ISSN 0012-9593

quatrième série - tome 48 fascicule 2 mars-avril 2015

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

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Tarifs

Europe : 515 €. Hors Europe : 545 €. Vente au numéro : 77 €.

ISSN 0012-9593

Directeur de la publication : Marc Peigné
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4e série, t. 48, 2015, p. 281 à 311

DEFORMATIONS OF LEVI FLAT HYPERSURFACES IN COMPLEX MANIFOLDS

BY PAOLO DE BARTOLOMEIS AND ANDREI IORDAN

ABSTRACT. – We first give a deformation theory of integrable distributions of codimension 1. This theory is used to study Levi-flat deformations: a Levi-flat deformation of a Levi flat hypersurface L in a complex manifold is a smooth mapping $\Psi : I\times M \to M$ such that $\Psi_t = \Psi(t,\cdot) \in \mathrm{Diff}(M)$, $L_t = \Psi_t L$ is a Levi flat hypersurface in M for every $t \in I$ and $L_0 = L$. We define a parametrization of families of smooth hypersurfaces near L such that the Levi flat deformations are given by the solutions of the Maurer-Cartan equation in a DGLA associated to the Levi foliation. We say that L is infinitesimally rigid if the tangent cone at the origin to the moduli space of Levi flat deformations of L is trivial. We prove the infinitesimal rigidity of compact transversally parallelizable Levi flat hypersurfaces in compact complex manifolds and give sufficient conditions for infinitesimal rigidity in Kähler manifolds. As an application, we prove the nonexistence of transversally parallelizable Levi flat hypersurfaces in a class of manifolds which contains $\mathbb{CP}_2$.

RÉSUMÉ. – Nous commençons par présenter une théorie des déformations de distributions intégrables de codimension 1. Cette théorie est utilisée pour étudier les déformations d'hypersurfaces Levi plates: une déformation Levi plate d'une hypersurface Levi plate L dans une variété complexe M est une application lisse $\Psi : I \times M \to M$ telle que $\Psi_t = \Psi(t,\cdot) \in \mathrm{Diff}(M)$, $L_t = \Psi_t L$ est une hypersurface Levi plate dans M pour tout $t \in I$ et $L_0 = L$. Nous définissons une paramétrisation des hypersurfaces Levi plates au voisinage de L telle que les déformations d'hypersurfaces Levi plates de L sont données par les solutions de l'équation de Maurer-Cartan dans une DGLA associée au feuilletage de Levi.

Nous disons que L est infinitésimalement rigide si le cône tangent à l'origine de l'espace de modules des déformations Levi plates de L est trivial. Nous prouvons que les hypersurfaces de Levi plates compactes transversalement parallélisables dans les variétés complexes compactes sont infinitésimalement rigides et nous donnons des conditions suffisantes pour la rigidité infinitésimale dans les variétés de Kähler. Comme application, nous démontrons la non existence d'hypersurfaces Levi plates transversalement parallélisables dans une classe de variétés qui contient l'espace projectif complexe de dimension $n \geqslant 2$.

The first author was supported by the M.I.U.R. project "Geometric Properties of Real and Complex Manifolds" and by G.N.S.A.G.A. of INdAM.

1. Introduction

Let M be a complex manifold and L a real hypersurface of class C^2 in M such that $M\backslash L = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \varnothing$. L is Levi flat if it satisfies one of the following equivalent conditions:

1) Ω_1 and Ω_2 are pseudoconvex domains.
2) L is foliated by complex hypersurfaces of M.
3) The Levi form of L vanishes.

It is well known that in general, if L is not of class C^2, we have only 3) $\implies$ 2) $\implies$ 1).

One of the oldest result concerning Levi flat hypersurfaces is a theorem of E. Cartan [3] which states that a real analytic Levi flat hypersurface is locally isomorphic to the set of vanishing of the real part of a holomorphic function. A generalization of this theorem for singular Levi flat hypersurfaces can be found in [9].

Recent research on Levi flat hypersurfaces in complex manifolds were motivated by the following conjecture of D. Cerveau [4]: there are no smooth Levi flat hypersurfaces in the complex projective space $\mathbb{CP}_n$, $n \geqslant 2$.

For $n \geqslant 3$, this conjecture was proved by Lins Neto for real analytic Levi flat hypersurfaces [15], by Y.-T. Siu for Levi flat hypersurfaces of class C^{12} [18] and by A. Iordan and F. Matthey for Lipschitz hypersurfaces of Sobolev class W^s, $s > 5/2$ [11]. Despite several attempts to prove this conjecture for $n = 2$, its proof is still incomplete.

Unlike $\mathbb{CP}_n$, $n \geqslant 2$, the complex tori $\mathbb{T}_n = \mathbb{C}^n/\Gamma$ contains the Levi flat hypersurfaces $\pi\left(\oplus_{j=1}^{2n-1}\mathbb{R}u_j + u\right)$ where $\pi : \mathbb{C}^n \to \mathbb{T}_n$ is the canonical projection, u_j, $j = 1, \ldots, 2n-1$, are $\mathbb{R}$-linearly independent vectors in Γ and $u \in \mathbb{C}^n$ [16]. It was conjectured in [16] that for every compact Levi flat hypersurface M in $\mathbb{T}_n$, $\pi^{-1}(M)$ is a union of affine hyperplanes.

In this paper we study the deformations of smooth Levi flat hypersurfaces in complex manifolds. The theory of deformations of complex manifolds was intensively studied from the 50s beginning with the famous results of Kodaira and Spencer [13] (see for ex. [12], [21]). In [17], Nijenhuis ans Richardson adapted a theory initiated by Gerstenhaber [6] and proved the connection between the deformations of complex analytic structures and the theory of differential graded Lie algebras (DGLA). This theory was developed following ideas of Deligne by Goldman and Millson [8].

The main results of this paper may be summarized as follows.

In the first chapter we consider integrable distributions of codimension 1 on smooth manifolds and we define a DGLA associated to the foliation such that the deformations of integrable distributions of codimension 1 are given by solutions of Maurer-Cartan equation in this algebra. As the examples show, this theory is highly non trivial and it seems to be interesting by itself. We mention that Kodaira and Spencer developed in [14] a theory of deformations of the so called multifoliate structures, which are more general than the foliate structures. Our approach in this paper for foliations of codimension 1 is different of theirs (see Remark 14) and allows us to study the Levi flat case.

In the second chapter we give a description of the deformations of a smooth Levi flat hypersurface L in a complex manifold by means of the Maurer-Cartan equation in the DGLA associated to the Levi foliation.

Then we establish the equations verified by the tangent to a regular familly of Levi flat deformations. We say that L is infinitesimally rigid (respectively strongly infinitesimally rigid) if the tangent cone at the origin to the moduli space of Levi flat deformations of L is trivial (respectively if the tangent cone at the origin to the solutions of the Maurer-Cartan equation in the DGLA associated to the Levi foliation is trivial) . We remark that Diederich and Ohsawa study in [5] the displacement rigidity of Levi flat hypersurfaces in disc bundle over compact Riemann surfaces. The definition of rigidity in [5] means that any small C^2 perturbation of a Levi flat hypersurface L is CR isomorphic with L, so L is strongly infinitesimally rigid.

We prove that a transversally parallelizable compact Levi flat hypersurface in a compact complex manifold is strongly infinitesimally rigid and we give a sufficient condition for infinitesimal rigidity in Kähler manifolds (Theorem 3). As an application, we prove that there are no compact transversally parallelizable Levi flat hypersurfaces in connected complex manifolds M such that for every $p \neq q \in M$ and every real hyperplane H_q in T_qM there exists a holomorphic vector field Y on M such that $Y(p) = 0$ and $Y(q) \oplus H_q = T_qM$. If $M = \mathbb{CP}_n$, $n \geq 2$, the hypotheses of the previous result are fulfilled.

The non existence of transversally parallelizable Levi flat hypersurfaces in $\mathbb{CP}_2$ can be obtained by different proofs. We chose here to give a proof by using the results of this paper. Another direct proof was furnished to the authors by Marco Brunella [2] who disappeared recently in a tragic accident. We want to pay tribute to the memory of Marco Brunella by giving also his proof of this result.

2. Deformation theory of integrable distribution of codimension 1

2.1. DGLA associated to an integrable distribution of codimension 1

DEFINITION 1. – *A differential graded Lie algebra (DGLA) is a triple* $(V^*, d, [\cdot,\cdot])$ *such that:*

1) $V^* = \oplus_{i\in\mathbb{N}} V^i$, *where* $\left(V^i\right)_{i\in\mathbb{N}}$ *is a family of* $\mathbb{C}$*-vector spaces and* $d : V^* \to V^*$ *is a graded homomorphism such that* $d^2 = 0$. *An element* $a \in V^k$ *is said to be homogeneous of degree* $k = \deg a$.

2) $[\cdot,\cdot] : V^* \times V^* \to V^*$ *defines a structure of graded Lie algebra i.e., for homogeneous elements we have*

$$[a,b] = -(-1)^{\deg a \deg b}[b,a] \tag{2.1}$$

and

$$[a,[b,c]] = [[a,b],c] + (-1)^{\deg a \deg b}[b,[a,c]]. \tag{2.2}$$

3) d *is compatible with the graded Lie algebra structure i.e.,*

$$d[a,b] = [da,b] + (-1)^{\deg a}[a,db]. \tag{2.3}$$

REMARK 1. – *If* (2.1) *is satisfied then* (2.2) *is equivalent to*

$$\mathfrak{S}_s(-1)^{\deg a \deg c}[a,[b,c]] = 0 \tag{2.4}$$

where $\mathfrak{S}_s$ *denotes the symmetric sum.*

DEFINITION 2. – *Let* $(V^*, d, [\cdot,\cdot])$ *be a DGLA and* $a \in V^1$. *We say that* a *verifies the Maurer-Cartan equation in* $(V^*, d, [\cdot,\cdot])$ *if*

$$da + \frac{1}{2}[a,a] = 0. \tag{2.5}$$

LEMMA 1. – *Let* $(V^*, d, [\cdot,\cdot])$ *be a DGLA and* $a \in V^1$. *Set* $d_a = d + [a,\cdot]$. *Then for every* $\omega \in V^*$ *we have*

$$d_a^2\omega = \left[da + \frac{1}{2}[a,a], \omega\right].$$

Proof. – Let $\omega \in V^k$. Since d satisfies (2.3) we have

$$\begin{aligned} d_a^2\omega &= (d + [a,\cdot])(d\omega + [a,\omega]) = d[a,\omega] + [a,d\omega] + [a,[a,\omega]] \\ &= [da,\omega] - [a,d\omega] + [a,d\omega] + [a,[a,\omega]] \\ &= [da,\omega] + [a,[a,\omega]]. \end{aligned}$$

But (2.2) gives

$$[a,[a,\omega]] = \frac{1}{2}[[a,a],\omega]$$

and the lemma follows. □

From Lemma 1 we obtain the following

COROLLARY 1. – *Let* $(V^*, d, [\cdot,\cdot])$ *be a DGLA and* $a \in V^1$ *verifying the Maurer-Cartan equation (2.5). Then* $d_a^2 = 0$. *Moreover, if* $Z(V^*) = \{0\}$, *where* $Z(V^*) = \{\beta \in V^* : [\beta,\alpha] = 0, \forall \alpha \in V^*\}$ *is the center of* $(V^*, d, [\cdot,\cdot])$, *then* a *verifies Maurer-Cartan equation* (2.5) *if and only if* $d_a^2 = 0$.

The starting point of the theory developed in this section is the following:

LEMMA 2. – *Let* L *be a* C^∞ *manifold and let* X *be a vector field on* L. *We denote by* $\Lambda^k(L)$ *the* k*-forms on* L *and* $\Lambda^*(L) = \oplus_{k\in\mathbb{N}}\Lambda^k(L)$. *For* $\alpha, \beta \in \Lambda^*(L)$, *set*

$$\{\alpha,\beta\} = \mathcal{L}_X\alpha \wedge \beta - \alpha \wedge \mathcal{L}_X\beta \tag{2.6}$$

where $\mathcal{L}_X$ *is the Lie derivative. Then* $(\Lambda^*(L), d, \{\cdot,\cdot\})$ *is a DGLA.*

Proof. – Since (2.1) is obvious we will verify (2.4). We have

$$\begin{aligned} \mathfrak{S}_s(-1)^{\deg a \deg c}\{a,\{b,c\}\} = \mathfrak{S}_s(-1)^{\deg a \deg c}(&\mathcal{L}_X a \wedge \mathcal{L}_X b \wedge c \\ &- \mathcal{L}_X a \wedge b \wedge \mathcal{L}_X c - a \wedge \mathcal{L}_X^2 b \wedge c + a \wedge b \wedge \mathcal{L}_X^2 c). \end{aligned}$$

Since

$$(-1)^{\deg c \deg a}\mathcal{L}_X a \wedge \mathcal{L}_X b \wedge c = (-1)^{\deg a \deg b}\mathcal{L}_X b \wedge c \wedge \mathcal{L}_X a$$

and

$$(-1)^{\deg a \deg c} a \wedge \mathcal{L}_X^2 b \wedge c = (-1)^{\deg b \deg c} c \wedge a \wedge \mathcal{L}_X^2 b,$$

it follows that

$$\mathfrak{S}_s(-1)^{\deg a \deg c}\{a,\{b,c\}\} = 0.$$

By using Cartan's formula

$$\mathcal{L}_X = \iota_X d + d\iota_X$$

we obtain

$$\begin{aligned} d\{a,b\} &= d\left(\left(\iota_X d + d\iota_X\right) a \wedge b - a \wedge \left(\iota_X d + d\iota_X\right) b\right) \\ &= d\iota_X da \wedge b + (-1)^{\deg a} \iota_X da \wedge db + (-1)^{\deg a} d\iota_X a \wedge db \\ &\quad - da \wedge \iota_X db - da \wedge d\iota_X b - (-1)^{\deg a} a \wedge d\iota_X db \\ &= \{da, b\} + (-1)^{\deg a} \{a, db\}. \end{aligned}$$

□

LEMMA 3. – *Let L be a C^∞ manifold and $\xi \subset T(L)$ a distribution of codimension 1. Let $\gamma \in \wedge^1(L)$ such that* $\ker \gamma = \xi$ *and let X be a vector field on L such that $\gamma(X) = 1$. Then the following are equivalent:*

i) *ξ is integrable;*

ii) *There exists $\alpha \in \wedge^1(L)$ such that $d\gamma = \alpha \wedge \gamma$;*

iii) *$d\gamma \wedge \gamma = 0$;*

iv) *$d\gamma = -\iota_X d\gamma \wedge \gamma$;*

v) *γ satisfies the Maurer-Cartan equation* (2.5) *in $(\Lambda^*(L), d, \{\cdot,\cdot\})$, where $\{\cdot,\cdot\}$ is defined in* (2.6).

