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COHOMOLOGY JUMP LOCI OF QUASI-PROJECTIVE VARIETIES

BY NERO BUDUR AND BOTONG WANG

ABSTRACT. – We prove that the cohomology jump loci in the space of rank one local systems over a smooth quasi-projective variety are finite unions of torsion translates of subtori. The main ingredients are a recent result of Dimca-Papadima, some techniques introduced by Simpson, together with properties of the moduli space of logarithmic connections constructed by Nitsure and Simpson.

RÉSUMÉ. – Dans cet article, on montre que les lieux de saut dans l'espace de systèmes locaux de rang un sur une variété lisse quasi-projective sont des réunions finies de subtores translatées par des éléments de torsion. Pour cela, nous utilisons un résultat récent de Dimca-Papadima, certaines techniques introduites par Simpson, ainsi que des propriétés de l'espace de moduli pour les connexions logarithmiques construit par Nitsure et Simpson.

1. Introduction

Let X be a connected, finite-type CW-complex. Define

$$\mathbf{M}_{\mathbf{B}}(X) = \mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$$

to be the variety of $\mathbb{C}^*$ representations of $\pi_1(X)$. Then $\mathbf{M}_{\mathbf{B}}(X)$ is a direct product of $(\mathbb{C}^*)^{b_1(X)}$ and a finite Abelian group. For each point $\rho \in \mathbf{M}_{\mathbf{B}}(X)$, there exists a unique rank one local system L_ρ, whose monodromy representation is ρ. The *cohomology jump loci* of X are the natural strata

$$\Sigma^i_k(X) = \{\rho \in \mathbf{M}_{\mathbf{B}}(X) \mid \dim_{\mathbb{C}} H^i(X, L_\rho) \geq k\}.$$

$\Sigma^i_k(X)$ is a Zariski closed subset of $\mathbf{M}_{\mathbf{B}}(X)$. A celebrated result of Simpson says that if X is a smooth projective variety defined over $\mathbb{C}$, then $\Sigma^i_k(X)$ is a union of torsion translates of subtori of $\mathbf{M}_{\mathbf{B}}(X)$.

In this paper, we generalize Simpson's result to quasi-projective varieties.

THEOREM 1.1. – *Suppose U is a smooth quasi-projective variety defined over $\mathbb{C}$. Then $\Sigma^i_k(U)$ is a finite union of torsion translates of subtori of $\mathbf{M}_{\mathbf{B}}(U)$.*

When U is compact, the theorem is proved in [7, 8], [1], [15], with the strongest form appearing in the latter. When $b_1(\bar{U}) = 0$, Arapura [2] showed that $\Sigma^i_k(U)$ are union of translates of subtori. The case of unitary rank one local systems on U has been considered in [3]. Dimca and Papadima were able to prove the following:

THEOREM 1.2 ([6, Theorem C]). – *Under the same assumption as Theorem 1.1, every irreducible component of $\Sigma^i_k(U)$ containing $\mathbf{1} \in \mathbf{M_B}(U)$ is a subtorus.*

The proof of this result reduces to the study of the infinitesimal deformations with cohomology constraints of the trivial local system. These are governed in general by infinite-dimensional models. In [6] it is shown that, in this case, the finite-dimensional Gysin model due to Morgan provides the necessary linear algebra description for the infinitesimal deformations.

The result of Dimca and Papadima serves as a key ingredient of our theorem. In Section 2, we will show that each irreducible component of $\Sigma^i_k(U)$ contains a torsion point. Then, in Section 3, we will see that, thanks to Theorem 1.2, having a torsion point on an irreducible component of $\Sigma^i_k(U)$ forces this component to be a translate of subtorus.

There are two other proofs of Simpson's theorem: one via positive characteristic methods [11], and one via D-modules [13, 12]. However, in this paper we follow the original approach of Simpson. There are no analogous results for higher rank local systems even in the projective case.

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2. Torsion points on the cohomology jump loci

Let X be a smooth complex projective variety, and let $D = \sum_{\lambda=1}^{n} D_\lambda$ be a simple normal crossing divisor on X with irreducible components D_λ. Let $U = X - D$. Thanks to Hironaka's theorem on resolution of singularities, every smooth quasi-projective variety U can be realized in this way. The goal of this section is to prove the following:

THEOREM 2.1. – *Each irreducible component of $\Sigma^i_k(U)$ contains a torsion point.*

First, we want to reduce to the case when X and each D_λ are defined over $\bar{\mathbb{Q}}$. This can be done using a technique which we have learnt from the proof of [15, Theorem 4.1]. We reproduce it here.

