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THE q-ANALOGUE OF THE WILD FUNDAMENTAL GROUP AND THE INVERSE PROBLEM OF THE GALOIS THEORY OF q-DIFFERENCE EQUATIONS

BY JEAN-PIERRE RAMIS AND JACQUES SAULOY

ABSTRACT. – In [23, 24], we defined q-analogues of alien derivations for linear analytic q-difference equations with integral slopes and proved a density theorem (in the Galois group) and a freeness theorem. In this paper, we completely describe the wild fundamental group and apply this result to the inverse problem in q-difference Galois theory.

RÉSUMÉ. – Nous avons défini dans [23, 24] des q-analogues des dérivations étrangères pour les équations aux q-différences linéaires analytiques à pentes entières, et prouvé un théorème de densité (dans le groupe de Galois) et un théorème de liberté. Dans cet article, nous décrivons complètement le groupe fondamental sauvage et appliquons ce résultat au problème inverse en théorie de Galois des équations aux q-différences.

1. Introduction

1.1. The problems

The main purpose of this paper is to give a *new* and probably definitive version of the local meromorphic classification of q-difference modules in the *integral* slopes case[(1)]. Using this result we shall get *a complete solution* of the inverse problem for the q-difference Galois theory in the local case, *for all* $q \in \mathbf{C}^*$, $|q| \neq 1$, and a solution of the inverse problem for *connected reductive algebraic groups* in the global case, also *for all* $q \in \mathbf{C}^*$, $|q| \neq 1$ (for the case of the *exceptional simple groups*, in particular, this result is new[(2)]).

(1) This is explained in Section 2.2. For the definition and properties of slopes, see Section 2 and [33].
(2) For the simple groups $\mathrm{SL}(n, \mathbf{C})$, $\mathrm{SO}(n, \mathbf{C})$, $\mathrm{Sp}(2n, \mathbf{C})$ there are *explicit* solutions with generalized q-hypergeometric difference equations due to J. Roques, cf. Section 5.1

1.1.1. *The q-wild fundamental group*. – In [25] we gave *three* versions of the local meromorphic classification of q-difference modules (in the integral slopes case). The first one uses algebraic normal forms and index theorems, it improves some results of Birkhoff and Guenther [3], there is no analog in the differential case. The second method uses a q-analog of Poincaré asymptotics expansions and the non Abelian cohomology $H^1(\mathbf{E}_q, \Lambda)$ of some sheaves Λ on the (loxodromic) elliptic curve $\mathbf{E}_q := \mathbf{C}^*/q^{\mathbf{Z}}$, it parallels some results of Malgrange and Sibuya (after Birkhoff, Balser-Jürkat-Lutz) in the differential case. The third method uses q-multisummability, it parallels [17] in the differential case.

The new version of the classification exposed here is based upon a "fundamental group" $\pi_{1,q,w,1}^{(0)}$ that we named the *q-wild fundamental group*[(3)], a q-analog of the *wild fundamental group* introduced by the first author in the differential case [7], [17]. There is an equivalence of (Tannakian) categories between the category of finite dimensional representations of this q-wild fundamental group and the category of q-difference modules (with integral slopes), moreover the image of a representation is "the" q-difference Galois group of the corresponding module (see Section 3.6 for a precise definition and statement). This classification is in the style of the *Riemann-Hilbert* correspondence for regular singular meromorphic linear differential equations and should have similar (important…) applications.

Of course there is a "trivial" candidate for a q-wild fundamental group satisfying our requirements: the Tannakian Galois group $\mathrm{Gal}(\mathscr{E}_1^{(0)})$ of the Tannakian category $\mathscr{E}_1^{(0)}$ of our q-modules, but this (proalgebraic) group is "too abstract and too big", our purpose was to get a *smaller* fundamental group (as small as possible !) which is Zariski dense in the Tannakian Galois group and to describe it *explicitly*. (As a byproduct, we shall get finally a complete description of the Tannakian Galois group itself.) It is important to notice that the Tannakian Galois group is an *algebraic object*, but that the construction of the smaller group is based upon *transcendental techniques* (complex analysis). This is similar to what happens with the Riemann-Hilbert correspondance.

We will see that it is possible to write:

$$\mathrm{Gal}(\mathscr{E}_1^{(0)}) = \mathfrak{St} \rtimes \mathrm{Gal}(\mathscr{E}_{p,1}^{(0)})$$

where[(4)], by definition, $\mathrm{Gal}(\mathscr{E}_{p,1}^{(0)}) := \mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*) \times \mathbf{C}$ and $\mathfrak{St}$ is a *prounipotent* group (named *the Stokes group*). We can replace $\mathrm{Gal}(\mathscr{E}_1^{(0)})$ by an equivalent datum, the action of $\mathrm{Gal}(\mathscr{E}_{p,1}^{(0)})$ on the Lie algebra $\mathfrak{st}$ of $\mathfrak{St}$. We denote this datum as a semi-direct product $\mathfrak{st} \rtimes \mathrm{Gal}(\mathscr{E}_{p,1}^{(0)})$.

We build a *free* Lie algebra L generated by an *infinite* family of symbols $\dot{\Delta}_i^{(\delta,\bar{c})}$ ($\delta \in \mathbf{N}^*$, $\bar{c} \in \mathbf{E}_q$, $i = 1, \ldots, \delta$) and $\dot{\Delta}^{(0)}$, the (pointed) q-alien derivations, endowed with an action of $\mathrm{Gal}(\mathscr{E}_{p,1}^{(0)})_s := \mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*)$, and a natural $\mathrm{Gal}(\mathscr{E}_{p,1}^{(0)})_s$-equivariant map

(3) In $\pi_{1,q,w,1}^{(0)}$, the subscript 1 is for the analogy with π_1, q is clear, w is for *wild*, the last 1 is for *integral* slopes case (i.e., with denominator 1) and the superscript $^{(0)}$ is for *local at* 0.

(4) This a priori strange notation is motivated by the fact that this group is the Galois group of the category of *pure* modules.

$L \to \tilde{\mathfrak{st}} := \mathfrak{st} \oplus \mathbf{C} \log \dot{\Delta}^{(0)}$. Then, *by definition*:

$$\pi^{(0)}_{1,q,w,1} := L \rtimes \mathrm{Gal}(\mathscr{E}^{(0)}_{p,1})_s$$

and we prove that the natural map

$$\mathrm{Rep}_{\mathbf{C}}(\mathrm{Gal}(\mathscr{E}^{(0)}_1)) \to \mathrm{Rep}_{\mathbf{C}}(\pi^{(0)}_{1,q,w,1})$$

is an *isomorphism*. To be more precise, $\mathrm{Rep}_{\mathbf{C}}(\mathrm{Gal}(\mathscr{E}^{(0)}_1))$ denotes the category of rational finite dimensional complex representations of the proalgebraic group $\mathrm{Gal}(\mathscr{E}^{(0)}_1)$ and $\mathrm{Rep}_{\mathbf{C}}(\pi^{(0)}_{1,q,w,1})$ the category of plain finite dimensional complex representations of the wild fundamental group $\pi^{(0)}_{1,q,w,1}$ (this will be made precise in Definition 3.9). The restriction of representations induces a functor $\mathrm{Rep}_{\mathbf{C}}(\mathrm{Gal}(\mathscr{E}^{(0)}_1)) \to \mathrm{Rep}_{\mathbf{C}}(\pi^{(0)}_{1,q,w,1})$ and this functor is an isomorphism, i.e., it is fully faithful and bijective on objects (see Theorem 3.10). Note that in the text all representations of algebraic or proalgebraic groups will be rational and we shall usually not bother to mention this explicitly.

As a byproduct, we prove that, for some convenient pronilpotent completion $L^\dagger$ (introduced in Section 3.6 and studied in the appendix) of the free Lie-algebra the map:

$$\exp(L^\dagger) \rtimes G^{(0)}_{p,1,s} \to \exp(\tilde{\mathfrak{st}}) \rtimes G^{(0)}_{p,1,s} = \mathfrak{St} \rtimes G^{(0)}_{p,1} = G^{(0)}_1$$

is an isomorphism of proalgebraic groups. It is an "explicit description" of the Tannakian group $G^{(0)}_1$.

The construction of L and the proof of its main properties is the outcome of a quite long process (in three steps: [23], [24] and the present article) and uses some deep results of [25]. In [23] we built some (pointed) q-alien derivations $\dot{\Delta}^{\delta}_{a}$ belonging to $\mathfrak{st}$[5], we interpreted them using q-Borel-Ramis transform and we got the "first level" of our construction (the "linear case" as in the two-slopes case). In [24] we proved the *Zariski density* of the Lie algebra generated by the q-alien derivations and we gave a first (awkward…) tentative of devissage in order to "free" a convenient *subset* of an extended set of alien derivations. Here we finally give "the good" devissage and we prove the *freeness theorem* (Theorem 3.8). The freeness property is *absolutely crucial*, it allows a very easy *computation* of the representations of the q-wild fundamental group and in particular the solution of the inverse problem.

The (q-Gevrey) devissage used in the present article is based upon the (q-Gevrey) devissage of the non-Abelian cohomology sets of some sheaves of unipotent groups on $\mathbf{E}_q$ and its relations with the q-alien derivations (this is explained in more detail in Sections 3.2 and 3.3). We think that this devissage *is interesting by itself* and will give later some relations between some $H^1(\mathbf{E}_q, \Lambda)$ and some (rational) representations of algebraic groups.

The underlying idea of our construction is that the knowledge of a q-difference module is equivalent to the knowledge of its *formal invariants* and of the corresponding q-Stokes phenomena (in the sense of [25]). This is similar to what happens in the differential case, but unfortunately there is a major difference, here the entries of the Stokes matrices are q-constants, that is elliptic functions on $\mathbf{E}_q$, and we would like instead some matrices belonging to $\mathrm{GL}_n(\mathbf{C})$ (the q-difference Galois groups are defined on $\mathbf{C}$). This motivates the

(5) The pointed q alien derivations are q-analog of the *algebraic* pointed alien derivation introduced in [16]. The name comes from the fact that *in the simplest cases* the Martinet-Ramis pointed alien derivations "coincide" with the derivations introduced before by J. Écalle under this name. For a proof, cf. [15].

replacement of Stokes matrices by q-alien derivations (using residues) introduced in [23]: a trick to *reduce* the field of constants from $\mathcal{M}(\mathbf{E}_q)$ to $\mathbf{C}$.

As a byproduct of our classification theorem we get a q-analog of the Ramis density theorem of the differential case [17].

At the end of the story there is a fascinating parallel between the differential and the q-difference case. However, it was impossible (in any case for us…) to mimic the differential approach which is essentially based upon the concept of *solution*, because in the q-difference case the solutions behave badly by tensor products. Hence we followed a new path using (roughly speaking) categories in place of solutions.

For more details about the analogies between the q-wild fundamental group and the wild fundamental group of the differential case the reader can have a look at the introduction of [23][6].

For each point $\bar{\alpha} \in \mathbf{E}_q$, we can consider the semi-direct product of the free Lie algebra generated by the symbols $\dot{\Delta}_{\bar{\alpha}}^{\delta,\bar{\alpha}^\delta}$ ($\delta \in \mathbf{N}^*$) by $\mathbf{C}^*$ (the action of $\mathbf{C}^*$ corresponding to the grading δ). The corresponding category of representations is isomorphic to the category of representations of a quotient of $\pi_{1,q,w,1}$. Similar groups appear in the linear differential case, in the non linear differential case (Lie algebras of Écalle pointed alien derivations [7]) and in the theory of the cosmic Galois group of Connes-Marcolli [5]. These groups are in some sense "motivic groups" (cf. also [1] 5. Coda[8]), therefore we can interpret our result as a "motivic version" of the local classification of the q-difference modules.

1.1.2. *The inverse problem of the Galois theory of q-difference equations.* – Using the q-wild fundamental group we can imitate the solution of the local inverse problem in the differential case due to the first author. The problem is to find necessary and sufficient conditions on a complex linear algebraic group in order that this group be the q-difference Galois group of a local meromorphic q-difference module with integral slopes ($q \in \mathbf{C}^*,\ |q| \neq 1$).

As in the differential case we get easily some *necessary conditions* using the algebraic group $V(G) := G/L(G)$ (where $L(G)$ is the invariant subgroup generated by all the maximal tori of G) and a Tannakian argument. In the differential case the corresponding conditions are sufficient, but here it is no longer the case, there appears a new necessary condition involving some type of co-weight on a maximal torus (existence of a Θ-structure[9]). Adding this condition we get a set of *necessary and sufficient conditions*. It follows in particular that a Borel subgroup of a reductive group is the q-difference Galois group of a local meromorphic q-difference module with integral slopes.

(6) In fact it is possible to get a perfect analogy if one replaces the free resurgent algebra of the wild fundamental group by a bigger free Lie algebra endowed with an action not only of $\mathbf{Z}$ but of its proalgebraic completion $\mathrm{Hom}_{gr}(\mathbf{C}^*, \mathbf{C}^*) \times \mathbf{C}$, we will return to this problem in a future paper.
(7) The Lie algebra generated by the Écalle pointed alien derivations $\{\dot{\Delta}_n\}_{n\in\mathbf{N}^*}$ is free, the grading corresponding to the rescaling of $e^{-1/x}$. There is a dictionary between Martinet-Ramis classification of saddle-nodes and some representations of this algebra [34].
(8) "Ce groupe d'une ubiquité stupéfiante", page 16.
(9) Cf. the Definition 5.14.

In [30] and [31] the second author proved a classification Theorem for *regular singular* q-difference modules, involving the local modules at 0 and ∞ and an invertible elliptic connection matrix (in Birkhoff style) and derived a description of the corresponding Galois group and of a Zariski dense subset of this group. We extend these results to the general case. Using this extension and the solution of the local inverse problem we get a partial solution of the global inverse problem. We prove in particular that every *connected reductive group* is the q-difference Galois group of a *rational* q-difference module.

1.2. Contents of the paper

We now briefly sketch the organisation of the paper. General notations and conventions are explained in the next Subsection 1.3.

Sections 2 to 4 are devoted to the "direct problem" of the description of the Galois group of a q-difference module (or system, or equation) with integral slopes. In Section 2, we review results from our previous work [23, 24] and adapt them to our present needs. In Section 3, we proceed to a complete description of the local Galois group; in particular, in 3.2 and 3.3 we explicitly describe the relation between local analytic classification according to the point of view of [25] and the point of view of representations here. In Section 4, we combine this with previous results from [31] to obtain a description of the global Galois group (when it makes sense); this is less complete that Section 3 but nevertheless sufficient for our use in Section 7.

Sections 5 to 7 are devoted to the inverse problem. This is introduced in Section 5, as well as an important technical tool, the notion of Θ-structure. In Section 6, the local inverse problem is solved. In Section 7, the global inverse problem is tackled.

1.3. General notations

Let $q \in \mathbf{C}$ be a complex number with modulus $|q| > 1$. We write σ_q the q-dilatation operator, so that, for any map f on an adequate domain in $\mathbf{C}$, one has: $\sigma_q f(z) = f(qz)$. Thus, σ_q defines a ring automorphism in each of the following rings: $\mathbf{C}\{z\}$ (convergent power series), $\mathbf{C}[[z]]$ (formal power series), $\mathcal{O}(\mathbf{C}^*)$ (holomorphic functions over $\mathbf{C}^*$), $\mathcal{O}(\mathbf{C}^*, 0)$ (germs at 0 of holomorphic functions over some punctured neighborhood of 0 in $\mathbf{C}^*$). Likewise, σ_q defines a field automorphism in each of their fields of fractions: $\mathbf{C}(\{z\})$ (convergent Laurent series), $\mathbf{C}((z))$ (formal Laurent series), $\mathcal{M}(\mathbf{C}^*)$ (meromorphic functions over $\mathbf{C}^*$), $\mathcal{M}(\mathbf{C}^*, 0)$ (germs at 0 of meromorphic functions over some punctured neighborhood of 0 in $\mathbf{C}^*$). The σ_q-invariants elements of $\mathcal{M}(\mathbf{C}^*, 0)$ actually belong to $\mathcal{M}(\mathbf{C}^*)$ and can be considered as meromorphic functions on the quotient Riemann surface $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$. Through the mapping $x \mapsto z = e^{2i\pi x}$, the latter is identified with the complex torus[10] $\mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$, where $q = e^{2i\pi\tau}$. Accordingly, we shall identify the fields $\mathcal{M}(\mathbf{C}^*, 0)^{\sigma_q}$, $\mathcal{M}(\mathbf{C}^*)^{\sigma_q}$ and $\mathcal{M}(\mathbf{E}_q)$. We shall write $a \mapsto \overline{a}$ the canonical projection map $\pi : \mathbf{C}^* \to \mathbf{E}_q$ and $[c; q] = \pi^{-1}(\overline{c}) = cq^{\mathbf{Z}}$ (a discrete logarithmic q-spiral). Last, we shall have use for the function $\theta \in \mathcal{O}(\mathbf{C}^*)$, a Jacobi Theta function such that $\sigma_q\theta = z\theta$ and θ has simple zeroes along $[-1; q]$. One then puts $\theta_c(z) = \theta(z/c)$, so that $\theta_c \in \mathcal{O}(\mathbf{C}^*)$ satisfies $\sigma_q\theta_c = (z/c)\theta_c$ and θ_c has simple zeroes along $[-c; q]$.

(10) Note however that we shall rather use the *multiplicative* notation for the group structure on $\mathbf{E}_q$.

For any two (pro)algebraic groups G, H, the set of morphisms from G to H is written $\mathrm{Hom}_{gralg}(G,H)$. When we want to consider all morphisms of abstract groups, forgetting the (pro)algebraic structure, we write $\mathrm{Hom}_{gr}(G,H)$.

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2. Previous results on the structure of the local Galois group

In this section, we recall the notations and results of [23, 24] and make more precise some of them.

A linear analytic q-difference equation at $0 \in \mathbf{C}$ is an equation:

$$\sigma_q X = AX, \tag{1}$$

where $A \in \mathrm{GL}_n(\mathbf{C}(\{z\}))$. We shall identify it with the q-difference module[11]:

$$M_A := (\mathbf{C}(\{z\})^n, \Phi_A), \text{ where } \Phi_A(X) := A^{-1}\sigma_q X. \tag{2}$$

If $B \in \mathrm{GL}_p(\mathbf{C}(\{z\}))$, morphisms from M_A to M_B are described by:

$$\mathrm{Hom}\big(M_A, M_B\big) = \{F \in \mathrm{Mat}_{p,n}(\mathbf{C}(\{z\})) \mid (\sigma_q F)A = BF\}. \tag{3}$$

The q-difference modules over $\mathbf{C}(\{z\})$ form a $\mathbf{C}$-linear neutral Tannakian category $\mathcal{E}^{(0)}$, of which we shall now distinguish some particular subcategories. First note that to each q-difference module is attached a Newton polygon, which can be described as a sequence $\mu_1 < \cdots < \mu_k$ of rational slopes coming with multiplicities $r_1, \ldots, r_k \in \mathbf{N}^*$. Modules with integral slopes form the full subcategory $\mathcal{E}_1^{(0)}$ of $\mathcal{E}^{(0)}$. Modules having only one slope are called pure isoclinic; direct sums of pure isoclinic modules are called pure and they form the full subcategory $\mathcal{E}_p^{(0)}$ of $\mathcal{E}^{(0)}$. Pure modules with integral slopes form the full subcategory $\mathcal{E}_{p,1}^{(0)}$ of both $\mathcal{E}_p^{(0)}$ and $\mathcal{E}_1^{(0)}$. Pure isoclinic modules of slope 0 are called Fuchsian; they form the full subcategory $\mathcal{E}_f^{(0)}$ of $\mathcal{E}_{p,1}^{(0)}$. All these categories are Tannakian subcategories of $\mathcal{E}^{(0)}$. Before describing their Galois groups, we shall have a look at their fiber functors.

For any q-difference module M, holomorphic solutions in σ_q-invariant open subsets of $(\mathbf{C}^*, 0)$ form a sheaf $\mathcal{F}_M$ over $\mathbf{E}_q$. This sheaf is locally free over the structural sheaf of $\mathbf{E}_q$ and thereby defines a holomorphic vector bundle which we also write $\mathcal{F}_M$. In case M is

[11] A difference module over a difference field (K,σ) (i.e., σ is an automorphism of the commutative field K) is a pair $M := (V, \Phi)$, where V is a finite dimensional vector space over K and Φ a σ-linear automorphism: $\forall a \in K,\ \forall x \in V,\ \Phi(ax) = \sigma(a)\Phi(x)$. Equivalently, M is a finite length left module over the ring $\mathcal{D}_{q,K} := K\left\langle \sigma, \sigma^{-1}\right\rangle$ of difference operators $\sum a_i \sigma^i$. Difference modules over $(\mathbf{C}(\{z\}), \sigma_q)$ are called q-difference modules.

given in matricial form $M_A = (\mathbf{C}(\{z\})^n, \Phi_A)$, these sheaf and bundle admit the following descriptions:

$$\mathcal{F}_M(V) = \{X \in \mathcal{O}\left(\pi^{-1}(V), 0\right)^n \mid \sigma_q X = AX\},$$
$$\mathcal{F}_M = \frac{(\mathbf{C}^*, 0) \times \mathbf{C}^n}{(z, X) \sim (qz, A(z)X)} \longrightarrow \frac{(\mathbf{C}^*, 0)}{z \sim qz} = \mathbf{E}_q.$$

In the right hand side of the first (resp. the second) equality, solutions $X \in \mathcal{O}\left(\pi^{-1}(V), 0\right)^n$ are taken to be germs at $0 \in \mathbf{C}^*$ (resp. the bundle $(\mathbf{C}^*, 0) \times \mathbf{C}^n$ to be quotiented is taken to be trivial over the germ of $\mathbf{C}^*$ at 0).

The functor $M \rightsquigarrow \mathcal{F}_M$ is exact, faithful and $\otimes$-compatible and provides a fiber functor on $\mathcal{E}^{(0)}$ over the base $\mathbf{E}_q$. Lifting $\mathcal{F}_M$ through π to an equivariant (trivial) bundle over $\mathbf{C}^*$, then taking fibers, we get a family $(\omega_a^{(0)})_{a \in \mathbf{C}^*}$ of fiber functors on $\mathcal{E}^{(0)}$ over $\mathbf{C}$, thus a Galois groupoid with base $\mathbf{C}^*$ over the field $\mathbf{C}$. (The reason to consider points in $\mathbf{C}^*$ rather than in $\mathbf{E}_q$ is that we want to use transcendental constructions[12].)

On the other hand, to each object M of $\mathcal{E}^{(0)}$ is associated a Newton polygon [33, def. 1.1.1], a formal invariant; this has rational slopes $\mu_1 < \cdots < \mu_k$ with multiplicities $r_1, \ldots, r_k \in \mathbf{N}^*$ and (contrary to the differential case) one can prove [33, th. 3.1.6] that there exists a unique tower of submodules $\{0\} = M_0 \subset \cdots \subset M_k = M$ such that each M_i/M_{i-1}, $i = 1, \ldots, k$, has only the slope μ_i (and then its rank is r_i). The functor $M \rightsquigarrow \operatorname{gr} M := \bigoplus M_i$ then has good Abelian and tensor properties [33, Section 3.2]. The resulting structure on $\mathcal{E}^{(0)}$ can be described as follows (note that it is stronger than the one described by Saavedra in [29, chap. IV, §2]). The category $\mathcal{E}^{(0)}$ is endowed with a family $(F_{\leq \mu})_{\mu \in \mathbf{Q}}$ of endofunctors such that, for each module M, the $F_{\leq \mu} M$ form a filtration of M by subobjects, with jumps at the slopes of M. The associated graded module:

$$\operatorname{gr} M := \bigoplus \frac{F_{\leq \mu} M}{F_{< \mu} M}$$

is pure and we get a functor $M \rightsquigarrow \operatorname{gr} M$ from $\mathcal{E}^{(0)}$ to $\mathcal{E}_p^{(0)}$, which is exact, faithful and $\otimes$-compatible. It is also a retraction of $\mathcal{E}_p^{(0)} \subset \mathcal{E}^{(0)}$. This yields a new family of fiber functors on $\mathcal{E}^{(0)}$:

$$\hat{\omega}_a^{(0)} := \omega_a^{(0)} \circ \operatorname{gr}.$$

In some sense, $\mathcal{E}_p^{(0)}$ is the "formalisation" of $\mathcal{E}^{(0)}$ and we see the $\hat{\omega}_a^{(0)}$, resp. the $\omega_a^{(0)}$, as points in a formal, resp. an analytic neighborhood of 0. (The reason for this is that, over the formal category, gr is isomorphic to the identity functor, see [33].)

Whatever the fiber functor used to define it, the Galois group[13] $\operatorname{Gal}(\mathcal{E}^{(0)})$ is the semi-direct product of the "formal" Galois group $\operatorname{Gal}(\mathcal{E}_p^{(0)})$ by a prounipotent group, the kernel

(12) It is not feasible in the setting of q-difference equations to define a fiber functor as the space of solutions in some big field K. Indeed, in order to get a fiber functor in this way, one has to take K rather big; then the fiber functor is defined over the field of constants of K, which will be bigger than $\mathbf{C}$. For instance, the natural choice $K = \mathcal{M}(\mathbf{C}^*)$ yields a Galois group over $\mathcal{M}(\mathbf{E}_q)$.

(13) We refer to [8] for the Tannakian formalism, except that, instead of using the group scheme $\underline{\operatorname{Gal}}(\mathcal{C}) := \underline{\operatorname{Aut}}^{\otimes}(\omega)$ (where ω is a fiber functor over $\mathbf{C}$ on the Tannakian category $\mathcal{C}$), we rather use the proalgebraic group $\operatorname{Gal}(\mathcal{C}) := \operatorname{Aut}^{\otimes}(\omega)$ of its $\mathbf{C}$-valued points, so that $\mathcal{C}$ is equivalent to the category $\operatorname{Rep}_{\mathbf{C}}(\operatorname{Gal}(\mathcal{C}))$ of rational representations of $\operatorname{Gal}(\mathcal{C})$.

of the morphism $i^* : \mathrm{Gal}(\mathcal{E}^{(0)}) \to \mathrm{Gal}(\mathcal{E}_p^{(0)})$ dual to the inclusion $i : \mathcal{E}_p^{(0)} \rightsquigarrow \mathcal{E}^{(0)}$; indeed, since $\mathrm{gr} \circ i$ is the identity of $\mathcal{E}_p^{(0)}$, we see that $i^* \circ \mathrm{gr}^*$ is the identity of $\mathrm{Gal}(\mathcal{E}_p^{(0)})$, and the fact that Ker i^* is prounipotent follows from the existence of the filtration. Restricting to $\mathcal{E}_1^{(0)}$, one gets:

$$\mathrm{Gal}(\mathcal{E}_1^{(0)}) = \mathfrak{St} \rtimes \mathrm{Gal}(\mathcal{E}_{p,1}^{(0)}), \tag{4}$$

where $\mathfrak{St}$ is a prounipotent group.

Definition 2.1. – We call $\mathfrak{St} := \mathrm{Ker}\, i^*$ the *Stokes group* and $\mathfrak{st} := \mathrm{Lie}(\mathfrak{St})$ the *Stokes Lie algebra*.

