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LOCAL INTEGRABILITY RESULTS IN HARMONIC ANALYSIS ON REDUCTIVE GROUPS IN LARGE POSITIVE CHARACTERISTIC

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Abstract. – Let $\mathbf{G}$ be a connected reductive algebraic group over a non-Archimedean local field $\mathbb{K}$, and let $\mathfrak{g}$ be its Lie algebra. By a theorem of Harish-Chandra, if $\mathbb{K}$ has characteristic zero, the Fourier transforms of orbital integrals are represented on the set of regular elements in $\mathfrak{g}(\mathbb{K})$ by locally constant functions, which, extended by zero to all of $\mathfrak{g}(\mathbb{K})$, are locally integrable. In this paper, we prove that these functions are in fact specializations of constructible motivic exponential functions. Combining this with the Transfer Principle for integrability of [8], we obtain that Harish-Chandra's theorem holds also when $\mathbb{K}$ is a non-Archimedean local field of sufficiently large positive characteristic. Under the hypothesis that mock exponential map exists, this also implies local integrability of Harish-Chandra characters of admissible representations of $\mathbf{G}(\mathbb{K})$, where $\mathbb{K}$ is an equicharacteristic field of sufficiently large (depending on the root datum of $\mathbf{G}$) characteristic.

Résumé. – Soit $\mathbf{G}$ un groupe algébrique réductif connexe au-dessus d'un corps local non archimédien $\mathbb{K}$, et soit $\mathfrak{g}$ son algèbre de Lie. D'après un théorème de Harish-Chandra, si $\mathbb{K}$ est de caractéristique zéro, alors les transformés de Fourier d'intégrales orbitales sont représentés, sur l'ensemble des éléments réguliers de $\mathfrak{g}(\mathbb{K})$, par des fonctions localement constantes, qui, si on les étend par zéro à tout $\mathfrak{g}(\mathbb{K})$, sont localement intégrables. Dans ce papier, nous démontrons que ces fonctions sont en fait des spécialisations de fonctions motiviques constructibles exponentielles. En combinant ceci avec le principe de transfert d'intégrabilité de [8], nous obtenons que le théorème de Harish-Chandra est valable aussi quand $\mathbb{K}$ est un corps local non archimédien de caractéristique positive suffisamment grande. Sous l'hypothèse que l'application exponentielle feinte existe, ceci implique aussi l'intégrabilité locale des caractères de Harish-Chandra de représentations admissibles de $\mathbf{G}(\mathbb{K})$, où $\mathbb{K}$ est un corps d'équi-caractéristique suffisamment grande (en fonction de la donnée radicielle de $\mathbf{G}$).

1. Introduction

In this paper we prove an extension of Harish-Chandra's theorems about local integrability of the functions representing various distributions arising in harmonic analysis on p-adic groups to the positive characteristic case, when the residue characteristic is large. Our method consists in transferring Harish-Chandra's results from characteristic zero to positive characteristic. In the recent years such transfer has become a prominent technique, culminating in the transfer of the Fundamental Lemma from positive characteristic to characteristic zero, [9], [38]. Two distinct ways of carrying out transfer have been described in the literature—one method is based on the idea of close local fields, due to D. Kazhdan and J.-L. Waldspurger, cf. [39]. The other method is based on the program outlined by T.C. Hales in [22] of making harmonic analysis on reductive groups over non-Archimedean local fields "field-independent" via the use of motivic integration, and this is the method we use.

We observe that the statements we are proving in this paper are much more analytic in nature than any of the statements previously handled by the transfer methods—namely, here we talk about L^1-integrability, as opposed to much more algebraic-type statements about equalities between integrals of functions that are known to be integrable. In this sense it is somewhat surprising that the transfer is still possible, and it requires a new type of transfer principle, which we prove in [8]. We note that the use of this very general transfer principle allows us to avoid substantial technical difficulties that one faces when using the method of transfer based on the technique of close local fields, at the cost, however, of not getting a precise lower bound on the characteristic of the fields for which our results apply.

Our main technical result is Theorem 5.8 showing that the functions representing the Fourier transforms of the orbital integrals form a family of so-called constructible motivic exponential functions. These functions were introduced by R. Cluckers and F. Loeser in [12]; they are defined in a field-independent manner by means of logic (in fact, we use a slight generalization; see §B.3.1). Theorem 5.8 implies that Transfer principles for integrability and boundedness apply to the Fourier transforms of all orbital integrals, and in particular, to the nilpotent ones. Once all the required properties of the nilpotent orbital integrals are transferred to the positive characteristic in Theorem 2.1, the analogues of many of the classical results for general distributions follow, thanks to the work of DeBacker [14], and J. Adler and J. Korman, [3]. Thus we obtain our main results: Theorems 2.2 and 5.9 (the former assumes the hypothesis on the existence of a mock exponential map, which we review in 2.2.1).

We note that for GL_n, the local integrability of characters was proved by Rodier [35] for $p > n$, and by a different method, by B. Lemaire [29] for arbitrary p. Lemaire also proved the local integrability of characters for the inner forms of GL_n and SL_n, and for twisted characters of GL_n, [30], [31].

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2. Results

2.1. Notation

For a discretely valued field $\mathbb{K}$, its ring of integers will be denoted by $\Omega_{\mathbb{K}}$, the maximal ideal by $\mathfrak{p}_{\mathbb{K}}$, and the residue field by $k_{\mathbb{K}}$.

Let $\mathcal{A}$ be the collection of all non-Archimedean local fields $\mathbb{K}$ of characteristic zero, with a chosen uniformizer $\varpi_{\mathbb{K}}$ of $\Omega_{\mathbb{K}}$, and let $\mathcal{B}$ be the collection of all local fields $\mathbb{K}$ of positive characteristic, with a uniformizer $\varpi_{\mathbb{K}}$ of $\Omega_{\mathbb{K}}$. The notation $\mathbb{K}$ will always stand for a local field that lies in $\mathcal{A} \cup \mathcal{B}$. For an integer $M > 0$, we will also often use the collections $\mathcal{A}_M$ and $\mathcal{B}_M$ of fields in $\mathcal{A}$ and $\mathcal{B}$ respectively, with residue characteristic greater than M.

We use Denef-Pas language $\mathcal{L}_{\mathbb{Z}}$ with coefficients in $\mathbb{Z}$—this is a first-order language of logic; roughly speaking, formulas in this language define subsets of affine spaces uniformly over all local fields $\mathbb{K} \in \mathcal{A} \cup \mathcal{B}$, (see Appendix B for precise definitions). By "definable" we shall mean, definable in the language $\mathcal{L}_{\mathbb{Z}}$. We survey all the definitions and theorems from the theory of motivic integration that we use in Appendix B. We note that if one wishes to work only with reductive groups defined over a fixed number field E (with a ring of integers Ω) and its completions, then one can use the language $\mathcal{L}_{\Omega}$ defined in Appendix B; all the results still apply since any language $\mathcal{L}_{\Omega}$ includes the language $\mathcal{L}_{\mathbb{Z}}$.

Throughout this paper, $\mathbf{G}$ stands for a connected reductive algebraic group over a local field $\mathbb{K}$, and $\mathfrak{g}$ for its Lie algebra. For $X \in \mathfrak{g}(\mathbb{K})$, $D_G(X)$ is the discriminant of X, see Appendix A for the definition.

Following Kottwitz, [27], we call a function $F(X)$, defined and locally constant on the set of regular elements $\mathfrak{g}(\mathbb{K})^{\mathrm{reg}}$, "nice" if it satisfies the following two requirements:

- when extended by zero to all of $\mathfrak{g}(\mathbb{K})$, it is locally integrable, and
- the function $|D_G(X)|^{1/2}F(X)$ is locally bounded on $\mathfrak{g}(\mathbb{K})$.

Similarly, call a function on $\mathbf{G}(\mathbb{K})$ "nice", if it satisfies the same conditions on $\mathbf{G}(\mathbb{K})$, with $D_G(X)$ replaced by its group version $D_G(g)$, namely, the coefficient at t^r (where r is the rank of $\mathbf{G}$) in the polynomial $\det((t+1)I - \mathrm{Ad}(g))$.

2.2. The statements

We refer to Appendix A for all the definitions (of orbital integrals, etc.) and a survey of the classical results.

Our main result states that the Fourier transforms of orbital integrals are represented by nice functions, in large positive characteristic.

THEOREM 2.1. – *There exists a constant $M_{\mathbf{G}}^{\mathrm{orb}} > 0$ that depends only on the absolute root datum of $\mathbf{G}$, such that for every $\mathbb{K} \in \mathcal{B}_{M_{\mathbf{G}}^{\mathrm{orb}}}$, for every $X \in \mathfrak{g}(\mathbb{K})$, the function $\widehat{\mu}_X$ is a nice function on $\mathfrak{g}(\mathbb{K})$.*

In this theorem and all similarly phrased statements below, our assertion that there exists a constant $M > 0$ that depends only on the absolute root datum of $\mathbf{G}$ such that so-and-so properties hold for $\mathbf{G}(\mathbb{K})$ with $\mathbb{K} \in \mathcal{A}_M \cup \mathcal{B}_M$, has the following meaning. As discussed in §3.1 below, given an absolute root datum Ψ (which is a field-independent construct), there exist finitely many possibilities for the root data of reductive groups over non-Archimedean local fields having the absolute root datum Ψ. We parametrize these possibilities by points of a definable set in §3.1. Then our statement says that there exists a constant M that depends only on Ψ, such that for every local field $\mathbb{K} \in \mathscr{A}_M \cup \mathscr{B}_M$, for all possible connected reductive groups $\mathbf{G}$ defined over $\mathbb{K}$ with absolute root datum Ψ, the assertions of the theorem hold.

Theorem 2.1 is proved below in §5.2.1.

Thanks to the local character expansion near a tame semisimple element, the above theorem implies that Harish-Chandra characters of admissible representations are represented by nice functions on the group, under the additional hypothesis on the existence of a so-called mock exponential map. Local character expansion in large positive characteristic is proved by DeBacker [14] near the identity, and by Adler-Korman [3] near a general tame semisimple element, if the mock exponential map exists. We start by quoting the hypothesis, which uses the notation defined in §3.3 below.

2.2.1. *The exponential map hypothesis (*[14, Hypothesis 3.2.1]*).* – *Suppose $r > 0$. There exists a bijective map $\mathbf{e} : \mathfrak{g}(\mathbb{K})_r \to \mathbf{G}(\mathbb{K})_r$ such that*

1. *for all pairs $x \in \mathscr{B}(\mathbf{G}, \mathbb{K})$, $s \in \mathbb{R}_{\geq r}$, we have*
   (a) $\mathbf{e}(\mathfrak{g}(\mathbb{K})_{x,s}) = \mathbf{G}(\mathbb{K})_{x,s}$,
   (b) *For all $X \in \mathfrak{g}(\mathbb{K})_{x,r}$ and for all $Y \in \mathfrak{g}(\mathbb{K})_{x,s}$, we have $\mathbf{e}(X)\mathbf{e}(Y) \equiv \mathbf{e}(X+Y) \bmod \mathbf{G}(\mathbb{K})_{x,s^+}$, and*
   (c) *$\mathbf{e}$ induces a group isomorphism of $\mathfrak{g}(\mathbb{K})_{x,s}/\mathfrak{g}(\mathbb{K})_{s,s^+}$ with $\mathbf{G}(\mathbb{K})_{x,s}/\mathbf{G}(\mathbb{K})_{s,s^+}$;*
2. *for all $g \in \mathbf{G}(\mathbb{K})$ we have $\mathrm{Int}(g) \circ \mathbf{e} = \mathbf{e} \circ \mathrm{Ad}(g)$;*
3. *$\mathbf{e}$ carries dX into dg (where dX and dg are Haar measures on $\mathfrak{g}(\mathbb{K})$ and $\mathbf{G}(\mathbb{K})$, respectively, associated with the same normalization of the Haar measure on $\mathbb{K}$, cf. §3.5).*

For classical groups one can take $\mathbf{e}$ to be the Cayley transform, for all $r > 0$.

THEOREM 2.2. – *There exists a constant $M_{\mathbf{G}} > 0$ that depends only on the absolute root datum of $\mathbf{G}$, such that if $\mathbb{K} \in \mathscr{B}_{M_{\mathbf{G}}}$ and Hypothesis* 2.2.1 *holds for $\mathbf{G}(\mathbb{K})$ with some $r > 0$, then for every admissible representation π of $\mathbf{G}(\mathbb{K})$, its Harish-Chandra character θ_π is a nice function on $\mathbf{G}(\mathbb{K})$; in particular, the integral $\int_{\mathbf{G}(\mathbb{K})} \theta_\pi(g) f(g)\, dg$ converges, and equals $\Theta_\pi(f)$, for all test functions $f \in C_c^\infty(\mathbf{G}(\mathbb{K}))$.*

We prove this theorem in §5.3.1 below.

REMARK 2.3. – DeBacker's result on the local character expansion that we use in the proof of this theorem requires, in its full strength, the assumption that Hypothesis 2.2.1 holds for $r \in \mathbb{R}$ such that $\mathfrak{g}_r = \mathfrak{g}_{\rho(\pi)^+}$, where $\rho(\pi)$ is the depth of π. Here we only use the fact that the local character expansion holds in some (definable) neighborhood of the identity, which is yielded by DeBacker's proof assuming just the existence of the mock exponential for some $r > 0$. Note also that we do not require the mock exponential map to be definable.

Finally, Theorem 2.1 also implies (thanks to a result of DeBacker) that Fourier transforms of general invariant distributions on $\mathfrak{g}(\mathbb{K})$ with support bounded modulo conjugation are represented by nice functions in a neighborhood of the origin. This is Theorem 5.9.

The rest of the main body of the paper is devoted to the proof of these theorems. Two appendices are provided for the reader's convenience—Appendix A contains a brief summary of the definitions and relevant classical results in harmonic analysis on p-adic groups, and

Appendix B summarizes the definitions and results from the theory of motivic integration, which is used in the proofs.

3. Definability of Moy-Prasad filtration subgroups

From now on we will freely use the language of definable subassignments, and constructible motivic functions; please see Appendix B for definitions and all related notation. We start by setting up the definition of the group, Lie algebra, and Moy-Prasad filtration subgroups in Denef-Pas language. As explained in Appendix B, for a group, specific subgroup, etc. to be definable, roughly speaking, means that it can be defined uniformly for all local fields $\mathbb{K}$ of sufficiently large residue characteristic, by formulas in a first-order language of logic, which do not depend on the field themselves.

3.1. Root datum and the group

The first step is to realize the group and its Lie algebra as definable subassignments, so that the methods of motivic integration apply. From now on we assume that the residue characteristic p is large enough so that the group $\mathbf{G}$ splits over a tamely ramified extension of $\mathbb{K}$.

Split reductive groups $\mathbf{G}$ are classified by the root data $\Psi = (X^*, \Phi, X_*, \Phi^\vee)$ consisting of the character group of a split maximal torus $\mathbf{T}$ in $\mathbf{G}$, the set of roots, the cocharacter group, and the set of coroots. The set of possible root data of this form is completely field-independent. Given a root datum Ψ, the group $\mathbf{G}(F)$ is a definable subset of $\mathrm{GL}_n(F)$, defined as the image of a definable embedding $\Xi : \mathbf{G} \hookrightarrow \mathrm{GL}_n$, defined over $\mathbb{Z}[1/R]$ for some large enough R (see [9, §4.1], where such an embedding is denoted by ρ_D, with D denoting the root datum).

