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OPTIMAL INTEGRAL PINCHING RESULTS

BY Vincent BOUR AND Gilles CARRON

ABSTRACT. – In this article, we generalize the classical Bochner-Weitzenböck theorem for manifolds satisfying an integral pinching on the curvature. We obtain the vanishing of Betti numbers under integral pinching assumptions on the curvature, and characterize the equality case. In particular, we reprove and extend to higher degrees and higher dimensions a number of integral pinching results obtained by M. Gursky for four-dimensional closed manifolds.

RÉSUMÉ. – La formule de Bochner-Weitzenböck implique qu'une variété riemannienne compacte dont l'opérateur de courbure est strictement positif a tous ses nombres de Betti triviaux. Nous obtenons un tel résultat d'annulation sous des hypothèses de pincement intégral sur la courbure. Nos résultats sont optimaux et nous analysons les cas d'égalités. Il s'agit d'une extension à la dimension supérieure d'un résultat de M. Gursky.

1. Introduction

The Bochner method has led to important relations between the topology and the geometry of Riemannian manifolds (see [5] for instance). The original theorem of S. Bochner asserts that a closed n-dimensional Riemannian manifold with nonnegative Ricci curvature has a first Betti number smaller than n. The technique used by S. Bochner has been refined, and the result extended to Betti numbers of higher degrees and to various notions of positive curvature. For instance S. Gallot and D. Meyer proved in [24] that the Betti numbers of a closed n-dimensional manifold with nonnegative curvature operator must be smaller than those of the torus of dimension n. More precisely, they proved that if a closed Riemannian manifold (M^n, g) has a nonnegative curvature operator, i.e., if

$$(1.1) \qquad \rho_g \leq \frac{1}{n(n-1)} R_g,$$

where $-\rho_g$ stands for the lowest eigenvalue of the traceless curvature operator and R_g is the scalar curvature of (M^n, g), then for all $1 \leq k \leq \frac{n}{2}$,

   – either its kth Betti number $b_k(M^n)$ vanishes,

– or equality holds in (1.1), $1 \leq b_k \leq \binom{n}{k}$ and every harmonic k-form is parallel.

Recently, using the Ricci flow, C. Böhm and B. Wilking proved that a Riemannian manifold with positive curvature operator (i.e., which satisfies the strict inequality in (1.1)) is not only a homological sphere, but is in fact diffeomorphic to a spherical space form ([6]). A little while later, S. Brendle and R. Schoen proved that this is still true for manifolds with 1/4-pinched sectional curvature ([13, 12]).

In 1998, in his paper [25], M. Gursky obtained several Bochner's type theorems in dimension four. The striking fact in his work is that the assumption on the curvature is only required in an integral sense. He later refined part of his results in [26].

Our formulation of M. Gursky's results will be given in term of the Yamabe invariant

$$
\mathrm{Y}(M,g) := \inf_{\substack{\varphi \in C_0^\infty(M) \\ \varphi \neq 0}} \frac{\int_M \left[\frac{4(n-1)}{n-2}|d\varphi|^2 + R_g\varphi^2 \right] dv_g}{\left(\int_M \varphi^{\frac{2n}{n-2}} dv_g \right)^{\frac{n-2}{n}}}.
$$

The Yamabe invariant is a conformal invariant: if u is a smooth function, then

$$
\mathrm{Y}(M,g) = \mathrm{Y}\left(M, e^{2u}g\right),
$$

hence it only depends on the conformal class $[g] = \left\{ e^{2u}g, \ u \in C^\infty(M) \right\}$ of the metric g.

When M is closed, the Yamabe invariant has the following geometric interpretation:

$$
\mathrm{Y}(M,[g]) = \inf_{\tilde{g} \in [g]} \left\{ \frac{1}{\mathrm{vol}(M,\tilde{g})^{1-\frac{2}{n}}} \int_M R_{\tilde{g}} dv_{\tilde{g}} \right\}.
$$

According to the work of H. Yamabe, N. Trudinger, T. Aubin and R. Schoen, we can always find a metric $\tilde{g} \in [g]$ conformally equivalent to g such that

$$
\frac{1}{\mathrm{vol}(M,\tilde{g})^{1-\frac{2}{n}}} \int_M R_{\tilde{g}} dv_{\tilde{g}} = \mathrm{Y}(M,[g]).
$$

The scalar curvature of such a metric $\tilde{g}$ is constant, and is equal to

$$
R_{\tilde{g}} = \frac{\mathrm{Y}(M^n,[g])}{\mathrm{vol}(M^n,g)^{\frac{2}{n}}}.
$$

We call such a metric a *Yamabe minimizer*. Using the Hölder inequality, we see that we always have

$$
\mathrm{Y}(M^n,[g]) \leq \|R_g\|_{L^{\frac{n}{2}}},
$$

with equality if and only if g is a Yamabe minimizer.

We can state two particular results of M. Gursky's articles [25] and [26] as follows:

THEOREM 1.1. – *Assume that (M^4,g) is a closed oriented manifold with positive Yamabe invariant.*

i) *If the traceless part of the Ricci curvature satisfies*

(1.2)
$$
\int_M \left| \overset{\circ}{\mathrm{Ric}}_g \right|^2 dv_g \leq \frac{1}{12}\mathrm{Y}(M^4,[g])^2,
$$

   then

   – *either its first Betti number $b_1(M^4)$ vanishes,*
   – *or equality holds in (1.2), $b_1 = 1$, g is a Yamabe minimizer and (M^4,g) is conformally equivalent to a quotient of $\mathbb{S}^3 \times \mathbb{R}$.*

ii) *If the Weyl curvature satisfies*

(1.3)
$$\int_M |W_g|^2 \, dv_g \leq \frac{1}{24} \, \mathrm{Y}(M^4, [g])^2,$$

then

- *either its second Betti number $b_2(M^4)$ vanishes,*
- *or equality holds in (1.3), $b_2 = 1$ and (M^4, g) is conformally equivalent to $\mathbb{P}^2(\mathbb{C})$ endowed with the Fubini-Study metric.*

The norms of the curvature tensors are taken by considering them as symmetric operators on differential forms, for instance with the Einstein summation convention we have

$$|W|^2 = \frac{1}{4} W_{ijkl} W^{ijkl} \quad \text{and} \quad |\mathrm{Ric}|^2 = \mathrm{Ric}_{ij} \mathrm{Ric}^{ij}.$$

M. Gursky proved these two results by finding a good metric in the conformal class of g, for which some pointwise pinching holds. Then, by combining a Bochner-Weitzenböck equation with the pointwise pinching, he was able to prove the vanishing of harmonic forms.

The purpose of the article is to prove several generalizations of M. Gursky's result. Instead of trying to obtain a pointwise pinching, we will take advantage of the Sobolev inequality induced by the positivity of the Yamabe invariant. We first prove an integral version of the classical Bochner-Weitzenböck theorem (Theorem 2.2), which allows us to show that a large part of the Bochner theorem of S. Gallot and D. Meyer on the Betti numbers of manifolds with nonnegative curvature operator remains true if we only make the assumption in an integral sense:

THEOREM A. – *If (M^n, g), $n \geq 4$ is a closed Riemannian manifold such that*

(1.4)
$$\|\rho_g\|_{L^{\frac{n}{2}}} \leq \frac{1}{n(n-1)} \, \mathrm{Y}(M^n, [g]),$$

then for all $1 \leq k \leq \frac{n-3}{2}$ or $k = \frac{n}{2}$,

- *either its kth Betti number $b_k(M^n)$ vanishes,*
- *or equality holds in (1.4) and (up to a conformal change in the case $k = \frac{n}{2}$) the pointwise equality $\rho_g = \frac{1}{n(n-1)} R_g$ holds, $1 \leq b_k \leq \binom{n}{k}$, every harmonic k-form is parallel and g is a Yamabe minimizer.*

REMARK 1.2. – In Theorem A, as well as in the other theorems of the article, the two cases are not mutually exclusive, i.e., equality can hold in (1.4) while a number of Betti numbers vanish.

We also obtain an alternative proof of Theorem 1.1 based on our integral Bochner-Weitzenböck theorem, and several generalizations of M. Gursky's result to higher dimensions and higher degrees. In particular, we prove the following extension to higher dimensions of the first part of Theorem 1.1:

THEOREM B. – *If (M^n, g), $n \geq 5$, is a compact Riemannian manifold with positive Yamabe invariant such that*

(1.5)
$$\left\| \mathring{\mathrm{Ric}}_g \right\|_{L^{\frac{n}{2}}} \leq \frac{1}{\sqrt{n(n-1)}} \, \mathrm{Y}(M^n, [g]),$$

then

- *either its first Betti number $b_1(M^n)$ vanishes,*
- *or equality holds in (1.5), $b_1 = 1$, and there exists an Einstein manifold (N^{n-1}, h) with positive scalar curvature such that (M^n, g) is isometric to a quotient of the Riemannian product*

$$(N^{n-1} \times \mathbb{R}, h + (dt)^2).$$

We prove an analogue of the second part of Theorem 1.1 in dimension 6:

THEOREM C. – *If (M^6, g) is a compact Riemannian manifold with positive Yamabe invariant such that*

$$(1.6) \qquad \|W_g\|_{L^3} \leq \frac{1}{2\sqrt{10}} \, Y(M^6, [g]),$$

then

- *either its third Betti number $b_3(M^6)$ vanishes,*
- *or equality holds in (1.6), $b_3 = 2$, and there exist two positive numbers a and b such that (M^6, g) is conformally equivalent to a quotient of $\left(\mathbb{S}^3 \times \mathbb{S}^3, a\, g_{\mathbb{S}^3} + b\, g_{\mathbb{S}^3}\right)$.*

And more generally, we obtain the following result (the constants $a_{n,k}$ and $b_{n,k}$ are defined in Section 3):

THEOREM D. – *If (M^n, g) is a compact Riemannian manifold with positive Yamabe invariant such that for some integer $1 \leq k \leq \frac{n}{2}$, $k \neq \frac{n-1}{2}$, the following pinching holds:*

$$(1.7) \qquad \left(a_{n,k} \|W\|_{\frac{n}{2}}^2 + b_{n,k} \|\mathring{\mathrm{Ric}}\|_{\frac{n}{2}}^2\right)^{\frac{1}{2}} \leq \frac{k(n-k)}{n(n-1)} \, Y(M, [g]),$$

then

- *either its kth Betti number $b_k(M^n)$ vanishes,*
- *or $n = 4$ and equality holds in Theorem 1.1,*
- *or $k = 1$ and equality holds in Theorem B,*
- *or $k = 2$, $n \geq 7$ and (M^n, g) is isometric to a quotient of: $a\left(\mathbb{S}^2 \times \frac{1}{n-5}\mathbb{S}^{n-2}\right)$,*
- *or $k = 3$, $n = 6$ and equality holds in Theorem C.*

It should be noticed that for a closed four-dimensional manifold (M^4, g), the condition

$$\int_M |W_g|^2 \, dv_g + \frac{1}{2} \int_M |\mathring{\mathrm{Ric}}_g|^2 \, dv_g \leq \frac{1}{24} \int_M R_g^2 \, dv_g$$

is conformally invariant, hence choosing g to be a Yamabe minimizer, one can state the following corollary of Theorem 1.1:

COROLLARY 1.3. – *If (M^4, g) is a closed Riemannian manifold with positive Yamabe invariant such that*

$$(1.8) \qquad \int_M |W_g|^2 \, dv_g + \frac{1}{2} \int_M |\mathring{\mathrm{Ric}}_g|^2 \, dv_g \leq \frac{1}{24} \int_M R_g^2 \, dv_g,$$

then

 i) *either the Betti numbers $b_1(M^4)$ and $b_2(M^4)$ vanish,*

ii) *or equality holds in* (1.8), *the first Betti number vanishes,* $b_2 = 1$ *and* (M^4, g) *is conformally equivalent to* $\mathbb{P}^2(\mathbb{C})$ *endowed with the Fubini-Study metric (up to orientation),*

iii) *or equality holds in* (1.8), *the second Betti number vanishes,* $b_1 = 1$, *and* (M^4, g) *is conformally equivalent to a quotient of* $\mathbb{S}^3 \times \mathbb{R}$.