The goal of this series of papers is the description of the Stokes group $\mathfrak{St}$ and the Stokes Lie algebra[14] $\mathfrak{st} := \mathrm{Lie}(\mathfrak{St})$ and its application to the inverse problem in q-difference Galois theory. The main tool on the side of q-difference equations is Theorem 3.10, which describes all Galois groups of systems with integral slopes in terms of representations of a *wild fundamental group*, actually, the semi-direct product of an infinite dimensional Lie algebra with a proalgebraic group, the Tannakian formal Galois group of the category of systems with integral slopes. We obtain it with the help of an explicit family of *Galoisian Stokes operators* built by the authors together with Changgui Zhang in [25] and used there to get an analytic classification of q-difference modules. It was proved in previous work [23, 24] that we thus obtain a generating family. The analytic classification and representations of $\mathfrak{st}$ are, in some sense, two models of the same thing, which allows us to give a precise description of the latter. In this comparison, the filtration above plays a crucial role and we shall now have a closer look at it.

Convention. – As already said, any object of $\mathcal{E}^{(0)}$ is equivalent to some M_A. It can moreover be shown that one may always choose A in so-called Birkhoff-Guenther normal form; in our case of interest, this is explained at the beginning of 2.2. This implies that $A \in \mathrm{GL}_n(\mathbf{C}[z, z^{-1}]) \subset \mathrm{GL}_n(\mathbf{C}(\{z\})) \cap \mathrm{GL}_n(\mathcal{O}(\mathbf{C}^*))$, so that the above definitions are simplified to:

$$\mathcal{F}_M(V) = \{X \in \mathcal{O}\left(\pi^{-1}(V)\right)^n \mid \sigma_q X = AX\},$$
$$\mathcal{F}_M = \frac{\mathbf{C}^* \times \mathbf{C}^n}{(z, X) \sim (qz, A(z)X)} \longrightarrow \frac{\mathbf{C}^*}{z \sim qz} = \mathbf{E}_q.$$

Moreover, starting from a module $M_A = (\mathbf{C}(\{z\})^n, \Phi_A)$ such that $A \in \mathrm{GL}_n(\mathbf{C}[z, z^{-1}])$, a module $M_B = (\mathbf{C}(\{z\})^p, \Phi_B)$ such that $B \in \mathrm{GL}_p(\mathbf{C}[z, z^{-1}])$, and a morphism $F : M_A \to M_B$, $F \in \mathrm{Mat}_{p,n}(\mathbf{C}(\{z\}))$, it follows from the relation $(\sigma_q F)A = BF \Rightarrow \sigma_q F = BFA^{-1}$ that F is holomorphic over $\mathbf{C}^*$ (the functional equation allows one to expand by a factor $|q| > 1$ any punctured disk of convergence). Thus, in order to have a more concrete description of the fiber functors $\omega_a^{(0)}$ and $\hat{\omega}_a^{(0)}$, we shall now restrict to the essential full Tannakian subcategory of $\mathcal{E}^{(0)}$ made of q-difference modules M_A such that $A \in \mathrm{GL}_n(\mathbf{C}[z, z^{-1}])$. We shall keep the notation $\mathcal{E}^{(0)}$ for this smaller (but equivalent) category. Then, one has canonical

[14] Actually, we shall extend here $\mathfrak{st}$ to a Lie algebra $\tilde{\mathfrak{st}}$ which contains the "Stokes operators of level 0", that is the unipotent part of the Fuchsian Galois group, corresponding to the q-logarithm.

identifications $\omega_a^{(0)}(M_A) = \mathbf{C}^n$, $\omega_a^{(0)}(M_B) = \mathbf{C}^p$ and $\omega_a^{(0)}(F) = F(a)$. A similar description of $\hat{\omega}_a^{(0)}$ will be given in 2.2.

2.1. Consequences of the slope filtration

We now combine the facts recalled above with some Tannakian general nonsense, always referring to [8, 29] for the formalism of Tannakian duality. In order to shorten notations, we temporarily write G for $\mathrm{Gal}(\mathscr{E}_1^{(0)})$, G_p for $\mathrm{Gal}(\mathscr{E}_{p,1}^{(0)})$, $S := \mathrm{Ker}\,(G \to G_p)$ for the Stokes group $\mathfrak{St}$ and $s := \mathrm{Lie}(S)$ for the Stokes Lie algebra.

From Tannakian theory, the proalgebraic group G is the inverse limit $\varprojlim G(M)$ of the algebraic groups $G(M) := \mathrm{Aut}^{\otimes}(\omega_{|\langle M\rangle})$, where M runs over the objects of $\mathscr{E}_1^{(0)}$, $\langle M\rangle$ denotes the Tannakian subcategory generated by M and $\omega_{|\langle M\rangle}$ the corresponding restriction of the relevant fiber functor ω. Actually, $G(M)$ is identified with an algebraic subgroup of $\mathrm{GL}(\omega(M)) = \mathrm{GL}_n(\mathbf{C})$ (since here $\omega(M) = \mathbf{C}^n$). The semi-direct decomposition $G = S \rtimes G_p$ refines into $G(M) = S(M) \rtimes G_p(M)$, with $G_p(M)$ consisting in block-diagonal matrices and $S(M)$ into upper triangular unipotent matrices (see 2.3 for a more precise description). We then have $G_p = \varprojlim G_p(M)$ and $S = \varprojlim S_p(M)$. Thus, $s(M) := \mathrm{Lie}(S(M))$ consists in upper triangular nilpotent matrices and, according to [6, A7], $s = \varprojlim s(M)$ is a pronilpotent Lie algebra.

Let us write $s_k(M)$ ($k \in \mathbf{N}^*$) the ideal of $s(M) \subset \mathrm{gl}_n(\mathbf{C})$ consisting in matrices $(a_{i,j})_{1\leq i,j\leq n}$ such that $a_{i,j} = 0$ for $j - i \leq k$. Then the sequence of $s_k := \varprojlim s_k(M)$ defines a descending filtration by ideals and the corresponding linear topology makes s a Hausdorff complete space. This yields the following lemma for which we shall have use later.

Lemma 2.2. – *Let $\tau \in G_p$ be such that each $\tau(M) \in G_p(M) \subset \mathrm{GL}_n(\mathbf{C})$ is an upper triangular unipotent matrix (such a τ will naturally appear in 2.4). Let $\nu := \log \tau$, and $\tilde{s} := s \oplus \mathbf{C}\nu$. Let $s' \subset s''$ be sub-Lie algebras such that $s'' = s' + [\mathbf{C}\nu, s'']$. Then s' topologically generates s''.*

Proof. – Note first that $[\mathbf{C}\nu, s] \subset s$, so that $\tilde{s}$ is a well defined Lie algebra. Setting $\Phi(s'') := [\mathbf{C}\nu, s'']$ and iterating, we find that $s'' = s' + \Phi^k(s'')$ for all $k > 0$. Since clearly $\Phi^k(s'') \subset s_k$, the conclusion follows. □

2.2. Overall structure and representations of $\mathrm{Gal}(\mathscr{E}_1^{(0)})$

We now make an important assumption:

From now on, we shall restrict to modules with integral slopes.

The reason is that we then have explicit normal forms, and we are going to use them heavily[15]. Indeed, any pure module M_0 with integral slopes $\mu_1 < \cdots < \mu_k$ and multiplicities

(15) In the general case of rational slopes, van der Put and Reversat obtained a precise description of pure modules and of the Galois group of $\mathscr{E}_p^{(0)}$, see [20]. Relying on these results, Virginie Bugeaud has started to extend the methods of the present series of papers to the case of two arbitrary slopes.

$r_1, \dots, r_k$ can be described as $M_{A_0} := (\mathbf{C}(\{z\})^n, \Phi_{A_0})$, and any module M such that $\mathrm{gr} M \approx M_0$ can be described as $M_A := (\mathbf{C}(\{z\})^n, \Phi_A)$ (see Equation (2)), with:

$$A_0 := \begin{pmatrix} z^{\mu_1} A_1 \dots\dots\dots & \dots \\ \dots \quad \dots\dots 0 & \dots \\ 0 \quad \dots\dots\dots & \dots \\ \dots \quad 0 \dots\dots & \dots \\ 0 \quad \dots \; 0 \; \dots & z^{\mu_k} A_k \end{pmatrix} \text{ and } A := \begin{pmatrix} z^{\mu_1} A_1 \dots\dots \dots & \dots \\ \dots \quad \dots\dots U_{i,j} & \dots \\ 0 \quad \dots\dots \dots & \dots \\ \dots \quad 0 \dots \dots & \dots \\ 0 \quad \dots \; 0 \; \dots & z^{\mu_k} A_k \end{pmatrix}, \tag{5}$$

where, for $1 \le i \le k$, $A_i \in \mathrm{GL}_{r_i}(\mathbf{C})$ and where, for $1 \le i < j \le k$, $U_{i,j} \in \mathrm{Mat}_{r_i,r_j}(\mathbf{C}(\{z\}))$; moreover, one can assume that the coefficients of each block $U_{i,j}$ belong to $\sum\limits_{\mu_i \le \ell < \mu_j} \mathbf{C} z^\ell$ (Birkhoff-Guenther normal form). Modules $M_A := (\mathbf{C}(\{z\})^n, \Phi_A)$ form an essential Tannakian subcategory of $\mathscr{E}_1^{(0)}$, so that we can restrict all our definitions and constructions to such objects.

The fiber functors $\hat{\omega}_a^{(0)}$ and $\omega_a^{(0)}$ admit the following concrete description. Let A, A_0 be as in (5) and write for short $M := M_A$, $M_0 := M_{A_0}$, so that $M_0 = \mathrm{gr} M$. Then $\hat{\omega}_a^{(0)}(M) = \omega_a^{(0)}(M) = \omega_a^{(0)}(M_0) = \mathbf{C}^n$. Now define similarly $B \in \mathrm{GL}_p(\mathbf{C}(\{z\}))$ in Birkhoff-Guenther normal form with slopes $\nu_1 < \cdots < \nu_l$ having multiplicities $s_1, \dots, s_l$ and B_0 its graded (block diagonal) component and put $N := M_B$, $N_0 := M_{B_0}$, so that $N_0 = \mathrm{gr} N$. Then any morphism $M \to N$ is a matrix $F \in \mathrm{Mat}_{p,n}(\mathbf{C}(\{z\}))$ such that $\sigma_q F = BFA^{-1}$, so that one easily shows that $F \in \mathrm{Mat}_{p,n}(\mathscr{O}(\mathbf{C}^*))$. The corresponding graded morphism $F_0 := \mathrm{gr}(F) \in \mathrm{Mat}_{p,n}(\mathbf{C}(\{z\})) \cap \mathrm{Mat}_{p,n}(\mathscr{O}(\mathbf{C}^*))$ has kl blocks of sizes $r_i \times s_j$, those such that $\mu_i = \nu_j$ coming from F, all the other ones being trivial. Then one has:

$$\omega_a^{(0)}(F) = F(a),$$
$$\hat{\omega}_a^{(0)}(F) = F_0(a).$$

The Galois groups of $\mathscr{E}_f^{(0)}$ and $\mathscr{E}_{p,1}^{(0)}$ are Abelian, so that we can use any fiber functor to describe them. Using the subscript “f” for “Fuchsian” and the subscript “p” for “pure”, we have:

$$G_f^{(0)} := \mathrm{Gal}(\mathscr{E}_f^{(0)}) = \mathrm{Hom}_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*) \times \mathbf{C},$$
$$G_{p,1}^{(0)} := \mathrm{Gal}(\mathscr{E}_{p,1}^{(0)}) = \mathbf{C}^* \times G_f^{(0)}.$$

(The notation Hom_{gr} was defined in 1.3.) We also write $G_{f,s}^{(0)} = \mathrm{Hom}_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*)$ the semi-simple component of the Fuchsian group $G_f^{(0)}$; its elements are identified with (abstract group) morphisms $\mathbf{C}^* \to \mathbf{C}^*$ that send q to 1. Likewise, we write $G_{f,u}^{(0)} = \mathbf{C}$ the unipotent component of $G_f^{(0)}$ and $T_1^{(0)} = \mathbf{C}^*$ the “theta torus” component of $G_{p,1}^{(0)}$; the latter should be compared[16] with the “exponential torus” component of the wild fundamental group of differential equations.

[16] For details on this analogy, see the introduction of [23] and the conclusion of [24].

Taking again A in form (5), the representation of $G_{p,1}^{(0)} = G_{f,s}^{(0)} \times G_{f,u}^{(0)} \times T_1^{(0)}$ corresponding to $M := M_A$ by Tannakian duality is the following:

$$(\gamma, \lambda, t) \mapsto \begin{pmatrix} t^{\mu_1}\gamma(A_{1,s})A_{1,u}^{\lambda} & \dots\dots\dots & \dots \\ \dots & \dots\dots\ 0 & \dots \\ 0 & \dots\dots\dots & \dots \\ \dots & 0\ \dots\dots & \dots \\ 0 & \dots\ 0\ \dots & t^{\mu_k}\gamma(A_{k,s})A_{k,u}^{\lambda} \end{pmatrix}.$$

We wrote $A_i = A_{i,s}A_{i,u}$ the Jordan decomposition into semi-simple and unipotent component, and $\gamma(A_{i,s})$ means γ operating on eigenvalues of A_i.

As explained before (see Definition 2.1), we write $\mathfrak{St}$ the kernel of $i^* : G_1^{(0)} \to G_{p,1}^{(0)}$, a prounipotent proalgebraic group, whence the semidirect decomposition of (4):

$$G_1^{(0)} = \mathfrak{St} \rtimes G_{p,1}^{(0)}.$$

We write $\mathfrak{st}$ the Lie algebra of $\mathfrak{St}$; it is pronilpotent, see Section 2.1.

2.3. First look at the structure of $\mathfrak{St}$ and $\mathfrak{st}$

Let us characterize *Stokes operators*, i.e., elements of the *Stokes group* $\mathfrak{St}$ and *alien derivations*, i.e., elements of the *Stokes Lie algebra* $\mathfrak{st}$. Let $s \in \mathfrak{St}$, resp. $D \in \mathfrak{st}$. Their respective images by the representation associated to matrix A (meaning: to module M_A) are

$$s(A) \in \mathfrak{St}(A) \subset \mathfrak{G}_{A_0}(\mathbf{C}) \subset \mathrm{GL}_n(\mathbf{C}), \text{ where } \mathfrak{St}(A) := \mathfrak{St}(M_A),$$
$$D(A) \in \mathfrak{st}(A) \subset \mathfrak{g}_{A_0}(\mathbf{C}) \subset \mathrm{gl}_n(\mathbf{C}), \text{ where } \mathfrak{st}(A) := \mathfrak{st}(M_A),$$

where we introduce the following unipotent algebraic group $\mathfrak{G}_{A_0}$ and its Lie algebra $\mathfrak{g}_{A_0}$:

$$\mathfrak{G}_{A_0} := \left\{ \begin{pmatrix} I_{r_1} & \dots\dots\dots\dots \\ \dots\dots\dots & \star\ \dots \\ 0 & \dots\dots\dots\dots \\ \dots\ 0 & \dots\dots\dots \\ 0\ \dots\ 0 & \dots\ I_{r_k} \end{pmatrix} \right\} \subset \mathrm{GL}_n,$$

$$\mathfrak{g}_{A_0} := \left\{ \begin{pmatrix} 0_{r_1} & \dots\dots\dots\dots \\ \dots\dots\dots & \star\ \dots \\ 0 & \dots\dots\dots\dots \\ \dots\ 0 & \dots\dots\dots \\ 0\ \dots\ 0 & \dots\ 0_{r_k} \end{pmatrix} \right\} \subset \mathrm{gl}_n = \mathrm{Mat}_n.$$

Here I_r and 0_r respectively denote the identity and the null matrix of size $r \times r$. The rectangular block $\star$ indexed by (i,j) such that $1 \leq i < j \leq k$ has size $r_i \times r_j$ and links the diagonal square blocks corresponding to slopes μ_i and μ_j.

Globally, s and D are characterized as follows. They must be functorial: if $(\sigma_q F)A = BF$, then

$$s(B)F_0(a) = F_0(a)s(A) \text{ and } D(B)F_0(a) = F_0(a)D(A)$$

(for the chosen base point $a \in \mathbf{C}^*$). They must be $\otimes$-compatible:

$$s(A \otimes B) = s(A) \otimes s(B) \text{ and } D(A \otimes B) = D(A) \otimes I_p + I_n \otimes D(B).$$

Last, they must be trivial on pure modules:

$$s(A_0) = I_n \text{ and } D(A_0) = 0_n.$$

The character group of the semi-simple component of $G_{p,1}^{(0)}$ is:

$$X\left(T_1^{(0)} \times G_{f,s}^{(0)}\right) = \mathbf{Z} \times \mathbf{E}_q.$$

To describe the adjoint action of this group on $\mathfrak{st}$ therefore amounts to give the decomposition in eigenspaces; note that for the projective limit $\mathfrak{st}$, we have to complete the direct sum:

$$\mathfrak{st} = \widehat{\bigoplus_{\delta \geq 1}} \mathfrak{st}^{(\delta)}, \text{ where } \mathfrak{st}^{(\delta)} = \widehat{\bigoplus_{\overline{c} \in \mathbf{E}_q}} \mathfrak{st}^{(\delta,\overline{c})}. \tag{6}$$

(Note that only the weights such that $\delta \geq 1$ are required, because of the triangular structure coming from the functorial filtration theorem.) This decomposition is expressed elementwise as a Fourier decomposition:

$$\forall D \in \mathfrak{st}\ ,\ \forall \sigma \in T_1^{(0)} \times G_{f,s}^{(0)}\ ,\ \sigma D \sigma^{-1} = \sum_{\chi \in X\left(T_1^{(0)} \times G_{f,s}^{(0)}\right)} \langle \chi, \sigma \rangle\, D^{(\chi)},$$

where, for $\chi = (\delta, \overline{c}) \in \mathbf{Z} \times \mathbf{E}_q$ and for $\sigma = (t, \gamma) \in \mathbf{C}^* \times \mathrm{Hom}_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*)$:

$$\langle \chi, \sigma \rangle = t^\delta \gamma(\overline{c}).$$

Thus, $D = \sum D^{(\delta,\overline{c})}$ (with unicity of the decomposition) and:

$$\sigma D^{(\delta,\overline{c})} \sigma^{-1} = t^\delta \gamma(\overline{c}) D^{(\delta,\overline{c})}.$$

Since $G_{p,1}^{(0)}$ is Abelian, conjugacy under elements of its unipotent component $G_{f,u}^{(0)}$ fixes each $\mathfrak{st}^{(\delta,\overline{c})}$. We shall write τ the (Zariski-) generator $1 \in \mathbf{C} = G_{f,u}^{(0)}$, so that:

$$\tau \mathfrak{st}^{(\delta,\overline{c})} \tau^{-1} = \mathfrak{st}^{(\delta,\overline{c})}.$$

2.4. First look at the representations of $\mathfrak{St}$ and $\mathfrak{st}$

More generally, the semi-simple component of $G_{p,1}^{(0)}$ operates on $\mathfrak{g}_{A_0}$ through $G_{p,1}^{(0)}(A) = G_{p,1}^{(0)}(A_0)$, whence a decomposition:

$$\mathfrak{g}_{A_0} = \bigoplus_{\delta \geq 1} {\mathfrak{g}_{A_0}}^{(\delta)}, \text{ where } {\mathfrak{g}_{A_0}}^{(\delta)} = \bigoplus_{\overline{c} \in \mathbf{E}_q} {\mathfrak{g}_{A_0}}^{(\delta,\overline{c})}.$$

(And, of course, $\mathfrak{st}^{(\delta)}(A) = \mathfrak{st}(A) \cap {\mathfrak{g}_{A_0}}^{(\delta)}$, etc.) More concretely, one can divide matrices in $\mathfrak{g}_{A_0}$ in rectangular blocks numbered (i,j) with $1 \leq i < j \leq k$; the block i,j has size $r_i \times r_j$ and links the (null) square diagonal blocks corresponding to slopes μ_i and μ_j. If one assumes moreover that the matrices A_i are divided into diagonal blocks corresponding to their eigenvalues, then one can further divide each block (i,j) into rectangular blocks numbered $(d,e) \in \mathrm{Sp}A_i \times \mathrm{Sp}A_j$. The action of $\sigma = (t,\gamma) \in T_1^{(0)} \times G_{f,s}^{(0)}$ (through its image in $\mathfrak{G}_{A_0}$) on the block $\big((i,j),(d,e)\big)$ is multiplication by the nonzero scalar $\frac{t^{\mu_i}}{t^{\mu_j}} \frac{\gamma(\overline{d})}{\gamma(\overline{e})}$.

Thus, the matrices of ${\mathfrak{g}_{A_0}}^{(\delta)}$ are those such that blocks with $\mu_j - \mu_i \neq \delta$ are all zero and the matrices of ${\mathfrak{g}_{A_0}}^{(\delta,\overline{c})}$ are those matrices of ${\mathfrak{g}_{A_0}}^{(\delta)}$ such that blocks with $d/e \not\equiv c \pmod{q^{\mathbf{Z}}}$ are all zero. We shall frequently identify ${\mathfrak{g}_{A_0}}^{(\delta)}$, resp. ${\mathfrak{g}_{A_0}}^{(\delta,\overline{c})}$ with the corresponding vector spaces of rectangular matrices, forgetting their null components. For instance, in the case of two slopes $\mu < \nu$ with multiplicities $r, s \in \mathbf{N}^*$, the (Abelian) Lie algebra $\mathfrak{g}_{A_0}$ has a single nontrivial component ${\mathfrak{g}_{A_0}}^{(\delta)}$, with $\delta = \nu - \mu$, and we identify it with $\mathrm{Mat}_{r,s}(\mathbf{C})$.

The conjugacy action of the unipotent component of $G^{(0)}_{p,1}(A)$ leaves stable each ${\mathfrak{g}_{A_0}}^{(\delta,\overline{c})}$. Recall its Zariski-generator τ defined at the very end of 2.3. Writing:

$$U := \tau(A) = \tau(A_0) = \begin{pmatrix} A_{1,u} & \cdots\cdots\cdots & \cdots \\ \cdots & \cdots\cdots\ 0 & \cdots \\ 0 & \cdots\cdots\cdots & \cdots \\ \cdots & 0\ \cdots\cdots & \cdots \\ 0 & \cdots\ 0 & \cdots\ A_{k,u} \end{pmatrix},$$

we see that:

$$U{\mathfrak{g}_{A_0}}^{(\delta,\overline{c})}U^{-1} = {\mathfrak{g}_{A_0}}^{(\delta,\overline{c})}.$$

Now fix M_0, A_0 in $\mathcal{E}^{(0)}_{p,1}$ as above and call ρ_0 the attached representation of $G^{(0)}_{p,1}$. We consider objects M, A in $\mathcal{E}^{(0)}_1$ above M_0, A_0 (that is, $\mathrm{gr} M = M_0$). By Tannakian duality, they correspond to representations ρ of $G^{(0)}_1 = \mathfrak{St} \rtimes G^{(0)}_{p,1}$ which restrict to ρ_0 on $G^{(0)}_{p,1}$. These representations ρ are in turn in one to one correspondance with representations of $\mathfrak{St}$ that are compatible with ρ_0. Translated in terms of representations of $\mathfrak{st}$, this gives:

PROPOSITION 2.3. – *Those representations of* $\mathfrak{st}$ *corresponding to objects* M, A *above* M_0, A_0 *are exactly those such that:*

1. *Each* $\mathfrak{st}^{(\delta,\overline{c})}$ *is mapped to* ${\mathfrak{g}_{A_0}}^{(\delta,\overline{c})}$*;*
2. *The conjugation by* τ *in* $\mathfrak{st}$ *is intertwined with the conjugation by* U *in* $\mathfrak{g}_{A_0}$*, i.e.,* $\rho(\tau D \tau^{-1}) = U\rho(D)U^{-1}$.

We write $\mathrm{Rep}_{A_0}(\mathfrak{st})$ *the set of these representations.*

Proof. – Indeed, the first condition expresses compatibility with the semi-simple component of the representation ρ_0. □

In this paper, we shall extend the definition of the Stokes Lie algebra to include the Fuchsian unipotent component and put:

$$\tilde{\mathfrak{st}} := \mathfrak{st} \rtimes \mathrm{Lie}(G^{(0)}_{f,u}) = \mathbf{C}\nu \oplus \mathfrak{st},$$

that is, $\tilde{\mathfrak{st}}$ is generated by $\mathfrak{st}$ and by $\mathrm{Lie}(G^{(0)}_{f,u}) = \mathbf{C}\nu$, where $\nu := \log \tau$. Since $G^{(0)}_{f,u}$ commutes with $T^{(0)}_1 \times G^{(0)}_{f,s}$, the adjoint action of this group on $\mathbf{C}\nu$ is trivial and we write $\tilde{\mathfrak{st}}^{(\chi)} := \mathfrak{st}^{(\chi)}$ and $\tilde{\mathfrak{st}}^{(0)} := \mathbf{C}\nu$.

COROLLARY 2.4. – *Those representations of* $\tilde{\mathfrak{st}}$ *corresponding to objects* M, A *above* M_0, A_0 *are exactly those such that:*

1. *Each* $\tilde{\mathfrak{st}}^{(\delta,\overline{c})}$ *is mapped to* ${\mathfrak{g}_{A_0}}^{(\delta,\overline{c})}$*;*
2. *The element* ν *is mapped to* $\log U$.

We write $\mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}})$ *the set of these representations.*

2.5. Explicit generators of $\mathfrak{St}$

Let A_0, A be as in (5). Then, there is a unique $F \in \mathfrak{G}_{A_0}(\mathbf{C}((z)))$ such that $F[A_0] = A$. We write it $\hat{F}_A$. The components of the (i,j) block of $\hat{F}_A$ have q-Gevrey level $\delta := \mu_j - \mu_i$, meaning that they are divergent formal series with coefficients a_k having a growth of order $q^{k^2/2\delta}$ (up to some $O(R^k)$ factor). Stokes operators, to be defined here below, are obtained by "summing" this formal object in various directions then taking quotients of such summations (ambiguities). We consider as candidate "directions of summation" the q-spirals $[c;q]$ in $\mathbf{C}^*$, equivalently, the points $\overline{c} \in \mathbf{E}_q$. Define:

$$\Sigma_{A_0} := \{\overline{c} \in \mathbf{E}_q \mid q^{\mathbf{Z}} c^{\mu_i} \mathrm{Sp}(A_i) \cap q^{\mathbf{Z}} c^{\mu_j} \mathrm{Sp}(A_j) \neq \varnothing \text{ for some } 1 \leq i < j \leq k\},$$

thus a finite subset of $\mathbf{E}_q$. Then [32]:

PROPOSITION 2.5. – *For all* $\overline{c} \in \mathbf{E}_q \setminus \Sigma_{A_0}$*, there is a unique* $F \in \mathfrak{G}_{A_0}(\mathcal{M}(\mathbf{C}^*))$ *such that* $F[A_0] = A$ *and subject to the following constraints: components of the* (i,j) *block are meromorphic over* $\mathbf{C}^*$ *with at worst poles over* $[-c;q]$*, of order* $\leq \mu_j - \mu_i$.