Proof. – ii) ⇒ iii) and iv) ⇒ ii) are evident.

iii) ⇒ iv). Suppose

$$d\gamma \wedge \gamma = 0.$$

Since

$$\iota_X(a \wedge b) = \iota_X a \wedge b + (-1)^{\deg(a)} a \wedge \iota_X b,\ a, b \in \Lambda^*(L),$$

we have

$$0 = \iota_X(d\gamma \wedge \gamma) = \iota_X(d\gamma) \wedge \gamma + (\iota_X\gamma)\, d\gamma = \iota_X(d\gamma) \wedge \gamma + d\gamma,$$

and so

$$d\gamma = -\iota_X(d\gamma) \wedge \gamma.$$

iv) ⇔ v). Since $\iota_X\gamma = 1$ we have

$$\{\gamma,\gamma\} = \mathcal{L}_X\gamma \wedge \gamma - \gamma \wedge \mathcal{L}_X\gamma = \iota_X d\gamma \wedge \gamma - \gamma \wedge \iota_X d\gamma = 2\iota_X d\gamma \wedge \gamma$$

so

$$d\gamma + \frac{1}{2}\{\gamma,\gamma\} = d\gamma + \iota_X d\gamma \wedge \gamma.$$

As i) ⇔ ii) is the theorem of Frobenius, the lemma is proved. □

By Lemma 2, Lemma 3 and Corollary 1 we obtain

COROLLARY 2. – *Let L be a C^∞ manifold and $\xi \subset T(L)$ an integrable distribution of codimension* 1. *Let $\gamma \in \wedge^1(L)$ such that* $\ker\gamma = \xi$ *and let X be a vector field on L such that $\gamma(X) = 1$. Set*

$$\delta = d_\gamma = d + \{\gamma, \cdot\}$$

where $\{\cdot,\cdot\}$ is defined in (2.6). *Then $(\Lambda^*(L), \delta, \{\cdot,\cdot\})$ is a DGLA.*

REMARK 2. – *Let $Z(\Lambda^*(L))$ be the center of $(\Lambda^*(L), \delta, \{\cdot,\cdot\})$. Then $Z(\Lambda^*(L)) = \{0\}$. Indeed, let $\alpha \in Z(\Lambda^*(L))$. Since $\{\alpha, 1\} = \mathscr{L}_X\alpha$ it follows that $\mathscr{L}_X\alpha = 0$. Let $x \in L$ and choose local coordinates $(x_1, \ldots, x_n)$ in a neighborhood U of x such that $X = \frac{\partial}{\partial x_1}$ on U. Let $\beta \in \Lambda^0(L)$ such that $\beta = x_1$ in a neighborhood of x. Then*

$$\{\alpha, \beta\}(X) = (\mathscr{L}_X\alpha \wedge \beta - \alpha \wedge \mathscr{L}_X\beta)(X) = -\alpha(X) = 0$$

and so $\alpha = 0$.

COROLLARY 3. – *Under the hypothesis of Corollary 2, we set*

$$\mathcal{Z}^*(L) = \{\alpha \in \Lambda^*(L) : \iota_X\alpha = 0\}.$$

Then $(\mathcal{Z}^*(L), \delta, \{\cdot,\cdot\})$ is a sub-DGLA of $(\Lambda^*(L), \delta, \{\cdot,\cdot\})$.

Proof. – Let $\alpha, \beta \in \mathcal{Z}^*(L)$. Since $\iota_X\alpha = 0$, $\iota_X\beta = 0$ and $\iota_X^2 = 0$ we have

$$\iota_X\delta\alpha = \iota_X(d\alpha + \iota_X d\gamma \wedge \alpha - \gamma \wedge \iota_X d\alpha) = \iota_X d\alpha - \iota_X d\alpha = 0$$

and

$$\begin{aligned}\iota_X\{\alpha, \beta\} = {} & \iota_X(\mathscr{L}_X\alpha \wedge \beta - \alpha \wedge \mathscr{L}_X\beta) = \iota_X\mathscr{L}_X\alpha \wedge \beta - (-1)^{\deg\alpha}\alpha \wedge \iota_X\mathscr{L}_X\beta \\ & = \iota_X(\iota_X d + d\iota_X)\alpha \wedge \beta - (-1)^{\deg\alpha}\alpha \wedge \iota_X(\iota_X d + d\iota_X)\beta = 0.\end{aligned}$$ □

REMARK 3. – *Let L be a C^∞ manifold and $\xi \subset T(L)$ an integrable distribution of codimension 1. Then there exists a 1-form γ on L such that $\xi = \ker\gamma$ if and only if ξ is co-orientable, i.e., the normal space to the foliation defined by ξ is orientable (see for ex.* [7]*).*

DEFINITION 3. – *Let L be a C^∞ manifold and $\xi \subset T(L)$ an integrable co-orientable distribution of codimension 1. A couple (γ, X) where $\gamma \in \wedge^1(L)$ and X is a vector field on L such that $\ker\gamma = \xi$ and $\gamma(X) = 1$ will be called a DGLA defining couple.*

REMARK 4. – *Let L be a C^∞ manifold and $\xi \subset T(L)$ an integrable distribution of codimension 1. Let (γ, X) be a DGLA defining couple for an integrable distribution ξ of codimension 1. Then (γ', X') is a DGLA defining couple for ξ if and only if $\gamma' = e^\lambda\gamma$, $\lambda \in C^\infty(M)$ and $X' = e^{-\lambda}X + V$, $V \in \xi$. Compare with the contact distribution case: the existence of a contact form ω on an odd dimensional manifold is equivalent with the co-orientability of the contact distribution* [10] *and it is unique up to a multiplication with a nonvanishing function. In this case the Reeb vector field R is uniquely defined by $\iota_R\omega = 1$ and $\iota_R d\omega = 0$. But contact distributions are nonintegrable.*

REMARK 5. – *Let $\alpha, \beta \in \mathcal{Z}^*(L)$ and let (γ, X) be a DGLA defining couple. Then*

$$\{\alpha, \beta\} = (\iota_X d + d\iota_X)\alpha \wedge \beta - \alpha \wedge (\iota_X d + d\iota_X)\beta = \iota_X d\alpha \wedge \beta - \alpha \wedge \iota_X d\beta \tag{2.7}$$

and

$$\{\gamma, \alpha\} = (\iota_X d + d\iota_X)\gamma \wedge \alpha - \gamma \wedge (\iota_X d + d\iota_X)\alpha = \iota_X d\gamma \wedge \alpha - \gamma \wedge \iota_X d\alpha. \tag{2.8}$$

DEFINITION 4. – *Let $(V^*, d_V, [\cdot,\cdot]_V)$, $(W^*, d_W, [\cdot,\cdot]_W)$ be DGLA and $\Phi : V^* \to W^*$ a graded morphism. We say that Φ is a DGVS-morphism (differential graded vector space morphism) if $\Phi d_V = d_W\Phi$. A DGVS-morphism Φ is a DGLA-morphism if $[\Phi(\alpha), \Phi(\beta)]_W = \Phi([\alpha, \beta]_V)$ for every $\alpha, \beta \in V^*$.*

REMARK 6. – *The DGLA structure of* $\mathcal{Z}^*(L)$ *depends on the choice of the DGLA defining couple* (γ, X). *In what follows, for given* ξ *we will fix* γ *and* X. *When it is necessary to emphasize this dependence we will write* $\left(\mathcal{Z}^*_{\gamma,X}(L), \delta_{\gamma,X}, \{\cdot,\cdot\}_{\gamma,X}\right)$.

The following proposition will describe shortly the effects of changing the defining couple:

PROPOSITION 1. – *Let* L *be a* C^∞ *manifold and* $\xi \subset T(L)$ *an integrable distribution of codimension* 1. *Let* (γ, X) *be a DGLA defining couple,* V *a* ξ*-valued vector field and* $\lambda \in C^\infty(L)$. *For* $\alpha \in \mathcal{Z}^*(L)$ *consider* $\Psi(\alpha) = \Psi_\lambda(\alpha) = e^\lambda \alpha$ *and* $\Theta(\alpha) = \Theta_V(\alpha) = \alpha + (-1)^{\deg \alpha} \iota_V \alpha \wedge \gamma$. *Then:*

i) $\Psi : \left(\mathcal{Z}^*_{\gamma,X}(L), \delta_{\gamma,X}, \{\cdot,\cdot\}_{\gamma,X}\right) \to \left(\mathcal{Z}^*_{e^\lambda\gamma, e^{-\lambda}X}(L), \delta_{e^\lambda\gamma,e^{-\lambda}X}, \{\cdot,\cdot\}_{e^\lambda\gamma,e^{-\lambda}X}\right)$ *is a DGLA-isomorphism.*

ii) $\Theta : \left(\mathcal{Z}^*_{\gamma,X}(L), \delta_{\gamma,X}\right) \to \left(\mathcal{Z}^*_{\gamma,X+V}(L), \delta_{\gamma,X+V}\right)$ *is a DGVS-isomorphism.*

Proof. – i) Let $\alpha, \beta \in \mathcal{Z}^*_{\gamma,X}(L)$. By (2.7) and (2.8) we have

$$\Psi\delta_{\gamma,X}\alpha = e^\lambda \left(da + \{\gamma,\alpha\}_{\gamma,X}\right) = e^\lambda \left(da + \iota_X d\gamma \wedge \alpha - \gamma \wedge \iota_X d\alpha\right) \tag{2.9}$$

and

$$\begin{aligned}
\left\{e^\lambda\gamma, e^\lambda\alpha\right\}_{e^\lambda\gamma,e^{-\lambda}X} &= \iota_{e^{-\lambda}X} d\left(e^\lambda\gamma\right) \wedge e^\lambda\alpha - e^\lambda\gamma \wedge \iota_{e^{-\lambda}X} d\left(e^\lambda\alpha\right)\\
&= \iota_X\left(e^\lambda d\lambda \wedge \gamma + e^\lambda d\gamma\right) \wedge \alpha - \gamma \wedge \iota_X d\left(e^\lambda\alpha\right)\\
&= e^\lambda[\iota_X(d\lambda)\gamma \wedge \alpha - d\lambda \wedge \alpha + \iota_X(d\gamma) \wedge \alpha - \gamma \wedge \iota_X(d\lambda \wedge \alpha)\\
&\quad - \iota_X(d\lambda)\gamma \wedge \alpha - \gamma \wedge \iota_X d\alpha]\\
&= e^\lambda\left[-d\lambda \wedge \alpha + \iota_X(d\gamma) \wedge \alpha - \gamma \wedge \iota_X d\alpha\right].
\end{aligned} \tag{2.10}$$

By replacing (2.10) in the formula

$$\delta_{e^\lambda\gamma,e^{-\lambda}X}\Psi\alpha = d\left(e^\lambda\alpha\right) + \left\{\gamma, e^\lambda\alpha\right\}_{e^\lambda\gamma,e^{-\lambda}X},$$

we deduce from (2.9) that

$$\Psi\delta_{\gamma,X} = \delta_{e^\lambda\gamma,e^{-\lambda}X}\Psi.$$

We have also

$$\begin{aligned}
\{\Psi\alpha, \Psi(\beta)\}_{e^\lambda\gamma,e^{-\lambda}X} &= \left\{e^\lambda\alpha, e^\lambda\beta\right\}_{e^\lambda\gamma,e^{-\lambda}X} = \iota_{e^{-\lambda}X} d\left(e^\lambda\alpha\right) \wedge e^\lambda\beta - e^\lambda\alpha \wedge \iota_{e^{-\lambda}X} d\left(e^\lambda\beta\right)\\
&= e^\lambda\left[\iota_X(d\lambda \wedge \alpha + d\alpha) \wedge \beta - \alpha \wedge \iota_X(d\lambda \wedge \beta + d\beta)\right]\\
&= e^\lambda\left[\iota_X(d\lambda)\alpha \wedge \beta + \iota_X d\alpha \wedge \beta - \iota_X(d\lambda)\alpha \wedge \beta - \alpha \wedge \iota_X d\beta\right]\\
&= e^\lambda\left[\iota_X d\alpha \wedge \beta - \alpha \wedge \iota_X d\beta\right] = \Psi\{\alpha,\beta\}_{\gamma,X}.
\end{aligned}$$

ii) Let $\alpha \in \mathcal{Z}^*_{\gamma,X}(L)$. Then

$$\begin{aligned}
\iota_{X+V}\Theta\alpha &= \iota_{X+V}\left(\alpha + (-1)^{\deg\alpha}\iota_V\alpha \wedge \gamma\right)\\
&= \iota_V\alpha + (-1)^{\deg\alpha}\iota_X(\iota_V\alpha \wedge \gamma) + (-1)^{\deg\alpha}\iota_V(\iota_V\alpha \wedge \gamma)\\
&= \iota_V\alpha + (-1)^{\deg\alpha}\iota_X\iota_V\alpha \wedge \gamma - \iota_V\alpha = 0.
\end{aligned}$$

It follows that Θ is well defined and the map $\Theta' : \mathcal{Z}^*_{\gamma,X+V}(L) \to \mathcal{Z}^*_{\gamma,X}(L)$ defined by $\Theta'(\alpha) = \alpha + (-1)^{\deg\alpha}\iota_{-V}\alpha \wedge \gamma$ is the inverse of Θ.