We showed that general reductive groups are definable (or, more precisely, appear as members of a constructible family), in [36, Appendix B]. This is based on the fact that every reductive group splits over the separable closure of F, and the F-forms of a group are in one-to-one correspondence with the Galois cohomology set $H^1(F, \mathbf{Aut}(\mathbf{G}))$ (see e.g., [37, §16.4.3]). Here we recall this construction briefly, also introducing the notation for the intermediate unramified extension of $\mathbb{K}$ that will be used below.

Recall that we are assuming that p is large enough so that $\mathbf{G}$ splits over a tamely ramified extension; let e be the ramification index. Then there exists an unramified extension $\mathbb{K}_f/\mathbb{K}$ of some degree f, such that $\mathbf{G}$ splits over a field L, which is a totally ramified Galois extension of $\mathbb{K}_f$. Let θ be a generator of $\mathrm{Gal}(\mathbb{K}_f/\mathbb{K})$ (the Frobenius element); and let $m = fe$ be the degree $[L : \mathbb{K}]$. Let $\Gamma = \mathrm{Gal}(L/\mathbb{K}) = \{\sigma_1, \dots \sigma_m\}$.

We have the exact sequence of Galois groups

$$1 \to \mathrm{Gal}(L/\mathbb{K}_f) \to \mathrm{Gal}(L/\mathbb{K}) \to \mathrm{Gal}(\mathbb{K}_f/\mathbb{K}) \to 1.$$

Let us assume that $\{\sigma_1, \dots \sigma_e\}$ is the subgroup of Γ fixing $\mathbb{K}_f$ and that σ_m projects to θ under the last map.

In [36, §B.4.2], we constructed a definable subassignment $S_{[\Gamma]} \subset h[m + m^3, 0, 0]$, with the following property. Given a local field $\mathbb{K}$ of sufficiently large residue characteristic, $S_{[\Gamma]}$ specializes to the set of tuples $(\bar{b}, \sigma_1, \dots, \sigma_m)$, where:

- $\bar{b}$ is a tuple of coefficients of a minimal polynomial over $\mathbb{K}$ that gives rise to a degree m extension, which we denote by $\mathbb{K}_{\bar{b}}$;
- $\sigma_1, \ldots, \sigma_m$ are $m \times m$ matrices, defining automorphisms of $\mathbb{K}_{\bar{b}}$ over $\mathbb{K}$, and
- the group $\{\sigma_1, \ldots, \sigma_m\}$ is isomorphic to Γ.

We also can, and do, add the condition that $\mathbb{K}_{\bar{b}}$ contains an unramified extension $\mathbb{K}_f$ of degree f, fixed by $\{\sigma_1, \ldots, \sigma_e\}$, and that σ_m projects to θ—the Frobenius element of $\mathbb{K}_f$, by stipulating that the restriction of σ_m is a generator of $\mathrm{Gal}(\mathbb{K}_f/\mathbb{K})$, which can be phrased using Denef-Pas language formulas. We are using $[\Gamma]$ as a subscript (as opposed to Γ) to emphasize that the subassignments $S_{[\Gamma]}$ and $Z_{[\Gamma]}$ depend only on the isomorphism class of Γ, and not on a specific group.

Suppose $\Psi = (X^*, \Phi, X_*, \Phi^\vee)$ is an absolute root datum as above. Then it defines a split reductive group $\mathbf{G}^{\mathrm{spl}}$ over $\mathbb{K}$, and therefore we get a definable subassignment $Z_{[\Gamma]}$ over $S_{[\Gamma]}$ that specializes to the set of 1-cocycles $Z^1(\Gamma, \mathrm{Aut}(\mathbf{G}^{\mathrm{spl}})(\mathbb{K}_{\bar{b}}))$. Finally, suppose $\mathbf{G}$ is a group defined over $\mathbb{K}$ that splits over an extension L as above. Then there exists a tuple $\bar{b}$ such that L is isomorphic to $\mathbb{K}_{\bar{b}}$. Let $\mathbf{G}^{\mathrm{spl}}$ be the split form of $\mathbf{G}$. Then we can think of $\mathbf{G}$ as the group $\mathbf{G}_z$ corresponding to a cocycle $z \in Z^1(\Gamma, \mathrm{Aut}(\mathbf{G}^{\mathrm{spl}})(\mathbb{K}_{\bar{b}}))$. It follows that $\mathbf{G}(\mathbb{K})$ appears as a fibre of a definable subassignment over $Z_{[\Gamma]}$ (by taking $\{z \cdot \sigma_1, \ldots, z \cdot \sigma_m\}$-fixed points, cf. [36, §B.4.3]).

Given an absolute root datum Ψ, there are finitely many possibilities for the root data of the groups $\mathbf{G}$ over $\mathbb{K}$ with the absolute root datum Ψ. Let M_Ψ be the constant such that when $\mathbb{K} \in \mathscr{A}_{M_\Psi} \cup \mathscr{B}_{M_\Psi}$, all possible reductive groups $\mathbf{G}$ over $\mathbb{K}$ with the absolute root datum Ψ (up to isomorphism) appear as fibres of definable subassignments over the subassignments $Z_{[\Gamma]}$, as $[\Gamma]$ runs over the finite set of all the possibilities relevant for Ψ.

3.2. Bruhat-Tits building

Here we follow the notation of [15] and [1] as much as possible. Let us first review this notation. Let $\mathscr{B} = \mathscr{B}(\mathbf{G}, \mathbb{K})$ denote the (enlarged) building of $\mathbf{G}(\mathbb{K})$. Fix a maximal unramified extension $\mathbb{K}^{\mathrm{unr}}$ of $\mathbb{K}$. Let $\mathbf{S}$ be a maximal $\mathbb{K}$-split torus of $\mathbf{G}$. Let $\mathbf{T}$ be the maximal $\mathbb{K}^{\mathrm{unr}}$-split torus of $\mathbf{G}$ containing $\mathbf{S}$. Let $\mathbf{Z}$ be the centralizer of $\mathbf{T}$ in $\mathbf{G}$; it is a maximal torus of $\mathbf{G}$, defined over $\mathbb{K}$. Let L be the extension over which $\mathbf{G}$ splits, as above, and let $\mathbb{K}_f = \mathbb{K}^{\mathrm{unr}} \cap L$ be its maximal unramified part. Then $\mathbf{T}$ splits over $\mathbb{K}_f$. Let $\mathscr{A}$ be the apartment of $\mathbf{T}(\mathbb{K}_f)$ in $\mathscr{B}(\mathbf{G}, \mathbb{K}_f)$. We can identify $\mathscr{A}(\mathbf{S}, \mathbb{K})$ with the $\mathrm{Gal}(\mathbb{K}_f/\mathbb{K})$-fixed points of $\mathscr{A}$.

Let Φ^{unr} be the set of roots of $\mathbf{G}$ relative to $\mathbf{T}$ and $\mathbb{K}_f$, and let $\tilde{\Phi}^{\mathrm{unr}}$ be the set of affine roots of $\mathbf{G}$ relative to $\mathbf{T}$, $\mathbb{K}_f$, and our choice of valuation on $\mathbb{K}$. We observe that Φ^{unr} can be recovered from the root datum and the action $z \cdot \sigma_i$, $1 \le i \le m$, (where z and σ_i are as above in §3.1). Hence, we can use Φ^{unr} and $\tilde{\Phi}^{\mathrm{unr}}$ in the constructions of subassignments over $S_{[\Gamma]}$. In this sense there is no harm in including the (possibly non-reduced) root system Φ^{unr} as part (though redundant) of the given root datum defining the group $\mathbf{G}$.

In this paper, we will only need to use a fixed alcove C in the apartment $\mathscr{A}$ such that $\mathrm{Gal}(\mathbb{K}_f/\mathbb{K})$-fixed points of its closure $\bar{C}$ (in the p-adic topology) contain an alcove of $\mathscr{B}$. Note that $\bar{C}$ is a poly-simplicial set. Moreover, the set $\bar{C}^{z\cdot\theta}$ of $\mathrm{Gal}(\mathbb{K}_f/\mathbb{K})$-fixed points of $\bar{C}$ is also a poly-simplicial set, since the Galois action is compatible with the poly-simplicial structure. In fact, we will need only the following information about the set $\bar{C}^{z\cdot\theta}$:

1. the list of its faces;
2. incidence relations between the faces;
3. a certain finite set of points in $\bar{C}$, called *optimal points*, discussed in the next subsection.

We observe that $\mathscr{A} = X_*(\mathbf{T}) \otimes \mathbb{R}$ is an affine space of dimension determined by Φ^{unr}, and the affine roots (which also are pre-computed from Φ^{unr}) define the hyperplanes in it, which, in turn, determine $\bar{C}$. Thus, the list of faces of $\bar{C}$ can be pre-computed once the root system Φ^{unr} is given. The action of $z \cdot \theta$ determines a permutation τ of Φ^{unr}, which, in turn, allows us to determine the list of faces of $\bar{C}^{z\cdot\theta}$. In summary, the information we need about $\bar{C}^{z\cdot\theta}$ is determined by the root datum Ψ and the permutation τ; and τ is determined by the parameter $z \in Z_{[\Gamma]}$ in a definable way. More precisely, given the root datum Ψ, there is a finite number of possibilities for the list of faces of $\bar{C}^{z\cdot\theta}$, and we can decompose $Z_{[\Gamma]}$ into a disjoint union of finitely many definable subsets, according to which possibility of $\bar{C}^{z\cdot\theta}$ a given cocycle z gives rise to. We will denote these subsets, indexed by the pairs $(\Phi^{\mathrm{unr}}, \tau)$, where Φ^{unr} is a root system and τ is a permutation acting on Φ^{unr}, by $Z_{\Phi^{\mathrm{unr}},\tau}$. Once we have done that, we can assume that the list of faces of $\bar{C}^{z\cdot\theta}$ is part of the data defining $\mathbf{G}$, and use it in the definitions of definable sets with parameters in $Z_{[\Gamma]}$.

Now let us turn to the set of optimal points. We will see that it can also be pre-computed from the root datum.

3.2.1. *Optimal points*. – In [33, §6.1], Moy and Prasad define the set $\mathcal{O}$ of the so-called optimal points; we will denote this set by $\mathfrak{P}$, since the notation $\mathcal{O}$ is reserved for the orbits.

Let C be the alcove in $\mathscr{A}$ that gave rise to the set $\bar{C}^{z\cdot\theta}$ as above. Let Σ be the set of affine roots $\psi \in \tilde{\Phi}^{\mathrm{unr}}$ that satisfy $\psi|_C > 0$, $(\psi - 1)|_C < 0$. This is a finite set that depends only on Φ^{unr}. Further, let $\mathfrak{C}_\Sigma$ be the collection of all the $\mathrm{Gal}(\mathbb{K}_f/\mathbb{K})$-invariant subsets $\mathfrak{S}$ of Σ; this finite collection depends only on the root datum Φ^{unr} and the permutation τ, as above.

Let $\mathfrak{S} \subset \Sigma$ be an element of $\mathfrak{C}_\Sigma$. Now we quote [1, §2.3], where it is stated that there exists a point $x_{\mathfrak{S}} \in \bar{C}$ such that:

(i) $\min_{\psi\in\mathfrak{S}} \psi(x_{\mathfrak{S}}) \geq \min_{\psi\in\mathfrak{S}} \psi(y)$ for all $y \in \bar{C}$;
(ii) $\psi(x_{\mathfrak{S}})$ is rational for all $\psi \in \tilde{\Phi}^{\mathrm{unr}}$;
(iii) $x_{\mathfrak{S}}$ is $\mathrm{Gal}(\mathbb{K}_f/\mathbb{K})$-invariant.

We observe that for the future constructions, we do not need the point $x_{\mathfrak{S}}$ itself, but rather the tuple of its "barycentric coordinates" $(\psi(x_{\mathfrak{S}}))_{\psi\in\Sigma}$. As pointed out in [33, §6.1], finding optimal points is a problem of linear programming. The input for this problem is the field-independent set of affine roots Σ; thus the output is also a field-independent tuple of rational coordinates $\psi(x_{\mathfrak{S}})$.

We denote by $\mathfrak{P}_{\Phi^{\mathrm{unr}},\tau}$ the set

$$\mathfrak{P}_{\Phi^{\mathrm{unr}},\tau} = \{(\psi(x_{\mathfrak{S}}))_{\psi\in\Sigma}\}_{\mathfrak{S}\in\mathfrak{C}_\Sigma}.$$

3.3. Moy-Prasad filtrations

In [33], Moy and Prasad associate with each pair (x, r), where $x \in \mathscr{B}(\mathbf{G}, \mathbb{K})$ and $r \geq 0$, (respectively, $r \in \mathbb{R}$):

– subgroups $\mathbf{G}(\mathbb{K})_{x,r^+} \subset \mathbf{G}(\mathbb{K})_{x,r}$ of $\mathbf{G}(\mathbb{K})$, for $r \geq 0$;
– lattices $\mathfrak{g}(\mathbb{K})_{x,r^+} \subset \mathfrak{g}(\mathbb{K})_{x,r}$ in $\mathfrak{g}(\mathbb{K})$, for $r \in \mathbb{R}$.

When $r = 0$, it is omitted from the notation; thus by definition, $\mathbf{G}(\mathbb{K})_x = \mathbf{G}(\mathbb{K})_{x,0}$, $\mathbf{G}(\mathbb{K})_x^+ = \mathbf{G}(\mathbb{K})_{x,0^+}$, $\mathfrak{g}(\mathbb{K})_x = \mathfrak{g}(\mathbb{K})_{x,0}$, $\mathfrak{g}(\mathbb{K})_x^+ = \mathfrak{g}(\mathbb{K})_{x,0^+}$. The groups $\mathbf{G}(\mathbb{K})_x$ and $\mathbf{G}(\mathbb{K})_x^+$ and the corresponding lattices in the Lie algebra depend only on the facet that contains the point x. Therefore, for a facet F we will denote them by $\mathbf{G}(\mathbb{K})_F$, $\mathbf{G}(\mathbb{K})_F^+$, and $\mathfrak{g}(\mathbb{K})_F$, $\mathfrak{g}(\mathbb{K})_F^+$, respectively.

We will use the fact that for a group that splits over a tamely ramified extension, the filtration subgroups with $r > 0$ can be obtained from its split form by taking Galois-fixed points. We first recall the definitions (this version is quoted from [1], see also [16]) for the split group $\mathbf{G}^{\mathrm{spl}}(L) = \mathbf{G}(L)$, where L is the extension that splits $\mathbf{G}$, as above. First, for any torus $\mathbf{T}$ defined over L, and for any extension E of L, define, for any $r \in \mathbb{R}$,

$$\mathfrak{t}(E)_r := \{H \in \mathfrak{t}(E) \mid \mathrm{ord}(d\chi(H)) \geq r \text{ for all } \chi \in X^*(\mathbf{T})\}.$$

For a torus $\mathbf{T}$ that is split over L, one can define the filtration subgroups of $\mathbf{T}(E)$ simply as follows: for $r \geq 0$, let

$$\mathbf{T}(E)_r := \{t \in \mathbf{T}(E) \mid \mathrm{ord}(\chi(t) - 1) \geq r \text{ for all } \chi \in X^*(\mathbf{T})\}.$$

Similarly, define

$$\mathfrak{t}(E)_{r^+} := \{H \in \mathfrak{t}(E) \mid \mathrm{ord}(d\chi(H)) > r \text{ for all } \chi \in X^*(\mathbf{T})\};$$
$$\mathbf{T}(E)_{r^+} := \{t \in \mathbf{T}(E) \mid \mathrm{ord}(\chi(t) - 1) > r \text{ for all } \chi \in X^*(\mathbf{T})\}.$$

Once and for all, fix a splitting $(\mathbf{B}, \mathbf{T}, \{x_\alpha\})$ of $\mathbf{G}^{\mathrm{spl}}$, defined over $\mathbb{Q}$. This splitting determines a well-defined subgroup $G_0 = \mathbf{G}(\Omega_L)$ of $\mathbf{G}(L)$. Let U_α be the one-parameter subgroup corresponding to x_α: $U_\alpha = 1 + Lx_\alpha$. Let $\psi = \alpha + n \in \tilde{\Phi}$ be an affine root. Define

$$U_\psi = \{g \in U_\alpha \mid g = 1 + tx_\alpha, \quad \mathrm{ord}(t) \geq n\}. \tag{3.1}$$

Note that $U_{\alpha+0} = U_\alpha \cap G_0$. Similarly, one can define the sublattices $\mathfrak{u}(L)_\psi \subset \mathfrak{g}(L)$ (with each $\mathfrak{u}(L)_\psi$ contained in the root subspace $\mathfrak{g}(L)_\alpha$, where α is the gradient of ψ).