One proves in [25] that, in some adequate sense, this F is asymptotic to $\hat{F}_A$. We write it $S_{\overline{c}}\hat{F}_A$ and we consider it as a *summation of* $S_{\overline{c}}\hat{F}_A$ *in the "direction"* $\overline{c} \in \mathbf{E}_q$. Thus, elements of Σ_{A_0} are *prohibited* directions of summation. The Stokes operators are then defined as:

$$S_{\overline{c},\overline{d}}\hat{F}_A := \left(S_{\overline{c}}\hat{F}_A\right)^{-1} S_{\overline{d}}\hat{F}_A.$$

These are meromorphic automorphisms of A_0, and they are Galois in the following sense: evaluating them at a fixed base point $a \in \mathbf{C}^*$ that is not a pole will yield elements of $\mathfrak{St}(A)$ for the corresponding fiber functor $\hat{\omega}_a^{(0)}$. More precisely [24]:

PROPOSITION 2.6. – *For all* $\overline{c}, \overline{d} \in \mathbf{E}_q \setminus \Sigma_{A_0}$ *such that* $a \notin [-c;q] \cup [-d;q]$ *(so that* a *is not a pole):*

$$S_{\overline{c},\overline{d}}\hat{F}_A(a) \in \mathfrak{St}(A)$$

and these elements, together with their conjugates under the action of $G_{p,1}^{(0)}(A)$*, are Zariski-generators of* $\mathfrak{St}(A)$.

Since $S_{\overline{c},\overline{d}}\hat{F}_A = (S_{\overline{c_0},\overline{c}}\hat{F}_A)^{-1} S_{\overline{c_0},\overline{d}}\hat{F}_A$, we may as well fix $\overline{c_0}$ and consider the family of all $S_{\overline{c_0},\overline{c}}\hat{F}_A(a)$. The question of their relations thus comes next.

2.6. Explicit generators of $\mathfrak{st}$

In order to try to"free" these generators, one goes to the Lie algebra. Fix an arbitrary $\overline{c_0} \in \mathbf{E}_q \setminus \Sigma_{A_0}$. For a given A, the map:

$$\overline{c} \mapsto \log S_{\overline{c_0},\overline{c}} \hat{F}_A(a)$$

is meromorphic on $\mathbf{E}_q$ with poles on Σ_{A_0}, with values in $\mathfrak{st}(A)$. Its residue at $\alpha \in \Sigma_{A_0}$ is written:

$$\Delta_\alpha(A) := \mathrm{Res}_{\beta=\alpha} \log S_{\overline{c_0},\beta} \hat{F}_A(a) \in \mathfrak{st}(A).$$

Residues at points $\alpha \notin \Sigma_{A_0}$ are null, except maybe at the particular point $\overline{a}$, where a encodes the fiber functor; but this one has no intrinsic significance and we shall have no use for it.

Now the above statement may be reinforced as follows. From [23, 24], it follows that the mapping $A \mapsto \Delta_\alpha(A)$ is functorial and tensor compatible in the sense of the Stokes Lie algebra (see Section 2.3) when defined on all operands; by continuity, this remains true without condition:

LEMMA 2.7. – *Each mapping $A \rightsquigarrow \Delta_\alpha(A)$ defines an element Δ_α of $\mathfrak{st}$.*

It was proved in [24, Theorem 3.5] (with slightly different notations) that:

PROPOSITION 2.8. – *The Δ_α, together with their conjugates under the action of $G_{p,1}^{(0)}$, are topological generators of $\mathfrak{st}$.*

According to (6) in 2.3, Δ_α admits a decomposition:

$$\Delta_\alpha = \hat{\bigoplus} \Delta_\alpha^{(\delta)}, \quad \Delta_\alpha^{(\delta)} = \hat{\bigoplus} \Delta_\alpha^{(\delta,\overline{c})}.$$

We see the components $\Delta_\alpha^{(\delta,\overline{c})}$ as q-analogs of alien derivations. From the preceding section, we draw:

THEOREM 2.9. – *The "q-alien derivations" $\Delta_\alpha^{(\delta,\overline{c})}$ together with ν generate topologically the Lie algebra $\tilde{\mathfrak{st}}$.*

REMARK 2.10. – It was conjectured at the end of [24] that those "q-alien derivations" $\Delta_\alpha^{(\delta,\overline{c})}$ such that $\alpha^\delta = \overline{c}$ (remember we use a multiplicative notation for the group $\mathbf{E}_q$), together with their conjugates under the action of $G_{p,1}^{(0)}$, are topological generators of $\mathfrak{st}$. This will be proved in Section 3.3. Therefore, those $\Delta_\alpha^{(\delta,\overline{c})}$ such that $\alpha^\delta = \overline{c}$ together with ν generate topologically the Lie algebra $\tilde{\mathfrak{st}}$. The condition on $\alpha, \delta, \overline{c}$ can be interpreted in terms of "directions of maximal growth" as in the theory of differential equations.

From considerations related to the classification theory (see Section 3.1), one can predict that these generators are not free: there should be δ of them for each pair $\delta, \overline{c}$, but there are δ^2. In this respect, the "freeness theorem" of [24] is quite incomplete. We shall here complete it by Theorem 3.8 at the end of 3.5.

2.7. q-Gevrey interpolation

Here, we use [24, §3.3.3]. For each level $\delta \in \mathbf{N} \cup \{\infty\}$, we define a category $\mathscr{E}^\delta$ with the same objects as $\mathscr{E}_1^{(0)}$ but morphisms having coefficients in the field of q-Gevrey series of level $> \delta$ (see definition at the beginning of 2.5). For $\delta = \infty$, the morphisms are analytic and $\mathscr{E}^\infty = \mathscr{E}_1^{(0)}$. For $\delta = 0$, any $\hat{F}_A$ is a morphism, so that any A is equivalent to A_0 and $\mathscr{E}^0 = \mathscr{E}_{p,1}^{(0)}$. In between, the interpolating categories $\mathscr{E}^\delta$ are related by essentially surjective and (not fully) faithful $\otimes$-compatible inclusion functors $\mathscr{E}^\delta \hookrightarrow \mathscr{E}^{\delta-1}$, whence the following diagram:

$$\begin{tikzcd}
\mathscr{E}_1^{(0)} \arrow[rrrr, "\mathrm{gr}"] \arrow[d, equal] & & & & \mathscr{E}_{p,1}^{(0)} \arrow[d, equal] \\
\mathscr{E}^\infty \arrow[r] \arrow[drr, "\hat{\omega}_a^{(0)}"'] & \cdots \arrow[r] & \mathscr{E}^\delta \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathscr{E}^0 \arrow[dll, "\omega_a^{(0)}"] \\
& & \mathrm{Vect}_{\mathbf{C}}^f. & &
\end{tikzcd}$$

Each $\mathscr{E}^\delta$ is Tannakian, with the same fiber functors as $\mathscr{E}_1^{(0)}$, and its Galois group is a closed subgroup of $G_1^{(0)}$ (its elements are $\otimes$-automorphisms of the fiber functor with more constraints imposed by functoriality since there are more morphisms; this is a particular case of [8, prop. 2.21 (b), p. 139]). Actually:

$$\mathrm{Gal}(\mathscr{E}^\delta) = \mathfrak{St}^{\leq \delta} \rtimes G_{p,1}^{(0)},$$

where $\mathfrak{St}^{\leq \delta}$ is the subgroup of $\mathfrak{St}$ with Lie algebra[17]:

$$Lie(\mathfrak{St}^{\leq \delta}) = \mathfrak{st}^{\leq \delta} := \sum_{\delta' \leq \delta} \mathfrak{st}^{(\delta')}.$$

Thus, $\mathfrak{st}^{\leq \delta}$ contains in particular all the $\Delta_\alpha^{(\delta', \overline{c})}$ for $\delta' \leq \delta$.

We now define:

$$\tilde{\mathfrak{st}}^{\leq \delta} := \sum_{\delta' \leq \delta} \tilde{\mathfrak{st}}^{(\delta')} = \mathbf{C}\nu \oplus \mathfrak{st}^{\leq \delta}.$$

Then, from what was said before and the grading, one draws:

Proposition 2.11. – *The Lie algebra $\tilde{\mathfrak{st}}^{\leq \delta}$ is generated by ν and the $\Delta_\alpha^{(\delta', \overline{c})}$ such that $\alpha^{\delta'} = \overline{c}$ for $\delta' \leq \delta$.*

Proof. – From Theorem 2.9, we know that the $\Delta_\alpha^{(\delta', \overline{c})}$ together with ν generate topologically the whole Lie algebra $\tilde{\mathfrak{st}}$. However, those with $\delta' > \delta$ cannot contribute to $\mathfrak{st}^{\leq \delta}$ (the grading being by the ordered monoid $\mathbf{N}$); and there is no need here for topological closure, since the degrees are bounded above, so there are no terms tending to 0. □

(17) This was denoted $\mathfrak{st}(\delta)$ in [24].

3. Structure of the Stokes component

In this section, we shall describe in detail the structure of $\mathfrak{st}$ and its representations. We first recall some necessary facts about classification.

3.1. Some useful results on local analytic classification

These results come from [25, 32]. Fix a pure module M_0 with matrix A_0 in form (5). The modules formally equivalent to M_0 are those such that $\mathrm{gr}M \approx M_0$. In order to classify them analytically, one rigidifies the situation by introducing "marked pairs" (M, g) made up of an analytic q-difference module M and an isomorphism $g : \mathrm{gr}(M) \to M_0$. We then define two such marked pairs (M, g) and (M', g') to be equivalent if there exists a morphism $f : M \to M'$ such that $g = g' \circ \mathrm{gr}(f)$. By standard commutative algebra, such a morphism f is automatically an isomorphism.

The set of equivalence classes of marked pairs is written $\mathcal{F}(M_0)$ and we see it as the space of *isoformal analytic classes in the formal class of* M_0. The corresponding classification problem was solved in [25] and we shall use it in 3.3 to get an alternative description of $\mathrm{Rep}_{A_0}(\mathfrak{st})$.

We define the sheaf $\Lambda_I(M_0)$ of *meromorphic automorphisms of* M_0 *infinitely tangent to identity* as:

$$\Lambda_I(M_0)(V) := \{F \in \mathfrak{G}_{A_0}(\mathcal{O}(\pi^{-1}(V))) \mid F[A_0] = A_0\}.$$

(V denoting an open subset of $\mathbf{E}_q$.) The reason for the name is that, according to [32, Lemma 2.7 and Section 4.1], for any F satisfying the above condition, $F - I_n$ is flat in the sense of q-Gevrey asymptotics. Then $\Lambda_I(M_0)$ is a sheaf of unipotent groups over $\mathbf{E}_q$, and it is Abelian only in the case that M_0 has one or two slopes; in the former case, it is trivial, in the latter case, it is a vector bundle [32, prop. 4.1].

Now let M in the formal class of M_0, with matrix A in form (5). The family of all the $S_{\overline{c},\overline{d}}\hat{F}_A$ for all $\overline{c}, \overline{d} \in \mathbf{E}_q \setminus \Sigma_{A_0}$ is a cocycle for the above sheaf:

$$(S_{\overline{c},\overline{d}}\hat{F}_A)_{\overline{c},\overline{d}} \in Z^1(\mathfrak{U}_{A_0}, \Lambda_I(M_0)).$$

Here, $\mathfrak{U}_{A_0}$ is the covering of $\mathbf{E}_q$ by the Zariski open sets $\mathbf{E}_q \setminus \{\overline{-c}\}$, $\overline{c} \in \mathbf{E}_q \setminus \Sigma_{A_0}$. The conditions on the poles of summations $S_{\overline{c}}\hat{F}_A$ imply that each $S_{\overline{c},\overline{d}}\hat{F}_A$ has only poles on $[-c; q] \cup [-d; q]$, with multiplicities $\leq \mu_j - \mu_i$ for the coefficients of the block (i, j). We call *privileged* such a cocycle and write $Z^1_{pr}(\mathfrak{U}_{A_0}, \Lambda_I(M_0))$ the space of privileged cocycles.

THEOREM 3.1 ([25, 32]). – *The maps sending A to this cocycle and the latter to its cohomology class induce isomorphisms of pointed sets:*

$$\mathcal{F}(M_0) \to Z^1_{pr}(\mathfrak{U}_{A_0}, \Lambda_I(M_0)) \to H^1(\mathbf{E}_q, \Lambda_I(M_0)).$$

We now describe a q-Gevrey interpolation of this classification. Write ${\mathfrak{G}_{A_0}}^{\geq\delta}$ the subgroup of $\mathfrak{G}_{A_0}$ defined by the vanishing of all blocks (i, j) such that $0 < \mu_j - \mu_i < \delta$. This is a normal subgroup of $\mathfrak{G}_{A_0}$ and each quotient ${\mathfrak{G}_{A_0}}^{\geq\delta}/{\mathfrak{G}_{A_0}}^{\geq\delta+1}$ is Abelian, indeed isomorphic to ${\mathfrak{g}_{A_0}}^{(\delta)}$, whence an exact sequence:

$$0 \to {\mathfrak{g}_{A_0}}^{(\delta)} \to \mathfrak{G}_{A_0}/{\mathfrak{G}_{A_0}}^{\geq\delta+1} \to \mathfrak{G}_{A_0}/{\mathfrak{G}_{A_0}}^{\geq\delta} \to 1.$$

This is actually a central extension. It induces a central extension of sheaves:

$$0 \to \lambda_I^{(\delta)}(M_0) \to \Lambda_I(M_0)/\Lambda_I^{\geq\delta+1}(M_0) \to \Lambda_I(M_0)/\Lambda_I^{\geq\delta}(M_0) \to 1,$$

where we write $\Lambda_I^{\geq\delta}(M_0)$ the subsheaf of $\Lambda_I(M_0)$ of Sections with values in ${\mathfrak{G}_{A_0}}^{\geq\delta}$. The sheaf $\lambda_I^{(\delta)}(M_0) := \Lambda_I^{\geq\delta}(M_0)/\Lambda_I^{\geq\delta+1}(M_0)$ is a sheaf of Abelian groups, actually a vector bundle over $\mathbf{E}_q$, corresponding by the construction at the beginning of Section 2 to a q-difference module that is pure isoclinic of slope δ: it is indeed the direct sum of the equations $\sigma_q f = (z^{\mu_i} A_i) f (z^{\mu_j} A_j)^{-1}$ for $\mu_j - \mu_i = \delta$. Now, using some non-Abelian cohomology from [11], one gets an exact sequence:

$$0 \to V^{(\delta)} \to \mathcal{F}^{\leq\delta}(M_0) \to \mathcal{F}^{\leq\delta-1}(M_0) \to 1. \tag{7}$$

The meaning of this sequence is the following:

1. The leftmost term $V^{(\delta)} := H^1(\mathbf{E}_q, \lambda_I^{(\delta)}(M_0))$ is a finite dimensional complex vector space (first cohomology of a vector bundle); its dimension is:
$$\dim_{\mathbf{C}} V^{(\delta)} = \delta \sum_{\mu_j - \mu_i = \delta} r_i r_j.$$
2. The group $V^{(\delta)}$ operates freely on the mid term, which is defined as the cohomology pointed set $\mathcal{F}^{\leq\delta}(M_0) := H^1(\mathbf{E}_q, \Lambda_I(M_0)/\Lambda_I^{\geq\delta+1}(M_0))$. (The special point of this pointed set is the class of the trivial cocycle all of whose components are the identity.)
3. The corresponding quotient map is the canonical arrow from $\mathcal{F}^{\leq\delta}(M_0)$ to the cohomology pointed set $\mathcal{F}^{\leq\delta-1}(M_0) := H^1(\mathbf{E}_q, \Lambda_I(M_0)/\Lambda_I^{\geq\delta}(M_0))$.

Thus, the fibers[18] of $\mathcal{F}^{\leq\delta}(M_0) \to \mathcal{F}^{\leq\delta-1}(M_0)$ inherit a natural structure of affine space over the vector space $V^{(\delta)}$. Accordingly, for $v \in V^{(\delta)}$, we shall write $\alpha \mapsto v \oplus \alpha$ the translation by v in $\mathcal{F}^{\leq\delta}(M_0)$ (that is, in each of the fibers just mentioned); and for two classes $\alpha, \alpha' \in \mathcal{F}^{\leq\delta}(M_0)$ having the same image in $\mathcal{F}^{\leq\delta-1}(M_0)$, we shall write $\alpha' \ominus \alpha$ the unique element of $V^{(\delta)}$ such that $\alpha' = v \oplus \alpha$.

The interpretation of $\mathcal{F}^{\leq\delta}(M_0)$ in terms of classification rests on the same interpolating categories $\mathcal{E}^\delta$ as in Subsection 2.7. An object of $\mathcal{E}^\delta$ can be identified with a matrix A in $\mathrm{GL}_n(\mathbf{C}(\{z\}))$, with undetermined blocks (i,j) for $\mu_j - \mu_i > \delta$, symbolized here by $\star$:

$$\begin{pmatrix} z^{\mu_1} A_1 \ldots\ldots & \star & \star & \star \\ \ldots \quad \ldots\ldots & \ldots & \star & \star \\ \ldots \quad \ldots\ldots & U_{i,j} & \ldots & \star \\ 0 \quad \ldots\ldots & \ldots & \ldots & \ldots \\ \ldots \quad 0 \ldots & \ldots & \ldots & \ldots \\ 0 \quad \ldots \; 0 & \ldots & \ldots & z^{\mu_k} A_k \end{pmatrix}.$$

The highest meaningful block diagonal consists in blocks $U_{i,j}$ with level $\mu_j - \mu_i = \delta$.

We fix a block diagonal matrix A_0 and we classify all matrices A with diagonal A_0, up to q-Gevrey gauge equivalence of level $> \delta$, that is under transforms in $\mathfrak{G}_{A_0}(\mathbf{C}((z)))$ all of whose coefficients are series of q-Gevrey level $> \delta$. This amounts to the same as fixing the pure module M_0 and doing q-Gevrey classification in its formal class. The space of isoformal classes above A_0 in $\mathcal{E}^\delta$ received a cohomological description in [25]: it is $\mathcal{F}^{\leq\delta}(M_0)$. Using

[18] Actually, each $\mathcal{F}^{\leq\delta}(M_0)$ can be endowed with an affine structure over the vector space $\bigoplus_{k\leq\delta} V^{(k)}$, but we shall not need this fact.

the Birkhoff-Guenther normal form (loc. cit.), one can moreover require null blocks (i,j) for $\mu_j - \mu_i > \delta$ and find its dimension as an affine space. We shall write $cl(A)$ the class of the module M_A in $\mathcal{F}^{\leq\delta}(A_0) := \mathcal{F}^{\leq\delta}(M_0)$.

In 3.3, we shall have use for the corresponding computational description of the exact sequence (7). Consider A, A' in $\mathcal{F}^{\leq\delta}(M_0)$ having the same image in $\mathcal{F}^{\leq\delta-1}(M_0)$. Then $\hat{F}_{A,A'} := \hat{F}_{A'}(\hat{F}_A)^{-1}$ lies in $\mathfrak{G}_{A_0}{}^{\geq\delta}(\mathbf{C}((z)))$, as well as its summations:

$$S_{\overline{c}}\hat{F}_{A,A'} := S_{\overline{c}}\hat{F}_{A'}(S_{\overline{c}}\hat{F}_A)^{-1}.$$

We get a cocycle:

$$S_{\overline{c},\overline{d}}\hat{F}_{A,A'} := \left(S_{\overline{c}}\hat{F}_{A,A'}\right)^{-1} S_{\overline{d}}\hat{F}_{A,A'}$$

of $\Lambda_I(M_0)$, in which the blocks for $\mu_j - \mu_i > \delta$ have no meaning and those for $0 < \mu_j - \mu_i < \delta$ vanish; thus, it yields a well defined privileged cocycle of $\lambda_I^{(\delta)}(M_0) := \Lambda_I^{\geq\delta}(M_0)/\Lambda_I^{\geq\delta+1}(M_0)$, whence a class in $V^{(\delta)} := H^1(\mathbf{E}_q, \lambda_I^{(\delta)}(M_0))$. This class is the element $cl(A') \ominus cl(A) \in V^{(\delta)}$ which sends the class of A to the class of A' in $\mathcal{F}^{\leq\delta}(M_0)$.

3.2. Linking representations of $\mathfrak{st}$ to isoformal analytic classes

Let M_0 be an object of $\mathcal{E}_{p,1}^{(0)}$ Its fiber by the functor gr from $\mathcal{E}_1^{(0)}$ to $\mathcal{E}_{p,1}^{(0)}$ can be identified with the category $\mathcal{C}(M_0)$ with objects the pairs (M,u), M an object of $\mathcal{E}_1^{(0)}$ and $u : \mathrm{gr}M \to M_0$ an isomorphism; and with morphisms $(M,u) \to (N,v)$ the morphisms $f : M \to N$ in $\mathcal{E}_1^{(0)}$ such that $u = v \circ \mathrm{gr}f$. Such a morphism is automatically an isomorphism so that $\mathcal{C}(M_0)$ is a groupoid and $\mathcal{F}(M_0)$ is the set $\pi_0(\mathcal{C}(M_0))$ of its connected components. Its cohomological description was explained in 3.1, we now use Tannakian duality to get a representation theoretic description.

To alleviate notations, in this section, we respectively write $\mathcal{C}, \mathcal{C}_0$ for $\mathcal{E}_1^{(0)}, \mathcal{E}_{p,1}^{(0)}$ and G, G_0 for their Galois groups $G_1^{(0)} = \mathrm{Gal}(\mathcal{E}_1^{(0)}), G_{p,1}^{(0)} = \mathrm{Gal}(\mathcal{E}_{p,1}^{(0)})$. We write $\mathrm{Rep}_{\mathbf{C}}(G), \mathrm{Rep}_{\mathbf{C}}(G_0)$ the categories of complex finite dimensional rational representations of these proalgebraic groups. The choice of the fiber functors is here irrelevant, all that we need is the equivalences of category $\mathcal{C}$ with $\mathrm{Rep}_{\mathbf{C}}(G)$ and of category $\mathcal{C}_0$ with $\mathrm{Rep}_{\mathbf{C}}(G_0)$.

We also introduce the auxiliary comma-category $\overline{\mathcal{C}}$ with objects the triples (M, M_0, u) where M, M_0 are objects of $\mathcal{C}, \mathcal{C}_0$ and where $u : \mathrm{gr}M \to M_0$ is an isomorphism; and with morphisms $(M, M_0, u) \to (N, N_0, v)$ the pairs (f, f_0) made up of a morphism $f : M \to N$ and of a morphism $f_0 : M_0 \to N_0$ such that $f_0 \circ u = v \circ \mathrm{gr}f$.

LEMMA 3.2. – *The category $\overline{\mathcal{C}}$ is equivalent to $\mathcal{C}$ and we can identify the fiber $\mathcal{C}(M_0)$ described above with the fiber $\overline{\mathcal{C}}(M_0)$.*

Proof. – Let F be the functor from $\mathcal{C}$ to $\overline{\mathcal{C}}$ defined by $M \rightsquigarrow (M, \mathrm{gr}M, \mathrm{Id}_{\mathrm{gr}M})$ and $f \rightsquigarrow (f, \mathrm{gr}f)$ and let G be the forgetful functor from $\overline{\mathcal{C}}$ to $\mathcal{C}$. Then $G \circ F$ is the identity functor of $\mathcal{C}$ and $F \circ G$ is isomorphic to the identity functor of $\overline{\mathcal{C}}$ by the natural transformation which sends $X = (M, M_0, u)$ to the morphism (Id_M, u) from $F \circ G(X) = (M, \mathrm{gr}M, \mathrm{Id}_{\mathrm{gr}M})$ to X. $\square$

We now carry on this construction to the equivalent categories $\mathrm{Rep}_{\mathbf{C}}(G)$ and $\mathrm{Rep}_{\mathbf{C}}(G_0)$. The setting is the same as in 2.1 with G_0 in place of G_p and we write $\pi := i^* : G \to G_0$. We shall use gr^* to identify G_0 to a (proalgebraic) subgroup of G. The Stokes group $\mathfrak{St} = \mathrm{Ker}\,\pi$ is such that $G = \mathfrak{St} \rtimes G_0$. Thus, G_0 acts upon $\mathfrak{St}$ by inner automorphisms, which we shall denote $s \mapsto s^g := g^{-1}sg$. We also shall denote $D \mapsto D^g$ the corresponding adjoint action on the Lie algebra $\mathfrak{st}$ of $\mathfrak{St}$.

The functor $\mathscr{C}_0 \hookrightarrow \mathscr{C}$ is thereby identified with the following functor from $\mathrm{Rep}_{\mathbf{C}}(G_0)$ to $\mathrm{Rep}_{\mathbf{C}}(G)$:

$$(\rho_0 : G_0 \to \mathrm{GL}(V)) \rightsquigarrow (\rho_0 \circ \pi : G \to \mathrm{GL}(V)),$$
$$(\phi : V \to V', \rho_0 \to \rho_0') \rightsquigarrow (\phi : V \to V', \rho_0 \circ \pi \to \rho_0' \circ \pi).$$

Similarly, the functor $\mathrm{gr} : \mathscr{C} \to \mathscr{C}_0$ is identified with the following functor from $\mathrm{Rep}_{\mathbf{C}}(G)$ to $\mathrm{Rep}_{\mathbf{C}}(G_0)$:

$$(\rho : G \to \mathrm{GL}(V)) \rightsquigarrow \left(\rho_{|G_0} : G_0 \to \mathrm{GL}(V)\right),$$
$$(\phi : V \to V', \rho \to \rho') \rightsquigarrow (\phi : V \to V', \rho_{|G_0} \to \rho'_{|G_0}).$$

Since $(\rho_0 \circ \pi)_{|G_0} = \rho_0$, the composition is the identity of $\mathrm{Rep}_{\mathbf{C}}(G_0)$ as it should. Then one checks that $\overline{\mathscr{C}}$ is identified to the category of triples (ρ, ρ_0, u), where $\rho : G \to \mathrm{GL}(V)$ and $\rho_0 : G_0 \to \mathrm{GL}(V_0)$ are rational representations and where $u : V \to V_0$ is an isomorphism from $\rho_{|G_0}$ to ρ_0, with morphisms from (ρ, ρ_0, u) to (ρ', ρ_0', u') the pairs (ϕ, ϕ') where $\phi : V \to V'$ and $\phi' : v' \to V_0'$ yield morphisms $\rho \to \rho'$ and $\rho_0 \to \rho_0'$ of representations and where moreover $\phi_0 \circ u = u' \circ \phi$. The equivalences of $\mathscr{C}$ and $\overline{\mathscr{C}}$ are easy to explicit.

Last, if M_0 "is" the representation $\rho_0 : G_0 \to \mathrm{GL}(V_0)$, the fiber $\overline{\mathscr{C}}(M_0)$ is identified with the category with objects the pairs (ρ, u) of a rational representation $\rho : G \to \mathrm{GL}(V)$ and a map $u : V \to V_0$ which is an isomorphism from $\rho_{|G_0}$ to ρ_0, with morphisms from (ρ, u) to (ρ', u') the maps $\phi : V \to V'$ which yield morphisms $\rho \to \rho'$ such that $u = u' \circ \phi$.

