sum_{m \geq 1} \Theta\left(\tilde{h}_m\left(\int([\mathscr{X}_x])\right)\right) T^m.$$

8.4. Rational series

Let R be a ring and let $\mathbb{A}$ be an invertible element in R. We consider the ring $R[T]_\dagger$ (resp. $R[T, T^{-1}]_\dagger$) which is the localization of $R[T]$ (resp. $R[T, T^{-1}]$) with respect to the multiplicative family generated by $1 - \mathbb{A}^a T^b$, $a \in \mathbb{Z}$, $b \geq 1$. By expanding into powers in T one gets a morphism

(8.4.1) $$e_T\colon R[T]_\dagger \longrightarrow R[\![T]\!]$$

which is easily checked to be injective. We shall identify an element in $R[T]_\dagger$ with its image in $R[\![T]\!]$. If $h = P/Q$ belongs to $R[T]_\dagger$, the difference $\deg(P) - \deg(Q)$ depends only on h, thus will be denoted $\deg(h)$. If $\deg(h) \leq 0$, we define $\lim_{T \to \infty} h$ as follows. If $\deg(h) < 0$,

we set $\lim_{T \to \infty} h = 0$. If $h = P/Q$ with P and Q of degree n, let p and q be the leading coefficients of P and Q. Since q is of the form $\varepsilon \mathbb{A}^a$ for some $a \in \mathbb{Z}$ and $\varepsilon \in \{-1, 1\}$, we may set $\lim_{T \to \infty} h = p\varepsilon \mathbb{A}^{-a}$, which is independent from the choice of P and Q.

Since

$$(8.4.2) \qquad \frac{1}{1 - \mathbb{A}^a T^b} = -\frac{\mathbb{A}^{-a} T^{-b}}{1 - \mathbb{A}^{-a} T^{-b}},$$

one may also expand elements of $R[T]_\dagger$ into powers of T^{-1}, giving rise to a morphism

$$(8.4.3) \qquad e_{T^{-1}} \colon R[T]_\dagger \longrightarrow R[\![T^{-1}]\!][T].$$

In particular, if h belongs to $R[T]_\dagger$, $\deg(h) \leq 0$ if and only if $e_{T^{-1}}(h)$ belongs to $R[\![T^{-1}]\!]$. Furthermore, in this case $\lim_{T \to \infty} h$ is equal to the constant term of $e_{T^{-1}}(h)$.

If $f(T) = \sum_{n \geq 0} a_n T^n$ and $g(T) = \sum_{n \geq 0} b_n T^n$ are two series in $R[\![T]\!]$ one defines their Hadamard product as $(f * g)(T) = \sum_{n \geq 0} a_n b_n T^n$.

LEMMA 8.4.1. – *Let h and h' belong to $R[T]_\dagger$. Set $\varphi = e_T(h)$, $\varphi' = e_T(h')$.*

(1) *There exists a (unique) element $\tilde{h}$ in $R[T]_\dagger$ such that $e_T(\tilde{h}) = \varphi * \varphi'$.*

(2) *Assume that φ and φ' belong to $T R[\![T]\!]$, and that $\deg(h), \deg(h') \leq 0$. Then $\deg(\tilde{h}) \leq 0$ and*

$$\lim_{T \to \infty} \tilde{h} = - \lim_{T \to \infty} h \cdot \lim_{T \to \infty} h'.$$

Proof. – Assertion (1) follows from Propositions 5.1.1 and 5.1.2 of [11] and their proofs. Indeed, by (the proof of) Proposition 5.1.1 of [11], there exists $\tilde{h} \in R[T, T^{-1}]_\dagger$ such that $e_T(\tilde{h}) = \varphi * \varphi'$ (with e_T extended to a morphism $R[T, T^{-1}]_\dagger \to R[\![T]\!][T^{-1}]$). But this forces $\tilde{h}$ to belong in fact to $R[T]_\dagger$. By (the proof of) Proposition 5.1.1 of [11], cf. also Proposition 5.1.2 of [11] and its proof, it follows from the assumptions in (2) that

$$(e_{T^{-1}}(\tilde{h}))(T^{-1}) = -(e_{T^{-1}}(h))(T^{-1}) * (e_{T^{-1}}(h'))(T^{-1}).$$

Thus $\deg(\tilde{h}) \leq 0$ and $\lim_{T \to \infty} \tilde{h} = - \lim_{T \to \infty} h \cdot \lim_{T \to \infty} h'$. $\qquad \square$

When $\varphi \in R[\![T]\!]$ is of the form $e_T(h)$ with $h \in R[T]_\dagger$, we shall say $\lim_{T \to \infty} \varphi$ *exists* if $\deg(h) \leq 0$, and set $\lim_{T \to \infty} \varphi = \lim_{T \to \infty} h$.

