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STATISTICAL PROPERTIES OF ONE-DIMENSIONAL MAPS UNDER WEAK HYPERBOLICITY ASSUMPTIONS

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ABSTRACT. – For a real or complex one-dimensional map satisfying a weak hyperbolicity assumption, we study the existence and statistical properties of physical measures, with respect to geometric reference measures. We also study geometric properties of these measures.

RÉSUMÉ. – Nous étudions l'existence et des propriétés statistiques des mesures physiques d'une application unidimensionnelle réelle ou complexe satisfaisant une hypothèse d'hyperbolicité faible, par rapport à une mesure de référence géométrique. Nous étudions aussi des propriétés géométriques de ces mesures.

1. Introduction

We study statistical properties of real and complex one-dimensional maps, under weak hyperbolicity assumptions. For such a map f we are interested in the existence and statistical properties of an invariant probability measure ν, supported on the Julia set of f, that is absolutely continuous with respect to a natural reference measure. The reference measure μ could be the Lebesgue measure on the phase space, or more generally a conformal measure supported on the Julia set. Such a measure ν, when ergodic, has the important property of being a *physical measure* with respect to μ. That is, for a subset E of the phase space that has positive measure with respect to μ, the measure ν describes the asymptotic distribution of each forward orbit of f starting at a point in E.

For maps that are uniformly hyperbolic on their Julia sets, the pioneering work of Sinaĭ, Ruelle, and Bowen [53, 4, 51] gives a satisfactory solution to these problems. See also [55] for an analysis closer to the approach here. However, a one-dimensional map with a critical point in its Julia set fails to be uniformly hyperbolic in a severe way. In order to control the

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effect of critical points in the Julia set, people often assume strong expansion along the orbits of critical values. See for example [8, 41, 2, 43, 23, 57, 5] in the real setting, and [44, 15, 16] in the complex setting. See [3] for a broad view.

For smooth interval maps, Bruin, Luzzatto and van Strien gave mixing rates upper bounds closely related to the growth of derivatives at the critical values [5]. Our results reveal that, rather surprisingly, the mixing rates can be much faster than the growth of derivatives at critical values: an interval map f satisfying the *Large Derivatives* condition

$$\lim_{n\to\infty} |Df^n(v)| = \infty, \text{ for each critical value } v \text{ of } f \text{ in the Julia set}$$

together with other mild conditions, has a super-polynomially mixing absolutely continuous invariant measure.

In the complex setting we show a similar result for a non-renormalizable polynomial f. These are the first non-exponential upper bounds for mixing rates in the complex setting. For a general rational map f without parabolic cycles we show that the summability condition with exponent 1 is enough to guarantee the existence of a super-polynomially mixing absolutely continuous invariant measure.

We shall now state two results, one for the real case and another for the complex case, and make comparisons with previous results. In order to avoid technicalities, we state these results in a more restricted situation than what we are able to handle. See § 2.1 for a more general formulation of our results and for precisions.

Recall that given a continuous map f acting on a compact metric space X, an f-invariant Borel probability measure ν is called *(strongly) mixing* if for all $\varphi, \psi \in L^2(X, \nu)$,

$$\mathscr{C}_n(\varphi, \psi) := \int_X \varphi \circ f^n \psi d\nu - \int_X \varphi d\nu \int_X \psi d\nu \to 0$$

as $n \to \infty$. Given $\gamma > 0$, we say that ν is *polynomially mixing of exponent* γ if for each essentially bounded function φ and each Hölder continuous function ψ, there exists a constant $C(\varphi, \psi) > 0$ such that

$$|\mathscr{C}_n(\varphi, \psi)| \le C(\varphi, \psi) n^{-\gamma}, \text{ for all } n = 1, 2, \ldots$$

Moreover, we say that ν is *super-polynomially mixing* if for all $\gamma > 0$ it is polynomially mixing of exponent γ.

THEOREM I. – *Let X be a compact interval and let $f : X \to X$ be a topologically exact C^3 multimodal map with non-flat critical points, having only hyperbolic repelling periodic points. Assume that for each critical value v of f we have*

$$\lim_{n\to\infty} |Df^n(v)| = \infty.$$

Then f has a unique invariant probability measure that is absolutely continuous with respect to Lebesgue measure. Moreover, this invariant measure is super-polynomially mixing.

The topological exactness is assumed to obtain uniqueness and the mixing property of the absolutely continuous invariant measure. For an interval map as in the theorem, the existence of the absolutely continuous invariant measure was proved before in [7, 6], although the argument in this paper provides an alternative proof. As mentioned above, our result on mixing rates significantly strengthens the previous result [5], where super-polynomial mixing rates were only proved under the condition that for each $\alpha > 0$ and each critical value v of f, we have $|Df^n(v)|/n^\alpha \to \infty$. In fact, only assuming $\liminf_{n\to\infty} |Df^n(v)|$ sufficiently large, our methods provide a definite polynomial mixing rate.

We now state a result for a complex rational map f of degree at least two. Often the Lebesgue measure of the Julia set $J(f)$ of f is zero. So the Lebesgue measure cannot be used as a reference measure in general. Instead people often use a conformal measure on the Julia set as a reference measure. Following Sullivan [55], we use conformal measures of exponent $\mathrm{HD}(J(f))$ as geometric reference measures, where $\mathrm{HD}(J(f))$ denotes the Hausdorff dimension of $J(f)$.

THEOREM II. – *Let f be either one of the following:*

1. *an at most finitely renormalizable polynomial of degree at least two, that has only hyperbolic periodic points, and such that for each critical value v of f in the Julia set,*
$$\lim_{n\to\infty} |Df^n(v)| = \infty;$$
2. *a complex rational map of degree at least two, without parabolic cycles, and such that for each critical value v of f in the Julia set,*
$$\sum_{n=1}^{\infty} \frac{1}{|Df^n(v)|} < \infty.$$

Then f has a unique conformal measure μ of exponent $\mathrm{HD}(J(f))$*; this measure is supported on the conical Julia set and its Hausdorff dimension is equal to* $\mathrm{HD}(J(f))$. *Furthermore, there is a unique invariant probability measure ν that is absolutely continuous with respect to μ, and the measure ν is super-polynomially mixing.*

Recall that for an integer $s \geq 1$, a complex polynomial f is *renormalizable of period s* if there are Jordan disks $U \Subset V$ such that the following hold:

- $f^s : U \to V$ is proper of degree at least two;
- the set $\{z \in U : f^{sn}(z) \in U \text{ for all } n = 1, 2, \ldots\}$ is a connected proper subset of $J(f)$;
- for each critical point c of f, there exists at most one $j \in \{0, 1, \ldots, s-1\}$ with $c \in f^j(U)$.

We say that f is *infinitely renormalizable* if there are infinitely many s for which f is renormalizable of period s.

For a complex polynomial f, hypothesis 1 of Theorem II is weaker than hypothesis 2.

Theorem II gives the first non-exponential mixing rates in the complex setting. As for the existence of the absolutely continuous invariant measure, this result gives a significant improvement of the previous result of Graczyk and Smirnov [16, Theorem 4]. Their result applies to a rational map f satisfying the following strong form of the summability condition, for a sufficiently small $\alpha \in (0, 1)$,

$$\sum_{n=1}^{\infty} \frac{n}{|Df^n(v)|^\alpha} < \infty, \text{ for every critical value } v \text{ of } f \text{ in } J(f). \tag{1}$$

For each $\alpha \in (0,1)$, the Fibonacci quadratic polynomial f_0 fails to satisfy this condition, although for every $\alpha > 0$

$$\sum_{n=1}^{\infty} \frac{1}{|Df_0^n(v)|^{\alpha}} < \infty, \text{ where } v \text{ is the finite critical value of } f_0,$$

see Remark 2.5. So Theorem II implies that the Fibonacci quadratic polynomial f_0 has a super-polynomially mixing absolutely continuous invariant measure.

REMARK 1.1. – In the proof of Theorems I and II we construct the absolutely continuous invariant measure by way of an inducing scheme with a super-polynomial tail estimate and some additional technical properties, see § 2.2. The results of [58] imply that this measure is super-polynomially mixing and that it satisfies the Central Limit Theorem for Hölder continuous observables. It also follows that the absolutely continuous invariant measure has other statistical properties, such as the Local Central Limit Theorem, and the Almost Sure Invariance Principle, see *e.g.,* [14, 35, 36, 56].

For a map f satisfying the hypotheses of Theorem I or Theorem II we show the density of the absolutely continuous invariant measure has the following regularity: if we denote by ℓ the maximal order of a critical point of f in the Julia set, then for each $p \in (0, \ell/(\ell-1))$ the invariant density belongs to the space L^p. We note that for each $p > \ell/(\ell-1)$ the invariant density does not belong to L^p, see Remark 2.17. In the real case the regularity of the invariant density was shown in [6, Main Theorem]; see also [43] for the case of unimodal maps satisfying a summability condition with a certain exponent. In the complex setting our result seems to be the first unconditional one. For rational maps satisfying a summability condition with a sufficiently small exponent, a similar result was shown in [16, Corollary 10.1] under an integrability assumption on the conformal measure μ that was first formulated in [44]. Actually, in the complex case we shall prove for each $\varepsilon > 0$ the following regularity of the conformal measure μ: for every sufficiently small $\delta > 0$ we have for every $x \in J(f)$,

$$\delta^{\mathrm{HD}(J(f))+\varepsilon} \leq \mu(B(x,\delta)) \leq \delta^{\mathrm{HD}(J(f))-\varepsilon}.$$

The lower bound is [28, Theorem 1], while the upper bound is new and implies the integrability condition for each exponent $\eta < \mathrm{HD}(J(f))$, see (3) in § 2.1.

Let us say a few words on our strategy. Prior to this work, it has been shown that a map satisfying the assumptions of Theorem I or of Theorem II has the following two expanding properties: "*expansion away from critical points*" and "*backward contraction*". Roughly speaking, the first property means that outside any given neighborhood of the critical points the map is uniformly hyperbolic; the second property means that a return domain to a ball of radius δ centered at a critical value is much smaller than δ. See § 2.1 for the precise definitions and references, as well as our "Main Theorem" stated for maps satisfying these two expanding properties.

In this paper, we provide a finer quantification of the expansion features of a map that satisfies the two properties stated above. Firstly, we show that the components of the preimages of a small ball intersecting the Julia set shrink *at least at a super-polynomial rate* (Theorem A in § 2.2.1). This unexpected result represents a significant improvement on the estimate of the same type in [16, Proposition 7.2], for rational maps satisfying the summability condition with a sufficiently small exponent. In our proof, the moduli of annuli are used to estimate diameter of sets. The situation in the real setting is much trickier than in the complex one, due to a significant control loss of the modulus of a thin annulus under pull-back, see Lemma 3.5 and the remark before it. We develop a "quasi-chain" construction to treat this problem. Secondly, we introduce a dimension-like parameter we call "*badness exponent*", that measures the combined size of all "bad pull-backs" of a suitably chosen small neighborhood of the critical points. A bad pull-back is a pull-back that is not contained in any diffeomorphic pull-back. This notion was first introduced in [46] and it has some resemblance with the pull-backs corresponding to a backward orbit of a critical point "with sequence 11…1", as used in [15, 16]. Using the local Markov structure (*nice sets*) provided by the backward contracting property, we show that the badness exponent is *zero* (Theorem B in § 2.2.2). A direct consequence of this result is that the conical Julia set has codimension zero in the Julia set (Corollary 6.3 in § 6.1).

These expanding properties are converted to statistical properties of the system through the construction of an induced Markov map. The approach is conventional but the construction is often technical. In this paper, this is done by applying techniques developed in [46, 47] with modification. We obtain a tail estimate in terms of the rate of shrinking of components of preimages of small sets, and of the badness exponent only (Theorem C in § 2.2.3). We also obtain the existence and regularity of a geometric conformal measure. The existence of a super-polynomially mixing absolutely continuous invariant measure then follows from a well-known result of Young [58].

The result on the regularity of the invariant density is obtained through an upper bound of the Poincaré series (Theorem E in § 2.2.5).

Finally let us mention a few by-products of our approach. For a map satisfying the hypotheses of Theorem I or of Theorem II, we show that several notions of fractal dimension of the Julia set coincide (Theorem F). For a complex polynomial that is expanding away from critical points and backward contracting, we show that the Julia set is locally connected when connected (Corollary 2.7), has Hausdorff dimension less than 2 (Corollary D) and is holomorphically removable (Theorem G).

2. The Main Theorem and reduced statements

In this section we recall the definition of the properties "expanding away from critical points" and "backward contracting", and then state our Main Theorem (§ 2.1) from which we deduce Theorems I and II as direct consequences. Then we state five intermediate results, Theorems A, B, C and E, and Corollary D (§ 2.2), and deduce the Main Theorem (§ 2.3). Finally, in § 2.4 we state some further results, which are proved in § 7.

The proofs of Theorems A, B, and C are independent, and are shown in §§ 4, 5, and 6, respectively. Corollary D is deduced from Theorem C in § 6.4. The proof of Theorem E depends on Corollary D, and it is given in § 7.

2.1. The Main Theorem

We say that a map $f : X \to X$ from a compact interval X of $\mathbb{R}$ into itself is *of class C^3 with non-flat critical points* if f is of class C^1 on X and satisfies the following properties:

- f is of class C^3 outside $\mathrm{Crit}(f) := \{x \in X : Df(x) = 0\}$;
- for each $c \in \mathrm{Crit}(f)$, there exist a number $\ell_c > 1$ (called *the order of f at c*) and diffeomorphisms ϕ, ψ of $\mathbb{R}$ of class C^3 with $\phi(c) = \psi(f(c)) = 0$ such that

$$|\psi \circ f(x)| = |\phi(x)|^{\ell_c}$$

holds on a neighborhood of c in X.

We use $\mathscr{A}_{\mathbb{R}}$ to denote the collection of all C^3 interval maps with non-flat critical points, without neutral periodic points, and that are *boundary-anchored*, *i.e.*,, for each $x \in \partial\,\mathrm{dom}(f)$, we have $f(x) \in \partial\,\mathrm{dom}(f)$ and $Df(x) \neq 0$. The last condition is convenient when considering pull-backs of sets.

We use $\mathscr{A}_{\mathbb{C}}$ to denote the collection of all rational maps of degree at least 2 without neutral periodic points. As in the real case, for $f \in \mathscr{A}_{\mathbb{C}}$ we denote by $\mathrm{Crit}(f)$ the set of critical points of f, and for each $c \in \mathrm{Crit}(f)$ we denote by ℓ_c the local degree of f at c, that we will also call *the order of f at c*.

For $f \in \mathscr{A} := \mathscr{A}_{\mathbb{R}} \cup \mathscr{A}_{\mathbb{C}}$, we denote by $\mathrm{dom}(f)$ the Riemann sphere $\overline{\mathbb{C}}$ if f is a complex map, and the compact interval where f is defined otherwise. The *Julia set* $J(f)$ of f is, by definition, the set of all points $x \in \mathrm{dom}(f)$ with the following property: for any neighborhood U of x, the family $\{f^n|U\}_{n=0}^{\infty}$ is not equicontinuous. This is a forward invariant compact set. We shall mainly be interested in the dynamics on $J(f)$, where the chaotic dynamical behavior concentrates. It is known that for $f \in \mathscr{A}_{\mathbb{C}}$, the set $J(f)$ is the closure of repelling periodic points and that for $f \in \mathscr{A}_{\mathbb{R}}$, the set $J(f)$ is the complement of the basins of periodic attractors. See [40] and [37] for more background.

For maps without critical points in the Julia set all of our results are either well-known or vacuous. So for $f \in \mathscr{A}$ we will implicitly assume that the set

$$\mathrm{Crit}'(f) := \mathrm{Crit}(f) \cap J(f)$$

is nonempty. We also put

$$\ell_{\max}(f) := \max\{\ell_c : c \in \mathrm{Crit}'(f)\}.$$

Given $\ell > 1$ we will denote by $\mathscr{A}(\ell)$ (resp. $\mathscr{A}_{\mathbb{R}}(\ell)$, $\mathscr{A}_{\mathbb{C}}(\ell)$) the class of all those $f \in \mathscr{A}$ (resp. $\mathscr{A}_{\mathbb{R}}$, $\mathscr{A}_{\mathbb{C}}$) such that $\ell_{\max}(f) \leq \ell$.

Definition 2.1. – We say that a map $f \in \mathscr{A}$ is *expanding away from critical points* , if for every neighborhood V' of $\mathrm{Crit}'(f)$ the map f is uniformly expanding on the set

$$A = \{z \in J(f) : \text{ for every } n \geq 0,\, f^n(z) \notin V'\},$$

i.e.,, there exist constants $C > 0$ and $\lambda > 1$ such that for any $z \in A$ and $n \geq 0$, we have $|Df^n(z)| \geq C\lambda^n$.

A theorem of Mañé asserts that every map $f \in \mathscr{A}_{\mathbb{R}}$ is expanding away from critical points, see [31]. Although the analogous statement for a map f in $\mathscr{A}_{\mathbb{C}}$ is false in general, it does hold if we assume in addition that f is a polynomial that is at most finitely renormalizable, see [25], and also [49] for the totally disconnected case.

We will now recall the "backward contraction property" introduced in [50] in the case of rational maps, and in [6] in the case of interval maps. Let $f \in \mathscr{A}$ be given. When studying $f \in \mathscr{A}_{\mathbb{R}}$, we use the standard metric on the interval $\operatorname{dom}(f)$, while when studying $f \in \mathscr{A}_{\mathbb{C}}$, we shall use the spherical metric on $\overline{\mathbb{C}}$. For a critical point c and $\delta > 0$ we denote by $\widetilde{B}(c, \delta)$ the connected component of $f^{-1}(B(f(c), \delta))$ containing c.

Definition 2.2. – Given a constant $r > 1$ we will say a map $f \in \mathscr{A}$ is *backward contracting with constant* r if there is a constant $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$, every $c \in \operatorname{Crit}'(f)$, every integer $m \geq 0$, and every connected component W of $f^{-m}(\widetilde{B}(c, r\delta))$,

$$\operatorname{dist}(W, f(\operatorname{Crit}(f))) \leq \delta \text{ implies } \operatorname{diam}(W) < \delta.$$

Furthermore, we say that f is *backward contracting* if, for every $r > 1$, it is backward contracting with constant r.

Remark 2.3. – The specific choice of metric we used is not important. If we use a different *conformal metric*, then we obtain a different class of backward contracting maps for which the results in this paper hold with the same proof. Observe that the Koebe principle is still valid independently of the choice of the conformal metric used to measure norms, since the ratio of two different conformal metrics is a positive continuous function. Furthermore, cross-ratios are only affected by a bounded multiplicative constant, so the results from § 3.3 remain essentially unchanged.

For a map f in $\mathscr{A}$ and $\alpha > 0$, a *conformal measure of exponent* α for f is a Borel probability measure on $\operatorname{dom}(f)$ such that for each Borel set U on which f is injective,

$$\mu(f(U)) = \int_U |Df|^\alpha d\mu.$$

On the other hand, the *conical Julia set* $J_{\mathrm{con}}(\cdot)$ *of* f is the set of all those points $x \in J(f)$ for which there is a constant $\delta > 0$ and infinitely many integers $m \geq 1$ satisfying the following property: f^m is a diffeomorphism between the connected component of $f^{-m}(B(f^m(x), \delta))$ containing x and $B(f^m(x), \delta)$.

For a subset A of $\mathbb{R}$, or of the Riemann sphere $\overline{\mathbb{C}}$, we denote by $\operatorname{HD}(A)$ the *Hausdorff dimension* of A and by $\overline{\operatorname{BD}}(A)$ (resp. $\underline{\operatorname{BD}}(A)$) the upper (resp. lower) box counting dimensions. See for example [13] for the definitions. The Hausdorff dimension of a Borel probability measure μ on $X = \mathbb{R}$ (or $\overline{\mathbb{C}}$) is defined as

$$\operatorname{HD}(\mu) := \inf\{\operatorname{HD}(Y) : Y \subset X, \mu(Y) = 1\}.$$

By definition, a *hyperbolic set* A of $f \in \mathscr{A}$ is a forward invariant compact set on which f is uniformly expanding. The *hyperbolic dimension* of a map $f \in \mathscr{A}$ is by definition,

$$\operatorname{HD}_{\mathrm{hyp}}(f) := \sup\{\operatorname{HD}(A) : A \text{ is a hyperbolic set}\}.$$

Recall that for $f \in \mathscr{A}_{\mathbb{C}}$, the map $f : J(f) \to J(f)$ is topologically exact, *i.e.*,, for any nonempty open subset U of $J(f)$ there exists an integer $N \geq 1$ such that $f^N(U) = J(f)$. It is however too restrictive to assume an interval map to be topologically exact on the Julia set and boundary-anchored simultaneously. For this reason we introduce the following definition. We say that a map $f \in \mathscr{A}_{\mathbb{R}}$ is *essentially topologically exact on* $J(f)$ if there exists a forward invariant compact interval X_0 containing all critical points of f such that $f : J(f|X_0) \to J(f|X_0)$ is topologically exact and such that the interior of the compact interval $\mathrm{dom}(f)$ is contained in $\bigcup_{n=0}^{\infty} f^{-n}(X_0)$.

Main Theorem. *For every $\ell > 1, h > 0, \varepsilon > 0,\ \gamma > 1$ and $p \in \left(0, \frac{\ell}{\ell-1}\right)$ there is a constant $r > 1$ such that the following properties hold. Let $f \in \mathscr{A}(\ell)$ be backward contracting with constant r, expanding away from critical points, and such that $\mathrm{HD}(J(f)) \geq h$. Suppose furthermore in the case $f \in \mathscr{A}_{\mathbb{R}}$ that f is essentially topologically exact on the Julia set. Then the following hold:*

1. *The hyperbolic dimension $\mathrm{HD}_{\mathrm{hyp}}(f)$ is equal to $\mathrm{HD}(J(f))$ and there is a conformal measure μ of exponent $\mathrm{HD}(J(f))$ for f that is ergodic, supported on $J_{\mathrm{con}}(f)$, satisfies $\mathrm{HD}(\mu) = \mathrm{HD}(J(f))$ and is such that for every sufficiently small $\delta > 0$ and every $x \in J(f)$,*

$$\mu(B(x, \delta)) \leq \delta^{\mathrm{HD}(J(f))-\varepsilon}. \tag{2}$$

   Furthermore, any other conformal measure for f supported on $J(f)$ is of exponent strictly larger than $\mathrm{HD}(J(f))$ and supported on a set of Hausdorff dimension less than h.

2. *There is a unique invariant probability measure ν that is absolutely continuous with respect to μ, and this invariant measure is polynomially mixing of exponent γ. Furthermore, the density of ν with respect to μ belongs to $L^p(\mu)$.*

Note that if $J(f)$ has positive Lebesgue measure, then the measure μ is proportional to the Lebesgue measure, since after suitable normalization the Lebesgue measure on $J(f)$ is clearly a conformal measure of exponent $\mathrm{HD}(J(f))$. In fact, this is already the case if $J(f)$ has the same Hausdorff dimension as the domain of f. See part 1 of Corollary D in § 2.2.4.

Note that (2) implies that for each $\eta \in (0, \mathrm{HD}(J(f)) - \varepsilon)$ the conformal measure μ satisfies the following integrability condition, first introduced in [44]; see also [16, §10]. There is a constant $C > 0$ such that for each $x_0 \in J(f)$,

$$\int_{J(f)} \mathrm{dist}(x, x_0)^{-\eta} d\mu(x) \leq C. \tag{3}$$

Let us now deduce Theorems I and II from the Main Theorem and the following fact.