We can assume X and each D_λ to be defined over a subring O of $\mathbb{C}$, which is finitely generated over $\mathbb{Q}$. Denote the embedding of O to $\mathbb{C}$ by $\sigma : O \to \mathbb{C}$. Each ring homomorphism $O \to \mathbb{C}$ corresponds to a point in $\mathrm{Spec}(O)(\mathbb{C})$. Denote by X^0 and D^0_λ the schemes over $\mathrm{Spec}(O)$ which give rise to X and D_λ respectively after tensoring with $\mathbb{C}$, that is $X = X^0 \times_{\mathrm{Spec}(O)} \mathrm{Spec}(\mathbb{C})$ and $D_\lambda = D^0_\lambda \times_{\mathrm{Spec}(O)} \mathrm{Spec}(\mathbb{C})$. By possibly replacing O by $O[\frac{1}{h}]$ for some $h \in O$, we can assume X^0 and every D^0_λ are smooth over $\mathrm{Spec}(O)$, and all the intersections of D^0_λ's are transverse. Since each connected component of $\mathrm{Spec}(O)(\mathbb{C})$ contains a $\bar{\mathbb{Q}}$ point, there exists a point $P \in \mathrm{Spec}(O)(\bar{\mathbb{Q}})$, and a continuous path from $\sigma \in \mathrm{Spec}(O)(\mathbb{C})$ to P in $\mathrm{Spec}(O)(\mathbb{C})^{\mathrm{top}}$. Then, according to Thom's First Isotopy Lemma [5, Ch. 1, Theorem 3.5], $X^0(\mathbb{C})$ together with its strata given by the $D^0_\lambda(\mathbb{C})$, is a topologically locally trivial fibration in the stratified sense over $\mathrm{Spec}(O)(\mathbb{C})^{\mathrm{top}}$. In particular, letting X' and D'_λ be the corresponding fibers over P, transporting along the path gives an isomorphism $(X-D)^{\mathrm{top}} \cong (X'-D')^{\mathrm{top}}$. Recall that $\mathbf{M}_{\mathrm{B}}(U)$ and $\Sigma^i_k(U)$ depend only on the topology of U. Hence replacing $U = X - D$ by $U' = X' - D'$, we may assume that X and each D_λ are defined over $\bar{\mathbb{Q}}$.

Next, we introduce the other side of the story, namely the logarithmic flat bundles on (X, D). A logarithmic flat bundle on (X, D) consists of a vector bundle E on X, and a logarithmic connection $\nabla : E \to E \otimes \Omega^1_X(\log D)$, satisfying the integrability condition $\nabla^2 = 0$. Given a logarithmic flat bundle (E, ∇), the flat sections of E on U (by which we will always mean on U^{top}) form a local system. And conversely, given any local system L on U (by which, as in the introduction, we will always mean a local system on U^{top}), it is always obtained from some logarithmic flat bundle (E, ∇). However, different logarithmic flat bundles may give the same local system. This correspondence between local systems on U and logarithmic flat bundles on (X, D) is very well understood (e.g., [4], [14], [9]).

For a vector bundle E on X, the structure of a logarithmic flat bundle (E, ∇) on (X, D) is the same as a $\mathcal{D}_X(\log D)$-module structure on E, where $\mathcal{D}_X(\log D)$ is the sheaf of logarithmic differentials.

Nitsure [10] and Simpson [16] constructed coarse moduli spaces, which are separated quasi-projective schemes, for Jordan-equivalence classes of semistable Λ-modules which are $\mathcal{O}_X$-coherent and torsion free, where Λ is a sheaf of rings of differential operators. The two examples of Λ which we are concerned with are $\mathcal{D}_X$, the usual sheaf of differential operators on X, and $\mathcal{D}_X(\log D)$, the sheaf of logarithmic differentials. We denote by $\mathbf{M}_{\mathrm{DR}}(X)$ and $\mathbf{M}_{\mathrm{DR}}(X/D)$ the moduli space of rank one $\mathcal{D}_X$-modules and the moduli space of rank one $\mathcal{D}_X(\log D)$-modules, respectively. In the rank one case, semistable is the same as stable and this condition is automatic as is the locally free condition, and Jordan-equivalence is the same as isomorphic. Thus, the points of $\mathbf{M}_{\mathrm{DR}}(X)$ and $\mathbf{M}_{\mathrm{DR}}(X/D)$ correspond to isomorphism classes of flat, respectively, logarithmic flat line bundles. Since we did not put any condition on the Chern class of the underlying line bundles, in general $\mathbf{M}_{\mathrm{DR}}(X/D)$ has infinitely many connected components. $\mathbf{M}_{\mathrm{DR}}(X)$, $\mathbf{M}_{\mathrm{DR}}(X/D)$, $\mathbf{M}_{\mathrm{B}}(X)$ and $\mathbf{M}_{\mathrm{B}}(U)$ are all algebraic groups, except $\mathbf{M}_{\mathrm{DR}}(X/D)$ may not be of finite type.