A. Chang, M. Gursky and P. Yang proved in [21, 20] that when the strict equality holds in (1.8), the manifold is not only a homological sphere, but is in fact diffeomorphic to a quotient of the round sphere $\mathbb{S}^4$.

In [7], the first author has been able to recover part of this result by using the gradient flow of a quadratic curvature functional. An important step in the proof is to rule out the formation of singularities by a blow-up analysis: if a singularity occurs along the flow, the curvature must blow up, and one can consider a sequence of metrics near the singular time with curvature going to infinity. After a suitable dilatation, this sequence actually converges to a "singularity model", which is a non-compact manifold satisfying an integral pinching condition. The classification of the singularities of those flows is therefore directly related to integral pinching results on non-compact manifolds. In Section 8, we will prove the following extension of Theorem B to non-compact manifolds:

THEOREM E. – *Let* (M^n, g), $n \geq 4$, *be a complete non-compact Riemannian manifold with positive Yamabe invariant. Assume that the lowest eigenvalue of the Ricci curvature satisfies* $\mathrm{Ric}_- \in L^p$ *for some* $p > \frac{n}{2}$, *and assume that* $R_g \in L^{\frac{n}{2}}$. *If*

$$(1.9) \qquad \left\| \mathring{\mathrm{Ric}}_g \right\|_{L^{\frac{n}{2}}} + \frac{n-4}{4\sqrt{n(n-1)}} \|R_g\|_{L^{\frac{n}{2}}} \leq \frac{n}{4} \frac{1}{\sqrt{n(n-1)}} \mathrm{Y}(M^n, [g]),$$

then

- *either* $H^1_c(M, \mathbb{Z}) = \{0\}$ *and in particular* M *has only one end,*
- *or equality holds in* (1.9), *and there exists an Einstein manifold* (N^{n-1}, h) *with positive scalar curvature and* $\alpha > 0$ *such that* (M^n, g) *or one of its two-fold coverings is isometric to*

$$\left(N^{n-1} \times \mathbb{R}, \, \alpha \cosh^2(t) \left(h + (dt)^2 \right) \right).$$

The structure of the paper is the following: in the next section, we recall the Bochner-Weitzenböck formula and state the extension of the Bochner-Weitzenböck theorem to manifolds with an integral pinched curvature. In Section 3, we give estimates on the lowest eigenvalue of the traceless Bochner-Weitzenböck curvature and analyze the equality case. In Section 4, we define a modified Yamabe invariant and prove a number of results related to the Yamabe invariant. In Section 5, we recall the refined Kato inequality for harmonic forms and its equality case for 1-forms. In Section 6, we prove the extended Bochner-Weitzenböck theorem. In Section 7, we prove the other optimal integral pinching theorems, and in the last section, we deal with non-compact manifolds and prove Theorem E.

2. The Bochner-Weitzenböck formula

We recall that a harmonic k-form ξ satisfies the Bochner-Weitzenböck formula

$$\langle \nabla^* \nabla \xi, \xi \rangle = -\langle \mathscr{R}_k \xi, \xi \rangle$$

where the Bochner-Weitzenböck curvature:

$$\mathscr{R}_k(x) \colon \Lambda^k T_x^* M \to \Lambda^k T_x^* M$$

is a symmetric operator that can be expressed by using the curvature operator. The trace of $\mathscr{R}_k$ is given by

$$\operatorname{tr}(\mathscr{R}_k) = \left(\dim \Lambda^k T_x^* M\right) \frac{k(n-k)}{n(n-1)} R_g.$$

We let $-r_k$ be the lowest eigenvalue of the traceless part of the Bochner-Weitzenböck curvature.

Since the nonnegativity of $\mathscr{R}_k$ is equivalent to

$$r_k \leq \frac{k(n-k)}{n(n-1)} R_g,$$

the classical Bochner-Weitzenböck theorem can be stated as follows:

THEOREM 2.1. – *Let (M^n, g), $n \geq 2$, be a compact Riemannian manifold. If*

$$(2.1) \qquad r_k \leq \frac{k(n-k)}{n(n-1)} R_g,$$

then

  – *either its kth Betti number $b_k(M^n)$ vanishes,*
  – *or equality holds in (2.1), $1 \leq b_k \leq \binom{n}{k}$ and every harmonic k-form is parallel.*

In Section 6, we will prove the following integral version of Theorem 2.1:

THEOREM 2.2. – *If (M^n, g), $n \geq 4$, is a compact Riemannian manifold such that for some integer $1 \leq k \leq \frac{n-3}{2}$ or $k = \frac{n}{2}$ the following pinching holds:*

$$(2.2) \qquad \|r_k\|_{L^{\frac{n}{2}}} \leq \frac{k(n-k)}{n(n-1)} \mathrm{Y}(M, [g]),$$

then

  – *either its kth Betti number $b_k(M^n)$ vanishes,*
  – *or equality holds in (2.2) and (up to a conformal change in the case $k = \frac{n}{2}$) the pointwise equality $r_k = \frac{k(n-k)}{n(n-1)} R_g$ holds, $1 \leq b_k \leq \binom{n}{k}$, every harmonic k-form is parallel and g is a Yamabe minimizer.*

According to [24], for all $1 \leq k \leq n-1$, we have $r_k \leq k(n-k)\rho_g$, thus Theorem A is a direct consequence of this theorem.

In dimension four, if we let w_g^+ be the largest eigenvalue of the self-dual part W_g^+ of the Weyl curvature and b_2^+ be the dimension of the self-dual harmonic 2-forms, we obtain the following result:

THEOREM 2.3. – *If (M^4, g) is a compact oriented Riemannian manifold such that*

(2.3) $$\left\| w_g^+ \right\|_{L^2} \leq \frac{1}{6} Y(M^4, [g]),$$

then

– *either $b_2^+(M^4) = 0$,*
– *or equality holds in (2.3), $1 \leq b_2^+ \leq 3$ and for every self-dual harmonic 2-form ω, there is a Yamabe minimizer $\tilde{g}$ in $[g]$ such that ω is Kähler for $\tilde{g}$.*

Conversely, according to [22], for any metric conformally equivalent to one which is Yamabe and Kähler, equality holds in (2.3).

2.1. Examples of manifolds for which equality holds in (2.2)

Equality holds in (2.2) for any metric with nonnegative $\mathscr{R}_k$ which is a positive Yamabe minimizer, as soon as $b_k \geq 1$. According to [24], we can construct examples of manifolds with nonnegative $\mathscr{R}_k$ by taking products of manifolds with nonnegative curvature operators. According to [10, IV.2], if the product is an Einstein manifold, it will be a Yamabe minimizer.

Let (M^n, g) be a product of round spheres and projective spaces

$$(\mathbb{S}^{n_1}, g_1) \times \cdots \times (\mathbb{S}^{n_p}, g_p) \times (\mathbb{P}^{m_1}(\mathbb{C}), h_1) \times \cdots \times (\mathbb{P}^{m_q}(\mathbb{C}), h_q),$$

with $n_i \geq 2$. Then (M, g) has a nonnegative curvature operator. For (M, g) to be Einstein, we have to take $R_{g_i} = \alpha \frac{n_i}{n}$ and $R_{h_i} = \alpha \frac{2m_i}{n}$ for some $\alpha > 0$.

If for some $0 \leq p' \leq p$ and $0 \leq m'_j \leq m_j$

$$\sum_{i=1}^{p'} n_i + 2 \sum_{j=1}^{q} m'_j = k,$$

then $b_k \geq 1$ and equality holds in (2.2). Hence, for all $k \geq 2$, there exist manifolds for which equality holds in (2.2).

For $k = 1$, according to [35], the quotients of $\mathbb{S}^{n-1} \times \mathbb{R}$ by a group of transformations generated by isometries of S^{n-1} and a translation of parameter $T > 0$ are Yamabe minimizing if and only if $T^2 \leq \frac{4\pi^2}{n-2}$. For those manifolds, the equality holds in (2.2) and in (1.5).

2.2. The Bochner-Weitzenböck curvature in more details

The symmetric operator $\mathscr{R}_k$ can be seen as a double form of degree (k, k) (see [29, 8, 30, 31]). Using the fact that the curvature can be seen as a double form of degree $(2, 2)$, the fact that the metric can be seen as a $(1, 1)$-form and the fact that the wedge product induces an algebra structure on the space of double forms (the Kulkarni-Nomizu product), there is a convenient way to write the Bochner-Weitzenböck curvature in degree $k \in [2, n/2]$:

$$\mathscr{R}_k = \mathscr{W}_k + \mathscr{Z}_k + \mathscr{A}_k$$

with

$$\mathscr{W}_k = -2 \frac{g^{k-2}}{(k-2)!} \cdot W_g, \quad \mathscr{Z}_k = \frac{n-2k}{n-2} \frac{g^{k-1}}{(k-1)!} \cdot \mathring{\mathrm{Ric}}_g, \quad \mathscr{A}_k = \frac{k(n-k)}{n(n-1)} R_g \operatorname{Id}_{\Lambda^k T^* M},$$

where $.$ is the product on double-forms, g^j is the metric to the power j with respect to this product, W_g is the Weyl curvature, and $\overset{\circ}{\mathrm{Ric}}_{\mathrm{g}} = \mathrm{Ric}_g - \frac{R_g}{n} g$ is the traceless part of the Ricci curvature. On 1-forms, we have

$$\mathscr{R}_1 = \mathrm{Ric}_g = \overset{\circ}{\mathrm{Ric}}_{\mathrm{g}} + \frac{R_g}{n} g.$$

3. Comparison between the first eigenvalue and the norm of curvature operators

In order to obtain estimates on r_k, we will use the following lemma:

LEMMA 3.1. – *If $A : E \to E$ is a traceless self-adjoint endomorphism on a Euclidean space E of dimension d, then its lowest eigenvalue a satisfies*

$$a^2 \leq \frac{d-1}{d} |A|^2 ,$$

and equality holds if and only if the spectrum of A is $\{-\nu, \frac{1}{d-1}\nu\}$ with $\nu \geq 0$ and $\frac{1}{d-1}\nu$ of multiplicity $d-1$.

Proof. – By a simple Lagrange multiplier argument, we see that:

$$\left(\inf\left\{ \lambda_1 , \sum_{i=1}^{d} \lambda_i^2 = 1 \text{ and } \sum_{i=1}^{d} \lambda_i = 0 \right\} \right)^2 = \frac{d-1}{d}. \qquad \square$$

For $1 \leq k \leq \frac{n-1}{2}$, let us define the constants $a_{n,k}$ and $b_{n,k}$ by

$$a_{n,k} = \left(\binom{n}{k} - 1 \right) \frac{k(n-k)}{n(n-1)} \frac{4(k-1)(n-k-1)}{(n-2)(n-3)},$$

$$b_{n,k} = \left(\binom{n}{k} - 1 \right) \frac{k(n-k)}{n(n-1)} \frac{(n-2k)^2}{(n-2)^2}.$$

LEMMA 3.2. – *If $1 \leq k \leq \frac{n-1}{2}$, then*

$$r_k^2 \leq a_{n,k} |W|^2 + b_{n,k} \big| \overset{\circ}{\mathrm{Ric}} \big|^2$$

and equality holds if and only if there exist a k-form u and a real number λ such that

$$\mathscr{R}_k = \lambda \,\mathrm{Id} - u \otimes u.$$

Proof. – We apply Lemma 3.1 to $\mathscr{W}_k + \mathscr{Z}_k$, and use the fact that for a traceless operator T on k-forms

$$\left| \frac{g^j}{j!} . T \right| = \frac{1}{j!} \left\langle \mathrm{c}^j \frac{g^j}{j!} . T \middle| T \right\rangle = \binom{n-2k}{j} |T|^2 ,$$

where c is the contraction operator defined in [30]. $\qquad \square$

When $k = n/2$ we can refine this inequality by using the fact that the Hodge star operator commutes with $\mathscr{R}_{n/2}$ and the fact that the square of the Hodge star operator on $n/2$-forms is $(-1)^{n/2} \,\mathrm{Id}$.