FACT 2.4. – *A map f is backward contracting if one of the following holds:*

1. *$f \in \mathscr{A}_{\mathbb{R}}$ and for all $c \in \mathrm{Crit}'(f)$, we have $|Df^n(f(c))| \to \infty$ as $n \to \infty$;*
2. *$f \in \mathscr{A}_{\mathbb{C}}$ is a polynomial that is at most finitely renormalizable and is such that for all $c \in \mathrm{Crit}'(f)$, we have $|Df^n(f(c))| \to \infty$ as $n \to \infty$;*

2′. *$f \in \mathscr{A}_{\mathbb{C}}$ is a rational map such that for all $c \in \mathrm{Crit}'(f)$, we have $\sum_{n=0}^{\infty} |Df^n(f(c))|^{-1} < \infty$.*

Proof. – These are [6, Theorem 1], [29, Theorem A] and [50, Theorem A], respectively. The first result is stated for maps without a periodic attractor, but the proof works without change under the current assumption. □

Proof of Theorem I. – We may extend $f : X \to X$ to be a boundary-anchored map $\tilde{f} : \widetilde{X} \to \widetilde{X}$ with all periodic points repelling, $\mathrm{Crit}(\tilde{f}) = \mathrm{Crit}(f)$ and such that $\mathrm{int}(\widetilde{X}) \subset \bigcup_{n=0}^{\infty} \tilde{f}^{-n}(X)$. Then $\tilde{f}$ is essentially topologically exact on its Julia set. By part 1 of Fact 2.4, $\tilde{f}$ is backward contracting. By Mañe's theorem, $\tilde{f}$ is expanding away from critical points. So, by the Main Theorem, $\tilde{f}$ has an invariant measure ν that is absolutely continuous with respect to the Lebesgue measure, and that is super-polynomially mixing. (Since $J(\tilde{f}) = \widetilde{X}$ has positive measure, the conformal measure μ is proportional to the Lebesgue measure on $\widetilde{X}$.) Note that ν is supported on X, so it is an invariant measure of f with the desired properties. □

Proof of Theorem II. – By parts 2 and 2′ of Fact 2.4, f is backward contracting. The fact that f is expanding away from critical points is known: this follows from either [25] or [50, Corollary 8.3] in the first case and from [44, proof of Lemma 3.1] in the second case. Since $\mathrm{HD}(J(f)) > 0$, applying the Main Theorem completes the proof. □

REMARK 2.5. – Let $f_0(z) = z^2 + c$ be the Fibonacci quadratic polynomial studied in [30]. This map satisfies the summability condition for every exponent $\alpha > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{|Df_0^n(c)|^{\alpha}} < \infty,$$

see [30, Lemma 5.9]. Using the results of [30] we will show that for every $\alpha \in (0,1)$,

$$\sum_{n=1}^{\infty} \frac{n}{|Df_0^n(c)|^{\alpha}} = \infty. \tag{4}$$

Let $\{u(k)\}_{k=1}^{\infty}$ be the *sequence of Fibonacci numbers* defined by $u(1) = 1$, $u(2) = 2$, and for $k \geq 3$ defined recursively by $u(k) = u(k-1) + u(k-2)$. If we put $\varphi = (1+\sqrt{5})/2$, then an induction argument shows that for every integer $k \geq 1$ we have $u(k) \geq \varphi^{k-1}$. By [30, Lemma 5.8] there is a constant $C > 0$ such that for every $k \geq 1$ we have $|Df^{u(k)}(c)| \leq C 2^{2k/3}$ and thus for each $\alpha \in (0,1)$,

$$\frac{u(k)}{|Df^{u(k)}(c)|^{\alpha}} \geq C\varphi^{k-1} 2^{-2\alpha k/3} \geq C\varphi^{-1}(\varphi 2^{-2/3})^k \geq C\varphi^{-1}.$$

This proves (4).

2.2. Reduced statements

2.2.1. *Polynomial Shrinking*

DEFINITION 2.6. – Given a sequence $\Theta = \{\theta_n\}_{n=1}^{\infty}$ of positive numbers, we say that a map $f \in \mathscr{A}$ satisfies the Θ-*Shrinking Condition*, if there exist constants $\rho > 0$ and $C > 0$ such that for every $x \in J(f)$ and every integer $m \geq 1$, the connected component W of $f^{-m}(B(f^m(x), \rho))$ containing x satisfies

$$\mathrm{diam}(W) \leq C\theta_m.$$

Given $\beta \geq 0$ we say that f satisfies the *Polynomial Shrinking Condition with exponent* β, if f satisfies the Θ-Shrinking Condition with $\Theta := \{n^{-\beta}\}_{n=1}^{\infty}$.

THEOREM A. – *For every $\ell > 1$ and $\beta > 0$ there is a constant $r > 1$ such that each map in $\mathscr{A}(\ell)$ that is expanding away from critical points and that is backward contracting with constant r satisfies the Polynomial Shrinking Condition with exponent β.*

In what follows, for a map $f \in \mathscr{A}$ we denote by $\beta_{\max}(f)$ the best polynomial shrinking exponent of f; i.e., the supremum of all $\beta \geq 0$ for which f satisfies the Polynomial Shrinking Condition with exponent β. So $\beta_{\max}(f) = 0$ means that f is not polynomially shrinking with any positive exponent.(1)

The following result is a direct consequence of Theorem A, of [38, Theorem 2], and of [50, Corollary 8.3].

COROLLARY 2.7 (Local connectivity). – *For every integer $\ell \geq 2$ there is an $r > 1$ such that, for any $f \in \mathscr{A}_{\mathbb{C}}(\ell)$ that is backward contracting with constant r the Julia set of f is locally connected when it is connected.*

2.2.2. *Badness exponent*. – Let us start by introducing “nice sets”.(2) For $f \in \mathscr{A}$, a set V, and an integer $m \geq 1$, each connected component of $f^{-m}(V)$ is called a *pull-back of V by f^m*.

DEFINITION 2.8. – For a map $f \in \mathscr{A}$, we will say that $V \subset \operatorname{dom}(f)$ is a *nice set* if the following hold:

- $\overline{V}$ is disjoint from the forward orbits of critical points not in $J(f)$ and periodic orbits not in $J(f)$;
- $V \supset \operatorname{Crit}'(f)$;
- each connected component of V is an open interval (resp. topological disk) and contains precisely one point in $\operatorname{Crit}'(f)$;
- for every integer $n \geq 1$ we have $f^n(\partial V) \cap V = \varnothing$.

For $c \in \operatorname{Crit}'(f)$ we denote by V^c the connected component of V containing c. A nice set V is called *symmetric* if for each $c \in \operatorname{Crit}'(f)$ we have $f(\partial V^c) \subset \partial f(V^c)$. Moreover, a *(symmetric) nice couple for f* is a pair of (symmetric) nice sets $(\widehat{V}, V)$ such that $\overline{V} \subset \widehat{V}$, and such that for every integer $n \geq 1$ we have $f^n(\partial V) \cap \widehat{V} = \varnothing$.

The following fact is proved for maps in $\mathscr{A}_{\mathbb{C}}$ in [50, Proposition 6.6]. See Lemma 3.13 for the general case.

FACT 2.9. – *For each $\ell > 1$ there is a constant $r > 1$ such that each $f \in \mathscr{A}(\ell)$ that is backward contracting with constant r possesses arbitrarily small (symmetric) nice couples.*

Fix $f \in \mathscr{A}$ and a set V. If W is a pull-back of V by f^m, we define an integer $d_V(W) \geq 1$ in the following way:

- If f is a rational map, then $d_V(W)$ is the degree of $f^m : W \to f^m(W)$, *i.e.,*, the maximal cardinality of $f^{-m}(x) \cap W$ for $x \in V$.

(1) Note that every map in $\mathscr{A}$ satisfies the Polynomial Shrinking Condition with exponent $\beta = 0$.

(2) In the case f is an interval map, the concept of nice set we use here differs from the usual concept of “nice interval”. A nice interval is an interval V such that for every integer $n \geq 1$, we have $f^n(\partial V) \cap V = \varnothing$. Thus, a nice set is a neighborhood of $\operatorname{Crit}'(f)$ formed by a union of nice intervals that satisfy some additional properties.

- If f is an interval map, then $d_V(W) := 2^N$, where N is the number of those $j \in \{0, \dots, m-1\}$ such that the connected component of $f^{-(m-j)}(V)$ containing $f^j(W)$ intersects $\mathrm{Crit}(f)$.

For a component W of V, we define $d_V(W) = 1$. When V is clear from the context, we shall often drop the subscript V, and write $d(W)$ instead of $d_V(W)$.

Let V be an open set and let W be a pull-back of V by f^m. If f^m is a diffeomorphism between W and a component of V, then we say that W is a *diffeomorphic pull-back of* V. Note that in the case when f is a rational map, this occurs if and only if f^m is univalent on W.

DEFINITION 2.10. – Given $f \in \mathscr{A}$ and an open set V, we will say that a pull-back W of V by f^m, $m \geq 1$, is *bad* if, for every integer $m' \in \{1, \dots, m\}$ such that $f^{m'}(W) \subset V$, the pull-back of V by $f^{m'}$ containing W is not diffeomorphic. Furthermore, we denote by $\mathfrak{B}_m(V)$ the collection of all bad pull-backs of V by f^m and put

$$\delta_{\mathrm{bad}}(V) := \inf \left\{ t > 0 : \sum_{m=1}^{\infty} \sum_{W \in \mathfrak{B}_m(V)} d_V(W) \operatorname{diam}(W)^t < \infty \right\}.$$

The *badness exponent of* f is defined as

$$\delta_{\mathrm{bad}}(f) := \inf\{\delta_{\mathrm{bad}}(V) : V \text{ is a nice set of } f\}. \tag{5}$$

We shall prove in Lemma 3.9 that $\delta_{\mathrm{bad}}(V) \leq \delta_{\mathrm{bad}}(V')$ for any nice sets $V \subset V'$. Thus if we have a sequence of nice sets $V_1 \supset V_2 \supset \cdots \searrow \mathrm{Crit}'(f)$, then $\delta_{\mathrm{bad}}(f) = \lim_{n\to\infty} \delta_{\mathrm{bad}}(V_n)$.

THEOREM B. – *For every $\ell > 1$ and $t > 0$ there is a constant $r \geq 2$ such that for each map $f \in \mathscr{A}(\ell)$ that is backward contracting with constant r, we have $\delta_{\mathrm{bad}}(f) < t$.*

REMARK 2.11. – Given $\ell > 2$ close to 2, let f be a Fibonacci unimodal map whose critical point c has order ℓ. Then f gives an example of a map that is backward contracting with a large constant and such that $\delta_{\mathrm{bad}}(f) > 0$. In fact, the results of [24] imply that such a map f is backward contracting with a large constant while the postcritical set $\omega(c)$ has positive Hausdorff dimension. On the other hand, f is persistently recurrent, so $\omega(c)$ is contained in $J(f) \setminus J_{\mathrm{con}}(f)$. Thus, by Lemma 6.7,

$$\delta_{\mathrm{bad}}(f) \geq \mathrm{HD}(J(f) \setminus J_{\mathrm{con}}(f)) \geq \mathrm{HD}(\omega(c)) > 0.$$

REMARK 2.12. – The arguments in Lemmas 7.1 and 7.2 of [46] show that the badness exponent of a rational map satisfying the Topological Collet-Eckmann (TCE) condition is zero. When restricted to the class of rational maps with a unique critical point in the Julia set, the TCE condition is equivalent to the Collet-Eckmann condition [45]. So Theorem B is significantly stronger within this class of maps.

In view of the results that follow, it would be interesting to have an answer for the following question:

QUESTION 2.13. – *For $f \in \mathscr{A}$, does $\beta_{\max}(f) = \infty$ imply $\delta_{\mathrm{bad}}(f) = 0$?*

2.2.3. *Canonical induced Markov map*. – Let $\mathscr{A}^*$ be the set of $f \in \mathscr{A}$ that satisfies the following:

(A1) f is expanding away from critical points;
(A2) $\mathrm{Crit}'(f) \neq \varnothing$ and f has arbitrarily small symmetric nice couples;
(A3) if $f \in \mathscr{A}_{\mathbb{R}}$, then f is essentially topologically exact on the Julia set.

Moreover, put $\mathscr{A}_{\mathbb{R}}^* := \mathscr{A}^* \cap \mathscr{A}_{\mathbb{R}}$ and $\mathscr{A}_{\mathbb{C}}^* := \mathscr{A}^* \cap \mathscr{A}_{\mathbb{C}}$.

Through an inducing scheme, we can convert Theorems A and B into statistical properties of maps $f \in \mathscr{A}^*$. The following definitions appeared first in [46].

Given a nice couple $(\widehat{V}, V)$ of f, we say that an integer $m \geq 1$ is *a good time* for a point x if $f^m(x) \in V$ and if the pull-back of $\widehat{V}$ containing x is diffeomorphic. We denote by D the set of all those points in V having a good time, and for each $x \in D$ we denote by $m(x)$ the least good time of x. Note that $m(x)$ is constant in any component W of D, so $m(W)$ makes sense. We say that $m(x)$ (resp. $m(W)$) is the *canonical inducing time* of x (resp. W) with respect to $(\widehat{V}, V)$. The *canonical induced map associated to the nice couple* $(\widehat{V}, V)$ is by definition the map $F : D \to V$ defined by $F(x) = f^{m(x)}(x)$. We denote by $J(F)$ the maximal invariant set of F; that is the set of all those points in V having infinitely many good times.

We say that a sequence $\{\theta_n\}_{n=1}^{\infty}$ of positive numbers is *slowly varying* if $\theta_n/\theta_{n+1} \to 1$ as $n \to \infty$. For instance, $\{n^{-\beta}\}_{n=1}^{\infty}$ and $\{\exp(-\sigma n^{\alpha})\}_{n=1}^{\infty}$ are slowly varying for any $\beta, \sigma, \alpha > 0$, but for each $\theta \in (0,1)$ the sequence $\{\theta^n\}_{n=1}^{\infty}$ is not slowly varying.

Theorem C. – *Fix* $f \in \mathscr{A}^*$. *If* $\delta_{\mathrm{bad}}(f) < \mathrm{HD}(J(f))$, *then* $\mathrm{HD}(J(f)) = \mathrm{HD}_{\mathrm{hyp}}(f)$. *Moreover, for each sufficiently small nice couple* $(\widehat{V}, V)$*, the associated canonical induced map* $F : D \to V$ *satisfies:*

$$\mathrm{HD}(J(F) \cap V^c) = \mathrm{HD}(J(f)), \text{ for all } c \in \mathrm{Crit}'(f).$$

Furthermore, fix $t \in (\delta_{\mathrm{bad}}(f), \mathrm{HD}(J(f)))$ *and assume that* f *satisfies the* Θ*-Shrinking Condition for some slowly varying and monotone decreasing sequence of positive numbers* $\Theta = \{\theta_n\}_{n=1}^{\infty}$. *Then for each sufficiently small symmetric nice couple* $(\widehat{V}, V)$*, there exists a constant* $\alpha_0 = \alpha_0(\widehat{V}, V) \in (t, \mathrm{HD}(J(f)))$ *such that, for all* $\alpha \geq \alpha_0$ *and* $\sigma \in [0, \alpha - t)$*, there is a constant* $C > 0$ *such that, for each* $Y \subset V$ *and each integer* $m \geq 1$*,*

$$\sum_{W \in \mathfrak{D} \,:\, m(W) \geq m,\, W \subset Y} \mathrm{diam}(W)^{\alpha} \leq C \,\mathrm{diam}(Y)^{\sigma} \sum_{n=m}^{\infty} \theta_n^{\alpha - t - \sigma},$$

where $\mathfrak{D}$ *is the collection of all components of* D.

Remark 2.14. – Of course, the latter part of the theorem is useful only when $\sum_{n=1}^{\infty} \theta_n^{\eta} < \infty$ for some $\eta \in (0, \mathrm{HD}(J(f)) - \delta_{\mathrm{bad}}(f))$.

Remark 2.15. – If for an exponentially decreasing sequence Θ the map f in Theorem C satisfies the Θ-Shrinking Condition, then Theorem C allows one to obtain an exponential tail estimate, as follows. As f is certainly super-polynomially shrinking, Theorem C shows that there exists a constant $\alpha \in (0, \mathrm{HD}(J(f)))$ such that $K := \sum_{W \in \mathfrak{D} \,:\, m(W) \geq 1} \mathrm{diam}(W)^{\alpha}$ is finite, and thus

$$\sum_{W \in \mathfrak{D} \,:\, m(W) \geq m} \mathrm{diam}(W)^{\mathrm{HD}(J(f))} \leq K \max_{W \in \mathfrak{D} \,:\, m(W) \geq m} \mathrm{diam}(W)^{\mathrm{HD}(J(f)) - \alpha}$$

is exponentially small in m. Notice that we lose control of the exponent significantly. A similar argument was used in the proof of [46, Theorem C].

2.2.4. *Conformal and invariant measures*. – Define

$$\gamma(f) := \beta_{\max}(f)\,(\mathrm{HD}(J(f)) - \delta_{\mathrm{bad}}(f))\,. \tag{6}$$

We use the following convention: the product of $+\infty$ with a real number a is $+\infty$ (resp. 0, $-\infty$) if $a > 0$ (resp. $a = 0$, $a < 0$). So $\gamma(f) > 0$ is equivalent to

$$\delta_{\mathrm{bad}}(f) < \mathrm{HD}(J(f)) \text{ and } \beta_{\max}(f) > 0.$$

Since for $f \in \mathscr{A}^*$ we have $\mathrm{HD}(J(f)) \geq \mathrm{HD}_{\mathrm{hyp}}(f) > 0$, Theorems A and B imply that when f is backward contracting we have $\gamma(f) = \infty$.

Corollary D. – *For $f \in \mathscr{A}^*$ the following properties hold.*

1. *If $\gamma(f) > 1$, then either $\mathrm{HD}(J(f)) < \mathrm{HD}(\mathrm{dom}(f))$ or $J(f)$ has a nonempty interior. Moreover, there exists a conformal measure μ of exponent $\mathrm{HD}(J(f))$ for f that is ergodic, supported on the conical Julia set, satisfies $\mathrm{HD}(\mu) = \mathrm{HD}(J(f))$, and is such that for each $\varepsilon > \delta_{\mathrm{bad}}(f) + \beta_{\max}(f)^{-1}$ the following holds: for each sufficiently small $\delta > 0$ we have for every $x \in J(f)$,*

$$\mu(B(x,\delta)) \leq \delta^{\mathrm{HD}(J(f))-\varepsilon}. \tag{7}$$

Furthermore, any other conformal measure for f supported on $J(f)$ is of exponent strictly larger than $\mathrm{HD}(J(f))$ and supported on a set of Hausdorff dimension less than or equal to $\delta_{\mathrm{bad}}(f)$.

2. *If $\gamma(f) > 2$, then there is an invariant probability measure ν that is absolutely continuous with respect to μ and this invariant measure ν is polynomially mixing of each exponent $\gamma \in (0, \gamma(f) - 2)$.*

2.2.5. *Regularity of the invariant density*. – Given $f \in \mathscr{A}$, let $q(f)$ be the infimum of those constants $q > 0$ for which there is a constant $C > 0$ such that the following property holds: for each $x \in J(f)$, $\delta > 0$, $m \geq 1$, and each pull-back W of $B(x,\delta)$ by f^m such that $f^m : W \to B(x,\delta)$ is a diffeomorphism whose distortion is bounded by 2, we have

$$\mathrm{diam}(W) \leq C\delta^{1/q}.$$

We clearly have $q(f) \geq \ell_{\max}(f)$. The following is a simple consequence of [28, Proposition 2], see Lemma 3.12.

Fact 2.16. – *For each $\ell > 1$ and $q > \ell$ there is a constant $r > 1$ such that if $f \in \mathscr{A}^*(\ell)$ is backward contracting with constant r, then $q(f) < q$. In particular, if f is backward contracting, then $q(f) = \ell_{\max}(f)$.*

Theorem E. – *Let $f \in \mathscr{A}^*$ be such that $\gamma(f) > 2$, and let μ be the conformal measure and ν the invariant measure given by Corollary D. Then for each*

$$p \in \left(1, q(f)\frac{1 - (\delta_{\mathrm{bad}}(f) + \beta_{\max}(f)^{-1})\,\mathrm{HD}(J(f))^{-1}}{q(f) - 1 + (\delta_{\mathrm{bad}}(f) + 2\beta_{\max}(f)^{-1})\,\mathrm{HD}(J(f))^{-1}}\right),$$

the density of ν with respect to μ belongs to $L^p(\mu)$.

REMARK 2.17. – For a map $f \in \mathscr{A}^*$ that is backward contracting, the theorem implies that the density of ν with respect to μ is in $L^p(\mu)$ for all

$$p < p(f) := q(f)/(q(f) - 1) = \ell_{\max}(f)/(\ell_{\max}(f) - 1).$$

If $J(f)$ has nonempty interior in $\operatorname{dom}(f)$, then this estimate is optimal in the sense that the density never belongs to the space $L^{p(f)}(\mu)$, as we shall now explain. In this case, μ is a rescaling (of a restriction) of the Lebesgue measure and the Lyapunov exponent of ν is strictly positive and its density is bounded from below by a positive constant almost everywhere in $J(f)$, see [26] or [12, Theorem 6] for the real case, and [27] or [11, Theorem 8] for the complex case. It thus follows from the invariance of ν and the conformality of μ that, if we denote by $h \in \{1, 2\}$ the dimension of $\operatorname{dom}(f)$ and by $c \in J(f)$ a critical point of f of maximal order, then there is a constant $C > 0$ such that the density is bounded from below by the function $C \operatorname{dist}(\cdot, f(c))^{-h/p(f)}$, on a set of full Lebesgue measure in $J(f)$. Thus the density cannot belong to $L^{p(f)}$.

Using the lower bound on the conformal measure given by [28, Theorem 1], a similar argument shows that if $f \in \mathscr{A}^*_{\mathbb{C}}$ is backward contracting and $p > p(f)$, then the invariant density does not belong to $L^p(\mu)$.

QUESTION 2.18. – *Suppose $f \in \mathscr{A}^*$ is such that $\gamma(f) = \infty$, and let μ and ν be as in Corollary D. Is it true that $d\nu/d\mu \notin L^{p(f)}(\mu)$?*

We state the following corollary for future reference. For the definition of the TCE condition, see for example [42] in the real case and [48] in the complex case.

COROLLARY 2.19. – *Let $f \in \mathscr{A}$ be a map satisfying the TCE condition and that it is not uniformly hyperbolic. In the real case, assume furthermore that f is essentially topologically exact on $J(f)$. Then f belongs to $\mathscr{A}^*$ and $\gamma(f) = \infty$. Moreover, if we denote by μ the conformal measure and ν the invariant measure given by Corollary D, then ν is exponentially mixing, for each $p \in (0, q(f)/(q(f) - 1))$ the density of ν with respect to μ belongs to $L^p(\mu)$, and for each $\varepsilon > 0$ we have for every sufficiently small $\delta > 0$ and every $x \in J(f)$*

$$\mu(B(x, \delta)) \leq \delta^{\mathrm{HD}(J(f)) - \varepsilon}.$$

When f is in $\mathscr{A}_{\mathbb{C}}$ the assertion that ν is exponentially mixing is shown in [46, Theorem C]; the remaining assertions of the corollary are new.