Finally, let $x \in \mathscr{A}(\mathbf{T}, L)$, $r \in \mathbb{R}$. Then one can define

$$\mathfrak{g}(L)_{x,r} = \mathfrak{t}(L)_r \oplus \sum_{\{\psi \in \tilde{\Phi} \mid \psi(x) \geq r\}} \mathfrak{u}(L)_\psi$$
$$\mathfrak{g}(L)_{x,r^+} = \mathfrak{t}(L)_{r^+} \oplus \sum_{\{\psi \in \tilde{\Phi} \mid \psi(x) > r\}} \mathfrak{u}(L)_\psi.$$

Similarly for the group, for $r \geq 0$, define $\mathbf{G}(L)_{x,r}$ as the subgroup of $\mathbf{G}(L)$ generated by $\mathbf{T}(L)_r$ and the subgroups U_ψ with $\psi(x) \geq r$, and $\mathbf{G}(L)_{x,r^+}$ as the subgroup of $\mathbf{G}(L)$ generated by $\mathbf{T}(L)_{r^+}$ and the subgroups U_ψ with $\psi(x) > r$.

Let $\tilde{\mathbb{R}}$ be the set $\mathbb{R} \cup \{s^+ \mid s \in \mathbb{R}\}$, with the natural ordering (see e.g., [2, §1.1] for details).

The key fact (quoted in this form from [2, Lemma 2.2.1, Remark 2.2.2]) we use is that since $L/\mathbb{K}$ is a tamely ramified Galois extension,

1. $\mathscr{B}(\mathbf{G}, L)^\Gamma = \mathscr{B}(\mathbf{G}, \mathbb{K})$, and
2. for $x \in \mathscr{B}(\mathbf{G}, \mathbb{K})$, $(\mathfrak{g}(L)_{x,r})^\Gamma = \mathfrak{g}(\mathbb{K})_{x,r}$, for $r \in \tilde{\mathbb{R}}$,
3. $(\mathbf{G}(L)_{x,r})^\Gamma = \mathbf{G}(\mathbb{K})_{x,r}$ for $r \in \tilde{\mathbb{R}}_{>0}$.

Note that if $L/\mathbb{K}$ is unramified, the equality in (3) holds for $r=0$ as well.

For a non-split group, we will use (2) as a definition of the filtration lattices $\mathfrak{g}(\mathbb{K})_{x,r}$, $r\in\tilde{\mathbb{R}}$, and use (3) as the definition of the filtration subgroups $\mathbf{G}(\mathbb{K})_{x,r}$, $r\in\tilde{\mathbb{R}}_{>0}$.

The definition of the parahoric subgroups $\mathbf{G}(\mathbb{K})_{x,0}$ for a group that splits over a ramified extension is more complicated, and does not readily translate to Denef-Pas language (which is our main goal in recalling the definitions). We will show below that for our purposes we can replace $\mathbf{G}(\mathbb{K})_{x,0}$ with the (in general, larger) set $(\mathbf{G}(L)_{x,0})^{\Gamma}$.

DEFINITION 3.1. – Define

$$\mathfrak{g}(\mathbb{K})_r := \bigcup_{x\in\mathscr{B}(\mathbf{G},\mathbb{K})} \mathfrak{g}(\mathbb{K})_{x,r}, \quad \text{and} \quad \mathbf{G}(\mathbb{K})_r := \bigcup_{x\in\mathscr{B}(\mathbf{G},\mathbb{K})} \mathbf{G}(\mathbb{K})_{x,r}.$$

Then the sets $\mathfrak{g}(\mathbb{K})_r$ and $\mathbf{G}(\mathbb{K})_r$ are open and closed, and are both $\mathbf{G}(\mathbb{K})$-domains.

3.4. Definability

Here we collect some basic statements about definability (or in one case, almost-definability) in Denef-Pas language of the filtration subgroups (respectively, the corresponding lattices in the Lie algebra) defined above.

LEMMA 3.2. – *Let $x\in\mathfrak{P}_\Psi$ be an optimal point. Then the sets $\mathfrak{g}(\mathbb{K})_{x,r}$ and $\mathfrak{g}(\mathbb{K})_{x,r^+}$ are definable using the parameter in $z\in Z_{[\Gamma]}$.*

Proof. – Consider the split case first. By definition,

$$\mathfrak{g}(\mathbb{K})_{x,r} = \mathfrak{t}(\mathbb{K})_r \oplus \sum_{\{\psi\in\tilde{\Phi}\mid\psi(x)\geq r\}} \mathfrak{u}(\mathbb{K})_\psi.$$

Since the set of values $\{\psi(x)\}_{\psi\in\Sigma}$ (where Σ is the set of affine roots from the definition of an optimal point) is field-independent by §3.2.1, the indexing set in the sum is a field-independent set determined by the point x; each set $\mathfrak{u}(\mathbb{K})_\psi$ is definable by definition, cf.(3.1). Note that due to natural inclusions between the sets $\mathfrak{u}(\mathbb{K})_\psi$ for the affine roots $\psi=\alpha+n$ with the same gradient α, the above sum in fact has finitely many non-redundant terms, and the number of these terms is field-independent.

The set $\mathfrak{t}(\mathbb{K})_r$ is clearly definable. Hence, the sum is definable. The same argument applies to $\mathfrak{g}(\mathbb{K})_{x,r^+}$. The non-split case follows from the split case. Indeed, by our definition, $\mathfrak{g}(\mathbb{K})_{x,r}$ is the set of Γ-fixed points of $\mathfrak{g}(L)_{x,r}$, which we just proved is definable. The group Γ acts by linear transformations (which depend on the parameter in $z\in Z_{[\Gamma]}$); hence, the set of fixed points is definable, using the parameter $z\in Z_{[\Gamma]}$. We note that in the split case, a similar lemma was first proved by J. Diwadkar, [18, Lemma 78]. □

COROLLARY 3.3. – 1. *Fix $r\in\mathbb{R}$. Then the sets $\mathfrak{g}(\mathbb{K})_r$ and $\mathfrak{g}(\mathbb{K})_{r^+}$ are definable with parameters in $Z_{[\Gamma]}$.*

2. *If we let l vary, then $\{\mathbf{1}_{\mathfrak{g}(\mathbb{K})_l}\}_{l\in\mathbb{Z}}$ is a constructible family of motivic functions indexed by $l\in\mathbb{Z}$.*

Proof. – (1) By [1, Lemma 2.3.2, and Remark 3.2.4], we have:

$$\mathfrak{g}(\mathbb{K})_r = \bigcup_{x \in \mathfrak{P}_{\Phi^{\mathrm{unr}},\tau}} {}^{\mathbf{G}(\mathbb{K})}\mathfrak{g}(\mathbb{K})_{x,r}.$$

The finite set of optimal points $\mathfrak{P}_{\Phi^{\mathrm{unr}},\tau}$ depends only on the parameter in $Z_{[\Gamma]}$ (more specifically, there are finitely many possibilities for this set, and the specific choice is determined by the definable subset $Z_{\Phi^{\mathrm{unr}},\tau} \subset Z_{[\Gamma]}$ from §3.2 that contains the cocycle z defining $\mathbf{G}$). Then by the previous lemma, $\mathfrak{g}(\mathbb{K})_r$ is a finite union (indexed by a field-independent set) of definable subsets, and hence, is definable. For $\mathfrak{g}(\mathbb{K})_{r+}$, there exists an $s \in \mathbb{R}$, such that $\mathfrak{g}(\mathbb{K})_{r+} = \mathfrak{g}(\mathbb{K})_s$ (cf. [1, Remark 3.2.4]); hence, the second statement follows from the first.

(2) Since the set of optimal points is independent of l, we only need to show that for an arbitrary optimal point $x \in \mathscr{B}(\mathbf{G}, \mathbb{K})$, the set $\mathfrak{g}(\mathbb{K})_{x,l}$ depends on l in a definable way. Recall that by our definition,

$$\mathfrak{g}(\mathbb{K})_{x,l} = \left(\mathfrak{t}(L)_l \oplus \sum_{\{\psi \in \tilde{\Phi} \mid \psi(x) \geq l\}} \mathfrak{u}(L)_\psi \right)^\Gamma,$$

where L is the Galois extension that splits $\mathbf{G}$ and $\Gamma = \mathrm{Gal}(L/\mathbb{K})$, as in §3.1. We see directly from the definitions that both the set $\mathfrak{t}(L)_l$, and the indexing set $\{\psi \in \tilde{\Phi} \mid \psi(x) \geq l\}$ are defined by inequalities with l on one side, and a definable function on the other, and thus, depend on l in a definable way. □

There are finitely many conjugacy classes of maximal parahoric subgroups in $\mathbf{G}(L)$, corresponding to the hyperspecial points of $\mathscr{B}(\mathbf{G}, L)$. One would like to prove that parahoric subgroups corresponding to special points in $\mathscr{B}(\mathbf{G}, \mathbb{K})$ are definable. Here we prove a weaker statement, sufficient for the purposes of his article. As always, when talking about definability, the residue characteristic of $\mathbb{K}$ is assumed to be sufficiently large. We note that the split case of the first statement of the following lemma first appeared in [18].

LEMMA 3.4. – *Let $x \in \mathscr{B}(\mathbf{G}, \mathbb{K})$ be a special point. Let L be a finite tamely ramified Galois extension such that $\mathbf{G}$ splits over L, and $\Gamma = \mathrm{Gal}(L/\mathbb{K})$, as above. Then*

1. *The set $K_0 := \mathbf{G}(L)_x^\Gamma$ is a definable (using a parameter $z \in Z_{[\Gamma]}$) subset of $\mathbf{G}(\mathbb{K})$, and*
2. *the set K_0 contains the parahoric subgroup $\mathbf{G}(\mathbb{K})_x$, and there exists a constant c that depends only on the root datum of $\mathbf{G}$ such that $[K_0 : \mathbf{G}(\mathbb{K})_x] \leq q^c$, where q is the cardinality of the residue field of $\mathbb{K}$. If $L/\mathbb{K}$ is unramified, then $K_0 = \mathbf{G}(\mathbb{K})_x$.*
3. *For every optimal point $x \in \bar{C}$ as in §3.2.1 above, for every $r > 0$, the subgroup $\mathbf{G}(\mathbb{K})_{x,r}$ is definable.*

Proof. – The proof of (1) is almost identical to the proof of Lemma 3.2 above. We start with the split case, and examine the definition of $\mathbf{G}(L)_x$. This subgroup depends only on the facet that contains x; and the values $\psi(x)$ of the affine roots are rational numbers (independent of the field) determined by the facet. Thus we have a finite, field-independent set of definable subgroups U_ψ, and a definable subgroup $T(L)_r$. To show that $\mathbf{G}(L)_x$, which, by definition, is generated by these subgroups, is definable, it remains to observe that there is a uniform bound on the length of the word of generators required to write down every element. In fact, it follows from Chevalley commutator relations that this length is bounded by $|\Phi| + r$, where r is the absolute rank of the group $\mathbf{G}$. Hence, $\mathbf{G}(L)_x$ is definable. Then $\mathbf{G}(L)_x^\Gamma$ is definable, using the parameter $z \in Z_{[\Gamma]}$, since Γ acts by definable automorphisms.

The statement (2) is Lemma B.14 in [36].

The proof of the statement (3) in the split case (i.e., for $\mathbf{G}(L)$) is identical to the proof of Part 1 of Lemma 3.3 above, with the lattices $\mathfrak{u}(L)_\psi$ replaced with the subgroups U_ψ, and using the same remark about the bound on the length of the word of generators as in part (1). The statement for $\mathbf{G}(\mathbb{K})_{x,r}$ follows immediately from the split case, since we assume that $r > 0$, by the condition (3) in §3.3. □

3.5. Haar measures

The functions representing the distributions that we study in this paper depend on the choice of the normalizations of the Haar measures on $\mathbf{G}(\mathbb{K})$ and $\mathfrak{g}(\mathbb{K})$. However, we note that the questions we are interested in, namely, those of local integrability and local boundedness, are not sensitive to scaling by a constant, hence, any normalization of Haar measure that makes it a specialization of a motivic measure will work for our purposes. Let us describe some aspects of motivic measures related to differential forms and our choices for the normalizations.

3.5.1. – Given a definable subassignment and a definable differential form on it, there is an associated motivic measure, see [11, §8] and [13, §12.3]. Let us explain how this works, focusing on what we need in this paper, namely uniformity in $\mathbb{K}$ and in families. A definable differential d-form ω on the affine space $\mathbb{K}^N$ with the coordinates $x_1, \ldots, x_N$ is given by a finite sum of terms of the form $f dx_{i_1} \wedge \cdots \wedge dx_{i_d}$ for $0 < i_1 < \cdots < i_d \leq N$ and where the coefficients f are definable functions from $\mathbb{K}^N$ to $\mathbb{K}$. For a d-dimensional $\mathbb{K}$-analytic definable submanifold $A \subset \mathbb{K}^N$, such a d-form ω gives a measure on A, usually denoted by $|\omega|$, since any definable function is $\mathbb{K}$-analytic away from a definable set of smaller dimension. This construction goes back to Weil and is detailed in [5]. Here we explain how to think of the measures defined by volume forms uniformly in $\mathbb{K}$ and in families.

By Lemma 11.3 of [13], and by observing the construction in its proof, the following holds for any definable set $A \subset \mathbb{K}^N$ of dimension d. There exist an integer $s \geq 0$ and a definable bijection $f : A' \subset k_{\mathbb{K}}^s \times A \to A$, induced by the coordinate projection, such that for each $\xi \in k_{\mathbb{K}}^s$, the fiber $A'_\xi := \{x \mid (\xi, x) \in A'\}$ is a $\mathbb{K}$-analytic manifold of dimension at most d for which there is a definable, isometric isomorphism of $\mathbb{K}$-analytic manifolds $\iota : A'_\xi \to B_\xi \subset \mathbb{K}^d$ which is induced by one of the coordinate projections $\mathbb{K}^N \to \mathbb{K}^d$. Clearly there is a definable differential d-form ω_ξ on $\mathbb{K}^d$ whose restriction to B_ξ coincides with the pullback of the restriction of ω to A'_ξ under ι^{-1}. Now ω_ξ is of the form $f dx_1 \wedge \cdots \wedge dx_d$ for some definable function f, and a measure can be defined on $\mathbb{K}^d$ (and hence on each of the B_ξ and the A'_ξ), by defining the measure of an open set U in $\mathbb{K}^d$ as the integral of $|f|$ over U against the Haar measure on $\mathbb{K}^d$, normalized so that the unit ball has measure 1. This gives the measure $|\omega|$ on A. This construction of forms and measures works uniformly in families and also when $\mathbb{K}$ varies, and corresponds to the motivic treatment of forms and measures in [11, §8] and [13, §12.3].