LEMMA 3.3. – *The fiber $\overline{\mathscr{C}}(M_0)$ can be identified with the set of representations $\rho : G \to \mathrm{GL}(V_0)$ such that $\rho_{|G_0} = \rho_0$.*

Here as in the next proposition, we mean that classes of representations are in bijective correspondance with this set; the bijection being explicit allows for an identification.

Proof. – This set is considered as a category having only identity morphisms. The identification comes from the functor which sends the object (ρ, u) to the representation $\rho_u : g \mapsto u \circ \rho(g) \circ u^{-1}$ and every morphism to the corresponding identity morphism. This is a retraction of the obvious inclusion, and an equivalence of categories. □

Now we return to our more concrete setting, with $G = \mathfrak{St} \rtimes G_0$. If $\rho_0 : G_0 \to \mathrm{GL}(V)$ is fixed, to specify a representation $\rho : G \to \mathrm{GL}(V_0)$ such that $\rho_{|G_0} = \rho_0$, we need only to give its restriction $\overline{\rho}$ to $\mathfrak{St}$, and this is subject to the necessary and sufficient condition:

$$\forall s \in \mathfrak{St}\,,\ \forall g \in G_0\,,\ \overline{\rho}(s^g) = (\overline{\rho}(s))^{\rho_0(g)}.$$

Since $\mathfrak{St}$ is connected and prounipotent, $\overline{\rho}$ is determined by the corresponding representation of the Lie algebra $\mathfrak{st}$. In the end, we have proved:

PROPOSITION 3.4. – *The fiber $\overline{\mathscr{C}}(M_0)$ can be identified with the set:*

$$\{\rho : \mathfrak{st} \to \mathrm{gl}(V_0) \mid \forall D \in \mathfrak{st}\ ,\ \forall g \in G_0\ ,\ \rho(D^g) = (\rho(D))^{\rho_0(g)}\}.$$

As explained in the introduction, this description of the set $\mathscr{F}(M_0)$ of isoformal analytic classes in terms of representations will allow to transport to representations of the Galois group the q-Gevrey interpolation obtained in [25]: this is the crucial (and deepest) step of our construction here.

3.3. Linking representations of $\mathfrak{st}$ with $H^1(\mathbf{E}_q, \lambda_I(M_0))$

The bijection of $H^1(\mathbf{E}_q, \lambda_I(M_0))$ with $\mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}})$ resulting from the two descriptions of $\mathscr{F}(M_0)$ (see Sections 2.4, 3.1 and 3.2) is obtained as follows: for any matrix A corresponding to a class in $\mathscr{F}(M_0)$, first compute the privileged cocycle $(S_{\overline{c},\overline{d}}\hat{F}_A) \in Z^1_{pr}(\mathfrak{U}_{A_0}, \Lambda_I(M_0))$. Write temporarily $h(A)$ its class in $H^1(\mathbf{E}_q, \lambda_I(M_0))$. On the other hand, write D_α the residue at $\beta = \alpha$ of the meromorphic function $\beta \mapsto \log S_{\overline{c_0},\beta}\hat{F}_A(z_0) \in \mathfrak{g}_{A_0}$, and $\Delta_\alpha^{(\delta,\overline{c})}(A)$ the components of D_α for $\alpha^\delta = \overline{c}$. Then call $\rho(A) \in \mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}})$ the unique representation which sends the "q-alien derivations" $\Delta_\alpha^{(\delta,\overline{c})}$ to the $\Delta_\alpha^{(\delta,\overline{c})}(A)$ and ν to U (the block-diagonal matrix of unipotent components of A); the unicity of this representation is a direct consequence of Theorem 2.9. The bijection puts in correspondance the class $h(A)$ with the representation $\rho(A)$.

Using $Z^1_{pr}(\mathfrak{U}_{A_0}, (\Lambda_I/\Lambda_I^{\geq \delta+1})(M_0))$, we get correspondingly a bijection of $H^1(\mathbf{E}_q, (\Lambda_I/\Lambda_I^{\geq\delta+1})(M_0))$ with $\mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}}^{\leq\delta})$, elements of this set being defined by the same two conditions used to define $\mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}})$ (see just after Corollary 2.4). This gives a commutative diagram with bijective horizontal arrows and surjective vertical arrows:

$$\begin{tikzcd} H^1(\mathbf{E}_q, \Lambda_I(M_0)) \arrow[r] \arrow[d] & \mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}}) \arrow[d] \\ H^1(\mathbf{E}_q, (\Lambda_I/\Lambda_I^{\geq\delta+1})(M_0)) \arrow[r] & \mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}}^{\leq\delta}). \end{tikzcd}$$

Just with this information, we shall now start to get structural information about $\tilde{\mathfrak{st}}$. Let $c \in \mathbf{C}^*$, $\delta \in \mathbf{N}^*$ and set $A_0 := \begin{pmatrix} 1 & 0 \\ 0 & cz^\delta \end{pmatrix}$. Then:

$$\mathfrak{g}_{A_0} = {\mathfrak{g}_{A_0}}^{(\delta,\overline{c})} = \begin{pmatrix} 0 & \mathbf{C} \\ 0 & 0 \end{pmatrix}.$$

Since $\mathfrak{g}_{A_0}$ is Abelian, $\mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}})$ can be identified with the dual space of $\left(\frac{\tilde{\mathfrak{st}}}{[\tilde{\mathfrak{st}},\tilde{\mathfrak{st}}]}\right)^{(\delta,\overline{c})}$. Now $\mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}})$ is a linear space and its bijection with $H^1(\mathbf{E}_q, \Lambda_I(M_0))$ is a linear isomorphism (it is a consequence of the q-Gevrey devissage in 3.4, but it is obvious in the particular case considered here). Since $H^1(\mathbf{E}_q, \Lambda_I(M_0)) = H^1(\mathbf{E}_q, \mathscr{F}_{cz^\delta})$ has dimension δ [25, 23], we conclude:

$$\dim_{\mathbf{C}} \left(\frac{\tilde{\mathfrak{st}}}{[\tilde{\mathfrak{st}}, \tilde{\mathfrak{st}}]}\right)^{(\delta,\overline{c})} = \delta.$$

Applying Theorem 2.9, we see that (the images of) the $\Delta_\alpha^{(\delta,\overline{c})}$ for arbitrary $\alpha \in \mathbf{E}_q$ generate the complex vector space $\left(\frac{\tilde{\mathfrak{st}}}{[\tilde{\mathfrak{st}},\tilde{\mathfrak{st}}]}\right)^{(\delta,\overline{c})}$. However, when computing the residues in the case of a matrix $A_u := \begin{pmatrix} 1 & u \\ 0 & cz^\delta \end{pmatrix}$, we only find poles at points such that $\alpha^\delta = \overline{c}$: for this calculation, see [23, Section 4.2]. Thus, if all the $\Delta_\alpha^{(\delta,\overline{c})}$ such that $\alpha^\delta = \overline{c}$ vanish on the class of A_u, this class is trivial. By duality, this means that (the images of) those $\Delta_\alpha^{(\delta,\overline{c})}$ such that $\alpha^\delta = \overline{c}$ generate $\left(\frac{\tilde{\mathfrak{st}}}{[\tilde{\mathfrak{st}},\tilde{\mathfrak{st}}]}\right)^{(\delta,\overline{c})}$.

Remark 3.5. – This is in essence the content of [23, Lemma 4.7 and Theorems 4.8, 4.9] (also see [25, prop. 3.11 and Section 6.2.1]) and it represents the basic step in the “Abelian (two slopes) case”, which served as a bootstrap for the general case. In [24] we attempted to obtain the general case by devissage of the filtration. Here we achieve this devissage in the next two subsections.

For each pair $(\delta, \overline{c}) \in \mathbf{N}^* \times \mathbf{E}_q$, we choose δ among the δ^2 points $\alpha \in \mathbf{E}_q$ such that $\alpha^\delta = \overline{c}$ in such a way that the images of the corresponding $\Delta_\alpha^{(\delta,\overline{c})}$ form a basis of $\frac{\tilde{\mathfrak{st}}}{[\tilde{\mathfrak{st}},\tilde{\mathfrak{st}}]}$. We write $\Delta_i^{(\delta,\overline{c})}$, $1 \leq i \leq \delta$, the corresponding q-alien derivations. In accordance with the analogy explained in the introduction, we see them as *“pointed”* and from now on they will be denoted $\dot{\Delta}_i^{(\delta,\overline{c})}$, and $\nu \in \mathfrak{st}^{(0)}$ will be denoted $\dot{\Delta}^{(0)}$.

Proposition 3.6. – *The family of all* $\dot{\Delta}_i^{(\delta,\overline{c})}$ *together with* $\dot{\Delta}^{(0)} := \nu$ *topologically generate* $\tilde{\mathfrak{st}}$.

Proof. – Call S the sub-Lie algebra generated by this family. It is naturally $\mathbf{N}$-graded and one has, for all $\delta \in \mathbf{N}$:

$$\tilde{\mathfrak{st}}^{(\delta)} = S^{(\delta)} + [\tilde{\mathfrak{st}}, \tilde{\mathfrak{st}}]^{(\delta)} = S^{(\delta)} + \sum_{i+j=\delta} [\tilde{\mathfrak{st}}^{(i)}, \tilde{\mathfrak{st}}^{(j)}].$$

We shall prove inductively that $S^{(\delta)} = \tilde{\mathfrak{st}}^{(\delta)}$ for all $\delta \in \mathbf{N}$, which will imply the conclusion. For $\delta = 0$, both sides are equal to $\mathbf{C}\nu$. Assuming it to be true for all degrees $< \delta$, we calculate:

$$\begin{aligned} \tilde{\mathfrak{st}}^{(\delta)} &= S^{(\delta)} + \sum_{i+j=\delta} [\tilde{\mathfrak{st}}^{(i)}, \tilde{\mathfrak{st}}^{(j)}] \\ &= S^{(\delta)} + [\mathbf{C}\nu, \tilde{\mathfrak{st}}^{(\delta)}] + \sum_{\substack{i+j=\delta \\ i,j<\delta}} [\tilde{\mathfrak{st}}^{(i)}, \tilde{\mathfrak{st}}^{(j)}] \\ &= S^{(\delta)} + [\mathbf{C}\nu, \tilde{\mathfrak{st}}^{(\delta)}] + \sum_{\substack{i+j=\delta \\ i,j<\delta}} [S^{(i)}, S^{(j)}] \\ &= S^{(\delta)} + [\mathbf{C}\nu, \tilde{\mathfrak{st}}^{(\delta)}] \end{aligned}$$

since $[S^{(i)}, S^{(j)}] \subset S^{(\delta)}$ when $i + j = \delta$. By Lemma 2.2 at the end of 2.1, this ends the proof. □

We will show in 3.5 that this family is in some sense free.

3.4. q-Gevrey devissage of the space of representations

From the q-Gevrey dévissage of $H^1(\mathbf{E}_q, \Lambda_I(M_0))$ and of $Z^1_{pr}(\mathfrak{U}_{A_0}, \Lambda_I(M_0))$, and from the identifications with $\mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}})$, we get the following commutative diagram of exact sequences (for concision, we do not indicate the dependency on M_0):

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathbf{E}_q, \lambda_I^{(\delta)}) \arrow[r] & H^1(\mathbf{E}_q, (\Lambda_I/\Lambda_I^{\geq \delta+1})) \arrow[r] & H^1(\mathbf{E}_q, (\Lambda_I/\Lambda_I^{\geq \delta})) \arrow[r] & 0 \\
0 \arrow[r] & Z^1_{pr}(\mathfrak{U}_{A_0}, \lambda_I^{(\delta)}) \arrow[r] \arrow[u] \arrow[d] & Z^1_{pr}(\mathfrak{U}_{A_0}, (\Lambda_I/\Lambda_I^{\geq \delta+1})) \arrow[r] \arrow[u] \arrow[d] & Z^1_{pr}(\mathfrak{U}_{A_0}, (\Lambda_I/\Lambda_I^{\geq \delta})) \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & W^{(\delta)} \arrow[r] & \mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}}^{\leq \delta}) \arrow[r] & \mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}}^{\leq \delta-1}) \arrow[r] & 0.
\end{tikzcd}$$

In the last line, being an exact sequence means that $W^{(\delta)}$ is a vector space acting on the middle term with quotient the rightmost term. We shall now describe this space and this action. For this, we recall the description given in [24, prop. 3.3.3] of the fibers of the surjection from $\mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}}^{\leq \delta})$ to $\mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}}^{\leq \delta-1})$ (modulo the change of notation from $\mathfrak{st}(\delta)$ there to $\tilde{\mathfrak{st}}^{\leq \delta}$ here). In loc. cit., the interpolation categories $\mathscr{C}^\delta$ were defined as having the same objects as $\mathscr{E}_1^{(0)}$, but through q-Gevrey conditions on the coefficients of the morphisms (see [24, bottom of p. 320 and Corollary 3.14]): these are actually the same as the categories $\mathscr{E}^\delta$ here introduced in Section 2.7. Let B be an element of $\mathscr{C}^{\delta-1}$ in Birkhoff-Guenther normal form having graded part A_0. Two elements A, A' of $\mathscr{C}^\delta$ lifting B are related by a unique formal gauge transform $\hat{F}_{A,A'} \in \mathfrak{G}_{A_0}(\mathbf{C}((z)))$. This matrix has null blocks for $0 < \mu_j - \mu_i < \delta$; the blocks corresponding to levels $\mu_j - \mu_i > \delta$ are irrelevant; and we call $\hat{f}_{A,A'}$ the part of $\hat{F}_{A,A'}$ corresponding to level $\mu_j - \mu_i = \delta$. The family of all the $S_{\overline{c},\overline{d}}\hat{f}_{A,A'}$ is a cocycle in $Z^1_{pr}(\mathfrak{U}_{A_0}, \lambda_I^{(\delta)})$. Then:

$$\Delta_\alpha^{(\delta)}(A, A') := \Delta_\alpha^{(\delta)}(A') - \Delta_\alpha^{(\delta)}(A) = Res_{\beta=\alpha} S_{\overline{c_0},\beta}\hat{f}_{A,A'}(z_0).$$

Moreover, all families $(\Delta_\alpha^{(\delta)})$ arising this way correspond to a difference $\Delta(A') - \Delta(A)$. We thus define:

$$W^{(\delta)} := \left\{ \left(\mathrm{Res}_{\beta=\alpha} S_{\overline{c_0},\beta} f(z_0)\right)_\alpha^{(\delta,\overline{c})} \mid f \in Z^1_{pr}(\mathfrak{U}_{A_0}, \lambda_I^{(\delta)}) \right\}.$$

If we encode a representation by the family of all the $\dot{\Delta}_i^{(\delta,\overline{c})}(A)$, we see that we do get an action of $W^{(\delta)}$ on $\mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}}^{\leq \delta})$ with quotient $\mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}}^{\leq \delta-1})$.

3.5. Freeing the alien derivations

Definition 3.7. – Let $\dot{\Delta}^{(0)}$ be a symbol corresponding to the element ν introduced in Section 2.4. For each $(\delta, \overline{c}) \in \mathbf{N}^* \times \mathbf{E}_q$ and $i = 1, \ldots, \delta$, let $\dot{\Delta}_i^{(\delta,\overline{c})}$ be a symbol corresponding to the actual alien derivation with the same notation. We call L the free Lie algebra generated by $\dot{\Delta}^{(0)}$ and all the $\dot{\Delta}_i^{(\delta,\overline{c})}$. We graduate it by the semi-group $\{0\} \cup \mathbf{N}^* \times \mathbf{E}_q$ by taking $\deg \dot{\Delta}^{(0)} = 0$ and $\deg \dot{\Delta}_i^{(\delta,\overline{c})} = (\delta, \overline{c})$. We also endow L with the following action of $T_1^{(0)} \times G_{f,1}^{(0)}$:

$$(\dot{\Delta}_i^{(\delta,\overline{c})})^{(t,\gamma)} := t^\delta \gamma(\overline{c}) \dot{\Delta}_i^{(\delta,\overline{c})},$$
$$(\dot{\Delta}^{(0)})^{(t,\gamma)} := \dot{\Delta}^{(0)}.$$

We write $\mathrm{Rep}_{A_0}(L)$ the set of representations from L to $\mathfrak{g}_{A_0}$ compatible with this action (and similarly for all stable sub-Lie algebras of L).

Write $L^{\leq\delta}$ the sub-Lie algebra generated by $\dot{\Delta}^{(0)}$ and all the $\dot{\Delta}_i^{(\delta',\overline{c})}$, $\delta' \leq \delta$; and $L^{>\delta}$ the ideal generated by all the $\dot{\Delta}_i^{(\delta',\overline{c})}$, $\delta' > \delta$. After [4, chap. 2, §2, no 9, prop. 10], $L^{\leq\delta}$ is free with basis the stated system of generators, while $L^{>\delta}$ is free with basis the family of all $(adD_1)\cdots(adD_k)\dot{\Delta}_i^{(\delta'',\overline{c})}$ where the D_i are $\dot{\Delta}_i^{(\delta',\overline{c})}$ with $\delta' \leq \delta$ and where $\delta'' > \delta$. Moreover, $L = L^{\leq\delta} \oplus L^{>\delta}$. Likewise, we have $L^{\leq\delta} = L^{\leq\delta-1} \oplus L^{(\delta)}$, where $L^{(\delta)}$ is the ideal generated by all the $\dot{\Delta}_i^{(\delta,\overline{c})}$ (i and $\overline{c}$ varying). We define likewise $L^{(\delta,\overline{c})}$ as the ideal generated by all the $\dot{\Delta}_i^{(\delta,\overline{c})}$ (i varying).

From the obvious dominant morphisms of Lie algebras $L \to \tilde{\mathfrak{st}}$ and $L^{\leq\delta} \to \tilde{\mathfrak{st}}^{\leq\delta}$, one deduces a commutative diagram with surjective horizontal maps and injective vertical maps:

$$\begin{tikzcd}
\mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}}) \arrow[r] \arrow[d] & \mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}}^{\leq\delta}) \arrow[d] \\
\mathrm{Rep}_{A_0}(L) \arrow[r] & \mathrm{Rep}_{A_0}(L^{\leq\delta}).
\end{tikzcd}$$

On the other hand, we have identifications:

$$\mathrm{Rep}_{A_0}(L) \approx \bigoplus_{(\delta,\overline{c})\in\mathbf{N}^*\times\mathbf{E}_q} \left(\mathfrak{g}_{A_0}{}^{(\delta,\overline{c})}\right)^{\delta}$$

$$\mathrm{Rep}_{A_0}(L^{\leq\delta}) \approx \bigoplus_{\substack{(\delta',\overline{c})\in\mathbf{N}^*\times\mathbf{E}_q \\ 1\leq\delta'\leq\delta}} \left(\mathfrak{g}_{A_0}{}^{(\delta',\overline{c})}\right)^{\delta'}$$

$$\mathrm{Rep}_{A_0}(L^{(\delta)}) \approx \bigoplus_{\overline{c}\in\mathbf{E}_q} \left(\mathfrak{g}_{A_0}{}^{(\delta,\overline{c})}\right)^{\delta}$$

$$\mathrm{Rep}_{A_0}(L^{(\delta,\overline{c})}) \approx \left(\mathfrak{g}_{A_0}{}^{(\delta,\overline{c})}\right)^{\delta}.$$

Indeed, the value of the generator ν is imposed since we consider representations in Rep_{A_0}, i.e., relative to the fixed A_0. Then, we can enrich as follows the previous diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathbf{E}_q, \lambda_I^{(\delta)}) \arrow[r] & H^1(\mathbf{E}_q, (\Lambda_I/\Lambda_I^{\geq\delta+1})) \arrow[r] & H^1(\mathbf{E}_q, (\Lambda_I/\Lambda_I^{\geq\delta})) \arrow[r] & 0 \\
0 \arrow[r] & Z^1_{pr}(\mathfrak{U}_{A_0}, \lambda_I^{(\delta)}) \arrow[r] \arrow[u] \arrow[d] & Z^1_{pr}(\mathfrak{U}_{A_0}, (\Lambda_I/\Lambda_I^{\geq\delta+1})) \arrow[r] \arrow[u] \arrow[d] & Z^1_{pr}(\mathfrak{U}_{A_0}, (\Lambda_I/\Lambda_I^{\geq\delta})) \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & W^{(\delta)} \arrow[r] \arrow[d] & \mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}}^{\leq\delta}) \arrow[r] \arrow[d] & \mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}}^{\leq\delta-1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Rep}_{A_0}(L^{(\delta)}) \arrow[r] & \mathrm{Rep}_{A_0}(L^{\leq\delta}) \arrow[r] & \mathrm{Rep}_{A_0}(L^{\leq\delta-1}) \arrow[r] & 0.
\end{tikzcd}$$

The new vertical arrows are a priori injections.

THEOREM 3.8 (Freeness theorem). – *The map* $\mathrm{Rep}_{A_0}(\tilde{\mathfrak{st}}) \to \mathrm{Rep}_{A_0}(L)$ *is bijective.*

Proof. – By induction, using the last two lines of the above diagram, it is enough to show that the leftmost vertical arrow is bijective. But it is linear and it sends injectively each $W^{(\delta,\overline{c})}$ to $\mathrm{Rep}_{A_0}(L^{(\delta,\overline{c})})$, which has the same dimension δ. □

3.6. First step in direction of the inverse problem

Recall from Section 2.2 the description of the pure (or formal) Galois group with integral slopes:

$$G_{p,1}^{(0)} = \mathrm{Gal}(\mathscr{E}_{p,1}^{(0)}) = \mathbf{C}^* \times \mathrm{Hom}_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*) \times \mathbf{C}.$$

In Section 2.4, we took off its unipotent component $\mathbf{C}$ and glued it with the Stokes group, so we now introduce its semi-simple component:

$$G_{p,1,s}^{(0)} := \mathbf{C}^* \times \mathrm{Hom}_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*).$$

It acts as follows on the free Lie algebra L: the action on $\mathbf{C}\nu$ is trivial; for each $(\delta, \overline{c}) \in \mathbf{N}^* \times \mathbf{E}_q$, the action on the component $L^{(\delta,\overline{c})}$ is multiplication by $t^\delta \gamma(\overline{c})$.

DEFINITION 3.9. – The *wild fundamental group* of $\mathscr{E}_1^{(0)}$ is the semi-direct product $\pi_{1,q,w,1}^{(0)} := L \rtimes G_{p,1,s}^{(0)}$. A *representation* of the wild fundamental group is the data of a rational linear representation of $G_{p,1,s}^{(0)}$ together with a representation of L, required to be compatible with the corresponding adjoint actions.

If L was nilpotent, it would be equivalent to consider the semi-direct product of groups $(\exp L) \rtimes G_{p,1,s}^{(0)}$, where $\exp L$ is just L endowed with the Campbell-Hausdorff group law ([6, A7]); and representations of this group with the conditions that they be rational on $G_{p,1,s}^{(0)}$. Here a similar description would be possible using the f-pronilpotent completion $L^\dagger$ studied in the Appendix 3, but we do not need it. At any rate, such representations make up a Tannakian category $\mathrm{Rep}_{\mathbf{C}}(\pi_{1,q,w,1}^{(0)}) = \mathrm{Rep}_{\mathbf{C}}(L \rtimes G_{p,1,s}^{(0)})$. To summarize, we have proved:

THEOREM 3.10. – (i) *The Tannakian categories* $\mathscr{E}_1^{(0)}$ *and* $\mathrm{Rep}_{\mathbf{C}}(G_1^{(0)})$ *are equivalent.*

(ii) *The restriction functor from* $\mathrm{Rep}_{\mathbf{C}}(G_1^{(0)})$ *to* $\mathrm{Rep}_{\mathbf{C}}(\pi_{1,q,w,1}^{(0)})$ *is an isomorphism, i.e., it is fully faithful and bijective on objects.*

(iii) *There is a natural bijection between isomorphism classes of representations of the wild fundamental group* $\pi_{1,q,w,1}^{(0)}$ *of* $\mathscr{E}_1^{(0)}$ *and isomorphism classes of objects of* $\mathscr{E}_1^{(0)}$. *All the Galois groups of such objects are the Zariski-closures of images of such representations.*

Proof. – Actually (i) is just Tannakian duality.

The restriction functor(which is plainly exact and $\otimes$-compatible) is obviously faithful. It is bijective on objects because of Theorem 3.8. To see that it is full, we consider two rational representations ρ, ρ' of $G_1^{(0)}$ in spaces V, V' and their restrictions ρ_0, ρ_0' to $\pi_{1,q,w,1}^{(0)}$. A morphism from ρ_0 to ρ_0' is a linear map $\phi : V \to V'$ that intertwines ρ_0, ρ_0', i.e., $\phi \circ \rho(s, g) = \rho'(s, g) \circ \phi$ when s is restricted to $\exp L$. But then this equality extends to the whole of $G_1^{(0)}$ by Zariski-density (Proposition 3.6) and ϕ is the image of a morphism from ρ to ρ'.

Then (iii) is a consequence of (i) and (ii). □

REMARK 3.11. – Let $\rho : L \rtimes G_{p,1,s}^{(0)} \to \mathrm{GL}(V)$ be a representation of the wild fundamental group in the sense of the above definition. It is easy to check that the restriction $d\rho'' : L \to \mathrm{End}(V)$ is nilpotent and vanishes on every alien derivation but perhaps a finite number. Therefore $d\rho''$ factors by $L^\dagger$, $L^\dagger$ being the f-pronilpotent completion of the free Lie algebra L (cf. the Appendix 3), which is a pronilpotent proalgebraic Lie algebra.

The natural morphism $L \to \tilde{\mathfrak{st}}$ factors into $L \to L^\dagger \to \tilde{\mathfrak{st}}$, the first morphism being injective and dominant (i.e., its image is dense)[19]. We deduce morphisms:

$$L \rtimes G_{p,1,s}^{(0)} \to L^\dagger \rtimes G_{p,1,s}^{(0)} \to \tilde{\mathfrak{st}} \rtimes G_{p,1,s}^{(0)},$$

and then functors:

$$\mathrm{Rep}(\tilde{\mathfrak{st}} \rtimes G_{p,1,s}^{(0)}) \to \mathrm{Rep}(L^\dagger \rtimes G_{p,1,s}^{(0)}) \to \mathrm{Rep}(L \rtimes G_{p,1,s}^{(0)}).$$

It follows from Theorem 3.8 that these are equivalences of categories, therefore:

$$L^\dagger \rtimes G_{p,1,s}^{(0)} \to \tilde{\mathfrak{st}} \rtimes G_{p,1,s}^{(0)}$$

is an isomorphism in the obvious proalgebraic sense (Proposition 3.3) and $L^\dagger \to \tilde{\mathfrak{st}}$ is an isomorphism of pronilpotent proalgebraic Lie algebras.

Finally we get an isomorphism of proalgebraic groups:

$$\exp(L^\dagger) \rtimes G_{p,1,s}^{(0)} \to \exp(\tilde{\mathfrak{st}}) \rtimes G_{p,1,s}^{(0)} = \mathfrak{St} \rtimes G_{p,1}^{(0)} = G_1^{(0)}.$$

This is an "explicit description" of the Tannakian group $G_1^{(0)}$.

4. Structure of the global Galois group

We consider here the Galois theory of equations with matrix in $\mathrm{GL}_n(\mathbf{C}(z))$. We shall not develop the theory in such general terms as we did in the previous sections, but just enough to be able to apply it to the inverse problem.