Proof. – By [42] in the real case and [48] in the complex one, there is an exponentially decreasing sequence Θ such that f satisfies the Θ-Shrinking Condition. Thus $\beta_{\max}(f) = \infty$ and f is expanding away from critical points. On the other hand, f has arbitrarily small nice couples by [46, Theorem E], and we also have $\delta_{\mathrm{bad}}(f) = 0$, as pointed out in Remark 2.12. This proves that f is in $\mathscr{A}^*$ and that $\gamma(f) = \infty$. In particular, f satisfies the hypotheses of Corollary D and Theorem E. Denote by μ the conformal measure and ν the invariant measure given by Corollary D. That ν is exponentially mixing is given by the exponential tail estimate in Remark 2.15, combined with well-known arguments (similar to those used in the proof of part 2 of Corollary D). The remaining assertions of the corollary are given by Corollary D and Theorem E. $\square$

2.3. Proof of the Main Theorem

Now we will complete the proof of the Main Theorem. If $\mathrm{Crit}'(f) = \varnothing$, then f is uniformly expanding on $J(f)$ and the statements of the Main Theorem are well-known. So we assume that $\mathrm{Crit}'(f) \neq \varnothing$.

Let $\ell > 1$, $h > 0$, $\varepsilon > 0$, $\gamma > 1$ and $p \in \left(0, \frac{\ell}{\ell-1}\right)$ be given, and let $q \in \left(\ell, \frac{p}{p-1}\right)$. By Theorems A and B and by Facts 2.9 and 2.16, there exists a constant $r > 2$ such that if $f \in \mathscr{A}(\ell)$ is backward contracting with constant r and satisfies the other assumptions of the Main Theorem, then $f \in \mathscr{A}^*(\ell)$, $\beta_{\max}(f) > 2(\gamma+2)/h$, $\delta_{\mathrm{bad}}(f) < h/2$, $\delta_{\mathrm{bad}}(f) + \beta_{\max}(f)^{-1} < \varepsilon$, $q(f) < q$ and

$$q \frac{1 - (\delta_{\mathrm{bad}}(f) + \beta_{\max}(f)^{-1})\, \mathrm{HD}(J(f))^{-1}}{q - 1 + (\delta_{\mathrm{bad}}(f) + 2\beta_{\max}(f)^{-1})\, \mathrm{HD}(J(f))^{-1}} > p.$$

Hence $\delta_{\mathrm{bad}}(f) < \mathrm{HD}(J(f))$ and by Theorem C we have $\mathrm{HD}(J(f)) = \mathrm{HD}_{\mathrm{hyp}}(f)$. On the other hand $\gamma(f) > \gamma + 2$, and by Corollary D and Theorem E there exist a conformal measure μ and an invariant measure ν with the desired properties. Moreover, the ergodicity of μ implies that ν is the only invariant measure of f that is absolutely continuous with respect to μ.

2.4. Fractal dimensions and holomorphic removability of Julia sets

In this section we state a result related to fractal dimensions (Theorem F), and another related to holomorphic removability of Julia sets in the complex setting (Theorem G). Both are independent of the Main Theorem and are shown in § 7.

To state our result on the equality of fractal dimensions, we make the following definition. Given $f \in \mathscr{A}$, $s > 0$ and a point $x_0 \in \mathrm{dom}(f)$ we define the *Poincaré series of f at x_0 with exponent s*, as

$$\mathscr{P}(x_0; s) := \sum_{n=1}^{\infty} \sum_{x \in f^{-n}(x_0)} |Df^n(x)|^{-s}.$$

We say that a point x is *exceptional* if the set $\bigcup_{n=0}^{\infty} f^{-n}(x)$ is finite, and we say that x is *asymptotically exceptional* if its α-limit set is finite. The *Poincaré exponent of f* is by definition,

$$\delta_{\mathrm{Poin}}(f) := \inf \{\{0\} \cup \{s > 0 : \mathscr{P}(x_0; s) < \infty \text{ for some } x_0 \text{ that is not asymptotically exceptional}\}\}.$$

Note that every point in the α-limit set of an asymptotically exceptional point is exceptional. It is well-known that for a rational map of degree at least 2 each asymptotically exceptional point is exceptional, that there are at most 2 exceptional points, and that they are not in the Julia set. Note however that for $f \in \mathscr{A}_{\mathbb{R}}$, any point in $\mathrm{dom}(f) \setminus X_0(f)$ is asymptotically exceptional, where $X_0(f)$ is the minimal forward invariant closed interval that contains $\mathrm{Crit}(f)$.

THEOREM F (Equality of fractal dimensions). – *If $f \in \mathscr{A}^*$ satisfies $\gamma(f) > 1$, then*

$$\delta_{\mathrm{Poin}}(f) = \overline{BD}(J(f)) = \mathrm{HD}(J(f)) = \mathrm{HD}_{\mathrm{hyp}}(f) > 0.$$

See (6) in § 2.2 for the definition of $\gamma(f)$ and Proposition 7.3 for some divergence/convergence properties of the Poincaré series.

Equalities of dimensions were shown in [28] for backward contracting rational maps without parabolic cycles, in [44] for rational maps whose derivatives at critical values grow at least as a stretched exponential function, in [16, Theorem 7] for rational maps satisfying a summability condition with a small exponent and without parabolic cycles, and in [10] for interval maps without recurrent critical points. These equalities were shown for a class of infinitely renormalizable quadratic polynomials in [1].

We will say that a compact subset J of the Riemann sphere is *holomorphically removable* if every homeomorphism $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ that is holomorphic outside J is a Möbius transformation.

THEOREM G (Holomorphic removability). – *If $f \in \mathscr{A}_{\mathbb{C}}^*$ is a polynomial such that*

$$\beta_{\max}(f)\,(2 - \delta_{\mathrm{bad}}(f)) > 1,$$

then the Julia set of f is holomorphically removable. In particular, for every integer $\ell \geq 2$, there is a constant $r > 1$ such that the Julia set of a complex polynomial $f \in \mathscr{A}_{\mathbb{C}}(\ell)$ that is backward contracting with constant r is holomorphically removable.

See [20, 22] and [16, Theorem 8] for other removability results of Julia sets.

3. Preliminaries

3.1. Notation

For $f \in \mathscr{A}$ and for a neighborhood V of $\mathrm{Crit}'(f)$ we put

$$K(V) := \{x \in \mathrm{dom}(f) : \text{ for every integer } n \geq 0,\ f^n(x) \notin V\}. \tag{8}$$

Denote by $\mathfrak{L}_V$ the collection of connected components of $\mathrm{dom}(f) \setminus K(V)$.

For $V \subset \mathrm{dom}(f)$ and an integer $m \geq 0$, let $\mathscr{M}_m(V)$ denote the collection of all components of $f^{-m}(V)$. Moreover, let $\mathscr{M}(V) := \bigcup_{m=0}^{\infty} \mathscr{M}_m(V)$.

3.2. Koebe distortion lemma

We shall frequently use the following Koebe distortion lemma.

LEMMA 3.1. – *For every $f \in \mathscr{A}$ there is a constant $\eta_* > 0$ such that for every $\varepsilon \in (0,1)$ there exists a constant $K(\varepsilon) > 1$ such that the following holds. Given $x \in \mathrm{dom}(f)$, $\eta \in (0, \eta_*)$, and $n \geq 1$, let W (resp. $W(\varepsilon)$) be the component of $f^{-n}(B(f^n(x), \eta))$ (resp. $f^{-n}(B(f^n(x), \varepsilon\eta))$) that contains x. Suppose that $f^n : W \to B(f^n(x), \eta)$ is a diffeomorphism. Suppose also that $\mathrm{dist}(f^n(x), J(f)) \leq \eta_*$ in the case $f \in \mathscr{A}_{\mathbb{R}}$. Then the distortion of f^n on $W(\varepsilon)$ is bounded by $K(\varepsilon)$. That is, for every $z_1, z_2 \in W(\varepsilon)$,*

$$|Df^n(z_1)|/|Df^n(z_2)| \leq K(\varepsilon).$$

Moreover, $K(\varepsilon) = 1 + O(\varepsilon)$ as $\varepsilon \to 0$.

Proof. – For the case $f \in \mathscr{A}_{\mathbb{C}}$, see for example [46, §2.4]. For the case $f \in \mathscr{A}_{\mathbb{R}}$, see [54, Theorem C (2)(ii)]. Recall that, by definition, maps in $\mathscr{A}_{\mathbb{R}}$ have no neutral cycles. □

3.3. Modulus

We shall use moduli of annuli to compare the size of nested sets. This method is popular in the complex setting. Recall that if $A \subset \overline{\mathbb{C}}$ is an annulus that is conformally isomorphic to the round annulus $\{z \in \mathbb{C} : 1 < |z| < R\}$, then $\operatorname{mod}(A) := \log R$. More generally, if $V \subsetneq \overline{\mathbb{C}}$ is open and $U \Subset V$, then we define $\operatorname{mod}(V; U)$ as the supremum of the moduli of those annuli contained in V that separate U from $\overline{\mathbb{C}} \setminus V$.

In order to deal with interval maps, let us introduce a similar notion in the real setting. If $J \Subset I$ are bounded intervals in $\mathbb{R}$, then we define $\operatorname{mod}(I; J) := \operatorname{mod}(D_*(I); D_*(J))$, where $D_*(I)$ denotes the round disk in $\mathbb{C}$ that has I as a diameter and $D_*(J)$ the corresponding disk for J. More generally, if V is a bounded open subset of $\mathbb{R}$, $U \Subset V$ is an interval, and if we denote by V_0 the connected component of V containing U, then we put

$$\operatorname{mod}(V; U) := \operatorname{mod}(V_0; U).$$

LEMMA 3.2. – *Let $V_2 \Supset V_1 \Supset V_0$ be either bounded intervals of $\mathbb{R}$, or open and connected proper subsets of $\mathbb{C}$. Then*

$$\operatorname{mod}(V_2; V_0) \geq \operatorname{mod}(V_2; V_1) + \operatorname{mod}(V_1; V_0).$$

Proof. – In the complex case, this lemma is known as Grotzch's inequality, see for example [39, Corollary A.5]. The real case follows from the complex one by definition. □

The following lemma relates modulus to diameter of sets.

LEMMA 3.3. – *There exists a constant $C_0 > 0$ such that the following property holds. Let $U \Subset V$ be either bounded intervals contained in $\mathbb{R}$, or open and connected subsets of $\overline{\mathbb{C}}$, such that $\operatorname{diam}(V) \leq \operatorname{diam}(\overline{\mathbb{C}})/2$. Then, letting $\mu = \operatorname{mod}(V; U)$,*

$$\operatorname{diam}(U) \leq C_0 \exp(-\mu) \operatorname{diam}(V).$$

Proof. – We only need to consider the complex case as the real case will then follow by definition. We may assume that μ is large. In this case, $V \setminus U$ contains a round annulus A with $\operatorname{mod}(A) = \mu - O(1)$. See for example [33, Theorem 2.1]. The lemma follows. □

We shall now consider distortion of modulus under pull-back. In the complex case, we have the following well-known lemma, see for example [17, Lemma 4.1.1]. A sequence $\{U_j\}_{j=0}^{s}$ of simply connected open sets is called *a chain* if for each $0 \leq j < s$, the set U_j is a component of $f^{-1}(U_{j+1})$ and $U_j \cap J(f) \neq \varnothing$.

LEMMA 3.4. – *Consider $f \in \mathscr{A}_{\mathbb{C}}$. Let $\{\widetilde{U}_j\}_{j=0}^{s}$ and $\{U_j\}_{j=0}^{s}$ be chains of topological disks such that $U_j \Subset \widetilde{U}_j$, and let*

$$\nu = \#\{0 \leq j < s : \widetilde{U}_j \text{ intersects } \operatorname{Crit}'(f)\}.$$

Then

$$\operatorname{mod}(\widetilde{U}_0; U_0) \geq \ell_{\max}(f)^{-\nu} \operatorname{mod}(\widetilde{U}_s; U_s).$$

In the real case, modulus is similarly distorted under a diffeomorphic pull-back, but the situation can be much worse under critical pull-back. For example, let $f(x) = x^2 + a$ be a real quadratic polynomial, and

$$\widetilde{U}_1 := (-\delta^2 + a, \delta^2 + a) \supset U_1 := (-\delta^2\varepsilon + a, \delta^2(1-\varepsilon) + a),$$

where $\delta > 0$ and $\varepsilon \in (0,1)$. Then a direct computation shows that, as $\varepsilon \to 0$,

$$\operatorname{mod}\left(f^{-1}(\widetilde{U}_1); f^{-1}(U_1)\right) \asymp \varepsilon \text{ and } \operatorname{mod}(\widetilde{U}_1; U_1) \asymp \varepsilon^{\frac{1}{2}}.$$

Nevertheless, the following Lemma 3.5 will be enough for our application.

Given $f \in \mathscr{A}_{\mathbb{R}}$, if a bounded open interval I contains a unique critical value v of f, we define

$$I^\sharp = (v - |I|, v + |I|); \tag{9}$$

otherwise, we write $I^\sharp = I$. Moreover, we say that a map $f \in \mathscr{A}_{\mathbb{R}}$ is *normalized near critical points* , if for each $c \in \operatorname{Crit}'(f)$, the equation $|f(x) - f(c)| = |x - c|^{\ell_c}$ holds in a neighborhood of c.

LEMMA 3.5. – *Consider $f \in \mathscr{A}_{\mathbb{R}}(\ell)$.*

1. *For any $\lambda \in (0,1)$ there exists a constant $\eta > 0$ such that if $\{\widetilde{U}_j\}_{j=0}^s$ and $\{U_j\}_{j=0}^s$ are chains of intervals such that $U_j \Subset \widetilde{U}_j$, such that $\widetilde{U}_j \cap \operatorname{Crit}(f) = \varnothing$ for all $j = 1, \ldots, s-1$ and such that $|\widetilde{U}_s| < \eta$, then*

$$\operatorname{mod}(\widetilde{U}_1; U_1) \geq \lambda \operatorname{mod}(\widetilde{U}_s; U_s).$$

Moreover, there exists a constant $K_0 > 0$ depending only on ℓ such that

$$\operatorname{mod}(\widetilde{U}_0; U_0) + K_0 \geq \ell^{-1} \operatorname{mod}(\widetilde{U}_1; U_1) \geq \lambda \ell^{-1} \operatorname{mod}(\widetilde{U}_s; U_s).$$

2. *Assume that f is normalized near critical points and let $c \in \operatorname{Crit}'(f)$. Let $V_1 \Supset U_1$ be intervals and let V_0 (resp. U_0) be a component of $f^{-1}(V_1^\sharp)$ (resp. $f^{-1}(U_1^\sharp)$) such that $U_0 \subset V_0$. If $c \in V_0$ and $|V_1| < \eta$, then*

$$\operatorname{mod}(V_0; U_0) \geq \ell_c^{-1} \operatorname{mod}(V_1; U_1),$$

where ℓ_c is the order of f at c.

We need two preparatory lemmas to prove this result. Recall that the *cross-ratio* of bounded intervals $J \Subset I$ of $\mathbb{R}$ is defined as

$$\operatorname{Cr}(I, J) := \frac{|I||J|}{|L||R|},$$

where L, R are the components of $I \setminus J$.

LEMMA 3.6. – *For bounded intervals $J \Subset I$ of $\mathbb{R}$,*

$$\operatorname{mod}(I; J) = 2 \log\left(\sqrt{\operatorname{Cr}(I,J)^{-1}} + \sqrt{1 + \operatorname{Cr}(I,J)^{-1}}\right).$$

Proof. – There exists a Möbius transformation σ such that $\sigma(I) = (-T, T)$ and $\sigma(J) = (-1, 1)$, where $T = \exp(\operatorname{mod}(I; J))$. Since

$$\operatorname{Cr}(I, J) = \operatorname{Cr}(\sigma(I), \sigma(J)) = 4T/(T-1)^2,$$

the lemma follows. □

LEMMA 3.7. – *For each* $\ell > 1$ *and* $0 \le a < b < 1$,
$$\mathrm{mod}((-1,1);(a,b)) \ge \ell^{-1}\,\mathrm{mod}((-1,1);(a^\ell,b^\ell)).$$

Proof. – Let us first consider the case $a = 0$. Let $\beta \in (0,1)$ be such that $\beta + \beta^{-1} = 2b^{-1}$, so that
$$\mathrm{mod}((-1,1);(-\beta,\beta)) = \mathrm{mod}((-1,1);(0,b)).$$
Thus, if we let $\hat{b} \in (0,1)$ be defined by $2\hat{b}^{-1} = \beta^{-\ell} + \beta^\ell$, then,
$$\begin{aligned}\mathrm{mod}((-1,1);(0,\hat{b})) &= \mathrm{mod}((-1,1);(-\beta^\ell,\beta^\ell))\\ &= \ell\,\mathrm{mod}((-1,1);(-\beta,\beta))\\ &= \ell\,\mathrm{mod}((-1,1);(0,b)).\end{aligned}$$
So we just need to show that $\hat{b} \le b^\ell$. This follows from the power mean inequality,
$$\hat{b}^{-1} = \frac{\beta^{-\ell}+\beta^\ell}{2} \ge \left(\frac{\beta^{-1}+\beta}{2}\right)^\ell = b^{-\ell},$$
see for example [19, **16**].

Now let us consider the case $a > 0$. Let t be the unique number in $(b,1)$ such that
$$\mathrm{mod}((-1,1);(a^\ell,b^\ell)) = \mathrm{mod}((-1,1);(0,t^\ell)) + \mathrm{mod}((0,t^\ell);(a^\ell,b^\ell)).$$
Then as above,
$$\mathrm{mod}((-1,1);(0,t)) \ge \ell^{-1}\,\mathrm{mod}((-1,1);(0,t^\ell)).$$
Note that
$$\mathrm{Cr}((0,t);(a,b)) = \frac{\frac{b}{a}-1}{1-\frac{b}{t}} \le \frac{\left(\frac{b}{a}\right)^\ell - 1}{1-\left(\frac{b}{t}\right)^\ell} = \mathrm{Cr}((0,t^\ell),(a^\ell,b^\ell)),$$
hence
$$\mathrm{mod}((0,t);(a,b)) \ge \mathrm{mod}((0,t^\ell);(a^\ell,b^\ell)) \ge \ell^{-1}\,\mathrm{mod}((0,t^\ell);(a^\ell,b^\ell)).$$
Finally, by Lemma 3.2,
$$\mathrm{mod}((-1,1);(a,b)) \ge \mathrm{mod}((-1,1);(0,t)) + \mathrm{mod}((0,t);(a,b)).$$
Combining these estimates, we complete the proof of the lemma. □

Proof of Lemma 3.5. – 1. For the first inequality, by Lemma 3.6, it suffices to prove that for any constant $\lambda' \in (0,1)$, we have $\mathrm{Cr}(\widetilde{U}_s,U_s) \ge \lambda'\,\mathrm{Cr}(\widetilde{U}_1,U_1)$, provided that $\mathrm{diam}(\widetilde{U}_s)$ is sufficiently small. But this is well-known: if f has negative Schwarzian derivative, we actually have $\mathrm{Cr}(\widetilde{U}_s,U_s) \ge \mathrm{Cr}(\widetilde{U}_1,U_1)$; otherwise, we may apply [54, Theorem C(3)] which claims that the first entry map to a small neighborhood of $f(\mathrm{Crit}'(f))$ has negative Schwarzian derivative, see the proof of [6, Proposition 1], for details.

For the second inequality we may assume $\widetilde{U}_0$ contains a critical point c and $\mathrm{mod}(\widetilde{U}_1;U_1)$ is large, *i.e.*,, $\mathrm{Cr}(\widetilde{U}_1,U_1)$ is small. Note that $|\widetilde{U}_s|$ small implies $|\widetilde{U}_0|$ small since f has no wandering interval. So by the non-flatness of critical points, we have $\mathrm{Cr}(\widetilde{U}_0,U_0) \le K_0'\,\mathrm{Cr}(\widetilde{U}_1,U_1)^{1/\ell_c}$ for some K_0' depending only on ℓ_c. The second inequality follows by Lemma 3.6 again.

2. It suffices to prove the following two inequalities:

$$\operatorname{mod}(V_1^\sharp; U_1^\sharp) \geq \operatorname{mod}(V_1; U_1); \tag{10}$$

$$\operatorname{mod}(V_0; U_0) \geq \ell_c^{-1} \operatorname{mod}(V_1^\sharp; U_1^\sharp). \tag{11}$$

Let us prove the inequality (10). If $f(c) \notin U_1$, then $V_1^\sharp \supset V^1$, $U_1^\sharp = U_1$, so this inequality clearly holds. Assume $f(c) \in U_1$ and let L, R denote the components of $V_1 \setminus U_1$. Then

$$\operatorname{Cr}(V_1, U_1) = \frac{|V_1||U_1|}{|L||R|} \geq \frac{4|V_1||U_1|}{(|L| + |R|)^2} = \frac{4|V_1||U_1|}{(|V_1| - |U_1|)^2} = \operatorname{Cr}(V_1^\sharp, U_1^\sharp),$$

which implies the inequality (10) by Lemma 3.6.

The inequality (11) follows from the local behavior of f near c. If $U_1 \ni f(c)$, then both $V_1^\sharp$ and $U_1^\sharp$ are centered at $f(c)$ and, by definition of modulus, we see that (11) holds with equality. If $U_1 \not\ni f(c)$, then the statement follows from Lemma 3.7. □

3.4. Bad pull-backs of a nice set

Let $f \in \mathscr{A}$ and let V be a nice set for f. It is easy to see that for each $W \in \mathscr{M}(V)$, either $W \cap V = \varnothing$ or $W \subset V$. Moreover, for any integers $0 \leq m_1 \leq m_2$ and $W_1 \in \mathscr{M}_{m_1}(V)$, $W_2 \in \mathscr{M}_{m_2}(V)$,

$$\text{either } W_1 \cap W_2 = \varnothing \text{ or } W_2 \subset W_1.$$

Recall that $W \in \mathscr{M}_m(V)$, $m \geq 1$ is called a bad pull-back of V by f^m if, for every integer $m' \in \{1, \ldots, m\}$ such that $f^{m'}(W) \subset V$, the pull-back of V by $f^{m'}$ containing W is not diffeomorphic. As before, we use $\mathfrak{B}_m(V)$ to denote the collection of all bad pull-backs of V by f^m.

LEMMA 3.8. – *Let V be a nice set of $f \in \mathscr{A}$ and let $W \in \mathscr{M}_m(V)$ with $m \geq 1$. Then the following are equivalent:*

1. *$W \in \mathfrak{B}_m(V)$;*
2. *for any $1 \leq m' \leq m$, W is not contained in any diffeomorphic pull-back of V by $f^{m'}$;*
3. *for any $1 \leq m' \leq m$, W is disjoint from any diffeomorphic pull-back of V by $f^{m'}$.*

Proof. – By definition, $1 \Leftrightarrow 2$. The assertion $3 \Rightarrow 2$ is trivial, while $2 \Rightarrow 3$ holds since any pull-back of V by $f^{m'}$ is either disjoint from W or contains W. □

LEMMA 3.9. – *For nice sets $V' \supset V$ for f, the following properties hold.*

1. *For every integer $m \geq 1$ and every bad pull-back W of V by f^m, the pull-back of V' by f^m containing W is bad.*
2. *$\delta_{\mathrm{bad}}(V') \geq \delta_{\mathrm{bad}}(V)$.*

Proof. – 1. Let W' be the pull-back of V' by f^m that contains W. Arguing by contradiction, assume that $W' \notin \mathfrak{B}_m(V')$. Then there exists a $m_0 \in \{1, 2, \ldots, m\}$ such that W' is contained in a diffeomorphic pull-back W_0 of V' by f^{m_0}. Then $f^{m_0}(W) \not\subset V$, so $m_0 < m$ and there exists a minimal $m_1 \in \{m_0 + 1, m' + 2, \ldots, m\}$ such that $f^{m_1}(W) \subset V$. By the minimality of m_1 and niceness of V, we obtain that $f^{m_1 - m_0}$ maps a simply connected set $U \supset f^{m_0}(W)$ diffeomorphically onto a component of V. By the niceness of V' we obtain that $U \subset V'$. So the pull-back of U by f^{m_0} that contains W is diffeomorphic. It follows that the pull-back of V by f^{m_1} that contains W is diffeomorphic, contradicting the assumption that $W \in \mathfrak{B}_m(V)$.