3.5.2. – For a split connected reductive group $\mathbf{G}^{\mathrm{spl}}$, one can explicitly write down a definable differential form, which we denote by ω^{spl}, that gives rise to a Haar measure on $\mathbf{G}^{\mathrm{spl}}(\mathbb{K})$ (see e.g., [9, §7.1]). For a non-split reductive group, we define the invariant differential form on it by pull-back from its split form, using the same construction as Gross uses for an inner form, cf. [21, (4.8)]. Since here we are working with not necessarily inner forms, we need to generalize this construction slightly to allow the volume form to have coefficients in an extension of the field $\mathbb{K}$, and in the end we do not generally get the canonical measure of Gross.

As in §3.1, we think of a general connected reductive group $\mathbf{G}$ as a fibre $\mathbf{G} = \mathbf{G}_z$ of a definable subassignment (constructed in §3.1) over a point $z \in Z_{[\Gamma]}$. Recall that the subassignment $Z_{[\Gamma]}$ specializes to the set of cocycles that give rise to forms of a given split reductive group. For every such form $\mathbf{G}_z$ with $z \in Z_{[\Gamma]}$, we have an isomorphism $\psi_z : L_z \otimes_{\mathbb{K}} \mathbf{G}_z \to L_z \otimes_{\mathbb{K}} \mathbf{G}^{\mathrm{spl}}$, where L_z is the finite extension over which $\mathbf{G}_z$ splits. Being an isomorphism of algebraic groups (defined over L_z), the map ψ_z is definable, using z as a parameter (recall that as discussed in §3.1, using the parameter z allows us to use the elements of L_z in all Denef-Pas formulas).

At the identity, the map ψ_z induces an isomorphism of 1-dimensional vector spaces (which we denote by the same symbol)

$$\psi_z : \wedge^{\dim \mathbf{G}}(L_z \otimes_{\mathbb{K}} \mathfrak{g}_z) \to \wedge^{\dim \mathbf{G}}(L_z \otimes_{\mathbb{K}} \mathfrak{g}^{\mathrm{spl}}),$$

defined over L_z and which is, clearly, definable using the parameter z. (We observe that in the case ψ_z is an inner twisting, this isomorphism is actually defined over $\mathbb{K}$ (cf. [28], pp. 68-69), but we do not need this fact here.) Similarly, for every $x \in \mathbf{G}(\mathbb{K})$, the map ψ_z induces an isomorphism (over L_z) of $\wedge^d T_x(\mathbf{G}) \otimes_{\mathbb{K}} L_z$ and $\wedge^d T_x(\mathbf{G}^{\mathrm{spl}}) \otimes_{\mathbb{K}} L_z$, where T_x denotes the tangent space at x. We still denote this isomorphism by ψ_z.

Let $\omega_z = \psi_z^*(\omega^{\mathrm{spl}})$. Then ω_z is a non-vanishing (since ψ_z is an isomorphism at every $x \in \mathbf{G}(\mathbb{K})$) top-degree differential form on $\mathbf{G}(\mathbb{K})$, of the form $f(x_1, \dots x_d) dx_1 \wedge \cdots \wedge dx_d$ in any coordinate chart on $\mathbf{G}(\mathbb{K})$, with f a regular L_z-valued function on the corresponding coordinate chart, whose coefficients are definable functions of z. Finally, ω_z is $\mathbf{G}(\mathbb{K})$-invariant, since it comes from a $\mathbf{G}(L_z)$-invariant form on $\mathbf{G}_z \otimes L_z$. We can define the measure $|\omega|_{L_z}$ on $\mathbf{G}(\mathbb{K})$ associated with the form ω_z by taking the L_z-absolute value of the function $f(x_1, \dots, x_d)$ on each coordinate chart as above, as in §3.5.1, except replacing the $\mathbb{K}$-absolute value with its unique extension to L_z.

We see that $(\omega_z)_{z \in Z_{[\Gamma]}}$ is a constructible family of definable differential forms on the fibres $\mathbf{G}_z$. Therefore, the measures $|\omega_z|_{L_z}$ form a family of (specializations of) motivic measures. These will be the Haar measures we consider on the groups $\mathbf{G}_z(\mathbb{K})$.

Remark 3.5. – We observe that the resulting Haar measure coincides with the canonical measure defined by Gross [21] in the case when $\mathbf{G}$ splits over an unramified extension, cf. [9, §7.1], as well as in the case when $\mathbf{G}$ is a non-quasi-split inner twist of $\mathbf{G}^{\mathrm{spl}}$, by definition in [21]; in the quasi-split ramified case this relationship still needs to be better understood, cf. [36, §B.5.2].

3.5.3. *Integrability of motivic functions.* – In this article, we occasionally integrate motivic (exponential) functions. We do that without further comment only in the situation when such integration amounts to integration of a function which is clearly integrable, such as a Schwartz-Bruhat function in the sense of [12, §7.5], or a product of a Schwartz-Bruhat function and an additive character. Such functions are known to be integrable in the sense of motivic integration by the results of [12], and the convergence of the p-adic integrals of their specializations is clear. In [8] we prove much stronger results about integrability for motivic functions, allowing us to handle questions of integrability of their specializations in the sense of L^1; here every time there is any issue with the convergence of the integral, we include a careful discussion, and invoke the corresponding transfer principles from [8].

4. Orbital integrals as "motivic distributions"

4.1. The linear dual of $\mathfrak{g}$, and assumptions on p

Let $\mathfrak{g}(\mathbb{K})^*$ denote the linear dual of $\mathfrak{g}(\mathbb{K})$. In [33, §3.5], Moy and Prasad define a filtration of $\mathfrak{g}(\mathbb{K})^*$ by lattices $\mathfrak{g}(\mathbb{K})^*_{x,r}$, where x is a point in the building $\mathscr{B}(\mathbf{G},\mathbb{K})$, and r is a real number, by

$$\mathfrak{g}(\mathbb{K})^*_{x,r} = \{\lambda \in \mathfrak{g}(\mathbb{K})^* \mid \lambda(\mathfrak{g}(\mathbb{K})_{x,(-r)^+}) \subset \mathfrak{p}_{\mathbb{K}}\}.$$

We will need a particularly nice non-degenerate bilinear form on $\mathfrak{g}(\mathbb{K})$; its existence is guaranteed by [4, Proposition 4.1]. We quote this proposition here omitting the details about the list of bad primes.

PROPOSITION 4.1 ([4, Proposition 4.1]). – *If the characteristic of $\mathbb{K}$ is outside a certain finite list of primes determined by the root datum of $\mathbf{G}$, then there exists a $\mathbb{K}$-valued, non-degenerate, bilinear, $\mathbf{G}(\mathbb{K})$-invariant, symmetric form $\langle , \rangle$ on $\mathfrak{g}(\mathbb{K})$ such that, under the associated identification of $\mathfrak{g}(\mathbb{K})$ with $\mathfrak{g}(\mathbb{K})^*$, for all $x \in \mathscr{B}(\mathbf{G},\mathbb{K})$ and all $r \in \mathbb{R}$, the lattice $\mathfrak{g}(\mathbb{K})_{x,r}$ is identified with $\mathfrak{g}(\mathbb{K})^*_{x,r}$.*

Note that the proof of [4, Proposition 4.1] is, in fact, constructive, so we can use the form constructed in the proof from now on to identify $\mathfrak{g}(\mathbb{K})^*$ with $\mathfrak{g}(\mathbb{K})$. This allows us to think of $\mathfrak{g}(\mathbb{K})^*$ as a definable set, identical to $\mathfrak{g}(\mathbb{K})$. Recall that in the beginning, we have fixed an embedding of $\mathfrak{g}^{\mathrm{spl}}$ into $\mathfrak{gl}_n$, defined over $\mathbb{Q}$, that leads to a consistent choice of coordinates on all its forms $\mathfrak{g} = \mathfrak{g}_z$; so let us, once and for all, fix the coordinates on the linear space $\mathfrak{g}(\mathbb{K})$. Let $X = (x_i)_{i=1}^d$ with respect to these coordinates, where d stands for $\dim \mathbf{G}$. Examining the proof of [4, Proposition 4.1], we observe that for $X, Y \in \mathfrak{g}(\mathbb{K})$, the value $\langle X, Y\rangle$ is a definable function of the coordinates of X and Y (using z as a parameter). Thus, to us, $\mathfrak{g}(\mathbb{K})^*$ is the same definable set as $\mathfrak{g}(\mathbb{K})$, with the same Haar measure, though we keep the $*$ for the convenience of interpretation, and sometimes denote this set by $\mathfrak{g}^*(\mathbb{K})$.

4.1.1. *Nilpotent orbits*. – In the rest of this paper, we will need to make assumptions on the characteristic of the field, which guarantee that the nilpotent orbital integrals are sufficiently well-behaved.

For an element of a Lie algebra $\mathfrak{g}(\mathbb{K})$, there are several definitions of "nilpotent" in the literature; it turns out that for large p (or in characteristic zero) they are all equivalent, but we will not use this fact here. We adopt the same definition as in [15], namely, we call an element $X \in \mathfrak{g}(\mathbb{K})$ *nilpotent* if there exists $\lambda \in X_*^{\mathbb{K}}(\mathbf{G})$ such that

$$\lim_{t\to 0} \mathrm{Ad}(\lambda(t))X = 0.$$

Following DeBacker, we denote by $\mathcal{O}(0,\mathbb{K})$ the set of orbits of nilpotent elements.

4.1.2. *The assumptions on* p. – Everywhere from now on, we need to assume that the characteristic of the field is sufficiently large so that all of the following conditions hold:

1. there are finitely many nilpotent orbits in $\mathfrak{g}$;
2. the nilpotent orbital integrals are distributions on $\mathfrak{g}(\mathbb{K})$;
3. the bilinear form from Proposition 4.1 exists.

It is proved in [32] and also [14] that when p is larger than some constant that can be computed from the absolute root datum of $\mathbf{G}$, the first two conditions hold. Let M_Ψ^{nilp} denote the constant such that for $p \geq M_\Psi^{\mathrm{nilp}}$ the above conditions hold. We also enlarge M_Ψ^{nilp} if necessary, and assume that $M_\Psi^{\mathrm{nilp}} \geq M_\Psi$, where M_ψ is the constant of §3.1.

4.2. Orbital integrals as motivic distributions

Let $X \in \mathfrak{g}(\mathbb{K})$. The definition of an orbital integral Φ_X, which is a distribution on the space $C_c^\infty(\mathfrak{g}(\mathbb{K}))$, is recalled in §A.1.2. In this paper, rather than use the approach to the definition of an orbital integral that requires us to fix a Haar measure on $\mathbf{G}(\mathbb{K})$ and a Haar measure on the centralizer $C_G(X)$, we will use a specific $\mathbf{G}(\mathbb{K})$-invariant differential form on the orbit of X. We start by recalling the construction of such a form, that goes back to Kirillov. The version we use here is quoted from [27, §17.3].

4.2.1. *Invariant volume forms on orbits*. – Let $\langle , \rangle$ be the (definable) non-degenerate, symmetric, $\mathbf{G}(\mathbb{K})$-invariant bilinear form on $\mathfrak{g}(\mathbb{K})$ from Proposition 4.1. We use this form to identify $\mathfrak{g}(\mathbb{K})$ with its linear dual $\mathfrak{g}^*(\mathbb{K})$, as above; this also identifies adjoint orbits with co-adjoint orbits, since the form $\langle , \rangle$ is $\mathbf{G}(\mathbb{K})$-invariant.

Let $X \in \mathfrak{g}(\mathbb{K})$, and let $\mathcal{O}_X$ be the adjoint orbit of X. The orbit $\mathcal{O}_X$, as a p-adic manifold, is identified with $\mathbf{G}(\mathbb{K})/C_G(X)$, and its tangent space at X is identified with $\mathfrak{g}/\mathfrak{g}_X$, where, with our identification of $\mathfrak{g}$ and $\mathfrak{g}^*$,

$$\mathfrak{g}_X = \{Y \in \mathfrak{g} \mid \langle X, [Y,Z]\rangle = 0 \text{ for all } Z \in \mathfrak{g}\}. \tag{4.1}$$

Consider the alternating form

$$\omega_X(Y,Z) = \langle X, [Y,Z]\rangle \tag{4.2}$$

on $\mathfrak{g}$. This form descends to a non-degenerate alternating form on $\mathfrak{g}/\mathfrak{g}_X$, [27, §17.3], and therefore, it gives a symplectic form on the tangent space to $\mathcal{O}_X$ at X. Let ω be the symplectic form on $\mathcal{O}_X$ defined by $\omega(X) = \omega_X$. Then ω is a non-degenerate symplectic form on $\mathcal{O}_X$. Note that the form ω depends only on the choice of the bilinear form $\langle , \rangle$. It follows that the orbit $\mathcal{O}_X$ is a symplectic manifold, and in particular, its dimension is even. Let $\dim \mathcal{O}_X = 2m$. Consider the form $\nu_m = \wedge^m \omega$—it is a non-vanishing $\mathbf{G}(\mathbb{K})$-invariant top degree differential form on $\mathcal{O}_X$, that is, a volume form, see [27, §17.3].

Definition 4.2. – In this paper, we denote by Φ_X the orbital integral (as a distribution) over the orbit of X equipped with the measure $|d\nu_m|$, where ν_m is the volume form on $\mathcal{O}_X$ as above (we assume that the bilinear form of Proposition 4.1 is fixed once and for all), and $m = \frac{1}{2}\dim \mathcal{O}_X$.

The next proposition states that for every integer m, $1 \le m \le (\dim \mathbf{G} - \operatorname{rank}\mathbf{G})/2$, the volume forms ν_m defined above form a family of definable volume forms on orbits of dimension $2m$. As a corollary, we obtain that the orbital integrals Φ_X form a family of "motivic distributions", in the sense that the result of the application of Φ_X to any constructible family of definable test functions is a motivic function in X and the parameters indexing the family. We note that a similar statement (just for semisimple elements, and with less detail) appears also in [36, §B.5.3].

Proposition 4.3. – *Fix an integer m, $1 \le m \le (\dim \mathbf{G} - \operatorname{rank}\mathbf{G})/2$. Then for the fields $\mathbb{K}$ of sufficiently large residue characteristic, the set of elements $X \in \mathfrak{g}(\mathbb{K})$ such that* $\dim \mathcal{O}_X = 2m$ *is definable, and by restricting the form ν_m to the orbits $\mathcal{O}_X$ of dimension $2m$, we obtain a family of definable $\mathbf{G}(\mathbb{K})$-invariant volume forms on these orbits.*

Proof. – First, observe that the vector spaces $\mathfrak{g}_X$ of (4.1) form a family of definable sets indexed by X, as is clear from their definitions, since the form $\langle , \rangle$ is definable, as discussed below Proposition 4.1. Now, for every integer k between the rank and the dimension of $\mathbf{G}$ of the same parity as $\dim \mathbf{G}$, let Let $V_k = \{X \in \mathfrak{g}(\mathbb{K}) \mid \dim \mathfrak{g}_X = k\}$. The sets V_k partition $\mathfrak{g}(\mathbb{K})$, and they are definable since for every positive integer k, the statement $\dim \mathfrak{g}_X = k$ can be expressed by means of Denef-Pas language formulas stating that there exists a collection of k vectors forming a basis of the linear space $\mathfrak{g}_X$ (cf. [19]).