Let $\Lambda_{\pm}^{n/2} T_x^* M$ be the eigenspaces of the Hodge star operator and $\mathscr{R}_{\pm,n/2}$ be the restriction of the Bochner-Weitzenböck curvature to $\Lambda_{\pm}^{n/2} T_x^* M$.

We define

$$a_{n,n/2} = \begin{cases} \frac{n(n-2)}{4(n-1)(n-3)} \left(\binom{n}{n/2} - 2 \right) & \text{if } n/2 \text{ is even} \\ \frac{n(n-2)}{8(n-1)(n-3)} \left(\binom{n}{n/2} - 2 \right) & \text{if } n/2 \text{ is odd.} \end{cases}$$

LEMMA 3.3. – *We have*

(3.1) $$r_{n/2}^2 \leq a_{n,n/2} |W|^2$$

and equality holds if and only if

- *when $n/2$ is odd: there exist an $n/2$-form u and a real number λ such that*

$$\mathscr{R}_{n/2} = \lambda \operatorname{Id} - u \otimes u - *u \otimes *u,$$

- *when $n/2$ is even: there is $\varepsilon \in \{-,+\}$ such that the traceless part of $R_{-\epsilon,n/2}$ vanishes and there exist an $n/2$-form u such that $*u = \varepsilon u$ and a real number λ such that*

$$\mathscr{R}_{\varepsilon,n/2} = \lambda \operatorname{Id} - u \otimes u.$$

Proof. – When $n/2$ is odd, all the eigenspaces of the Bochner-Weitzenböck curvature are stable by the Hodge star operator hence they come with an even multiplicity. And when $n/2$ is even we obtain that $r_{n/2}$ is less than the lowest eigenvalue of $\mathscr{R}_{\varepsilon,n/2}$. $\square$

Characterization of the equality case

An important feature of the Bochner-Weitzenböck curvature is that it satisfies the first Bianchi identity. Seeing once again $\mathscr{R}_k$ as a symmetric operator

$$\mathscr{R}_k \colon \Lambda^k T_x^* M \to \Lambda^k T_x^* M,$$

the first Bianchi identity asserts that if $(\theta_i)_i$ is an orthonormal basis of $(T_x^* M, g)$ then

$$\forall \alpha \in \Lambda^{k-1} T_x^* M, \quad \sum_i \theta_i \wedge \mathscr{R}_k (\theta_i \wedge \alpha) = 0.$$

We now assume that there exist a real number λ and a k-form $u \in \Lambda^k T_x^* M$ such that

$$\mathscr{R}_k = \lambda \operatorname{Id} - u \otimes u.$$

We get that for any orthonormal basis $(e_i)_i$, if we let $(\theta_i)_i$ be its dual basis, then (see [29])

$$\sum_i u \wedge \theta_i \otimes e_i \llcorner u = 0.$$

We introduce the orthogonal decomposition $T_x M = V \oplus V^\perp$ where

$$V^\perp = \{v, v \llcorner u = 0\},$$

and choose an orthonormal basis $(e_i)_i$ of $T_x M$ diagonalizing the quadratic form

$$v \mapsto |v \llcorner u|^2,$$

and such that $(e_i)_{1 \leq i \leq \ell}$ is a basis of V. Then $\{e_i \llcorner u\}_{1 \leq i \leq \ell}$ is an orthogonal family of $\Lambda^{k-1} T^* M$.

From the identity

$$\sum_i u \wedge \theta_i \otimes e_i \llcorner u = 0,$$

we deduce that $i \in \{1, \ldots, \ell\} \Rightarrow u \wedge \theta_i = 0$. Hence $\ell = k$ and

$$u = |u| \theta_1 \wedge \cdots \wedge \theta_k = |u| dv_V.$$

We can go one step further. Indeed if $k \in [2, \frac{n-1}{2}]$, the curvature operator is uniquely determined by the Bochner-Weitzenböck curvature: the components of the curvature operator can be expressed by taking contractions of $\mathscr{R}_k$ (see [31, Theorem 4.4]).

We first see that if $T_x M = V \oplus V^\perp$ and $u = dv_V$ then

$$c(u \otimes u) = *_V g_V,$$

where g_V is the metric on V viewed as a double $(1,1)$-form on V and

$$*_V : \Lambda^{(1,1)} V^* \to \Lambda^{(k-1,k-1)} V^*$$

is the Hodge star acting on double forms of V. The computations of [31, Theorem 4.4] imply that the traceless part of $c^{k-1}(\mathscr{R}_k)$ is proportional to the traceless part of the Ricci curvature, hence the Ricci curvature is a linear combination of g_V and $g_{V^\perp}$; we also get that $c^{k-2}(\mathscr{R}_k)$ is a linear combination of $g.$Ric, of g^2 and of the curvature operator. Hence in our case, we easily get that there are numbers $\alpha = \alpha(x)$, $\beta = \beta(x)$ and $\gamma = \gamma(x)$ such that the curvature operator at x is

$$\alpha \frac{g_V^2}{2} + \beta \frac{g_{V^\perp}^2}{2} + \gamma \frac{g^2}{2}.$$

Hence, using the orthogonal decomposition

$$\Lambda^k T^* \tilde{M} = \bigoplus_{j=0}^{k} \Lambda^{k-j} V^* \otimes \Lambda^j \left(V^\perp \right)^*,$$

we find that the eigenvalues of the Bochner-Weitzenböck curvature $\mathscr{R}_k$ are

$$\alpha j(k-j) + \beta j(n-k-j) + \gamma k(n-k),$$

with multiplicity $\binom{k}{j}\binom{n-k}{j}$, where $j \in \{0, \dots, k\}$. But the assumption asserts that $\mathscr{R}_k$ has only two eigenvalues and that the lowest one has multiplicity 1. The only possible case is $k = 2$ and $\alpha = (n-5)\beta$. Moreover, $\beta \geq 0$, since the lowest eigenvalue of the traceless part of $\mathscr{R}_k$ is a negative multiple of β. Consequently we have:

PROPOSITION 3.4. – *If there is a non-zero k-form u such that*

$$\mathscr{R}_k(x) = \lambda Id - u \otimes u$$

then $k = 2$ and $T_x M$ has an orthogonal decomposition

$$T_x M = V \oplus V^\perp,$$

with $V^\perp = \{v, v \lrcorner u = 0\}$ of codimension 2. Moreover, u is colinear to the volume form of V, and the curvature operator is of the form

$$(n-5)\beta \frac{g_V^2}{2} + \beta \frac{g_{V^\perp}^2}{2} + \gamma \frac{g^2}{2},$$

with $\beta \geq 0$.

When $n/2$ is odd, we use the complex structure given by the Hodge star operator on $n/2$-forms and obtain:

PROPOSITION 3.5. – *If $n/2$ is odd and if there is a non-zero $n/2$-form u such that*

$$\mathcal{R}_{n/2} = \lambda \operatorname{Id} - u \otimes u - *u \otimes *u,$$

then $T_x M$ has an orthogonal decomposition

$$T_x M = V \oplus V^\perp,$$

with $V = \{v, v \wedge u = 0\}$ and $V^\perp = \{v, v \llcorner u = 0\}$ of dimension $n/2$.

Moreover, u is colinear to the volume form of V and $*u$ is colinear to the volume form of $V^\perp$.

Indeed, with the same orthogonal decomposition $T_x M = V \oplus V^\perp$ as before, with

$$V^\perp = \{v, v \llcorner u = 0\},$$

we get that for any vector $w \in V^\perp$,

$$w^\flat \wedge *u = *(w \llcorner u) = 0.$$

Hence there is a $(\ell - n/2)$-form $\psi \in \Lambda^{\ell - n/2} V^*$ such that $*u = \psi \wedge dv_{V^\perp}$ and $u = *_V \psi$.

The Bianchi identity implies that

$$0 = \sum_{i=1}^{\ell} *_V \psi \wedge \theta_i \otimes e_i \llcorner *_V \psi \pm \sum_{i=1}^{\ell} \psi \wedge dv_{V^\perp} \wedge \theta_i \otimes e_i \llcorner (\psi \wedge dv_{V^\perp}).$$

Because $(e_i \llcorner *_V \psi)_{1 \le i \le \ell} \cup \{e_i \llcorner (\psi \wedge dv_{V^\perp})\}_{1 \le i \le \ell}$ is an orthogonal family, we conclude that $\ell = n/2$ and $\psi = 1$.

And when $n/2$ is even, we have:

PROPOSITION 3.6. – *Assume that $n/2$ is even, that for $\varepsilon \in \{-, +\}$ the traceless part of $R_{-\epsilon, n/2}$ vanishes and that there is a non-zero $n/2$-form u such that $*u = \varepsilon u$ and*

$$\mathcal{R}_{\varepsilon, n/2} = \lambda \operatorname{Id} - u \otimes u.$$

Then $n = 4$ and u is colinear to $g(J., .)$ where J is a unitary complex structure on $T_x M$.

Indeed, we obtain that the Bianchi operator applied to $u \otimes u$ is a multiple of the Bianchi operator applied to the Hodge star operator. But the Bianchi operator applied to the Hodge star operator is a multiple of the Hodge star operator. Hence, if (e_i) is an orthonormal basis of $T_x M$ then $(e_i \llcorner u)_i$ is a basis of $\Lambda^{n/2-1} T_x^* M$.

This can only occur when $n = 4$ and when $u = |u| g(J., .)$, where J is a unitary complex structure on $T_x M$.

4. The Yamabe invariant

We recall that when M is closed, there always exists a positive smooth function φ such that

(4.1) $$\int_M \left[\frac{4(n-1)}{n-2} |d\varphi|^2 + R_g \varphi^2 \right] dv_g = \mathrm{Y}(M, [g]), \text{ and } \int_M \varphi^{\frac{2n}{n-2}} dv_g = 1.$$

Moreover, since C_0^∞ is dense in $H_1^2(M)$ (see [3]), the infimum defining the Yamabe invariant can also be taken over $H_1^2(M)$, and any function $\varphi \in H_1^2(M)$ with $\|\varphi\|_{L^{\frac{2n}{n-2}}} = 1$ attaining the infimum is smooth, positive and solution to the Yamabe equation:

(4.2) $$\frac{4(n-1)}{n-2} \Delta_g \varphi + R_g \varphi = \mathrm{Y}(M, [g]) \varphi^{\frac{n+2}{n-2}}$$

(see [3]). We can also write that equation

$$L_g(\varphi) = \mathrm{Y}(M, [g]) \varphi^{\frac{n+2}{n-2}},$$

where L_g denotes the conformal Laplacian

$$L_g = \frac{4(n-1)}{n-2} \Delta_g + R_g,$$

and satisfies the following conformal covariance property: if $\tilde{g} = u^{\frac{4}{n-2}} g$, with u a smooth positive function, then

$$u^{\frac{n+2}{n-2}} L_{\tilde{g}}(\varphi) = L_g(u\varphi).$$

It follows in particular that

$$u^{\frac{n+2}{n-2}} R_{\tilde{g}} = \frac{4(n-1)}{n-2} \Delta_g u + R_g u.$$

Therefore, if u is a positive smooth solution of (4.2), then the metric $\tilde{g} = u^{\frac{4}{n-2}} g$ is a Yamabe minimizer.