2. By part 1 each $W \in \mathfrak{B}_m(V)$ is contained in an element of $\mathfrak{B}_m(V')$. Clearly, for each $W' \in \mathfrak{B}_m(V')$, and any $t \geq 0$,

$$\sum_{W \in \mathfrak{B}_m(V)\,:\, W \subset W'} d_V(W)\,\mathrm{diam}(W)^t \leq d_{V'}(W')\,\mathrm{diam}(W')^t.$$

It follows that

$$\sum_{m=1}^{\infty} \sum_{W \in \mathfrak{B}_m(V)} d_V(W)\,\mathrm{diam}(W)^t \leq \sum_{m=1}^{\infty} \sum_{W' \in \mathfrak{B}_m(V')} d_{V'}(W')\,\mathrm{diam}(W')^t,$$

hence $\delta_{\mathrm{bad}}(V') \geq \delta_{\mathrm{bad}}(V)$. □

3.5. Expansion away from critical points

LEMMA 3.10. – *Let $f \in \mathscr{A}$ be expanding away from critical points, let $\rho > 0$ be given, and let $(\widehat{V}, V)$ be a nice couple such that for each $c \in \mathrm{Crit}'(f)$ we have $\mathrm{diam}(\widehat{V}^c) < \rho$. Then there are constants $\kappa_0 > 1$, $K_0 > 1$ and $\rho_0 \in (0, \rho)$, such that the following property holds. Let D be the domain of the canonical induced map associated to $(\widehat{V}, V)$. Then for each $x \in J(f)$ and $\delta \in (0, \rho/2)$ there is an integer $n \geq 0$ such that one of the following properties holds:*

1. *the distortion of f^n on $B(x, \delta)$ is bounded by K_0, and*
$$\rho_0 < \mathrm{diam}(f^n(B(x, \delta))) < \rho;$$
2. *$f^n(B(x, \kappa_0\delta)) \subset V$, $|Df^n(x)| \geq \rho_0$, the distortion of f^n on $B(x, \kappa_0\delta)$ is bounded by K_0, and every connected component of D intersecting $f^n(B(x, \delta))$ is contained in $f^n(B(x, \kappa_0\delta))$.*

Proof. – Fix a compact neighborhood $\widetilde{V}$ of $\overline{V}$ contained in $\widehat{V}$. For each $m \geq 1$ and each component W of $f^{-m}(V)$, let $\widehat{W} \supset \widetilde{W}$ be the components of $f^{-m}(\widehat{V}) \supset f^{-m}(\widetilde{V})$ that contain W. By the Koebe principle there are constants $\kappa_0 > 1$, $K_1 > 1$ and $\rho_1 > 0$ such that if $f^m|\widehat{W}$ is a diffeomorphism onto a connected component of $\widehat{V}$, then the following properties hold:

- the distortion of $f^m|\widetilde{W}$ is bounded by K_1;
- for each $y \in W$ we have $|Df^m(y)| \geq \rho_1$;
- $\mathrm{dist}(\partial\widetilde{W}, W) \geq 2(\kappa_0 - 1)^{-1}\,\mathrm{diam}(W)$.

Since f is uniformly expanding on $K(V) \cap J(f)$, it follows that there is a constant $\rho_2 > 0$ such that for each $x' \in J(f)$ and $\delta' \in (0, \rho/2)$ such that $B(x', \kappa_0\delta')$ intersects $K(V)$, there is an integer $n \geq 0$ such that $f^n(B(x', \delta')) \supset B(f^n(x'), \rho_2)$, such that $\mathrm{diam}(f^n(x', \delta')) < \rho$, and such that the distortion of f^n on $B(x', \kappa_0\delta')$ is bounded by 2. Replacing ρ_2 by a smaller constant if necessary, we assume that $(\kappa_0^2 - 1)\rho_2$ is less than the minimal diameter of the components of V.

We prove the assertion of the lemma with

$$K_0 = 2K_1 \text{ and } \rho_0 = \min\{\rho_1, K_1^{-1}\rho_2\}.$$

Let $x \in J(f)$ and $\delta > 0$ be given. If $B(x, \kappa_0\delta)$ intersects $K(V)$, then assertion 1 holds by definition of ρ_2. So we assume that $B(x, \kappa_0\delta)$ does not intersect $K(V)$. If

$$B(x, \kappa_0\delta) \subset V \text{ and } B(x, \kappa_0\delta) \not\subset D,$$

then we put $m = 0$, and we denote by W_0 the connected component of V containing $B(x, \kappa_0\delta)$. Otherwise there is an integer $m \geq 1$ such that $f^m(B(x, \kappa_0\delta)) \subset V$ and such that f^m maps a neighborhood of $B(x, \kappa_0\delta)$ diffeomorphically onto a connected component of $\widehat{V}$. We assume that m is the largest integer with this property, and let W_0 be the pull-back of V by f^m that contains x. In both cases the distortion of f^m on $B(x, \kappa_0\delta)$ is bounded by K_1, and

$$f^m(B(x, \kappa_0\delta)) \subset V \text{ and } f^m(B(x, \kappa_0\delta)) \not\subset D.$$

If every connected component of D intersecting $f^m(B(x, \delta))$ is contained in $f^m(B(x, \kappa_0\delta))$, then assertion 2 holds with $n = m$. Otherwise there is a connected component W of D that intersects $f^m(B(x, \delta))$ but is not contained in $f^m(B(x, \kappa_0\delta))$. Let us prove that assertion 1 holds for some $n \geq m + m(W)$ in this case, where $m(W)$ is the canonical inducing time of W with respect to $(\widehat{V}, V)$. Indeed, $W' := (f^m|W_0)^{-1}(W)$ is a pull-back of V by $f^{m+m(W)}$ such that $f^{m+m(W)}$ maps a neighborhood of W' diffeomorphically onto a connected component of $\widehat{V}$. Since $W' \cap B(x, \delta) \neq \varnothing$, from the definition of κ_0 it follows that $B(x, \delta) \subset \widetilde{W'}$, so the distortion of $f^{m+m(W)}|B(x, \delta)$ is bounded by K_1. On the other hand, by the choice of m, we have $B(x, \kappa_0\delta) \not\subset W'$. Suppose $\operatorname{diam}(W') \leq (\kappa_0^2 - 1)\delta$, so that

$$\frac{\operatorname{diam}(B(x, \delta))}{\operatorname{diam}(W')} \geq \frac{1}{\kappa_0^2 - 1}.$$

Letting $n := m + m(W)$, the set $f^n(W')$ is a connected component of V, so

$$\operatorname{diam}(f^n(B(x, \delta))) \geq K_1^{-1} \frac{1}{\kappa_0^2 - 1} \operatorname{diam}(f^n(W')) \geq K_1^{-1} \rho_2.$$

This proves assertion 1 with $n = m + m(W)$ in the case $\operatorname{diam}(W') \leq (\kappa_0^2 - 1)\delta$. Suppose now $\operatorname{diam}(W') > (\kappa_0^2 - 1)\delta$. Then $B(x, \kappa_0\delta) \subset \widetilde{W'}$, so $f^{m+m(W)}|B(x, \kappa_0\delta)$ has distortion bounded by K_1. Moreover, the set $f^{m+m(W)}(B(x, \kappa_0\delta))$ intersects $\partial V \subset K(V)$, so assertion 1 holds by the choice of ρ_2. □

In the case of complex rational maps, the following lemma is an easy consequence of [46, Lemma 6.3]. The proof extends to the case of interval maps without change. Recall that for a nice set V we denote by $\mathfrak{L}_V$ the collection of components of $\operatorname{dom}(f) \setminus K(V)$, where $K(V)$ is as in (8), § 3.1. For each element U of $\mathfrak{L}_V$, there exists a unique integer $l(U) \geq 0$ such that $f^{l(U)}$ maps U diffeomorphically onto a component of V.

Lemma 3.11. – *Let $f \in \mathscr{A}$ be expanding away from critical points. Then for each nice set V for f there exist constants $\alpha_0 \in (0, \mathrm{HD}_{\mathrm{hyp}}(f))$, C_0 and $\varepsilon_0 > 0$ such that, for every integer $m \geq 0$,*

$$\sum_{U \in \mathfrak{L}_V : l(U) \geq m} \operatorname{diam}(U)^{\alpha_0} < C_0 \exp(-\varepsilon_0 m). \tag{12}$$

Moreover, if $\mathrm{Crit}'(f) \neq \varnothing$*, then for each conformal measure* μ *supported on* $J(f)$*, there exist constants* $C' > 0$ *and* $\kappa \in (0,1)$ *such that for each integer* $m \geq 0$*,*

$$\mu\left(\{z \in J(f) : z, f(z), \ldots, f^{m-1}(z) \notin V\}\right) \leq C'\kappa^m.$$

Proof. – Let V_0 be a sufficiently small neighborhood of $\mathrm{Crit}'(f)$ contained in V, such that for each $c \in \mathrm{Crit}'(f)$ the set $K := K(V_0) \cap J(f)$ intersects V^c. Thus each element of $\mathfrak{L}_V$ intersects K. Since by hypothesis f is uniformly expanding on K, it follows that there are constants $C_1 > 0$ and $\varepsilon_1 > 0$ such that for each $U \in \mathfrak{L}_V$ we have $\mathrm{diam}(U) \leq C_1 \exp(-\varepsilon_1 l(U))$.

By [46, Lemma 6.3] there is a constant $\alpha_1 \in (0, \mathrm{HD_{hyp}}(J(f)))$ such that,

$$C_2 := \sum_{U \in \mathfrak{L}_V} \mathrm{diam}(U)^{\alpha_1},$$

is finite. Fix $\alpha_0 \in (\alpha_1, \mathrm{HD_{hyp}}(J(f)))$ and put $\varepsilon_0 := \varepsilon_1(\alpha_0 - \alpha_1)$. We thus have for every integer $m \geq 0$,

$$\begin{aligned}\sum_{\substack{U \in \mathfrak{L}_V \\ l(U) \geq m}} \mathrm{diam}(U)^{\alpha_0} &\leq \max_{\substack{U \in \mathfrak{L}_V \\ l(U) \geq m}} \mathrm{diam}(U)^{\alpha_0 - \alpha_1} \sum_{\substack{U \in \mathfrak{L}_V \\ l(U) \geq m}} \mathrm{diam}(U)^{\alpha_1} \\ &\leq C_2 C_1^{\alpha_0 - \alpha_1} \exp(-\varepsilon_0 m).\end{aligned}$$

To prove the last assertion of the lemma, notice first that the exponent α of μ satisfies $\alpha \geq \mathrm{HD_{hyp}}(f)$ [34], so $\alpha \geq \alpha_0$. Thus for every $m \geq 1$

$$\sum_{U \in \mathfrak{L}_V : l(U) \geq m} \mathrm{diam}(U)^{\alpha} \leq \max_{U \in \mathfrak{L}_V} \mathrm{diam}(U)^{\alpha - \alpha_0} \sum_{U \in \mathfrak{L}_V : l(U) \geq m} \mathrm{diam}(U)^{\alpha_0}$$

is exponentially small in m. Moreover $\mu(K(V)) = 0$ because f is uniformly expanding on $K(V) \cap J(f)$. On the other hand, by the Koebe principle there is a constant $C_3 > 0$ such that for each $U \in \mathfrak{L}_V$ we have $\mu(U) \leq C_3 \,\mathrm{diam}(U)^{\alpha}$. Thus, the last assertion of the lemma follows from the first from the inclusion,

$$\{z \in J(f) : z, f(z), \ldots, f^{m-1}(z) \notin V\} \subset \left(K(V) \cup \bigcup_{U \in \mathfrak{L}_V : l(U) \geq m} W\right). \qquad \square$$

3.6. Backward contracting maps

The following lemma is simple consequence of [28, Proposition 2].

LEMMA 3.12. – *For each* $\ell > 1$ *and* $\kappa \in (0, \ell^{-1})$ *there exists a constant* $r > 1$ *such that, if* $f \in \mathscr{A}^*(\ell)$ *is backward contracting with constant* r*, then there is a constant* $C > 0$ *such that for each subset* Q *of* $\mathrm{dom}(f)$ *intersecting* $J(f)$ *and each pull-back* P *of* Q*,*

$$\mathrm{diam}(P) \leq C \,\mathrm{diam}(f(Q))^{\kappa}.$$

Proof. – Fix ℓ and κ and assume that $f \in \mathscr{A}^*(\ell)$ is backward contracting with a sufficiently large constant r. By (the proof of) [28, Proposition 2], there exists a constant $C' > 0$ such that when $Q = \widetilde{B}(c, \varepsilon)$ for some $c \in \mathrm{Crit}'(f)$ and $\varepsilon > 0$, then each pull-back P of Q satisfies $\mathrm{diam}(P) \leq C'\varepsilon^{\kappa}$.

For the general case, let us fix a small constant $\varepsilon_0 > 0$. Since f is uniformly expanding on $J(f) \cap K(\widetilde{B}(\mathrm{Crit}'(f); \varepsilon_0/2))$, we may assume without loss of generality that Q is connected and contained in $\widetilde{B}(c; \varepsilon_0)$, for some $c \in \mathrm{Crit}'(f)$. Let $\varepsilon_1 \in (0, \varepsilon_0]$ be minimal such that Q is contained in the closure of $\widetilde{B}(c, \varepsilon_1)$, and write $Q' = \widetilde{B}(c, \varepsilon_1)$, $Q'' = \widetilde{B}(c, 2\varepsilon_1)$. Provided that ε_0 was chosen small enough,

$$\frac{\mathrm{diam}(f(Q))}{\mathrm{diam}(Q)} \asymp \frac{\mathrm{diam}(f(Q'))}{\mathrm{diam}(Q')} \asymp \varepsilon_1^{\ell_c - 1}. \tag{13}$$

Let $P \in \mathscr{M}_n(Q)$ for some $n \geq 1$, and consider the chains $\{P_j\}_{j=0}^n$, $\{P'_j\}_{j=0}^n$ and $\{P''_j\}_{j=0}^n$ with $P = P_0 \subset P'_0 \subset P''_0$ and $P''_n = Q''$, $P'_n = Q'$, $P_n = Q$. If $f^n : P''_0 \to Q''$ is a diffeomorphism, then $f^n | P'_0$ has bounded distortion, so

$$\frac{\mathrm{diam}(P)}{\mathrm{diam}(P'_0)} \asymp \frac{\mathrm{diam}(Q)}{\mathrm{diam}(Q')} \asymp \frac{\mathrm{diam}(f(Q))}{\mathrm{diam}(f(Q'))} \leq \left(\frac{\mathrm{diam}(f(Q))}{\mathrm{diam}(f(Q'))} \right)^{\kappa},$$

which implies that $\mathrm{diam}(P) \leq C \, \mathrm{diam}(f(Q))^{\kappa}$ since $\mathrm{diam}(P'_0) \leq C' \varepsilon_1^{\kappa}$. Otherwise, let $m < n$ be maximal such that P''_m contains a critical point, say c'. By the backward contracting property, we have $\mathrm{diam}(P''_{m+1}) \leq 2\varepsilon_1 r^{-1} < \varepsilon_0$. Since $f^{n-m-1} | P'_{m+1}$ has bounded distortion, using (13) we obtain

$$\mathrm{diam}(P_{m+1}) \asymp \mathrm{diam}(P'_{m+1}) \frac{\mathrm{diam}(f(Q))}{\mathrm{diam}(f(Q'))} \leq \mathrm{diam}(f(Q)) r^{-1},$$

which implies that $\mathrm{diam}(P_{m+1}) < \mathrm{diam}(f(Q))$ provided that r is large enough. Since $P_m \subset \widetilde{B}(c', \varepsilon_0)$, we may repeat the above argument with Q replaced by P_m. By an induction on n, we complete the proof of the lemma. $\square$

Given a nice set V, a component of the set $f^{-1}(\mathrm{dom}(f) \setminus K(V)) \cap V$ is called a *return domain* . These are maximal pull-backs of V that are contained in V. Given $\lambda > 0$ we will say that V is λ-*nice* if for return domain W of V we have $\overline{W} \subset V$ and,

$$\mathrm{mod}(V; W) \geq \lambda.$$

The following is essentially a combination of [50, Proposition 6.6] and [6, Lemma 3].

LEMMA 3.13. – *Given $\ell > 1$ and $\lambda > 0$, there is a constant $r > 4$ such that for every $f \in \mathscr{A}(\ell)$ that is backward contracting with constant r, the following property holds. For every sufficiently small $\delta > 0$, there is a symmetric nice couple $(\widehat{V}, V)$, such that V is λ-nice, and such that for each $c \in \mathrm{Crit}'(f)$,*

$$\widetilde{B}(c, r\delta/2) \subset \widehat{V}^c \subset \widetilde{B}(c, r\delta), \quad \widetilde{B}(c, \delta) \subset V^c \subset \widetilde{B}(c, 2\delta).$$

Proof. – We assume that f is backward contracting with constant $r \geq 2$. Then there is a symmetric nice set $\widehat{V}$ for f such that

$$\widetilde{B}(c, r\delta/2) \subset \widehat{V}^c \subset \widetilde{B}(c, r\delta),$$

see [6, Proposition 3] in the case of interval maps, and [50, Proposition 6.5] in the case of rational maps. For each $c \in \mathrm{Crit}'(f)$, let $V_{c,*}$ be the union of $\widetilde{B}(c, \delta)$ and all the return domains of $\widehat{V}$ that intersect $\widetilde{B}(c, \delta)$. By definition, $f(\partial V_{c,*}) \subset \partial f(V_{c,*})$, and by the backward contraction assumption, $V_{c,*} \subset \widetilde{B}(c, 2\delta)$. In the real case, let $V^c = V_{c,*}$ and in the complex case, let V^c be the filling of $V_{c,*}$, *i.e.,*, the union of $V_{c,*}$ and the components of $\overline{\mathbb{C}} \setminus V_{c,*}$ that are contained in $\widehat{V}^c$. Then, in both cases, V^c is simply connected,

$$f(\partial V^c) \subset \partial f(V^c), \text{ and } V^c \subset \widetilde{B}(c, 2\delta) \Subset \widehat{V}^c.$$

Let $V := \bigcup_{c \in \mathrm{Crit}'(f)} V^c$. Note that for each $x \in \partial V$ and $k \geq 1$, $f^k(x) \notin \widehat{V}$, hence $(\widehat{V}, V)$ is a symmetric nice couple. Provided r is large enough,

$$\mathrm{mod}(\widehat{V}^c; V^c) \geq (2\ell_{\max}(f))^{-1} \log(r/4)$$

is large. It follows that V is a λ-nice set for a large λ. Indeed, if U is a return domain of V with return time s, then the pull-back of $\widehat{V}$ by f^s that contains U is either diffeomorphic or unicritical, and it is contained in V. By either Lemma 3.4 or Lemma 3.5, we obtain that $\mathrm{mod}(V; U)$ is large. □

Definition 3.14. – For a map $f \in \mathscr{A}$ and an integer $m \geq 1$ we will say that a pull-back W of an open set V by f^m is a *child of* V if it contains precisely one critical point of f, and if f^{m-1} maps a neighborhood of $f(W)$ diffeomorphically onto a component of of V. We shall write $m_V(W) = m$.

In the case of interval maps the following lemma is a variant of [6, Lemma 4].

Lemma 3.15. – *For each $s > 0$ and $\ell > 1$ there is a constant $r > 4$ such that for every $f \in \mathscr{A}(\ell)$ that is backward contracting with constant r, the following property holds. For each sufficiently small $\delta > 0$ there is a nice set $V = \bigcup_{c \in \mathrm{Crit}'(f)} V^c$ such that for each $c \in \mathrm{Crit}'(f)$,*

$$\widetilde{B}(c, \delta) \subset V^c \subset \widetilde{B}(c, 2\delta),$$

and such that

$$\sum_{\text{children } Y \text{ of } V} \mathrm{diam}(f(Y))^s \leq \delta^s.$$

Proof. – Assume that f is backward contracting with a large constant r. By Lemma 3.13, for each sufficiently small $\delta > 0$ there exists a λ-nice set $V = \bigcup_{c \in \mathrm{Crit}'(f)} V^c$ such that for each $c \in \mathrm{Crit}'(f)$,

$$\widetilde{B}(c, \delta) \subset V^c \subset \widetilde{B}(c, 2\delta),$$

where $\lambda > 0$ is a large constant. Let us prove that the conclusion of the lemma holds for this choice of V.

Take $c \in \mathrm{Crit}'(f)$ and let $Y_k(c)$ be the k-th largest child of V containing c. By the backward contracting property, we have $Y_1(c) \subset \widetilde{B}(c, 2r^{-1}\delta)$. Let $s_k = m_V(Y_k(c))$. Then $f^{s_k}(Y_{k+1}(c))$ is contained in a return domain of V, hence $\mathrm{mod}(V; f^{s_k}(Y_{k+1}(c))) \geq \lambda$. By the definition of child, $Y_k(c)$ is a unicritical pull-back of V. Thus by Lemma 3.4 or Lemma 3.5, we obtain that $\mathrm{mod}(Y_k(c); Y_{k+1}(c)) \geq \lambda'$, where $\lambda' \to \infty$ as $\lambda \to \infty$. By Lemma 3.3, it follows that $\mathrm{diam}(Y_{k+1}(c)) / \mathrm{diam}(Y_k(c))$ is small provided that r is large. The conclusion of the lemma follows. □

4. Polynomial shrinking of components

In this section, we study the size of pull-backs of a small set. The main result is the following proposition, from which we shall derive Theorem A.

PROPOSITION 4.1. – *For each $\ell > 1$ and $\kappa_0 \in (0, \ell^{-1})$, there exists a constant $R > 3$ such that if $f \in \mathscr{A}(\ell)$ is expanding away from critical points and backward contracting with a constant $r > R$, then the following holds. For any $\eta_0 > 0$ sufficiently small there exist constants $C_0, A_0 > 0$ and for any chain $\{W_j\}_{j=0}^s$ with $W_s \subset \widetilde{B}(\mathrm{Crit}'(f), \eta_0)$, there exists an integer $\nu \geq 0$ such that*

$$\mathrm{diam}(W_0) \leq C_0 \min\left\{r^{-\kappa_0 \nu}, \exp\left(-\kappa_0^{\nu}(A_0 s + \mu)\right)\right\}, \tag{14}$$

where $\mu = \mathrm{mod}(\widetilde{B}(\mathrm{Crit}'(f), 3\eta_0); W_s)$.

The proof of this proposition in the real case is more complicated than in the complex case. We shall state and prove a preparatory lemma for the real case. The readers who are only interested in the complex case may skip this part.

DEFINITION 4.2. – Consider $f \in \mathscr{A}_{\mathbb{R}}$. A sequence $\{U_j\}_{j=0}^s$ of open intervals is called a *quasi-chain* if for each $0 \leq j < s$, the set U_j contains a component of $f^{-1}(U_{j+1})$. The *order* of the quasi-chain is the number of $j \in \{0, 1, \ldots, s-1\}$ such that U_j contains a critical point.