Let ω be the differential 2-form on $\mathfrak{g}(\mathbb{K})$, which, at every X, coincides with the alternating form ω_X of (4.2), viewed as an element of $\wedge^2\mathfrak{g}(\mathbb{K})^*$. Recall that we have chosen the coordinates $(x_i)_{i=1}^d$ on $\mathfrak{g}(K)$ (see the discussion after Proposition 4.1). We observe that the coefficients of the form ω are linear functions of x_i, with coefficients that are definable $\mathbb{K}$-valued functions of the parameter z, since the form $\langle , \rangle$ is definable using z as a parameter. For every integer $m \le (\dim \mathbf{G} - \operatorname{rank}\mathbf{G})/2$, consider the $2m$-form $\nu_m := \wedge^m \omega$ on $\mathfrak{g}$. Then it is a definable $2m$-form on the definable subassignment $\mathfrak{g}$, and it has the form $\nu_m = \sum_S f_S \wedge_{i\in S} dx_i$, where S runs over the subsets of cardinality $2m$ of $\{1, \dots, n\}$, and f_S are polynomials with coefficients that are definable functions of z.

By restricting the definable form ν_m from $\mathfrak{g}$ to the definable family of orbits of elements of V_{d-2m}, we obtain a family (indexed by X and z) of definable $2m$-forms on the orbits $\mathcal{O}_X$ with $X \in V_{d-2m}$. By [27, §17.3] (as summarized above), the specialization of the definable form ν_d to $\mathbb{K}$ coincides with a $\mathbf{G}(\mathbb{K})$-invariant volume form on the orbit of X. Thus, we obtain that the measures on the orbits that we have defined above come from a family of definable volume forms. □

Corollary 4.4. – *Given a family of motivic exponential test functions* $\{f_a\}_{a\in S}$*, such that* $f_{a,\mathbb{K}} \in C_c^\infty(\mathbf{G}(\mathbb{K}))$*, where* $S \in$ *Def is some definable subassignment, there exist a constant* $M > 0$*, and a motivic exponential function* F *on* $\mathfrak{g} \times S$*, such that for all local fields* $\mathbb{K} \in \mathcal{A}_M \cup \mathcal{B}_M$*, for every* $X \in \mathfrak{g}_\mathbb{K}$*, we have*

$$\Phi_X(f_{a,\mathbb{K}}) = F_\mathbb{K}(X, a) \quad \text{for all } a \in S_\mathbb{K}.$$

(Note that naturally, both F *and* M *depend on the family* $\{f_a\}_{a\in S}$*.)*

Proof. – The statement follows immediately from the above proposition, the constructions of §3.5.1 and §3.5.2 and the main theorems on motivic integration of Sect. 4 in [12]. □

5. Local integrability in large positive characteristic

5.1. The function η

In [2, Appendix A], R. Huntsinger proved an integral formula for the function representing the Fourier transform of an invariant distribution, which plays a crucial role in our proof. We need to quote some definitions from [2, Appendix A]. Let Λ be an additive character of $\mathbb{K}$ with conductor $\mathfrak{p}_\mathbb{K}$. Here we will make use of the notation defined in Appendix A, see §A.3 below; we will also use the Fourier transform on $\mathfrak{g}$; the definition is recalled in §A.1.3.

Definition 5.1 ([2, Definition A.1.1]). – Let K be an open compact subgroup of $\mathbf{G}(\mathbb{K})$. For $\lambda \in \mathfrak{g}^*(\mathbb{K})$ and $X \in \mathfrak{g}(\mathbb{K})$, define

$$\eta_X(\lambda) = \int_K \Lambda\left(\lambda(\mathrm{Ad}(k)X)\right)\, dk,$$

where dk is the Haar measure on K normalized so that the volume of K is 1.

In this definition, the subgroup K is arbitrary. Later we will need to assume that it is definable. Once and for all, let us pick a definable open compact subgroup, for example, take $K = \mathbf{G}(\mathbb{K})_{x_0,r}$ for some fixed optimal point x_0 in an alcove $\bar{C}$ as in §3.3 (i.e., more precisely, we pick an arbitrary tuple of barycentric coordinates of x_0, that is, an element of the set $\mathfrak{P}_{\Phi^{\mathrm{unr}},\tau}$ of §3.2.1), and an arbitrary $r > 0$, say, $r = 1$ (such a subgroup is definable by Lemma 3.4). Everywhere below, we will use this subgroup K. Let $c_\mathbb{K}$ be the volume of this subgroup $K = \mathbf{G}(\mathbb{K})_{x_0,1} \subset \mathbf{G}(\mathbb{K})$ with respect to the measure on $\mathbf{G}(\mathbb{K})$ defined in §3.5. Then $c_\mathbb{K}$ is a "motivic constant", that is, a motivic function on a point. Since $c_\mathbb{K}$ might not be invertible in the ring of motivic functions, we cannot say that the motivic measure will be normalized to give K volume 1; instead, the constant $c_\mathbb{K}$ will appear as a denominator every time we need to replace the integral over K with a motivic integral.

Definition 5.2. – Fix $r \in \mathbb{R}$. Let $\mathbf{1}_{\mathfrak{g}^*(\mathbb{K})_r}$ denote the characteristic function of the set $\mathfrak{g}^*(\mathbb{K})_r \subset \mathfrak{g}^*(\mathbb{K})$. Let

$$\eta_{X,r} := \eta_X \mathbf{1}_{\mathfrak{g}^*(\mathbb{K})_r}.$$

For every $r \in \mathbb{R}$, the function $\eta_{X,r}$ belongs to the space $C_c^\infty(\mathfrak{g}^*(\mathbb{K})_r)$, see [2, Corollary A.3.4]. Then, given a distribution $\mu \in J(\mathfrak{g}^*(\mathbb{K})_r)$ (where we use the notation recalled in §A.3 below), it makes sense to write $\mu(\eta_X) = \mu(\eta_{X,r})$. Moreover, the map $X \mapsto \eta_{X,r}$ is locally constant in X. The main theorem of [2, Appendix A] is:

THEOREM 5.3 ([2, Theorem A.1.2]). – *Fix $r \in \mathbb{R}$. Let $\mu \in J(\mathfrak{g}^*(\mathbb{K})_r)$. Then $\widehat{\mu}$ is represented on $\mathfrak{g}^{\mathrm{reg}}$ by the locally constant function $X \mapsto \mu(\eta_X)$.*

Using the bilinear form $\langle , \rangle$ to identify $\mathfrak{g}(\mathbb{K})$ with $\mathfrak{g}^*(\mathbb{K})$, we can transport the function η to $\mathfrak{g}(\mathbb{K})$.

DEFINITION 5.4. – Let $\tilde{\eta}$ be the function on $\mathfrak{g}(\mathbb{K})$ defined by:

$$\tilde{\eta}_X(Y) = \int_K \Lambda(\langle \mathrm{Ad}(k)X, Y\rangle)\, dk,$$

and let

$$\tilde{\eta}_{X,r} := \tilde{\eta}_X \mathbf{1}_{\mathfrak{g}(\mathbb{K})_r}.$$

Now, observe that if we identify $\mathfrak{g}(\mathbb{K})$ with $\mathfrak{g}^*(\mathbb{K})$ using the form $\langle , \rangle$, then the space of distributions $J(\mathfrak{g}^*(\mathbb{K}))$ is identified with the space $J(\mathfrak{g}(\mathbb{K}))$. Since the set $\mathfrak{g}^*(\mathbb{K})_r$ is identified with $\mathfrak{g}(\mathbb{K})_r$ for all $r \in \mathbb{R}$, the space $J(\mathfrak{g}^*(\mathbb{K})_r)$ is identified with $J(\mathfrak{g}(\mathbb{K})_r)$. Now Theorem 5.3 can be restated as:

COROLLARY 5.5. – *Fix $r \in \mathbb{R}$. Let $\mu \in J(\mathfrak{g}(\mathbb{K})_r)$. Then $\widehat{\mu}$ is represented on $\mathfrak{g}^{\mathrm{reg}}$ by the locally constant function $X \mapsto \mu(\tilde{\eta}_X)$.*

5.2. Fourier transforms of orbital integrals

In this section, we use the bilinear form of Proposition 4.1. We continue to work with the function $\tilde{\eta}_X$ from Definition 5.4 above.

LEMMA 5.6. – *Up to a constant, the functions $Y \mapsto \tilde{\eta}_X(Y)$ form a constructible family of motivic exponential functions (indexed by $X \in \mathfrak{g}$). More precisely, there exist a motivic exponential function Υ on $\mathfrak{g} \times \mathfrak{g}$, and a constant M_Υ such that for all $\mathbb{K} \in \mathcal{A}_{M_\Upsilon} \cup \mathcal{B}_{M_\Upsilon}$, we have*

$$\frac{1}{c_{\mathbb{K}}} \Upsilon_{\mathbb{K}}(X, Y) = \tilde{\eta}_X(Y),$$

for all $(X, Y) \in \mathfrak{g}(\mathbb{K}) \times \mathfrak{g}(\mathbb{K})$, where $c_{\mathbb{K}}$ is the motivic constant that is defined below Definition 5.1.

Proof. – In the definition of the function $\tilde{\eta}_X$, the compact open subgroup K is arbitrary. Pick the definable subgroup K discussed after the Definition 5.1. Then the statement follows immediately from the main theorem about motivic integrals of motivic exponential functions, [12, Theorem 4.1.1], which states that if we integrate a motivic exponential function with respect to some of its variables, with respect to a motivic measure, the result is a motivic exponential function of the remaining variables. Note that here the integral over K is with respect to the measure that comes from the differential form on $\mathbf{G}(\mathbb{K})$ defined in §3.5, which requires a slight generalization of the framework of motivic integration, which is described in §B.3.1. □

Recall that the locally constant function on $\mathfrak{g}^{\mathrm{reg}}$ representing the Fourier transform of Φ_X is denoted by $\widehat{\mu}_X$ (cf. Appendix A). In order to prove the next theorem about local integrability of the functions $\widehat{\mu}_X$ in positive characteristic, we need a family version of Lemma 5.6 for the functions $\tilde{\eta}_{X,r}$ as r varies. More precisely, it will be sufficient to consider the family of functions $\tilde{\eta}_{X,l}$, as l runs through the integers. Here we use the subscript 'u' in all the notation to emphasize that this is a *uniform in a family* version of the corresponding earlier constructions.

LEMMA 5.7. – *Up to a constant, the functions $Y \mapsto \tilde{\eta}_{X,l}(Y)$ form a constructible family of motivic exponential functions (indexed by $X \in \mathfrak{g}$ and $l \in \mathbb{Z}$). More precisely, there exist a motivic exponential function Υ^u on $\mathfrak{g} \times h[0,0,1] \times \mathfrak{g}$, and a constant M^u_Υ such that for all $\mathbb{K} \in \mathcal{A}_{M^u_\Upsilon} \cup \mathcal{B}_{M^u_\Upsilon}$, we have*

$$\frac{1}{c_{\mathbb{K}}} \Upsilon^u_{\mathbb{K}}(X, l, Y) = \tilde{\eta}_{X,l}(Y),$$

for all $(X, l, Y) \in \mathfrak{g}(\mathbb{K}) \times \mathbb{Z} \times \mathfrak{g}(\mathbb{K})$.

Proof. – By definition, $\tilde{\eta}_{X,l} = \tilde{\eta}_X \mathbf{1}_{\mathfrak{g}(\mathbb{K})_l}$. By Lemma 5.6, the family $\{\tilde{\eta}_X\}_{X\in\mathfrak{g}}$ is a constructible family of motivic exponential functions, and by Corollary 3.3, Part 2, the functions $\mathbf{1}_{\mathfrak{g}(\mathbb{K})_l}$ also form a constructible family indexed by $l \in \mathbb{Z}$; thus, $\tilde{\eta}_{X,l}$ is a constructible family of motivic exponential functions, indexed by $\mathfrak{g} \times \mathbb{Z}$. □

Now we are ready to prove the main theorem of this section.

THEOREM 5.8. – *There exist a constant M depending only on the root datum of $\mathbf{G}$, and a motivic exponential function H on $\mathfrak{g} \times \mathfrak{g}^{\mathrm{reg}}$ such that for every local field $\mathbb{K} \in \mathcal{A}_M \cup \mathcal{B}_M$, for every $X \in \mathfrak{g}$, we have*

$$\widehat{\mu}_X(Y) = \frac{1}{c_{\mathbb{K}}} H_{\mathbb{K}}(X, Y)$$

for all $Y \in \mathfrak{g}^{\mathrm{reg}}_{\mathbb{K}}$, where $c_{\mathbb{K}}$ is the volume of the subgroup K, see the discussion after Definition 5.1.

Proof. – We recall that $\mathfrak{g}(\mathbb{K}) = \bigcup_{r<0} \mathfrak{g}(\mathbb{K})_r$.

For the moment, fix a field $\mathbb{K}$; let $X \in \mathfrak{g}(\mathbb{K})$, and let $r \in \mathbb{R}$ be an arbitrary real number such that $X \in \mathfrak{g}(\mathbb{K})_r$. Then the distribution Φ_X lies in the space $J(\mathfrak{g}(\mathbb{K})_r)$, and by Huntsinger's formula (of Corollary 5.5), we have $\widehat{\mu}_X(Y) = \Phi_X(\tilde{\eta}_{Y,r})$, where $\tilde{\eta}_{Y,r}$ are the functions from §5.1. Note that the right-hand side does not depend on r as long as $X \in \mathfrak{g}(\mathbb{K})_r$. Thus, for every integer l, we have

$$\widehat{\mu}_X(Y) = \Phi_X(\tilde{\eta}_{Y,l}) \text{ for } X \in \mathfrak{g}(\mathbb{K})_l,\ Y \in \mathfrak{g}(\mathbb{K})^{\mathrm{reg}}.$$

By Lemma 5.7, for $\mathbb{K} \in \mathcal{A}_{M^u_\Upsilon} \cup \mathcal{B}_{M^u_\Upsilon}$, we have that $c_{\mathbb{K}}\tilde{\eta}_{Y,l}$ is the specialization to $\mathbb{K}$ of the motivic exponential function Υ^u. Take the family $\tilde{\eta}_{Y,l}$ (indexed by $Y \in \mathfrak{g}^{\mathrm{reg}}$ and $l \in \mathbb{Z}$) as the family of test functions. The theorem now follows by applying Proposition 4.4 to this family. □

5.2.1. *Proof of Theorem 2.1.* – The statement now follows from Theorem 5.8 and Harish-Chandra's theorems [25] (reproduced as Statement (2) in §A.2), by the Transfer of local integrability and Transfer of boundedness principles, [8], quoted here as Theorems B.2 and B.3. Indeed, Harish-Chandra's result asserts nice-ness of $\widehat{\mu}_X$ when the characteristic of $\mathbb{K}$ is zero, and the transfer results of [8], which apply thanks to Theorem 5.8, yield that the conditions of nice-ness (local integrability and local boundedness) are independent of the characteristic of $\mathbb{K}$, once the residue field characteristic is sufficiently large. □

Let us summarize the origins of the constant $M_{\mathbf{G}}^{\mathrm{orb}}$ that provides the restriction on the characteristic in this theorem.