4.1. The global Fuchsian Galois group

We recall here results from [31], mostly its Subsection 3.2. Unfortunately, some of the results that we need are not completely proven there: details can be found in the thesis "Théorie de Galois des équations aux q-différences fuchsiennes", available at `http://www.math.univ-toulouse.fr/~sauloy/PAPIERS/these.pdf`. We shall slightly adapt the notations of loc. cit. so that they extend more easily to our case of interest in the next section.

Let $\mathcal{E}_f$ be the category with objects the matrices $A \in \mathrm{GL}_n(\mathbf{C}(z))$ which are Fuchsian[20] at 0 and at ∞, and with morphisms $F : A \to B$ the matrices $F \in \mathrm{Mat}_{p,n}(\mathbf{C}(z))$ such that $(\sigma_q F)A = BF$. It is endowed with a natural tensor structure[21] which makes it a neutral Tannakian category. Each object A of $\mathcal{E}_f$ can be written, non canonically:

$$A = M^{(0)}[A^{(0)}] = M^{(\infty)}[A^{(\infty)}],$$

(19) We are going to prove below that the second morphism is an isomorphism.
(20) All definitions and constructions given at 0 can be applied at ∞ by using the coordinate $w := 1/z$.
(21) The conventions used to obtain a matrix (and not a quadritensor) as the result of tensoring two matrices are detailed in loc. cit.

where:

$$A^{(0)}, A^{(\infty)} \in \mathrm{GL}_n(\mathbf{C}) \text{ and } M^{(0)} \in \mathrm{GL}_n(\mathbf{C}(\{z\})), M^{(\infty)} \in \mathrm{GL}_n(\mathbf{C}(\{w\})).$$

For the constant matrix $C \in \mathrm{GL}_n(\mathbf{C})$, one builds a canonical fundamental solution of $\sigma_q X = CX$ in the following way. First, special functions are built from theta functions that satisfy the following elementary equations: $\sigma_q l_q = l_q + 1$; and, for all $c \in \mathbf{C}^*$: $\sigma_q e_c = c e_c$. All these functions are meromorphic over $\mathbf{C}^*$; moreover, we have $e_1 = 1$ and $e_{qc} = z e_c$. Then, from the Jordan decomposition $C = C_s C_u$, where $C_s = P\mathrm{Diag}(c_1, \ldots, c_n)P^{-1}$, one draws $e_{C_u} := C_u^{l_q}$ and $e_{C_s} := P\mathrm{Diag}(e_{c_1}, \ldots, e_{c_n})P^{-1}$. Last, $e_C := e_{C_s} e_{C_u}$. Thus $\sigma_q X = AX$ admits the following non canonical fundamental solutions:

$$\mathcal{X}^{(0)} := M^{(0)} e_{A^{(0)}} \text{ and } \mathcal{X}^{(\infty)} := M^{(\infty)} e_{A^{(\infty)}}.$$

The *Birkhoff connection matrix* is then defined as:

$$P := (\mathcal{X}^{(\infty)})^{-1} \mathcal{X}^{(0)} \in \mathrm{GL}_n(\mathcal{M}(\mathbf{E}_q)).$$

In order to give it a functorial and even Galois meaning, we record two basic facts. First [31, Lemma 1.2.4.1, p. 935], if $F^{(0)}$ is a meromorphic (at 0) morphism from $A^{(0)}$ to $B^{(0)}$, then of course $F^{(0)} e_{A^{(0)}} = e_{B^{(0)}} R^{(0)}$ where $R^{(0)}$ is elliptic. But more is true: from the special form of our solutions, one can deduce that $R^{(0)} \in \mathrm{Mat}_{p,n}(\mathbf{C})$. (Similarly at ∞.) This is used in the context of the following commutative diagram:

$$\begin{tikzcd}
I_n \arrow[r, "e_{A^{(\infty)}}"] \arrow[d, "R^{(\infty)}"] & A^{(\infty)} \arrow[r, "M^{(\infty)}"] \arrow[d, "F^{(\infty)}"] & A \arrow[d, "F"] & A^{(0)} \arrow[l, "M^{(0)}"'] \arrow[d, "F^{(0)}"] & I_n \arrow[l, "e_{A^{(0)}}"'] \arrow[d, "R^{(0)}"] \\
I_p \arrow[r, "e_{B^{(\infty)}}"] & B^{(\infty)} \arrow[r, "N^{(\infty)}"] & B & B^{(0)} \arrow[l, "N^{(0)}"'] & I_p. \arrow[l, "e_{B^{(0)}}"']
\end{tikzcd}$$

One can start from F and complete it outwards, or start from $R^{(0)}$ and $R^{(\infty)}$ and complete it inwards.

As for the tensor properties, the basic fact is that it is impossible to choose the family of functions e_c so that $e_c e_d = e_{cd}$. Thus we are led to introduce the cocycle of elliptic functions $\phi(c, d) := \frac{e_c e_d}{e_{cd}}$ and to extend it to matrices (through their eigenvalues) so as to have the formula:

$$e_{C_1} \otimes e_{C_2} = e_{C_1 \otimes C_2} \Phi(C_1, C_2).$$

Note that for unipotent matrices there is no twisting since $e_1 = 1$.

4.1.0.1. *The tensor category $\mathcal{C}_f$ of connection data.* – Its objects are triples $(A^{(0)}, P, A^{(\infty)})$, where $A^{(0)}, A^{(\infty)} \in \mathrm{GL}_n(\mathbf{C})$ and $P \in \mathrm{GL}_n(\mathcal{M}(\mathbf{E}_q))$. Morphisms from $(A^{(0)}, P, A^{(\infty)})$ to $(B^{(0)}, Q, B^{(\infty)})$ are pairs $(R^{(0)}, R^{(\infty)}) \in \mathrm{Mat}_{p,n}(\mathbf{C})^2$ such that:

$$\begin{aligned}
R^{(\infty)} P &= Q R^{(0)}, \\
F^{(0)} &:= e_{B^{(0)}} R^{(0)} (e_{A^{(0)}})^{-1} && \text{is meromorphic at } 0 \\
F^{(\infty)} &:= e_{B^{(\infty)}} R^{(\infty)} (e_{A^{(\infty)}})^{-1} && \text{is meromorphic at } \infty.
\end{aligned}$$

In loc. cit. an explicit condition is given ensuring these meromorphies, but we shall not need it. (It is used to guarantee that the following constructions do work.)

Now the tensor product has to be twisted in order to get the theorem we need. For morphisms, and for the left and right components of objects, we use the usual tensor product. For the middle component, we shall use the twisted tensor product, defined as follows:

$$(A_1^{(0)}, P_1, A_1^{(\infty)}) \otimes (A_2^{(0)}, P_2, A_2^{(\infty)}) := (A_1^{(0)} \otimes A_2^{(0)}, P_1 \underline{\otimes} P_2, A_1^{(\infty)} \otimes A_2^{(\infty)}),$$

where:

$$P_1 \underline{\otimes} P_2 := \Phi(A_1^{(\infty)}, A_2^{(\infty)})(P_1 \otimes P_2)\big(\Phi(A_1^{(0)}, A_2^{(0)})\big)^{-1}.$$

THEOREM 4.1. – *The tensor categories $\mathcal{E}_f$ and $\mathcal{C}_f$ are equivalent.*

Proof. – Because of the non canonical choice, one does not define a functor from one of these categories to the other. Instead, one defines yet another category $\mathcal{S}_f$ with objects $(A^{(0)}, M^{(0)}, A^{(\infty)}, M^{(\infty)})$ and with morphisms $(R^{(0)}, R^{(\infty)})$, all being subject to adequate conditions. The tensor structure on $\mathcal{S}_f$ is the natural one. Then functors from $\mathcal{S}_f$ to $\mathcal{E}_f$ and $\mathcal{C}_f$ are easily defined. Note that the proof of the essential surjectivity of the second functor is essentially due to Birkhoff (it rests on his theorem of factorisation of analytic matrices). □

4.1.0.2. *The Galois group of $\mathcal{E}_f$ and $\mathcal{C}_f$.* – From the description of $\mathcal{C}_f$, it is clear how to define fiber functors $\omega_f^{(0)}$ and $\omega_f^{(\infty)}$ on it. These extend to the local categories obtained by keeping only the 0 or ∞ component, and by allowing meromorphic morphisms. One thus obtains the local Galois groups $G_f^{(0)}$ and $G_f^{(\infty)}$ that were described in Section 2.2. We want to use P to connect them. More precisely, we should like each value $P(a) \in \mathrm{GL}_n(\mathbf{C})$ to behave like a "connection formula" in Riemann-Hilbert correspondance, and so be a Galois isomorphism from $\omega^{(0)}(A)$ to $\omega^{(\infty)}(A)$. This does not work because the formation of the Birkhoff matrix is not $\otimes$-compatible: that is, $(P_1 \underline{\otimes} P_2)(a) \neq P_1(a) \otimes P_2(a)$. We shall therefore twist P in order to obtain tensor-compatibility and also functoriality. This is done as follows.

One can define explicitly a family of (abstract) group morphisms g_a from $\mathbf{C}^*$ to itself such that[22] $g_a(q) = a$ for all $a \in \mathbf{C}^*$. Then we set $\psi_a(c) := \frac{e_c(a)}{g_a(c)}$ and we extend each function ψ_a to a function Ψ_a on matrices, through their eigenvalues. Last, we define:

$$\check{P}(a) := \big(\Psi_a(A^{(\infty)})\big)^{-1} P(a) \Psi_a(A^{(0)}),$$

and can prove that, for each $a \in \mathbf{C}^*$, one has an isomorphism of fiber functors $(A^{(0)}, P, A^{(\infty)}) \rightsquigarrow \check{P}(a)$ from $\omega_f^{(0)}$ to $\omega_f^{(\infty)}$. Since $\check{P}(a)$ is not defined for all a, this actually applies to a smaller category than $\mathcal{E}_f$, but any given object belongs to "most" of these subcategories.

THEOREM 4.2. – *The group generated by $G_f^{(0)}$, one particular conjugate $\big(\check{P}(a)\big)^{-1} G_f^{(\infty)} \check{P}(a)$ and the set of all defined values $\big(\check{P}(b)\big)^{-1} \check{P}(a)$ is Zariski-dense in the global Galois group of A.*

The proof uses Chevalley criterion: any line in any tensor construction that is fixed by the smaller group is fixed by the bigger one. It rests on the following useful fact: if x is an eigenvector for $G_f^{(0)}$, then it is an eigenvector for $\Psi_a(A^{(0)})$. We shall sketch the proof in our case of interest in the next subsection.

[22] In [31, 3.2.2.2], the stated condition is $g_a(q) = 1$, but it is a typographical error.

4.2. The global Galois group with integral slopes

We now extend the results above to the case of irregular equations with integral slopes. As the extension involves no new idea, our presentation will be concise. The category $\mathcal{E}_1$ of interest has as objects systems with matrix $A \in \mathrm{GL}_n(\mathbf{C}(z))$ such that their slopes at 0 and at ∞ are integral; and as morphisms $A \to B$ matrices $F \in \mathrm{Mat}_{p,n}(\mathbf{C}(z))$ such that $(\sigma_q F)A = BF$. The tensor product is the natural one and makes it a neutral Tannakian category. Each object A of $\mathcal{E}_1$ can be written, non canonically:

$$A = M^{(0)}[A^{(0)}] = M^{(\infty)}[A^{(\infty)}],$$

where $M^{(0)} \in \mathrm{GL}_n(\mathbf{C}(\{z\}))$, $M^{(\infty)} \in \mathrm{GL}_n(\mathbf{C}(\{w\}))$ and $A^{(0)}$, $A^{(\infty)}$ are in Birkhoff-Guenther normal form.

To define solutions, we choose once and for all a function θ such that $\sigma_q\theta = z\theta$ and an arbitrary direction of summation in $\mathbf{E}_q$. Because of this, the following constructions are only valid on a subcategory of $\mathcal{E}_1$, but each particular object of $\mathcal{E}_1$ belongs to "most" of these subcategories. We shall call $A_p^{(0)}$, $A_p^{(\infty)}$ the pure systems associated to $A^{(0)}$, $A^{(\infty)}$ by the gr functor (hence there block-diagonal parts). Let $S^{(0)}$ be the meromorphic isomorphism from $A_p^{(0)}$ to $A^{(0)}$ obtained by summation along the selected direction mentioned above; and similarly at infinity. Then, calling $\mu_1, \dots, \mu_k$ the slopes of $A^{(0)}$ and $r_1, \dots, r_k$ their multiplicities, let $\Gamma^{(0)} := \mathrm{Diag}(\theta^{\mu_1} I_{r_1}, \dots, \theta^{\mu_k} I_{r_k})$. We have $A_p^{(0)} = \Gamma^{(0)}[A_f^{(0)}]$ with $A_f^{(0)} \in \mathrm{GL}_n(\mathbf{C})$. In the end, using the similar notations at infinity, we put:

$$e_{A^{(0)}} := S^{(0)}\Gamma^{(0)} e_{A_f^{(0)}} \text{ and } e_{A^{(\infty)}} := S^{(\infty)}\Gamma^{(\infty)} e_{A_f^{(\infty)}}.$$

Thus $\sigma_q X = AX$ admits the following non canonical fundamental solutions:

$$\mathcal{X}^{(0)} := M^{(0)} e_{A^{(0)}} \text{ and } \mathcal{X}^{(\infty)} := M^{(\infty)} e_{A^{(\infty)}}.$$

The *Birkhoff connection matrix* is then defined as:

$$P := (\mathcal{X}^{(\infty)})^{-1}\mathcal{X}^{(0)} \in \mathrm{GL}_n(\mathcal{M}(\mathbf{E}_q)).$$

Its tensor behaviour is exactly similar to that observed in the Fuchsian case and we shall set, in appropriate context:

$$(A_1^{(0)}, P_1, A_1^{(\infty)}) \otimes (A_2^{(0)}, P_2, A_2^{(\infty)}) := (A_1^{(0)} \otimes A_2^{(0)}, P_1 \underline{\otimes} P_2, A_1^{(\infty)} \otimes A_2^{(\infty)}),$$

where:

$$P_1 \underline{\otimes} P_2 := \Phi((A_1)_f^{(\infty)}, (A_2)_f^{(\infty)})(P_1 \otimes P_2)\big(\Phi((A_1)_f^{(0)}, (A_2)_f^{(0)})\big)^{-1}.$$

The functorial behaviour requires some more comments. Let B be an object of rank p in $\mathcal{E}_1$ and $B^{(0)}$, $N^{(0)}$, $B_p^{(0)}$, $T^{(0)}$, $B_f^{(0)}$, $\Delta^{(0)}$, $\mathcal{Y}^{(0)}$, $\mathcal{Y}^{(\infty)}$, and Q the associated data corresponding respectively to $A^{(0)}$, $M^{(0)}$, $A_p^{(0)}$, $S^{(0)}$, $A_f^{(0)}$, $\Gamma^{(0)}$, $\mathcal{X}^{(0)}$, $\mathcal{X}^{(\infty)}$ and P. Let F be a morphism from A to B. Then we have a commutative diagram:

$$\begin{tikzcd}
I_n \arrow[r, "e_{A_f^{(\infty)}}"] \arrow[d, "R^{(\infty)}"] & A_f^{(\infty)} \arrow[r, "\Gamma^{(\infty)}"] \arrow[d, "F_f^{(\infty)}"] & A_p^{(\infty)} \arrow[r, "S^{(\infty)}"] \arrow[d, "F_p^{(\infty)}"] & A^{(\infty)} \arrow[r, "M^{(\infty)}"] \arrow[d, "F^{(\infty)}"] & A \arrow[d, "F"] & A^{(0)} \arrow[l, "M^{(0)}"'] \arrow[d, "F^{(0)}"] & A_p^{(0)} \arrow[l, "S^{(0)}"'] \arrow[d, "F_p^{(0)}"] & A_f^{(0)} \arrow[l, "\Gamma^{(0)}"'] \arrow[d, "F_f^{(0)}"] & I_n \arrow[l, "e_{A_f^{(0)}}"'] \arrow[d, "R^{(0)}"] \\
I_p \arrow[r, "e_{B_f^{(\infty)}}"'] & B_f^{(\infty)} \arrow[r, "\Delta^{(\infty)}"'] & B_p^{(\infty)} \arrow[r, "T^{(\infty)}"'] & B^{(\infty)} \arrow[r, "N^{(\infty)}"'] & B & B^{(0)} \arrow[l, "N^{(0)}"] & B_p^{(0)} \arrow[l, "T^{(0)}"] & B_f^{(0)} \arrow[l, "\Delta^{(0)}"] & I_p. \arrow[l, "e_{B_f^{(0)}}"]
\end{tikzcd}$$

Of course, all vertical arrows can be defined from F. For instance,

$$F^{(0)} := (N^{(0)})^{-1} \circ F \circ M^{(0)} \in \mathrm{Mat}_{p,n}(\mathbf{C}(\{z\}))$$

is a morphism in $\mathscr{E}_1^{(0)}$, and similarly at ∞. Then one can see that

$$F_p^{(0)} := (T^{(0)})^{-1} \circ F^{(0)} \circ S^{(0)}$$

is actually $\mathrm{gr} F^{(0)}$ (and similarly at ∞); and, from the block-diagonal structures of the involved matrices, one can see that $F_f^{(0)} := (\Delta^{(0)})^{-1} \circ F_p^{(0)} \circ \Gamma^{(0)}$ is actually equal to $F_p^{(0)} = \mathrm{gr} F^{(0)}$, the block-diagonal of $F^{(0)}$. Then, from the lemma already quoted [31, Lemma 1.2.4.1, p. 935], we see that $R^{(0)} := (e_{B_f^{(0)}})^{-1} \circ F_f^{(0)} \circ e_{A_f^{(0)}} \in \mathrm{Mat}_{p,n}(\mathbf{C})$ and similarly at ∞.

Conversely, if we are given the two lines and the most external vertical arrows $R^{(0)}$, $R^{(\infty)}$, the condition to be able to go inwards and fill in the other vertical arrows to get a commutative diagram is that $QR^{(0)} = R^{(\infty)}P$. The condition to get a rational F is that $F^{(0)} \in \mathrm{Mat}_{p,n}(\mathbf{C}(\{z\}))$ and similarly at ∞. Indeed, from the functional equation $\sigma_q F^{(0)} = B^{(0)} F^{(0)} (A^{(0)})^{-1}$ and the fact that $A^{(0)}, B^{(0)}$ are in Birkhoff-Guenther normal form, one deduces that $F^{(0)}$ is meromorphic on $\mathbf{C}$, and similarly at ∞, so that F is actually meromorphic on the Riemann sphere, thus rational.

4.2.0.3. *The tensor category $\mathscr{C}_1$ of connection data.* – Its objects are triples $(A^{(0)}, P, A^{(\infty)})$, where $A^{(0)}, A^{(\infty)} \in \mathrm{GL}_n(\mathbf{C}(\{z\}))$ are in Birkhoff-Guenther normal form and $P \in \mathrm{GL}_n(\mathscr{M}(\mathbf{E}_q))$. Morphisms from $(A^{(0)}, P, A^{(\infty)})$ to $(B^{(0)}, Q, B^{(\infty)})$ are pairs $(R^{(0)}, R^{(\infty)}) \in \mathrm{Mat}_{p,n}(\mathbf{C})^2$ such that:

$$R^{(\infty)} P = Q R^{(0)},$$
$$F^{(0)} := e_{B^{(0)}} R^{(0)} (e_{A^{(0)}})^{-1} \text{ is meromorphic at } 0$$
$$F^{(\infty)} := e_{B^{(\infty)}} R^{(\infty)} (e_{A^{(\infty)}})^{-1} \text{ is meromorphic at } \infty.$$

REMARK 4.3. – We saw in 4.1.0.1 that there was an explicit condition (although we did not state it) on $R^{(0)}$ for $F_p^{(0)} = F_f^{(0)}$ to be meromorphic at 0. Here, we must add a new condition to ensure that $F^{(0)}$ is also meromorphic at 0. This condition is obviously related to the summations $S^{(0)}$ and $T^{(0)}$. We have not so far an explicit criterion, but it could be related to the way $F_p^{(0)}$ links the classifying cohomology class in $H^1(\mathbf{E}_q, \mathscr{F}_{A_p^{(0)}})$ corresponding to $A^{(0)}$ to the classifying cohomology class in $H^1(\mathbf{E}_q, \mathscr{F}_{B_p^{(0)}})$ corresponding to $B^{(0)}$.

The tensor structure is defined as follows. For morphisms, and for the left and right components of objects, we use the usual tensor product. For the middle component, we shall use the twisted tensor product, defined as follows:

$$(A_1^{(0)}, P_1, A_1^{(\infty)}) \otimes (A_2^{(0)}, P_2, A_2^{(\infty)}) := (A_1^{(0)} \otimes A_2^{(0)}, P_1 \underline{\otimes} P_2, A_1^{(\infty)} \otimes A_2^{(\infty)}),$$

where:

$$P_1 \underline{\otimes} P_2 := \Phi(A_1^{(\infty)}, A_2^{(\infty)})(P_1 \otimes P_2)\big(\Phi(A_1^{(0)}, A_2^{(0)})\big)^{-1}.$$

Recall that we have extended the definition of Φ to this setting.

THEOREM 4.4. – *The tensor categories $\mathscr{E}_1$ and $\mathscr{C}_1$ are equivalent.*

Proof. – The method and the proof are the same as in the Fuchsian case: we use an enriched category $\mathcal{S}_1$ with objects $(A^{(0)}, M^{(0)}, A^{(\infty)}, M^{(\infty)})$ and with morphisms $(R^{(0)}, R^{(\infty)})$, all being subject to obvious conditions. The tensor structure on $\mathcal{S}_1$ is the natural one. Then functors from $\mathcal{S}_1$ to $\mathcal{E}_1$ and $\mathcal{C}_1$ are defined and proved to be $\otimes$-equivalences exactly as in the Fuchsian case. □

4.2.0.4. *The Galois group of $\mathcal{E}_1$ and $\mathcal{C}_1$*. – From the description of $\mathcal{C}_1$, it is clear how to define fiber functors $\omega_1^{(0)}$ and $\omega_1^{(\infty)}$ on it and that their extension to the local categories $\mathcal{E}_1^{(0)}$ and $\mathcal{E}_1^{(\infty)}$ yields local Galois groups which are precisely the Galois group $G_1^{(0)}$ studied in this paper and its counterpart $G_1^{(\infty)}$ at ∞.

Also the formula:

$$\check{P}(a) := \big(\Psi_a(A^{(\infty)})\big)^{-1} P(a)\Psi_a(A^{(0)})$$

extends here with the only adaptation that $\Psi_a(A^{(0)})$ means $\Psi_a(A_f^{(0)})$, and similarly at ∞. Again, one finds that, for each $a \in \mathbf{C}^*$, one has an isomorphism of fiber functors $(A^{(0)}, P, A^{(\infty)}) \rightsquigarrow \check{P}(a)$ from $\omega_1^{(0)}$ to $\omega_1^{(\infty)}$ (again, on appropriate subcategories).

Theorem 4.5. – *The group generated by $G_1^{(0)}$, one particular conjugate $\big(\check{P}(a)\big)^{-1} G_1^{(\infty)} \check{P}(a)$ and the set of all defined values $\big(\check{P}(b)\big)^{-1}\check{P}(a)$ is Zariski-dense in the global Galois group of A.*

Proof. – The proof uses again Chevalley criterion in a similar way to loc. cit. Suppose we have two lines $D^{(0)}$ and $D^{(\infty)}$ that are respectively fixed by $G_1^{(0)}$ and $G_1^{(\infty)}$ and such that each $\check{P}(a)$ sends $D^{(0)}$ to $D^{(\infty)}$. Taking generators $x^{(0)}, x^{(\infty)}$, we see by Tannakian duality that they define rank one subobjects $x^{(0)} : a^{(0)} \to A^{(0)}$ and $x^{(\infty)} : a^{(\infty)} \to A^{(\infty)}$. By the lemma quoted at the end of the previous subsection, the fact that $x^{(0)}, x^{(\infty)}$ are respectively eigenvectors of $G_1^{(0)}, G_1^{(\infty)}$ implies that the value $P(a)$ of the non-twisted connection matrix sends $D^{(0)}$ to $D^{(\infty)}$, so that $P(a)x^{(0)} = p(a)x^{(\infty)}$ for some $p(a) \in \mathbf{C}$. But then p is a non-trivial elliptic function, $(a^{(0)}, p, a^{(\infty)})$ is a rank one object of $\mathcal{C}_1$ and $(x^{(0)}, x^{(\infty)})$ an embedding of this object as a subobject of $(A^{(0)}, P, A^{(\infty)})$. Then, by functoriality, all elements of the global Galois group must fix this subobject, whence the two lines. □

Corollary 4.6. – *Topological generators of the Stokes Lie algebra at* 0 *and* ∞ *together with topological generators of the local pure Galois groups and the values of $\check{P}(a)$ are together topological generators of the global Galois group.*

5. The inverse problem

5.1. Known results

To our knowledge there existed before almost no result on the *local* inverse problem that we shall solve below (for the integral slope case). We will review the known results on the *global* inverse problem.

As far as we know, the first significant result on the global inverse problem of the q-difference Galois theory is due to P. Etingof [10] (Proposition 3.4, page 7). We recall that the system $\sigma_q Y = AY$ is said to be *regular* if $A(0) = A(\infty) = I_n$.

PROPOSITION 5.1. – *Let G be any connected complex linear algebraic group, there exists $\delta > 0$ (depending on G) such that, for all $0 < |q| < \delta$, there exists a rational* regular *difference system $\sigma_q Y = AY$ whose q-difference Galois group is G.*

The proof of Proposition 5.1 is related to the following result (cf. [35]).

LEMMA 5.2. – *Let G be any complex linear algebraic group, then there exists $g_1, \dots, g_m \in G$ such that the subgroup generated by $g_1, \dots, g_m$ is Zariski-dense in G.*

We recall that the Tretkoffs used this lemma (and the Riemann-Hilbert correspondence) to solve the inverse problem of the Galois differential theory with regular singular systems. Actually, the condition that q is small enough can be relaxed, according to the following argument, which was shown to us by Julien Roques: the Galois group of a regular system does not change by ramification, as follows easily from the Theorem of Etingof that the values of $P(a)^{-1}P(b)$ generate it (P being the connection matrix of Birkhoff)[23].

However, the proof of Etingof is extremely sketchy and we have not been able to fill in the details. Indeed, it relies on two arguments, the first (presented as obvious) yielding an even stronger statement than the above result of the Tretkoffs. So we prefer not to rely on Etingof's result. Moreover, our method here seems more "economical" in creating singularities.

In [21, Corollary 12.17], van der Put and Singer give the following sufficient condition for $G \subset \mathrm{GL}_n(\mathbf{C})$ to be the Galois group of a q-difference equation over $\mathbf{C}(z)$: G contains a finite commutative subgroup Z lying in the connected component of the normalizer of G^0 in $\mathrm{GL}_n(\mathbf{C})$, and moreover mapping surjectively onto G/G^0.

If G is *Abelian*, it is possible to improve the Proposition 5.1 in the following way.