Given a chain $\{V_j\}_{j=0}^t$, we can construct a quasi-chain $\{\widehat{V}_j\}_{j=0}^t$ with $\widehat{V}_j \supset V_j$ as follows. First of all, $\widehat{V}_t = V_t$. Once $\widehat{V}_j$ has been defined for some $1 \leq j \leq t$, let V'_{j-1} be the component of $f^{-1}(\widehat{V}_j)$ that contains V_{j-1}, and let

$$\widehat{V}_{j-1} = \begin{cases} \widetilde{B}(c; |\widehat{V}_j|) & \text{if } V'_{j-1} \text{ contains a unique critical point } c; \\ V'_{j-1} & \text{otherwise.} \end{cases}$$

Note that $\widehat{V}_{j-1}$ contains a component of $f^{-1}(\widehat{V}_j)$ and in the former case, $\widehat{V}_{j-1}$ is the component of $f^{-1}((\widehat{V}_j)^\sharp)$ that contains c, where $\widehat{V}_j^\sharp$ is as in (9), § 3.3. We shall say that $\{\widehat{V}_j\}_{j=0}^t$ is the *preferred quasi-chain* for the chain $\{V_j\}_{j=0}^t$.

LEMMA 4.3. – *Consider a map $f \in \mathscr{A}_{\mathbb{R}}(\ell)$ that is normalized near critical points and fix $\kappa_0 \in (0, \ell^{-1})$. For each $\eta_0 > 0$ small enough the following holds. Let $\{V_j\}_{j=0}^s$ and $\{W_j\}_{j=0}^s$ be chains with $V_j \Supset W_j$, for $j = 0, 1, \ldots, s$, and such that $V_s \subset \widetilde{B}(\mathrm{Crit}'(f), 3\eta_0)$, and let $\{\widehat{V}_j\}_{j=0}^s$ and $\{\widehat{W}_j\}_{j=0}^s$ be the corresponding preferred quasi-chains. Assume that $V_j \cap \mathrm{Crit}(f) = \varnothing$ for all $1 \leq j < s$. Then*

$$\mathrm{mod}(\widehat{V}_0; \widehat{W}_0) \geq \kappa_0 \, \mathrm{mod}(V_s; W_s).$$

Proof. – Let $\lambda = \ell\kappa_0 \in (0, 1)$ and let $\eta > 0$ be the constant given by part 1 of Lemma 3.5. Assuming that η_0 is sufficiently small, we have $|V_s|, |V_1| < \eta$ since f has no wandering interval. By construction, $\widehat{V}_j = V_j$, $\widehat{W}_j = W_j$ for all $j = 1, 2, \ldots, s$. If $V_0 \cap \mathrm{Crit}(f) = \varnothing$, then $\widehat{V}_0 = V_0$, $\widehat{W}_0 = W_0$, and $f^s : V_0 \to V_s$ is a diffeomorphism, so the desired inequality follows from part 1 of Lemma 3.5. In the case $V_0 \cap \mathrm{Crit}(f) \neq \varnothing$,

$$\mathrm{mod}(V_1; W_1) \geq \lambda \, \mathrm{mod}(V_s; W_s).$$

Moreover, by part 2 of Lemma 3.5,

$$\operatorname{mod}(\widehat{V}_0; \widehat{W}_0) \geq \ell_c^{-1} \operatorname{mod}(V_1; W_1). \tag{15}$$

Combining these two inequalities above gives us the desired estimate. □

Proof of Proposition 4.1. – Fix $\kappa_0 \in (0, \ell^{-1})$ and assume f is backward contracting with a sufficiently large constant r that the conclusion of Lemma 3.12 holds with $\kappa = \kappa_0$. Let $\eta_0 > 0$ be a small constant such that for all $\delta \in (0, \eta_0)$, each pull-back W of $\widetilde{B}(\mathrm{Crit}'(f), r\delta)$ with $\mathrm{dist}(W, f(\mathrm{Crit}'(f))) < \delta$ satisfies $\mathrm{diam}(W) < \delta$. Moreover, when considering an interval map, we assume that f is normalized near critical points (after a C^3 conjugacy) and reduce η_0 if necessary so that Lemma 4.3 holds.

Let $t_1 < t_2 < \ldots < t_k = s$ be all the positive integers such that

$$f^{t_i}(W_0) \cap \widetilde{B}(\mathrm{Crit}'(f), \eta_0) \neq \varnothing.$$

By the backward contraction property, $f^{t_i}(W_0) \subset \widetilde{B}(c_i, 3\eta_0)$ for $i = 1, \ldots, k$. For each i, let c_i be the critical point in $\mathrm{Crit}'(f)$ closest to $f^{t_i}(W)$ and let $\{Y_i^j\}_{j=0}^{t_i}$ be the chain with $Y_i^{t_i} = \widetilde{B}(c_i, 3\eta_0)$ and $Y_i^0 \supset W_0$ and write $Y_i = Y_i^0$. For $1 \leq i \leq k$ and $0 \leq j \leq t_i$, let $\widehat{Y}_i^j = Y_i^j$ in the complex case; and let $\{\widehat{Y}_i^j\}_{j=0}^{t_i}$ be the preferred quasi-chain for $\{Y_i^j\}_{j=0}^{t_i}$ in the real case. Moreover, let $\widehat{W}_j = W_j$, $j = 0, 1, \ldots, s$ in the complex case; and let $\{\widehat{W}_j\}_{j=0}^{s}$ be the preferred quasi-chain for $\{W_j\}_{j=0}^{s}$ in the real case.

Let ν_i be the order of the (quasi-)chain $\{\widehat{Y}_i^j\}_{j=0}^{t_i}$ and let $\nu = \max_{i=1}^{k} \nu_i$. We first prove there exists a constant $C_1 > 0$ such that

$$\mathrm{diam}(W_0) \leq \mathrm{diam}(\widehat{W}_0) \leq C_1 r^{-\kappa_0 \nu}. \tag{16}$$

Indeed, take $i_0 \in \{1, 2, \ldots, k\}$ such that $\nu = \nu_{i_0}$ and let $0 \leq j_0 < j_1 < \cdots < j_{\nu-1} < j_\nu = t_{i_0}$ be the integers such that $\widehat{Y}_{i_0}^{j_m}$ intersects $\mathrm{Crit}'(f)$, $m = 0, 1, \ldots, \nu$. Then by the backward contracting property (and the construction of the quasi-chains in the real case), we prove inductively that $\widehat{Y}_{i_0}^{j_{\nu-m}} \subset \widetilde{B}(\mathrm{Crit}'(f), 3r^{-m}\eta_0)$. In particular, $\widehat{Y}_{i_0}^{j_0} \subset \widetilde{B}(\mathrm{Crit}'(f), 3r^{-\nu}\eta_0)$. By Lemma 3.12, we obtain (16).

Next, let us prove there exist constants $C_2, A_0 > 0$ such that

$$\mathrm{diam}(W_0) \leq C_2 \exp\{-\kappa_0^{\nu}(A_0 s + \mu)\}. \tag{17}$$

To this end, let $\widehat{D}_i := \widehat{Y}_{i+1}^{t_i}$ for $i = 1, 2, \ldots, k-1$, let $\widehat{D}_k = W_s$, and put

$$\mu_i = \operatorname{mod}(\widetilde{B}(\mathrm{Crit}'(f), 3\eta_0); \widehat{D}_i), \text{ for } i = 1, 2, \ldots, k.$$

So $\mu_k = \mu$.

For $i \in \{1, \ldots, k-1\}$ and $t_i < j < t_{i+1}$, the set Y_{i+1}^j is disjoint from $\widetilde{B}(\mathrm{Crit}'(f), \eta_0)$ so we have $\widehat{Y}_{i+1}^j = Y_{i+1}^j$. Hence by the backward contraction property,

$$\widehat{D}_i \subset \widetilde{B}(\mathrm{Crit}'(f), 2\eta_0) \Subset \widetilde{B}(\mathrm{Crit}'(f), 3\eta_0).$$

Moreover, since f is uniformly expanding outside $\widetilde{B}(\mathrm{Crit}'(f), \eta_0)$, $\mathrm{diam}(\widehat{D}_i)$ is exponentially small in terms of $t_{i+1} - t_i$. For a similar reason, $\mathrm{diam}(\widehat{Y}_1)$ is exponentially small in terms of t_1. Thus there are constants $A_0 > 0$ and $C_3 > 0$ such that

$$\mu_i \geq A_0(t_{i+1} - t_i), i = 1, 2, \ldots, k-1 \tag{18}$$

and

$$\operatorname{diam}(\widehat{Y}_1) \leq C_3 \exp(-A_0 t_1). \tag{19}$$

In the complex case, by Lemma 3.4 we obtain that

$$\operatorname{mod}(\widehat{Y}_i; \widehat{Y}_{i+1}) \geq \kappa_0^{\nu_i} \operatorname{mod}(\widetilde{B}(\operatorname{Crit}'(f), 3\eta_0); \widehat{D}_i) \geq \kappa_0^{\nu} \mu_i,$$

where $\widehat{Y}_{k+1} := \widehat{W_0}$. This equality also holds in the real case by repeatedly applying Lemma 4.3. Indeed, if

$$0 \leq j_0 < j_1 < \cdots < j_{\nu_i - 1} < j_{\nu_i} = t_i$$

are the integers such that $\widehat{Y}_i^{j_m}$ intersects $\operatorname{Crit}'(f)$, then for any $m = 1, 2, \ldots, \nu_i$, $\widehat{Y}_i^{j_m} \subset \widetilde{B}(c_0, 3\eta_0)$ so that

$$\operatorname{mod}(\widehat{Y}_i^{j_m}; \widehat{Y}_{i+1}^{j_m}) \geq \kappa_0 \operatorname{mod}(\widehat{Y}_i^{j_{m-1}}; \widehat{Y}_{i+1}^{j_{m-1}}).$$

Thus, in both cases we have by Lemma 3.2 that

$$\operatorname{mod}(\widehat{Y}_1; \widehat{Y}_{k+1}) \geq \sum_{i=1}^{k} \operatorname{mod}(\widehat{Y}_i; \widehat{Y}_{i+1}) \geq \kappa_0^{\nu} \sum_{i=1}^{k} \mu_i.$$

By (18), this implies

$$\operatorname{mod}(\widehat{Y}_1; \widehat{W_0}) = \operatorname{mod}(\widehat{Y}_1; \widehat{Y}_{k+1}) \geq \kappa_0^{\nu}(A_0(s - t_1) + \mu).$$

Using (19) and applying Lemma 3.3, we obtain (17).

Combining the inequalities (17) and (16), we obtain the inequality (14) with $C_0 = \max(C_1, C_2)$. □

Proof of Theorem A. – Fix a small constant $\eta_0 > 0$. By assumption, f is uniformly expanding on the maximal invariant set K of f in $J(f) \cap \widetilde{B}(\operatorname{Crit}'(f), \eta_0/2)$. It follows that there are constants $\rho > 0$ and $A_1 > 0$ so that the following property holds: for every $y \in J(f)$, every integer $t \geq 1$ and every chain $\{W'_j\}_{j=0}^{t}$ satisfying

$$W'_t = B(y, \rho), W'_0 \cap \widetilde{B}(\operatorname{Crit}'(f), \eta_0/2) \neq \varnothing,$$

and such that for every $j \in \{1, \ldots, t-1\}$ the set W'_j is disjoint from $\widetilde{B}(\operatorname{Crit}'(f), \eta_0/2)$, we have

$$W'_0 \subset \widetilde{B}(\operatorname{Crit}'(f), \eta_0), \text{ and } \operatorname{mod}(\widetilde{B}(\operatorname{Crit}'(f), 3\eta_0); W'_0) \geq A_1 t.$$

Let $x \in J(f)$, $m \geq 1$ and let W be the component of $f^{-m}(B(f^m(x), \rho))$ containing x. We shall prove that

$$\operatorname{diam}(W) \leq C \min\left\{r^{-\kappa_0 \nu}, \exp\left(-\kappa_0^{\nu} A m\right)\right\}, \tag{20}$$

holds for some integer $\nu \geq 0$, where $C, A > 0$ are constants.

Consider the chain $\{W_j\}_{j=0}^{m}$ with $W_m = B(f^m(x), \rho)$ and $W_0 \ni x$. If for every $s \in \{0, \ldots, m\}$ the set W_s is disjoint from $\widetilde{B}(\operatorname{Crit}'(f), \eta_0/2)$, then the desired inequality follows with $\nu = 0$ from the assumption that f is uniformly expanding on K. So we suppose that there is an integer $s \in \{0, \ldots, m\}$ such that W_s intersects $\widetilde{B}(\operatorname{Crit}'(f), \eta_0/2)$, and assume that s is maximal with this property. By our choice of ρ we have $W_s \subset \widetilde{B}(\operatorname{Crit}'(f), \eta_0)$, and

$$\mu := \operatorname{mod}(\widetilde{B}(\operatorname{Crit}'(f), 3\eta_0); W_s) \geq A_1(m - s). \tag{21}$$

Applying Proposition 4.1 to the chain $\{W_j\}_{j=0}^{s}$, we obtain a non-negative integer ν such that

$$\operatorname{diam}(W) \le C_0 \min\left\{r^{-\kappa_0 \nu}, \exp\left(-\kappa_0^{\nu}(A_0 s + \mu)\right)\right\},$$

which together with (21) implies that (20) holds with $A = \min(A_0, A_1)$.

To conclude the proof, let $\beta > 0$ satisfy $\beta < \log r/(\kappa_0^{-1}\log \kappa_0^{-1})$, so that

$$\varepsilon := 1 - \beta(\kappa_0^{-1}\log \kappa_0^{-1})/\log r > 0.$$

If ν is such that $r^{-\kappa_0\nu} \ge m^{-\beta}$, then

$$\nu \le \beta \log m/(\kappa_0 \log r) \text{ and } \exp(-\kappa_0^{\nu} A m) \le \exp(-m^{\varepsilon} A).$$

Thus, (20) implies that f satisfies the Polynomial Shrinking Condition with exponent β. □

5. Bounding the badness exponent

This section is devoted to the proof of Theorem B. We shall prove a recursive formula for the size of *relatively bad* pull-backs.

5.1. Relatively bad pull-backs

Let $f \in \mathscr{A}$ and let V_0 be a nice set for f. Recall that given an integer $m \ge 1$ we say that $W \in \mathscr{M}_m(V_0)$ is a bad pull-back of V_0 by f^m, if for every integer $m' \in \{1, \ldots, m\}$ such that $f^{m'}(W) \subset V_0$ the pull-back of V_0 by $f^{m'}$ containing W is not diffeomorphic.

For a subset V of V_0 and an integer $m \ge 1$, we say that a pull-back W of V by f^m is *bad relative to* V_0, if the pull-back of V_0 by f^m containing W is bad. Denote by $\mathfrak{B}_0^{\mathrm{rel}}(V) := \mathscr{M}_0(V)$ the collection of components of V; and for $m \ge 1$, denote by $\mathfrak{B}_m^{\mathrm{rel}}(V)$ the collection of all pull-backs of V by f^m that are bad relative to V_0. Moreover, denote by $\mathfrak{B}_{m,o}^{\mathrm{rel}}(V)$ the collection of all elements W of $\mathfrak{B}_m^{\mathrm{rel}}(V)$ for which f^m maps W diffeomorphically onto a component of V. Clearly, for any integer $m \ge 1$ the following properties hold:

- $W \in \mathfrak{B}_m^{\mathrm{rel}}(V_0)$ if and only if W is a bad pull-back of V_0 by f^m;
- for $V \subset V_0$ and $W \in \mathscr{M}_m(V)$, $W \in \mathfrak{B}_m^{\mathrm{rel}}(V)$ if and only if for any $m' \in \{1, 2, \ldots, m\}$, W is not contained in any diffeomorphic pull-back of V_0 by $f^{m'}$;
- if $\widetilde{V} \subset V \subset V_0$ and $\widetilde{W} \in \mathfrak{B}_m^{\mathrm{rel}}(\widetilde{V})$, then the component of $f^{-m}(V)$ containing $\widetilde{W}$ belongs to $\mathfrak{B}_m^{\mathrm{rel}}(V)$.

LEMMA 5.1. – *For every $V \subset V_0$ and every $m \ge 1$,*

$$\mathfrak{B}_m^{\mathrm{rel}}(V) = \mathfrak{B}_{m,o}^{\mathrm{rel}}(V) \cup \left(\bigcup_{\text{children } Y \text{ of } V} \mathfrak{B}_{m-m(Y)}^{\mathrm{rel}}(Y)\right),$$

where $m(Y) = m_V(Y)$ is as in Definition 3.14.

Proof. – For each $W \in \mathfrak{B}_m^{\mathrm{rel}}(V) \setminus \mathfrak{B}_{m,o}^{\mathrm{rel}}(V)$, there is $m' \in \{1, \ldots, m\}$ such that the connected component Y of $f^{-m'}(V)$ containing $f^{m-m'}(W)$ contains a critical point of f. If m' is the minimal integer with this property, then Y is a child of V and $m' = m(Y)$. If $m' = m$ then $W = Y$; otherwise, we have $W \in \mathfrak{B}_{m-m'}^{\mathrm{rel}}(Y)$ since $Y \subset V_0$. This proves that the set on the left-hand side is contained in the set on the right-hand side.

To prove the other direction, we first note that by definition $\mathfrak{B}^{\mathrm{rel}}_{m,o}(V) \subset \mathfrak{B}^{\mathrm{rel}}_{m}(V)$. It remains to show that for a child Y of V we have $\mathfrak{B}^{\mathrm{rel}}_{m-m(Y)}(Y) \subset \mathfrak{B}^{\mathrm{rel}}_{m}(V)$. Indeed, since Y contains a critical point of f we have $Y \in \mathfrak{B}^{\mathrm{rel}}_{m(Y)}(V)$, so the conclusion holds if $m(Y) = m$. Now assume that $m_0 := m - m(Y) > 0$ and consider $W \in \mathfrak{B}^{\mathrm{rel}}_{m_0}(Y)$. To prove $W \in \mathfrak{B}^{\mathrm{rel}}_{m}(V)$ let W_0 be the pull-back of V_0 by f^m that contains W. Let $m' \in \{1, \ldots, m\}$ be such that $f^{m'}(W_0) \subset V_0$. If $m' \leq m_0$ then the pull-back V_0 by $f^{m'}$ containing W_0 is not univalent because $W \in \mathfrak{B}^{\mathrm{rel}}_{m_0}(Y)$ and $Y \subset V_0$. If $m' \geq m_0 + 1$, then the pull-back of V_0 by $f^{m'-m_0}$ containing $f^{m_0}(W)$ is not diffeomorphic because it contains Y. This shows that $W \in \mathfrak{B}^{\mathrm{rel}}_{m}(V)$. Thus $\mathfrak{B}^{\mathrm{rel}}_{m-m(Y)}(Y) \subset \mathfrak{B}^{\mathrm{rel}}_{m}(V)$. □

5.2. Proof of Theorem B

Fix $f \in \mathscr{A}$, put $\tau := 2^{-\ell_{\max}(f)}$ and fix $\delta_0 > 0$ sufficiently small so that for every $\delta \in (0, \delta_0]$, every integer $i \geq 1$, if W is a pull-back of $\widetilde{B}(\mathrm{Crit}'(f), 2\tau^i\delta)$ by f^m for some $m \geq 1$ and the pull-back $\widetilde{W}$ of $\widetilde{B}(\mathrm{Crit}'(f), \delta)$ by f^m containing W is diffeomorphic, then we have $\mathrm{diam}(\widetilde{W}) \geq A^{-1} \cdot 2^i \,\mathrm{diam}(W)$, where $A > 1$ is a universal (Koebe) constant.

In the rest of this section we fix $t > 0$, and put $\tilde{\ell} = \ell_{\max}(f)$ in the complex case and $\tilde{\ell} = 2$ in the real case. We assume that f is backward contracting with a large constant r so that the conclusion of Lemma 3.15 holds with

$$s = t/(4\ell_{\max}(f)) \tag{22}$$

and so that

$$(2r^{-1})^s \leq \varepsilon := \tilde{\ell}^{-1} A^{-t} 2^{-t} \left(1 - 2^{-t/2}\right). \tag{23}$$

So, reducing $\delta_0 > 0$ if necessary, for each integer $n \geq 0$ there exists a nice set $V_n = \bigcup_{c \in \mathrm{Crit}'(f)} V_n^c$ such that for each $c \in \mathrm{Crit}'(f)$,

$$\widetilde{B}(c, \tau^n\delta_0) \subset V_n^c \subset \widetilde{B}(c, 2\tau^n\delta_0),$$

and

$$\sum_{\text{children } Y \text{ of } V_n} \mathrm{diam}(f(Y))^s \leq (\tau^n\delta_0)^s.$$

Note that for each integer $n \geq 0$, and each child Y of V_n, we have $\mathrm{diam}(f(Y)) \leq (2r^{-1})\tau^n\delta_0$, hence

$$\sum_{\text{children } Y \text{ of } V_n} \mathrm{diam}(f(Y))^{2s} \leq (2r^{-1}\tau^n\delta_0)^s \sum_{\text{children } Y \text{ of } V_n} \mathrm{diam}(f(Y))^s.$$

It follows that for each integer $n \geq 0$,

$$\sum_{\text{children } Y \text{ of } V_n} \mathrm{diam}(f(Y))^{2s} \leq \varepsilon(\tau^n\delta_0)^{2s}. \tag{24}$$

Given an integer $m \geq 0$ and a subset V of V_0 we put

$$\Xi_t(V, m) := \sum_{j=0}^{m-1} \sum_{W \in \mathfrak{B}^{\mathrm{rel}}_j(V)} d_V(W)\,\mathrm{diam}(W)^t,$$

and

$$\Xi_t^*(V,m) := \sum_{j=0}^{m-1} \sum_{W \in \mathfrak{B}_j^{\mathrm{rel}}(V) \setminus \mathfrak{B}_{j,o}^{\mathrm{rel}}(V)} d_V(W)\,\mathrm{diam}(W)^t.$$

Note that by definition $\Xi_t(V,0)=0$ and that for $\widetilde{V} \subset V \subset V_0$ such that every connected component of V contains at most one connected component of $\widetilde{V}$, we have that for each $j \geq 0$ and $W \in \mathscr{M}_j(V)$,

$$\sum_{\widetilde{W} \in \mathscr{M}_j(\widetilde{V}) \,:\, \widetilde{W} \subset W} d_{\widetilde{V}}(\widetilde{W}) \leq d_V(W). \tag{25}$$

In particular, we have that for each $m \geq 0$,

$$\Xi_t(\widetilde{V}, m) \leq \Xi_t(V,m).$$

Lemma 5.2. – *Under the above circumstances, for each integer $m \geq 1$,*

$$\Xi_t(V_0,m) \leq \sum_{c \in \mathrm{Crit}'(f)} \mathrm{diam}(V_0^c)^t + \tilde{\ell} \sum_{\textit{children } Y \textit{ of } V_0} \Xi_t(Y, m-1) \tag{26}$$

and, for each $n \geq 1$,

$$\Xi_t(V_n,m) \leq A^t \tilde{\ell} \sum_{i=0}^{n-1} 2^{(i+1-n)t} \sum_{\textit{children } Y \textit{ of } V_i} \Xi_t(Y, m-1). \tag{27}$$

Proof. – 1. To prove the first assertion observe that by definition no univalent pull-back of V_0 is bad relative to V_0, so $\mathfrak{B}_{j,o}^{\mathrm{rel}}(V_0) = \varnothing$ for all $j \geq 1$. Thus, by Lemma 5.1,

$$\Xi_t(V_0,m) \leq \sum_{W \in \mathfrak{B}_0^{\mathrm{rel}}(V_0)} \mathrm{diam}(W)^t + \sum_{\text{children } Y \text{ of } V_0} d(Y)\Xi_t(Y, m - m_{V_0}(Y)),$$

where $m_{V_0}(Y) \geq 1$ is as in Definition 3.14. Since $\mathfrak{B}_0^{\mathrm{rel}}(V_0)$ is the collection of components of V_0, the desired inequality follows.