1. $M_{\mathbf{G}}^{\mathrm{orb}} \geq M_\Psi$ (where M_Ψ is the constant defined in §3.1), so that the group $\mathbf{G}(\mathbb{K})$ indeed appears as an element in the constructible family for some parameter in $Z_{[\Gamma]}$ with a suitable Γ.
2. We assume that $M_{\mathbf{G}}^{\mathrm{orb}} \geq M_\Psi^{\mathrm{nilp}}$, which ensures that it is large enough so that there are finitely many nilpotent orbits, and nilpotent orbital integrals are well-defined distributions. It is also large enough so that the bilinear form $\langle , \rangle$ of §4.1 exists.
3. $M_{\mathbf{G}}^{\mathrm{orb}} \geq M_\Upsilon^u$, where M_Υ^u is the constant defined in Lemma 5.7, so that the family of motivic functions of Lemma 5.7 specializes to Huntsinger's functions $\tilde{\eta}_{X,l}$.
4. $M_{\mathbf{G}}^{\mathrm{orb}}$ needs to be large enough so that for the family of functions $\tilde{\eta}_{X,l}$, the motivic integrals over the orbits specialize to the orbital integrals, see Proposition 4.4.
5. Finally, $M_{\mathbf{G}}^{\mathrm{orb}}$ might need to be enlarged further so that transfer of integrability holds for the motivic exponential function $H(X, Y)$ that specializes to $c_{\mathbb{K}}\widehat{\mu}_X(Y)$.

5.3. Harish-Chandra characters

Let $M_{\mathbf{G}}$ be a constant, determined by the root datum of $\mathbf{G}$, such that all the hypotheses listed in §A.5 hold. That is, we take $M_{\mathbf{G}}$ to be the maximum of the constants $M_{\mathbf{M}}^{\mathrm{orb}}$ of Theorem 2.1 for every $\mathbf{M}$ on the list of possible reductive groups that can arise as the connected component of a centralizer of a semisimple element of $\mathbf{G}$.

5.3.1. *Proof of Theorem 2.2.* – Let $\mathbb{K} \in \mathscr{B}_{M_{\mathbf{G}}}$, where $M_{\mathbf{G}}$ is defined above, and let π be an admissible representation of $\mathbf{G}(\mathbb{K})$. First, we prove this theorem in a neighborhood of the identity, more precisely, on the set $\mathbf{G}(\mathbb{K})_r$, where r is chosen so that $\mathfrak{g}_r = \mathfrak{g}_{\rho(\pi)+}$, where $\rho(\pi)$ is the depth of π, or any larger number such that Hypothesis 2.2.1 holds for r. The niceness of θ_π restricted to this neighborhood is immediate; indeed, by Theorem A.7, on $\mathbf{G}(\mathbb{K})_r$ the function θ_π is a finite linear combination of the functions $\widehat{\mu}_{\mathcal{O}}$, and these functions are nice by Theorem 2.1, Part (1). The next statement is an easy technical point: since θ_π is nice, the integral $\int_{\mathbf{G}(\mathbb{K})} \theta_\pi(g) f(g)\, dg$ converges for all test functions f with support contained in $\mathbf{G}(\mathbb{K})_r$, not just those with support contained in the set of regular elements. One still needs to show that this integral coincides with the value of the distribution $\Theta_\pi(f)$ for such functions. This is almost a tautology, based on careful reading of the work of DeBacker. Indeed, even though the local character expansion is stated in [14] as an equality of functions defined on the regular set, in fact it is proved at the level of distributions, without the assumption that the support of the test function is contained in the regular set; see the proof of [14, Theorem 3.5.2], where (using our notation for the orbital integral) it is shown that for any $f \in C_c^\infty(\mathfrak{g}(\mathbb{K})_r)$,

$$\Theta_\pi(f \circ \mathbf{e}^{-1}) = \sum_{\mathcal{O} \in \mathcal{O}(0,\mathbb{K})} c_{\mathcal{O}}(\pi) \widehat{\Phi}_{\mathcal{O}}(f).$$

Then we have, for all $f \in C_c^\infty(\mathbf{G}(\mathbb{K})_r)$:

$$\begin{aligned}\Theta_\pi(f) &= \sum_{\mathcal{O} \in \mathcal{O}(0,\mathbb{K})} c_{\mathcal{O}}(\pi) \widehat{\Phi}_{\mathcal{O}}(f \circ \mathbf{e}) = \sum_{\mathcal{O} \in \mathcal{O}(0,\mathbb{K})} c_{\mathcal{O}}(\pi) \int_{\mathfrak{g}(\mathbb{K})} (f \circ \mathbf{e})(X) \widehat{\mu}_{\mathcal{O}}(X)\, dX \\ &= \int_{\mathbf{G}(\mathbb{K})} \theta_\pi(g) f(g)\, dg,\end{aligned}$$

where now we know that all the integrals converge, by Theorem 2.1.

Now let us prove that θ_π is nice away from the identity as well. Our strategy, roughly speaking, is to prove that θ_π is nice in a neighborhood of every semisimple element, and that any compact set in $\mathbf{G}(\mathbb{K})$ can be covered with finitely many such neighborhoods. Harish-Chandra's descent [23, Chapter 6] allows to reduce the statement about θ_π in a $\mathbf{G}(\mathbb{K})$-neighborhood of a semisimple element γ to a statement about a related distribution θ defined on a neighborhood of γ inside the centralizer $M = C_G(\gamma)$ of γ. Finally, on a suitable neighborhood in M, niceness of θ follows from the local character expansion due to Adler and Korman [3] and the fact that Fourier transforms of nilpotent orbital integrals are nice, as we have shown above.

We proceed with the proof. By our choice of the constant $M_{\mathbf{G}}$, all the hypotheses listed in §A.5 hold. Let $\gamma \in \mathbf{G}(\mathbb{K})$ be an arbitrary semisimple element, let $\mathbf{M}$ be the algebraic group such that $C_G(\gamma) = \mathbf{M}(\mathbb{K})$, let $M = \mathbf{M}(\mathbb{K}) = C_G(\gamma)$, and let all the remaining notation be as in §A.5. Let $r > \max\{\rho(\pi), 2s(\gamma)\}$.

Let θ be the distribution on M defined in [35, Proposition 1] (cf. also [3, §7], where the same definition is explained for the restriction of θ to M_r). By [23, Corollary from Theorem 11, p. 49], if we show that θ is represented by a nice function (which is also denoted by θ, by slight abuse of notation) on the set M, it will follow that the function θ_π is nice on $\mathbf{G}(\mathbb{K})$. So, it remains to prove that θ is a nice function on M.

By Theorem A.8 (and using the notation of that theorem), for $Y \in \mathfrak{m}_r''$, the function $\theta(\gamma \mathbf{e}(Y))$ is a finite linear combination of the functions $\widehat{\mu}_{\mathcal{O}}(Y)$, as $\mathcal{O}$ runs over the set of nilpotent orbits in $\mathfrak{m}$. Let us extend both sides by zero to γM_r. Then we have the equality on γM_r. By Theorem 2.1, $\widehat{\mu}_{\mathcal{O}}(Y)$ are nice functions on $\mathfrak{m}(\mathbb{K})$. Hence, θ is a nice function as a function on γM_r. Now we would like to show that θ is a nice function on M. For this, since it is conjugation-invariant, it suffices to show that the sets ${}^M\gamma M_r$ (which, by definition, are open in M) cover M, as γ runs over the set of semisimple elements in M.

We observe that all semisimple elements of M are covered automatically. Now suppose m is an arbitrary element of M. Our assumptions on the characteristic of $\mathbb{K}$ guarantee that $\mathbf{M}(\mathbb{K})$ contains semisimple and unipotent parts of its elements. Then we have $m = \gamma_s \gamma_u$, with $\gamma_u \in C_M(\gamma_s)$. Then we can conjugate m by an element of $C_M(\gamma_s)$ so that γ_u gets replaced by a conjugate that is as close to the identity as we wish; in particular, we can ensure that it is in M_r, which completes this part of the proof.

Finally, an argument identical to that shown above for a neighborhood of the identity shows that the equality

$$\Theta_\pi(f) = \int_{\mathbf{G}(\mathbb{K})} \theta_\pi(g) f(g)\, dg$$

holds for *all* test functions $f \in C_c^\infty(\mathbf{G}(\mathbb{K}))$. □

5.4. General invariant distributions near the origin

Combining our Theorem 2.1 with DeBacker's results summarized in Appendix A as Theorem A.6, we obtain a partial extension to the large positive characteristic case (in a neighborhood of the origin) of Harish-Chandra's theorem about invariant distributions with support bounded modulo conjugation.

THEOREM 5.9. – *Let* $\mathbf{G}$ *be a connected unramified reductive group with the Lie algebra* $\mathfrak{g}$. *Let* $M_{\mathbf{G}}^{\mathrm{orb}}$ *be the constant from Theorem 2.1. Let* $\mathbb{K} \in \mathcal{B}_{M_{\mathbf{G}}^{\mathrm{orb}}}$, *and let* T *be an invariant distribution on* $\mathfrak{g}(\mathbb{K})$ *with support bounded modulo conjugation. Suppose that the support of* T *is contained in* $\mathfrak{g}(\mathbb{K})_{(-r)^+}$ *with some* $r \in \mathbb{R}$. *Then the restriction of* $\hat{T}$ *to* $\mathfrak{g}(\mathbb{K})_r$ *is represented by a nice function* ϑ_T *on* $\mathfrak{g}(\mathbb{K})_r$.

Proof. – Let $\mathbb{K} \in \mathcal{B}_{M_{\mathbf{G}}^{\mathrm{orb}}}$, and let T be an invariant distribution on $\mathfrak{g}(\mathbb{K})$ with support bounded modulo conjugation. Then the support of T is contained in $\mathfrak{g}(\mathbb{K})_{(-r)^+}$ for sufficiently large $r > 0$. Fix such an r, and let f be an arbitrary test function with support contained in $\mathfrak{g}(\mathbb{K})_r$. Then by Remark A.5, the function $\hat{f}$ belongs to the space $\mathcal{D}_{(-r)^+}$. By Theorem A.6, the restriction of T to the space $\mathcal{D}_{(-r)^+}$ is a linear combination of the nilpotent orbital integrals. Since by definition, $\widehat{T}(f) = T(\hat{f})$, this implies that for all functions $f \in C_c^\infty(\mathfrak{g}(\mathbb{K})_r)$,

$$\widehat{T}(f) = T(\hat{f}) = \sum_{\mathcal{O} \in \mathcal{O}(0,\mathbb{K})} c_{\mathcal{O}} \Phi_{\mathcal{O}}(\hat{f}) = \sum_{\mathcal{O} \in \mathcal{O}(0,\mathbb{K})} c_{\mathcal{O}} \widehat{\Phi}_{\mathcal{O}}(f),$$

with some constants $c_{\mathcal{O}}$. Therefore on $\mathfrak{g}(\mathbb{K})_r$, the distribution $\widehat{T}$ is represented by the function $\vartheta_T = \sum_{\mathcal{O}} c_{\mathcal{O}} \widehat{\mu}_{\mathcal{O}}$, which is nice, by Theorem 2.1. □

Appendix A

Invariant distributions: classical results

In this section we review the notation, definitions, and some of the classical results of harmonic analysis on p-adic groups that are relevant to the present paper.

A.1. Definitions

As before, $\mathbb{K}$ is a non-Archimedean local field (with no assumption on its characteristic), $\mathbf{G}$ is a reductive algebraic group over $\mathbb{K}$, and $\mathfrak{g}$ is its Lie algebra.

A.1.1. *Characters.* – Let (π, V) be an irreducible admissible representation of $\mathbf{G}(\mathbb{K})$. Then the *distribution character* of π is the distribution on the space $C_c^\infty(\mathbf{G}(\mathbb{K}))$ of locally constant, compactly supported functions on $\mathbf{G}(\mathbb{K})$ defined by

$$\Theta_\pi(f) = \mathrm{Tr} \int_{\mathbf{G}(\mathbb{K})} f(g)\pi(g)\, dg$$

(since π is admissible, the linear operator on the right-hand side is of finite rank, and hence its trace is well defined).

It was proved by Harish-Chandra in characteristic zero, and in positive characteristic, by G. Prasad [1, Appendix B] for connected groups, and by J. Adler and J. Korman in general [3, Appendix] that there exists a locally constant function θ_π defined on the set of regular elements $\mathbf{G}(\mathbb{K})^{\mathrm{reg}}$ that represents the distribution character:

$$\Theta_\pi(f) = \int_{\mathbf{G}(\mathbb{K})} \theta_\pi(g) f(g)\, dg, \tag{A.1}$$

for all $f \in C_c^\infty(\mathbf{G}(\mathbb{K})^{\mathrm{reg}})$. The function θ_π is called the Harish-Chandra character.

A.1.2. *Orbital integrals.* – Let $X \in \mathfrak{g}(\mathbb{K})$; we denote by $\mathcal{O}_X$ its adjoint orbit $\mathcal{O}_X = \{\mathrm{Ad}(g)X \mid g \in \mathbf{G}(\mathbb{K})\}$. Then $\mathcal{O}_X$ (with the p-adic topology) is homeomorphic to $\mathbf{G}(\mathbb{K})/C_G(X)$ (where $C_G(X)$ is the stabilizer of X); given Haar measures on $\mathbf{G}(\mathbb{K})$ and $C_G(X)$, the space $\mathbf{G}(\mathbb{K})/C_G(X)$ carries a $\mathbf{G}(\mathbb{K})$-invariant quotient measure. For the fields $\mathbb{K}$ of characteristic zero, it was proved by Deligne and Ranga Rao [34] that when transported to the orbit of X, this measure is a Radon measure on $\mathfrak{g}(\mathbb{K})$, i.e., it is finite on compact subsets of $\mathfrak{g}(\mathbb{K})$ (strictly speaking, it is the group version of this statement that is proved in [34], but in characteristic zero this is equivalent to the Lie algebra version). We denote this quotient measure on $\mathbf{G}(\mathbb{K})/C_G(X)$ by d^*g; then the *orbital integral* at X is the distribution Φ_X on $C_c^\infty(\mathfrak{g}(\mathbb{K}))$ defined by

$$\Phi_X(f) = \int_{\mathbf{G}(\mathbb{K})/C_G(X)} f(\mathrm{Ad}(g)X) d^*g.$$

For the fields of good positive characteristic, convergence of this integral is proved by McNinch [32]. We emphasize that the orbital integral, as a distribution, depends on the normalization of Haar measures on $\mathbf{G}(\mathbb{K})$ and on $C_G(X)$, which together determine a normalization of the invariant measure on the orbit of X. In this paper, instead of fixing these normalizations, we use the normalization of the measures on the orbits that comes from a family of volume forms obtained from the symplectic forms on co-adjoint orbits, see § 4.2.

A.1.3. *Fourier transform.* – Given an additive character Λ of $\mathbb{K}$, we can define the Fourier transform on the Lie algebra $\mathfrak{g}(\mathbb{K})$, which maps functions on $\mathfrak{g}(\mathbb{K})$ to functions on $\mathfrak{g}^*(\mathbb{K})$.