4.1. The modified Yamabe invariant

For $\beta \geq 0$, we introduce the modified Yamabe invariant:

(4.3) $$\mathrm{Y}_g(\beta) := \inf_{\substack{\varphi \in C_0^\infty(M) \\ \varphi \neq 0}} \frac{\int_M \left(\frac{4(n-1)}{n-2} |d\varphi|^2 + \beta R_g \varphi^2 \right) dv_g}{\left(\int_M \varphi^{\frac{2n}{n-2}} dv_g \right)^{\frac{n-2}{n}}}.$$

In particular, for $\beta = 1$, this modified Yamabe invariant is the Yamabe invariant:

$$\mathrm{Y}_g(1) = \mathrm{Y}(M, [g]).$$

The function $\beta \to \mathrm{Y}_g(\beta)$ is an infimum of affine functions of β, hence it is concave and for all $\beta \in [0, 1]$ we obtain

$$(1 - \beta) \mathrm{Y}_g(0) + \beta \mathrm{Y}(M, [g]) \leq \mathrm{Y}_g(\beta).$$

When (M, g) is closed, $\mathrm{Y}_g(0) = 0$ and we have:

PROPOSITION 4.1. – *If (M^n, g) is a closed Riemannian manifold, then*

$$(4.4) \qquad \beta \, \mathrm{Y}(M, [g]) \le \mathrm{Y}_g(\beta).$$

If $\beta \in (0, 1)$, then equality holds in this inequality if and only if g is a Yamabe minimizer, and the only functions attaining the infimum in (4.3) are constant functions.

Proof. – Since $\beta \to \mathrm{Y}_g(\beta)$ is concave, it is equal to its chord $\beta \, \mathrm{Y}(M, [g])$ at an interior point $\beta \in (0, 1)$ if and only if it is affine. Then, if for some u and some $\beta \in (0, 1)$, equality is attained in (4.3), the affine function of β corresponding to u is above the function Y_g and is equal to $\mathrm{Y}_g(\beta)$ at β, hence it must be equal to Y_g on $[0, 1]$. Therefore the function u realizes the infimum in (4.3) for all $\beta \in [0, 1]$. Taking $\beta = 0$ yields $\int_M |du|^2 \, dv_g = 0$, hence u is constant. Then, taking $\beta = 1$ shows that g is a Yamabe minimizer. $\qquad \square$

4.2. The Yamabe invariant on complete non-compact manifolds

We still define the Yamabe invariant by

$$\mathrm{Y}(M, g) := \inf_{\substack{\varphi \in C_0^\infty(M) \\ \varphi \ne 0}} \frac{\int_M \left(\frac{4(n-1)}{n-2} \, |d\varphi|^2 + R_g \varphi^2 \right) dv_g}{\left(\int_M \varphi^{\frac{2n}{n-2}} \, dv_g \right)^{\frac{n-2}{n}}}$$

and we still have for any smooth function u:

$$\mathrm{Y}(M, g) = \mathrm{Y}(M, e^{2u} g).$$

The Yamabe functional

$$\mathscr{Y}_g(\varphi) = \frac{\int_M \left(\frac{4(n-1)}{n-2} \, |d\varphi|^2 + R_g \varphi^2 \right) dv_g}{\left(\int_M \varphi^{\frac{2n}{n-2}} \, dv_g \right)^{\frac{n-2}{n}}}$$

is also well-defined when R_g is in $L^{\frac{n}{2}}(M, g)$, φ is in $L^{\frac{2n}{n-2}}(M, g)$ and $d\varphi$ is in $L^2(M, g)$. Moreover, when g is complete, $C_0^\infty(M)$ is dense in the space

$$H = \{ \varphi \in L^{\frac{2n}{n-2}}(M, g), \, |d\varphi| \in L^2(M, g) \}.$$

Therefore, we also have

$$\mathrm{Y}(M, [g]) = \inf_H \mathscr{Y}_g,$$

and any function in H with $\|\varphi\|_{L^{\frac{2n}{n-2}}} = 1$ attaining the infimum is a weak solution to the Yamabe Equation (4.2). If in addition φ is in $C^{0,\alpha}$, then by classical regularity theorems φ is smooth, and by maximum principle it is positive (see [3]).

The following Lemma is inspired by [18, Proposition 2.3]:

LEMMA 4.2. – *If (M, g) is a complete non-compact Riemannian manifold with a positive Yamabe constant and scalar curvature in $L^{\frac{n}{2}}$, then it has infinite volume and satisfies a Sobolev inequality*

$$(4.5) \qquad \|\varphi\|_{L^{\frac{2n}{n-2}}}^2 \le C \, \|d\varphi\|_{L^2}^2,$$

for some $C > 0$ and for all $\varphi \in C_0^\infty(M)$.

Proof. – Let us fix some ball $B(x_0, r) \subset M$. Since $Y(M, [g]) > 0$ and by using the Hölder inequality, for any smooth functions with support outside the ball $B(x_0, r)$, we have

$$\|\varphi\|^2_{L^{\frac{2n}{n-2}}} \leq \frac{1}{Y(M,[g])} \left(\frac{4(n-1)}{n-2} \|d\varphi\|^2_{L^2} + \left(\int_{M \setminus B(x_0,r)} |R_g|^{n/2} \, dv_g \right)^{\frac{2}{n}} \|\varphi\|^2_{L^{\frac{2n}{n-2}}} \right).$$

Since R_g is in $L^{\frac{n}{2}}(M, g)$, we can take r such that

$$\left(\int_{M \setminus B(x_0,r)} |R_g|^{n/2} \, dv_g \right)^{\frac{2}{n}} \leq \frac{Y(M,[g])}{2},$$

and we obtain the Sobolev inequality on $M \setminus B(x_0, r)$.

According to [28, Lemma 3.2], there exists a uniform bound from below on the volume of any ball $B(y, 1) \subset M \setminus B(x_0, r)$. Since (M, g) is not compact, we can find a sequence of points x_k in $M \setminus B(x_0, r)$ such that x_k is in $B(x_0, k+1) \setminus B(x_0, k)$. Then the balls $B(x_{3k}, 1)$ are two by two disjoint, and thus

$$\text{vol}_g(M) \geq \sum_{k \geq 0} \text{vol}_g(B(x_{3k}, 1)) = \infty.$$

Therefore, according to [16, Proposition 2.5], there exists C' such that the Sobolev inequality (4.5) holds on M. □

PROPOSITION 4.3. – *If (N^{n-1}, h) is an Einstein manifold with scalar curvature*

$$R_h = (n-2)(n-1),$$

then

$$\frac{Y(N \times \mathbb{R}, [h + ds^2])}{\text{vol}(N, h)^{\frac{2}{n}}} = \frac{Y(\mathbb{S}^n)}{\text{vol}(\mathbb{S}^{n-1})^{\frac{2}{n}}}.$$

Proof. – We note that equality holds if (N, h) is the round sphere $\mathbb{S}^{n-1}$. If (N, h) is not the round sphere, then by the Bishop-Gromov inequality

$$\text{vol}(N, h) < \text{vol}(\mathbb{S}^{n-1}).$$

Moreover the conformal class of $g = h + ds^2$ contains the metric $g_0 = \frac{1}{\cosh^2(s)}(h + ds^2)$, which is isometric to the spherical suspension of (N, h):

$$(N \times (0, 2\pi), \sin^2(r)h + dr^2).$$

The metric g_0 is Einstein, has constant scalar curvature equal to $n(n-1)$ and its volume is

$$\text{vol}(N, h) \frac{\text{vol}(\mathbb{S}^n)}{\text{vol}(\mathbb{S}^{n-1})}.$$

The Yamabe invariant of the cylindrical metric $h + ds^2$ hence satisfies

$$Y(N \times \mathbb{R}, [h + ds^2]) \leq \left(\text{vol}(N, h) \frac{\text{vol}(\mathbb{S}^n)}{\text{vol}(\mathbb{S}^{n-1})} \right)^{2/n} n(n-1) < Y(\mathbb{S}^n).$$

According to [1, Theorem C], there is a bounded smooth function φ on $N \times \mathbb{R}$ such that

$$\int_M \frac{4(n-1)}{n-2} |\nabla \varphi|^2 + R_g \varphi^2 dv_g = Y(M, [g]) \left(\int_M \varphi^{\frac{2n}{n-2}} dv_g \right)^{1 - \frac{2}{n}}$$

and there exist some positive real numbers c, C, α, β such that for all $x \in N$ and $s \in \mathbb{R}$

$$ce^{-\alpha|s|} \le \varphi(x, s) \le Ce^{-\beta|s|}.$$

We let $\tilde{g} = \varphi^{\frac{4}{n-2}}(h + ds^2)$ and

$$\rho(x, s) = \left(\varphi^{\frac{2}{n-2}}(x, s) \cosh(s)\right)^{-1},$$

so that

$$\tilde{g} = \rho^{-2} g_0.$$

We can now run the proof of Obata (see [10] or [34]) and show that since g_0 is Einstein, $\tilde{g}$ must also be Einstein. Indeed, if we note $\gamma_g(X) = L_X g - \frac{2}{n}(\delta_g X)g$, we have

$$\gamma_{\tilde{g}}^* \left(\frac{1}{\rho}\gamma_{g_0}(\nabla_{g_0}\rho)\right) = \frac{2}{n-2}\gamma_{\tilde{g}}^* \left(\mathring{\mathrm{Ric}}_{\tilde{g}}\right) = 0,$$

hence, with $\vec{\nu} = \frac{\nabla s}{|\nabla s|_{\tilde{g}}}$, we get for $T > 0$

$$\int_{N \times [-T, T]} \frac{1}{\rho^3} |\gamma_{g_0}(\nabla_{g_0}\rho)|_{\tilde{g}}^2 \, dv_{\tilde{g}} \le C \int_{\partial(N \times [-T, T])} \frac{1}{\rho} |\gamma_{g_0}(\nabla_{g_0}\rho)|_{\tilde{g}} |\nabla_{g_0}\rho|_{\tilde{g}} \, i_{\vec{\nu}}(dv_{\tilde{g}})$$

$$\le 2C \max_{\{-T; T\}} \int_N |\gamma_{g_0}(\nabla_{g_0}\rho)|_{g_0} |d\rho|_{g_0} \varphi^{2\frac{n-1}{n-2}} \, dv_h.$$

According to [2], the function φ is polyhomogeneous, and the boundary term goes to zero as T goes to infinity.

Then, $\nabla_{g_0}\rho$ is a conformal Killing field. But a conformal vector field of the cylindrical metric is a sum of a conformal vector field on (N, h) and a generator of the translation. If (N, h) is not the round sphere, the conformal Killing fields of (N, h) are Killing fields, hence the conformal factor must be radial. Then we have (see [1, Claim 2.13])

$$\frac{Y(N \times \mathbb{R}, [h + ds^2])}{\mathrm{vol}(N, h)^{\frac{2}{n}}} = \frac{Y(\mathbb{S}^n)}{\mathrm{vol}(\mathbb{S}^{n-1})^{\frac{2}{n}}}. \qquad \Box$$

5. The refined Kato inequality

The classical Kato inequality asserts that if ξ is a smooth k-form on a Riemannian manifold (M^n, g), then

$$|d|\xi||^2 \le |\nabla \xi|^2.$$

When ξ is moreover assumed to be harmonic, i.e., closed and co-closed

$$d\xi = d^*\xi = 0,$$

then for $k \in [0, n/2]$, the Kato inequality can be refined as

(5.1) $$\frac{n+1-k}{n-k} |d|\xi||^2 \le |\nabla \xi|^2.$$

See [9], and [11, 14] for the computation of the refined Kato constant.