2. To prove the second assertion fix $n \geq 1$. For $W \in \mathfrak{B}_j^{\mathrm{rel}}(V_n)$ and $i \in \{0,\ldots,n-1\}$, denote by W^i the component of $f^{-j}(V_i)$ containing W. Thus $W^i \in \mathfrak{B}_j^{\mathrm{rel}}(V_i)$, and by definition $W^0 \notin \mathfrak{B}_{j,o}^{\mathrm{rel}}(V_0)$. We denote by $i(W)$ the largest integer $i \in \{0,\ldots,n-1\}$ such that $W^i \notin \mathfrak{B}_{j,o}^{\mathrm{rel}}(V_i)$.

2.1. Let us prove that for each $i \in \{0,1,\ldots,n-1\}$,

$$\sum_{j=0}^{m-1} \sum_{W \in \mathfrak{B}_j^{\mathrm{rel}}(V_n) \,:\, i(W)=i} d_{V_n}(W)\,\mathrm{diam}(W)^t \leq A^t \cdot 2^{(i+1-n)t} \Xi_t^*(V_i,m). \tag{28}$$

To this end, we first prove that for each $W \in \mathfrak{B}_j^{\mathrm{rel}}(V_n)$ with $i(W)=i$,

$$\mathrm{diam}(W) \leq A \cdot 2^{i+1-n}\,\mathrm{diam}(W^i). \tag{29}$$

Indeed, this is trivial if $i = n-1$. If $i < n-1$, then W^{i+1} is a diffeomorphic pull-back of V_{i+1}, so, by the definition of A and the inclusion $W^{i+1} \subset W^i$,

$$\mathrm{diam}(W) \leq A \cdot 2^{i+1-n}\,\mathrm{diam}(W^{i+1}) \leq A \cdot 2^{i+1-n}\,\mathrm{diam}(W^i).$$

Together with (25), the inequality (29) implies that for each $j \geq 0$ and each $W' \in \mathfrak{B}_j^{\mathrm{rel}}(V_i)$,

$$\sum_{\substack{W \in \mathfrak{B}_j^{\mathrm{rel}}(V_n): \\ i(W)=i, W^i = W'}} d_{V_n}(W) \operatorname{diam}(W)^t \leq d_{V_i}(W') \operatorname{diam}(W')^t A^t \cdot 2^{(i+1-n)t}.$$

Summing over all $W' \subset \mathfrak{B}_j^{\mathrm{rel}}(V_n) \setminus \mathfrak{B}_{j,o}^{\mathrm{rel}}(V_n)$, we obtain the inequality (28).

2.2. In view of Lemma 5.1, for each $i \in \{0, \ldots, n-1\}$,

$$\begin{aligned} \Xi_t^*(V_i, m) &\leq \sum_{\text{children } Y \text{ of } V_i} d_{V_i}(Y) \Xi_t(Y, m - m(Y)) \\ &\leq \tilde{\ell} \sum_{\text{children } Y \text{ of } V_i} \Xi_t(Y, m-1). \end{aligned}$$

Together with (28), this implies,

$$\Xi_t(V_n, m) \leq A^t \tilde{\ell} \sum_{i=0}^{n-1} 2^{(i+1-n)t} \sum_{\text{children } Y \text{ of } V_i} \Xi_t(Y, m-1). \qquad \square$$

Proof of Theorem B. – Let $C > 0$ be a sufficiently large constant that

$$\sum_{c \in \mathrm{Crit}'(f)} \operatorname{diam}(V_0^c)^t \leq C \delta_0^{2s} \left(1 - \varepsilon \tilde{\ell} \tau^{-2s}\right). \tag{30}$$

With the notation introduced above we need to show that

$$\lim_{m \to \infty} \Xi_t(V_0, m) < \infty.$$

We will prove by induction in $m \geq 0$ that for every $n \geq 0$,

$$\Xi_t(V_n, m) \leq C(\tau^n \delta_0)^{2s}, \tag{31}$$

which clearly implies the desired assertion.

Since for each integer $n \geq 0$ we have $\Xi_t(V_n, 0) = 0$, when $m = 0$ inequality (31) holds trivially for every $n \geq 0$. Let $m \geq 1$ be given and assume by induction that inequality (31) holds for every $n \geq 0$, replacing m by $m - 1$.

Fix $i \geq 0$. Let us prove that

$$\sum_{\text{children } Y \text{ of } V_i} \Xi_t(Y, m-1) \leq C\varepsilon(\tau^{i-1}\delta_0)^{2s}. \tag{32}$$

Indeed, for a child Y of V_i, letting k be the largest integer such that $Y \subset V_k$, we have $\operatorname{diam}(f(Y)) \geq \tau^{k+1}\delta_0$. By the induction hypothesis, it follows that,

$$\Xi_t(Y, m-1) \leq \Xi_t(V_k, m-1) \leq C(\tau^k \delta_0)^{2s} \leq C(\tau^{-1} \operatorname{diam}(f(Y)))^{2s},$$

thus

$$\sum_{\text{children } Y \text{ of } V_i} \Xi_t(Y, m-1) \leq C \sum_{\text{children } Y \text{ of } V_i} (\tau^{-1} \operatorname{diam}(f(Y)))^{2s},$$

which implies (32) by (24).

Taking $i = 0$ in (32), we obtain by (30) and (26),

$$\Xi_t(V_0, m) \leq C\delta_0^{2s},$$

which proves (31) for $n = 0$. By (32) and (27), for a given integer $n \geq 1$ we obtain,

$$\begin{aligned}\Xi_t(V_n, m) &\leq A^t \tilde{\ell} \sum_{i=0}^{n-1} 2^{(i+1-n)t} C\varepsilon (\tau^{(i-1)}\delta_0)^{2s} \\ &\leq C(\tau^n \delta_0)^{2s} \varepsilon A^t \tilde{\ell} \tau^{-4s} \sum_{i=0}^{n-1} (2^t \tau^{2s})^{i+1-n} \\ &\leq C(\tau^n \delta_0)^{2s} \varepsilon A^t \tilde{\ell} 2^{-t} (1 - 2^{-t/2})^{-1},\end{aligned}$$

where we used $\tau = 2^{-\ell_{\max}(f)}$ and $s = t/(4\ell_{\max}(f))$. Inequality (31) follows by applying (23), thus completing the proof of the theorem. □

6. Induced Markov maps

This section is devoted to the proof of Theorem C and Corollary D. We shall first prove in § 6.1 the desired dimension estimate applying arguments in [46], see Proposition 6.1. Then we proceed to the tail estimate, where a Whitney decomposition type argument originated in [47] plays an important role. We first reduce the proof of Theorem C to Proposition 6.6 in § 6.2, and then give the proof of this proposition in § 6.3. We also deduce Corollary D from Theorem C in § 6.4.

We fix throughout this section a map f in the class $\mathscr{A}^*$ defined in § 2.2.3. Moreover, we denote by $\mathrm{dom}(f)$ the domain of f. In this section we use the hyperbolic dimension $\mathrm{HD}_{\mathrm{hyp}}(f)$ and the conical Julia set $J_{\mathrm{con}}(f)$ defined in § 2.1, as well as the badness exponent $\delta_{\mathrm{bad}}(f)$ defined in Definition 2.10.

6.1. Dimension estimate

Let us first prove the following:

PROPOSITION 6.1. – *Assume that $f \in \mathscr{A}^*$ satisfies $\delta_{\mathrm{bad}}(f) < \mathrm{HD}(J(f))$. Then,*

$$\mathrm{HD}(J(f)) = \mathrm{HD}(J_{\mathrm{con}}(f)) = \mathrm{HD}_{\mathrm{hyp}}(f),$$

and for each sufficiently small nice couple $(\widehat{V}, V)$ and each $c \in \mathrm{Crit}'(f)$,

$$\mathrm{HD}(J(F) \cap V^c) = \mathrm{HD}(J(f)),$$

where F denotes the canonical induced map associated with $(\widehat{V}, V)$, defined in § 2.2.3.

For an open neighborhood V of $\mathrm{Crit}'(f)$, let $K(V)$ be as in § 3.1. It follows from the definition of $\mathscr{A}^*$ that $K(V) \cap J(f)$ is a hyperbolic set for f. For a nice set V and $m \geq 1$, we use $\mathfrak{B}_m(V)$ to denote the collection of all bad pull-backs of V by f^m, see Definition 2.10.

We need the following lemma.

LEMMA 6.2. – *For each $f \in \mathscr{A}^*$,*

$$\mathrm{HD}(J(f) \setminus J_{\mathrm{con}}(f)) \leq \delta_{\mathrm{bad}}(f).$$

Furthermore, for each nice couple $(\widehat{V}, V)$ of f such that $\delta_{\mathrm{bad}}(\widehat{V}) < \mathrm{HD}(J(f))$,

$$\mathrm{HD}((J(f) \cap V) \setminus J(F)) < \mathrm{HD}(J(f)),$$

where F denotes the canonical induced map associated with $(\widehat{V}, V)$.

Proof. – To prove the first inequality it is enough to prove that for each nice couple $(\widehat{V}, V)$ for f,

$$\mathrm{HD}(J(f) \setminus J_{\mathrm{con}}(f)) \le \delta_{\mathrm{bad}}(\widehat{V}). \tag{33}$$

To this end, let N be the subset of $(V \cap J(f)) \setminus J(F)$ of those points that return at most finitely many times to V under forward iteration, and let

$$I := ((V \cap J(f)) \setminus J(F)) \setminus N.$$

To estimate $\mathrm{HD}(I)$, let $\widetilde{I}$ be the subset of I of those points x such that for every integer $m \ge 1$ with $f^m(x) \in V$, the pull-back of $\widehat{V}$ by f^m containing x is not diffeomorphic. This implies that every pull-back of $\widehat{V}$ containing x is bad. Therefore, for every integer $m \ge 1$,

$$\widetilde{I} \subset \bigcup_{j=m}^{\infty} \bigcup_{\widetilde{W} \in \mathfrak{B}_j(\widehat{V})} \widetilde{W}.$$

The definition of badness exponent implies that $\mathrm{HD}(\widetilde{I}) \le \delta_{\mathrm{bad}}(\widehat{V})$. Noting that for every y in I there is an integer $n \ge 0$ such that $f^n(y)$ is in $\widetilde{I}$, we conclude that $\mathrm{HD}(I) \le \mathrm{HD}(\widetilde{I}) \le \delta_{\mathrm{bad}}(\widehat{V})$.

Let us prove that $N \setminus J_{\mathrm{con}}(f)$ is at most countable. Indeed, $K(V) \cap J(f)$ is a hyperbolic set, hence $K(V) \cap J(f) \subset J_{\mathrm{con}}(f)$. Since

$$N \subset \bigcup_{n=1}^{\infty} f^{-n}(K(V) \cap J(f)) \tag{34}$$

and since $f^{-1}(J_{\mathrm{con}}(f)) \subset J_{\mathrm{con}}(f) \cup \mathrm{Crit}'(f)$, we conclude that

$$N \setminus J_{\mathrm{con}}(f) \subset \bigcup_{n=0}^{\infty} f^{-n}(\mathrm{Crit}'(f))$$

is at most countable.

Since $J(F) \subset J_{\mathrm{con}}(f)$ and $K(V) \cap J(f) \subset J_{\mathrm{con}}(f)$, it follows that

$$\mathrm{HD}(J(f) \setminus J_{\mathrm{con}}(f)) = \mathrm{HD}((J(f) \setminus J_{\mathrm{con}}(f)) \cap V) \le \mathrm{HD}(I) \le \delta_{\mathrm{bad}}(\widehat{V}).$$

This proves (33), hence the first equality of the lemma.

To prove the last inequality we need the following result: for any hyperbolic set A of f, $\mathrm{HD}(A) < \mathrm{HD}_{\mathrm{hyp}}(f)$. This is proved in [46, Lemmas 6.2] in the complex case, as a consequence of the (essentially) topologically exact property of the Julia set. The proof works without change for maps $f \in \mathscr{A}_{\mathbb{R}}^*$. Since $K(V) \cap J(f)$ is a hyperbolic set, by (34) we conclude that $\mathrm{HD}(N) < \mathrm{HD}(J(f))$. Since

$$\mathrm{HD}((J(f) \cap V) \setminus J(F)) = \mathrm{HD}(N \cup I) \le \max\{\mathrm{HD}(N), \delta_{\mathrm{bad}}(\widehat{V})\},$$

$\delta_{\mathrm{bad}}(\widehat{V}) < \mathrm{HD}(J(f))$ implies $\mathrm{HD}((J(f) \cap V) \setminus J(F)) < \mathrm{HD}(J(f))$. □

Proof of Proposition 6.1. – By Lemma 3.9, for a sufficiently small nice couple $(\widehat{V}, V)$ we have $\delta_{\mathrm{bad}}(\widehat{V}) < \mathrm{HD}(J(f))$. By Lemma 6.2, it follows that $\mathrm{HD}(J(f)) = \mathrm{HD}(J_{\mathrm{con}}(f))$, and that

$$\mathrm{HD}((J(f) \cap V) \setminus J(F)) < \mathrm{HD}(J(f)). \tag{35}$$

Hence $\mathrm{HD}(J(F) \cap V^c) = \mathrm{HD}(J(f))$ for each $c \in \mathrm{Crit}'(f)$.

It remains to show that $\mathrm{HD}_{\mathrm{hyp}}(f) \geq \mathrm{HD}(J(f))$. To do this let D be the domain of F and consider an enumeration $(W_n)_{n\geq 1}$ of the connected components of D. For each integer $n_0 \geq 1$ let F_{n_0} be the restriction of F to $\bigcup_{n=1}^{n_0} W_n$. Then the maximal invariant set $J(F_n)$ of F_n is contained in a uniformly hyperbolic set of f. Together with [32, Theorem 4.2.13] this implies that

$$\mathrm{HD}_{\mathrm{hyp}}(f) \geq \lim_{n\to\infty} \mathrm{HD}(J(F_n)) = \mathrm{HD}(J(F)) = \mathrm{HD}(J(f)). \qquad \square$$

Let us mention the following consequence of Lemma 6.2 and Theorem B to conclude this section.

Corollary 6.3. – *Assume that $f \in \mathscr{A}$ is expanding away from critical points and backward contracting. In the real case, assume furthermore that f is essentially topologically exact on the Julia set. Then*

$$\mathrm{HD}(J(f) \setminus J_{\mathrm{con}}(f)) = 0.$$

Proof. – If $\mathrm{Crit}'(f) = \varnothing$ then f is uniformly hyperbolic and the result is immediate. Otherwise the assumptions imply that $f \in \mathscr{A}^*$ by Fact 2.9. By Theorem B, $\delta_{\mathrm{bad}}(f) = 0$ so the assertion follows by Lemma 6.2. $\square$

This was shown in [16, Proposition 7.3] for rational maps satisfying the summability condition with each positive exponent, and in [46, §1.4] for rational maps satisfying the TCE condition. See also [52] for related results in the case of Collet-Eckmann interval maps.

Remark 6.4. – A direct consequence of Corollary 6.3 and of [18, Theorem 0.2] is that a backward contracting rational map that is expanding away from critical points, and that is not a Lattès example, has no invariant line fields and is quasi-conformally rigid. In fact, the conclusion of Corollary 6.3 shows that such a map is "uniformly weakly hyperbolic" in the sense of [18].

6.2. Tail estimate

Let us start with some notation. Let $(\widehat{V}, V)$ be a nice couple for f. Recall that for each integer $m \geq 0$, we denote by $\mathscr{M}_m(\widehat{V})$ the collection of connected components of $f^{-m}(\widehat{V})$ (§ 3.1). Moreover, for $m \geq 1$ we denote by $\mathfrak{B}_m(\widehat{V})$ the collection of bad pull-backs of $\widehat{V}$ by f^m (Definition 2.10). In what follows, $\mathfrak{B}_0(\widehat{V}) := \mathscr{M}_0(\widehat{V})$.

Let $\mathfrak{L}_V$ be the collection of components of $\mathrm{dom}(f) \setminus K(V)$. For $U \in \mathfrak{L}_V$, let $l(U) = l_V(U)$ denote the landing time of U into V. Then for each $U \in \mathfrak{L}_V$, there exists a set $\widehat{U} \supset U$ such that $f^{l(U)}$ maps $\widehat{U}$ diffeomorphically onto a component of $\widehat{V}$. Moreover, if $U \not\subset V$, then $\widehat{U} \cap V = \varnothing$.

For each $\widetilde{Y} \in \mathscr{M}_{\widetilde{m}}(\widehat{V})$ with $\widetilde{m} \geq 0$, we use $\mathfrak{D}_{\widetilde{Y}}$ to denote the collection of all simply connected sets W for which the following holds: there exist $\widetilde{Y} \supset \widehat{W} \supset W$ and $U \in \mathfrak{L}_V$ such that $U \subset f(V)$ and such that $f^{\widetilde{m}+1}$ maps W diffeomorphically onto U and maps $\widehat{W}$ diffeomorphically onto $\widehat{U}$.

We will need the following lemma, which is [47, Lemma 3.5]. It is worth noticing that this is the only place where we use a nice couple, as opposed to a nested pair of nice sets.

LEMMA 6.5. – *Let $F : D \to V$ be the canonical induced map associated to $(\widehat{V}, V)$, let $\mathfrak{D}$ be the collection of all connected components of D and let $m(x)$ be the canonical inducing time of $x \in D$. Then*

$$\mathfrak{D} = \bigcup_{\widetilde{m}=0}^{\infty} \bigcup_{\widetilde{Y} \in \mathfrak{B}_{\widetilde{m}}(\widehat{V})} \mathfrak{D}_{\widetilde{Y}},$$

and for each $\widetilde{m} \geq 0$, $\widetilde{Y} \in \mathfrak{B}_{\widetilde{m}}(\widehat{V})$ and $x \in D \cap \widetilde{Y}$,

$$m(x) = \widetilde{m} + 1 + l(f^{\widetilde{m}+1}(x)). \tag{36}$$

Proof. – Clearly for each $\widetilde{Y} \in \mathscr{M}_{\widetilde{m}}(\widehat{V})$ and $x \in W \in \mathfrak{D}_{\widetilde{Y}}$ we have $x \in D$ and $m(x) \leq \widetilde{m} + 1 + l(f^{\widetilde{m}+1}(x))$. Moreover, if $\widetilde{Y} \in \mathfrak{B}_{\widetilde{m}}(\widehat{V})$, then $m(x) > \widetilde{m}$, since $\widetilde{Y}$ is disjoint from any diffeomorphic pull-back of $\widehat{V}$ by $f^{m'}$ for any $m' \leq \widetilde{m}$. It follows that for $\widetilde{Y} \in \mathfrak{B}_{\widetilde{m}}(\widehat{V})$, (36) holds for all $x \in D \cap \widetilde{Y}$ and $\mathfrak{D}_{\widetilde{Y}} \subset \mathfrak{D}$.

It remains to prove that a connected component W of D belongs to $\mathfrak{D}_{\widetilde{Y}}$ for some $\widetilde{Y} \in \mathfrak{B}_{\widetilde{m}}(\widehat{V})$. If the canonical inducing time $m(W)$ is the first return time of W to V, then $f(W) \in \mathfrak{L}_V$ and, if we denote by $\widetilde{Y}$ the connected component of $\widehat{V}$ containing W, then $W \in \mathfrak{D}_{\widetilde{Y}}$. Suppose now that $m(W)$ is not the first return time of W to V, let $n \in \{1, \ldots, m(W) - 1\}$ be the penultimate return time of W to V, and put $W' := f^n(W)$. As we clearly have $f(W') \in \mathfrak{L}_V$, we just need to show that the pull-back $\widetilde{Y}$ of $\widehat{V}$ by f^n containing W is bad. Arguing by contradiction, assume the contrary. Then by Lemma 3.8, $\widetilde{Y}$ is contained in a diffeomorphic pull-back of $\widehat{V}$ by f^j for some $j \in \{1, \ldots, n\}$. Then $f^j(W)$ is contained in a component U of $\mathrm{dom}(f) \setminus K(V)$. Clearly $j + l(U) \leq n$, $f^{j+l(U)}(W) \subset V$ and $f^{j+l(U)}$ maps a neighborhood of W diffeomorphically onto a component $\widehat{V}$. This implies that the canonical time of W is not greater than n, which is a contradiction. □

The following proposition is a crucial estimate.

PROPOSITION 6.6. – *Assume that $f \in \mathscr{A}^*$ satisfies the Θ-Shrinking Condition for some slowly varying and monotone decreasing sequence of positive numbers $\Theta = \{\theta_n\}_{n=1}^{\infty}$. Then for each sufficiently small symmetric nice couple $(\widehat{V}, V)$ for f, with $\delta_{\mathrm{bad}}(\widehat{V}) < \mathrm{HD}_{\mathrm{hyp}}(f)$, there exists a constant $\alpha_0 \in (\delta_{\mathrm{bad}}(\widehat{V}), \mathrm{HD}_{\mathrm{hyp}}(f))$ such that, for real numbers α, t, with*

$$\alpha \geq \alpha_0, \ t \in (\delta_{\mathrm{bad}}(\widehat{V}), \alpha),$$

the following holds: there is a constant $C_1 > 0$ such that for $Y \subset \widetilde{Y} \in \mathscr{M}_{\widetilde{m}}(\widehat{V})$ with $\widetilde{m} \geq 0$ and each integer $m \geq 1$, if we put

$$D(\widetilde{Y}) := d_{\widehat{V}}(\widetilde{Y}) \left(\log d_{\widehat{V}}(\widetilde{Y}) + 1 \right),$$

then

$$\sum_{W \in \mathfrak{D}_{\widetilde{Y}} : W \subset Y, m(W) \geq m} \mathrm{diam}(W)^{\alpha} \leq C_1 D(\widetilde{Y}) \, \mathrm{diam}(Y)^t \left(\sum_{i=m}^{\infty} \theta_i^{\alpha - t} \right), \tag{37}$$

where $m(W)$ is the canonical inducing time on W with respect to $(\widehat{V}, V)$.

To prove this proposition, we will apply a technique based on a *Whitney decomposition* of the complement of the critical values of $f^{\widetilde{m}+1} : \widetilde{Y} \to f(\widehat{V})$. The proof is rather long and we suspend it to § 6.3 and complete the proof of Theorem C now.