Definition A.1 ([14, §3.1]). – Let dX be a Haar measure on $\mathfrak{g}(\mathbb{K})$. For any $f \in C_c^\infty(\mathfrak{g}(\mathbb{K}))$, let

$$\hat{f}(\lambda) = \int_{\mathfrak{g}(\mathbb{K})} f(X)\Lambda(\lambda(X))\, dX,$$

where $\lambda \in \mathfrak{g}^*(\mathbb{K})$.

The Fourier transform on $\mathfrak{g}^*(\mathbb{K})$ is defined similarly.

REMARK A.2. – As pointed out in [2, §0], there are in fact three objects appearing here: $\mathfrak{g}(\mathbb{K})$, its linear dual $\mathfrak{g}^*(\mathbb{K})$, and its Pontryagin dual $\widehat{\mathfrak{g}}(\mathbb{K})$. The choice of the character Λ is equivalent to the choice of an identification of $\mathfrak{g}^*(\mathbb{K})$ with $\widehat{\mathfrak{g}}(\mathbb{K})$.

From now on, we will assume that the characteristic of $\mathbb{K}$ is large enough so that Proposition 4.1 holds. Then one can use the bilinear form $\langle , \rangle$ from Proposition 4.1 to identify $\mathfrak{g}(\mathbb{K})$ with $\mathfrak{g}^*(\mathbb{K})$. With this identification, the definition of Fourier transform for a function $f \in C_c^\infty(\mathfrak{g}(\mathbb{K}))$ takes the form:

$$\hat{f}(Y) = \int_{\mathfrak{g}(\mathbb{K})} f(X)\Lambda(\langle X, Y\rangle)\, dX,$$

and $\hat{f}$ is again a locally constant compactly supported function on $\mathfrak{g}(\mathbb{K})$.

With the identification of $\mathfrak{g}(\mathbb{K})$ with $\mathfrak{g}^*(\mathbb{K})$ given by the form $\langle , \rangle$, for a distribution T on $C_c^\infty(\mathfrak{g}(\mathbb{K}))$, its Fourier transform is defined to be

$$\widehat{T}(f) = T(\hat{f}).$$

A.2. Local integrability theorems

When the field $\mathbb{K}$ has *characteristic zero*, and $\mathbf{G}$ is connected, the following facts are due to Howe [26] and Harish-Chandra [24], [25]:

1. For an admissible representation π of $\mathbf{G}(\mathbb{K})$, its Harish-Chandra character θ_π is a nice function on $\mathbf{G}(\mathbb{K})$; and in particular, the representation of the distribution character (A.1) holds for *all* $f \in C_c^\infty(\mathbf{G}(\mathbb{K}))$.
2. For an arbitrary element $X \in \mathfrak{g}(\mathbb{K})$ the Fourier transform of the orbital integral Φ_X is represented by a locally constant function $\widehat{\mu}_X$ supported on $\mathfrak{g}(\mathbb{K})^{\mathrm{reg}}$:

$$\Phi_X(\widehat{f}) = \int_{\mathfrak{g}(\mathbb{K})} f(g)\widehat{\mu}_X(g)\, dg$$

for $f \in C_c^\infty(\mathfrak{g}(\mathbb{K}))$; and the function $\widehat{\mu}_X$ is nice.

Clozel [6] extended these results to the case of nonconnected $\mathbf{G}$, still in characteristic zero.

In positive characteristic, the existence of the locally constant function $\widehat{\mu}_X$ of (2) on $\mathfrak{g}^{\mathrm{reg}}$, such that the integral in (2) converges for the test functions f *with support contained in* $\mathbf{G}(\mathbb{K})^{\mathrm{reg}}$ is proved by R. Huntsinger [2, Appendix A], assuming the characteristic is large enough so that the orbital integrals are, indeed, distributions.

For GL_n, SL_n, and their inner forms, the statements (1) and (2) in positive characteristic were proved by Rodier [35] and Lemaire [29], [30], [31]. For general groups, we prove the analogues of these statements in this paper (as they have been out of reach up to now).

A.3. Some spaces of distributions

Everything in this short section is quoted from [14]. Here we state the key result about the distributions with bounded support, which, in this precise quantitative version and this generality is due to DeBacker. Recall the definitions first.

Definition A.3. – Let $J(\mathfrak{g}(\mathbb{K}))$ denote the space of $\mathbf{G}(\mathbb{K})$-invariant distributions on $\mathfrak{g}(\mathbb{K})$, and $J(\mathfrak{g}(\mathbb{K})_r)$ denote the space of $\mathbf{G}(\mathbb{K})$-invariant distributions on $\mathfrak{g}(\mathbb{K})$ with support in $\mathfrak{g}(\mathbb{K})_r$ (where $\mathfrak{g}(\mathbb{K})_r$ is the G-domain defined in §3.3 as a union of Moy-Prasad filtration lattices). We use the similar notation $J(\mathfrak{g}^*(\mathbb{K}))$, $J(\mathfrak{g}^*(\mathbb{K})_r)$ for the dual Lie algebra. Let $J(\mathcal{N})$ denote the space of $\mathbf{G}(\mathbb{K})$-invariant distributions whose support is contained in the set of nilpotent elements $\mathcal{N}$. Let $\mathcal{O}(0,\mathbb{K})$ denote the set of nilpotent orbits in $\mathfrak{g}(\mathbb{K})$.

Definition A.4. – Let $\mathcal{D}_r$ be the space of functions on $\mathfrak{g}(\mathbb{K})$ that can be represented as a finite sum $f = \sum f_i$, where f_i is a complex-valued, compactly supported function on $\mathfrak{g}(\mathbb{K})$, invariant under $\mathfrak{g}(\mathbb{K})_{y_i,r}$ for some $y_i \in \mathcal{B}(\mathbf{G},\mathbb{K})$.

Remark A.5. – We observe that with our choice of the conductor of the character Λ (see §3.5), if a test function f on $\mathfrak{g}(\mathbb{K})$ lies in the space $\mathcal{D}_r$, then the support of its Fourier transform $\hat{f}$ is contained in $\mathfrak{g}(\mathbb{K})_{(-r)^+}$, and if the support of f is contained in $\mathfrak{g}(\mathbb{K})_r$, then $\hat{f} \in \mathcal{D}_{(-r)^+}$.

The following statement is the summary of the part of the main result of [14] that is used in this paper.

Theorem A.6 ([14, Theorem 2.1.5, Corollary 3.4.6 and Remark 2.1.7])

Suppose all the hypotheses mentioned in §4.1 hold. If $r \in \mathbb{R}$, then the distributions $\{\operatorname{res}_{\mathcal{D}_r} \Phi_{\mathcal{O}}\}_{\mathcal{O}\in\mathcal{O}(0,\mathbb{K})}$ form a basis of $\operatorname{res}_{\mathcal{D}_r} J(\mathcal{N})$, and

$$\operatorname{res}_{\mathcal{D}_r} J(\mathfrak{g}(\mathbb{K})_r) = \operatorname{res}_{\mathcal{D}_r} J(\mathcal{N}).$$

A.4. Local character expansion

For an admissible representation π of $\mathbf{G}(\mathbb{K})$, we denote its *depth* (defined in [33, Theorem 5.2]) by $\rho(\pi)$.

Theorem A.7 ([14, Theorem 3.5.2]). – *Let $\mathbb{K}$ be a complete non-Archimedean local field with finite residue field of characteristic p. Let π be an admissible representation of $\mathbf{G}(\mathbb{K})$. Choose r such that $\mathfrak{g}_r = \mathfrak{g}_{\rho(\pi)^+}$. Suppose p is sufficiently large so that the hypotheses from §4.1 are satisfied. Suppose also that Hypothesis* 2.2.1 *holds. Then there exist constants $c_{\mathcal{O}}(\pi) \in \mathbb{C}$ indexed by $\mathcal{O}(0,\mathbb{K})$ such that*

$$\theta_\pi(\mathbf{e}(X)) = \sum_{\mathcal{O}\in\mathcal{O}(0,\mathbb{K})} c_{\mathcal{O}}(\pi)\widehat{\mu}_{\mathcal{O}}(X)$$

for all $X \in \mathfrak{g}(\mathbb{K})_r \cap \mathfrak{g}(\mathbb{K})^{\mathrm{reg}}$.

We observe that the coefficients $c_{\mathcal{O}}(\pi)$ are defined only after the field $\mathbb{K}$ is fixed; at present we do not have any general approach that would yield information about the way they depend on the field, since such an approach to begin with would require a field-independent way to parametrize representations. For toral very supercuspidal representations the beginnings of such a parametrization are discussed in [7].

A.5. Local character expansion near a tame semisimple element

Let $\mathbf{G}$, $\mathbb{K}$, and an admissible representation π of $\mathbf{G}(\mathbb{K})$ be as above. For a semisimple element $\gamma \in \mathbf{G}(\mathbb{K})$, its centralizer $C_G(\gamma)$ is a reductive (not necessarily connected) algebraic group over $\mathbb{K}$. There is a finite list (depending only on the root datum of $\mathbf{G}$) of the possible root data for the (connected components of) the centralizers of semisimple elements in $\mathbf{G}(\mathbb{K})$. We will denote a connected reductive group on this list by $\mathbf{M}^\circ$, and its Lie algebra by $\mathfrak{m}$.

Assume that the characteristic of $\mathbb{K}$ is large enough so that all the hypotheses of 4.1 hold for every possible $\mathbf{M}^\circ$ (the connected component of the centralizer of a semisimple element of $\mathbf{G}$) in the place of $\mathbf{G}$. We also need to assume Hypothesis 2.2.1 for every such $\mathbf{M}^\circ$ (more precisely, we need the slightly weaker Hypothesis 8.5 from [3]). We observe that when the characteristic of $\mathbb{K}$ is large enough, then both $\mathbf{G}^\circ$ and $\mathbf{M}^\circ$ split over the same tame extension; thus, Hypothesis 8.3 of [3] holds; therefore, the restriction of the mock exponential map for $\mathbf{G}(\mathbb{K})$ satisfies the conditions of Hypothesis 8.5 from [3]. Thus, when the residue characteristic of $\mathbb{K}$ is large enough, it is sufficient to assume our Hypothesis 2.2.1, for some $r > 0$.

We need to introduce some more notation from [3].

Let $\gamma \in \mathbf{G}^{\mathrm{ss}}(\mathbb{K})$, and $C_G(\gamma) = \mathbf{M}(\mathbb{K})$ as above. We can consider Moy-Prasad filtration subgroups and the corresponding lattices in $\mathfrak{m}$ (as defined in §3.3, with $\mathbf{M}^\circ$ in place of $\mathbf{G}$); so we have the subgroups $\mathbf{M}^\circ(\mathbb{K})_{x,r}$ for $x \in \mathscr{B}(\mathbf{M}^\circ, \mathbb{K})$, etc. Let $M_r = \mathbf{M}(\mathbb{K})_r$. Following [3, §4], define, for $m \in M$:

$$D_{G/M}(m) = \det\left((\mathrm{Ad}(m) - 1)|_{\mathfrak{g}/\mathfrak{m}}\right)$$

(with the convention that when $M = G$, $D_{G/M} \equiv 1$). Further, for $r \geq 0$, let

$$M'_r = \{m \in M_r \mid D_{G/M}(\gamma m) \neq 0\}$$
$$M''_r = \{m \in M_r \mid \gamma m \in \mathbf{G}(\mathbb{K})^{\mathrm{reg}}\}.$$

Then $M''_r \subset M'_r$ are dense open subsets of M_r.

For an element $\gamma \in \mathbf{G}(\mathbb{K})^{\mathrm{ss}}$, Adler and Korman introduced the notion of *singular depth* $s(\gamma)$ (see [3, Definition 4.1]); we will not need the precise definition here. The main result we need is the following theorem (we are using our earlier notation θ_π for the function representing the distribution character of the representation π).

Let θ be the distribution on M_r obtained from Θ_π via descent, as explained in [3, §7]. It is represented on M''_r by a locally constant function θ, see [3, Lemma 7.5]. Then for θ, an analogue of the local character expansion (in terms of the Fourier transforms of nilpotent orbital integrals on $\mathbf{M}$) holds:

THEOREM A.8 ([3, Corollary 12.10]). – *Let* $r > \max\{\rho(\pi), 2s(\gamma)\}$. *Then*

$$\theta(\gamma \mathbf{e}(Y)) = \sum_{\mathcal{O} \in \mathcal{O}_{\mathfrak{m}}} c_{\mathcal{O}} \widehat{\mu}_{\mathcal{O}}(Y)$$

for all $Y \in \mathfrak{m}''_r := \mathbf{e}^{-1}(M''_r)$, *for some complex coefficients* $c_{\mathcal{O}}$ *that depend on the representation* π, *and where* $\mathcal{O}_{\mathfrak{m}}$ *is the set of nilpotent orbits in* $\mathfrak{m}$.

Appendix B
Constructible exponential functions

Here we recall briefly the main notions and notation used in motivic integration; we refer to the original articles [11], [12], [13] for complete details, and to [10], [20], and especially [9] for exposition.

B.1. Denef-Pas language and definable subassignments

Denef-Pas language is a first order language of logic designed for working with valued fields. The formulas in this language can have variables of three sorts: the valued field sort, the residue field sort, and the value group sort (in our setting, the value group is always assumed to be $\mathbb{Z}$, so we call this sort the $\mathbb{Z}$-sort). Here is the list of symbols used to denote operations and binary relations in this language:

- In the valued field sort: $+$ and $\times$ for the binary operations of addition and multiplication; $\mathrm{ord}(\cdot)$ for the valuation (it is a function from the valued field sort to the $\mathbb{Z}$-sort), and $\overline{\mathrm{ac}}(\cdot)$ for the so-called angular component—a function from the valued field sort to the residue field sort (more about this function below).
- In the residue field sort: $+$ and $\times$ for addition and multiplication.
- In the $\mathbb{Z}$-sort: $+$ for addition; the binary relations $\geq$, and $\equiv_n$ for the congruence modulo n for every $n \in \mathbb{N}$.
- There is also the binary relation $=$ in every sort.

Initially, the symbols for the constants are just 0 and 1 in every sort, and the symbol ∞ in the $\mathbb{Z}$-sort to denote the valuation of 0 (with the natural rules with respect to ∞ and all the operations and relations, such as $\infty \geq n$ is true for all n, etc.).

Given a number field E with the ring of integers Ω, one can make a variant of Denef-Pas language with coefficients in $\Omega[\![t]\!]$ in the valued field sort. This means that a constant symbol is formally added to the valued field sort for every element of $\Omega[\![t]\!]$. We denote this language by $\mathcal{L}_\Omega$. In this paper we use the language $\mathcal{L}_{\mathbb{Z}}$; however, since there might be applications where one wishes to work over a fixed number field E that is different from $\mathbb{Q}$, here we discuss this slightly more general setting.

The formulas in $\mathcal{L}_\Omega$ are built from the symbols for variables in every sort and constant symbols, using the listed above operations and relations, and conjunction, disjunction, negation, and the quantifiers $\forall$ and $\exists$.

Given a valued field $\mathbb{K}$ that is an algebra over Ω *with the choice of the uniformizer of the valuation* ϖ, one can interpret the formulas in $\mathcal{L}_\Omega$ by letting the variables range, respectively, over $\mathbb{K}$, the residue field $k_{\mathbb{K}}$ of $\mathbb{K}$, and $\mathbb{Z}$ (which is the value group of $\mathbb{K}$). The function symbols $\mathrm{ord}(x)$ and $\overline{\mathrm{ac}}(x)$ are interpreted as follows: $\mathrm{ord}(x)$ denotes the valuation of x, and $\overline{\mathrm{ac}}(x)$ denotes the so-called angular component of x: if x is a unit, then $\overline{\mathrm{ac}}(x)$ is the residue of x modulo ϖ (thus, an element of the residue field); for a general $x \neq 0$ define $\overline{\mathrm{ac}}(x) = \overline{\mathrm{ac}}(\varpi^{-\operatorname{ord}(x)}x)$; thus, $\overline{\mathrm{ac}}(x)$ is the first non-zero coefficient of the ϖ-adic expansion of x. By definition, $\overline{\mathrm{ac}}(x) = 0$ if and only if $x = 0$.