Since $\iota_V \gamma = 0$ and $d\gamma = -\iota_X d\gamma \wedge \gamma$, by using the expression of $\delta_{\gamma,X}$ from (2.9), we obtain

$$\begin{aligned}\Theta\delta_{\gamma,X}\alpha &= \delta_{\gamma,X}\alpha - (-1)^{\deg\alpha}\,\iota_V\left(d\alpha + \iota_X d\gamma \wedge \alpha - \gamma \wedge \iota_X d\alpha\right)\wedge\gamma \\ &= \delta_{\gamma,X}\alpha - (-1)^{\deg\alpha}\,\iota_V d\alpha \wedge \gamma - (-1)^{\deg\alpha}\left(\iota_V\iota_X d\gamma\right)\wedge\alpha\wedge\gamma \\ &\quad + (-1)^{\deg\alpha}\,\iota_X d\gamma \wedge \iota_V\alpha\wedge\gamma \\ &= \delta_{\gamma,X}\alpha - \gamma\wedge\iota_V d\alpha - (-1)^{\deg\alpha}\left(\iota_V\iota_X d\gamma\right)\wedge\alpha\wedge\gamma + d\gamma\wedge\iota_V\alpha. \qquad (2.11)\end{aligned}$$

We have

$$\begin{aligned}\left\{\gamma, \alpha + (-1)^{\deg\alpha}\,\iota_V\alpha\wedge\gamma\right\}_{\gamma,X+V} &= \iota_{X+V}d\gamma\wedge\left(\alpha + (-1)^{\deg\alpha}\,\iota_V\alpha\wedge\gamma\right) \\ &\quad - \gamma\wedge\iota_{X+V}d\left(\alpha + (-1)^{\deg\alpha}\,\iota_V\alpha\wedge\gamma\right) \\ &= \iota_X d\gamma\wedge\alpha + (-1)^{\deg\alpha}\,\iota_X d\gamma\wedge\iota_V\alpha\wedge\gamma \\ &\quad + \iota_V d\gamma\wedge\alpha + (-1)^{\deg\alpha}\,\iota_V d\gamma\wedge\iota_V\alpha\wedge\gamma \\ &\quad - \gamma\wedge\iota_X d\alpha - (-1)^{\deg\alpha}\,\gamma\wedge\iota_X d\left(\iota_V\alpha\wedge\gamma\right) \\ &\quad - \gamma\wedge\iota_V d\alpha - (-1)^{\deg\alpha}\,\gamma\wedge\iota_V d\left(\iota_V\alpha\wedge\gamma\right)\end{aligned}$$

and

$$d\left(\alpha + (-1)^{\deg\alpha}\,\iota_V\alpha\wedge\gamma\right) = d\alpha + (-1)^{\deg\alpha}\,d\iota_V\alpha\wedge\gamma - \iota_V\alpha\wedge d\gamma.$$

So

$$\begin{aligned}\delta_{\gamma,X+V}\Theta\alpha &= d\alpha + (-1)^{\deg\alpha}\,d\iota_V\alpha\wedge\gamma - \iota_V\alpha\wedge d\gamma \\ &\quad + \iota_X d\gamma\wedge\alpha + (-1)^{\deg\alpha}\,\iota_X d\gamma\wedge\iota_V\alpha\wedge\gamma + \iota_V d\gamma\wedge\alpha \\ &\quad + (-1)^{\deg\alpha}\,\iota_V d\gamma\wedge\iota_V\alpha\wedge\gamma - \gamma\wedge\iota_X d\alpha - (-1)^{\deg\alpha}\,\gamma\wedge\iota_X d\left(\iota_V\alpha\wedge\gamma\right) \\ &\quad - \gamma\wedge\iota_V d\alpha - (-1)^{\deg\alpha}\,\gamma\wedge\iota_V d\left(\iota_V\alpha\wedge\gamma\right). \qquad (2.12)\end{aligned}$$

Since

$$\begin{aligned}\gamma\wedge\iota_X d\left(\iota_V\alpha\wedge\gamma\right) &= \gamma\wedge\iota_X\left(d\iota_V\alpha\wedge\gamma + (-1)^{\deg\alpha-1}\,\iota_V\alpha\wedge d\gamma\right) \\ &= (-1)^{\deg\alpha}\,\gamma\wedge d\iota_V\alpha + \gamma\wedge\iota_V\alpha\wedge\iota_X d\gamma \\ &= (-1)^{\deg\alpha}\left(\gamma\wedge d\iota_V\alpha - \iota_V\alpha\wedge d\gamma\right)\end{aligned}$$

and

$$\begin{aligned}\gamma\wedge\iota_V d\left(\iota_V\alpha\wedge\gamma\right) &= \gamma\wedge\iota_V\left(d\iota_V\alpha\wedge\gamma + (-1)^{\deg\alpha-1}\,\iota_V\alpha\wedge d\gamma\right) \\ &= \gamma\wedge\iota_V\alpha\wedge\iota_V d\gamma,\end{aligned}$$

(2.12) gives

$$\begin{aligned}\delta_{\gamma,X+V}\Theta\alpha &= \delta_{\gamma,X}\alpha + (-1)^{\deg\alpha}\,d\iota_V\alpha\wedge\gamma - \iota_V\alpha\wedge d\gamma \\ &\quad + (-1)^{\deg\alpha}\,\iota_X d\gamma\wedge\iota_V\alpha\wedge\gamma + \iota_V d\gamma\wedge\alpha\end{aligned}$$

$$+ (-1)^{\deg \alpha} \iota_V d\gamma \wedge \iota_V \alpha \wedge \gamma - \gamma \wedge d\iota_V \alpha + \iota_V \alpha \wedge d\gamma$$
$$- \gamma \wedge \iota_V d\alpha - (-1)^{\deg \alpha} \gamma \wedge \iota_V \alpha \wedge \iota_V d\gamma$$
$$= \delta_{\gamma,X} \alpha + d\gamma \wedge \iota_V \alpha + \iota_V d\gamma \wedge \alpha - \gamma \wedge \iota_V d\alpha. \tag{2.13}$$

Finally, from (2.11) and (2.13) it follows that

$$\begin{aligned}\delta_{\gamma,X+V} \Theta\alpha - \Theta\delta_{\gamma,X}\alpha &= \iota_V d\gamma \wedge \alpha + (-1)^{\deg \alpha} (\iota_V \iota_X d\gamma)\, \alpha \wedge \gamma \\ &= -\iota_V (\iota_X d\gamma \wedge \gamma) \wedge \alpha + (-1)^{\deg \alpha} (\iota_V \iota_X d\gamma)\, \alpha \wedge \gamma \\ &= -\iota_V (\iota_X d\gamma)\, \gamma \wedge \alpha + (-1)^{\deg \alpha} (\iota_V \iota_X d\gamma)\, \alpha \wedge \gamma = 0.\end{aligned}$$ □

2.2. Moduli space of deformations of integrable distributions of codimension 1

Let L be a C^∞ manifold and $\xi \subset T(L)$ an integrable co-orientable distribution of codimension 1. We fix a DGLA defining couple (γ, X) and we consider the DGLA $(\mathcal{Z}^*(L), \delta, \{\cdot,\cdot\})$ previously defined.

Lemma 4. – *Let $\alpha \in \mathcal{Z}^1(L)$. The following are equivalent:*

i) *The distribution $\xi_\alpha = \ker(\gamma + \alpha)$ is integrable.*

ii) *α satisfies the Maurer-Cartan equation* (2.5) *in* $(\mathcal{Z}^*(L), \delta, \{\cdot,\cdot\})$.

Proof. – By Lemma 3 the distribution $\ker(\gamma + \alpha)$ is integrable if and only if $\gamma+\alpha$ satisfies (2.5) in $(\Lambda^*(L), d, \{\cdot,\cdot\})$. Since γ satisfies (2.5) we have

$$\begin{aligned} d(\gamma+\alpha) + \frac{1}{2}\{\gamma+\alpha, \gamma+\alpha\} &= d\alpha + \{\gamma, \alpha\} + \frac{1}{2}\{\alpha,\alpha\} \\ &= \delta\alpha + \frac{1}{2}\{\alpha,\alpha\}\end{aligned}$$

and the lemma follows. □

Notation 1. –

$$\mathfrak{MC}_\delta(L) = \left\{ \alpha \in \mathcal{Z}^1(L) : \ \delta a + \frac{1}{2}\{\alpha,\alpha\} = 0 \right\}.$$

Following [14] we define:

Definition 5. – *By a differentiable family of deformations of an integrable distribution ξ we mean a differentiable family $\omega : \mathcal{D} = (\xi_t)_{t\in I} \mapsto t \in I =]-a, a[,\ a > 0,$ of integrable distributions such that $\xi_0 = \omega^{-1}(0) = \xi$. By a differentiable family of small deformations of an integrable distribution ξ we mean the restriction $\mathcal{D}|I_\varepsilon = \omega^{-1}(I_\varepsilon)$ of a differentiable family of $\omega : \mathcal{D} \to I_\varepsilon =]-\varepsilon, \varepsilon[$ of deformations of $\xi = \omega^{-1}(0)$ to a sufficiently small neighborhood of* 0 *in I.*

Remark 7. – *By Lemma 4 a differentiable family of deformations of an integrable distribution is given by a differentiable family $(\gamma+\alpha_t)_{t\in I}$ in $\mathcal{Z}^1(L)$ such that $\xi_t = \ker(\gamma+\alpha_t)$ and $\alpha_0 = 0$.*

Definition 6. – *Let $\mathcal{U}$ be a neighborhood of the identity in the group $\mathcal{G} = \mathrm{Diff}(L)$ of diffeomorphisms of L and $\mathcal{V}$ be a neighborhood of* 0 *in $\mathcal{Z}^1(L)$ such that $\Phi^*(\gamma+\alpha)(X) \neq 0$, $\left(\Phi^{-1}\right)^*(\gamma+\alpha)(X) \neq 0$ for every $(\Phi,\alpha) \in \mathcal{U}\times\mathcal{V}$. We define*

(2.14)

$$(\Phi,\alpha) \in \mathcal{U}\times\mathcal{V} \subset \mathcal{G}\times\mathcal{Z}^1(L) \to \mathcal{Z}^1(L) \ni \chi(\Phi)(\alpha) = \left(\Phi^*(\gamma+\alpha)(X)\right)^{-1}\Phi^*(\gamma+\alpha)-\gamma.$$

Remark 8. – *The previous definition is adapted for small deformations. If $\beta = \chi(\Phi)(\alpha)$, $\xi_{\chi(\Phi)(\alpha)} = \Phi^*\xi_\alpha$. This means that ξ_α is integrable if and only if $\xi_{\chi(\Phi)(\alpha)}$ is integrable. By Lemma 4 we deduce that α satisfies the Maurer-Cartan equation* (2.5) *in the DGLA $(\mathcal{Z}^*(L), \delta, \{\cdot,\cdot\})$ if and only if $\chi(\Phi)(\alpha)$ does.*

Remark 9. – *We consider the right action of $\mathcal{G}$ on the set $\mathcal{D}$ of distributions of codimension* 1 *on L given by*

$$\tau(\Phi)(\xi) = \Phi_*^{-1}\xi,\ \Phi \in \mathcal{G},\ \xi \in \mathcal{D}. \tag{2.15}$$

Denote by $\mathcal{I}$ the subset of $\mathcal{D}$ given by the coorientable integrable distributions. Since $\xi = \ker\beta$ if and only if $\tau(\Phi)(\xi) = \ker\Phi^*\beta$ it follows that $\mathcal{I}$ is $\mathcal{G}$-invariant.

Definition 7. – i) *$\mathcal{I}/\mathcal{G}$ is the moduli space of integrable distributions of codimension* 1 *on L.*

ii) *We consider the one-to-one mapping*

$$\mathcal{Z}^1(L) \ni \alpha \mapsto \zeta_\alpha = \ker(\gamma+\alpha) \in \mathcal{R}, \tag{2.16}$$

where $\mathcal{R} = \left\{\zeta \in \mathcal{D} :\ \zeta = \ker(\gamma+\beta),\ \beta \in \mathcal{Z}^1(L)\right\} \subset \mathcal{D}$. The moduli space of deformations of integrable distributions of codimension 1 *of ξ is $\pi^{-1}(\pi(\mathcal{I}\cap\mathcal{R}))/\mathcal{G}$, where $\pi : \mathcal{D} \to \mathcal{D}/\mathcal{I}$ is the canonical map.*

Remark 10. – *Let $\nu \in \pi^{-1}(\pi(\mathcal{I}\cap\mathcal{R}))/\mathcal{G}$, $\nu = \pi(\zeta)$, where $\zeta \in \mathcal{I}\cap\mathcal{R}$. By Lemma 4 there exists $\alpha \in \mathfrak{MC}_\delta(L)$ such that $\zeta = \zeta_\alpha = \ker(\gamma+\alpha)$. Then if $\Phi \in \mathcal{G}$ is sufficiently close to the identity we have*

$$\tau(\Phi)(\zeta_\alpha) = \Phi_*^{-1}\zeta_\alpha = \ker\Phi^*(\gamma+\alpha) = \ker(\gamma+\chi(\Phi)(\alpha)) = \zeta_{\chi(\Phi)(\alpha)},$$

so $\nu = \pi\left(\zeta_{\chi(\Phi)(\alpha)}\right)$ and the action given by (2.14) *is the local description of the global action given by* (2.15) *via the correspondence* (2.16).