The diagram of Fig. 1 (p. 230) plays an essential role in our proof.

Let us first explain how the arrows are defined. Since every $\mathcal{D}_X$-module is naturally a $\mathcal{D}_X(\log D)$-module, there is a natural embedding $\mathbf{M}_{\mathrm{DR}}(X) \hookrightarrow \mathbf{M}_{\mathrm{DR}}(X/D)$. On the other hand, the embedding $U \hookrightarrow X$ induces a surjective map on the fundamental group $\pi_1(U) \to \pi_1(X)$. Composing this map with the representations, we have $\mathbf{M}_{\mathrm{B}}(X) \hookrightarrow \mathbf{M}_{\mathrm{B}}(U)$. For every rank one logarithmic flat bundle (E, ∇), taking the residue along each D_λ is the map res. In other words, $\mathrm{res}((E, \nabla)) = \{\mathrm{res}_{D_\lambda}(\nabla)\}_{1 \leq \lambda \leq n}$. Around each D_λ, we can take a small loop γ_λ. The map ev is the evaluation at the loops γ_λ. More precisely $\mathrm{ev}(\rho) = \{\rho(\gamma_\lambda)\}_{1 \leq \lambda \leq n}$.

For the horizontal arrows, $RH : \mathbf{M}_{\mathrm{DR}}(X) \to \mathbf{M}_{\mathrm{B}}(X)$ is taking the monodromy representations for flat bundles. Since every logarithmic flat bundle on (X, D) restricts to a flat bundle on U, taking the monodromy representation on U is $RH : \mathbf{M}_{\mathrm{DR}}(X/D) \to \mathbf{M}_{\mathrm{B}}(U)$. The map $\exp : \mathbb{C}^n \to (\mathbb{C}^*)^n$ is component-wise defined to be multiplying by $2\pi\sqrt{-1}$, then taking exponential. On $\mathbf{M}_{\mathrm{DR}}(X/D)$, there are some special elements. Let $(\mathcal{O}_X, d)$ be the

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathbb{Z}^n & = & \mathbb{Z}^n \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbf{M}_{\mathbf{DR}}(X) & \longrightarrow & \mathbf{M}_{\mathbf{DR}}(X/D) & \xrightarrow{\text{res}} & \mathbb{C}^n \\
 & & \downarrow{\scriptstyle RH} & & \downarrow{\scriptstyle RH} & & \downarrow{\scriptstyle \exp} \\
0 & \longrightarrow & \mathbf{M}_{\mathbf{B}}(X) & \longrightarrow & \mathbf{M}_{\mathbf{B}}(U) & \xrightarrow{\text{ev}} & (\mathbb{C}^*)^n \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
\end{array}$$

FIGURE 1.

trivial rank one logarithmic flat bundle on (X, D). Notice that $\mathcal{O}_X(-D_\lambda)$ is preserved under d, that is, there is an induced map $d : \mathcal{O}_X(-D_\lambda) \to \mathcal{O}_X(-D_\lambda) \otimes \Omega^i_X(\log D)$. Therefore, $(\mathcal{O}_X(-D_\lambda), d)$ is also a logarithmic flat bundle on (X, D). The map $\mathbb{Z}^n \to \mathbf{M}_{\mathbf{DR}}(X/D)$ is defined by $\{m_\lambda\}_{1\leq\lambda\leq n} \mapsto \bigotimes_{1\leq\lambda\leq n}(\mathcal{O}_X(-D_\lambda), d)^{\otimes m_\lambda}$. The map $\mathbb{Z}^n \to \mathbb{C}^n$ is the natural inclusion map.

Notice that all the maps are group homomorphisms, all the rows and columns are exact. The first map RH is an analytic isomorphism, since $\mathbf{M}_{\mathbf{DR}}(X)$ and $\mathbf{M}_{\mathbf{B}}(X)$ analytically represent the same functor ([17]). Similarly, the quotient $\mathbf{M}_{\mathbf{DR}}(X/D)/\mathbb{Z}^n$ and $\mathbf{M}_{\mathbf{B}}(U)$ represent the same functor from the category of analytic spaces to the category of sets. Therefore, by Yoneda's lemma, $RH : \mathbf{M}_{\mathbf{DR}}(X/D) \to \mathbf{M}_{\mathbf{B}}(U)$ is an analytic covering map with transformation group $\mathbb{Z}^n$. The map exp is obviously an analytic covering map.