In this way, any formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_r)$ with n free (that is, not bound by quantifiers) variables of the valued field sort, m free variables of the residue field sort, and r free variables of the $\mathbb{Z}$-sort yields a subset of $\mathbb{K}^n \times k_{\mathbb{K}}^m \times \mathbb{Z}^r$, namely those points $(x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_r) \in \mathbb{K}^n \times k_{\mathbb{K}}^m \times \mathbb{Z}^r$ where ϕ takes the value "true". Sets of this form for some $\mathcal{L}_\Omega$-formula ϕ are called *definable*. A function is called definable if its graph is a definable set.

Let us (temporarily) denote the category of fields L which admit an injective ring homomorphism from Ω to L by $\underline{\mathrm{Flds}}_\Omega$. We write $h[n, m, r]$ for the functor from $\underline{\mathrm{Flds}}_\Omega$ to $\underline{\mathrm{Sets}}$ that sends L to $L((t))^n \times L^m \times \mathbb{Z}^r$. Any formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_r)$ as above in particular induces a map sending any $L \in \underline{\mathrm{Flds}}_\Omega$ to a subset of $L((t))^n \times L^m \times \mathbb{Z}^r$. A map obtained in this way from an $\mathcal{L}_\Omega$-formula is called a *definable subassignment of* $h[n, m, r]$ (or simply a *definable subassignment* if we do not want to specify n, m, r). A similar notion of assignments was first introduced in [17].

A morphism of definable subassignments consists of a family of maps between the corresponding definable sets for each $L \in \underline{\mathrm{Flds}}_\Omega$, such that the family of graphs of these maps is a definable subassignment.

DEFINITION B.1. – The category of definable (in the language $\mathcal{L}_\Omega$) subassignments of $h[n, m, r]$ with some integers $n, m, r \geq 0$ is denoted by Def. The category of definable subassignments of $h[0, m, 0]$ for some $m > 0$ is denoted by RDef (thus, the subassignments in RDef are defined by formulas that can *only* have free variables of the residue field sort).

We also need the "relative" situation: suppose $S \in \mathrm{Def}$ is a definable subassignment. Then one can define Def_S—the category of definable subassignments over S—to be the category of definable subassignments with a fixed morphism to S (with morphisms, naturally, defined to be morphisms over S). The category RDef_S consists of subassignments of $S \times h[0, n, 0]$ with the projection onto the first coordinate as the fixed morphism to S. If X is a definable subassignment, we write $X[m, n, r]$ for $X \times h[m, n, r]$.

B.2. Specialization

The main point of using the language $\mathcal{L}_\Omega$ is *specialization*, which we survey briefly, while referring to [10, §6.7] or [20, §5] and [13] for a more extensive exposition. Let $\mathcal{A}$ be the collection of completions of algebraic extensions of the base field E, and let $\mathcal{B}$ be the collection of positive-characteristic local fields that admit a homomorphism from Ω—these are the collections of fields to which we would like to apply a transfer principle. Strictly speaking, we should write $\mathcal{A}_\Omega$ and $\mathcal{B}_\Omega$, but Ω is usually clear from the context; and in the main body of this paper $\Omega = \mathbb{Z}$, so we drop this subscript.

Let S be a definable subassignment of $h[n, m, r]$ for some m, n, and r; suppose that S is defined by an $\mathcal{L}_\Omega$-formula ϕ. Let $\mathbb{K} \in \mathcal{A} \cup \mathcal{B}$ be a discretely valued field, with a choice of the uniformizer of the valuation ϖ. Then the formula ϕ can be interpreted in $\mathbb{K}$ to give a subset $S_{\mathbb{K}}$ of $\mathbb{K}^n \times k_{\mathbb{K}}^m \times \mathbb{Z}^r$. The set $S_{\mathbb{K}}$ is called the *specialization* of the subassignment S to $\mathbb{K}$.

For two formulas ϕ_1 and ϕ_2 defining the same subassignment S, there exists a constant M, such that for $\mathbb{K} \in \mathscr{A}_M \cup \mathscr{B}_M$ their specializations to $\mathbb{K}$ give the same set regardless of which formula we use. We emphasize that a definable subassignment can be specialized *both* to the fields of characteristic zero and those of positive characteristic, and the specialization is well defined as long as the residue characteristic is sufficiently large.

B.3. Motivic exponential functions

For a definable subassignment X, the ring of the so-called *constructible motivic functions* on X, denoted by $\mathscr{C}(X)$, is defined in [11]. We will use a slight generalization of this, described below in §B.3.1. The elements of $\mathscr{C}(X)$ are, essentially, formal constructions defined using the language $\mathscr{L}_\Omega$. For the sake of brevity (and consistency with [8]), we drop the word "constructible" everywhere from now on, and refer to the elements of $\mathscr{C}(X)$ as "motivic functions". An important feature of motivic functions is specialization to functions on definable subsets of affine spaces over discretely valued fields. Namely, let $F \in \mathscr{C}(X)$. Let $\mathbb{K} \in \mathscr{A} \cup \mathscr{B}$ be a non-Archimedean local field. Let ϖ be the uniformizer of the valuation on $\mathbb{K}$. Then the motivic function F specializes to a $\mathbb{Q}$-valued function $F_{\mathbb{K}}$ on $X_{\mathbb{K}}$, for all fields $\mathbb{K}$ of residue characteristic bigger than a constant that depends only on the choice of the $\mathscr{L}_\Omega$-formulas defining F and X. As explained in [9, §2.9], one can tensor the ring $\mathscr{C}(X)$ with $\mathbb{C}$, and then the specializations $F_{\mathbb{K}}$ of elements of $\mathscr{C}(X) \otimes \mathbb{C}$ form a $\mathbb{C}$-algebra of functions on $X_{\mathbb{K}}$, which we denote by $\mathscr{C}_{\mathbb{K}}(X_{\mathbb{K}})$. See [8, §4.2.5] for a general form of a motivic function.

Further, for a subassignment X as above, the ring of motivic constructible exponential functions $\mathscr{C}^{\exp}(X)$ is defined in [12]. The elements of this ring specialize to what we call (p-adic) constructible exponential functions. In the motivic setting, we also drop the word "constructible" from now on. In order to get a specialization of a motivic exponential function, one needs to choose, in addition to a local field $\mathbb{K}$ with uniformizer ϖ, an additive character Λ of $\mathbb{K}$ satisfying the condition

$$\Lambda(x) = \exp\left(\frac{2\pi i}{p} \mathrm{Tr}_{k_{\mathbb{K}}}(\bar{x})\right) \tag{B.1}$$

for $x \in \Omega_{\mathbb{K}}$. Here, p is the characteristic of $k_{\mathbb{K}}$, $\bar{x} \in k_{\mathbb{K}}$ is the reduction of x modulo ϖ, and $\mathrm{Tr}_{k_{\mathbb{K}}}$ is the trace of $k_{\mathbb{K}}$ over its prime subfield (see [8, §§4.1, 4.2.6] for details). The set of characters of $\mathbb{K}$ satisfying the condition (B.1) is denoted by $\mathscr{D}_{\mathbb{K}}$.

Given a field $\mathbb{K} \in \mathscr{A} \cup \mathscr{B}$ as above, with a uniformizer ϖ and an additive character Λ as in (B.1), we consider the $\mathbb{Q}$-algebra of functions on $X_{\mathbb{K}}$ generated by the specializations of motivic exponential functions. As above, we can tensor it with $\mathbb{C}$; we denote the resulting $\mathbb{C}$-algebra by $\mathscr{C}^{\exp}_{\mathbb{K},\Lambda}(X_{\mathbb{K}})$. See [8, §4.2.9, §3.2] for details.

We often need to talk about motivic (respectively, motivic exponential) functions on the set of $\mathbb{K}$-points of an algebraic group $\mathbf{G}$ or its Lie algebra $\mathfrak{g}$. We observe that any affine algebraic variety V (for example, $V = \mathbf{G}$ or $V = \mathfrak{g}$) naturally gives a definable subassignment of $h[m, 0, 0]$ with some m; let us for the moment denote this subassignment by $\tilde{V}$. Then $\tilde{V}_{\mathbb{K}} = V(\mathbb{K})$, for all non-Archimedean local fields $\mathbb{K}$ of sufficiently large residue characteristic. However, to keep notation simple, we simply talk about motivic functions on $V(\mathbb{K})$ for a variety V, implying that we replace $V(\mathbb{K})$ with $\tilde{V}_{\mathbb{K}}$; it is in this sense that we talk about motivic functions on $\mathbf{G}(\mathbb{K})$ or $\mathfrak{g}(\mathbb{K})$ in this paper.

In [11], Cluckers and Loeser defined a class $\mathrm{I}C(X)$ of *integrable* motivic functions, which is closed under integration with respect to parameters (where integration is with respect to the *motivic measure*). Given a local field $\mathbb{K}$ with a choice of the uniformizer, these functions specialize to integrable (in the classical sense) functions on $X_{\mathbb{K}}$, and motivic integration specializes to the classical integration with respect to an appropriately normalized Haar measure, when the residue characteristic of $\mathbb{K}$ is sufficiently large. In [12] the definition of "integrable" and the notion of motivic integration are extended to motivic *exponential* functions. Moreover, there is a notion of "motivic" Fourier transform that specializes to the classical Fourier transform. In [8], we provide a more general treatment of the issues of integrability, essentially, proving that any motivic exponential function whose specializations are L^1-integrable for almost all $\mathbb{K} \in \mathscr{C}$ can be "interpolated" by a motivic exponential function integrable in the sense of [12] (where by "interpolation" we mean that it has the same specilaizations for every $\mathbb{K} \in \mathscr{C}_M \cup \mathscr{B}_M$ for a sufficiently large M), cf. [8, Theorem 4.3.3].

B.3.1. *A generalization of motivic exponential functions*. – In some places in this paper, we have to take roots of q (the cardinality of the residue field), e.g., in the notion of "nice" in §2.1, and in the definition of the measure on $\mathbf{G}(\mathbb{K})$ in §3.5. To this end, we generalize the notion of motivic (exponential) functions as follows. Let F be a motivic (exponential) function on some S, let $f : S \to \mathbb{Z}$ be a definable morphism, and let $r \geq 1$ be an integer. We call any expression H of the form $F\mathbb{L}^{\frac{1}{r}f}$ a motivic (exponential) function on S, and we call the functions $F_{\mathbb{K}} q_{\mathbb{K}}^{\frac{1}{r}f_{\mathbb{K}}}$ on $S_{\mathbb{K}}$ the specializations $H_{\mathbb{K}}$ of H for $\mathbb{K} \in \mathscr{C}_M \cup \mathscr{B}_M$ of large residue field characteristic $q_{\mathbb{K}}$. We also allow finite linear combinations of such expressions. All classical results about motivic (exponential) functions easily generalize to this setting, by splitting S into r disjoint parts according to $f \mod r$.

B.3.2. *Conventions*. – For the sake of brevity, we use the term "motivic (exponential) function" a little loosely, in the sense that we sometimes refer to a p-adic function by this collection of adjectives if it is obtained by specialization from a motivic exponential function. Precisely, we say that a function f on some subset of an affine space over a non-Archimedean local field $\mathbb{K}$ is a motivic (exponential) function if the following conditions hold:

1. the domain of f is a specialization $S_{\mathbb{K}}$ of some definable subassignment $S \in \mathrm{Def}$; and
2. there exists a motivic (exponential) function F on S, and in the case when "exponential" is relevant, an additive character $\Lambda \in \mathscr{D}_{\mathbb{K}}$, such that $f = F_{\mathbb{K}}$ (respectively, $f = F_{\mathbb{K},\Lambda}$).
3. If $\mathbb{K}$ is allowed to vary, then the definable subassignment S and the motivic (exponential) function F can be taken independently of $\mathbb{K}$.

A similar convention applies to integration and Fourier transform; for example, when we integrate the specialization of a motivic exponential function (with respect to a p-adic Haar measure), we think of the integral as the specialization of the corresponding motivic integral.

Sometimes, we find it convenient to talk about families of motivic functions, of definable sets, or even of definable volume forms. We occasionally use the term "constructible family" to emphasize that the objects in question depend on the parameter indexing the family in a definable way. Thus, by a constructible family of motivic (exponential) functions $\{f_a\}_{a\in S}$

on X, where X and S are definable subassignments, we mean nothing but a motivic (exponential) function on $S \times X$.

B.4. Motivic exponential functions and representatives

As noted briefly in §B.2 above, and explained in [8] in detail, the specialization of a subassignment (and therefore, of a motivic exponential function) depends, in principle, on the choice of specific formulas used to define the subassignment and the function in question. Given one such choice of formulas, there exists a constant $M > 0$ such that for the fields $\mathbb{K} \in \mathcal{A}_M \cup \mathcal{B}_M$, the specialization to $\mathbb{K}$ is well defined. In [8], the choice of formulas is referred to as "the choice of representatives", meaning that a subassignment is thought of as an equivalence class of formulas.

We observe that in this paper (as well as in all applications of motivic integration so far) whenever we prove that a certain object or function is "motivic", it automatically comes with a collection of formulas defining it; that is, the motivic objects always appear with the choice of representatives in the sense of [8, §4.2.2] (we emphasize again that the choice of representatives amounts to a choice of specific formulas defining the given subassignment). Since all our definable objects come with a choice of formulas defining them, we can assume that this is the choice of representatives built into all the constants that provide the lower bounds on residue characteristic in all our results.

B.5. Transfer of integrability and boundedness

We quote the transfer of integrability and transfer of boundedness principles from [8]. (For simplicity, we quote the version without parameters, that is, we take the parametrizing space X to be a point in [8, Theorem 4.4.1] and [8, Theorem 4.4.2].)

Theorem B.2 ([8, Theorem 4.4.1]). – *Let F be a motivic exponential function on $h[n, 0, 0]$. Then there exists $M > 0$, such that for the fields $\mathbb{K} \in \mathcal{A}_M \cup \mathcal{B}_M$, the truth of the statement that $F_{\mathbb{K},\Lambda}$ is (locally) integrable for all $\Lambda \in \mathcal{D}_{\mathbb{K}}$ depends only on the isomorphism class of the residue field of $\mathbb{K}$.*

Theorem B.3 ([8, Theorem 4.4.2]). – *Let F be a motivic exponential function on $h[n, 0, 0]$. Then, for some $M > 0$, for the fields $\mathbb{K} \in \mathcal{A}_M \cup \mathcal{B}_M$, the truth of the statement that $F_{\mathbb{K},\Lambda}$ is (locally) bounded for all $\Lambda \in \mathcal{D}_{\mathbb{K}}$ depends only on the isomorphism class of the residue field of $\mathbb{K}$.*

The main technical result of this paper is that Fourier transforms of orbital integrals are represented on the set of regular elements by motivic exponential functions. Thus, the transfer principles apply, yielding local integrability (respectively, local boundedness) for $\mathbb{K} \in \mathcal{B}_M$ for large M.

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