Notation 2. – *We will denote the moduli space of deformations of integrable distributions of codimension* 1 *of ξ by $\mathfrak{MC}_\delta(L)/\sim_{\mathcal{G}}$.*

Remark 11. – *Let $\mathcal{G}^0$ be the identity component of $\mathcal{G}$, $\Lambda^1(L)'$ the set of nowhere vanishing* 1*-forms on L and $\Lambda^1(L)'/e^{\Lambda^0(L)}$ the set of cooriented distributions. Then we have the group action*

$$\mathcal{G}^0 \times \Lambda^1(L)'/e^{\Lambda^0(L)} \ni (\Phi, \ker\gamma_\alpha) \to \ker\chi(\Phi)(\alpha) \in \Lambda^1(L)'/e^{\Lambda^0(L)}$$

of $\mathcal{G}^0$ on $\Lambda^1(L)'/e^{\Lambda^0(L)}$ and consider $\left[\mathfrak{MC}_\delta(L)/\mathcal{G}^0\right]$ the associated transformation groupoid (see [8] *for the definition of transformation groupoids): another possibility of defining $\mathfrak{MC}_\delta(L)/\sim_{\mathcal{G}}$ is to take the germ at (Id_L, ξ).*

The moduli space of deformations of integrable distributions of codimension 1 depends a priori on the DGLA defining couple. We will now prove that the moduli space $\mathfrak{MC}_{\delta_{\gamma,X}}(L)/\sim_{\mathcal{G}}$ and $\mathfrak{MC}_{\delta_{\widehat{\gamma},\widehat{X}}}(L)/\sim_{\mathcal{G}}$ of deformations of integrable distributions of codimension 1 corresponding to defining couples (γ, X) and $(\widehat{\gamma}, \widehat{X})$ are canonically isomorphic:

PROPOSITION 2. – *Let L be a C^∞ manifold and $\xi \subset T(L)$ an integrable distribution of codimension 1. Let (γ, X) be a DGLA defining couple and $V \neq 0$ a ξ-valued vector field on L. Let $\mathfrak{U}_V = \{\alpha \in Z^1(L) : (1 + \iota_V\alpha)(x) \neq 0, x \in L\}$. For $\alpha \in \mathfrak{U}_V$ define $F_V\alpha = (1 + \iota_V\alpha)^{-1}(\alpha - (\iota_V\alpha)\gamma)$. Then $F_V : \mathfrak{MC}_{\delta_{\gamma,X}}(L) \cap \mathfrak{U}_V \to \mathfrak{MC}_{\delta_{\gamma,X+V}}(L) \cap \mathfrak{U}_V$ is an isomorphism which induces an isomorphism*

$$\widetilde{F_V} : \mathfrak{MC}_{\delta_{\gamma,X}}(L) \cap \mathfrak{U}_V/\sim_{\mathcal{G}} \to \mathfrak{MC}_{\delta_{\gamma,X+V}}(L) \cap \mathfrak{U}_V/\sim_{\mathcal{G}}.$$

Proof. – Let $\alpha \in \mathfrak{MC}_{\delta_{\gamma,X}}(L) \cap \mathfrak{U}_V$. The Lemma 4 implies that $\operatorname{Ker}(\gamma + \alpha)$ is integrable. Since

$$\iota_{X+V}F_V\alpha = (1 + \iota_V\alpha)^{-1}\iota_{X+V}(\alpha - (\iota_V\alpha)\gamma) = (1 + \iota_V\alpha)^{-1}(\iota_V\alpha - \iota_V\alpha) = 0,$$

it follows that $F_V\alpha \in Z^1_{\gamma,X+V}(L)$. From Proposition 1 it follows that F_V is the restriction to $\mathfrak{MC}_{\delta_{\gamma,X}}(L) \cap \mathfrak{U}_V$ of the DGVS-isomorphism $(1 + \iota_V\alpha)^{-1}\Theta_V$, where Θ_V was defined in Proposition 1. We have

$$(\gamma + F_V\alpha) = \left(\gamma + (1 + \iota_V\alpha)^{-1}(\alpha - (\iota_V\alpha)\gamma)\right) = \gamma + \alpha,$$

so $\operatorname{Ker}(\gamma + \alpha) = \operatorname{Ker}(\gamma + F_V\alpha)$ and by using again the Lemma 4 we obtain $F_V\alpha \in \mathfrak{MC}_{\delta_{\gamma,X+V}}(L)$.

The invariance of F_V follows by Remark 8. □

From Proposition 1 and Proposition 2 we obtain

COROLLARY 4. – *Let L be a C^∞ manifold and $\xi \subset T(L)$ an integrable distribution of codimension 1. Let (γ, X), $\left(\widehat{\gamma}, \widehat{X}\right)$ be DGLA defining couples, $\widehat{\gamma} = \pm e^\lambda\gamma$, $\widehat{X} = \pm e^{-\lambda}X + V$ with $\lambda \in C^\infty(L)$ and V a ξ-valued vector field. Then there exists a canonical isomorphism $F : \mathfrak{MC}_{\delta_{\gamma,X}}(L)/\sim_{\mathcal{G}} \to \mathfrak{MC}_{\delta_{\widehat{\gamma},\widehat{X}}}(L)/\sim_{\mathcal{G}}$ between the moduli space of deformations of integrable distributions of codimension 1 of ξ, $F = \Theta_V \circ \Psi_\lambda$, $\Psi_\lambda : \mathcal{Z}^*_{\gamma,X}(L) \to \mathcal{Z}^*_{\widehat{\gamma},e^{-\lambda}X}(L)$, $\Theta_V : \mathcal{Z}^*_{\widehat{\gamma},e^{-\lambda}X}(L) \to \mathcal{Z}^*_{(\widehat{\gamma},\widehat{X})}(L)$, $\Psi_\lambda(\alpha) = e^\lambda\alpha$ and $\Theta(\alpha) = \Theta_V(\alpha) = \alpha + (-1)^{\deg\alpha}\iota_V\alpha \wedge \gamma$.*

LEMMA 5. – *Let Y be a vector field on L and Φ^Y the flow of Y. Then*

$$\frac{d\chi\left(\Phi^Y_t\right)}{dt}_{|t=0}(0) = -\delta(\iota_Y\gamma).$$

Proof. – We have

$$\begin{aligned}
\frac{d\chi\left(\Phi_t^Y\right)}{dt}_{|t=0}(0) &= \frac{d\left(\left(\left(\left(\Phi_t^Y\right)^{-1}\right)^*(\gamma)(X)\right)^{-1}\left(\left(\left(\Phi_t^Y\right)^{-1}\right)^*(\gamma)-\gamma\right)\right)}{dt}_{|t=0}\\
&= \left(\left(\Phi_t^Y\right)^{-1}\right)^*(\gamma)\,\frac{d\left(\left(\left(\Phi_t^Y\right)^{-1}\right)^*(\gamma)(X)^{-1}\right)}{dt}_{|t=0}(0)\\
&\quad+\left(\left(\Phi_t^Y\right)^{-1}\right)^*(\gamma)(X)^{-1}\,\frac{d\left(\left(\Phi_t^Y\right)^{-1}\right)^*}{dt}_{|t=0}\\
&= \frac{d\left(\left(\left(\left(\Phi_t^Y\right)^{-1}\right)^*(\gamma)(X)\right)^{-1}\right)}{dt}_{|t=0}\gamma+\frac{d\left(\left(\left(\Phi_t^Y\right)^{-1}\right)^*(\gamma)\right)}{dt}_{|t=0}\\
&= \mathcal{L}_Y(\gamma)(X)\,\gamma-\mathcal{L}_Y\gamma\\
&= (d\iota_Y\gamma)(X)\,\gamma+\iota_Y d\gamma(X)\,\gamma-d\iota_Y\gamma-\iota_Y d\gamma.
\end{aligned}$$

By Lemma 3 iv)

$$\begin{aligned}
\iota_Y d\gamma &= -\iota_Y(\iota_X d\gamma\wedge\gamma) = -(\iota_Y(\iota_X d\gamma))\,\gamma+(\iota_Y\gamma)\,\iota_X d\gamma\\
&= -(d\gamma(X,Y))\,\gamma+(\iota_Y\gamma)\,\iota_X d\gamma,
\end{aligned}$$

so

$$\begin{aligned}
\frac{d\chi\left(\Phi_t^Y\right)}{dt}_{|t=0}(0) &= (d\iota_Y\gamma)(X)\,\gamma-d\gamma(Y,X)\,\gamma-d\iota_Y\gamma\\
&\quad+(d\gamma(X,Y))\,\gamma-(\iota_Y\gamma)\,\iota_X d\gamma\\
&= (\iota_X d\iota_Y\gamma)\,\gamma-d\iota_Y\gamma-(\iota_Y\gamma)\,\iota_X d\gamma. \qquad (2.17)
\end{aligned}$$

Since

$$\mathcal{L}_X\gamma = d\iota_X\gamma+\iota_X d\gamma = \iota_X d\gamma$$

it follows that

$$\begin{aligned}
\delta\iota_Y\gamma &= d\iota_Y\gamma+\{\gamma,\iota_Y\gamma\} = d\iota_Y\gamma+\mathcal{L}_X\gamma\wedge\iota_Y\gamma-\gamma\wedge\mathcal{L}_X\iota_Y\gamma \qquad (2.18)\\
&= d\iota_Y\gamma+(\iota_Y\gamma)\,\iota_X d\gamma-X(\iota_Y\gamma)\,\gamma.
\end{aligned}$$

From (2.17) and (2.18) we obtain

$$\frac{d\chi\left(\Phi_t^Y\right)}{dt}_{|t=0}(0) = -\delta\iota_Y\gamma.$$

□

Definition 8. – *A $\mathfrak{MC}_\delta(L)$-valued curve through the origin is a continuous mapping $\lambda:[-a,a]\to\mathfrak{MC}_\delta(L)$, $a>0$, such that $\lambda(0)=0$. We say that α is the tangent vector at the origin of the $\mathfrak{MC}_\delta(L)$-valued curve λ through the origin to $\mathfrak{MC}_\delta(L)$ if $\alpha=\lim_{t\to0}\frac{\lambda(t)}{t}=\frac{d\lambda}{dt}_{|t=0}$.*

Proposition 3. – *Let α be the tangent vector at the origin of a $\mathfrak{MC}_\delta(L)$-valued curve through the origin λ, Y a vector field on L and Φ^Y the flow of Y. Set $\mu(t)=\chi\left(\Phi_t^Y\right)(\lambda(t))$. Then:*

i) $\delta\alpha=0$.

ii) *The tangent vector* β *at the origin of the* $\mathfrak{MC}_\delta(L)$*-valued curve* μ *is*

$$\beta = \alpha - \delta \iota_Y \gamma.$$

Proof. – i) By Lemma 4, $\lambda(t)$ verifies the Maurer-Cartan equation for every t. Since $\lambda(t) = \alpha t + o(t)$, we have $\delta\alpha = 0$.

ii) $\beta = \frac{d\mu}{dt}_{|t=0} = \frac{d}{dt}\chi\left(\Phi_t^Y(\lambda(t))\right)_{|t=0} = \frac{d\chi\left(\Phi_t^Y\right)}{dt}_{|t=0}(0) + \alpha.$

The Proposition 3 follows now by Lemma 5. □

The Proposition 3 justifies the following definition:

DEFINITION 9. – *The tangent cone* $T_{[0]}\left(\mathfrak{MC}_\delta(L)/\sim_{\mathcal{G}}\right)$ *at* $[0]$ *to* $\mathfrak{MC}_\delta(L)/\sim_{\mathcal{G}}$ *is the collection of cohomology classes in* $H^1(\mathcal{Z}(L), \delta)$ *of the tangent vectors at* 0 *to* $\mathfrak{MC}_\delta(L)$*-valued curves.*

DEFINITION 10. – *We say that the deformation theory is not obstructed at* $[0]$ *if*

$$T_{[0]}\left(\mathfrak{MC}_\delta(L)/\sim_{\mathcal{G}}\right) = H^1(\mathcal{Z}(L), \delta).$$

REMARK 12. – *In general, to establish unobstructedness of a deformation theory is a very hard problem and conditions as the vanishing of*

$$q : H^1(\mathcal{Z}(L), \delta) \to H^2(\mathcal{Z}(L), \delta),\ q(a) = \{a, a\},$$

will provide only curves of formal solutions to the Maurer-Cartan equation with prescribed tangent vectors at 0 (see for ex. [1]*).*

REMARK 13. – *There exists a natural isomorphism* $\Theta : \Lambda^*(\xi) \to \mathcal{Z}^*(L)$*: for* $\alpha \in \Lambda^1(\xi)$ *set* $\Theta(\alpha)(X) = 0$*,* $\Theta(\alpha)(Y) = \alpha(Y)$ *if* $Y \in \xi$ *and extend by linearity. Let* $d_b : \Lambda^*(\xi) \to \Lambda^*(\xi)$ *be the differential along the leaves of* ξ*. By using this isomorphism we consider* $d_b : \mathcal{Z}^*(L) \to \mathcal{Z}^*(L)$ *and for every* $\alpha \in \mathcal{Z}^*(L)$ *we have*

$$d_b\alpha = \iota_X(\gamma \wedge d\alpha) = d\alpha - \gamma \wedge \iota_X d\alpha. \tag{2.19}$$

Indeed let $\alpha \in \Lambda^p(\xi)$ *and* $X_1, \ldots, X_{p+1} \in \xi$*. Since* $\gamma(X_j) = 0$*,* $j = 1, \ldots, p+1$ *and* $\gamma(X) = 1$*, we have*

$$\iota_X(\gamma \wedge d\alpha)(X_1, \ldots, X_{p+1}) = (\gamma \wedge d\alpha)(X, X_1, \ldots, X_{p+1}) = d\alpha(X_1, \ldots, X_{p+1}).$$

LEMMA 6. – *The form* $\iota_X d\gamma$ *is* d_b*-closed.*

Proof. – From Lemma 3 *iii*) we obtain

$$0 = d(\gamma \wedge \iota_X d\gamma) = d\gamma \wedge \iota_X d\gamma - \gamma \wedge d\iota_X d\gamma = -\gamma \wedge d\iota_X d\gamma$$

so $\iota_X(\gamma \wedge d\iota_X d\gamma) = 0$ and the Lemma follows by (2.19). □

NOTATION 3. – *The cohomology class* $[\iota_X d\gamma] \in H^1(\Lambda^*(\xi), d_b)$ *which depends only on* ξ *will be denoted by* $c(\xi)$*.*

LEMMA 7. – *Let* $\alpha \in \mathcal{Z}^p(L)$. *Then*

$$\delta\alpha = d_b\alpha + \iota_X d\gamma \wedge \alpha. \tag{2.20}$$

In particular

$$d_b\alpha = \delta\alpha \iff \iota_X d\gamma \wedge \alpha = 0.$$

Proof. – By (2.19) and (2.8) we have

$$\delta\alpha = d\alpha + \{\gamma, \alpha\} = d\alpha + \iota_X d\gamma \wedge \alpha - \gamma \wedge \iota_X d\alpha = d_b\alpha + \iota_X d\gamma \wedge \alpha$$

and the lemma follows. □

REMARK 14. – *We would like to mention that Kodaira and Spencer developed in* [14] *a theory of deformations of the so called multifoliate structures, which are more general than the foliate structures. A multifoliate structure on an orientable manifold* X *of dimension* n *is an atlas* $\left(U_i, (x_i^\alpha)_{\alpha=1,\dots,n}\right)$ *such that the changes of coordinates verify*

$$\frac{\partial x_i^\alpha}{\partial x_k^\beta} = 0 \; for \; \beta \not\gtrsim \alpha,$$

where $(\mathcal{P}, \geqq)$ *is a finite partially ordered set,* $\{\alpha\}$ *a set of integers such that there is given a map* $\{\alpha\} \mapsto [\alpha]$ *of* α *onto* $\mathcal{P}$ *and the order relation "* $\gtrapprox$ *" is defined by* $\alpha > \beta$ *if and only if* $[\alpha] > [\beta]$, $\alpha \sim \beta$ *if and only if* $[\alpha] = [\beta]$. *An usual foliation is the particular case when* $\mathcal{P} = \{a, b\}$, $a > b$.