According to the discussion following Theorem 2.1, we can assume X and each D_λ to be defined over $\bar{\mathbb{Q}}$ without loss of generality. Then $\mathbf{M}_{\mathbf{DR}}(X/D)$ and $\mathbf{M}_{\mathbf{DR}}(X)$ are also defined over $\bar{\mathbb{Q}}$. The representation varieties $\mathbf{M}_{\mathbf{B}}(U)$ and $\mathbf{M}_{\mathbf{B}}(X)$ are always defined over $\mathbb{Q}$. Therefore, all the horizontal arrows in the above diagram are maps defined over $\bar{\mathbb{Q}}$. From now on, we should think of $\mathbb{C}^n$ and $(\mathbb{C}^*)^n$ as varieties defined over $\bar{\mathbb{Q}}$, or in other words, as $\mathbb{A}^n_{\mathbb{C}} = \mathbb{A}^n_{\bar{\mathbb{Q}}} \times_{\bar{\mathbb{Q}}} \mathbb{C}$ and $(\mathbb{G}_{m,\mathbb{C}})^n = (\mathbb{G}_{m,\bar{\mathbb{Q}}})^n \times_{\bar{\mathbb{Q}}} \mathbb{C}$, respectively.

LEMMA 2.2. – *Suppose* $Z \subset \mathbb{C}^n$ *is a non-empty Zariski constructible subset defined over* $\bar{\mathbb{Q}}$. *Suppose* $\exp(Z) \subset (\mathbb{C}^*)^n$ *is also a Zariski constructible subset defined over* $\bar{\mathbb{Q}}$. *Then* $\exp(Z)$ *contains a torsion point.*

Proof. – When $n = 1$, this follows from the Gelfond-Schneider theorem, which says if α and $e^{2\pi\sqrt{-1}\alpha}$ are both algebraic numbers, then $\alpha \in \mathbb{Q}$.

We use induction on n. Suppose the lemma is true for $\mathbb{C}^{n-1}$. Let $p_1 : \mathbb{C}^n \to \mathbb{C}^{n-1}$ and $p_2 : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^{n-1}$ be the projections to the first $n-1$ factors. Then $p_1(Z) \subset \mathbb{C}^{n-1}$ and $p_2(\exp(Z)) \subset (\mathbb{C}^*)^{n-1}$ are both defined over $\bar{\mathbb{Q}}$. Since $\exp(p_1(Z)) = p_2(\exp(Z))$, by induction hypothesis, $\exp(p_1(Z))$ contains a torsion point τ in $(\mathbb{C}^*)^{n-1}$.

Let $M = p_2^{-1}(\tau)$. Then $\exp^{-1}(M)$ is a disjoint union of infinitely many copies of $\mathbb{C}$. Choose one copy of those, which intersects with Z. Denote this copy by N. Since τ is a torsion point in $(\mathbb{C}^*)^{n-1}$, N is defined by equations with $\mathbb{Q}$ coefficients. Consider the following diagram,

$$\begin{array}{ccc} N & \xrightarrow{q_1} & \mathbb{C} \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ M & \xrightarrow{q_2} & \mathbb{C}^*, \end{array}$$

where q_1 and q_2 are projections to the last coordinates respectively. Then q_1 and q_2 are isomorphisms defined over $\bar{\mathbb{Q}}$. If $M \subset \exp(Z)$, then every torsion point in $\mathbb{C}^*$ via q_2^{-1} gives a torsion point in $\exp(Z)$. If $M \not\subseteq \exp(Z)$, then $M \cap \exp(Z)$ contains finitely many points. Hence, $N \cap Z$ also contains finitely many points. In this case, let σ be any point in $N \cap Z$, $q_1(\sigma) \in \mathbb{C}$ is defined over $\bar{\mathbb{Q}}$. On the other hand, $\exp(\sigma)$ is a point in $M \cap \exp(Z)$, and hence defined over $\bar{\mathbb{Q}}$. Thus, $q_2(\exp(\sigma)) = \exp(q_1(\sigma)) \in \mathbb{C}^*$ is defined over $\bar{\mathbb{Q}}$. Now, the Gelfond-Schneider theorem implies that $p_2(\exp(\sigma))$ is torsion in $\mathbb{C}^*$. Since $q_2(\exp(\sigma))$ is torsion in $\mathbb{C}^*$ and $p_2(\exp(\sigma)) = \tau$ is torsion in $(\mathbb{C}^*)^{n-1}$, $\exp(\sigma) \in \exp(Z)$ is a torsion point in $(\mathbb{C}^*)^n$. □

REMARK 2.3. – In fact, Jiu-Kang Yu has pointed out to us that, using Hilbert's irreducibility theorem one, can prove that if Z and $\exp(Z)$ are closed irreducible subvarieties defined over $\bar{\mathbb{Q}}$, then $\exp(Z)$ is a torsion translate of subtorus. We give the proof in the appendix.

Remember that we assume that X and each D_λ are defined over $\bar{\mathbb{Q}}$.