Proof of Theorem C. – By Proposition 6.1, the first part of the theorem holds. Now fix $t \in (\delta_{\mathrm{bad}}(f), \mathrm{HD}(J(f)))$, and assume that f satisfies the Θ-Shrinking Condition for some slowly varying and monotone decreasing sequence of positive numbers $\Theta = \{\theta_n\}_{n=1}^{\infty}$. Let $(\widehat{V}, V)$ be a sufficiently small nice couple so that the conclusion of Proposition 6.6 holds and such that $\delta_{\mathrm{bad}}(\widehat{V}) < t$. Such a nice couple exists by Lemma 3.9. Then there exists $\eta > 0$ such that

$$\sum_{\widetilde{m}=0}^{\infty} \sum_{\widetilde{Y} \in \mathfrak{B}_{\widetilde{m}}(\widehat{V})} d_{\widehat{V}}(\widetilde{Y}) \operatorname{diam}(\widetilde{Y})^{t-\eta} < \infty.$$

As $D(\widetilde{Y})/d_{\widehat{V}}(\widetilde{Y})^{t/(t-\eta)}$ is bounded from above, it follows that

$$C_0 := \sum_{\widetilde{m}=0}^{\infty} \sum_{\widetilde{Y} \in \mathfrak{B}_{\widetilde{m}}(\widehat{V})} D(\widetilde{Y}) \operatorname{diam}(\widetilde{Y})^{t} < \infty.$$

Fix an integer $m \geq 1$, let D be the domain of the canonical induced map associated to $(\widehat{V}, V)$, and let $\mathfrak{D}$ be the collection of its connected components. By Lemma 6.5,

$$\sum_{\substack{W \in \mathfrak{D}:\\ W \subset Y, m(W) \geq m}} \operatorname{diam}(W)^{\alpha} = \sum_{\widetilde{m}=0}^{\infty} \sum_{\widetilde{Y} \in \mathfrak{B}_{\widetilde{m}}(\widehat{V})} \sum_{\substack{W \in \mathfrak{D}_{\widetilde{Y}}:\\ W \subset Y, m(W) \geq m}} \operatorname{diam}(W)^{\alpha}. \tag{38}$$

Applying Proposition 6.6 with t replaced by $t + \sigma$, we obtain that there is a constant $C_1 > 0$ such that for each integer $\widetilde{m} \geq 0$ and each $\widetilde{Y} \in \mathfrak{B}_{\widetilde{m}}(\widehat{V})$,

$$\sum_{\substack{W \in \mathfrak{D}_{\widetilde{Y}}:\\ W \subset Y, m(W) \geq m}} \operatorname{diam}(W)^{\alpha} \leq C_1 D(\widetilde{Y}) \operatorname{diam}(Y \cap \widetilde{Y})^{t+\sigma} \left(\sum_{i=m}^{\infty} \theta_i^{\alpha - t - \sigma} \right).$$

Combined with (38) and the inequality $\operatorname{diam}(Y \cap \widetilde{Y})^{t+\sigma} \leq \operatorname{diam}(\widetilde{Y})^{t} \operatorname{diam}(Y)^{\sigma}$, we obtain,

$$\sum_{\substack{W \in \mathfrak{D}:\\ W \subset Y, m(W) \geq m}} \operatorname{diam}(W)^{\alpha} \leq C_1 \left(\sum_{\widetilde{m}=0}^{\infty} \sum_{\widetilde{Y} \in \mathfrak{B}_{\widetilde{m}}(\widehat{V})} D(\widetilde{Y}) \operatorname{diam}(\widetilde{Y})^{t} \right) \operatorname{diam}(Y)^{\sigma} \left(\sum_{i=m}^{\infty} \theta_i^{\alpha - t - \sigma} \right).$$

This proves the desired upper bound with $C = C_0 C_1$. □

6.3. Proof of Proposition 6.6

The whole section is devoted to the proof of Proposition 6.6. By assumption, there exist constants $C_1 > 0$ and $\rho > 0$ such that for any $x \in J(f)$ and any $n \geq 1$, the component of $f^{-n}(B(f^n(x), \rho))$ that contains x has diameter not greater than $C_1 \theta_n$. Let $(\widehat{V}, V)$ be a symmetric nice couple for f so that

$$\delta_{\mathrm{bad}}(\widehat{V}) < \mathrm{HD}_{\mathrm{hyp}}(f) \text{ and } \widehat{V} \subset \widetilde{B}(\mathrm{Crit}'(f), \rho/4).$$

Furthermore, let $K_0 > 1$ and $\rho_0 \in (0, \rho)$ be the constants given by Lemma 3.10 for this choice of $(\widehat{V}, V)$ and ρ and let $\alpha_0 \in (0, \mathrm{HD}_{\mathrm{hyp}}(f))$ and $C_0, \varepsilon_0 > 0$ be the constants given by Lemma 3.11 for the choice of V. We fix $\alpha \geq \alpha_0$, $t \in (\delta_{\mathrm{bad}}(\widehat{V}), \alpha)$, an integer $\widetilde{m} \geq 0$, a connected component $\widetilde{Y}$ of $f^{-\widetilde{m}}(V)$, a subset Y of $\widetilde{Y}$, and an integer $m \geq 1$. Put $s := \min\{m - \widetilde{m} - 1, 0\}$.

Let E be the set of all critical values of $f^{\widetilde{m}+1} : \widetilde{Y} \to f(\widehat{V})$. Since this map is a composition of unicritical maps, we have that for some $C_2 > 0$ independent of $\widetilde{Y}$,

$$\#E \leq C_2(\log d(\widetilde{Y}) + 1). \tag{39}$$

We shall define a family $\mathscr{Q}$ of intervals/squares (of Whitney type) that cover $J(f) \cap f(V) \setminus E$ and then pull it back by $f^{\widetilde{m}+1}$ to obtain a family $\mathscr{P}$ of subsets of $\widetilde{Y}$. For each $P \in \mathscr{P}$, we shall estimate the total size of elements of $\mathfrak{D}_{\widetilde{Y}}$ contained in Y, which are roughly contained in P. For technical reasons, in the case that f is a rational map, we shall assume that $f(\widehat{V})$ is bounded in $\mathbb{C}$. This assumption causes no loss of generality since we may conjugate f by a rotation in such a way that ∞ is not in the closure of $f(\widehat{V})$.

We identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way. For an integer n, by a *dyadic interval of (geometric) depth* n, we mean an interval of $\mathbb{R}$ of the following form: $[k \cdot 2^{-n}, (k+1) \cdot 2^{-n})$, where k is an integer. A *dyadic square of (geometric) depth* n is the product of two dyadic intervals of the same depth in $\mathbb{C}$. For a dyadic interval (resp. square) Q, we use $\mathrm{dep}(Q)$ to denote its depth. Moreover, we use Q'' and Q' to denote the closed concentric interval (resp. square) such that

$$\mathrm{diam}(Q'') = 2\,\mathrm{diam}(Q') = 4\,\mathrm{diam}(Q),$$

and use $\widehat{Q}$ to denote the smallest dyadic interval/square with $\widehat{Q} \supsetneq Q$.

1. In the real (resp. complex) case, let $\mathscr{Q}$ be the collection of all dyadic intervals (resp. squares) Q such that

$$Q \cap f(V) \cap J(f) \neq \varnothing, Q'' \subset f(\widehat{V}) \setminus E,$$

and such that $\widehat{Q}$ does not satisfy these properties. Note that the elements of $\mathscr{Q}$ are pairwise disjoint and that

$$\bigcup_{Q \in \mathscr{Q}} Q \supset (f(V) \cap J(f)) \setminus E. \tag{40}$$

On the other hand, by the maximality in the definition of $\mathscr{Q}$, it follows that there is a constant $C_3 > 0$ independent of $\widetilde{Y}$ such that for each $Q \in \mathscr{Q}$ we have either

$$\widehat{Q}'' \cap E \neq \varnothing \text{ or } \mathrm{diam}(Q) \geq C_3 \min_{c \in \mathrm{Crit}'(f)} \mathrm{diam}(\widehat{V}^c). \tag{41}$$

For each $Q \in \mathscr{Q}$, let $\mathscr{P}(Q)$ be the collection of all components of $f^{-\widetilde{m}-1}(Q) \cap \widetilde{Y}$ and let $\mathscr{P} = \bigcup_{Q \in \mathscr{Q}} \mathscr{P}(Q)$. Furthermore, for each $P \in \mathscr{P}(Q)$ we denote by P' the pull-back of Q' by $f^{\widetilde{m}+1}$ containing P.

2. We will now complete the proof of the proposition in the special case where there exist $Q \in \mathscr{Q}$ and $P \in \mathscr{P}(Q)$ such that $Y \subset P'$. We assume that there is at least one element of $\mathfrak{D}_{\widetilde{Y}}$ contained in Y, otherwise the desired estimate is trivial. Let n be given by Lemma 3.10 with some $x \in Z := f^{\widetilde{m}+1}(Y)$ and with $\delta = \mathrm{diam}(Z)$. Since there is at least one element of $\mathfrak{D}_{\widetilde{Y}}$ contained in Y, it follows that Z contains an element of $\mathfrak{L}_V$. So we must fall into the first case of this lemma. Thus, the distortion of f^n on $f^{\widetilde{m}+1}(Y) \subset B(x, \mathrm{diam}(Z))$ is bounded by K_0, and

$$\rho_0/(2K_0) < \mathrm{diam}(f^{n+\widetilde{m}+1}(Y)) < \rho.$$

Since our hypotheses imply that the distortion of $f^{\widetilde{m}+1}$ on Y is uniformly bounded, it follows that there is a constant $C_4 > 0$ independent of $\widetilde{Y}$ such that

$$\sum_{\substack{W \in \mathfrak{D}_{\widetilde{Y}}:\\ W \subset Y, m(W) \geq m}} \operatorname{diam}(W)^{\alpha} \leq C_4 \operatorname{diam}(Y)^{\alpha} \sum_{\substack{U \in \mathfrak{L}_V:\\ U \subset f^{n+\widetilde{m}+1}(Y),\\ l(U) \geq m-(n+\widetilde{m}+1)}} \operatorname{diam}(U)^{\alpha}.$$

By Lemma 3.11, if we put $m_0 := \max\{m - (n + \widetilde{m} + 1), 0\}$, then,

$$\sum_{W \in \mathfrak{D}_{\widetilde{Y}} : W \subset Y, m(W) \geq m} \operatorname{diam}(W)^{\alpha} \leq C_4 C_0 \operatorname{diam}(Y)^{\alpha} \exp(-\varepsilon_0 m_0).$$

Since $\operatorname{diam}(Y) \leq C_1 \theta_{n+\widetilde{m}+1}$, the desired estimate follows in this case from the inequality $\operatorname{diam}(Y)^{\alpha} \leq C_1^{\alpha-t} \operatorname{diam}(Y)^t \theta_{n+\widetilde{m}+1}^{\alpha-t}$, and from the fact that Θ is slowly varying.

3. From now on we assume that for each $P \in \mathscr{P}$ the set Y is not contained in P'. This implies that there is a constant $C_5 > 0$ independent of $\widetilde{Y}$ such that for each $P \in \mathscr{P}$ intersecting Y,

$$\operatorname{diam}(P) \leq C_5 \operatorname{diam}(Y). \tag{42}$$

Fix a neighborhood V_0 of $\operatorname{Crit}'(f)$ with $V_0 \Subset V$. For each $U \in \mathfrak{L}_V$, choose a point $z_U \in U \setminus E$ with $f^{l(U)}(z_U) \in V_0$. By the Koebe principle, there exists a constant $\kappa > 0$ such that for all $U \in \mathfrak{L}_V$,

$$U \supset B(z_U, \kappa \operatorname{diam}(U)). \tag{43}$$

Recall that we have fixed an integer $m \geq 1$ and that $s = \min\{m - \widetilde{m} - 1, 0\}$. For $Q \in \mathscr{Q}$ and $P \in \mathscr{P}(Q)$, let

$$\mathfrak{L}(Q; s) := \{U \in \mathfrak{L}_V : z_U \in Q, U \subset f(V), l(U) \geq s\},$$
$$\mathfrak{D}_Y(P; s) := \{W \in \mathfrak{D}_{\widetilde{Y}} : W \subset Y, W \cap P \neq \varnothing, f^{\widetilde{m}+1}(W) \in \mathfrak{L}(Q; s)\},$$
$$Q^* := Q \cup \left(\bigcup_{U \in \mathfrak{L}(Q;s)} U \right), \text{ and } P_Y^* := P \cup \left(\bigcup_{W \in \mathfrak{D}_Y(P;s)} W \right).$$

Clearly $f^{\widetilde{m}+1}(P_Y^*) \subset Q^*$ and by (42),

$$\operatorname{diam}(P_Y^*) \leq (C_5 + 1) \operatorname{diam}(Y). \tag{44}$$

Furthermore, we put

$$\mathscr{Q}^{\sharp} := \{Q \in \mathscr{Q} : \text{ there is } P \in \mathscr{P}(Q) \text{ such that } \mathfrak{D}_Y(P; s) \neq \varnothing\}.$$

Clearly each $Q \in \mathscr{Q}^{\sharp}$ is such that $\mathfrak{L}(Q; s) \neq \varnothing$. On the other hand, by (40) for each $U \in \mathfrak{L}_V$ contained in $f(V)$ and with $l(U) \geq s$, the point z_U is contained in a unique $Q \in \mathscr{Q}$. Therefore

$$\{W \in \mathfrak{D}_{\widetilde{Y}} : W \subset Y, m(W) \geq m\} = \bigcup_{Q \in \mathscr{Q}^{\sharp}} \bigcup_{P \in \mathscr{P}(Q)} \mathfrak{D}_Y(P; s). \tag{45}$$

4. For each $Q \in \mathscr{Q}^{\sharp}$ fix $x_Q \in Q$ and let $n_Q \geq 0$ be the integer given by Lemma 3.10 with $x = x_Q$ and $\delta = \operatorname{diam}(\widehat{Q}'' \cup Q^*)$. Since Q^* contains an element of $\mathfrak{L}_V$, we must fall into

the first case of the lemma. So the distortion of f^{n_Q} on the ball $B_Q := B(x_Q, \operatorname{diam}(\widehat{Q}'' \cup Q^*))$, and hence on $\widehat{Q}'' \cup Q^*$, is bounded by K_0 and we have,

$$\rho > \operatorname{diam}(f^{n_Q}(B_Q)) > \rho_0.$$

Since $\operatorname{diam}(Q')/\operatorname{diam}(\widehat{Q}'')$ is bounded independently of $\widetilde{Y}$, it follows that there is a constant $\rho_1 > 0$ independent of $\widetilde{Y}$ such that,

$$\rho > \operatorname{diam}(f^{n_Q}(Q^*)) > \rho_1. \tag{46}$$

5. For each $n \geq 0$ let $\mathscr{Q}_n^\sharp := \{Q \in \mathscr{Q}^\sharp : n_Q = n\}$. We will prove that there is a constant $C_6 > 0$ independent of $\widetilde{Y}$ such that for each integer n,

$$\#\mathscr{Q}_n^\sharp \leq C_6(\#E + 1). \tag{47}$$

To prove this, we decompose $\mathscr{Q}_n^\sharp$ into the following subsets:

$$\mathscr{Q}_n^1 := \{Q \in \mathscr{Q}_n^\sharp : \widehat{Q}'' \cap E = \varnothing\},$$
$$\mathscr{Q}_n^2 := \{Q \in \mathscr{Q}_n^\sharp \setminus \mathscr{Q}_n^1 : \text{there is } U \in \mathfrak{L}(Q;s) \text{ such that } U \supset \widehat{Q}''\},$$
$$\mathscr{Q}_n^3 := \mathscr{Q}_n^\sharp \setminus (\mathscr{Q}_n^1 \cup \mathscr{Q}_n^2).$$

We first observe that from (41), and from the fact that the elements of $\mathscr{Q}$ are pairwise disjoint, it follows that $\#\mathscr{Q}_n^1$ is bounded from above by a constant independent of $\widetilde{Y}$.

For $Q \in \mathscr{Q}_n \setminus \mathscr{Q}_n^1$, there exists an $e \in E \cap \widehat{Q}''$. Clearly, for each $e \in E$, $\mathscr{Q}_n^2$ contains at most one element Q with $\widehat{Q}'' \ni e$. Thus $\#\mathscr{Q}_n^2 \leq \#E$.

To complete the proof of (47), it suffices to prove that for each $e \in E$, the cardinality of $\mathscr{Q}_n^3(e) = \{Q \in \mathscr{Q}_n^3 : \widehat{Q}'' \ni e\}$ is bounded from above independently of $\widetilde{Y}$. Since $\operatorname{dist}(e, Q)/\operatorname{diam}(Q) \leq \operatorname{diam}(\widehat{Q}'')/\operatorname{diam}(Q)$ is uniformly bounded for $Q \in \mathscr{Q}_n^3(e)$, the statement follows once we prove that any two elements Q_1, Q_2 of $\mathscr{Q}_n^3(e)$ have comparable diameters. To prove this we first observe that, by (43), for each $Q \in \mathscr{Q}_n^3$ the quotient $\operatorname{diam}(Q^*)/\operatorname{diam}(Q)$ is uniformly bounded. On the other hand, for $Q_1, Q_2 \in \mathscr{Q}_n^3(e)$ the sets $\widehat{Q}_1''$ and $\widehat{Q}_1''$ both contain e, so the distortion of f^n on $\widehat{Q}_1'' \cup Q_1^* \cup \widehat{Q}_2'' \cup Q_2^*$ is uniformly bounded. Since furthermore, $\operatorname{diam}(f^n(Q_1^*)) \asymp \operatorname{diam}(f^n(Q_2^*)) \asymp 1$, we have $\operatorname{diam}(Q_1^*) \asymp \operatorname{diam}(Q_2^*)$, and hence $\operatorname{diam}(Q_1) \asymp \operatorname{diam}(Q_2)$. This completes the proof of (47).

6. For $Q \in \mathscr{Q}^\sharp$, put

$$s_Q := \inf\{l(U) : U \in \mathfrak{L}(Q;s)\} \in \{s, s+1, \ldots\}. \tag{48}$$

For each $U \in \mathfrak{L}(Q;s)$, we have $l(U) \geq n_Q$, since $f^{l(U)}(U)$ contains a critical point, while $U \subset Q^*$, so $f^{n_Q} : U \to f^{n_Q}(U)$ is a diffeomorphism. Thus

$$s_Q \geq n_Q, f^{n_Q}(U) \in \mathfrak{L}_V \text{ and } l(f^{n_Q}(U)) \geq s_Q - n_Q.$$

Let us prove that there exists a constant $C_7 > 0$ such that for each $Q \in \mathscr{Q}^\sharp$ and $P \in \mathscr{P}(Q)$,

$$\sum_{W \in \mathfrak{D}_Y(P;s)} \operatorname{diam}(W)^\alpha \leq C_7 \operatorname{diam}(P_Y^*)^\alpha \exp(-\varepsilon_0(s_Q - n_Q)). \tag{49}$$

To this end, we first show that $f^{n_Q+\widetilde{m}+1}|P_Y^*$ has uniformly bounded distortion. Indeed, since $f^{\widetilde{m}+1}(P_Y^*) \subset Q^*$, and $f^{n_Q}|Q^*$ has bounded distortion, it suffices to prove that

$f^{\widetilde{m}+1}|P_Y^*$ has bounded distortion. Since $Q'' \cap E = \varnothing$, the pull-back of Q'' by $f^{\widetilde{m}+1}$ that contains P is diffeomorphic, so by the Koebe principle, $f^{\widetilde{m}+1}|P$ has uniformly bounded distortion. Moreover, for each $W \in \mathfrak{D}_{\widetilde{Y}}$, we have $U := f^{\widetilde{m}+1}(W) \in \mathfrak{L}_V$ and $f^{\widetilde{m}+1+l(U)-j}|f^j(W)$ has uniformly bounded distortion for $j = 0, 1, \ldots, \widetilde{m}+1+l(U)$, since it extends to a diffeomorphism onto the component of $\widehat{V}$ that contains $f^{l(U)}(U)(\subset V)$. Therefore, $f^{\widetilde{m}+1}|W$ has uniformly bounded distortion. It follows that $f^{\widetilde{m}+1}|P_Y^*$ has uniform bounded distortion.

Consequently, there is a constant $C_8 > 0$ independent of $\widetilde{Y}$ such that

$$\sum_{W \in \mathfrak{D}_Y(P;s)} \operatorname{diam}(W)^\alpha \le C_8 \frac{\operatorname{diam}(P_Y^*)^\alpha}{\operatorname{diam}(f^{n_Q}(Q^*))^\alpha} \sum_{U \in \mathfrak{L}(Q;s)} \operatorname{diam}(f^{n_Q}(U))^\alpha.$$

Together with (46) and Lemma 3.11, this implies (49).

7. We are ready to complete the proof of the proposition. For $P \in \mathscr{P}$ with $\mathfrak{D}_Y(P;s) \neq \varnothing$, let $Q = f^{\widetilde{m}+1}(P) \in \mathscr{Q}^\sharp$. Since

$$f^{n_Q+\widetilde{m}+1}(P_Y^*) \subset f^{n_Q}(Q^*) \text{ and } \operatorname{diam}(f^{n_Q}(Q^*)) < \rho,$$

we have

$$\operatorname{diam}(P_Y^*) \le C_1 \theta_{n_Q+\widetilde{m}+1}. \tag{50}$$

So by (44) and (49), if we put $C_9 = C_7(C_5+1)^t C_1^{\alpha-t}$, then

$$\sum_{W \in \mathfrak{D}_Y(P;s)} \operatorname{diam}(W)^\alpha \le C_9 \operatorname{diam}(Y)^t \theta_{n_Q+\widetilde{m}+1}^{\alpha-t} \exp(-\varepsilon_0(s_Q - n_Q)).$$

This inequality certainly holds also in the case $\mathfrak{D}_Y(P;s) = \varnothing$. For each $Q \in \mathscr{Q}^\sharp$, we have $\#\mathscr{P}(Q) \le d_{\widehat{V}}(\widetilde{Y})$, so

$$\sum_{P \in \mathscr{P}(Q)} \sum_{W \in \mathfrak{D}_Y(P;s)} \operatorname{diam}(W)^\alpha \le C_9 d_{\widehat{V}}(\widetilde{Y}) \operatorname{diam}(Y)^t \theta_{n_Q+\widetilde{m}+1}^{\alpha-t} \exp(-\varepsilon_0(s_Q - n_Q)).$$

Recall that for each $Q \in \mathscr{Q}^\sharp$, we have $s_Q \ge \max(n_Q, s)$. Thus, by (45),

$$\begin{aligned}
\sum_{W \in \mathfrak{D}_{\widetilde{Y}}(s)\,:\,W \subset Y} \operatorname{diam}(W)^\alpha &= \sum_{n=0}^{\infty} \sum_{Q \in \mathscr{Q}_n^\sharp} \sum_{P \in \mathscr{P}(Q)} \sum_{W \in \mathfrak{D}_Y(P;s)} \operatorname{diam}(W)^\alpha \\
&\le C_9 d(\widetilde{Y}) \operatorname{diam}(Y)^t \sum_{n=0}^{\infty} \#\mathscr{Q}_n^\sharp \theta_{n+\widetilde{m}+1}^{\alpha-t} \exp(-\varepsilon_0 \max(0, s-n)). \\
&\le C_9 C_6 (C_2+1) D(\widetilde{Y}) \operatorname{diam}(Y)^t \sum_{n=0}^{\infty} \theta_{n+\widetilde{m}+1}^{\alpha-t} \exp(-\varepsilon_0 \max(0, s-n)),
\end{aligned}$$

where in the last inequality we used (47) and (39). The desired inequality follows from the fact that the sequence $\{\theta_n\}_{n=1}^\infty$ is slowly varying.

The proof of Proposition 6.6 is completed.

6.4. Proof of Corollary D

The whole section is devoted to the proof of Corollary D. The crucial step is to prove existence of a conformal measure supported on $J_{\rm con}(f)$ and the uniform estimate on its local dimension. The rest is a rather simple application of Young's result. We shall use the following lemma, which is proved in [46, Theorem 2] in the complex case.

LEMMA 6.7. – *Assume that $f \in \mathscr{A}^*$ has a nice couple $(\widehat{V}, V)$ such that the associated canonical induced map $F : D \to V$ satisfies the following:*

1. *For every $c \in \mathrm{Crit}'(f)$, we have $\mathrm{HD}(J(F) \cap V^c) = \mathrm{HD}(J(f))$.*
2. *There exists a constant $\alpha \in (0, \mathrm{HD}(J(f)))$ such that $\sum_{W \in \mathfrak{D}} \mathrm{diam}(W)^\alpha < \infty$, where $\mathfrak{D}$ is the collection of components of D.*

Then there is a conformal measure μ of exponent $\mathrm{HD}(J(f))$ for f that is ergodic, supported on $J_{\rm con}(f)$, and satisfies

$$\mathrm{HD}(\mu) = \mathrm{HD}(J(f)) \text{ and } \mu(V \setminus D) = 0. \tag{51}$$

Furthermore, any other conformal measure for f supported on $J(f)$ is of exponent strictly larger than $\mathrm{HD}(J(f))$ and supported on $J(f) \setminus J_{\rm con}(f)$.