Kodaira and Spencer define in [14] *subsheafs* $\Phi^p_{\mathcal{P}}$, $p \in \mathbb{N}$, *of the sheaf of germs of jet forms of degree* p *on* X *which are compatible with the multifoliate structure and a differential* D *such that*

$$0 \to \Theta_{\mathcal{P}} \xrightarrow{D} \Phi^1_{\mathcal{P}} \xrightarrow{D} \Phi^2_{\mathcal{P}} \xrightarrow{D} \cdots \xrightarrow{D} \Phi^n_{\mathcal{P}} \to 0$$

is a resolution of the sheaf $\Theta_{\mathcal{P}}$ *of the vector fields tangent to the multifoliate structure. They define also a Lie bracket* $[\cdot, \cdot]$ *on jet forms such that* $\left(\left(\oplus_{p=1}^n \ker D\right)(X),\ D, [\cdot, \cdot]\right)$ *is a DGLA and every small deformation of the multifoliate structure is given by a family* $\{v(t)\} \subset \Phi^1_{\mathcal{P}}(X)$ *verifying* $[v(t), v(t)] = 0$ *and* $v(0) = d$. *So* $v(t) + d$ *verifies the Maurer-Cartan equation. Moreover* $\frac{\partial v}{\partial t}_{|t=0} \in Z\left(\Phi^1_{\mathcal{P}}\right)$ *and the class* $\left[\frac{\partial v}{\partial t}_{|t=0}\right] \in H^1(X, \Theta_{\mathcal{P}})$ *represents the infinitesimal deformation of the multifoliate structure along a tangent vector* $\frac{\partial}{\partial t}$.

In our approach, defined only for deformation of foliations of codimension 1, *the DGLA algebra* $(\mathcal{Z}^*(L), \delta, \{\cdot, \cdot\})$ *associated to a foliation on a cooriented manifold* L *is a subalgebra of the algebra* $(\Lambda^*(L), \delta, \{\cdot, \cdot\})$ *of forms on* L. *Its definition depends on the choice of a DGLA defining couple, but the cohomology class of this algebra does not depend on its choice. The deformations are given by forms in* $\mathcal{Z}^1(L)$ *verifying the Maurer-Cartan equation and the moduli space takes into account the diffeomorphic deformations. The infinitesimal deformations along curves are subsets of the first cohomology group of the DGLA* $(\mathcal{Z}^*(L), \delta, \{\cdot, \cdot\})$.

2.3. Transversally parallelizable foliations

Recall the following

DEFINITION 11. – *Let L be a C^∞ manifold and $\xi \subset T(L)$ a distribution of codimension* 1. *ξ is called transversally parallelizable if there exists a* 1*-form ω on L such that $\xi = \ker \omega$ and* $d\omega = 0$.

PROPOSITION 4. – *Let L be a C^∞ manifold, $\xi \subset T(L)$ a distribution of codimension* 1 *and (γ, X) a DGLA defining couple. The following assertions are equivalent:*

i) *ξ is transversally parallelizable.*

ii) $c(\xi) = 0$.

iii) *There exists $\lambda \in C^\infty(L)$ such that $\iota_X d\left(e^\lambda \gamma\right) = 0$.*

iv) *There exists a DGLA defining couple $\left(\widehat{\gamma}, \widehat{X}\right)$ such that $\delta_{\widehat{\gamma}, \widehat{X}} = d_b$.*

Proof. – The assertion i) $\Longrightarrow$ iii) is obvious and iii) $\Longleftrightarrow$ iv) by Lemma 7.

iv) $\Longrightarrow$ i). We may suppose that $\lambda \in C^\infty(L)$ such that $\widehat{\gamma} = e^\lambda \gamma$ and $\widehat{X} = e^{-\lambda} X + V$, $V \in \xi$. The Lemma 7 applied to 0-forms implies

$$\iota_{\widehat{X}} d\left(e^\lambda \gamma\right) = 0$$

and by Lemma 3 iv) it follows that

$$d\left(e^\lambda \gamma\right) = -\iota_{\widehat{X}} d\left(e^\lambda \gamma\right) \wedge e^\lambda \gamma = 0.$$

i) $\Longrightarrow$ ii). Let $\lambda \in C^\infty(L)$ such that $d\left(e^\lambda \gamma\right) = 0$. Since

$$d\left(e^\lambda \gamma\right) = e^\lambda (d\gamma + d\lambda \wedge \gamma) = e^\lambda (-\iota_X d\gamma \wedge \gamma + d\lambda \wedge \gamma) = 0$$

it follows that

$$d\lambda \wedge \gamma = \iota_X d\gamma \wedge \gamma. \tag{2.21}$$

We have

$$\iota_X (d\lambda \wedge \gamma) = (\iota_X d\lambda)\gamma - d\lambda \tag{2.22}$$

and

$$\iota_X (\iota_X d\gamma \wedge \gamma) = -\iota_X d\gamma, \tag{2.23}$$

so by (2.21), (2.22) and (2.23) we obtain

$$(\iota_X d\lambda)\gamma - d\lambda = -\iota_X d\gamma. \tag{2.24}$$

From (2.19) and (2.24) it follows that

$$d_b \lambda = d\lambda - (\iota_X d\lambda)\gamma = \iota_X d\gamma,$$

so $c(\xi) = 0$.

ii) $\Longrightarrow$ i). Let $\lambda \in C^\infty(L)$ such that

$$d_b \lambda = \iota_X d\gamma = d\lambda - (\iota_X d\lambda)\gamma.$$

Then

$$d\left(e^{\lambda}\gamma\right)=e^{\lambda}\left(d\gamma+d\lambda\wedge\gamma\right)=e^{\lambda}\left(-\iota_X d\gamma\wedge\gamma+d\lambda\wedge\gamma\right)$$
$$=e^{\lambda}\left(\left(-\iota_X d\gamma+d\lambda\right)\wedge\gamma\right)=e^{\lambda}\left(\left(\iota_X d\lambda\right)\gamma\right)\wedge\gamma=0. \qquad \square$$

EXAMPLE 1. – *Let M be a compact manifold and $H^k_{DR}(M)$ its de Rham cohomology group of degree k. Suppose that there exist $\tau_1,\dots,\tau_p$ closed 1-forms on M such that their classes $[\tau_1],\dots,[\tau_p]$ form a basis of $H^1_{DR}(M)$ and such that $[\tau_j\wedge\tau_k],\, j,k=1,\dots,p,\ j<k$, are linearly independent in $H^2_{DR}(M)$. Let $L=S^1\times M$ endowed with the product foliation given by $\xi=\ker ds$ where (s,x) are variables in $S^1\times M$. The following assertions are equivalent:*

i) $\beta(s,x)=a(s)\sum_{j=1}^{p}c_j\tau_j(x)$, $c_j\in\mathbb{R}$, $(s,x)\in L$.

ii) *There exists a curve Γ with values in $\mathfrak{MC}_\delta(L)/\sim_{\mathcal{G}}$ such that the tangent to Γ at the origin is $[\beta]$.*

In particular $T_{[0]}\left(\mathfrak{MC}_\delta(L)/\sim_{\mathcal{G}}\right)=C^\infty\left(S^1\right)\times H^1_{DR}(M)/\mathbb{R}^*$ where the action of $\mathbb{R}^*$ is given by $\lambda(a,h)=\left(\lambda a,\lambda^{-1}h\right)$.

Proof. – We consider the DGLA defining couple $(\gamma,X)=\left(ds,\frac{\partial}{\partial s}\right)$.

i) $\Longrightarrow$ ii). Let $\beta(s,x)=a(s)\sum_{j=1}^{p}c_j\tau_j(x)$. Take $\alpha_t=\beta t$. Then $\alpha_t\in Z^1(L)$ and $\delta\beta=d_b\beta=d_x\beta=0$.

Moreover

$$\{\beta,\beta\}=2\iota_X d\beta\wedge\beta=2\iota_{\frac{\partial}{\partial s}}\left(a'ds\wedge\sum_{j=1}^{p}c_j\tau_j+a\sum_{j=1}^{p}c_j d\tau_j\right)\wedge a\sum_{j=1}^{p}c_j\tau_j=0.$$

So $\alpha_t\in\mathfrak{MC}_\delta(L)$ and we can consider $\Gamma: t\to[\alpha_t]\in\mathfrak{MC}_\delta(L)/\sim_{\mathcal{G}}$.

ii) $\Longrightarrow$ i). Let $\alpha_t=t\beta+t^2\sigma+o\left(t^2\right)\in\mathfrak{MC}_\delta(L)$. Then

$$\{\alpha_t,\alpha_t\}=t^2\{\beta,\beta\}+o\left(t^2\right)$$

and

$$\delta\alpha_t=d_b\alpha_t=td_x\beta+t^2d_x\sigma+o\left(t^2\right).$$

Since $\alpha_t\in\mathfrak{MC}_\delta(L)$, we obtain $d_x\beta=0$ and $\{\beta,\beta\}+2d_x\sigma=0$, so $[\beta]\in H^1_{DR}(M)$ and $[\{\beta,\beta\}]=0\in H^2_{DR}(M)$.

Since $\iota_{\frac{\partial}{\partial s}}\beta=0$ we have

$$\beta(s,x)=\sum_{j=1}^{p}\beta_j(s)\,\tau_j+d_xf(s,x),\ f\in C^\infty(L).$$

By Proposition 3 we may suppose $\beta(s,x)=\sum_{j=1}^{p}\beta_j(s)\,\tau_j(x)$. Then

$$d\beta=\sum_{j=1}^{p}\beta_j'ds\wedge\tau_j$$

and

$$\{\beta,\beta\}=2\iota_{\frac{\partial}{\partial s}}d\beta\wedge\beta=2\left(\sum_{j=1}^{p}\beta_j'\tau_j\right)\wedge\left(\sum_{j=1}^{p}\beta_j\tau_j\right)=2\sum_{j\neq k}\beta_j'\beta_k\tau_j\wedge\tau_k.$$

But

$$[\{\beta,\beta\}] = 2\sum_{j<k} \left(\beta'_j\beta_k - \beta'_k\beta_j\right)[\tau_j \wedge \tau_k] = 0 \in H^2_{DR}(M)$$

and from the assumption of linear independence it follows that $\beta'_j\beta_k - \beta'_k\beta_j = 0$ for every $1 \leq j < k \leq p$. This means that $\beta_j = c_j a$, $c_j \in \mathbb{R}$, $a \in C^\infty\left(S^1\right)$ and $\beta(s,x) = a(s)\sum_{j=1}^{p} c_j\tau_j(x)$, $(s,x) \in L$. □

Remark 15. – *In the previous example we have* $T_{[0]}\left(\mathfrak{MC}_\delta(L)/\sim_{\mathscr{G}}\right) \neq H^1(\mathcal{Z}(L),\delta)$ *so the deformation theory is obstructed at* $[0]$. *The hypotheses are fulfilled in the particular case where* M *is a torus.*

3. Deformations of Levi-flat hypersurfaces

3.1. Maurer-Cartan equation for Levi-flat deformations

Let M be a complex manifold and L a Levi flat hypersurface of class C^∞ in M such that the Levi foliation of M is co-orientable. In this case there exists $r \in C^\infty(M)$, $dr \neq 0$ on L such that $L = \{z \in M : r(z) = 0\}$ and set $j : L \to M$ the natural inclusion. As $dr \neq 0$ on a neighborhood of L in M we will suppose in the sequel that $dr \neq 0$ on M.

We denote by J the complex structure on M. Then the distribution $\xi = T(L) \cap JT(L)$ is integrable and $\xi = \ker\gamma$, where $\gamma = j^*(d^c_J r)$. Since $d^c_J = J^{-1}dJ$, we have $d^c_J r = -Jdr$.

Let g be a fixed Hermitian metric on M and $Z = \mathrm{grad}_g r / \left\|\mathrm{grad}_g r\right\|^2_g$. Then the vector field $X = JZ$ is tangent to L and verifies

$$\gamma(X) = d^c_J r(JZ) = 1.$$

It follows that the couple (γ, X) defined above is a DGLA defining couple for the Levi foliation. For a given defining function, we will fix this DGLA defining couple and when its dependence on the defining function r has to be emphasized, we will say the DGLA defining couple associated to r.

Let U be a tubular neighborhood of L in M and $\pi : U \to L$ the projection on L along the integral curves of Z. As we are interested in infinitesimal deformations we may suppose $U = M$.

We will now parametrize the real hypersurfaces near L and diffeomorphic to L as graphs over L:

Let $\mathscr{F} = C^\infty(L;\mathbb{R})$ and $a \in \mathscr{F}$. Denote

$$L_a = \{z \in M : r(z) = a(\pi(z))\}.$$

Since Z is transverse to L, L_a is a hypersurface in M. Consider the map $\Phi_a : M \to M$ defined by $\Phi_a(p) = q$, where

$$\pi(q) = \pi(p),\ r(q) = r(p) + a(\pi(p)). \tag{3.1}$$

U is a tubular neighborhood of L, so Φ_a is a diffeomorphism of M such that $\Phi_a(L) = L_a$ and $\Phi_a^{-1} = \pi|_{L_a}$.

Conversely, let $\Psi \in \mathcal{U} \subset \mathcal{G} =$Diff (M), where $\mathcal{U}$ is a suitable neighborhood of the identity in $\mathcal{G}$ as in Definition 6. Then there exists $a \in \mathcal{F}$ such that $\Psi(L) = L_a$. Indeed, for $x \in L$, let $q(x) \in \Psi(L)$ such that $\pi(q(x)) = x$. By defining $a(x) = r(q(x))$, we obtain $\Psi(L) = L_a$.

So we have the following:

Lemma 8. – *Let $\Psi \in \mathcal{U}$. Then there exists a unique $a \in \mathcal{F}$ such that $\Psi(L) = L_a$.*

It follows that a neighborhood $\mathcal{V}_{\mathcal{F}}$ of 0 in $\mathcal{F}$ is a set of parametrization of hypersurfaces close to L.