LEMMA 2.4. – *Let T be an irreducible component of $\Sigma^i_k(U)$. Then there exists an irreducible subvariety S of $\mathbf{M}_{\mathrm{DR}}(X/D)$ defined over $\bar{\mathbb{Q}}$ such that $RH(S) = T$.*

Proof. – For any $\rho \in \mathbf{M}_{\mathrm{B}}(U)$, $RH^{-1}(\rho)$ contains all the possible extensions of L_ρ to a logarithmic flat bundle over (X, D). Suppose $(E, \nabla) \in RH^{-1}(L)$, and suppose ∇ does not have any residue being equal to a positive integer, that is, $\mathrm{res}((E, \nabla))$ does not have any positive integer in its coordinates. Then by a theorem of Deligne [4, II, 6.10], the hypercohomology of the algebraic de Rham complex

$$E \otimes \Omega^\bullet_X(\log D) = [E \xrightarrow{\nabla} E \otimes \Omega^1_X(\log D) \xrightarrow{\nabla} E \otimes \Omega^2_X(\log D) \xrightarrow{\nabla} \cdots]$$

computes the cohomology of the local system L, i.e., $\mathbb{H}^i(X, E \otimes \Omega^\bullet_X(\log D)) \cong H^i(U, L_\rho)$.

Define the bad locus $BL \subset \mathbf{M}_{\mathrm{DR}}(X/D)$ to be the locus where one of the residues of ∇ is a positive integer. Then BL is the preimage of infinitely many hyperplanes in $\mathbb{C}^n$ via res. Define

$$\Sigma^i_k(X/D) = \{(E, \nabla) \in \mathbf{M}_{\mathrm{DR}}(X/D) \mid \dim \mathbb{H}^i(X, E \otimes \Omega^\bullet_X(\log D)) \geq k\}.$$

Given any point ρ_0 in $\Sigma^i_k(U)$, one can always find an extension $(E_0, \nabla_0) \in \mathbf{M}_{\mathrm{DR}}(X/D)$ of L_{ρ_0}, which is not in BL, e.g., the Deligne extension. Then $RH(\Sigma^i_k(X/D) - BL) = \Sigma^i_k(U)$.

Now, given $T \subset \Sigma^i_k(U)$ as an irreducible component, take any point ρ_0 in T. Since RH is analytically a covering map, there is a unique irreducible component S of $RH^{-1}(T)$ containing the Deligne extension (E_0, ∇_0) of L_{ρ_0}. Since $S \not\subseteq BL$ and since RH is analytically a covering map, we have $RH(S) = T$. By semicontinuity theorem, $\Sigma^i_k(X/D) \subset \mathbf{M}_{\mathrm{DR}}(X/D)$

is closed and defined over $\bar{\mathbb{Q}}$. Since S is an irreducible component of $\Sigma^i_k(X/D)$, S is closed and defined over $\bar{\mathbb{Q}}$. □

Now, we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. – Let T be an irreducible component of $\Sigma^i_k(U)$. By [6, Lemma 9.2], $\Sigma^i_k(U)$ is defined by some Fitting ideal coming from the CW-complex structure of U. Thus, $\Sigma^i_k(U) \subset \mathbf{M_B}(U)$ is defined over $\mathbb{Q}$. Hence, as an irreducible component of $\Sigma^i_k(U)$, T is defined over $\bar{\mathbb{Q}}$. According to Lemma 2.4, there exists $S \subset \mathbf{M_{DR}}(X/D)$ defined over $\bar{\mathbb{Q}}$ such that $RH(S) = T$. Then $\mathrm{res}(S) \subset \mathbb{C}^n$ and $\mathrm{ev}(T) \subset (\mathbb{C}^*)^n$ are defined over $\bar{\mathbb{Q}}$, and moreover, $\exp(\mathrm{res}(S)) = \mathrm{ev}(T)$. According to Lemma 2.2, $\mathrm{ev}(T)$ contains a torsion point τ.

Since $\tau \in (\mathbb{C}^*)^n$ is torsion, we can take $l \in \mathbb{Z}_+$ such that $\tau^l = \mathbf{1} \in (\mathbb{C}^*)^n$. Then the image of the l-power map $(\,\cdot\,)^l : \mathrm{ev}^{-1}(\tau) \longrightarrow \mathbf{M_B}(U)$ is equal to $\mathbf{M_B}(X)$. Choose $\eta \in \mathrm{ev}^{-1}(\tau)$, such that $\eta^l = \mathbf{1}$. Every $\xi \in RH^{-1}(\eta)$ is a $\bar{\mathbb{Q}}$ point in $\mathbf{M_{DR}}(X/D)$. In fact, since $\eta^l = \mathbf{1}$ in $\mathbf{M_B}(U)$, ξ^l is in the image of $\mathbb{Z}^n \to \mathbf{M_{DR}}(X/D)$. Recall that the image of $\{m_\alpha\}_{1\leq\alpha\leq n}$ under $\mathbb{Z}^n \to \mathbf{M_{DR}}(X/D)$ is $\bigotimes_{1\leq\lambda\leq n}(\mathcal{O}_X(-D_\lambda), d)^{\otimes m_\lambda}$, which is clearly a $\bar{\mathbb{Q}}$ point in $\mathbf{M_{DR}}(X/D)$. Therefore, ξ, as an l-th root of a $\bar{\mathbb{Q}}$ point, has to be a $\bar{\mathbb{Q}}$ point.