PROPOSITION 5.3. – *Let G be any* Abelian *connected complex linear algebraic group, then,* for all $q \in \mathbf{C}^*$, $|q| \neq 1$, *there exists a rational* regular *difference system $\sigma_q Y = AY$ whose q-difference Galois group is G.*

This proposition follows from [31], using the following lemma.

LEMMA 5.4. – *Let G be any* Abelian *connected complex linear algebraic group, then there exists a rational* dominant *map $f : \mathbf{E}_q \to G$.*

Here $\mathbf{E}_q$ is seen as a projective algebraic curve (an elliptic curve). The proof of this lemma follows from the existence of an isomorphism $G \approx (G_m)^k \times (G_u)^l$.

From Propositions 5.1 and 5.3 one could conjecture that for every connected complex linear algebraic group G and for all $q \in \mathbf{C}^*$, $|q| \neq 1$, there exists a rational *regular* difference system $\sigma_q Y = AY$ whose q-difference Galois group is G.

Such a system will have in general "a lot of singularities". Below we will attack the global inverse problem in the opposite direction, searching a system with a *minimal* number of singularities in the spirit of a q-analog of the Abhyankar conjecture.

Another source of solutions of the *inverse* problem are of course the known solutions of the *direct* problem, in particular from the computation of the q-difference Galois groups of the *generalized q-hypergeometric equations* (regular singular or not). One can find a complete

[23] If one uses instead the stronger density Theorem 4.2, one sees more generally that the connected component does not change by ramification (for a Fuchsian system).

solution of this last problem in a series of papers of J. Roques [26, 27, 28]. Limiting ourselves to the cases of *simple* groups, the complete list obtained by J. Roques is: $\mathrm{SL}(n, \mathbf{C})$, $\mathrm{SO}(n, \mathbf{C})$, $\mathrm{Sp}(2n, \mathbf{C})$.

It is interesting to compare with the differential case (cf. [2], [9], [13], [18]). The simple groups which are differential groups of *generalized hypergeometric differential equations* (regular singular or not) are: $\mathrm{SL}(n, \mathbf{C})$, $\mathrm{SO}(n, \mathbf{C})$, $\mathrm{Sp}(2n, \mathbf{C})$ and ... the group G_2 ! Therefore the *only* difference between the q-difference case and the differential case is the exceptional group G_2.

5.2. Linear algebraic groups: reminders and complements

5.2.1. *Notations and definitions. Levi decomposition.* – In the following *all* the algebraic groups are *complex linear* algebraic groups. In general G is a linear algebraic group, $\mathfrak{g}$ is its Lie algebra, $T \subset G$ is a torus, $\mathfrak{t}$ the Lie algebra of T, and $D \subset G$ is an Abelian semi-simple group.

An algebraic group G contains a unique *maximal normal solvable subgroup*, this subgroup is closed. Its identity component is called the *radical* $R(G)$ of G.

We will denote $R_u(G)$ the *unipotent radical* of G (i.e., the set of unipotent elements of $R(G)$). A group G is *reductive* if and only if $R_u(G) = \{e\}$.

Definition 5.5. – A *Levi subgroup* of a linear algebraic group G is a maximal reductive subgroup.

We have an exact sequence:

$$\{e\} \to R_u(G) \to G \to G/R_u(G) \to \{e\}$$

and, if $H \subset G$ is a Levi subgroup, then the quotient map $G \to G/R_u(G)$ induces an isomorphism $H \to G/R_u(G)$. More precisely we have the following result (essentially due to Mostow).

Proposition 5.6. – *Let G be a linear algebraic group.*

(i) *If $H \subset G$ is a Levi subgroup, then G is a semi-direct product: $G = R_u(G) \rtimes H$.*
(ii) *Any two Levi subgroups of G are conjugate under an inner automorphism.*
(iii) *If $H \subset G$ is a subgroup and if the quotient map $G \to G/R_u(G)$ induces an isomorphism $H \to G/R_u(G)$, then H is a Levi subgroup. (As noted before the proposition, the converse is true.)*

Proof. – For (i) and (ii), cf. [19] (a subgroup is fully reducible if and only if it is reductive).

Let $H \subset G$ be a subgroup such that the quotient map $G \to G/R_u(G)$ induces an isomorphism $H \to G/R_u(G)$, H is reductive, therefore it is contained in a maximal reductive subgroup H' and $H = H'$. □

Definition 5.7. – A *Levi decomposition* of a linear algebraic group G is an isomorphism $G \approx U \rtimes S$, where S is *reductive* and U is *unipotent*.

5.2.2. *Diagonalisable and triangularizable groups*. – We shall recall the notions of *diagonalisable* and of *triangularizable* algebraic group. The properties of the diagonalisable groups and of the triangularizable *connected* groups are well known, but for the triangularizable *non connected* groups we do not know good references, hence, for sake of completeness, we shall detail the necessary results.

An algebraic group G is *diagonalisable* if and only if it is Abelian and semi-simple ($G = G_s$). If G is diagonalisable, then every representation of G is diagonalisable in the matrix sense. An algebraic group G is diagonalisable if and only if there exists a faithful representation of G which is diagonalisable in the matrix sense.

We will say that a linear algebraic group G is *triangularizable* if there exists a faithful triangular representation. A triangularizable group is *solvable*.

A solvable *connected* linear algebraic group is triangularizable (Lie-Kolchin theorem). In particular a unipotent group is triangularizable.

PROPOSITION 5.8. – *A linear algebraic group G is triangularizable if and only if $G \approx U \rtimes D$, where U is* unipotent *and D is Abelian and semi-simple. Then $U = R_u(G)$ and $D \approx G/R_u(G)$.*

The Levi subgroups of a triangularizable algebraic group G are the maximal Abelian semi-simple subgroups. If G is connected, the Levi subgroups are the maximal tori.

Proof. – If $G \approx U \rtimes D$, where U is *unipotent* and D is Abelian and semi-simple, G is triangularizable by [14, I.7, lemma, p. 20].

We suppose that G is triangularizable, there exists a faithful representation $\rho : G \to \mathrm{GL}_n(\mathbf{C})$, such that $\rho(G)$ is an upper triangular subgroup of $\mathrm{GL}_n(\mathbf{C})$, a subgroup of the upper triangular subgroup T_n. We denote U_n (resp. D_n) the unipotent upper-triangular subgroup (resp. the diagonal subgroup) of $\mathrm{GL}_n(\mathbf{C})$, then $T_n = U_n \rtimes D_n$, $T_n/U_n = D_n$.

There exists a Levi decomposition $G = U \rtimes D$, where U is *unipotent* and D reductive. Then $\rho(U)$ is unipotent, therefore it is a subgroup of U_n and ρ induces an injective morphism $D = G/U \to T_n/U_n = D_n$. The group D_n is Abelian semi-simple, and D is isomorphic to a subgroup, therefore D is Abelian semi-simple.

The Levi subgroups are Abelian semi-simple and any Abelian semi-simple subgroup is reductive, the result follows. □

5.3. Θ-structures on linear algebraic groups

5.3.1. *Weights and coweights*

DEFINITION 5.9. – Let G be an Abelian semi-simple group. The *weight group* $G^\bullet$ of G is the group of homomorphisms of algebraic groups $G \to \mathbf{C}^*$.

A weight on G is usually called a *character* on G, but we shall use the words "weight" and "coweight" to emphasize the relation with the infinitesimal point of view.

The group $G^\bullet$ is an Abelian finitely generated group.

The weight functor $G \rightsquigarrow G^\bullet$ is an antiequivalence of categories between Abelian semi-simple algebraic groups and finitely generated Abelian groups. The quasi inverse of the weight functor is $\mathrm{Hom}_{gr}(., \mathbf{C}^*)$.

DEFINITION 5.10. – Let D be an Abelian semi-simple group. A coweight on D is a homomorphism of algebraic groups $\mathbf{C}^* \to D$.

A coweight on D is also called a *one parameter subgroup* of D.

If $\chi : \mathbf{C}^* \to D$ is a coweight on D, its image is contained in the maximal torus $T \subset D$, therefore it is also a coweight on T.

Let $f : \mathbf{C}^* \to \mathbf{C}^*$ be a homomorphism of algebraic groups, then $f : z \mapsto z^n$, $n \in \mathbf{Z}$, is the *degree* of f and we denote $\deg f = n$.

For every weight ξ and every coweight χ on an algebraic Abelian semi-simple group D, we set

$$\langle \xi, \chi \rangle := \deg(\xi \circ \chi).$$

DEFINITION 5.11. – Let T be a complex algebraic torus. The *weight lattice* $T^\bullet$ of T is the group of weights $T \to \mathbf{C}^*$, and the *coweight lattice* $T_\bullet$ of T is the group of coweights $\mathbf{C}^* \to T$.

The groups $T^\bullet$ and $T_\bullet$ are both *free Abelian* groups whose rank is the dimension of T. The map $(\xi, \chi) \mapsto \langle \xi, \chi \rangle := \deg(\xi \circ \chi)$ is a canonical *non degenerate pairing* $T^\bullet \times T_\bullet \to \mathbf{Z}$ (cf. for example [12, 16.1]).

The weight functor $T \rightsquigarrow T^\bullet$, resp. the coweight functor $T \rightsquigarrow T_\bullet$ is an antiequivalence, resp. an equivalence of categories between algebraic tori and finitely generated free Abelian groups. The quasi inverse of the weight functor is $\mathrm{Hom}_{gr}(., \mathbf{C}^*)$.

An isomorphism of algebraic torus $\Phi : (\mathbf{C}^*)^\mu \to T$ gives a $\mathbf{Z}$-basis of $T_\bullet$ and the inverse isomorphism $\Phi^{-1} : T \to (\mathbf{C}^*)^\mu$ gives a $\mathbf{Z}$-basis of $T^\bullet$.

To a weight $\xi : T \to \mathbf{C}^*$ we associate its infinitesimal counterpart $L\xi : \mathfrak{t} \to \mathbf{C}$ (remember that $\mathfrak{t}$ denotes the Lie algebra of T). If ξ is defined by the formula $(z_1, \ldots, z_k) \mapsto z_1^{n_1} \cdots z_k^{n_k}$, then $L\xi$ is defined as $(\zeta_1, \ldots, \zeta_k) \mapsto n_1\zeta_1 + \cdots + n_k\zeta_k$.

We will sometimes "identify" the group of weights and the group of infinitesimal weights and we will interpret $T^\bullet$ as a $\mathbf{Z}$-submodule of the complex dual space $\mathfrak{t}^*$ of $\mathfrak{t}$. According to the tradition, we will freely use the *additive* notation for the weights. We will denote $\mathscr{W}_{\mathbf{R}}$ the real vector space $\mathbf{R} \otimes_{\mathbf{Z}} T^\bullet$.

For $\xi \in T^\bullet$, $\chi \in T_\bullet$, we define:

$$\langle L\xi, L\chi \rangle := L\xi \circ L\chi(1) = \langle \xi, \chi \rangle.$$

Let G be a linear algebraic group and let D be an *Abelian semi-simple group*. We recall that the *roots* of D are the *non trivial* weights on D for the adjoint action of D on the Lie algebra $\mathfrak{g}$. We denote $\mathfrak{g}_\xi$ the root space associated to the root ξ:

$$\mathfrak{g}_\xi := \{x \in \mathfrak{g} \mid \forall \lambda \in D\ ,\ (\mathrm{Ad}\lambda)(x) = \xi(\lambda)x\}.$$

We have $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\xi \in \mathscr{R}} \mathfrak{g}_\xi$, the sum being on the set of roots $\mathscr{R}$ and $\mathfrak{g}_0$ being the space of elements *invariant* by D.

If $D = T$ is connected (a torus), then

$$\mathfrak{g}_\xi = \{x \in \mathfrak{g} \mid \forall \tau \in \mathfrak{t}\ ,\ (\mathrm{ad}\tau)(x) = L\xi(\tau)x\}.$$

Let χ be a *non trivial* coweight on a torus $T \subset G$, it induces a *grading* of Lie algebras on $\mathfrak{g}$:

$$\forall k \in \mathbf{Z}\ ,\ gr_\chi^k := \{x \in \mathfrak{g} \mid \forall t \in \mathbf{C}^*\ ,\ (\mathrm{Ad}\chi(t))(x) = t^k x\} = \{x \in \mathfrak{g} \mid [L\chi(1), x] = kx\}.$$

If ξ is a weight on T, then there exists a unique $k \in \mathbf{Z}$ such that $\mathfrak{g}_\xi \subset gr_\chi^k$ and we have $k = \langle \xi, \chi \rangle$. In particular $\mathfrak{g}_0 \subset gr_\chi^0$. We have:

$$gr_\chi^k = \bigoplus_{\langle \xi, \chi \rangle = k} \mathfrak{g}_\xi.$$

5.3.2. *Θ-coweights and Θ-structures*

Definition 5.12. – Let $D \subset G$ be an Abelian semi-simple subgroup and $\mathscr{P} := \{\xi_i\}_{i \in I}$ a finite family of non trivial weights on D. We will say that a coweight χ on D is *positive* (resp. *negative*) on $\mathscr{P}$ if, for every $i \in I$, $\langle \xi_i, \chi \rangle > 0$ (resp. $\langle \xi_i, \chi \rangle < 0$).

Definition 5.13. – Let G be a *triangularizable* linear algebraic group. Let D be a *Levi subgroup of G*. By Proposition 5.8, D is Abelian semi-simple. A Θ-*coweight* on D is a coweight on D which is *negative* on the family of *roots* for the adjoint action of D on the Lie algebra $\mathfrak{g}$ of G.

If χ is a Θ-*coweight* on D, then $\mathfrak{g}_0 = gr_\chi^0$ and $\mathfrak{g} = \bigoplus_{k \in -\mathbf{N}} gr_\chi^k$.

5.3.2.1. Θ-*structures*

Definition 5.14. – We will say that a linear algebraic group G admits a Θ-structure if it is triangularizable and if there exists a Θ-*coweight* on a Levi subgroup of G.

Then, by conjugation, there exists a Θ-*coweight* on any Levi subgroup of G.

Definition 5.15. – Let G be a linear algebraic group, let $D \subset G$ be an Abelian semi-simple subgroup and χ a Θ-coweight on D. We will say that χ is *dominant* if, for every root ζ on D, we have $\dim \mathfrak{g}_\zeta \leq -\langle \zeta, \chi \rangle$.

Lemma 5.16. – *Let G be a* triangularizable *complex linear algebraic group, let $D \subset G$ be an Abelian semi-simple subgroup. We suppose that χ is a Θ-coweight on D. Then there exists a Θ-coweight on D which is* dominant.

Proof. – Let $m \in \mathbf{N}^*$ and $\varphi_m : \mathbf{C}^* \to \mathbf{C}^*$ defined by $\varphi_m : t \mapsto t^m$. Then $\chi_m := \chi \circ \varphi_m$ is a coweight on T, and for every root ξ, we have $\langle \xi, \chi \rangle < 0$, that is $\langle \xi, \chi \rangle \leq -1$, whence $\langle \xi, \chi_m \rangle = m \langle \xi, \chi \rangle \leq -m$, and χ_m is a Θ-coweight. Then, for a sufficiently big m ($m \geq \max_{\xi \in \mathscr{R}}(\dim \mathfrak{g}_\xi)$), χ_m is dominant. □

5.3.3. *Existence of a Θ-structure*

Remark 5.17. –
1. If $G = U \rtimes D$ is a Levi decomposition *such that the semidirect product is not direct* and if D is an Abelian *finite* group, then there exists no Θ-structure on G.
2. We suppose that there exists a Θ-structure on a linear algebraic group G. If ξ is a root, then ξ^{-1} is *not* a root.

3. There exists a triangularizable connected linear algebraic group G such that there exists no Θ-structure on G. Let :
$$G := \left\{ \left(\begin{smallmatrix} 1 & \alpha & \beta \\ 0 & t & \gamma \\ 0 & 0 & 1 \end{smallmatrix} \right) \,\Big|\, t \in \mathbf{C}^*, \alpha, \beta, \gamma \in \mathbf{C} \right\},$$
it is triangular and it admits the infinitesimal roots 1 and -1, therefore there exists no Θ-structure on G.

A triangularizable group being given it seems difficult to find a practical criterion to decide if it admits a Θ-structure. We shall give now a *sufficient condition* (we will use it below for the case of Borel subgroups of reductive groups).

DEFINITION 5.18. – Let G be a linear algebraic group and $T \subset G$ a *torus*. A *good system* of roots for the adjoint action of T on $\mathfrak{g}$ is a set Σ of roots such that

(i) Σ is a $\mathbf{R}$-*free* subset of $\mathscr{W}_{\mathbf{R}}$;
(ii) every root $\xi \in \mathscr{R}$ can be written $\xi = \sum_{k \in I} a_i \xi_i$, with, for every $i \in I$, $a_i \in \mathbf{R}_+$ and $\xi_i \in \Sigma$.

PROPOSITION 5.19. – *Let G be a connected triangularizable group and $T \subset G$ a* maximal torus. *If there exists a good system of roots for the adjoint action of T on $\mathfrak{g}$, then there exists a Θ-structure on G.*

Proof. – We prove firstly a preliminary lemma (part (ii) of this lemma will be used later).

LEMMA 5.20. – (i) *Let $f_1, \ldots, f_{\mu'}$ be* independent *$\mathbf{R}$-linear forms on $\mathbf{R}^\mu$, there exists $p = (p_1, \ldots, p_\mu) \in \mathbf{Z}^\mu$ such that $f_i(p) < 0$ for all $i = 1, \ldots, \mu'$.*
(ii) *Let $f_1, \ldots, f_{\mu'}$ be non-trivial $\mathbf{R}$-linear forms on $\mathbf{R}^\mu$, there exists $p = (p_1, \ldots, p_\mu) \in \mathbf{Z}^\mu$ such that $f_i(p) \neq 0$ for all $i = 1, \ldots, \mu'$.*

Proof. – (i) The set $U := \{y \in \mathbf{R}^\mu \mid f_i(y) < 0,\ i = 1, \ldots, \mu'\}$ is a *non-void* open subset of $\mathbf{R}^\mu$, therefore there exists $p' = (p'_1, \ldots, p'_\mu) \in \mathbf{Q}^\mu \cap U$. If $y \in U$ and $a \in \mathbf{N}^*$, then $ay \in U$, the result follows.
(ii) The set $U := \{y \in \mathbf{R}^\mu \mid f_i(y) \neq 0,\ i = 1, \ldots, \mu'\}$ is a *non-void* open subset of $\mathbf{R}^\mu$, therefore there exists $p' = (p'_1, \ldots, p'_\mu) \in \mathbf{Q}^\mu \cap U$. If $y \in U$ and $a \in \mathbf{N}^*$, then $ay \in U$, the result follows.

□

We can now prove the proposition.

Let $\Phi : (\mathbf{C}^*)^\mu \to T$ be an isomorphism of tori.

Let $\Sigma = \{\xi_1, \ldots, \xi_{\mu'}\}$ be a *good system* of roots of G. For $i = 1, \ldots, \mu'$, we set $f_i := L\xi_i \circ L\Phi$. We interpret $f_1, \ldots, f_{\mu'}$ as linear forms on $\mathbf{R}^\mu$, by hypothesis they are independent, therefore we can apply the Lemma 5.20 above. There exists $p = (p_1, \ldots, p_\mu) \in \mathbf{Z}^\mu$ such that $f_i(p) < 0$ for all $i = 1, \ldots, \mu'$. We define a morphism $\chi : \mathbf{C}^* \to T$ by $\Phi^{-1} \circ \chi : t \to (t_1 := t^{p_1}, \ldots, t_\mu := t^{p_\mu})$, for $i = 1, \ldots, \mu'$, then we set $v_i := f_i \circ L(\Phi^{-1} \circ \chi) := L\xi_i \circ L\chi$. We have $v_i(1) = f_i(p) < 0$. If ξ is a root, then $\langle L\xi, L\chi \rangle = \sum_{i=1,\ldots,\mu'} a_i v_i(1)$ with $a_i \geq 0$, $a_1 + \cdots + a_{\mu'} > 0$ and therefore $\langle L\xi, L\chi \rangle = \langle \xi, \chi \rangle < 0$. □

In the following proposition, T is a maximal torus of G. One implication is Proposition 5.19.

PROPOSITION 5.21. – *If the dimension of T is one, then G admits a Θ-structure if and only if there exists a good system of roots.*

Proof. – If G admits a Θ-structure, then there exists a surjective morphism $\eta : \mathbf{C}^* \to T$ such that, for every root ξ, $\langle \xi, \eta \rangle$ is negative. Let ξ_1 be a root, then, for every root ξ, we have $L\xi = aL\xi_1$ with $a > 0$, therefore $\{\xi_1\}$ is a good system of roots. □

For basic definitions on Borel subgroups, positive systems of roots..., cf. [12].

PROPOSITION 5.22. – *If G^+ is a* Borel subgroup *of a* connected reductive *algebraic group, then there exists a Θ-structure on G^+.*

Proof. – Let G^+ be a Borel subgroup of the connected reductive algebraic group G. Let T be a maximal torus of G contained in G^+, then G^+ corresponds to a *positive* system of roots $\mathscr{R}^+$ of G' ($\mathfrak{g} = \mathfrak{t} + \bigoplus_{\xi \in \mathscr{R}^+} \mathfrak{g}_\rho$). We denote by $\mathscr{B} := (\xi_1, \dots, \xi_\mu)$ a *basis* (or system of simple roots) of this system $\mathscr{R}^+$ (such a basis exists). Then every root in $\mathscr{R}^+$ is a linear combination of the roots of this basis with positive coefficients (they are integers) and therefore $\mathscr{B}$ is a *good system* of roots. Then the result follows from the Proposition 5.19. □

5.4. Some complements on linear algebraic groups

We shall use later this part for the solution of the local inverse problem and in our study of the global inverse problem. Similar tools were introduced by the first author in order to solve inverse problems in the *differential case*. For the missing proofs, cf. [22, 11.3,11.4].

We denote by $L(G)$ the subgroup of an algebraic group G generated by all the maximal tori of G, it is a connected algebraic normal subgroup and the maximal torus of the algebraic group $V(G) := G/L(G)$ is reduced to the identity.

LEMMA 5.23. – *The Lie algebra $\mathfrak{L}$ of $L(G)$ is generated by $\mathfrak{t}$ (the Lie algebra of a maximal torus) and the root-spaces $\mathfrak{g}_\xi$.*

The group $R_u(G)/\big(G^0, R_u(G)\big)$ is a *commutative unipotent* group, therefore it can be identified with a finite dimensional complex vector space. The finite group G/G^0 acts naturally on $R_u(G)/\big(G^0, R_u(G)\big)$.

We set $S(G) := R_u(G)/\big(G^0, R_u(G)\big) \rtimes G/G^0$. Due to a result of the first author [22, Proposition 1.8, page 276] , there is an isomorphism of algebraic groups:

$$S(G) \to V(G)/\big(V(G)^0, V(G)^0\big).$$

LEMMA 5.24. – *The linear algebraic groups $S(G)$, $V(G)$ and $V(G)/(V(G)^0, V(G)^0)$ have the same number m of topological generators.*

We have $\dim R_u(G)/\big(G^0, R_u(G)\big) \le m$ and it is an equality if G is connected.

If G is topologically generated by s elements, then $m \le s$.

LEMMA 5.25. – *Let G be an algebraic group endowed with a Θ-structure defined by a Θ-coweight χ on a Levi subgroup $D \subset G$. Let $T \subset D$ be the maximal torus of D. We set $U := R_u(G)$ and denote by $\mathfrak{u}$ its Lie algebra. Then:*

(i) $\mathfrak{u} = \mathfrak{u}_0 \oplus \bigoplus_{\xi\in\mathcal{R}} \mathfrak{g}_\xi$ *and* $\bigoplus_{\xi\in\mathcal{R}} \mathfrak{g}_\xi$ *is a sub-Lie algebra of* $\mathfrak{u}$*;*
(ii) $\mathfrak{L} = \mathfrak{t} \oplus \bigoplus_{\xi\in\mathcal{R}} \mathfrak{g}_\xi$, $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\xi\in\mathcal{R}} \mathfrak{g}_\xi$, $\mathfrak{g}_0 = \mathfrak{u}_0 \oplus \mathfrak{t}$, $\mathfrak{g} = \mathfrak{u}_0 \oplus \mathfrak{L}$*;*

Proof. – (i) For any weight ξ on D, $\mathfrak{u}_\xi \subset \mathfrak{g}_\xi$ and if ξ is a root, $\mathfrak{g}_\xi = \mathfrak{u}_\xi$.
Let $\alpha, \beta \in \mathcal{R}$, we have $\mathfrak{g}_{\alpha+\beta} = (0)$ or $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. As $\alpha + \beta \neq 0$, in the second case $\alpha + \beta$ is a root. Hence $\bigoplus_{\xi\in\mathcal{R}} \mathfrak{g}_\xi$ is a sub-Lie algebra of $\mathfrak{u}$.
(ii) For all $\xi \in \mathcal{R}$, $[\mathfrak{t}, \mathfrak{g}_\xi] = \mathfrak{g}_\xi$, therefore, using (i) and the Lemma 5.23, we get $\mathfrak{L} = \mathfrak{t} \oplus \bigoplus_{\xi\in\mathcal{R}} \mathfrak{g}_\xi$. □

6. The local inverse problem

6.1. The regular singular case

6.1.1. *Universal groups and representations. Necessary conditions*. – The universal group for the local regular singular case (at 0) is the *commutative* proalgebraic group:

$$G_f^{(0)} = G_{f,s}^{(0)} \times G_{f,u}^{(0)},$$

with:

$$G_{f,s}^{(0)} = \mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*) \text{ and } G_{f,s}^{(0)} = \mathbf{C}.$$

To a germ (at the origin) of meromorphic q-difference system $\Delta : \sigma_q Y = AY$, up to meromorphic equivalence, corresponds a rational representation:

$$\rho_f : G_f^{(0)} \to \mathrm{GL}_n(\mathbf{C})$$

and conversely. The q-difference Galois group of Δ is $G = \mathrm{Im}\rho_f$. It is *Abelian*.

The knowledge of the representation ρ is equivalent to the knowledge of a pair of *commuting* representations:

$$\rho_{f,s} : G_{f,s}^{(0)} \to \mathrm{GL}_n(\mathbf{C}) \quad \rho_{f,u} : G_{f,u}^{(0)} \to \mathrm{GL}_n(\mathbf{C}).$$

We have $G_s = \mathrm{Im}\rho_{f,s}$ and $G_u = \mathrm{Im}\rho_{f,u}$ and our commutation condition means that each element of G_s commutes with each element of G_u.

The commutative unipotent group G_u being the image of $\mathbf{C}$ by $\rho_{f,u}$ its dimension is *at most one*.

Note that the group $\mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*)$ is *topologically generated* by (exactly) two elements [31] and $\mathbf{C}$ is topologically generated by one element.

PROPOSITION 6.1. – *Let G be the Galois group of a local* regular singular *q-difference system, then:*

- *The group G is* Abelian *and* topologically generated *by two elements;*
- *The finite group G/G^0 is* algebraically generated *by at most two elements;*
- $\dim_{\mathbf{C}} G_u \leq 1$.