For $a \in \mathcal{V}_{\mathcal{F}}$, consider the almost complex structure $J_a = \left(\Phi_a^{-1}\right)_* \circ J \circ (\Phi_a)_*$ on M and denote

$$\alpha_a = \left(d^c_{J_a} r(X)\right)^{-1} j^* \left(d^c_{J_a} r\right) - \gamma. \tag{3.2}$$

Then $\alpha_a \in \mathcal{Z}^1(L)$ and

$$\ker(\gamma + \alpha_a) = \ker j^* \left(d^c_{J_a} r\right) = TL \cap J_a TL. \tag{3.3}$$

Let $V \in TL \cap J_a TL$. Then $V = Y + \theta X$ with $Y \in TL \cap JTL$ and θ a real function on L. By (3.3) we have

$$d^c_{J_a} r(V) = j^* d^c_{J_a} r(Y) + \theta j^* d^c_{J_a} r(X) = 0,$$

so

$$\theta = -\left(d^c_{J_a} r(X)\right)^{-1} d^c_{J_a} r(Y) = -\alpha_a(Y)$$

and it follows that

$$TL \cap J_a TL = \{Y - (\alpha_a(Y)) X : Y \in TL \cap JTL\}. \tag{3.4}$$

Since

$$\pi_*(TL_a \cap JTL_a) = \left(\Phi_a^{-1}\right)_* (TL_a \cap JTL_a) = TL \cap \left(\Phi_a^{-1}\right)_* (J(\Phi_a)_* TL) = TL \cap J_a TL \tag{3.5}$$

from (3.3), (3.4) and (3.5) we obtain the following

Lemma 9. – *For every $a \in \mathcal{V}_{\mathcal{F}}$ the form α_a is the unique form in $\mathcal{Z}^1(L)$ verifying*

$$\ker(\gamma + \alpha_a) = \pi_*(TL_a \cap JTL_a).$$

Moreover,

$$\begin{aligned}\ker(\gamma + \alpha_a) &= \ker j^* \left(d^c_{J_a} r\right) = \pi_*(TL_a \cap JTL_a) = TL \cap J_a TL \\ &= \{Y - (\alpha_a(Y)) JZ : Y \in TL \cap JTL\}.\end{aligned}$$

By using Lemma 9 and Lemma 4 we can state the following

Corollary 5. – *For every $a \in \mathcal{V}_{\mathcal{F}}$, the following assertions are equivalent:*

i) *L_a is Levi flat.*

ii) *α_a satisfies the Maurer-Cartan equation in $(\mathcal{Z}^*(L), \delta, \{\cdot,\cdot\})$ i.e.,*

$$\delta\alpha_a + \frac{1}{2}\{\alpha_a, \alpha_a\} = 0. \tag{3.6}$$

Remark 16. – *Suppose now that $a, b \in \mathcal{V}_{\mathcal{F}}$, $\Phi \in \mathcal{G} =$ Diff (L) and $\chi(\Phi)(\alpha_a) = \alpha_b$, where $\chi(\Phi)$ is the group action defined in* (2.14). *From Lemma 9 and Remark 8 it follows that L_a is Levi flat if and only if L_b is Levi flat.*

Notation 4. – *Set* $\mathcal{E} = \{\alpha_a : a \in \mathcal{V}_{\mathcal{F}}\}$.

Remark 17. – *$\mathcal{E}$ parametrizes the codimension* 1 *distributions close to* $TL \cap JTL$ *which are of the form* $TL \cap \widetilde{J}TL$ *for* $\widetilde{J}$ *complex structure (possible non integrable) close to* J, *where* $\widetilde{J} = (I+S)\, J\, (I+S)^{-1}$ *with* $S \in \Lambda_J^{0,1}(M) \otimes T(M)$ *close to* 0.

By using the notations of Definition 7 , we are now able to put in evidence the moduli space of deformations of Levi-flat manifolds of L:

Definition 12. – *Let* $\mathcal{R}_{\mathcal{F}} = \{\zeta \in \mathcal{D} : \zeta = \ker(\gamma + \beta),\ \beta \in \mathcal{E}\}$. *The moduli space of deformations of Levi-flat manifolds of L is* $\pi^{-1}(\pi(\mathcal{I} \cap \mathcal{R}_{\mathcal{F}}))/\mathcal{G}$.

Remark 18. – *From Remark 10 it follows that the corresponding local action of* $\mathcal{G}$ *on* $\mathcal{E}$ *is given by* $\alpha_b = \chi(\Phi)(\alpha_a)$, *where* $a, b \in \mathcal{V}_{\mathcal{F}}$ *and* $\Phi \in \mathcal{G}$ *is sufficiently close to the identity. If* $r, r' \in C^\infty(M)$, $dr \neq 0$, $dr' \neq 0$ *on L such that* $L = \{z \in M : r(z) = 0\} = \{z \in M : r'(z) = 0\}$, $r = hr'$ *with* $h > 0$ *of class* C^∞ *in a neighborhood L. So* $\{z \in M : r(z) = a(\pi(z))\} = \{z \in M : r'(z) = h^{-1}(z)\, a(\pi(z))\}$. *It follows that the previous definition does not depend on the choice of the defining function r of L and by Proposition 1 it follows that it does not depend on the choice of the metric g either. We remark also that the moduli space of deformations of Levi-flat manifolds of L identifies Levi flat hypersurfaces up to a foliated diffeomorphism and not up to a CR diffeomorphism.*

3.2. Equations for infinitesimal Levi-flat deformations

Let M be a complex manifold, J the complex structure on M, L a Levi flat hypersurface in M and I an open interval in $\mathbb{R}$ containing the origin. A 1-dimensional Levi-flat deformation of L is a smooth mapping $\Psi : I \times M \to M$ such that $\Psi_t = \Psi(t,\cdot) \in \mathrm{Diff}(M)$, $L_t = \Psi_t L$ is a Levi flat hypersurface in M for every $t \in I$ and $L_0 = L$. By the previous subsection there exists a family $(a_t)_{t\in I}$ in $\mathcal{V}_{\mathcal{F}}$ such that $\pi_*(TL_{a_t} \cap JTL_{a_t}) = \ker(\gamma + \alpha_{a_t})$ and α_{a_t} satisfies the Maurer-Cartan equation (3.6) in $(\mathcal{Z}^*(L), \delta, \{\cdot,\cdot\})$ for every t. We will say that the family $(a_t)_{t\in I}$ is a family in $\mathcal{V}_{\mathcal{F}}$ defining a Levi-flat deformation of L.

We define now $\delta^c : \mathcal{Z}^*(L) \to \mathcal{Z}^*(L)$: for $\alpha \in \mathcal{Z}^p(L)$ and $V_1, \ldots, V_{p+1} \in T(L) \cap JT(L)$ set $\delta^c\alpha(V_1, \ldots, V_{p+1}) = J^{-1}\delta J\alpha(V_1, \ldots, V_{p+1})$ and $\delta^c\alpha(X, V_1, \ldots, V_p) = 0$. By extending this definition by linearity we obtain $\delta^c\alpha \in \mathcal{Z}^{p+1}(L)$.

Recall that (γ, X) is a DGLA defining couple, where $\gamma = j^*(d_J^c r)$ and $X = JZ = J\left(\mathrm{grad}_g r / \|\mathrm{grad}_g r\|^2\right)$, r is a defining function for L and g a Hermitian metric on M.

Proposition 5. – *Let L be a Levi flat hypersurface in a complex manifold M,* $(a_t)_{t\in I}$ *a family in* $\mathcal{V}_{\mathcal{F}}$ *defining a Levi-flat deformation of L and* $p = \frac{da_t}{dt}{}_{|t=0}$. *Then*

$$\frac{d\alpha_{a_t}}{dt}{}_{|t=0} = \delta^c p.$$

Proof. – Since $\alpha_{a_t}(X) = 0$ for every t it follows that

$$\frac{d\alpha_{a_t}}{dt}_{|t=0}(X) = 0 = (\delta^c p)(X). \tag{3.7}$$

Let V be a section of $TL \cap JTL$, which will be identified for simplicity with $j_* V$. Then (3.2) gives

$$\frac{d\alpha_{a_t}}{dt}_{|t=0}(V) = \frac{d}{dt}_{|t=0}\left(\left(d^c_{J_{a_t}} r(X)\right)^{-1}\right) j^*\left(d^c_{J_{a_0}} r\right)(V) + \left(d^c_{J_{a_0}}(JZ)\right)^{-1} \frac{d}{dt}_{|t=0} j^*\left(d^c_{J_{a_t}} r\right)(V).$$

But

$$j^*\left(d^c_{J_{a_0}} r\right)(V) = j^*\left(d^c_J r\right)(V) = 0$$

and

$$\left(d^c_{J_{a_0}} r(X)\right)^{-1} = \left(d^c_J r(X)\right)^{-1} = 1,$$

so

$$\begin{aligned}\frac{d\alpha_{a_t}}{dt}_{|t=0}(V) &= \frac{d}{dt}_{|t=0} j^*\left(d^c_{J_{a_t}} r\right)(V) = \frac{d}{dt}_{|t=0}\left(-J_{a_t} dr\right)(V)\\ &= -(dr)\frac{d}{dt}_{|t=0}\left(J_{a_t} V\right).\end{aligned} \tag{3.8}$$

We have

$$\begin{aligned}\frac{d}{dt}_{|t=0}\left(J_{a_t} V\right) &= \frac{d}{dt}_{|t=0}\left(\left(\Phi^{-1}_{a_t}\right)_* \circ J \circ \left(\Phi_{a_t}\right)_*\right)(V)\\ &= \frac{d}{dt}_{|t=0}\left(\Phi^{-1}_{a_t}\right)_*(JV) + J\frac{d}{dt}_{|t=0}\left(\Phi_{a_t}\right)_*(V).\end{aligned} \tag{3.9}$$

By using the definition (3.1) of Φ_{a_t} we have

$$r\left(\Phi_{a_t}(z)\right) = r(z) + a_t(\pi(z)) = r(z) + tp(\pi(z)) + o(t),$$

where π is the projection along the integral curves of Z. It follows that

$$\frac{d\left(\Phi_{a_t}\right)_*}{dt}_{|t=0} = (p \circ \pi) Z. \tag{3.10}$$

If we consider a smooth extension $\widetilde{p}$ of p to M and the flow $\Phi^{\widetilde{p}Z}$ of $\widetilde{p}Z$, we have

$$\frac{d\Phi^{\widetilde{p}Z}_t}{dt}(z) = (\widetilde{p}Z)\left(\Phi^{\widetilde{p}Z}_t(z)\right)$$

and restricting to L, by (3.10) we obtain

$$\frac{d\left(\Phi_{a_t}\right)_*}{dt}_{|t=0} = \frac{d\left(\Phi^{\widetilde{p}Z}_t\right)_*}{dt}_{|t=0} = pZ. \tag{3.11}$$

So (3.9) and (3.11) give

$$\begin{aligned}\frac{d}{dt}_{|t=0}(J_{a_t}V) &= \frac{d}{dt}_{|t=0}\left(\Phi^{pZ}_{-t}\right)_*(JV) + J\frac{d}{dt}_{|t=0}\left(\Phi^{pZ}_{t}\right)_*(V)\\ &= -\mathscr{L}_{pZ}(JV) + J\mathscr{L}_{pZ}(V)\\ &= -[pZ, JV] + J[pZ, V]\\ &= -p[Z, JV] + JV(p)Z + pJ[Z, V] - V(p)JZ.\end{aligned}$$

Replacing this formula in (3.8) we obtain

$$\frac{d}{dt}_{|t=0}\alpha_{a_t}(V) = -(dr)(-p[Z, JV] + JV(p)Z + pJ[Z, V] - V(p)JZ).$$

Since $dr(JZ) = 0$ and $dr(Z) = 1$ it follows that

$$\begin{aligned}\frac{d}{dt}_{|t=0}\alpha_{a_t}(V) &= (dr)(p[Z, JV]) - JV(p) - p(dr)J[Z, V]\\ &= pdr([Z, JV]) - JV(p) + p(d^c r)[Z, V]. \qquad (3.12)\end{aligned}$$

By using

$$0 = ddr(Z, JV) = Z(dr(JV)) - JV(dr(Z)) - dr[Z, JV]$$

we obtain

$$dr[Z, JV] = 0$$

and (3.12) becomes

$$\frac{d}{dt}_{|t=0}\alpha_{a_t}(V) = -JV(p) + p(d^c r)[Z, V]. \qquad (3.13)$$

Since $d^c r(V) = -dr(JV) = 0$ and $d^c r(Z) = -dr(JZ) = 0$, it follows that

$$dd^c r(Z, V) = Z(d^c r(V)) - V(d^c r(Z)) - d^c r([Z, V]) = -d^c r([Z, V])$$

and from (3.13) we deduce

$$\frac{d}{dt}_{|t=0}\alpha_{a_t}(V) = -JV(p) - pdd^c r(Z, V) = d^c p(V) - pJ(\iota_{JZ}dd^c r)(V). \qquad (3.14)$$

Now

$$(\delta^c p)(V) = -\delta p(JV) = d^c p(V) - \{\gamma, p\}(JV) \qquad (3.15)$$

and

$$\{\gamma, p\}(JV) = p\mathscr{L}_X\gamma(JV) - (\mathscr{L}_X p)\gamma(JV). \qquad (3.16)$$

Since $\gamma(JV) = 0$ and $\iota_X\gamma = 1$, (3.16) becomes

$$\{\gamma, p\}(JV) = p\iota_X d\gamma(JV). \qquad (3.17)$$

Therefore, recalling now that $\gamma = j^*(d^c r)$ and $X = JZ$, from (3.17) we obtain

$$\{\gamma, p\}(JV) = p(\iota_{JZ}dd^c r)(JV)$$

and from (3.15) it follows that

$$(\delta^c p)(V) = d^c p(V) - p(\iota_{JZ}dd^c r)(JV). \qquad (3.18)$$

Finally, by (3.14), (3.18) and (3.7) we conclude

$$\frac{d\alpha_{a_t}}{dt}_{|t=0} = \delta^c p\,. \tag{3.19}$$

□

NOTATION 5. – *For a DGLA defining couple (γ, X) we denote $\mathfrak{b} = \iota_X d\gamma$. By Lemma 6, $\mathfrak{b}$ is d_b-closed and $c\,(T\,(L) \cap JT\,(L)) = [\mathfrak{b}] \in H^1\,(\Lambda^*\,(\xi)\,, d_b)$. Let F be a compact leaf of the Levi foliation. Then there exists a unique harmonic form $\mathfrak{b}_F \in \Lambda^1\,(F)$ with respect to the fixed metric g such that $\left[\mathfrak{b}_{|F}\right] = [\mathfrak{b}_F] \in H^1\,(F, d_b)$, where $\mathfrak{b}_{|F}$ is the restriction of $\mathfrak{b}$ to F.*

COROLLARY 6. – *Let L be a Levi flat hypersurface in a complex manifold M, $(a_t)_{t\in I}$ a family in $\mathcal{V}_{\mathcal{J}}$ defining a Levi-flat deformation of L and $p = \frac{da_t}{dt}_{|t=0}$. Then:*

$$\delta\delta^c p = 0 \tag{3.20}$$

or equivalently

$$d_b d_b^c p - d_b p \wedge J\mathfrak{b} - d_b^c p \wedge \mathfrak{b} - pJd_b^c \mathfrak{b} - p\mathfrak{b} \wedge J\mathfrak{b} = 0. \tag{3.21}$$

Proof. – α_{a_t} verifies the Maurer-Cartan equation (3.6) in $(\mathcal{Z}^*\,(L)\,, \delta, \{\cdot,\cdot\})$ so

$$\delta\alpha_{a_t} + \frac{1}{2}\left\{\alpha_{a_t}, \alpha_{a_t}\right\} = 0$$

for every t. Since

$$\frac{d}{dt}_{|t=0}\left\{\alpha_{a_t}, \alpha_{a_t}\right\} = 0,$$

(3.20) follows from (3.19).