Notice that

$$(\mathrm{ev}\circ RH)^{-1}(\tau) = \bigcup_{\xi\in RH^{-1}(\eta)} (\xi \cdot \mathbf{M_{DR}}(X)).$$

Moreover, since $RH(S) = T$,

$$\begin{aligned} T \cap \mathrm{ev}^{-1}(\tau) &= RH(S) \cap \mathrm{ev}^{-1}(\tau) \\ &= RH\big(S \cap (RH \circ \mathrm{ev})^{-1}(\tau)\big) \\ &= \bigcup_{\xi\in RH^{-1}(\eta)} RH\Big(S \cap \big(\xi \cdot \mathbf{M_{DR}}(X)\big)\Big). \end{aligned}$$

Each $RH(S \cap (\xi \cdot \mathbf{M_{DR}}(X)))$ is closed in $\mathbf{M_B}(U)$, and $T \cap \mathrm{ev}^{-1}(\tau)$ is a noetherian topological space. Hence, for some $\xi_0 \in RH^{-1}(\eta)$, $RH(S \cap (\xi_0 \cdot \mathbf{M_{DR}}(X)))$ contains an irreducible component of $T \cap \mathrm{ev}^{-1}(\tau)$. Since $RH(\xi_0) = \eta$, $RH((\xi_0^{-1} \cdot S) \cap \mathbf{M_{DR}}(X))$ contains an irreducible component of $\eta^{-1} \cdot (T \cap \mathrm{ev}^{-1}(\tau))$. Recall that $\eta \in \mathrm{ev}^{-1}(\tau)$. Thus, $\eta^{-1} \cdot (T \cap \mathrm{ev}^{-1}(\tau)) \subset \mathbf{M_B}(X)$.

Now, RH maps an irreducible component W of $(\xi_0^{-1} \cdot S) \cap \mathbf{M_{DR}}(X)$ to an irreducible component $RH(W)$ of $\eta^{-1} \cdot (T \cap \mathrm{ev}^{-1}(\tau)) \subset \mathbf{M_B}(X)$. Both of these irreducible components are defined over $\bar{\mathbb{Q}}$. Indeed, since ξ_0 and S are defined over $\bar{\mathbb{Q}}$ in $\mathbf{M_{DR}}(X/D)$, and η, T, $\mathrm{ev}^{-1}(\tau)$ are defined over $\bar{\mathbb{Q}}$ in $\mathbf{M_B}(U)$, $(\xi_0^{-1} \cdot S) \cap \mathbf{M_{DR}}(X)$ and $\eta^{-1} \cdot (T \cap \mathrm{ev}^{-1}(\tau)) \subset \mathbf{M_B}(X)$ are defined over $\bar{\mathbb{Q}}$ in $\mathbf{M_{DR}}(X/D)$ and $\mathbf{M_B}(U)$, respectively. Hence, the same is true for their irreducible components. Thus, we can apply [15, Theorem 3.3] which says that this irreducible component $RH(W)$ of $\eta^{-1} \cdot (T \cap \mathrm{ev}^{-1}(\tau)) \subset \mathbf{M_B}(X)$ is a torsion translate of a subtorus. In particular, $\eta^{-1} \cdot (T \cap \mathrm{ev}^{-1}(\tau)) \subset \mathbf{M_B}(X)$ contains a torsion point. Since η is also a torsion point, T must contain a torsion point. □

3. Finite Abelian covers

First, we consider a more general situation. Let U be a connected, finite-type CW-complex, and let $\mathbf{M_B}(U) = \mathrm{Hom}(\pi_1(U), \mathbb{C}^*)$ be the moduli space of rank one local systems on U, which is naturally an algebraic group. Suppose $\tau \in \mathbf{M_B}(U)$ is a torsion point. Denote the universal cover of U by $\tilde{U}$, and let H be the kernel of $\tau : \pi_1(U) \to \mathbb{C}^*$. Then H acts on $\tilde{U}$ and we denote the quotient $\tilde{U}/H$ by V. Now, $\langle\tau\rangle$, the subgroup of $\mathbf{M_B}(U)$ generated by τ, acts on V, and the quotient $V/\langle\tau\rangle = U$. Denote this quotient by $f : V \to U$. Composing with $f_* : \pi_1(V) \to \pi_1(U)$, f induces a homomorphism of algebraic groups $f^\star : \mathbf{M_B}(U) \to \mathbf{M_B}(V)$. Under this construction, we immediately have $f^\star(\tau) = \mathbf{1} \in \mathbf{M_B}(V)$ is the identity element, i.e., $f^\star(\tau)$ maps every element in $\pi_1(V)$ to 1. The main result of this section is the following:

PROPOSITION 3.1. – *Fixing i, suppose that for every $k \in \mathbb{Z}_+$, each irreducible component of $\Sigma^i_k(V)$ containing $\mathbf{1}$ is a subtorus. Then for every $k \in \mathbb{Z}_+$, each irreducible component of $\Sigma^i_k(U)$ containing τ is a translate of subtorus.*

Proof. – Denote the order of τ in $\mathbf{M_B}(X)$ by r. For any local system L on U,

$$f_*f^*(L) \cong \bigoplus_{j=0}^{r-1} L \otimes_{\mathbb{C}} L_\tau^{\otimes j}.$$

According to the projection formula, $H^i(V, f^*(L)) \cong H^i(U, f_*f^*(L))$. Therefore,

$$\dim H^i(V, f^*(L)) = \sum_{j=0}^{r-1} \dim H^i(U, L \otimes L_\tau^{\otimes j}). \tag{1}$$

Let T be an irreducible component of $\Sigma^i_k(U)$ containing τ, and let ρ be a general point in T. Define $\beta_j = \dim H^i(U, L_\rho \otimes L_\tau^{\otimes j})$, for $0 \leq j \leq r-1$, and $\beta = \sum_{0\leq j\leq r-1} \beta_j$. It is possible that $T \subset \Sigma^i_{k+1}(U)$, and in this case, $\beta > k$.

CLAIM. – *$f^\star(T)$ is an irreducible component of $\Sigma^i_\beta(V)$.*

Proof of Claim. – By the definition of ρ and β, it is clear that $f^\star(T) \subset \Sigma^i_\beta(V)$. Let S be the irreducible component of $\Sigma^i_\beta(V)$ containing $f^\star(T)$. We want to show that $S = f^\star(T)$. Let $\tilde{S}$ be a connected component of $(f^\star)^{-1}(S)$ containing T. Since $f^\star$ is a covering map, $\tilde{S}$ is irreducible and is a covering space of S.

Suppose $T \subsetneq \tilde{S}$. Take a general point ρ' in $\tilde{S}$. Since $\tilde{S}$ is irreducible, and since T is an irreducible component of $\Sigma^i_k(U)$, we can assume $\rho' \notin \Sigma^i_k(U)$. Therefore, $\dim H^i(U, L_{\rho'}) < \dim H^i(U, L_\rho)$. Since ρ' is more general than ρ, $\dim H^i(U, L_{\rho'} \otimes L_\tau^{\otimes j}) \leq \dim H^i(U, L_\rho \otimes L_\tau^{\otimes j})$, for every $1 \leq j \leq r-1$. Thus,

$$\sum_{j=0}^{r-1} \dim H^i(U, L_{\rho'} \otimes L_\tau^{\otimes j}) < \sum_{j=0}^{r-1} \dim H^i(U, L_\rho \otimes L_\tau^{\otimes j}).$$

Now, equality (1) implies that $\dim H^i(V, f^*(L_{\rho'})) < \beta$, and hence $f^\star(\rho')$ is not contained in $\Sigma^i_\beta(V)$. This is a contradiction to the definition of ρ' and $\tilde{S}$. So we have proved $T = \tilde{S}$. Therefore, $f^\star(T) = S$ is an irreducible component of $\Sigma^i_\beta(V)$. □

Since $\tau \in T$, $f^\star(T)$ contains $\mathbf{1}$. By the assumption of the theorem, $f^\star(T)$ is a subtorus in $\mathbf{M}_\mathbf{B}(V)$. Since $f^\star$ is a covering map, obviously T must be a translate of subtorus. We finished the proof of the proposition. □

Theorem 1.1 is a direct consequence of Theorem 2.1, Proposition 3.1, and Theorem 1.2.

4. Appendix

We prove the following strengthening of Lemma 2.2 pointed out to us by Jiu-Kang Yu.