6.1.1.1. *A description of* $\mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*)$. – We recall the description of $\mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*)$. We choose $\tau \in \mathbf{C}$ such that $e^{-2i\pi\tau} = q$ ($\mathrm{Im}\tau > 0$). The map $w \mapsto z := e^{2i\pi w}$ induces an isomorphism of $\mathbf{C}/(\mathbf{Z} \oplus \mathbf{Z}\tau)$ on $\mathbf{E}_q$. We consider $\mathbf{C}$ as a $\mathbf{Q}$-vector space, we can write it as a direct sum of $\mathbf{Q}$-vector spaces $\mathbf{C} = \mathbf{Q} \oplus \mathbf{Q}\tau \oplus \mathbf{L}$, then we have a product of $\mathbf{Z}$-modules $\mathbf{C}/(\mathbf{Z} \oplus \mathbf{Z}\tau) \approx (\mathbf{Q}/\mathbf{Z}) \times (\mathbf{Q}\tau/\mathbf{Z}\tau) \times \mathbf{L}$ and the corresponding image is the product of $\mathbf{Z}$-modules:

$$\mathbf{E}_q = \underline{\mu} \times \underline{\mu}_q \times \mathbf{L},$$

where $\underline{\mu} := e^{2i\pi\mathbf{Q}}$ is the group of the roots of the unity, $\underline{\mu}_q = q^{\mathbf{Q}}/q^{\mathbf{Z}}$ is the image in $\mathbf{E}_q$ of the subgroup $q^{\mathbf{Q}} \subset \mathbf{C}^*$ ($\underline{\mu}_q \approx \underline{\mu} \approx \mathbf{Q}/\mathbf{Z}$) and $\mathbf{L}$ is a torsion free subgroup (the "universal lattice", defined up to isomorphism).

We shall consider each Abelian group as the *inductive limit* of its finitely generated subgroups.

Recall that the groups written $G^{(0)}$ are (universal) local Galois groups at 0, while G^0 means the neutral component of any proalgebraic group G. We have a short exact sequence of proalgebraic groups:

$$\big(G^{(0)}_{f,s}\big)^0 \to G^{(0)}_{f,s} \to G^{(0)}_{f,s}/\big(G^{(0)}_{f,s}\big)^0,$$

we get it applying the exact contravariant functor $\mathrm{Hom}_{gr}(.,\mathbf{C}^*)$ to the short exact sequence of groups:

$$\underline{\mu} \times \underline{\mu}_q \to \mathbf{E}_q \to \mathbf{E}_q/(\underline{\mu} \times \underline{\mu}_q).$$

We have $\mathbf{E}_q \to \mathbf{E}_q/(\underline{\mu} \times \underline{\mu}_q) \approx \mathbf{L}$, therefore:

$$\mathrm{Hom}_{gr}\big(\mathbf{E}_q/(\underline{\mu} \times \underline{\mu}_q), \mathbf{C}^*\big) \approx \mathrm{Hom}_{gr}\big(\mathbf{L}, \mathbf{C}^*\big).$$

Hence $\big(G^{(0)}_{f,s}\big)^0 \approx \mathrm{Hom}_{gr}\big(\mathbf{E}_q/(\underline{\mu} \times \underline{\mu}_q), \mathbf{C}^*\big)$ is a *protorus*, we will call it *the Fuchsian protorus* and we will denote it $\mathbf{T_f}$.

We recall that $\mathrm{Hom}_{gr}(\mathbf{Q}/\mathbf{Z}, \mathbf{C}^*) = \hat{\mathbf{Z}}$. Then $\mathrm{Hom}(\underline{\mu}, \mathbf{C}^*) = \hat{\mathbf{Z}}(1)$ ($\hat{\mathbf{Z}}(1)$ is the multiplicative notation for $\hat{\mathbf{Z}}$).

We have $G^{(0)}_{f,s}/\mathbf{T_f} \approx \hat{\mathbf{Z}}(1) \times \hat{\mathbf{Z}}(1)$.

Considering $G^{(0)}_{f,s}$ as a proalgebraic group, we get $\mathbf{E}_q = \mathrm{Hom}(G^{(0)}_{f,s}, \mathbf{C}^*)$ (here Hom is for *morphisms of proalgebraic groups*, i.e., rational homomorphisms), as the inductive limit of its *finitely generated* subgroups. Then we can consider $\mathbf{E}_q$ as the *group of weights* of $G^{(0)}_{f,s}$. More precisely, if $\bar{c} \in \mathbf{E}_q$, then the map $\psi_{\bar{c}} : G^{(0)}_{f,s} = \mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*) \to \mathbf{C}^*$ defined by $f \in \mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*) \mapsto f(\bar{c})$ is a weight on $G^{(0)}_{f,s}$ and conversely if φ is a weight on $G^{(0)}_{f,s}$, there exists a unique $\bar{c} \in \mathbf{E}_q$ such that $\varphi = \psi_{\bar{c}}$.

Applying the functor $\mathrm{Hom}(.,\mathbf{C}^*)$ to the (non-canonical) decomposition $G^{(0)}_{f,s} = \hat{\mathbf{Z}}(1) \times \hat{\mathbf{Z}}(1) \times \mathbf{T_f}$, we get the (non-canonical) decomposition $\mathbf{E}_q = \underline{\mu} \times \underline{\mu}_q \times \mathbf{L}$.

6.1.2. *The inverse problem for the regular singular case, a Tannakian solution.* – We solve the inverse problem for the regular singular case using the Tannakian mechanism, proving that the conditions of the Proposition 6.1 are sufficient. Afterwards we will give an elementary proof.

PROPOSITION 6.2. – *Let G be an* Abelian *complex linear algebraic group such that:*

(i) *G is* topologically *generated by at most* two *elements;*

(ii) $\dim_{\mathbf{C}} G_u \leq 1$.

Then G is the local Galois group of a local regular singular *meromorphic linear q-difference system.*

The condition (i) *can be replaced by the following (a priori weaker) condition:*

(iii) *G/G^0 is generated by at most* two *elements.*

Proof. – We will give a Tannakian proof, defining a *surjective* morphism $\rho : G^{(0)}_{f,s} \to G$. Then, if $r : G \to \mathrm{GL}_n(\mathbf{C})$ is a faithful representation, the morphism $r \circ \rho : G^{(0)}_{f,s} \to \mathrm{GL}_n(\mathbf{C})$ defines a system of rank n whose Galois group is $r(G)$.

Let G be an *Abelian* linear algebraic group satisfying conditions (ii) and (iii), then $G = G_u \times G_s$. The natural map $G_s/G_s^0 \to G/G^0$ is an isomorphism, therefore there exists an isomorphism $G_s \approx \mathbf{Z}/p_1\mathbf{Z} \times \mathbf{Z}/p_2\mathbf{Z} \times (\mathbf{C}^*)^\nu$ ($p_1, p_2 \in \mathbf{N}^*$). The degenerate case where p_1 or $p_2 = 1$ is easy and left to the reader.

Using a sub-lattice of rank n of $\mathbf{L}$, we get a surjective morphism $\mathbf{T_f} \to (\mathbf{C}^*)^\nu$. There exists also a surjective morphism $\underline{\mu} \times \underline{\mu}_q \to \mathbf{Z}/p_1\mathbf{Z} \times \mathbf{Z}/p_2\mathbf{Z}$. Hence we get a surjective morphism

$$\rho_s : G^{(0)}_{f,s} \approx \underline{\mu} \times \underline{\mu}_q \times \mathbf{T_f} \to \mathbf{Z}/p_1\mathbf{Z} \times \mathbf{Z}/p_2\mathbf{Z} \times (\mathbf{C}^*)^\nu.$$

The Lie algebra $\mathfrak{u}$ of G_u is of dimension at most one. Therefore there exists $N \in \mathfrak{u}$ such that $G_u = \{\exp tN \mid t \in \mathbf{C}\}$. If $N = 0$, the end of the proof is trivial. Otherwise, the map $\rho_u : G^{(0)}_{f,u} \approx \mathbf{C} \to G_u$ defined by $t \mapsto \exp tN$ is an isomorphism (of algebraic groups).

The representations ρ_s and ρ_u clearly commute and the morphism $\rho := (\rho_s, \rho_u) : G^{(0)}_f = G_{f,s} \times G_{f,u} \to G$ is onto. That ends the proof. □

6.1.3. *Explicit descriptions and elementary proof.* – We shall recall how to compute the Galois group of a local regular singular q-difference system and shall deduce a(n) (elementary) proof of Proposition 6.2 from this computation.

Up to a meromorphic gauge transformation, it is sufficient to consider the case of a *constant coefficient* system $\Delta : \sigma_q Y = AY$, $A \in \mathrm{GL}_n(\mathbf{C})$.

We suppose that the matrix A is in upper triangular Jordan form. The representation ρ of the universal group $G^{(0)}_f = G^{(0)}_{f,s} \times G^{(0)}_{f,u}$ associated to the system Δ is:

$$\rho = (\rho_s, \rho_u) : \ (\gamma, \lambda) \mapsto \gamma(A_s) A_u^\lambda. \tag{8}$$

We have $A_s = \mathrm{Diag}\,(a_1, \ldots, a_n)$, then $\gamma(A_{i,s}) = \mathrm{Diag}\,\big(\gamma(\overline{a}_1), \ldots, \gamma(\overline{a}_n)\big)$, $\overline{a}_i$ being the image of a_i in $\mathbf{E}_q$.

Let H be the subgroup of $\mathbf{E}_q$ generated by the image of Sp A. Using the decomposition $\mathbf{E}_q = \underline{\mu} \times \underline{\mu} \times \mathbf{L}$, we get (up to the isomorphism $\underline{\mu}_q \approx \underline{\mu}$) $H = \mathbf{Z}/p_1\mathbf{Z} \times \mathbf{Z}/p_2\mathbf{Z} \times \Lambda$, where Λ is a lattice of rank μ.

The algebraic group $\mathrm{Hom}_{gr}(H, \mathbf{C}^*)$ is an algebraic quotient of $\mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*)$ (using the canonical injection $H \to \mathbf{E}_q$) and the semi-simple component G_s of the Galois group G of Δ is the image of the quotient map, that is $\mathrm{Hom}_{gr}(H, \mathbf{C}^*)$, then:

$$G_s = \mathbf{Z}/p_1\mathbf{Z} \times \mathbf{Z}/p_2\mathbf{Z} \times \mathrm{Hom}_{gr}(H, \mathbf{C}^*) \approx \mathbf{Z}/p_1\mathbf{Z} \times \mathbf{Z}/p_2\mathbf{Z} \times (\mathbf{C}^*)^\mu.$$

More precisely we get the representation of G_s in $\mathrm{GL}_n(\mathbf{C})$ corresponding to (8) using the interpretation of H as the group of weights of G_s. This representation is given by the diagonal weights $(\overline{a}_i)_{i=1,\dots,n}$ $(\overline{a}_i \in H)$.

We can now solve explicitly the inverse problem.

Let G be an *Abelian* complex linear algebraic group satisfying the conditions of the Proposition 6.2, we will compute a matrix $A \in \mathrm{GL}_n(\mathbf{C})$ such that the system $\Delta : \sigma_q Y = AY$ admits G as Galois group.

More precisely, we start from a faithful *representation* of the *Abelian* group G in GL_n in upper triangular form. Then G_s is diagonal and (due to condition *(ii)*) there exists a *unipotent* matrix $N \in \mathcal{M}_n(\mathbf{C})$ such that $G_u = \{N^\lambda \mid \lambda \in \mathbf{C}\}$.

The Abelian linear algebraic group G_s is isomorphic to the product of a finite group (the quotient G/G^0) by a torus of dimension μ, and the finite component is generated by at most two elements. Then $G_s \approx \mathbf{Z}/p_1\mathbf{Z} \times \mathbf{Z}/p_2\mathbf{Z} \times (\mathbf{C}^*)^\mu$. The dual group (group of weights) of $\mathbf{Z}/p_1\mathbf{Z} \times \mathbf{Z}/p_2\mathbf{Z} \times (\mathbf{C}^*)^\mu$ is $\mathbf{Z}/p_1\mathbf{Z} \times \mathbf{Z}/p_2\mathbf{Z} \times \mathbf{Z}^\mu$

Using the decomposition $\mathbf{E}_q = \underline{\mu} \times \underline{\mu}_q \times \mathbf{L}$, we get an isomorphism between $\mathbf{Z}/p_1\mathbf{Z} \times \mathbf{Z}/p_2\mathbf{Z} \times \mathbf{Z}^\mu$ and a subgroup H of $\mathbf{E}_q$. We can therefore interpret H as the group of weights on the diagonal group G_s.

We denote $\varpi_1, \dots, \varpi_n$ the *diagonal weights* of the diagonal group G_s, they are elements of H. Let $a_1, \dots, a_n \in \mathbf{C}^*$ such that their natural images in $\mathbf{E}_q$ are $\varpi_1, \dots, \varpi_n$. We moreover require these choices to be consistent in the following sense: each time $\varpi_i = \varpi_j$, we take $a_i = a_j$. Then H is generated by $\overline{a}_1 = \varpi_1, \dots, \overline{a}_n = \varpi_n$.

We can now define $A \in \mathrm{GL}_n(\mathbf{C})$:

$$A_s := \mathrm{Diag}\,(a_1, \dots, a_n) \quad \text{and} \quad A_u := N.$$

Indeed, because of our consistent choices above, A_s and A_u do commute. Then the Galois group of $\Delta : \sigma_q Y = AY$ is G.

6.2. The pure case with integral slopes

6.2.1. *Universal groups and representations. Necessary conditions*. – The universal group for the pure case with integral slopes (at 0) is the *commutative* proalgebraic group:

$$G_{p,1}^{(0)} = G_{f,s}^{(0)} \times G_{f,u}^{(0)} \times T_1^{(0)},$$

with:

$$G_{f,s}^{(0)} = \mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*), \quad G_{f,s}^{(0)} = \mathbf{C} \ \text{and}\ T_1^{(0)} = \mathbf{C}^*.$$

To a germ (at the origin) of meromorphic q-difference system, pure with integral slopes, up to meromorphic equivalence, corresponds a morphism:

$$\rho : G_{p,1}^{(0)} \to \mathrm{GL}_n(\mathbf{C});$$

$G = \mathrm{Im}\rho$ is the Galois group of the system, it is *commutative*.

The knowledge of the representation ρ is equivalent to the knowledge of a triple of pairwise commuting representations:

$$\rho_{f,s} : G_{f,s}^{(0)} \to \mathrm{GL}_n(\mathbf{C}), \quad \rho_{f,u} : G_{f,u}^{(0)} \to \mathrm{GL}_n(\mathbf{C}), \quad \rho_\theta : T_1^{(0)} \to \mathrm{GL}_n(\mathbf{C}).$$

We have (up to the obvious reordering of the factors) $G_s = \mathrm{Im}(\rho_{f,s}, \rho_\theta)$, $G_u = \mathrm{Im}\rho_{f,u}$.

As in the regular singular case, we get the following result.

PROPOSITION 6.3. – *Let G be the Galois group of a local* pure *q-difference system, then:*

- *G is* Abelian *and* topologically generated *by two elements;*
- *G/G^0 is* algebraically generated *by at most two elements;*
- $\dim_{\mathbf{C}} G_u \leq 1$.

We recall that we have a (non-canonical) decomposition $G_{f,s}^{(0)} = \hat{\mathbf{Z}}(1) \times \hat{\mathbf{Z}}(1) \times \mathbf{T_f}$, where the Fuchsian universal protorus is a *subgroup* of $G_{f,s}^{(0)}$.

We will denote T_f the image of $\mathbf{T_f}$ by $\rho_{f,s}$ and we will call it the *Fuchsian torus* of G. We will denote T_θ the image of ${\mathbf{T_1}}^{(0)}$ by $\rho_{f,s}$ and we will call it the θ-torus of G. The θ-torus and the Fuchsian torus of G generate the *maximal torus* of G.

6.2.2. *Sufficient conditions*

PROPOSITION 6.4. – *Let G be an* Abelian *complex linear algebraic group and a non trivial coweight $\chi : \mathbf{C}^* \to G_s$. We suppose that:*

(i) *G is* topologically *generated by at most* two *elements;*
(ii) $\dim_{\mathbf{C}} G_u \leq 1$.

Then G is the local Galois group of a local pure *meromorphic linear q-difference system with integral slopes such that $\chi = \rho_\theta$ (where $\rho = (\rho_f, \rho_\theta)$ is the representation defining the system). The condition (i) can be replaced by the following (a priori weaker) condition:*

(iii) *G/G^0 is generated by at most* two *elements.*

Proof. – We will prove the existence of a system such that its Fuchsian torus T_f is a maximal torus, or equivalently such that $T_\theta \subset T_f$. The proof is Tannakian and it is only a slight modification of the proof of the Proposition 6.2.

Let T be the maximal torus of G, it contains the image of χ.

We build as above a *surjective* representation:

$$\rho_f = (\rho_{f,s}, \rho_{f,u}) : \ G_f^{(0)} = G_{f,s}^{(0)} \times G_{f,u}^{(0)} \to G.$$

Then using $\mathbf{T}_1^{(0)} = \mathbf{C}^*$, we define a representation

$$\rho = (\rho_f, \rho_\theta) : \ G_{p,1}^{(0)} = G_f^{(0)} \times \mathbf{T}_1^{(0)} \to G,$$

by $\rho_\theta = \chi$. (The component representations automatically commute.) It is a surjective morphism and it answers the question. □

6.2.3. *Explicit descriptions.* – We recall how to compute the Galois group of a local pure q-difference system with integral slopes and deduce a new (elementary) proof of Proposition 6.4 from this computation.

Up to a meromorphic gauge transformation, it is sufficient to consider the case of a system $\Delta : \sigma_q Y = AY$, such that the matrix A is in upper triangular normal form:

$$A := \begin{pmatrix} z^{\mu_1} A_1 & \ldots\ldots & \ldots & \ldots \\ \ldots & \ldots\ldots & U_{i,j} & \ldots \\ 0 & \ldots\ldots & \ldots & \ldots \\ \ldots & 0 \;\ldots & \ldots & \ldots \\ 0 & \ldots \; 0 & \ldots & z^{\mu_k} A_k \end{pmatrix}, \tag{9}$$

where, for $1 \leq i \leq k$, $A_i \in \mathrm{GL}_{r_i}(\mathbf{C})$ is in Jordan form, and $\mu_1, \ldots, \mu_k \in \mathbf{Z}$. (Usually we take $\mu_1 < \cdots < \mu_k$, although this has no consequence in the *formal* case.)

The representation ρ of the universal group $G_{p,1}^{(0)} = G_{f,s}^{(0)} \times G_{f,u}^{(0)} \times T_1^{(0)}$ associated to the system Δ is:

$$\rho = (\rho_s, \rho_u, \rho_\theta) : \; (\gamma, \lambda, t) \mapsto \begin{pmatrix} t^{\mu_1} \gamma(A_{1,s}) A_{1,u}^{\lambda} & \ldots\ldots\ldots & \ldots \\ \ldots & \ldots\ldots \; 0 & \ldots \\ 0 & \ldots\ldots\ldots & \ldots \\ \ldots & 0 \; \ldots\ldots & \ldots \\ 0 & \ldots \; 0 \; \ldots & t^{\mu_k} \gamma(A_{k,s}) A_{k,u}^{\lambda} \end{pmatrix}. \tag{10}$$

We can now give a new proof of Proposition 6.4. We start from the Abelian group G and the one-parameter subgroup χ. We can assume that it is diagonalized:

$$\forall t \in \mathbf{C}^* \; , \; \chi(t) = \begin{pmatrix} t^{\mu_1} I_{r_1} & \ldots\ldots\ldots & \ldots \\ \ldots & \ldots\ldots \; 0 & \ldots \\ 0 & \ldots\ldots\ldots & \ldots \\ \ldots & 0 \; \ldots\ldots & \ldots \\ 0 & \ldots \; 0 \; \ldots & t^{\mu_k} I_{r_k} \end{pmatrix},$$

and we apply the explicit proof of Proposition 6.2 to each of the regular singular blocks of ranks r_i, yielding matrices A_i with constant coefficients. Then we set:

$$A := \begin{pmatrix} z^{\mu_1} A_1 & \ldots\ldots\ldots & \ldots \\ \ldots & \ldots\ldots \; 0 & \ldots \\ 0 & \ldots\ldots\ldots & \ldots \\ \ldots & 0 \; \ldots\ldots & \ldots \\ 0 & \ldots \; 0 \; \ldots & z^{\mu_k} A_k \end{pmatrix}.$$

The image of χ is contained in G, therefore the Galois group of the system $\sigma_q Y = AY$ is G and we have $\chi = \rho_\theta$.

6.3. The local inverse problem: the general case with integral slopes

6.3.1. *Necessary conditions*

THEOREM 6.5. – *Let G be a complex linear algebraic subgroup. If G is the local Galois group of a meromorphic linear q-difference equation, then:*

(i) *G is triangularizable;*
(ii) *$G/L(G)$ is Abelian and* topologically *generated by at most* two *elements;*
(iii) *the finite group G/G^0 is Abelian and generated by at most* two *elements;*
(iv) *the group G/G^0 acts* trivially *on $R_u(G)/\big(G^0, R_u(G)\big)$ and the dimension of the vector space $R_u(G)/\big(G^0, R_u(G)\big)$ is at most* one.

Proof. – (i) Trivial.

(ii) We will use a Tannakian argument which is a variant of an idea due to O. Gabber in the differential case [13].

Let $G^{(0)}$ be the Tannakian group of the Tannakian category $\mathcal{E}^{(0)}$. To a q-difference system of rank n, meromorphic at the origin, corresponds a (rational) representation $\rho : G^{(0)} \to \mathrm{GL}_n(\mathbf{C})$ and conversely. If $G = \mathrm{Gal}_{\mathbf{C}(\{z\})}(\Delta)$ is the Galois group of Δ, then $G = \mathrm{Im}\rho$.

Let $\pi : G \to G/L(G)$ be the canonical map, let $\iota : G/L(G) \to \mathrm{GL}_{n'}(\mathbf{C})$ be a faithful linear representation of $G/L(G)$, then we get a continuous linear representation $\rho' : \iota\circ\pi\circ\rho : G^{(0)} \to \mathrm{GL}_{n'}(\mathbf{C})$.

To the representation ρ' corresponds a q-difference system Δ' of rank n' and $G' := \iota(G/V(G)) = \mathrm{Gal}_{\mathbf{C}(\{z\})}(\Delta') = \mathrm{Im}\rho'$.

The maximal torus of G' is reduced to the identity, therefore the θ-torus of Δ' is trivial and Δ' is *regular singular*.

Hence the Galois group G' of Δ' is *Abelian* and *topologically generated* by at most *two* elements [31]. Moreover $G' = G'_s G'_u$, where the unipotent group G'_u is *topologically generated* by at most *one* element [31].

(iii) We have $G/L(G) = V(G) \approx G'$. The group G/G^0 is a quotient of $V(G)$ therefore it is *Abelian* and topologically generated by at most two elements, as it is *finite* it is *algebraically* generated by at most *two* elements.

(iv) We set as in Section 5.4 $S(G) := R_u(G)/\big(G^0, R_u(G)\big) \rtimes G/G^0$, we recall that there is an isomorphism of algebraic groups $S(G) \to V(G)/\big(V(G)^0, V(G)^0\big)$. The group $V(G)$ being commutative, we get an isomorphism $S(G) \to V(G)$, $S(G)$ is commutative and the action of G/G^0 on $R_u(G)/\big(G^0, R_u(G)\big)$ is *trivial*.

We have an isomorphism $S(G)_u = R_u(G)/\big(G^0, R_u(G)\big) \to V(G)_u$. As $V(G)_u$, $S(G)_u$ is topologically generated by at most *one* generator. Then $\dim_{\mathbf{C}} R_u(G)/\big(G^0, R_u(G)\big) \leq 1$. $\square$

We think that the four *necessary* conditions of the above theorem are *not sufficient*. Anyway if we want to realize G as the Galois group of a meromorphic linear q-difference system whose Newton polygon has *integral* slopes, then there is a *new* necessary condition ((vi) of the following theorem). This condition is not trivial: there exists a solvable linear algebraic group satisfying the conditions (ii), (iii), (iv) of Theorem 6.5 which does not satisfy the condition (vi) of Theorem 6.6 below (cf. 3 of Remark 5.17, page 206).

THEOREM 6.6. – *Let G be a complex linear algebraic subgroup. If G is the local Galois group of a meromorphic linear q-difference system whose Newton polygon has* integral slopes*, then:*

(i) *G is triangularizable;*
(ii) *$G/L(G)$ is Abelian and* topologically *generated by at most* two *elements;*
(iii) *G/G^0 is Abelian and generated by at most* two *elements;*
(iv) *the dimension of the unipotent component ot the Abelian group $G/L(G)$ is* at most one*;*
(v) *the dimension of $R_u(G)/\big(G^0, R_u(G)\big)$ is at most* one*;*
(vi) *there exists a Θ-structure on G.*

Proof. – Assertions (i) to (v) follow from the Proposition 6.1 and Theorem 6.5.

It remains to prove (vi).

Every system with integral slopes admits, up to meromorphic equivalence, a Birkhoff-Guenther normal form, therefore it is sufficient to prove the result for a system $\sigma_q Y = AY$ in Birkhoff-Guenther normal form:

$$A = A_U := \begin{pmatrix} B_1 & \ldots\ldots & \ldots & \ldots \\ \ldots\ldots & \ldots & U_{i,j} & \ldots \\ 0 & \ldots\ldots & \ldots & \ldots \\ \ldots & 0 & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots\ B_k \end{pmatrix}, \tag{11}$$

where, for $1 \leq i < j \leq k$, $U_{i,j} \in \mathrm{Mat}_{r_i,r_j}(\mathbf{C}(\{z\}))$. Here, U stands short for $(U_{i,j})_{1\leq i<j\leq k} \in \prod\limits_{1\leq i<j\leq k} \mathrm{Mat}_{r_i,r_j}(\mathbf{C}(\{z\}))$. (This requirement is actually weaker than the true Birkhoff-Guenther normal form, where the $U_{i,j}$ would have polynomial coefficients, cf. Section 2.2.)

We suppose that:

$$B_i = z^{\mu_i} A_i,\ A_i \in \mathrm{GL}_{r_i}(\mathbf{C}),\ \mu_1 < \cdots < \mu_i < \cdots < \mu_k,$$

and we set (5):

$$A_0 := \begin{pmatrix} B_1 & \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots & 0 \quad \ldots \\ 0 & \ldots\ldots\ldots\ldots \\ \ldots\ 0 & \ldots\ldots\ldots \\ 0\ \ldots\ 0 & \ldots\ B_k \end{pmatrix}.$$

We firstly consider the differential Galois group G_0 of the pure system $\sigma_q Y = A_0 Y$. This group is Abelian, in upper triangular form, and its semi-simple component $D := (G_0)_s$ is *diagonal*.

We define a coweight of D by $\chi : t \mapsto (t^{\mu_1} I_{r_1}, \ldots, t^{\mu_k} I_{r_k})$ and we denote its image by T_θ (the theta-torus). Then the maximal torus T of D is generated by the "Fuchsian torus" T_f and T_θ.

We consider now the differential Galois group G of the system $\sigma_q Y = AY$. It is in upper triangular form, it contains G_0 as a subgroup, moreover D is a Levi subgroup of G.