The equality case in the refined Kato inequality for 1-forms

Assume that (M^n, g) is a complete Riemannian manifold, that $\xi \in C^\infty(T^*M)$ is a harmonic 1-form and that equality holds almost everywhere in the refined Kato inequality:

$$|d|\xi||^2 = \frac{n-1}{n}|\nabla\xi|^2.$$

We can locally find a primitive Φ of ξ:

$$d\Phi = \xi.$$

Then Φ is a harmonic function and in this case, the refined Kato inequality is in fact the Yau inequality for harmonic functions ([38, Lemma 2]). Moreover, passing to the normal covering $\pi \colon \widehat{M} \to M$ associated to the kernel of the homomorphism

$$\gamma \in \pi_1(M) \mapsto \int_\gamma \xi,$$

we have $\pi^*\xi = d\Phi$ for a harmonic function $\Phi \in C^\infty(\widehat{M})$ (see for instance [27]). We will now review the proof of a result of P. Li and J. Wang ([32]).

PROPOSITION 5.1. – *Assume that (M^n, g) is a complete Riemannian manifold carrying a non-constant harmonic function Φ such that almost everywhere*

$$|d|d\Phi||^2 = \frac{n-1}{n}|\nabla d\Phi|^2.$$

Then there exists a complete Riemannian manifold (N^{n-1}, h) such that (M^n, g) is isometric to $N^{n-1} \times \mathbb{R}$ endowed with a warped product metric $\eta^2(t)h + (dt)^2$. Moreover, there are constants c_1, c_2 such that:

$$\Phi(x, t) = c_1 + c_2 \int_0^t \frac{dr}{\eta(r)^{n-1}}.$$

Proof. – We assume that on $U = \{x \in M^n, d\Phi(x) \neq 0\}$

$$|d|d\Phi||^2 = \frac{n-1}{n}|\nabla d\Phi|^2.$$

Moreover, we add a constant to Φ such that the set

$$N = \{x \in U, \Phi(x) = 0\}$$

is not empty.

The equality case in the Yau's inequality implies that there is a function $a \colon U \to \mathbb{R}$ such that if we let $\vec{\nu} = \frac{\nabla\Phi}{|\nabla\Phi|}$, then in the orthogonal decomposition $T_x M^n = \ker(d\Phi) \oplus \mathbb{R}\vec{\nu}$, we have

$$\nabla d\Phi = \begin{pmatrix} a\,\mathrm{Id} & 0 \\ 0 & -(n-1)a \end{pmatrix}.$$

Therefore, we see that

$$\vec{u} \in \ker(d\Phi) \implies d_{\vec{u}}\left(|d\Phi|^2\right) = 0.$$

Hence the length of $d\Phi$ is locally constant on the regular level sets of Φ. Moreover, we have

$$\nabla_{\vec{u}}(\vec{\nu}) = \frac{1}{|d\Phi|}\left(\nabla_{\vec{u}}(\nabla\Phi) - \langle\nabla_{\vec{u}}(\nabla\Phi)|\vec{\nu}\rangle\,\vec{\nu}\right),$$

and since $\nabla_{\vec{\nu}}(\nabla\Phi)$ is in $\mathbb{R}\vec{\nu}$, $\nabla_{\vec{\nu}}(\vec{\nu}) = 0$ and the integral curves of the vector field $\vec{\nu}$ are geodesics.

We consider the map $E\colon N \times \mathbb{R} \to M^n$ given by

$$E(x,t) = \exp_x(t\vec{\nu}(x)).$$

For $x \in N$, and $\vec{u} \in \vec{\nu}^{\perp}$, $\nabla_{\vec{u}}E(x,t)$ is a Jacobi field along $E(x,t)$, hence is orthogonal to $\vec{\nu}$ for all $t \in \mathbb{R}$. Consequently, $\Phi(E(x,t))$ only depends on t and there exists a function $\psi\colon \mathbb{R} \to \mathbb{R}$ such that $\psi(0) = 0$ and for all $(x,t) \in N \times \mathbb{R}$,

$$\Phi(E(x,t)) = \psi(t).$$

We fix $K \subset N$ a compact subset, and we let (α, ω) be the maximal open set containing 0 such that $E\colon K \times (\alpha, \omega) \to M^n$ is a local diffeomorphism. Then, on $K \times (\alpha, \omega)$, we have

$$E^*(\nabla d\Phi) = \psi'' dt \otimes dt + \psi' \nabla dt.$$

Consequently,

$$\psi'' = -(n-1)(a \circ E) \quad \text{and} \quad \psi' \nabla dt = (a \circ E)(E^*g).$$

Hence $a \circ E$ only depends on t, and the hypersurfaces $K \times \{t\} \subset (K \times (\alpha, \omega), E^*g)$ are totally umbilical. Therefore, we get that on $K \times (\alpha, \omega)$,

$$E^*g = \eta^2(t)h + (dt)^2,$$

with

(5.2) $$a \circ E = \frac{\eta'}{\eta} \quad \text{and} \quad \psi(t) = c \int_0^t \frac{dr}{\eta(r)^{n-1}}.$$

Now, if ω is finite, then for some $x \in K$, $(E^*g)(x,w)$ is not invertible, hence we must have $\lim_{t \to \omega} \eta(t) = 0$. According to (5.2), we also have $\lim_{t \to \omega} \eta'(t) = 0$, thus $\eta(t) = o(w-t)$ and

$$\psi(w) = c \int_0^\omega \frac{dr}{\eta(r)^{n-1}} = +\infty,$$

which is not possible. Hence $\omega = +\infty$ and the same argument shows that $\alpha = -\infty$.

Therefore, $E\colon N \times \mathbb{R} \to M^n$ is an immersion. Since $d\Phi$ is locally constant on the level sets of Φ, N is a connected component of the closed set $\{x \in M, \phi(x) = 0\}$, thus is closed. Then, as E is a local isometry, $E(N \times \mathbb{R})$ is complete, hence closed in M^n, and open, thus E is a surjection.

Moreover, if $E(x,s) = E(y,t)$, then $\psi(s) = \psi(t)$ hence $s = t$, and following the flow of $-\vec{\nu}$ from $E(x,t)$ or $E(y,t)$ for a time t, we see that $x = y$. Therefore, E is also injective. $\square$

6. The integral Bochner-Weitzenböck Theorem

In this section, we prove Theorem 2.2 and Theorem 2.3. Suppose that ξ is a non-trivial harmonic k-form on a complete manifold (M,g), not necessarily compact.

For $\varepsilon > 0$ we introduce

$$f_\varepsilon = \sqrt{|\xi|^2 + \varepsilon^2}.$$

Elementary computations lead to

$$f_\varepsilon \Delta f_\varepsilon - |df_\varepsilon|^2 = \langle \nabla^* \nabla \xi, \xi \rangle - |\nabla \xi|^2,$$

and for $p > 0$ we get

$$\Delta f_\varepsilon^p = p f_\varepsilon^{p-2} \left(f_\varepsilon \Delta f_\varepsilon - (p-1)|df_\varepsilon|^2 \right)$$

$$= p f_\varepsilon^{p-2} \left(\langle \nabla^* \nabla \xi, \xi \rangle - |\nabla \xi|^2 + (2-p)|df_\varepsilon|^2 \right)$$

$$\leq p f_\varepsilon^{p-2} \left(\langle \nabla^* \nabla \xi, \xi \rangle - \frac{n+1-k}{n-k} |d|\xi||^2 + (2-p)|df_\varepsilon|^2 \right).$$

Note that we have

$$|df_\varepsilon|^2 = \frac{|\xi|^2 |d|\xi||^2}{|\xi|^2 + \varepsilon^2} \leq |d|\xi||^2.$$

Hence, choosing $p = \frac{n-1-k}{n-k}$, we have $p \in (0, 2)$ and

$$2 - p = \frac{n+1-k}{n-k}.$$

We obtain

$$\Delta f_\varepsilon^p \leq p f_\varepsilon^{p-2} \langle \nabla^* \nabla \xi, \xi \rangle.$$

According to the Bochner-Weitzenböck formula $\langle \nabla^* \nabla \xi, \xi \rangle = -\langle \mathscr{R}_k \xi, \xi \rangle$, we then have

$$\Delta f_\varepsilon^p \leq -p f_\varepsilon^{p-2} \langle \mathscr{R}_k \xi, \xi \rangle$$

$$\leq -p \frac{k(n-k)}{n(n-1)} R_g f_\varepsilon^{p-2} |\xi|^2 + p \, r_k f_\varepsilon^{p-2} |\xi|^2,$$

and we finally get

$$(6.1) \qquad \Delta f_\varepsilon^p + \frac{k(n-1-k)}{n(n-1)} R_g f_\varepsilon^{p-2} |\xi|^2 \leq \frac{n-1-k}{n-k} r_k f_\varepsilon^{p-2} |\xi|^2.$$

6.1. The vanishing result

If the manifold (M, g) is closed, by multiplying this inequality by f_ε^p and integrating over M, we obtain

$$\int_M |d(f_\varepsilon^p)|^2 \, dv_g + \frac{k(n-1-k)}{n(n-1)} \int_M R_g f_\varepsilon^{2(p-1)} |\xi|^2 \, dv_g \leq \frac{n-1-k}{n-k} \int_M r_k f_\varepsilon^{2(p-1)} |\xi|^2 \, dv_g.$$

We define $v = |\xi|^{\frac{n-1-k}{n-k}}$. Since $f_\varepsilon^{2(p-1)} |\xi|^2 \leq |\xi|^{2p}$, by Fatou's Lemma we see that v is in $H_1^2(M)$, and by letting ε go to zero, we get by Lebesgue's dominated convergence theorem that

$$\int_M \left[|dv|^2 + \frac{k(n-1-k)}{n(n-1)} R_g v^2 \right] dv_g \leq \frac{n-1-k}{n-k} \int_M r_k v^2 \, dv_g.$$

When $k \neq \frac{n-1}{2}$, we have

$$\beta = \frac{4k(n-1-k)}{n(n-2)} \in (0, 1],$$

hence we obtain

$$\frac{n-2}{4(n-1)} Y_g(\beta) \|v\|_{L^{\frac{2n}{n-2}}}^2 \leq \frac{n-1-k}{n-k} \int_M r_k v^2 \, dv_g,$$

and according to the Hölder inequality,

$$(6.2) \qquad \frac{n-2}{4(n-1)} Y_g(\beta) \|v\|_{L^{\frac{2n}{n-2}}}^2 \leq \frac{n-1-k}{n-k} \|r_k\|_{L^{n/2}} \|v\|_{L^{\frac{2n}{n-2}}}^2.$$

Therefore, either v vanishes on M, or

$$\frac{n-2}{4(n-1)} \, Y_g(\beta) \leq \frac{n-1-k}{n-k} \, \|r_k\|_{L^{n/2}} \, .$$

In that case, according to Proposition 4.1, we obtain

$$\beta \frac{n-2}{4(n-1)} \, Y(M, [g]) \leq \frac{n-1-k}{n-k} \, \|r_k\|_{L^{n/2}} \, ,$$

hence

(6.3) $$\|r_k\|_{L^{n/2}} \geq \frac{k(n-k)}{n(n-1)} \, Y(M, [g]).$$

For the middle degree $n/2$ when $n/2$ is even, the Hodge star operator $*$ induces a parallel decomposition $\Lambda^{\frac{n}{2}} T^* M = \Lambda^{\frac{n}{2}}_+ T^* M \oplus \Lambda^{\frac{n}{2}}_- T^* M$. And since the traceless Bochner-Weitzenböck curvature $\mathcal{W}_{\frac{n}{2}}$ commutes with $*$, it admits a decomposition

$$\mathcal{W}_{\frac{n}{2}} = \mathcal{W}^+_{\frac{n}{2}} \oplus \mathcal{W}^-_{\frac{n}{2}}.$$

If ξ is a self-dual form, i.e., if $*\xi = \xi$, and if $-r^+_{n/2}$ is the lowest eigenvalue of $\mathcal{W}^+_{n/2}$, we get

(6.4) $$\left\|r^+_{n/2}\right\|_{L^{n/2}} \geq \frac{n}{4(n-1)} \, Y(M, [g]).$$

6.2. The equality case

We will now characterize the equality case of (6.3).