8.5. Expressing the motivic Milnor fiber

We consider the rings $K^{\hat{\mu}}(\mathrm{Var}_F)[[\mathbb{A}^1]^{-1}][T]_\dagger$ and $K^{\hat{\mu}}(\mathrm{Var}_F)_{\mathrm{loc}}[T]_\dagger$ with $\mathbb{A} = [\mathbb{A}^1]$. More generally, in this section, when we write $R[T]_\dagger$ it will always be with $\mathbb{A} = [\mathbb{A}^1]$.

It is known that the motivic zeta function $Z_{f,x}(T)$ belongs to $K^{\hat{\mu}}(\mathrm{Var}_F)[[\mathbb{A}^1]^{-1}][T]_\dagger$ and that $\lim_{T \to \infty} Z_{f,x}(T)$ exists, cf. [12], [27].

One sets

$$(8.5.1) \qquad \mathscr{S}_{f,x} = - \lim_{T \to \infty} Z_{f,x}(T).$$

This element of $K^{\hat{\mu}}(\mathrm{Var}_F)[[\mathbb{A}^1]^{-1}]$ is the motivic Milnor fiber considered in [12], [27]. We shall show in Corollary 8.5.3 how one may extract directly the image of $\mathscr{S}_{f,x}$ in $K^{\hat{\mu}}(\mathrm{Var}_F)_{\mathrm{loc}}$ from $\int([\mathscr{X}_x])$.

Let χ denote the o-minimal Euler characteristic. There exists a unique morphism

$$(8.5.2) \qquad \alpha \colon K(\mathrm{vol}\Gamma[*]) \longrightarrow {!}K(\mathrm{RES})([\mathbb{A}^1]^{-1})$$

which, for every $n \geq 0$, sends the class of Δ in $K(\mathrm{vol}\Gamma[n])$ to $\chi(\Delta)([\mathbb{A}^1] - 1)^n$, and a unique morphism

$$(8.5.3) \qquad \beta \colon K(\mathrm{volRES}[*]) \longrightarrow {!}K(\mathrm{RES})([\mathbb{A}^1]^{-1})$$

which, for every $n \geq 0$, sends the class of Y in $K(\mathrm{volRES}[n])$ to $[Y]$.

Taking the tensor product of α and β one gets a morphism

$$(8.5.4) \qquad \Upsilon \colon K(\mathrm{volRV}[*]) \longrightarrow {!}K(\mathrm{RES})([\mathbb{A}^1]^{-1})$$

since the relations (8.1.5) in the kernel of the morphism (8.1.7) are respected. One defines similarly a morphism

$$(8.5.5) \qquad \Upsilon \colon K(\mathrm{volRV}^{\mathrm{2bdd}}[*]) \longrightarrow {!}K(\mathrm{RES})([\mathbb{A}^1]^{-1}).$$

PROPOSITION 8.5.1. – *Let Y be in $K(\mathrm{volRV}^{\mathrm{2bdd}}[*])$. The series*

$$Z(Y)(T) = \sum_{m \geq 1} h_m(Y) T^m$$

in ${!}K(\mathrm{RES})([\mathbb{A}^1]^{-1})\llbracket T \rrbracket$ belongs to ${!}K(\mathrm{RES})([\mathbb{A}^1]^{-1})[T]_\dagger$, $\lim_{T \to \infty} Z(Y)(T)$ exists and