By (2.20) we have

$$\delta^c p = -J\delta p = -J\left(d_b p + p\iota_X d\gamma\right) = d_b^c p - pJ\mathfrak{b}$$

and

$$\begin{aligned}\delta\delta^c p &= \delta\left(d_b^c p - pJ\mathfrak{b}\right) = d_b\left(d_b^c p - pJ\mathfrak{b}\right) + \mathfrak{b} \wedge \left(d_b^c p - pJ\mathfrak{b}\right)\\ &= d_b d_b^c p - d_b p \wedge J\mathfrak{b} - p d_b J\mathfrak{b} - d_b^c p \wedge \mathfrak{b} - p\mathfrak{b} \wedge J\mathfrak{b}.\end{aligned}$$

So (3.20) is equivalent to (3.21). □

PROPOSITION 6. – *Let M be a complex manifold and L a C^∞ Levi flat hypersurface in M. Let F be a compact leaf of the Levi foliation. Then there exists a defining function ρ of L such that the DGLA defining couple $\left(\widehat{\gamma}, \widehat{X}\right)$ associated to ρ verifies*

$$\mathfrak{b}_F = \iota_{\widehat{X}} d\widehat{\gamma}_{|F} = \iota_{\widehat{X}}\left(d_b d_b^c \rho\right)_{|F}\,. \tag{3.22}$$

Proof. – Let r be a C^∞ defining function for L and (γ, X) the DGLA defining couple associated to r.

Since $\left[\mathfrak{b}_{|F}\right] = [\mathfrak{b}_F] \in H^1\,(F, d_b)$, there exists $\lambda \in C^\infty\,(F)$ such that

$$\mathfrak{b}_F = \mathfrak{b}_{|F} + d_b\lambda.$$

By using (2.19) we obtain

$$\mathfrak{b}_F = \iota_X d_b d_b^c r_{|F} + d\lambda - \left(\iota_X d\lambda\right) j^*\left(d^c r\right)_{|F}\,. \tag{3.23}$$

We choose a smooth extension of λ on M which we denote by λ too, and set $\rho = e^{-\lambda} r$.

We have

$$d^c\left(e^{-\lambda}r\right)=e^{-\lambda}\left(d^cr-rd^c\lambda\right)$$

and

$$dd^c\left(e^{-\lambda}r\right)=e^{-\lambda}\left(-d\lambda\wedge d^cr+rd\lambda\wedge d^c\lambda+dd^cr-dr\wedge d^c\lambda-rdd^c\lambda\right). \tag{3.24}$$

Let V be a section of $TL\cap JTL$. Since $r=0$ on L, $j^*d^cr\,(X)=1$ and $j^*d^cr\,(V)=0$, from (3.24) we obtain

$$\begin{aligned}\iota_{e^\lambda X}dd^c\left(e^{-\lambda}r\right)(V)&=dd^c\left(e^{-\lambda}r\right)\left(e^\lambda X,V\right)\\&=e^{-\lambda}\left(\left(-d\lambda\wedge d^cr\right)\left(e^\lambda X,V\right)\right)+dd^cr\left(e^\lambda X,V\right)\\&\quad-dr\wedge d^c\lambda\left(\left(e^\lambda X,V\right)\right)\\&=\left(\left(-d\lambda\wedge d^cr\right)(X,V)+\iota_{e^\lambda X}dd^cr\,(V)\right)\\&=\left(d\lambda\,(V)+\iota_{e^\lambda X}dd^cr\,(V)\right).\end{aligned}\tag{3.25}$$

But (3.23) and (3.25) give

$$\iota_{e^\lambda X}dd^c\left(e^{-\lambda}r\right)(V)=\mathfrak{b}_F\,(V)\ \ on\ F$$

and this equality proves (3.22). □

PROPOSITION 7. – *Let L be a Levi flat hypersurface in a Kähler manifold M, $(a_t)_{t\in I}$ a family in $\mathcal{V}_{\mathcal{F}}$ defining a Levi-flat deformation of L and $p=\frac{da_t}{dt}_{|t=0}$. Let F be a compact leaf of the Levi foliation and ∂_b, $\overline{\partial}_b$ the tangential operators along the leaves. Then*

$$d_bd_b^cp-d_bp\wedge J\mathfrak{b}_F-d_b^cp\wedge\mathfrak{b}_F-p\mathfrak{b}_F\wedge J\mathfrak{b}_F=0 \tag{3.26}$$

or equivalently

$$\partial_b\overline{\partial}_bp+\partial_bp\wedge\overline{\theta}_F-\overline{\partial}_bp\wedge\theta_F-p\overline{\theta}_F\wedge\theta_F=0 \tag{3.27}$$

where

$$\theta_F=\mathfrak{b}_F^{1,0}=\frac{1}{2}\left(\mathfrak{b}_F-iJ\mathfrak{b}_F\right).$$

Proof. – We choose a defining function of L as in Proposition 6. We consider on F the metric induced by the Kähler metric of M. Since $\mathfrak{b}_F$ is a harmonic form on F with respect to this Kähler metric, it follows that $J\mathfrak{b}_F$ is also a harmonic form. So $d_bJ\mathfrak{b}_F=d_b^cJ\mathfrak{b}_F=0$ and (3.26), (3.27) follow from (3.21). □

3.3. A uniqueness theorem for partial differential equations

In this section we prove a uniqueness theorem for second order partial differential equations on compact Kähler manifolds which will be used in the next sections to give infinitesimal rigidity results for Levi flat hypersurfaces.

For $\varphi,\psi\in\Lambda^k(M)$, we use the notations

$$\langle\varphi,\psi\rangle=\varphi\wedge*\overline{\psi},\ \langle\langle\varphi,\psi\rangle\rangle=\int_M\langle\varphi,\psi\rangle\,,\|\varphi\|^2=\langle\langle\varphi,\varphi\rangle\rangle\,,\ \|\varphi\|_\infty^2=\sup_M*\langle\varphi,\varphi\rangle\,,$$

where $*$ is the Hodge operator. If $T\in End\,(\Lambda^*M)$, we denote $T^c=J^{-1}TJ$, where J is the complex structure of M.

THEOREM 1. – *Let M be a compact Kähler manifold and $\beta \neq 0$ a harmonic 1-form on M. Let $A \in End(\Lambda^* M)$ defined by $A\alpha = \beta \wedge \alpha$ and $P = d + A$. Suppose that $\Delta - A^*A$ is positive defined on a subspace $E \subset \Lambda^0 M$, where Δ is the Laplace operator on M. Then $f = 0$ is the unique solution of the equation $PP^c f = 0$, $f \in E$. In particular $\Delta - A^*A$ is positive defined if $\|\beta\|^2_\infty < \lambda^1_\Delta$, where λ^1_Δ is the smallest strictly positive eigenvalue of the Dirichlet form $f \mapsto \langle\langle \triangle f, f\rangle\rangle$ and the conclusion of the theorem is valid in this case.*

Proof. – Let $f \in E$ such that

$$PP^c f = dP^c f + \beta \wedge P^c f = 0. \tag{3.28}$$

Let ω be the Kähler form on M and $\Lambda : \Lambda^{k+2} M \to \Lambda^k M$ the adjoint of the exterior multiplication by ω, $\Lambda\alpha = *^{-1}(\omega \wedge *\overline{\alpha})$. Then (3.28) gives

$$\Lambda dP^c f = -\Lambda(\beta \wedge P^c f) = -\langle \omega, \beta \wedge P^c f\rangle. \tag{3.29}$$

Step 1. – We have

$$\langle \omega, \beta \wedge P^c f\rangle = \langle J\beta, P^c f\rangle. \tag{3.30}$$

Indeed, let $(\theta_1, \ldots, \theta_n, J\theta_1, \ldots, J\theta_n)$ be a local orthonormal basis at z for $\Lambda^1(M)$ such that $\omega(z) = \sum_j d\theta_j \wedge dJ\theta$. Then by writing $\beta = \sum_j a_j d\theta_j + \sum_j b_j dJ\theta_j$, $P^c f = \sum_j c_j d\theta_j + \sum_j d_j dJ\theta_j$, we have

$$\langle \omega, \beta \wedge P^c f\rangle(z) = \sum_j (a_j d_j - b_j c_j)(z)\, dV = \langle J\beta, P^c f\rangle(z).$$

Step 2. – Let $B = d^c - P^c$. Then $(\Lambda d + B^*) P^c f = 0$.

We will compute B^* on $\Lambda^0(M)$: let $\varphi \in \Lambda^0(M), \psi \in \Lambda^1(M)$. Since $B\alpha = -J^{-1}AJ\alpha = -J^{-1}\beta \wedge J\alpha$, we have

$$\langle\langle B\varphi, \psi\rangle\rangle = \int_M \varphi J\beta \wedge *\psi = \langle\langle \varphi, B^*\psi\rangle\rangle = \int_M \varphi * B^*\psi \tag{3.31}$$

and it follows that

$$B^*\psi = *(J\beta \wedge *\psi),\ \psi \in \Lambda^1(M).$$

In particular $B^* P^c f = *(J\beta \wedge *P^c f) = *\langle J\beta, P^c f\rangle$ and from (3.29) and (3.30) we obtain

$$(\Lambda d + B^*) P^c f = 0. \tag{3.32}$$

Step 3. – $(P^c)^\# P^c f = 0$ where $(P^c)^\# = - * P^c *$.

We have

$$(P^c)^\# = - * (d^c - B) * = (d^c)^* + B^* = (d^c - B)^* + 2B^* = (P^c)^* + 2B^*. \tag{3.33}$$

Since M is Kähler, by using (3.33) we have

$$[d, \Lambda] = -(d^c)^* = -(P^c)^\# + B^*$$

so

$$(P^c)^\# P^c f = (-[d, \Lambda] + B^*) P^c f = (\Lambda d + B^*) P^c f.$$

From (3.32) we conclude that

$$(P^c)^\# P^c f = 0. \tag{3.34}$$

Step 4. – $\|df\| = \|f\beta\|$.

By (3.33) and (3.34) we have

$$\left\langle\left\langle (P^c)^{\#} P^c f, f\right\rangle\right\rangle = \langle\langle\left((P^c)^* + 2B^*\right) P^c f, f\rangle\rangle = \|P^c f\|^2 + 2\,\langle\langle P^c f, Bf\rangle\rangle = 0. \tag{3.35}$$

But

$$\begin{aligned}\langle\langle P^c f, Bf\rangle\rangle &= \langle\langle P^c f, fJ\beta\rangle\rangle = \langle\langle P^c f, fJ\beta\rangle\rangle = \langle\langle -JPf, fJ\beta\rangle\rangle \\ &= \langle\langle -Pf, f\beta\rangle\rangle = \langle\langle -\left(d+A\right)f, Af\rangle\rangle = -\,\langle\langle df, Af\rangle\rangle - \|Af\|^2\end{aligned} \tag{3.36}$$

and

$$\langle\langle df, Af\rangle\rangle = \int_M f df \wedge *\beta = \frac{1}{2}\int_M df^2 \wedge *\beta = -\frac{1}{2}\int_M f^2 d\left(*\beta\right) = 0 \tag{3.37}$$

because β is harmonic and

$$\|d\left(*\beta\right)\| = \|d^*\beta\| = 0.$$

From (3.35), (3.36) and (3.37) it follows that

$$\|P^c f\|^2 - 2\,\|Af\|^2 = 0. \tag{3.38}$$

But

$$\|P^c f\|^2 = \|Pf\|^2 = \langle\langle\left(d+A\right)f, \left(d+A\right)f\rangle\rangle = \|df\|^2 + \|Af\|^2$$

and by replacing this expression of $\|P^c f\|^2$ in (3.38) we complete the proof of step 4.

Step 5. – $f = 0$ and the case $\sup_M * \langle\beta,\beta\rangle < \lambda^1_\Delta$.

Since

$$\|df\|^2 = \langle\langle df, df\rangle\rangle = \langle\langle d^* df, f\rangle\rangle = \langle\langle \Delta f, f\rangle\rangle$$

and

$$\|f\beta\|^2 = \|Af\|^2 = \langle\langle A^* Af, f\rangle\rangle$$

by the step 4 it follows that

$$\langle\langle\left(\Delta - A^* A\right)f, f\rangle\rangle = 0$$

which implies $f = 0$.

Finally, as in the computation (3.31) of B^* we obtain

$$A^*\psi = *\,\langle\beta,\psi\rangle\,,\ \psi \in \Lambda^1\left(M\right)$$

and so

$$A^* Af = *f\,\langle\beta,\beta\rangle\,.$$

In particular

$$\langle\langle\left(\Delta - A^* A\right)f, f\rangle\rangle = \langle\langle\Delta f, f\rangle\rangle - \langle\langle *f\,\langle\beta,\beta\rangle\,, f\rangle\rangle \geq \left(\lambda^1_\Delta - \sup_M *\,\langle\beta,\beta\rangle\right)\|f\|^2\,.$$

So if $\|\beta\|^2_\infty < \lambda^1_\Delta$, the operator $\Delta - A^* A$ is positive definite and the theorem is proved. □

3.4. Infinitesimal rigidity results for Levi flat hypersurfaces

By using Corollary 3 and Corollary 5 it is natural to give the following definition:

DEFINITION 13. – *Let L be a Levi flat hypersurface in a complex manifold M. We say that L is infinitesimally rigid (respectively strongly infinitesimally rigid), if for any family* $(a_t)_{t\in I}$ *in* $\mathcal{V}_{\mathcal{J}}$ *defining a Levi-flat deformation of L*

$$\left[\frac{d\alpha_{a_t}}{dt}_{|t=0}\right] = 0 \in H^1\left(\mathcal{Z}\left(L\right),\delta\right),$$

respectively

$$\frac{d\alpha_{a_t}}{dt}_{|t=0} = 0.$$

THEOREM 2. – *Let M be a smooth complex manifold and L a compact connected transversally parallelizable compact Levi flat hypersurface in M. Then L is strongly infinitesimally rigid.*

Proof. – Since L is transversally parallelizable, every leaf of the Levi foliation is compact or every leaf of the Levi foliation is dense (see for example [7] for the properties of transversally parallelizable manifolds). By Proposition 4 we can consider a DGLA defining couple (γ, X) such that $\mathfrak{b} = \iota_X d\gamma = 0$ and $\delta = d_b$.