When $1 \leq k \leq \frac{n-3}{2}$

If equality holds in (6.3) and v does not vanish, then equality must hold everywhere. In particular, we have $Y_g(\beta) = \beta \, Y(M, [g])$, and the function v attains the infimum in (4.4). When $k \leq \frac{n-3}{2}$, since we have

$$0 < \frac{4k(n-1-k)}{n(n-2)} < 1,$$

according to Proposition 4.1, g is a Yamabe minimizer and v is constant, hence $|\xi|$ is constant. Furthermore, the equality for the Hölder inequality in (6.2) implies that r_k is constant, hence

$$r_k = \frac{k(n-k)}{n(n-1)} R_g.$$

By Theorem 2.1, every harmonic k-form is parallel and $b_k \leq \binom{n}{k}$.

The middle degree

If equality holds in (6.3), as $\beta = 1$, g is not necessarily a Yamabe minimizer. However, since v must realize the infimum of the Yamabe functional, the metric $\tilde{g} = v^{\frac{4}{n-2}} g$ is a Yamabe minimizer. Then, the form ξ is still harmonic for $\tilde{g}$ but has constant $\tilde{g}$−length

$$|\xi|_{\tilde{g}} = 1.$$

And since the traceless Bochner-Weitzenböck curvature $\mathcal{W}_{\frac{n}{2}}$ only depends on the Weyl curvature, the pinching is conformally invariant and equality also holds for $\tilde{g}$. Then equality in (6.2) implies that $r_{n/2}(\tilde{g})$ is constant, hence $r_{n/2}(\tilde{g}) = \frac{n}{4(n-1)} R_{\tilde{g}}$ and by Theorem 2.1, every $\tilde{g}$-harmonic $n/2$-form is $\tilde{g}$-parallel and $b_{n/2} \leq \binom{n}{n/2}$.

The middle degree in dimension 4

If $b_2^+ \neq 0$, then according to (6.4) we have

(6.5) $$\frac{1}{3} \mathrm{Y}(M, [g]) \leq \left\| r_2^+ \right\|_{L^2} = 2 \left\| w^+ \right\|_{L^2}.$$

If equality holds in (6.5) and if there exists a non-trivial self-dual harmonic 2-form ξ, then according to the study of the middle degree case, there is a Yamabe minimizer $\tilde{g} \in [g]$ such that ξ is $\tilde{g}$-parallel with $|\xi|_{\tilde{g}}^2 = 2$. Then ξ is a Kähler form on $(M, \tilde{g})$.

7. Pinching involving the norm of the curvature

On a closed manifold, according to (6.3) and the inequalities of Section 3, if $b_k \neq 0$, we have

(7.1) $$\left(a_{n,k} \|W\|_{\frac{n}{2}}^2 + b_{n,k} \|\mathring{\mathrm{Ric}}\|_{\frac{n}{2}}^2 \right)^{\frac{1}{2}} \geq \|r_k\|_{L^{n/2}} \geq \frac{k(n-k)}{n(n-1)} \mathrm{Y}(M, [g]).$$

We will now characterize the equality case in (7.1).

7.1. For one-forms in dimension greater than 5

If $b_1 \neq 0$ and if

$$\left\| \mathring{\mathrm{Ric}} \right\|_{L^{\frac{n}{2}}} = \frac{1}{\sqrt{n(n-1)}} \mathrm{Y}(M, [g]),$$

then according to Subsection 6.2, Ric_g is nonnegative with b_1 zero eigenvalues which correspond to b_1 parallel vector fields. According to the de Rham splitting theorem, the universal cover of (M, g) splits as a Riemannian product $(N^{n-b_1} \times \mathbb{R}^{b_1}, h + (dt)^2)$. But according to Lemma 3.2, Ric_g has only two distinct eigenvalues, hence $b_1 = 1$ and (N, h) is Einstein with positive scalar curvature.

7.2. For one-forms in dimension 4

Since there must be equality in (6.3), there must also be equality in the Kato inequality. Then according to Proposition 5.1, M has a normal cover $\widehat{M} = N^3 \times \mathbb{R}$ with a warped product metric

$$\hat{g} = \eta^2(t)h + (dt)^2,$$

where for some $T > 0$, η is a T-periodic function and the deck transformation group is generated by

$$\gamma(x, t) = (\phi(x), t + T),$$

with $\phi \colon N \to N$ a h-isometry.

We can write that $\hat{g}$ is isometric to $\tilde{g} = e^{-2f(s)}(h + ds^2)$. Then,

$$\mathring{\mathrm{Ric}}_{\tilde{g}} = \mathring{\mathrm{Ric}}_h + \frac{1}{2}(1 - f'' - (f')^2)\left(h - 3ds^2\right).$$

Since equality holds in the inequality between the first eigenvalue and the norm of $\mathring{\mathrm{Ric}}_{\tilde{g}}$, we have

$$\mathring{\mathrm{Ric}}_{\tilde{g}} = r_1 e^{-2f}\left(ds^2 - \frac{1}{3}h\right),$$

then

$$\mathring{\mathrm{Ric}}_h = \left(r_1 e^{-2f} + \frac{1}{2}(1 - f'' - (f')^2)\right)(3ds^2 - h)$$

and by taking the trace on $TN \subset T\widehat{M}$, we see that $\mathring{\mathrm{Ric}}_h$ must vanish. Thus (N^3, h) is Einstein hence of constant sectional curvature, and (M, g) is conformally equivalent to a quotient of $\mathbb{S}^3 \times \mathbb{R}$. We recover Theorem 1.1 i).

REMARKS 7.1. – i) If the translation parameter T is too large then the product metric cannot be a Yamabe minimizer. Indeed the second variation of the Yamabe functional has a negative eigenvalue at the product metric when

$$T^2 > \frac{4\pi^2(n-1)}{R_h}.$$

Conversely, on $\mathbb{S}^{n-1} \times \mathbb{S}^1$, the product metric is a Yamabe minimizer as soon as

$$(7.2) \qquad\qquad T^2 \leq \frac{4\pi^2}{n-2}$$

(cf. [35]). Therefore, in dimension 4, if $b_1 \neq 0$, equality holds in (1.2) if and only if (M, g) is conformally equivalent to a quotient of $\mathbb{S}^3 \times \mathbb{R}$ with translation parameter satisfying (7.2).

ii) If (M, g) satisfies the pinching

$$\int_M |\mathring{\mathrm{Ric}}_g|^2 dv_g \leq \frac{1}{12}\int_M R_g^2 dv_g$$

which is conformally invariant according to the Gauss-Bonnet formula, we can suppose (up to a conformal change) that g is a Yamabe minimizer and satisfies (1.2).

7.3. For two-forms in dimension 4

If we have the equality

$$\|W_g\|_{L^2} = \frac{1}{2\sqrt{6}} \, Y(M, [g]),$$

and $b_2 \neq 0$, then by taking a two-fold covering if M is not orientable and choosing the right orientation, we have equality in (6.5), $b_2^+(M) = 0$ and $W_g^- = 0$. Hence (M, g) is conformally equivalent to a Kähler self-dual manifold with constant scalar curvature. According to [8, 22], (M, g) is conformally equivalent to $\mathbb{P}^2(\mathbb{C})$ endowed with the Fubini-Study metric, and we recover Theorem 1.1 ii).

7.4. In degree k $\in [2, \frac{n-2}{2}]$ when $n \geq 7$

If equality holds in (7.1) and if there exists a non-trivial harmonic k-form, then according to Proposition 3.4 we must have $k = 2$.

When $n \geq 7$, we have $k \leq \frac{n-3}{2}$, and according to Subsection 6.2, the metric g is a Yamabe minimizer and ξ is parallel. According to Proposition 3.4, we obtain a parallel decomposition $T^*M = V \oplus V^\perp$, and the universal cover of (M, g) splits as a Riemannian product

$$\pi \colon \tilde{M} = X_1 \times X_2 \to M$$

where X_1 has dimension 2 and $\pi^*\xi$ is colinear to λdv_{X_1}. Still from Proposition 3.4, we see that $\gamma = 0$, that X_2 has constant positive sectional curvature, that we can normalize to be 1, and that X_1 has constant sectional curvature, hence is a 2-sphere of curvature $n - 5$.

7.5. For 2-forms in dimension 6

We consider a closed manifold (M^6, g) with $b_2 \neq 0$ which satisfies

$$\|r_2\|_{L^3} = \frac{4}{15} \, Y(M, [g])$$

and

$$|r_2|^2 = a_{6,2} \, |W|^2 + b_{6,2} \big| \overset{\circ}{\text{Ric}} \big|^2.$$

In this case, there is a harmonic 2-form ξ for which equality holds in the refined Kato inequality, and the curvature operator is

$$\beta \left(\frac{g_V^2}{2} + \frac{g_{V^\perp}^2}{2} \right) + \gamma \frac{g^2}{2},$$

where at each point

$$T_x M = V \oplus V^\perp.$$

Following the computations done in [17], we introduce a local orthonormal frame $(e_1, e_2, e_3, \ldots, e_6)$ and its dual frame $(\theta^1, \ldots, \theta^6)$, with $V = \text{Vect}(e_1, e_2)$. We can write that

$$d|\xi| = \rho|\xi|\theta^1 \quad \text{and} \quad \xi = |\xi|\theta^1 \wedge \theta^2.$$

The computation leads to

$$\nabla_{e_1}\xi = \rho\xi, \quad \nabla_{e_2}\xi = 0 \quad \text{and} \quad \nabla_{e_j}\xi = -\frac{1}{4}\rho|\xi|\theta^j \wedge \theta^2,$$

for $j \geq 3$. Hence, writing $\Omega = \frac{\xi}{|\xi|}$, we obtain $\nabla_{e_1}\Omega = \nabla_{e_2}\Omega = 0$,

$$R(e_1, e_2)\Omega = (\beta + \gamma)\Omega \quad \text{and} \quad R(e_1, e_2)\Omega = -\nabla_{[e_1, e_2]}\Omega.$$

This implies that

$$[e_1, e_2] \in V = \text{Vect}(e_1, e_2).$$

Hence $\nabla_{[e_1,e_2]}\Omega = 0$ and thus $\beta + \gamma = 0$. However, the scalar curvature of g is

$$R_g = 14\beta + 30\gamma = -16\beta.$$

This is not possible, since $\beta \geq 0$ and since we have assumed that the Yamabe invariant of (M, g) is positive.

7.6. The middle degree

We consider a closed manifold (M^n, g) with $b_{\frac{n}{2}} \neq 0$ and such that the following equality holds

$$a_{n,n/2} \left(\int_M |W|^{\frac{n}{2}} dv_g \right)^{\frac{2}{n}} = \frac{n}{4(n-1)} Y(M, [g]).$$

If $n/2$ is even, by Proposition 3.6 we must have $n = 4$ and (M, g) conformally equivalent to $\mathbb{P}^2(\mathbb{C})$ endowed with the Fubini-Study metric.

If $n/2$ is odd, then according to the middle degree case in Section 6, up to a conformal change $\tilde{g} = |\xi|^{\frac{4}{n}} g$ on the metric, we can suppose that g is a Yamabe minimizer and that ξ is parallel.