$$\lim_{T \to \infty} Z(Y)(T) = -\Upsilon(Y).$$

Proof. – We may assume Y is of the form $\Psi([W] \otimes [\Delta])$ with W in $\mathrm{RES}[p]$ and Δ in $\Gamma[q]$. By Lemma 8.4.1, $Z(Y)(T)$ is the Hadamard product of $Z(\Psi([W] \otimes 1))(T)$ and $Z(\Psi(1 \otimes [\Delta]))(T)$. Thus it is enough to prove the statement for $\Psi([W] \otimes 1)$ and $\Psi(1 \otimes [\Delta])$. By construction,

$$(8.5.6) \qquad Z(\Psi([W] \otimes 1))(T) = [W] \sum_{m \geq 1} [\mathbb{A}^1]^{-\alpha m} T^{\beta m}$$

for some integers $\alpha \in \mathbb{Z}$ and $\beta \geq 1$. Hence $Z(\Psi([W] \otimes 1))(T)$ belongs to ${!}K(\mathrm{RES})([\mathbb{A}^1]^{-1})[T]_\dagger$, $\lim_{T \to \infty} Z(\Psi([W] \otimes 1))(T)$ exists and is equal to $-[W] \quad = -\Upsilon(\Psi([W] \otimes 1))$.

The statement for $\Psi(1 \otimes [\Delta])$ follows from Lemma 8.5.2, using the morphism $\mathbb{Z}[U, U^{-1}] \to {!}K(\mathrm{RES})([\mathbb{A}^1]^{-1})$ sending U to $[\mathbb{A}^1]^{-1}$. $\qquad\square$

LEMMA 8.5.2. – *Let Δ be a bounded definable subset of Γ^n. Let $\ell \colon \Delta \to \Gamma$ be piecewise (i.e., on each piece of a finite definable partition) of the form $x = (x_i) \mapsto \sum a_i x_i + b$, with the a_i's and b in $\mathbb{Z}$. For every integer $m \geq 1$, set*

$$s_m(\Delta, \ell) = \sum_{(\gamma_1, \dots, \gamma_n) \in \Delta \cap (m^{-1}\mathbb{Z})^n} U^{-m\ell(\gamma)},$$

in $\mathbb{Z}[U, U^{-1}]$ and set

$$Z(\Delta, \ell)(T) = \sum_{m \geq 1} s_m(\Delta, \ell) T^m$$

in $\mathbb{Z}[U, U^{-1}]\llbracket T \rrbracket$. Then, the series $Z(\Delta, \ell)$ belongs to $\mathbb{Z}[U, U^{-1}][T]_\dagger$, $\lim_{T \to \infty} Z(\Delta, \ell)(T)$ exists and

$$\lim_{T \to \infty} Z(\Delta, \ell)(T) = -\chi(\Delta).$$

Proof. – Let Δ be a bounded definable subset of Γ^n. We shall say the lemma holds for Δ if it holds for Δ and any ℓ. If Δ' is another bounded definable subset of Γ^n such that $[\Delta] = [\Delta']$ in $K_+(\Gamma_{\mathbb{Z}}^{\mathrm{bdd}})[n]$, then the lemma holds for Δ if and only if it holds for Δ'. Thus the property for Δ depends only on its class in $K_+(\Gamma_{\mathbb{Z}}^{\mathrm{bdd}})[n]$. Localisation with respect to $[0]_1$ is harmless here, and one deduces that the property of satisfying the lemma for Δ depends only on its class $[\Delta]/[0]_1^n$ in $K_+^{df}(\Gamma_{\mathbb{Z}}^{\mathrm{bdd}})$. We shall say $[\Delta]/[0]_1^n$ satisfies the lemma if Δ does.