Let $(a_t)_{t\in I}$ be a family in $\mathcal{V}_{\mathcal{J}}$ defining a Levi-flat deformation of L and $p = \frac{da_t}{dt}_{|t=0}$. Then (3.21) becomes

$$d_b d_b^c p = 0. \tag{3.39}$$

Suppose that every leaf of the Levi foliation of L is compact. By (3.39) it follows that p is constant on each leaf, so $\delta^c p = 0$. By Proposition 5 it follows that L is strongly infinitesimally rigid.

Suppose now that every leaf of the Levi foliation is dense. Let $z_0 \in L$ such that $p(z_0) = \sup_L p$ and let L_{z_0} be the leaf of the Levi foliation through z_0. By (3.39) it follows that p is constant on L_{z_0}. Since L_{z_0} is dense, p is constant on L and L is strongly infinitesimally rigid. □

Now we study the case of infinitesimal rigidity of general Levi flat hypersurfaces in smooth compact connected Kähler manifolds.

LEMMA 10. – *Let M be an n-dimensional Kähler manifold, L a Levi flat hypersurface in M and F a compact leaf of the Levi foliation. Let* $(a_t)_{t\in I}$ *a family in* $\mathcal{V}_{\mathcal{J}}$ *defining a Levi-flat deformation of L and* $p = \frac{da_t}{dt}_{|t=0}$. *Then*

$$\int_F p\mathfrak{b}_F \wedge J\mathfrak{b}_F \wedge \omega^{n-2} = 0$$

where ω is a Kähler form on M and J the complex structure of M.

Proof. – From (3.27) it follows that

$$\int_F \partial_b\overline{\partial}_b p\wedge\omega^{n-2}+\int_F \partial_b p\wedge\overline{\theta}_F\wedge\omega^{n-2}-\int_F \overline{\partial}_b p\wedge\theta_F\wedge\omega^{n-2}-\int_F p\overline{\theta}_F\wedge\theta_F\wedge\omega^{n-2}=0.$$

Since $\partial_b\theta_F=\overline{\partial}_b\theta_F=0$, we have

$$\int_F \partial_b\overline{\partial}_b p\wedge\omega^{n-2}=\int_F d_b\left(\overline{\partial}_b p\wedge\omega^{n-2}\right)=0,$$
$$\int_F \partial_b p\wedge\overline{\theta}_F\wedge\omega^{n-2}=\int_F \partial_b\left(p\overline{\theta}_F\right)\wedge\omega^{n-2}=\int_F d_b\left(p\overline{\theta}_F\wedge\omega^{n-2}\right)=0,$$
$$\int_F \overline{\partial}_b p\wedge\theta_F\wedge\omega^{n-2}=\int_F \overline{\partial}_b\left(p\theta_F\right)\wedge\omega^{n-2}=\int_F d_b\left(p\theta_F\wedge\omega^{n-2}\right)=0,$$

and the lemma is proved. □

THEOREM 3. – *Let M be an n-dimensional Kähler manifold, J the complex structure of M, ω a Kähler form on M and L a Levi flat hypersurface in M with compact leaves. Suppose that for every leaf F of the Levi foliation such that $\mathfrak{b}_F\neq 0$, Δ_F-T_F is positive definite on $\mathfrak{B}_F$, where Δ_F is the Laplace operator on F, $T_F\in End\left(\Lambda^0(F)\right)$ is the operator defined by $T_F\varphi=*\varphi\langle\mathfrak{b}_F,\mathfrak{b}_F\rangle$ and*

$$\mathfrak{B}_F=\left\{f\in C^\infty(M):\ \int_F f\mathfrak{b}_F\wedge J\mathfrak{b}_F\wedge\omega^{n-2}=0\right\}.$$

Then L is strongly infinitesimally rigid. In particular this is true if $\|\mathfrak{b}_F\|_\infty^2<\lambda_F$ for every leaf F of L, where λ_F is the smallest strictly positive eigenvalue of the Dirichlet form $f\mapsto\int_F|\nabla f|^2$ restricted to $\mathfrak{B}_F$ and $\|\mathfrak{b}_F\|_\infty^2=\sup_F *\langle\mathfrak{b}_F,\mathfrak{b}_F\rangle$.

Proof. – Let $(a_t)_{t\in I}$ be a family in $\mathcal{V}_{\mathcal{F}}$ defining a Levi-flat deformation of L and $p=\frac{da_t}{dt}_{|t=0}$. Let F be a leaf of the Levi foliation. We recall that by (2.20) we have $\delta\alpha=d_b\alpha+\mathfrak{b}_F\wedge\alpha$.

If $\mathfrak{b}_F=0$, (3.20) implies that $dd^cp=0$ and it follows that p is constant on F.

Suppose now that $\mathfrak{b}_F\neq 0$. By (3.20) we have $\delta\delta^cp=0$ and by Lemma 10 $p\in\mathfrak{B}_F$. We can apply the uniqueness Theorem 1 on F for $\beta=\mathfrak{b}_F$ and it follows that $p=0$ on F.

So $\delta^cp=0$ on L and by Proposition 5 L is strongly infinitesimally rigid. The last assertion follows also by Theorem 1. □

REMARK 19. – *Note that in general $\mathfrak{b}_F$ is not continuous with respect to F.*

3.5. Non existence of Levi flat transversally parallelizable hypersurfaces in $\mathbb{CP}_n$, $n\geq 2$

One of the basic questions in the theory of foliations is the following: Let $\mathcal{F}$ be a singular holomorphic foliation of codimension 1 of $\mathbb{CP}_2$. Does every leaf of $\mathcal{F}$ accumulate to the singular set of $\mathcal{F}$? This question led to the conjecture of the non-existence of smooth Levi flat hypersurfaces in $\mathbb{CP}_n$, $n\geqslant 2$, and under suitable hypotheses, in compact complex manifolds.

We recall that for $\mathbb{CP}_n$, $n\geqslant 3$, the positive answer to this question was given in [15] and [18]. For $n=2$ the problem is still open. In this paragraph we prove the non existence of transversally paralelizable Levi flat hypersurfaces in:

a) connected complex manifolds M such that for every $p \neq q \in M$ and every real hyperplane H_q in T_qM there exists a holomorphic vector field Y on M such that $Y(p) = 0$ and $Y(q) \oplus H_q = T_qM$ (Theorem 5). The proof uses techniques developed in this paper.

b) complex compact Kähler surfaces M such that $\dim H^2(M) = 1$ (Theorem 6). The proof of this result was communicated to us by M. Brunella [2].

Both Theorems 5 and 6 imply that there are no transversally parallelisable Levi flat hypersurfaces in $\mathbb{CP}_2$ (Theorem 4).

THEOREM 4. – *There are no transversally parallelizable C^2 Levi flat hypersurfaces in $\mathbb{CP}_n$, $n \geq 2$.*

Proof. – Recall that Y.-T. Siu's theorem [19] and [11] prove the non existence of C^2 Levi flat hypersurfaces in $\mathbb{CP}_n$, $n \geq 3$.

Let L be a transversally parallelizable Levi flat hypersurface in $\mathbb{CP}_2$. Suppose that Y is a holomorphic vector field on M. Then $\left(\Phi_t^Y(L)\right)_t$ is a Levi-flat deformation of L and let $(a_t)_{t\in I}$ be a family in $\mathcal{V}_{\mathcal{F}}$ defining this Levi-flat deformation of L. Set $p = \frac{da_t}{dt}_{|t=0}$.

By (3.19) we have

$$\frac{d}{dt}_{|t=0}\alpha_{a_t} = \delta^c p\,.$$

Theorem 2 implies that L is strongly infinitesimally rigid and it follows that $\delta^c p = 0$. By Lemma 4, we may suppose that $\delta = d_b$, so $d_b^c p = 0$.

As a Levi flat hypersurface in $\mathbb{CP}_2$ has no compact leaves, every leaf is dense in L and it follows that p is constant.

Let g be a fixed Hermitian metric on $\mathbb{CP}_2$ and $Z = \operatorname{grad}_g r / \left\|\operatorname{grad}_g r\right\|_g^2$. As in 3.1, $a_t(X) = r(X(t))$, $X \in \mathbb{CP}_2$ with $X(t) = \gamma_{Z,X} \cap \Phi_t^Y(L)$ and $\gamma_{Z,X}$ the integral curve of Z passing through X. We have

$$Y = Y_n + Y_t$$

where

$$Y_n = dr(Y)\,Z,\ Y_t(r) = Y - dr(Y)\,Z$$

are the normal and tangential components of Y. Since $a_t(X) = r\left(\Phi_t^{Y_n}(X)\right)$ it follows that

$$p = \frac{da_t}{dt}_{|t=0} = dr(Y_n) = Y_n(r)\,.$$

As $Y_n = \langle Z, Y\rangle_g Z$, where $\langle\cdot,\cdot\rangle_g$ is the scalar product induced by g we obtain that $p = \langle Z, Y\rangle_g$ and we conclude that $\langle Z, Y\rangle_g$ is constant on L for every holomorphic vector field on M.

Let $X \in L$ and consider homogeneous coordinates $[z_0, z_1, z_2]$ in $\mathbb{CP}_2$ such that $X = [1,0,0]$ and the Euler vector field Y such that $Y([1,0,0]) = 0$. Since

$$\langle Z, Y\rangle_g(X) = \langle Z([1,0,0]), Y[1,0,0]\rangle_g = 0,$$

it follows that $\langle Z, Y\rangle_g = 0$ and this means that Y is tangent to L. But by Siu's Proposition 2.3 [19], this gives a contradiction. □

This theorem can be generalized and proved without using Y.-T. Siu's Proposition 2.3 from [19] :

THEOREM 5. – *Let M be a connected complex manifold such that for every $p \neq q \in M$ and every real hyperplane H_q in T_qM there exists a holomorphic vector field Y on M such that $Y(p) = 0$ and $Y(q) \oplus H_q = T_qM$. Then there are no compact transversally parallelizable Levi flat hypersurfaces in M. The hypotheses are fulfilled if $M = \mathbb{CP}_n$, $n \geq 2$.*

Proof. – Let L be a transversally parallelizable Levi flat hypersurface in M. As in the proof of Theorem 4, $\langle Z, Y\rangle_g$ is constant on every leaf of the Levi foliation of L for every holomorphic vector field on M. Let $p \in L$ and let q be a distinct point of the leaf F passing through p. Let Y be a holomorphic vector field on M such that $Y(p) = 0$ and $Y(q) \oplus T_qL = T_qM$. Since $Y(p) = 0$ it follows that Y is tangent to F and we obtain a contradiction. □

LEMMA 11. – *Let L be a real hypersurface in a complex compact Kähler surface M such that $M \backslash L = U_1 \cup U_2$ where U_1, U_2 are open disjoint subsets of M and let ω be the $(1,1)$-form associated to the Kähler metric of M. Suppose that* $\dim H^2(M) = 1$. *Then*

i) *ω is exact on U_1 or on U_2;*

ii) *the restriction of ω to L is exact.*

Proof. – i) Let ψ be a cycle such that $H^2(M) = \mathbb{C}[\psi]$. Suppose that ω is neither exact on U_1 nor on U_2. Then there exist 2-cycles $\varphi_j \subset U_j$ such that $\int_{\varphi_j} \omega \neq 0$, $j = 1, 2$. But $[\varphi_j] = c_j[\psi]$, $j = 1, 2$ and $[\varphi_1][\varphi_2] = 0$. Contradiction.

ii) Suppose that ω is exact on U_1. Let φ be a 2-cycle φ on L. We can approximate φ by 2-cycles φ_ε on U_1. Since $\int_{\varphi_\varepsilon} \omega = 0$, it follows that $\int_\varphi \omega = 0$,. □

COROLLARY 7. – *Under the hypotheses of Lemma 11 we have $\int_L \gamma \wedge \omega = 0$ for every closed 1-form γ on L.*

Proof. – By Lemma 11, $\omega = d\alpha$ on L, so

$$\int_L \gamma \wedge \omega = \int_L d(\gamma \wedge \alpha) = 0. \qquad \square$$

THEOREM 6. – *Let L be a real hypersurface in a complex compact Kähler surface M such that $M \backslash L = U_1 \cup U_2$ where U_1, U_2 are open disjoint subsets of M such that* $\dim H^2(M) = 1$. *There are no transversally parallelizable Levi flat hypersurfaces in M.*

Proof. – Let ω be the $(1,1)$-form associated to the Kähler metric of M. Let L be a Levi flat transversally parallelizable hypersurface in M such that the Levi foliation of L is given by the 1-form γ. Then $\gamma \wedge \omega(x) \neq 0$ for every $x \in L$. Indeed, let $x \in L$ and choose local coordinates (t_1, t_2, t_3) in a neighborhood of x such that $x = 0$, $\gamma = \alpha(t_1)\, dt_1$ and $(0, t_2, t_3)$ are coordinates on the leaf L_x through x. There exist local holomorphic coordinates $z \in (z_1, z_2)$ in a neighborhood V of x such that $L_x = \{z \in V : z_2 = 0\}$. It follows that $\alpha(0)\, dt_1 \wedge dz_1 \wedge d\overline{z}_1 \neq 0$. Consequently $\int_L \gamma \wedge \omega > 0$ or $\int_L \gamma \wedge \omega < 0$ and we obtain a contradiction by Corollary 7. □

Acknowledgements

We would like to thank the referee of our paper who indicated us the reference [14], noticed an error in the first version of the manuscript and did a lot of remarks that improved the quality of the paper. He also remarked that the proof of Theorem 4 follows by the fact that a closed form defining the Levi foliation of L defines a holonomy invariant Lebesgue-class measure on transversals. Thus, by a Theorem of D. Sullivan [20], there is a closed current $T \neq 0$ directed by the Levi foliation. But as L can be isotoped on L', which is still foliated and disjoint of L, we obtain a contradiction. The authors would also like to thank T.-C. Dinh for useful discussions.

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(Manuscrit reçu le 1er novembre 2011 ;
accepté, après révision, le 23 janvier 2014.)

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