According to Proposition 3.5, the universal cover of (M, g) splits as a Riemannian product $X_1 \times X_2$ where X_1 and X_2 have dimension $n/2$. Moreover, in the orthogonal decomposition

$$\Lambda^{\frac{n}{2}} T^*(X_1 \times X_2) = \bigoplus_{j=0}^{\frac{n}{2}} \Lambda^j T^* X_1 \otimes \Lambda^{\frac{n}{2}-j} T^* X_2,$$

the Bochner-Weitzenböck curvature has the decomposition

$$\mathscr{R}_{\frac{n}{2}} = \sum_{j=0}^{\frac{n}{2}} \left(\mathscr{R}_{\frac{n}{2}-j}^{X_1} \otimes \text{Id}_{\Lambda^j T^* X_2} + \text{Id}_{\Lambda^{\frac{n}{2}-j} T^* X_1} \otimes \mathscr{R}_j^{X_2} \right).$$

Hence for $j \in \{0, \ldots, \frac{n}{2}\}$, $\mathscr{R}_{\frac{n}{2}-j}^{X_1}$ and $\mathscr{R}_j^{X_2}$ are multiple of the identity. In particular $\mathscr{R}_{\frac{n}{2}-2}^{X_1}$ and $\mathscr{R}_2^{X_2}$ are multiple of the identity, and by [36] or [31], this implies that X_1 and X_2 have constant sectional curvature.

Moreover, the eigenvalues of $\mathscr{R}_{\frac{n}{2}}$ are

$$j \left(\frac{n}{2} - j \right) \frac{R_g}{n(n-1)},$$

with multiplicity $\binom{n/2}{j}^2$, where $j \in \{0, \ldots, \frac{n}{2}\}$. The only possibility to have only two distinct eigenvalues is when $n = 6$. Then X_1 and X_2 are two round spheres.

REMARK 7.2. – If X_1 and X_2 are two round spheres of same radius, then the product is Einstein. According to [10] it is a Yamabe minimizer, and thus equality really holds in (1.6).

8. The non-compact case

We will prove the following result, which implies Theorem E:

THEOREM 8.1. – *Let (M^n, g), $n \geq 4$, be a complete non-compact Riemannian manifold with positive Yamabe invariant. Assume that the lowest eigenvalue of the Ricci curvature satisfies* $\mathrm{Ric}_- \in L^p$ *for some $p > \frac{n}{2}$, and assume that $R_g \in L^{\frac{n}{2}}$. If*

$$(8.1) \qquad \left\| r_1 \right\|_{L^{\frac{n}{2}}} + \frac{n-4}{4n} \left\| R_g \right\|_{L^{\frac{n}{2}}} \leq \frac{1}{4} \, \mathrm{Y}(M, [g]),$$

then

- *either $H_c^1(M, \mathbb{Z}) = \{0\}$ and in particular M has only one end,*
- *or equality holds in (8.1) and there exists an Einstein manifold (N^{n-1}, h) with positive scalar curvature and $\alpha > 0$ such that (M^n, g) or one of its two-fold coverings is isometric to*

$$\left(N^{n-1} \times \mathbb{R}, \, \alpha \, \cosh^2(t) \left(h + (dt)^2 \right) \right).$$

Proof. – According to Lemma 4.2, there exists C such that the following Sobolev inequality holds:

$$(8.2) \qquad \forall \varphi \in C_0^\infty(M) \qquad \left\| \varphi \right\|_{L^{\frac{2n}{n-2}}}^2 \leq C \left\| d\varphi \right\|_{L^2}^2 .$$

Then, according to [19, Proposition 5.2], if $H_c^1(M, \mathbb{Z}) \neq \{0\}$, then M or one of its two-fold coverings has at least two ends.

If M has at least two ends, then according to [15, Theorem 2], we can find a compact set $K \subset M$ with

$$M \setminus K = \Omega_- \cup \Omega_+,$$

and with both Ω_- and Ω_+ unbounded, and a harmonic function $\Phi \colon M \mapsto (-1, 1)$ such that $d\Phi \in L^2$,

$$\lim_{\substack{x \to \infty \\ x \in \Omega_-}} \Phi(x) = -1 \quad \text{and} \quad \lim_{\substack{x \to \infty \\ x \in \Omega_+}} \Phi(x) = 1.$$

In particular $\xi := d\Phi$ is an L^2 harmonic 1-form on (M, g).

If M has only one end and $\check{\pi} \colon \check{M} \to M$ is a two-fold covering of M with at least two ends, then $(\check{M}, \check{\pi}^* g)$ satisfies the Sobolev inequality (8.2), and we can find a compact set $K \subset M$ such that $\Omega := M \setminus K$ is connected and $\check{M} \setminus \pi^{-1}(K) = \check{\Omega}_- \cup \check{\Omega}_+$ with $\check{\Omega}_-$ and $\check{\Omega}_+$ unbounded. Then we can find a harmonic function $\check{\Phi} \colon \check{M} \to (-1, 1)$ such that $d\check{\Phi} \in L^2$,

$$\lim_{\substack{x \to \infty \\ x \in \check{\Omega}_-}} \check{\Phi}(x) = -1 \quad \text{and} \quad \lim_{\substack{x \to \infty \\ x \in \check{\Omega}_+}} \check{\Phi}(x) = 1.$$

Moreover this function is unique by maximum principle, hence the image of $\check{\Phi}$ by a deck transformation of $\check{\pi} \colon \check{M} \to M$ is either $\check{\Phi}$ or $-\check{\Phi}$. In particular, the function $|\xi| = |d\check{\Phi}|$ is well defined on M and is in $L^2(M, g)$.

Furthermore, since Ric_- is in L^p for some $p > n/2$, and since

$$\Delta |\xi| \leq -\mathrm{Ric}_- |\xi|,$$

we get by DeGiorgi-Nash-Moser iterative scheme (see for instance [37, Theorem B.1]) that ξ is in L^∞, and that

$$(8.3) \qquad \lim_{x \to \infty} |\xi| = 0.$$

In particular, the function $v = |\xi|^{\frac{n-2}{n-1}}$ is in $L^{\frac{2n}{n-2}}(M, g)$.

If χ is a Lipschitz function with compact support and h is a smooth function, we have the integration by parts formula

$$\int_M |d(\chi h)|^2 \, dv_g = \int_M \left[|d\chi|^2 \, h^2 + \chi^2 h \Delta h \right] dv_g.$$

By multiplying inequality (6.1) by $\chi^2 f_\varepsilon^p$ and integrating over M, we obtain:

$$\int_M |d(\chi f_\varepsilon^p)|^2 \, dv_g + \frac{n-2}{n(n-1)} \int_M R_g f_\varepsilon^{2p-2} |\xi|^2 \chi^2 dv_g$$
$$\leq p \int_M r_1 \, f_\varepsilon^{2p-2} |\xi|^2 \chi^2 dv_g + \int_M |d\chi|^2 \, f_\varepsilon^{2p} dv_g,$$

where $p = \frac{n-2}{n-1}$. If we let ε go to zero, we get by Fatou's Lemma and Lebesgue's dominated convergence theorem that

$$\int_M |d(\chi v)|^2 \, dv_g + \frac{n-2}{n(n-1)} \int_M R_g (\chi v)^2 dv_g \leq \frac{n-2}{n-1} \int_M r_1 (\chi v)^2 dv_g + \int_M |d\chi|^2 \, v^2 dv_g,$$

hence

$$\int_M \left(\frac{4(n-1)}{n-2} |d(\chi v)|^2 + R_g (\chi v)^2 \right) dv_g$$
$$\leq 4 \int_M \left(r_1 + \frac{n-4}{4n} R_g \right) (\chi v)^2 dv_g + \frac{4(n-1)}{n-2} \int_M |d\chi|^2 \, v^2 dv_g.$$

For $R > 0$, we introduce the functions:

$$\chi_R(x) = \begin{cases} 1 & \text{on } B(x_0, R) \\ 2 - \frac{d(x, x_0)}{R} & \text{on } B(x_0, 2R) \setminus B(x_0, R) \\ 0 & \text{on } M \setminus B(x_0, 2R) \end{cases}$$

where $x_0 \in M$ is a fixed point.

According to the Hölder inequality,

$$\int_M \left(\frac{4(n-1)}{n-2} |d(\chi_R v)|^2 + R_g (\chi_R v)^2 \right) dv_g$$
$$\leq 4 \left\| r_1 + \frac{n-4}{4n} R_g \right\|_{L^{\frac{n}{2}}} \|\chi_R v\|_{L^{\frac{2n}{n-2}}}^2 + \epsilon(R) \|d\chi_R\|_{L^{2(n-1)}}^2,$$

where

$$\epsilon(R) = \frac{4(n-1)}{n-2} \left(\int_{B(x_0, 2R) \setminus B(x_0, R)} |\xi|^2 \, dv_g \right)^{\frac{n-2}{n-1}}.$$

Therefore

$$Y(M, [g]) \|\chi_R v\|^2_{L^{\frac{2n}{n-2}}} \leq \int_M \left(\frac{4(n-1)}{n-2} |d(\chi_R v)|^2 + R_g (\chi_R v)^2 \right) dv_g$$

$$\leq 4 \left\| r_1 + \frac{n-4}{4n} R_g \right\|_{L^{\frac{n}{2}}} \|\chi_R v\|^2_{L^{\frac{2n}{n-2}}} + \epsilon(R) \|d\chi_R\|^2_{L^{2(n-1)}} .$$

We have

$$\|d\chi_R\|^2_{L^{2(n-1)}} \leq \frac{1}{R^2} \operatorname{vol}(B(x_0, 2R))^{\frac{1}{n-1}},$$

and according to [23, Theorem 1] (see also [33, 4]), when the lowest eigenvalue Ric_- of the Ricci curvature is in L^q for some $\frac{n}{2} < q \leq n-1$, then

$$(8.4) \qquad\qquad \operatorname{vol} B(x_0, R) = O\left(R^{2(n-1)} \right).$$

Since $\operatorname{Ric}_- = -r_1 + \frac{R_g}{n}$, Ric_- is in $L^{n/2} \cap L^p$ for some $p > n/2$, and we have (8.4), hence

$$\epsilon(R) \|d\chi_R\|^2_{L^{2(n-1)}} \xrightarrow[R \to \infty]{} 0.$$

Since v is in $L^{\frac{2n}{n-2}}$ and R_g is in $L^{\frac{n}{2}}$, we get by Lebesgue's dominated convergence theorem that the function v satisfies

$$Y(M, [g]) \|v\|^2_{L^{\frac{2n}{n-2}}} \leq \frac{4(n-1)}{n-2} \int_M |dv|^2 dv_g + \int_M R_g v^2 dv_g$$

$$\leq 4 \left\| r_1 + \frac{n-4}{4n} R_g \right\|_{L^{\frac{n}{2}}} \|v\|^2_{L^{\frac{2n}{n-2}}} .$$

Therefore, if v does not vanish, then

$$(8.5) \qquad \frac{1}{4} Y(M, [g]) \leq \left\| r_1 + \frac{n-4}{4n} R \right\|_{L^{\frac{n}{2}}} \leq \|r_1\|_{L^{\frac{n}{2}}} + \frac{n-4}{4n} \|R\|_{L^{\frac{n}{2}}} .$$

If furthermore equality holds, then v is a minimizer for the Yamabe functional, thus satisfies the Yamabe equation

$$\frac{4(n-1)}{n-2} \Delta_g v + R_g v = Y(M, [g]) v^{\frac{n+2}{n-2}},$$

and we can suppose that

$$\|v\|_{L^{\frac{2n}{n-2}}} = 1.$$

As $v \in C^{0, \frac{n-2}{n-1}}$, it is smooth and positive, and the metric $\tilde{g} = v^{\frac{4}{n-2}} g$ has constant scalar curvature equal to $Y(M, [g])$.