Let I be a definable bounded interval in Γ and $\ell' \colon \Gamma \to \Gamma$ a linear form $x \mapsto ax + b$ with a and b in $\mathbb{Z}$. Then, by a direct geometric series computation one gets that the lemma holds for I and ℓ'. It follows that the lemma holds for I and any ℓ. In particular, the lemma holds for the subsets $[0, \gamma)$ and $\{\gamma\}$ of Γ, with γ in $\mathbb{Q}$. Let $K_+^{df}(\Gamma_{\mathbb{Z}}^{\mathrm{bdd}})'$ be the sub-semi-ring of $K_+^{df}(\Gamma_{\mathbb{Z}}^{\mathrm{bdd}})$ generated by $[\gamma]_1/[0]_1$ and $[0, \gamma)_1/[0]_1$, for γ in $\mathbb{Q}$. It follows from Lemma 8.4.1 that the lemma holds for all elements in $K_+^{df}(\Gamma_{\mathbb{Z}}^{\mathrm{bdd}})'$ since it holds for the generators $[\gamma]_1/[0]_1$ and $[0, \gamma)_1/[0]_1$. By Lemma 2.21 of [21], for any element a in $K_+^{df}(\Gamma_{\mathbb{Z}}^{\mathrm{bdd}})$ there exists a nonzero $m \in \mathbb{N}$, b and c in $K_+^{df}(\Gamma_{\mathbb{Z}}^{\mathrm{bdd}})'$ such that $ma + b = c$. Since the lemma holds for b and c, it follows that the lemma holds for ma, hence for a, and the statement follows. $\square$

COROLLARY 8.5.3. – *Let X be a smooth connected variety over F of dimension d, f be a regular function on X which is non identically zero, and x be a closed point of $f^{-1}(0)$. Then the image of $\mathscr{S}_{f,x}$ in $K^{\hat{\mu}}(\mathrm{Var}_F)_{\mathrm{loc}}$ is equal to*

$$\Theta\Big(\Upsilon\Big(\int([\mathscr{X}_x])\Big)\Big).$$

Proof. – This follows directly from Corollary 8.3.2 and Proposition 8.5.1. $\square$

REMARK 8.5.4. – It is not known whether the localisation morphisms $\iota\colon K(\mathrm{Var}_F)[[\mathbb{A}^1]^{-1}] \to K(\mathrm{Var}_F)_{\mathrm{loc}}$ and $\iota\colon K^{\hat{\mu}}(\mathrm{Var}_F)[[\mathbb{A}^1]^{-1}] \to K^{\hat{\mu}}(\mathrm{Var}_F)_{\mathrm{loc}}$ are injective. However, the morphism $H\colon K(\mathrm{Var}_F)[[\mathbb{A}^1]^{-1}] \to \mathbb{Z}[u, v, u^{-1}, v^{-1}]$ induced by the Hodge-Deligne polynomial vanishes on the kernel of ι, hence factors through the image of ι. In particular, the Euler characteristic with compact supports $\chi_c\colon K(\mathrm{Var}_F)[[\mathbb{A}^1]^{-1}] \to \mathbb{Z}$ factors through the image of ι. This extends to the equivariant setting. In particular one can recover the Hodge-Steenbrink spectrum of f at x from the image of $\mathscr{S}_{f,x}$ in $K^{\hat{\mu}}(\mathrm{Var}_F)_{\mathrm{loc}}$, cf. [12], [27].

REMARK 8.5.5. – When $F = \mathbb{C}$, Theorem 1.1.1 together with Corollary 8.5.3 provides a proof avoiding resolution of singularities that the topological Milnor fiber F_x and the motivic Milnor fiber $\mathscr{S}_{f,x}$ have the same Euler characteristic with compact supports, namely that $\chi_c(F_x) = \chi_c(\mathscr{S}_{f,x})$. Indeed, by Remark 8.5.4 one may apply χ_c to (8.5.1), thus getting $\chi_c(\mathscr{S}_{f,x}) = -\lim_{T \to \infty} \sum_{m \geq 1} \chi_c(\mathscr{X}_{m,x}) T^m$, which may be rewritten, by Theorem 1.1.1, as $\chi_c(\mathscr{S}_{f,x}) = -\lim_{T \to \infty} \sum_{m \geq 1} \Lambda(M_x^m) T^m$. By quasi-unipotence of local monodromy (a statement for which there exist proofs not using resolution of singularities, see, e.g., SGA 7 I 1.3), there is an integer m_0 such that all eigenvalues of M_x on the cohomology groups of F_x have order dividing m_0. Thus $\sum_{m \geq 1} \Lambda(M_x^m) T^m$ can be rewritten as $\sum_{1 \leq i \leq m_0} \Lambda(M_x^i) \frac{T^i}{1-T^{m_0}}$ and the equality $\chi_c(\mathscr{S}_{f,x}) = \Lambda(M_x^{m_0}) = \chi_c(F_x)$ follows.

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