We denote $\varpi_1, \ldots, \varpi_n$ the *diagonal weights* of D. The root for the adjoint action of D on $\mathfrak{g}$ are elements of the set $\{\varpi_i \varpi_j^{-1} \mid i < j\}$. If ξ is a root, the corresponding root space is not trivial: there exists $x \in \mathfrak{g}_\xi$ such that $x \neq 0$. Therefore there exist i, j, with $i < j$, such that $x_{ij} \neq 0$, then $\langle \xi, \chi \rangle = \mu_i - \mu_j < 0$ ($\varpi_i \circ \chi(t) = t^{\mu_i}$). Hence χ is a Θ-structure on G. □

6.3.2. *Sufficient conditions.* – We will prove in this part that the conditions of the Theorem 6.6 are sufficient.

From Lemma 5.25 we deduce the following preliminary result.

LEMMA 6.7. – *Let G be a complex linear algebraic group admitting a Θ-structure. Then the following conditions are equivalent:*

(i) *the dimension of $R_u(G)/\big(G^0, R_u(G)\big)$ is at most one;*

(ii) *if $G = U \rtimes D$ is a Levi decomposition, then the dimension of $\mathfrak{u}_0$ (which was defined in Lemma 5.25) is at most one;*

(iii) *if D is a Levi subgroup of G, then* $\dim C_G(D) \leq \dim D + 1$ *(we write $C_G(D)$ the centralizer of D).*

If these conditions are satisfied, then:

$$\dim R_u(G)/\big(G^0, R_u(G)\big) = \dim \mathfrak{u}_0 = \dim C_G(D) - \dim D.$$

THEOREM 6.8. – *Let G be a complex linear algebraic group such that:*

(i) *$G/L(G)$ is Abelian and* topologically *generated by at most* two *elements;*

(ii) *the dimension of $R_u(G)/\big(G^0, R_u(G)\big)$ is at most one;*

(iii) *G admits a Θ-structure,*

then G is the local Galois group of a meromorphic linear q-difference system whose Newton polygon has integral slopes.

More precisely, if $r : G \to \mathrm{GL}_n(\mathbf{C})$ is a faithful representation, it is possible to find a meromorphic linear q-difference system whose Newton polygon has integral slopes *and whose Galois group is $r(G)$.*

It is possible to replace the condition *(ii)* by the following:

(ii′) *the dimension of the unipotent component of the Abelian group $G/L(G)$ is at most one.*

Proof. – The proof is Tannakian, starting from an algebraic group G, we will obtain the system as a rational representation of the total Galois group with integral slopes:

$$\rho = (\rho_w, \rho_{p,1}) : G_1^{(0)} = \mathfrak{St} \rtimes G_{p,1}^{(0)} \to G$$

whose image is G.

We will build this representation using our main result on the description of the representations of the Tannakian group $G_1^{(0)}$ via the representations of the *wild fundamental group* $L \rtimes G_{p,1,s}^{(0)}$. We recall (cf. Sections 2 and 3) that the knowledge of ρ_w is equivalent to the knowledge of its infinitesimal counterpart $L\rho_w$ and that the knowledge of $L\rho_w$ is equivalent to the knowledge of a representation of L: $\lambda : L \to \mathfrak{g}$, compatible with the corresponding adjoint actions of $G_{p,1,s}^{(0)}$ and $\rho_{p,1}\big(G_{p,1,s}^{(0)}\big)$ (cf. Section 2.4). Moreover we have $\mathrm{Im} L\rho_w = \mathrm{Im}\lambda$.

Let G be a *triangularizable* complex linear algebraic goup. Let χ be a Θ-coweight on a Levi subgroup D of G. Using Lemma 5.16, we can suppose that χ is *dominant*.

We will build the representation ρ in three steps:

– we will define a rational representation $\rho_f : G_f^{(0)} \to G$, whose image is the centralizer $C_G(D)$;
– using the coweight $\chi : \mathbf{C}^* \to D$ and the canonical injection $D \to G$, we get a morphism $\rho_\theta : \mathbf{C}^* \to G$ and we define a rational representation $\rho_{p,1} = (\rho_\theta, \rho_f) : G_{p,1}^{(0)} \to G$;
– we will define a representation $\lambda : L \to \mathfrak{g}$ such that, if $L\rho_w : \mathfrak{st} \to \mathfrak{g}$ is the associated representation, then $\rho := (\rho_w, \rho_{p,1}) : G_1^{(0)} \to G$ is onto.

6.3.2.1. *Definition of ρ_f and $\rho_{p,1}$*. – We consider the centralizer $C_G(D)$, its Lie algebra is $\mathfrak{u}_0 \oplus \mathfrak{t}$ and, according to the hypothesis and to Lemma 6.7, $\dim \mathfrak{u}_0 \leq 1$. We choose a generator N of the vector space $\mathfrak{u}_0$, then $C_G(D) = U_0 \times D$, where $U_0 = \exp \mathfrak{u}_0 = \{\exp tN \mid t \in \mathbf{C}\}$, in particular $C_G(D)$ is Abelian.

We consider the Abelian algebraic group $C_G(D)$ and the coweight χ on $D \subset C_G(D)$. They satisfy the conditions of Proposition 6.3 ($C_G(D)/(C_G(D))^0 \approx G/G^0$), therefore there exists a representation

$$\rho'_{p,1} : G_{p,1}^{(0)} \to C_G(D)$$

such that $\mathrm{Im}\rho'_{p,1} = C_G(D)$ and such that the corestriction of $\rho'_\theta : \mathbf{T}_1^{(0)} \to C_G(D)$ to D is equal to the coweight χ.

By composition of $\rho'_{p,1}$ by the canonical injection $C_G(D) \to G$, we get a representation:

$$\rho_{p,1} = \rho'_{p,1} \circ r : G_{p,1}^{(0)} \to G.$$

Its image is topologically generated by D and $\exp N$.

6.3.2.2. *Definition of ρ_w and ρ*. – We want to extend the representation $\rho_{p,1}$ into a *surjective* representation

$$\rho = (\rho_w, \rho_{p,1}) : G_1^{(0)} = \mathfrak{S} \rtimes G_{p,1}^{(0)} \to G.$$

As we recalled above, the knowledge of $\rho_w : \mathfrak{St} \to G$ is equivalent to the knowledge of a representation:

$$\lambda : L \to \mathfrak{g},$$

the images of λ and $L\rho_w$ being equal.

We will define λ such that its image contains all the root spaces $\mathfrak{g}_\xi$, $\xi \in \mathscr{R}$.

We have a surjective map

$$\rho_{f,s} : G_{f,s}^{(0)} = \mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*) \to D.$$

Let ξ be a root on D, then $\xi \circ \rho_{f,s}$ defines a weight on $\mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*)$, that is an element $\overline{c} \in \mathbf{E}_q$. We set $\langle \xi, \chi \rangle =: -\delta$, $\delta \in \mathbf{N}^*$. Therefore to each root ξ we associate a label $(\delta, \overline{c}) \in \mathbf{N}^* \times \mathbf{E}_q$. We denote by $\Sigma \subset \mathbf{N}^* \times \mathbf{E}_q$ the *finite* subset of labels obtained from the roots by this procedure.

If $(\delta, \overline{c}) \notin \Sigma$, for all $i = 1, \ldots, \delta$, we set $\lambda(\dot{\Delta}_i^{(\delta,\overline{c})}) := 0$. It remains to define $\lambda(\dot{\Delta}_i^{(\delta,\overline{c})})$ for $(\delta, \overline{c}) \in \Sigma$ and for all $i = 1, \ldots, \delta$.

We set $d_\xi := \dim \mathfrak{g}_\xi$. The Θ-coweight χ is *dominant*, therefore $d_\xi \leq \delta$. We choose a *basis* $(e_{\xi,1}, \ldots, e_{\xi,d_\xi})$ of the vector space $\mathfrak{g}_\xi$ and we set $\lambda(\dot{\Delta}^{(\delta,\overline{c})}) := e_{\xi,i}$ if $i = 1, \ldots, d_\xi$ and $\dot{\Delta}_i^{(\delta,\overline{c})} := 0$ if $i = d_\xi + 1, \ldots, \delta$.

Then, for every root $\xi \in \mathscr{R}$, the image of λ contains the root space $\mathfrak{g}_\xi$.

6.3.2.3. *End of the proof*. – By construction, the image of λ, and therefore the image of $L\rho_w$ contains the sum of the root spaces $\bigoplus_{\xi\in\mathcal{R}}\mathfrak{g}_\xi = \bigoplus_{\xi\in\mathcal{R}}\mathfrak{u}_\xi$ and the image of $L\rho_{f,u}$ is u_0. Therefore the image of $L\rho$ contains $\mathfrak{u}$ and the image of ρ contains $U = R_u(G)$. The image of ρ contains also $C_G(D)$ and a fortiori D. Finally the image of ρ is G. □

Using Proposition 5.22 we get the following result.

COROLLARY 6.9. – *Let G be a* Borel subgroup *of a* connected reductive *algebraic group, then it is the local Galois group of a meromorphic linear q-difference system whose Newton polygon has* integral slopes.

7. About the global inverse problem

We have a "glueing" lemma.

LEMMA 7.1. – (i) *Let $A^{(0)}$ (resp. $A^{(\infty)}$) be an object of $\mathcal{E}_1^{(0)}$ (resp. $\mathcal{E}_1^{(\infty)}$). We suppose that $A^{(0)}$ and $A^{(\infty)}$ are in Birkhoff-Guenther normal form and that $A_f^{(0)} = A_f^{(\infty)} \in \mathrm{GL}_n(\mathbf{C})$. Let $G_1^{(0)}$ (resp. $G_1^{(\infty)}$) be the Galois group of $A^{(0)}$ (resp. $A^{(\infty)}$) and G the Galois group of the global system defined by $(A^{(0)}, I_n, A^{(\infty)})$. Then G is the Zariski closure in* $\mathrm{GL}_n(\mathbf{C})$ *of the subgroup generated by $G_1^{(0)}$ and $G_1^{(\infty)}$.*

(ii) *Let G^+ and G^- be two* connected *algebraic subgroups of* $\mathrm{GL}_n(\mathbf{C})$ *satisfying the conditions of Theorem 6.8 (or equivalently such that they are local Galois group of meromorphic linear q-difference systems whose Newton polygon have* integral slopes*). We suppose that G^+ and G^- admit a* same *maximal torus. We denote G the Zariski closure in* $\mathrm{GL}_n(\mathbf{C})$ *of the subgroup generated by G^+ and G^-. Then G is the global Galois group of a meromorphic linear q-difference system whose Newton polygon has* integral slopes *at* 0 *and* ∞.

Proof. – (i) follows easily from Theorem 4.5.

(ii) Going back to the proof of Theorem 6.8, we can find $A^{(0)}$ (resp. $A^{(\infty)}$) such that G^+ (resp. G^-) is the Galois group of $A^{(0)}$ (resp. $A^{(\infty)}$) and such that $A_f^{(0)} = A_f^{(\infty)}$ (we choose $A_f^{(0)}$ such that the subgroup generated by its semi-simple part is Zariski dense in T). Then the result follows from (i). □

PROPOSITION 7.2. – *Let G be a* connected reductive *linear algebraic group, then G is the global Galois group of a meromorphic linear q-difference system whose Newton polygons at* 0 *and* ∞ *have* integral slopes.

Moreover it is possible to get a q-difference system admitting G as a Galois group with a trivial *(generalized) Birkhoff connection matrix and such that the local groups at* 0 *and* ∞ *are* Borel subgroups.

Proof. – If the maximal torus of G is *trivial*, then the conditions of the Proposition 6.1 are satisfied, therefore G is the Galois group of a *local* regular singular equation. It is easy to conclude using [31].

We can suppose that $G \subset \mathrm{GL}_n(\mathbf{C})$ and that the maximal torus T of G is *not trivial* and in diagonal form.

We denote G^+ and G^- two opposite Borel subgroups of G and we choose as explained above a coweight χ of T such that χ is a Θ-coweight for G^+ and χ^{-1} is a Θ-coweight for G^-. Using 6.9 we prove that G^+ (resp. G^-) is the local Galois group of a meromorphic linear q-difference system whose Newton polygon has integral slopes. We end the proof using Lemma 7.1. □

THEOREM 7.3. – *Let G be a* connected *linear algebraic group. We suppose that the dimension of the vector space $R_u(G)/(G, R_u(G))$ is at most* 2. *Then G is the global Galois group of a rational linear q-difference system whose Newton polygons at* 0 *and* ∞ *have* integral slopes.

In particular we can apply this result to a connected group. It generalizes Proposition 7.2.

Proof. – If the maximal torus of G is *trivial*, then the conditions of Proposition 6.1 are satisfied, therefore G is the Galois group of a *local* regular singular equation. It is easy to conclude using [31].

We can suppose that $G \subset \mathrm{GL}_n(\mathbf{C})$ and that the maximal torus T of G is *not trivial* and in diagonal form.

LEMMA 7.4. – *There exists a coweight χ on T which is* non null *on each* root ξ *for the adjoint action of T on the Lie algebra $\mathfrak{g}$ of G: $\langle \xi, \chi \rangle \neq 0$.*

Proof. – The proof is a variant of an argument used above.

Let $\Phi : (\mathbf{C}^*)^\mu \to T$ be an isomorphism of tori.

Let $\Sigma = \{\xi_1, \ldots, \xi_\nu\}$ be the set of roots of G. For $i = 1, \ldots, \nu$, we set $f_i := L\xi_i \circ L\Phi$. We interpret $f_1, \ldots, f_{\mu'}$ as linear forms on $\mathbf{R}^\mu$. There exists $p = (p_1, \ldots, p_\mu) \in \mathbf{Z}^\mu$ such that $f_i(p) \neq 0$ for all $i = 1, \ldots, \nu$ (cf. Lemma 5.20). We define a coweight $\chi : \mathbf{C}^* \to T$ by $\Phi^{-1} \circ \chi : t \mapsto (t_1 := t^{p_1}, \ldots, t_\mu := t^{p_\mu})$, then, for $i = 1, \ldots, \nu$, we set $v_i := f_i \circ L(\Phi^{-1} \circ \chi) := L\xi_i \circ L\chi$. We have $v_i(1) = f_i(p) \neq 0$, then $\langle L\xi_i, L\chi \rangle = \langle \xi_i, \chi \rangle \neq 0$. □

We return to the proof of the theorem. We will suppose that we are in the "worst case" that is $\dim R_u(G)/(G, R_u(G)) = 2$, the reader will easily adapt the proof to the other cases.

We denote by $\mathcal{R} \subset \mathfrak{g}^*$ the set of roots.

The commutative group $V(G) \approx R_u(G)/(G, R_u(G))$ is topologically generated by two elements (cf. Lemma 5.24).

The Lie algebra of $V(G)$ is the image of $\mathfrak{g}_0^n$ induced by the quotient map (cf. Lemma 5.23). Hence there exist $N^+, N^- \in \mathfrak{g}_0^n$ whose images generate the Lie algebra of $V(G)$. Then the Lie algebra $\mathfrak{g}$ is generated by $\mathbf{C}N^+, \mathbf{C}N^-$ and the Lie algebra of $L(G)$, therefore by $\mathbf{C}N^+, \mathbf{C}N^-$, $\mathfrak{t}$ and the root spaces $\mathfrak{g}_\xi$, $\xi \in \mathcal{R}$ (cf. Lemma 5.23).

We set $\mathcal{R}^+ := \{\xi \in \mathcal{R} \mid \langle \xi, \chi \rangle < 0\}$ and $\mathcal{R}^- := \{\xi \in \mathcal{R} \mid \langle \xi, \chi \rangle > 0\}$. We have a partition $\mathcal{R} = \mathcal{R}^+ \cup \mathcal{R}^-$.

We denote by G^+ (resp. G^-) the algebraic subgroup of G topologically generated by, T, exp $(\mathbf{C}N^+)$ and the exp $\mathfrak{g}_\xi$, $\xi \in \mathcal{R}^+$ (resp. T, exp $(\mathbf{C}N^-)$ and the exp $\mathfrak{g}_\xi$, $\xi \in \mathcal{R}^-$). The group G is clearly topologically generated by G^+ and G^-.

Then χ defines a Θ-structure on G^+ and χ^{-1} defines a Θ-structure on G^-. Using 6.8 we prove that G^+ (resp. G^-) is the local Galois group of a meromorphic linear q-difference system whose Newton polygon has integral slopes. We end the proof using Lemma 7.1. □

REMARK 7.5. – In fact as we noticed above, we proved more than what is stated in the proposition. In some sense the only singularities of the constructed equation are 0 and ∞ (cf. [30]). This is a first step towards a q-analog version of the Abhyankar conjecture. The reader will compare with the solution of the differential Abhyankar Conjecture due to the first author.

THEOREM 7.6. – *If a complex linear algebraic group G is the q-difference Galois group of a rational system, then $V(G) := G/L(G)$ is the q-difference Galois group of a rational regular singular system.*

The proof is "Tannakian" and similar to the first part of the proof of the Theorem 6.5.

Conversely we can conjecture that, using a variant of the proof of the Proposition 7.3, the condition of the theorem is not only necessary but that it is also sufficient (the reader will compare with the proof of the corresponding result in the differential case by the first author).

Appendix

Pronilpotent completions

To a family $(x_i)_{i\in I}$, we associate the *free Lie algebra* $\mathrm{Lib}\big((x_i)_{i\in I}\big)$ generated over $\mathbf{C}$. We will denote $\widehat{\mathrm{Lib}}\big((x_i)_{i\in I}\big)$ the completion of $\mathrm{Lib}\big((x_i)_{i\in I}\big)$ for the descending central filtration:

$$L^\wedge := \widehat{\mathrm{Lib}}\big((x_i)_{i\in I}\big) = \varprojlim_{n\in\mathbf{N}} L/L^n,$$

with $L := \mathrm{Lib}\big((x_i)_{i\in I}\big)$ and $L^1 := L$, $L^{n+1} := [L, L^n]$.

If I is *finite*, we refer to [6] for the following properties. Then each L/L^n is a finite dimensional nilpotent complex Lie algebra, therefore it is an *algebraic* Lie algebra and $L^\wedge$ is a pronilpotent proalgebraic Lie algebra.

The functor "Lie algebra" is an equivalence between the category of unipotent algebraic groups and the category of finite dimensional nilpotent Lie algebras. We shall denote exp the inverse equivalence.

We set:

$$\exp(L^\wedge) := \varprojlim_{n\in\mathbf{N}} \exp(L/L^n).$$

It is a prounipotent algebraic group, whose Lie algebra is $L^\wedge$.

If I is *infinite*, then the situation is more complicated. The dimension of each nilpotent Lie algebra L/L^n is infinite and the pronilpotent completion $L^\wedge$ is not satisfying for our purposes. Therefore we will introduce another completion of L, the f-pronilpotent completion $L^\dagger$.

Let $J \subset I$ be a *finite* subset. We have a natural map of Lie algebras:

$$p_J : \mathrm{Lib}\big((x_i)_{i\in I}\big) \to \mathrm{Lib}\big((x_j)_{j\in J}\big),$$

defined by $p_J(x_i) := 0$ if $i \notin J$ and $p_J(x_i) := x_i$ if $i \in J$. We define similarly maps $p_{J_1,J_2} : \mathrm{Lib}\big((x_i)_{i\in J_2}\big) \to \mathrm{Lib}\big((x_j)_{j\in J_1}\big)$ if $J_1 \subset J_2 \subset I$ (J_2 finite).

Going to the nilpotent completions, we get maps:

$$\hat{p}_J : \widehat{\mathrm{Lib}}\big((x_i)_{i\in I}\big) \to \widehat{\mathrm{Lib}}\big((x_j)_{j\in J}\big), \quad p_{J_1,J_2} : \widehat{\mathrm{Lib}}\big((x_i)_{i\in J_2}\big) \to \widehat{\mathrm{Lib}}\big((x_j)_{j\in J_1}\big).$$

The $\hat{\mathrm{Lib}}\big((x_j)_{j\in J}\big)$ ($J \subset I$, J finite) are pronilpotent proalgebraic Lie algebras and the p_{J_1,J_2} ($J_2 \subset I$ finite, $J1 \subset J_2$) are morphisms of proalgebraic Lie algebras.

We thus get a projective system of prounipotent proalgebraic Lie algebras and, by definition, the f-pronilpotent completion $L^\dagger := \mathrm{Lib}^\dagger\big((x_i)_{i\in I}\big)$ of $L := \mathrm{Lib}\big((x_i)_{i\in I}\big)$ is the projective limit of this system,

$$L^\dagger := \varprojlim_{J\subset I} \hat{\mathrm{Lib}}\big((x_j)_{j\in J}\big), \quad J \text{ finite}.$$

It can be interpreted as a projective limit of prounipotent proalgebraic Lie algebras. Then we can pass to groups, using the functor exp, and we can define a projective limit of unipotent groups $\exp L^\dagger$, whose Lie algebra is $L^\dagger$.

The natural map $L \to L^\dagger$ is injective and dominant (its image is dense).

REMARK .1. – If I is *finite*, then $L^\dagger = L^\wedge$.

If I is *infinite*, then we have maps $L \to L^\wedge \to L^\dagger$ and $L^\wedge \to L^\dagger$ is not an isomorphism.

We shall consider now some actions of an *Abelian* proalgebraic group G on a free Lie algebra L and the corresponding "semi-direct products" $L \rtimes G$.

In what follows we will suppose that each one-dimensional complex vector space $\mathbf{C}x_i$ is stable under the action of G and that the action of G on $\mathbf{C}x_i$ is, for all $i \in I$, algebraic. Therefore the representations $\rho_i : G \to \mathbf{C}^*$, given by $g \in G \mapsto \rho_i(g)$, with $g(x_i) = \rho_i(g)x_i$ are rational, they are *weights* on G.

By definition a *representation* ρ of $L \rtimes G$ is the data of a rational linear representation ρ' of G ($\rho' : G \to \mathrm{GL}(V)$), together with a representation $d\rho''$ of L in the same space ($d\rho'' : L \to \mathrm{End}(V)$), required to be compatible with the corresponding adjoint actions. We consider the corresponding Tannakian category $\mathrm{Rep}(L \rtimes G)$.

In what follows we will suppose that:

(i) for all weight on G, there exists only a *finite set* of $i \in I$ such that $\rho_i = \rho$;
(ii) for every representation $\rho = (\rho', d\rho'')$ of $L \rtimes G$, the image of $d\rho''$ is a *nilpotent* subalgebra of $\mathrm{End}(V)$.

LEMMA .2. – *Let $\rho = (\rho', d\rho'')$ be a representation of $L \rtimes G$. Then there exists only a finite set of $i \in I$ such that $d\rho''(x_i) \neq 0$.*

Proof. – Let $\rho = (\rho', d\rho'')$ be a representation of $L \rtimes G$ in a finite dimensional space V. Let $i \in I$, for all $g \in G$:

$$\mathrm{Ad}_{\rho'(g)}\big(d\rho''(x_i)\big) = d\rho''\big(g(x_i)\big) = d\rho''\big(\rho_i(g)x_i\big) = \rho_i(g)d\rho''(x_i).$$

We suppose that $d\rho''(x_i) \neq 0$. There exists $g_0 \in G$ such that $\rho_i(g_0) \neq 1$, then $\mathrm{Ad}_{\rho'(g_0)}\big(d\rho''(x_i)\big) = \rho_i(g_0)d\rho''(x_i)$, therefore there exists a *root* ξ for the adjoint action of $\rho_1(G)$ on $\mathrm{End}V$ such that $d\rho''(x_i)$ belongs to the corresponding root space and we have $\rho_i = \xi \circ \rho'$. The number of roots ξ is finite, the result follows, using the condition (i). □

If $J \subset I$ is a finite subset such that, for all $i \in I \setminus J$, $d\rho''(x_i) = 0$, then the representation $d\rho''$ factors by $\mathrm{Lib}\big((x_j)_{j\in J}\big)$ and, as the image of $d\rho''$ is nilpotent, it factors by $\hat{\mathrm{Lib}}\big((x_j)_{j\in J}\big)$. Therefore the natural map:

$$\mathrm{Lib}\big((x_i)_{i\in I}\big) \to \mathrm{Lib}^\dagger\big((x_i)_{i\in I}\big)$$

induces an isomorphism:

$$\mathrm{Rep}\Big(\mathrm{Lib}^\dagger\big((x_i)_{i\in I}\big) \rtimes G\Big) \to \mathrm{Rep}\Big(\mathrm{Lib}\big((x_i)_{i\in I}\big) \rtimes G\Big).$$

PROPOSITION .3. – *Under the above conditions, the Tannakian group of the Tannakian category Rep*$\Big(\mathrm{Lib}\big((x_i)_{i\in I}\big) \rtimes G\Big)$ *is isomorphic to* $\mathrm{Lib}^\dagger\big((x_i)_{i\in I}\big) \rtimes G$. *More precisely, if we have a G-equivariant morphism of prounipotent proalgebraic Lie algebras* $\varphi : \mathrm{Lib}^\dagger\big((x_i)_{i\in I} \to \Lambda$ *inducing an isomorphism:*

$$\mathrm{Rep}\Big(\Lambda \rtimes G\Big) \to \mathrm{Rep}\Big(\mathrm{Lib}^\dagger\big((x_i)_{i\in I}\big) \rtimes G\Big),$$

then φ *is an isomorphism.*

EXAMPLE .4. – Our main Example is:

$$I := \{\iota = (\delta, \bar{c}, i) | (\delta, \bar{c}) \in \mathbf{N}^* \times \mathbf{E}_q, i = 1, \ldots, \delta\} \cup \{0\},$$

with $x_\iota := \dot{\Delta}_i^{(\delta,\bar{c})}$ if $\iota \neq 0$ and $x_0 := \dot{\Delta}^{(0)}$. Then $L := \mathrm{Lib}\big((x_\iota)_{\iota\in I}\big)$, $G := G_{p,1,s}^{(0)}$. The weights ρ_i are defined by:

$$\rho_\iota := \delta\bar{c},$$

$\mathbf{E}_q$ being interpreted as the group of weights on $\mathrm{Hom}_{gr}(\mathbf{E}_q, \mathbf{C}^*))$ if $\iota \neq 0$ and $\rho_1 := 1$.

It is easy to check that the conditions (i), (ii) are satisfied.

Using the Proposition .3, we prove that

$$L^\dagger \to \tilde{\mathfrak{st}}$$

is an isomorphism of pronilpotent proalgebraic Lie algebras. It follows that

$$\exp(L^\dagger) \rtimes G_{p,1,s}^{(0)} \to \exp(\tilde{\mathfrak{st}}) \rtimes G_{p,1,s}^{(0)} = G_1^{(0)}$$

is an isomorphism of proalgebraic groups, giving a *transcendental* explicit description of the q-difference universal local Galois group $G_1^{(0)}$.

Variations. – It is possible to use (more complicated) variants of the above formalism for various problems of local classification of dynamical systems.

1. Local classification of meromorphic linear differential equations. In that case condition (ii) is not satisfied.
2. Local classification of meromorphic linear difference equations.
3. Local classification of meromorphic saddle nodes in the plane. In that case it is necessary to use some *infinite dimensional* representations. As an exercise the reader can explicit this example using the dictionary between the Martinet-Ramis classification and the Écalle resurgent classification detailed in [34].

There are also some analogies with the wild ramification phenomena in the classical Galois theory of local fields, but that is another story.

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