LEMMA 4.1. – *Suppose $S \subset \mathbb{C}^n$ is a Zariski closed subset defined over $\bar{\mathbb{Q}}$. Suppose $T \subset (\mathbb{C}^*)^n$ is also a Zariski closed subset defined over $\bar{\mathbb{Q}}$ such that* $\dim S = \dim T$ *and* $\exp(S) \subset T$. *Then T is a torsion translate of a subtorus.*

Proof. – First we prove the lemma for the case $\operatorname{codim}(S) = 1$. Denote the projections to the first $n-1$ coordinates by $p_1 : \mathbb{C}^n \to \mathbb{C}^{n-1}$ and $p_2 : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^{n-1}$. After a change of bases, we can assume that $\dim(p_1(S)) = \dim(p_2(T)) = n-1$.

Let $\rho \in \mathbb{Q}^{n-1} \subset \mathbb{C}^{n-1}$ be a point with rational coordinates. Denote $p_1^{-1}(\rho)$ and $p_2^{-1}(\exp(\rho))$ by A_ρ and B_ρ, respectively. Since $\dim(p_2(T)) = n-1$, for a general ρ, $B_\rho \cap T$ consists of finitely many points. Since $\exp(S) \subset T$ and $\exp(A_\rho) = B_\rho$, we have

$$\exp(A_\rho \cap S) \subset B_\rho \cap T.$$

The projection to the last coordinate defines an isomorphism $A_\rho \cong \mathbb{C}$. Similarly we have $B_\rho \cong \mathbb{C}^*$. Under these isomorphisms, $A_\rho \cap S$ and $B_\rho \cap T$ are both defined over $\bar{\mathbb{Q}}$. This means any point in $A_\rho \cap S$ is a $\bar{\mathbb{Q}}$ point, and its image under the exponential is also a $\bar{\mathbb{Q}}$ point. Now, according to Gelfond-Schneider theorem, the points in $A_\rho \cap S$ must be rational points.

We have shown that for a general $\rho \in \mathbb{C}^{n-1}$, $A_\rho \cap S$ consists of only points with rational coordinates. Suppose S is defined by a polynomial $f(x_1, \ldots, x_n) = 0$ with coefficients in $\bar{\mathbb{Q}}$. Since S is irreducible, f is irreducible over $\bar{\mathbb{Q}}$. Let $\bar{f}$ be the irreducible polynomial defined over $\mathbb{Q}$ that has f as a factor over $\bar{\mathbb{Q}}$. Now, for a general $\rho \in \mathbb{Q}^{n-1}$, the intersection of the zero locus of $\bar{f}$ and A_ρ must contain at least one point with rational coordinates. This means that by plugging in a general $(n-1)$-tuple of rational numbers into the first $n-1$ variables, $\bar{f}(x_1, \ldots, x_n) = 0$ has at least one solution $x_n \in \mathbb{Q}$. However, by Hilbert irreducibility theorem, after plugging in such a general $(n-1)$-tuple of rational numbers, $\bar{f}$ is irreducible over $\mathbb{Q}$ as a polynomial in x_n. Therefore, $\bar{f}$ must be of degree one in x_n. Since the coordinates can be chosen generically, $\bar{f}$ itself is of degree one. Now, it is obvious that S is a translate of a linear subspace defined over $\mathbb{Q}$, and T is a translate of a subtorus by a torsion point.

Next, we use induction on the codimension of S. Suppose $\operatorname{codim}(S) \geq 2$. We define the projections $p_1 : \mathbb{C}^n \to \mathbb{C}^{n-1}$ and $p_2 : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^{n-1}$ as before. After a change of bases, we can assume that $\dim(p_1(S)) = \dim(S)$. Then, $\dim(p_2(T)) = \dim(T) = \dim(S)$.

Let $S' = \overline{p_1(S)}$ and $T' = \overline{p_2(T)}$ be the closures in the usual Euclidean topology. Since $p_1(S)$ and $p_2(T)$ are Zariski constructible sets, both the Zariski topology and the usual topology define the same closure. Hence S' and T' are defined over $\bar{\mathbb{Q}}$. Since the exponential map is continuous in the usual topology,

$$\exp\left(\overline{p_1(S)}\right) \subset \overline{\exp(p_1(S))}.$$

Since $\exp(S) \subset T$ and $\exp(p_1(S)) = p_2(\exp(S))$, we have

$$\exp(S') = \exp\left(\overline{p_1(S)}\right) \subset \overline{\exp(p_1(S))} = \overline{p_2(\exp(S))} \subset \overline{p_2(T)} = T'.$$

Using the induction hypothesis on the pair $S' \subset \mathbb{C}^{n-1}$ and $T' \subset (\mathbb{C}^*)^{n-1}$, we conclude that T' is a torsion translate of a subtorus. Now, by choosing a torsion point of T' as origin, we can identify T' as $(\mathbb{C}^*)^{\dim(T)}$. Taking the connected component of $\exp^{-1}(p_2^{-1}(T'))$ containing S and choosing a compatible origin on this connected component, the problem is reduced to a codimension one case, which is already solved. ☐

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