Moreover, since equality must hold in the refined Kato inequality, then according to Proposition 5.1, M or one of its two-fold coverings is isometric to $N \times \mathbb{R}$ endowed with a metric

$$\hat{g} = \eta^2(t) h + dt^2.$$

If we take the new coordinate $s = \int_0^t \eta^{-1}(\tau) d\tau$, we can write that

$$\hat{g} = e^{-2f(s)} (h + ds^2),$$

where s is in (s_-, s_+), with

$$s_+ = \int_0^{+\infty} \frac{dt}{\eta(t)} \quad \text{and} \quad s_- = \int_0^{-\infty} \frac{dt}{\eta(t)} .$$

Since $v = e^{(n-2)f}$ is a solution of the Yamabe equation for the metric $\hat{g}$, the function $w = e^{\frac{n-2}{2}f}$ is a solution of the Yamabe equation for the metric $h + (ds)^2$, hence satisfies

$$(8.6) \qquad -4\frac{n-1}{n-2}w''(s) + R_h w(s) = Y(M,[g])w(s)^{\frac{n+2}{n-2}}.$$

In particular, we see that R_h only depends on s, hence is constant.

We can now prove that $s_+ = +\infty$. Recall that

$$|\xi| = \eta^{1-n} = e^{(n-1)f} = w^{2\frac{n-1}{n-2}}.$$

If s_+ is finite, then because of (8.3), we get

$$\lim_{s \to s_+} w = 0.$$

The differential equation (8.6) implies that w' must have a non-zero limit when $s \to s_+$. Therefore, there exists $c > 0$ such that

$$w \underset{s \to s_+}{\sim} c(s_+ - s).$$

And since the metric $\hat{g} = w^{-\frac{4}{n-2}}(h + ds^2)$ is complete, we must have

$$\int_0^{s^+} w(s)^{-\frac{2}{n-2}} ds = +\infty,$$

hence $n = 4$. But according to (4), when $n = 4$, the scalar curvature of $\hat{g}$ satisfies

$$\frac{1}{w^3}R_{\hat{g}} = -6\left(\frac{1}{w}\right)'' + \frac{1}{w}R_h,$$

thus $R_{\hat{g}}$ goes to $-12c^2$ when $s \to s_+$, and therefore is not in $L^{n/2}(M, \hat{g})$. Consequently, $s_+ = +\infty$, and the same argument shows that $s_- = -\infty$.

From (8.6), we deduce that there is a constant c such that

$$-4\frac{n-1}{n-2}(w')^2 + R_h w^2 = \frac{n-2}{n}Y(M,[g])w^{\frac{2n}{n-2}} + c.$$

Since $\lim_{s \to \pm\infty} w = 0$, we must have $c = 0$. Moreover, since $Y(M,[g])$ is positive, and w is a positive function we must also have $R_h > 0$. Up to a change of time variable and a scaling on $\hat{g}$, we can suppose that

$$R_h = (n-2)(n-1).$$

Let $\varphi = e^{-f} = w^{-\frac{2}{n-2}}$. We obtain

$$-(\varphi')^2 + \varphi^2 = \frac{Y(M,[g])}{4n(n-1)}.$$

Therefore, for some s_0, we have

$$\varphi(s) = \sqrt{\frac{Y(M,[g])}{4n(n-1)}}\cosh(s - s_0).$$

Conversely, if (N^{n-1}, h) is a closed manifold with positive scalar curvature

$$R_h = (n-2)(n-1),$$

and if

$$(M, g) = \left(N^{n-1} \times \mathbb{R}, \alpha\cosh^2(t)\left(h + (dt)^2\right)\right),$$

then

$$R_g = (n-1)(n-4)\frac{1}{\alpha \cosh^4(t)}$$

$$\mathring{\mathrm{Ric}}_g = \mathring{\mathrm{Ric}}_h + 2\frac{n-2}{n}\left(h - (n-1)ds^2\right).$$

We see that the lowest eigenvalue of $\mathring{\mathrm{Ric}}_g$ satisfies

$$r_1(g) = \frac{1}{\alpha \cosh^4(t)}\left(r_1(h) + \frac{2(n-2)(n-1)}{n}\right),$$

and thus we obtain

$$\|r_1\|_{L^{\frac{n}{2}}} + \frac{n-4}{4n}\|R_g\|_{L^{\frac{n}{2}}} = \frac{C}{4}\,\mathrm{Y}(M,[g]),$$

with

$$C = \frac{n(n-1) + 4r_1(h)}{\mathrm{Y}(M,[g])}\,\mathrm{vol}((N,h))^{\frac{2}{n}}\left(\int_{\mathbb{R}}\frac{dt}{\cosh^n(t)}\right)^{\frac{2}{n}}$$

$$= \left(1 + \frac{4r_1(h)}{n(n-1)}\right)\left(\frac{\mathrm{vol}((N,h))}{\mathrm{vol}(\mathbb{S}^{n-1})}\right)^{\frac{2}{n}}\frac{\mathrm{Y}(\mathbb{S}^n)}{\mathrm{Y}(N \times \mathbb{R}, [h + dt^2])}.$$

According to [1, Proposition 2.12], we always have

$$\left(\frac{\mathrm{vol}((N,h))}{\mathrm{vol}(\mathbb{S}^{n-1})}\right)^{\frac{2}{n}}\frac{\mathrm{Y}(\mathbb{S}^n)}{\mathrm{Y}(N \times \mathbb{R}, [h + dt^2])} \geq 1.$$

Hence, for C to be equal to 1, $r_1(h)$ must vanish, i.e., h must be Einstein. Then, according to Proposition 4.3, we have

$$\left(\frac{\mathrm{vol}((N,h))}{\mathrm{vol}(\mathbb{S}^{n-1})}\right)^{\frac{2}{n}}\frac{\mathrm{Y}(\mathbb{S}^n)}{\mathrm{Y}(N \times \mathbb{R}, [h + dt^2])} = 1,$$

and it follows that equality holds in (8.1). □

BIBLIOGRAPHY

[1] K. AKUTAGAWA, B. BOTVINNIK, Yamabe metrics on cylindrical manifolds, *Geom. Funct. Anal.* **13** (2003), 259–333.

[2] K. AKUTAGAWA, G. CARRON, R. MAZZEO, The Yamabe problem on stratified spaces, *Geom. Funct. Anal.* **24** (2014), 1039–1079.

[3] T. AUBIN, *Some nonlinear problems in Riemannian geometry*, Monographs in Math., Springer, 1998.

[4] E. AUBRY, Finiteness of π_1 and geometric inequalities in almost positive Ricci curvature, *Ann. Sci. École Norm. Sup.* **40** (2007), 675–695.

[5] P. H. BÉRARD, From vanishing theorems to estimating theorems: the Bochner technique revisited, *Bull. Amer. Math. Soc.* **19** (1988), 371–406.

[6] C. BÖHM, B. WILKING, Manifolds with positive curvature operators are space forms, *Ann. of Math.* **167** (2008), 1079–1097.

[7] V. BOUR, Fourth order curvature flows and geometric applications, preprint arXiv:1012.0342.

[8] J.-P. BOURGUIGNON, Les variétés de dimension 4 à signature non nulle dont la courbure est harmonique sont d'Einstein, *Invent. Math.* **63** (1981), 263–286.

[9] J.-P. BOURGUIGNON, The magic of Weitzenböck formulas, in *Variational methods* (I. Ekeland, J.-M. Coron, H. Berestycki, eds.), Birkhäuser, 1990.

[10] J.-P. BOURGUIGNON, J.-P. EZIN, Scalar curvature functions in a conformal class of metrics and conformal transformations, *Trans. Amer. Math. Soc.* **301** (1987), 723–736.

[11] T. BRANSON, Kato constants in Riemannian geometry, *Math. Res. Lett.* **7** (2000), 245–261.

[12] S. BRENDLE, R. SCHOEN, Manifolds with 1/4-pinched curvature are space forms, *J. Amer. Math. Soc.* **22** (2009), 287–307.

[13] S. BRENDLE, R. M. SCHOEN, Classification of manifolds with weakly 1/4-pinched curvatures, *Acta Math.* **200** (2008), 1–13.

[14] D. M. J. CALDERBANK, P. GAUDUCHON, M. HERZLICH, Refined Kato inequalities and conformal weights in Riemannian geometry, *J. Funct. Anal.* **173** (2000), 214–255.

[15] H.-D. CAO, Y. SHEN, S. ZHU, The structure of stable minimal hypersurfaces in $\mathbf{R}^{n+1}$, *Math. Res. Lett.* **4** (1997), 637–644.

[16] G. CARRON, Une suite exacte en L^2-cohomologie, *Duke Math. J.* **95** (1998), 343–372.

[17] G. CARRON, Rigidity and L^2 cohomology of hyperbolic manifolds, *Ann. Inst. Fourier (Grenoble)* **60** (2010), 2307–2331.

[18] G. CARRON, M. HERZLICH, The Huber theorem for non-compact conformally flat manifolds, *Comment. Math. Helv.* **77** (2002), 192–220.

[19] G. CARRON, E. PEDON, On the differential form spectrum of hyperbolic manifolds, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **3** (2004), 705–747.

[20] S.-Y. A. CHANG, M. GURKSY, P. C. YANG, A conformally invariant sphere theorem in four dimensions, *Publ. Math. IHÉS* **98** (2003), 105–143.

[21] S.-Y. A. CHANG, M. J. GURSKY, P. C. YANG, An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature, *Ann. of Math.* **155** (2002), 709–787.

[22] A. DERDZIŃSKI, Self-dual Kähler manifolds and Einstein manifolds of dimension four, *Compositio Math.* **49** (1983), 405–433.

[23] S. GALLOT, Isoperimetric inequalities based on integral norms of Ricci curvature, *Astérisque* **157-158** (1988), 191–216.

[24] S. GALLOT, D. MEYER, Opérateur de courbure et laplacien des formes différentielles d'une variété riemannienne, *J. Math. Pures Appl.* **54** (1975), 259–284.

[25] M. J. GURSKY, The Weyl functional, de Rham cohomology, and Kähler-Einstein metrics, *Ann. of Math.* **148** (1998), 315–337.

[26] M. J. GURSKY, Four-manifolds with $\delta W^+ = 0$ and Einstein constants of the sphere, *Math. Ann.* **318** (2000), 417–431.

[27] A. HATCHER, *Algebraic topology*, Cambridge Univ. Press, Cambridge, 2002.

[28] E. HEBEY, *Sobolev spaces on Riemannian manifolds*, Lecture Notes in Math. **1635**, Springer, 1996.

[29] R. S. Kᴜʟᴋᴀʀɴɪ, On the Bianchi Identities, *Math. Ann.* **199** (1972), 175–204.

[30] M.-L. Lᴀʙʙɪ, Double forms, curvature structures and the (p, q)-curvatures, *Trans. Amer. Math. Soc.* **357** (2005), 3971–3992.

[31] M.-L. Lᴀʙʙɪ, On Weitzenböck curvature operators, preprint arXiv:1010.0733, 2006.

[32] P. Lɪ, J. Wᴀɴɢ, Weighted Poincaré inequality and rigidity of complete manifolds, *Ann. Sci. École Norm. Sup.* **39** (2006), 921–982.

[33] P. Lɪ, S.-T. Yᴀᴜ, Curvature and holomorphic mappings of complete Kähler manifolds, *Compositio Math.* **73** (1990), 125–144.

[34] M. Oʙᴀᴛᴀ, The conjectures on conformal transformations of Riemannian manifolds, *J. Differential Geometry* **6** (1971/72), 247–258.

[35] R. M. Sᴄʜᴏᴇɴ, Variational theory for the total scalar curvature functional for Riemannian metrics and related topics, in *Topics in calculus of variations (Montecatini Terme, 1987)*, Lecture Notes in Math. **1365**, Springer, 1989, 120–154.

[36] S. Tᴀᴄʜɪʙᴀɴᴀ, The mean curvature for p-plane, *J. Diff. Geom.* **8** (1973), 47–52.

[37] D. Yᴀɴɢ, Convergence of Riemannian manifolds with integral bounds on curvature. II, *Ann. Sci. École Norm. Sup.* **25** (1992), 179–199.

[38] S. T. Yᴀᴜ, Harmonic functions on complete Riemannian manifolds, *Comm. Pure Appl. Math.* **28** (1975), 201–228.

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