Proof. – First of all, it suffices to prove that there exists a conformal measure μ of exponent $\mathrm{HD}(J(f))$ that is supported on $J_{\rm con}(f)$ and that satisfies (51). The ergodicity of μ, as well as the assertions concerning the other conformal measures, follow from the fact that μ is supported on $J_{\rm con}(f)$, see [9] or [34], where only the complex case was considered, but the proof extends without change to the real case.

The following is a slight modification of the proof of [46, Theorem 2], given for rational maps. The modification is necessary for the real case since it is *a priori* unknown whether f has a conformal measure of exponent $\mathrm{HD}(J(f))$. As in the complex case, the assumptions imply that F has a conformal measure ν of exponent $t_0 := \mathrm{HD}(J(F)) = \mathrm{HD}(J(f))$, with $\nu(J(F)) = 1$, and $\mathrm{HD}(\nu) = \mathrm{HD}(J(F))$. Note that we do not need F to be topologically mixing since $\mathrm{HD}(J(F) \cap V^c)$ does not depend on c.

Let $G : \mathrm{dom}(f) \setminus K(V) \to V$ denote the first landing map to V, *i.e.,*, for each $x \in \mathrm{dom}(f) \setminus K(V)$, $G(x) = f^s(x)$, where s is the minimal non-negative integer such that $f^s(x) \in V$. For each component W of $\mathrm{dom}(f) \setminus K(V)$, we define a measure ν_W as follows:

$$\nu_W(E) = \int_{G(E)} |D(G|W)^{-1}|^{t_0} d\mu, \text{ for } E \subset W.$$

Since f is expanding outside the critical points, the distortion of $G|W$ is bounded from above by a constant independent of W. Thus there is a constant $C > 0$ such that $\nu_W(W) \le C\,\mathrm{diam}(W)^{t_0}$ for every component W of $\mathrm{dom}(f) \setminus K(V)$. Since $\sum_W \mathrm{diam}(W)^\alpha < \infty$ holds for some $\alpha < \mathrm{HD}_{\rm hyp}(f)$ (Lemma 3.11 with $m = 1$), the measure $\mu_0 := \sum_W \nu_W$ is finite. Let μ be the normalization of μ_0. Then μ is supported on $J_{\rm con}(f)$, satisfying (51). It remains to show that μ is a conformal measure of exponent t_0: for any Borel set $A \subset J(f)$ for which $f \mid A$ is injective,

$$\mu(f(A)) = \int_A |Df|^{t_0} d\mu.$$

Indeed, by writing A as a finite union of subsets, we only need to consider the following cases:

Case 1. $A \cap (K(V) \cup (V \setminus D)) = \varnothing$. Then this equality holds, as shown in Cases 1 and 2 of the proof of Proposition B.2 of [46].

Case 2. $A \subset K(V)$. Then the equality holds since both sides are equal to 0.

Case 3. A is a finite set. As clearly μ has no atom, the equality holds.

Case 4. $A \subset V \setminus (D \cup \mathrm{Crit}'(f))$. In this case, $\mu(A) = 0$, so we only need to prove $\mu(f(A)) = 0$. Since $x \in A$ has no good time, $f(x)$ can only have at most finitely many good times, so either $f(x) \in K(V)$ or $G(f(x)) \notin J(F)$. Thus $G(f(A) \setminus K(V)) \subset V \setminus J(F)$, so $\mu(f(A)) = 0$.

This proof is completed. □

Proof of Corollary D. – 1. Assume $\gamma(f) > 1$. Choose

$$\sigma_0 \in (\mathrm{HD}(J(f)) - \varepsilon, \mathrm{HD}(J(f)) - \delta_{\mathrm{bad}}(f) - \beta_{\max}(f)^{-1}),$$

$t \in (\delta_{\mathrm{bad}}(f), \mathrm{HD}(J(f)))$ and $\beta \in (0, \beta_{\max}(f))$, so that

$$\beta(\mathrm{HD}(J(f)) - t - \sigma_0) > 1.$$

Let $(\widehat{V}, V)$ be a nice couple for f given by Theorem C for this choice of t and for $\Theta = \{n^{-\beta}\}_{n=1}^{\infty}$. Applying this theorem with $\alpha = \mathrm{HD}(J(f)), \sigma = 0, m = 1$, and with Y equal to each of the connected components of V, we conclude that the nice couple $(\widehat{V}, V)$ satisfies the hypotheses of Lemma 6.7. So, there exists a conformal measure μ of exponent $\mathrm{HD}(J(f))$ satisfying all the desired properties except that (7) is to be shown. To do this, take $\rho > 0$ and let ρ_0, κ_0 and K_0 be given by Lemma 3.10 for this choice of ρ and $(\widehat{V}, V)$. Given $\delta > 0$ and $x \in J(f)$, let $n \geq 0$ be given by this lemma. In the first case of this lemma, it follows from the conformality of μ and the distortion bound of f^n on $B(x, \delta)$, that there is a constant $C_0 > 0$ independent of δ and x such that,

$$\mu(B(x, \delta)) \leq C_0 \delta^{\mathrm{HD}(J(f))}.$$

Suppose now that we are in the second case of Lemma 3.10, and denote by $\mathfrak{D}$ the collection of connected components of D. Then, using $\mu(V \setminus D) = 0$ (Lemma 6.7),

$$\mu(f^n(B(x, \delta))) \leq \sum_{W \in \mathfrak{D} \,:\, W \cap f^n(B(x,\delta)) \neq \varnothing} \mu(W) \leq \sum_{W \in \mathfrak{D} \,:\, W \subset f^n(B(x, \kappa_0 \delta))} \mu(W).$$

Since for each $W \in \mathfrak{D}$ the map $f^{m(W)}|W$ extends to a diffeomorphism onto a connected component of $\widehat{V}$, it follows from the Koebe principle that there is a constant $C_1 > 0$ such that for each $W \in \mathfrak{D}$,

$$\mu(W) \leq C_1 \operatorname{diam}(W)^{\mathrm{HD}(J(f))}. \tag{52}$$

We thus have,

$$\mu(f^n(B(x, \delta))) \leq C_1 \sum_{W \in \mathfrak{D} \,:\, W \subset f^n(B(x, \kappa_0 \delta))} \operatorname{diam}(W)^{\mathrm{HD}(J(f))}.$$

Applying Theorem C with $\alpha = \mathrm{HD}(J(f))$, $\sigma = \sigma_0$, $Y = f^n(B(x, \kappa_0 \delta))$, and $m = 1$, we conclude that there is a constant $C_2 > 0$ independent of δ and x such that,

$$\mu(f^n(B(x, \delta))) \leq C_2 \operatorname{diam}(f^n(B(x, \kappa_0 \delta)))^{\sigma_0} \leq C_2' \operatorname{diam}(f^n(B(x, \delta)))^{\sigma_0},$$

where $C_2' = C_2(\kappa_0 K_0)^{\sigma_0}$. By the conformality of μ, the distortion bound of f^n on $B(x,\delta)$, and the fact that $|Df^n(y)| \geq \rho_0$ for all $y \in B(x,\delta)$, we conclude that there is a constant $C_3 > 0$ independent of δ and x such that $\mu(B(x,\delta)) \leq C_3\delta^{\sigma_0}$. This completes the proof of (7).

The fact that either $\mathrm{HD}(J(f)) < \mathrm{HD}(\mathrm{dom}(f))$ or $J(f)$ has nonempty interior follows from the existence of a conformal measure supported on $J_{\mathrm{con}}(f)$, see for example [16, §8.2].

The assertions concerning conformal measures that are not proportional to μ follow from Lemma 6.2 and Lemma 6.7.

2. Assume $\gamma(f) > 2$, fix $\gamma \in (0, \gamma(f) - 2)$, and put $\widetilde{\gamma} = \gamma + 2$. Taking $t > \delta_{\mathrm{bad}}(f)$ closer to $\delta_{\mathrm{bad}}(f)$, and $\beta \in (0, \beta_{\max}(f))$ closer to $\beta_{\max}(f)$ if necessary, we assume that $\beta(\mathrm{HD}(J(f)) - t) > \widetilde{\gamma}$. Applying Theorem C with $\alpha = \mathrm{HD}(J(f))$, $\sigma = 0$ and with Y equal to each of the connected components of V, we conclude that there exists a constant $C_4 > 0$ such that for each $m \geq 1$,

$$\sum_{W \in \mathfrak{D} \,:\, m(W) \geq m} \mathrm{diam}(W)^{\mathrm{HD}(J(f))} \leq C_4 \sum_{n=m}^{\infty} n^{-\beta(\mathrm{HD}(J(f))-t)} \leq C_4 \sum_{n=m}^{\infty} n^{-\widetilde{\gamma}} \leq 2C_4 m^{-\widetilde{\gamma}+1}.$$

Thus by (52) we obtain,

$$\mu(\{x \in D : m(x) \geq m\}) \leq 2C_1C_4 m^{-\widetilde{\gamma}+1}.$$

Taking $(\widehat{V}, V)$ smaller if necessary, we may assume that for some $\widetilde{c} \in \mathrm{Crit}'(f)$ the set

$$\{m(W) : W \text{ connected component of } D \cap V^{\widetilde{c}} \text{ such that } F(W) = V^{\widetilde{c}}\},$$

is nonempty and its greatest common divisor is equal to 1. This last result is proven in [46, Lemma 4.1] for rational maps and its proof works without change for interval maps in $\mathscr{A}^*$. Then we proceed in a similar way as in the proof of Theorem B and Theorem C of [46], applying L. S. Young's results in [58] to the first return map of F to $V^{\widetilde{c}}$. □

7. Poincaré series

In this section we give the proofs of Theorems E, F and G, based on estimates of the Poincaré series and their integrated versions, that we state as Propositions 7.1 and 7.2.

We fix throughout this section a map f in $\mathscr{A}$ and denote by $\mathrm{dom}(f)$ its domain. Recall that for $s > 0$ and for a point $x_0 \in \mathrm{dom}(f)$, the Poincaré series of f at x_0 with exponent s, is defined as

$$\mathscr{P}(x_0; s) = \sum_{m=0}^{\infty} \mathscr{P}_m(x_0; s),$$

where

$$\mathscr{P}_m(x_0; s) = \sum_{x \in f^{-m}(x_0)} |Df^m(x)|^{-s}.$$

Clearly, if μ is a conformal measure of exponent s without an atom, then $d((f^m)_*\mu)/d\mu = \mathscr{P}_m(\,\cdot\,; s)$ on a set of full measure with respect to μ.

Recall that for a subset Q of $\operatorname{dom}(f)$ and an integer $m \geq 0$, we denote by $\mathscr{M}_m(Q)$ the collection of all pull-backs of Q by f^n. Let

$$\theta_m(Q) := \sup\{\operatorname{diam}(P) : P \in \mathscr{M}_m(Q)\}, \text{ and } \theta(Q) := \sup_{m=0}^{\infty} \theta_m(Q).$$

Moreover, for $s \geq 0$ we let

$$\mathscr{L}_m(Q; s) = \sum_{P \in \mathscr{M}_m(Q)} d_Q(P) \operatorname{diam}(P)^s, \text{ and } \mathscr{L}(Q; s) = \sum_{m=0}^{\infty} \mathscr{L}_m(Q; s),$$

where $d_Q(P)$ is defined as in § 2.2.2.

Note that if $x \in J(f)$ is disjoint from the critical orbits, then

$$\mathscr{P}_m(x; s) = \lim_{\varepsilon \to 0} \frac{\mathscr{L}_m(B(x, \varepsilon); s)}{\operatorname{diam}(B(x, \varepsilon))^s}.$$

For $z \in J(f)$ and $m \geq 0$, let

$$\Delta_m(z) := \operatorname{dist}\left(z, \bigcup_{j=0}^{m} f^j(\operatorname{Crit}'(f))\right).$$

Let $\varepsilon_0 \in (0, 1/2)$ be sufficiently small so that the constant $K(2\varepsilon_0)$ given by the Koebe principle (Lemma 3.1) satisfies $K(2\varepsilon_0) \leq 2$, and put

$$\xi_m(z) := \theta_m\left(B\left(z, \varepsilon_0 \Delta_m(z)\right)\right).$$

Given a nice set $\widehat{V}$, let $\mathfrak{B}_0(\widehat{V}) = \mathscr{M}_0(\widehat{V})$, and for $m \geq 1$, let $\mathfrak{B}_m(\widehat{V})$ denote the collection of all elements $\widetilde{Y} \in \mathscr{M}_m(\widehat{V})$ that are bad pull-backs of $\widehat{V}$. Moreover, for $s \geq 0$, let

$$\mathscr{L}_m^{\text{bad}}(\widehat{V}; s) := \sum_{\widetilde{Y} \in \mathfrak{B}_m(\widehat{V})} d_{\widehat{V}}(\widetilde{Y}) \operatorname{diam}(\widetilde{Y})^s \text{ and } \mathscr{L}^{\text{bad}}(\widehat{V}; s) := \sum_{m=0}^{\infty} \mathscr{L}_m^{\text{bad}}(\widehat{V}; s).$$

Our main technical results of this section are the following:

PROPOSITION 7.1. – *Assume that $f \in \mathscr{A}^*$ has a conformal measure of exponent $h_0 > \delta_{\text{bad}}(f)$ charging every neighborhood of each point in $\operatorname{Crit}'(f)$. Then for each sufficiently small nice couple $(\widehat{V}, V)$, the following hold:*

1. *For any $s > h_0$ and $t \in (0, s)$, there exists a constant $C > 0$ such that for each $z \in V \cap J(f)$,*

$$\mathscr{P}(z; s) \leq C \sum_{m=0}^{\infty} \mathscr{L}_m^{\text{bad}}(\widehat{V}; t) \xi_m(z)^{s-t} \Delta_m(z)^{-s}. \tag{53}$$

2. *For each $t \in (0, h_0)$ there exists a constant $C > 0$ such that for each $z \in V \cap J(f)$ and each integer $n \geq 1$,*

$$\mathscr{P}_n(z; h_0) \leq C \sum_{m=0}^{n} \mathscr{L}_m^{\text{bad}}(\widehat{V}; t) \xi_m(z)^{h_0 - t} \Delta_m(z)^{-h_0}. \tag{54}$$

In the following proposition we use the conical Julia set $J_{\text{con}}(f)$ defined in § 2.1, the best polynomial shrinking exponent $\beta_{\max}(f)$ defined in § 2.6, and the badness exponent $\delta_{\text{bad}}(f)$ defined in Definition 2.10.

PROPOSITION 7.2. – *Assume that $f \in \mathscr{A}^*$ has a conformal measure μ of exponent $h_0 > \delta_{\mathrm{bad}}(f)$ such that $\beta_{\max}(f)(h_0 - \delta_{\mathrm{bad}}(f)) > 1$ and such that for each open set U intersecting $\mathrm{Crit}'(f)$ we have $\mu(U) > 0$. Then there exists a constant $\delta_0 > 0$ such that for each $z \in J(f)$ and each $s > h_0$,*

$$\mathscr{L}(B(z, \delta_0); s) < \infty.$$

Moreover, if $\mu(J_{\mathrm{con}}(f)) = 0$, then we also have $\mathscr{L}(B(z, \delta_0); h_0) < \infty$ for each $z \in J(f)$.

Notice that in the proposition above the conformal measure μ might not charge $J(f)$.

We shall suspend the proof of these propositions until § 7.4. Let us now deduce from them Theorems E, F and G, in §§ 7.1, 7.2 and 7.3, respectively.

7.1. Proof of Theorem E

This section is devoted to the proof of Theorem E. So assume that $f \in \mathscr{A}^*$ satisfies $\gamma(f) > 2$. We put $h_0 = \mathrm{HD}(J(f))$.

Let $\beta \in (0, \beta_{\max}(f))$, $t > \delta_{\mathrm{bad}}(f)$, $q > q(f)$ and $q' < p/(p-1)$ be such that $t + 2\beta^{-1} < \mathrm{HD}(J(f))$, and

$$h := \frac{q'}{q'-1}\left(h_0 - (h_0 - t - 2\beta^{-1})/q\right) < h_0 - \delta_{\mathrm{bad}}(f) - \beta_{\max}(f)^{-1}.$$

We will prove that there is a constant $C_0 > 0$ such that for each Borel subset A of $J(f)$, and each integer $n \geq 0$,

$$\mu(f^{-n}(A)) \leq C_0 \mu(f(A))^{1/q'}. \tag{55}$$

Note that this will complete the proof of the theorem. Indeed, this implies that any accumulation point ν' of the sequence of measures $\left\{\frac{1}{n}\sum_{i=0}^{n-1} f_*^i \mu\right\}_{n=0}^{\infty}$ is such that for each Borel subset A of $J(f)$, we have $\nu'(A) \leq C_0 \mu(A)^{1/q'}$. Thus ν' is an invariant probability measure that is absolutely continuous with respect to μ, and since $p < q'/(q'-1)$ by our choice of q', we also have that the density of ν' with respect to μ belongs to $L^p(\mu)$. By ergodicity of μ, we have $\nu = \nu'$.

1. We first prove that there exists a constant a $C_1 > 0$ such that for each Borel subset B of $\mathrm{dom}(f)$,

$$\int_B \Delta_m^{-h(1-1/q')} d\mu \leq C_1 (m+1) \mu(B)^{1/q'}. \tag{56}$$

Indeed, since $h < h_0 - \delta_{\mathrm{bad}}(f) - \beta_{\max}(f)^{-1}$, by part 1 of Corollary D there exists a constant $C_2 > 0$ such that for each $z_0 \in J(f)$,

$$\int_{J(f)} \mathrm{dist}(z_0, z)^{-h} d\mu(z) \leq C_2.$$

By Hölder inequality, for each Borel subset B of $\mathrm{dom}(f)$,

$$\int_B \mathrm{dist}(z_0, z)^{-h(1-1/q')} d\mu(z) \leq C_2^{1-1/q'} \mu(B)^{1/q'}.$$

Thus, the desired inequality holds with $C_1 = \# \mathrm{Crit}'(f) C_2^{1-1/q'}$.

2. Now let $(\widehat{V}, V)$ be a sufficiently small nice couple so that $\delta_{\mathrm{bad}}(\widehat{V}) < t$. Since f has a conformal measure of exponent $h_0 = \mathrm{HD}(J(f))$, by Proposition 7.1 there exists a constant $C_3 > 0$ such that for each $z \in V$

$$\mathscr{P}_n(z; h_0) \le C_3 \sum_{m=0}^{n} \mathscr{L}_m^{\mathrm{bad}}(\widehat{V}; t + 2\beta^{-1}) \xi_m(z)^{h_0 - t - 2\beta^{-1}} \Delta_m(z)^{-h_0}. \tag{57}$$

Let us prove that there exists a constant $C_4 > 0$ such that

$$\frac{\mathscr{P}_n(z; h_0)}{|Df(z)|^{h_0}} \le C_4 \sum_{m=0}^{n} \mathscr{L}_m^{\mathrm{bad}}(\widehat{V}; t + 2\beta^{-1}) \Delta_{m+1}(f(z))^{-h(1-1/q')}. \tag{58}$$

Indeed, there exists a constant $C_5 > 1$ such that for each $z \in V$,

$$\Delta_{m+1}(f(z)) \le C_5 |Df(z)| \Delta_m(z). \tag{59}$$

Since $q > q(f)$, by our choice of ε_0, we have for some constant $C_6 > 0$,

$$\xi_m(z) \le \theta_{m+1}(B(f(z), 2\varepsilon_0 |Df(z)| \Delta_m(z))) \le C_6 (|Df(z)| \Delta_m(z))^{1/q}.$$

Inequality (58) follows using (59), and the definition of h.

3. Let $(\widehat{V}, V)$ be as above. We prove that (55) holds for all Borel sets $A \subset V$. Without loss of generality, we may assume that $f|A$ is injective. Then

$$\mu(f^{-n}(A)) = \int_A \mathscr{P}_n(z; h_0) d\mu(z) = \int_{f(A)} \frac{\mathscr{P}_n(z; h_0)}{|Df(z)|^{h_0}} d\mu(w),$$

where $w = f(z)$. By (58) and (56), this implies

$$\begin{aligned}
\mu(f^{-n}(A)) &\le C_4 \sum_{m=0}^{n} \mathscr{L}_m^{\mathrm{bad}}(\widehat{V}, t + 2\beta^{-1}) \int_{f(A)} \Delta_{m+1}(w)^{-h(1-1/q')} d\mu(w) \\
&\le C_4 C_1 \sum_{m=0}^{n} \mathscr{L}_m^{\mathrm{bad}}(\widehat{V}, t + 2\beta^{-1})(m+2) \mu(f(A))^{1/q'}.
\end{aligned}$$

Now fix $\beta' \in (\beta, \beta_{\max}(f))$. Then there is a constant $C_7 > 0$ such that for each $\widetilde{Y} \in \mathfrak{B}_m(\widehat{V})$,

$$d(\widetilde{Y}) \operatorname{diam}(\widetilde{Y})^{t + 2\beta^{-1}} \le C_7 d(\widetilde{Y}) \operatorname{diam}(\widetilde{Y})^t (m+2)^{-2\beta'\beta^{-1}},$$

which implies

$$\begin{aligned}
C_8 := \sum_{m=0}^{\infty} \mathscr{L}_m^{\mathrm{bad}}(\widehat{V}; t + 2\beta^{-1})(m+2) &\le C_7 \sum_{m=0}^{\infty} \mathscr{L}_m^{\mathrm{bad}}(\widehat{V}; t)(m+2)^{1-2\beta'\beta^{-1}} \\
&\le C_7 \mathscr{L}^{\mathrm{bad}}(\widehat{V}; t) \sum_{m=0}^{\infty} (m+2)^{1-2\beta'\beta^{-1}} < \infty.
\end{aligned}$$

This proves that for Borel sets $A \subset V$, the inequality (55) holds with $C_0 = C_1 C_4 C_8$.

4. It remains to prove (55) holds for all Borel sets $A \subset J(f) \setminus V$. The case $n = 0$ is trivial, so we shall assume $n \ge 1$. For $m \ge 1$, let

$$X_m := \{z \in J(f) : z, f(z), \ldots, f^{m-1}(z) \notin V\}$$

and $A_m = \{z \in X_m : f^{m-1}(z) \in A\}$. Then for any $n \geq 1$,

$$f^{-n}(A) = A_{n+1} \cup \left(\bigcup_{m=1}^{n} f^{-(n-m)}(f^{-1}(A_m) \cap V) \right).$$

By what we have proved in part 3, it suffices to show there exist constants $C_9 > 0$ and $\kappa \in (0, 1)$ such that

$$\mu(A_m) \leq C_9 \kappa^m \mu(f(A)). \tag{60}$$

To this end, let $V' \Subset V$ be a nice set and let

$$X'_m := \{z \in J(f) : z, f(z), \ldots, f^{m-1}(z) \notin V'\}.$$

Then by the latter part of Lemma 3.11, $\mu(X'_m)$ is exponentially small in m. Clearly, there exists a small constant $\rho > 0$ such that for each $z \in X_m$, the map f^m maps a neighborhood $U(z)$ of z diffeomorphically onto $B(f^m(z), \rho)$ with uniformly bounded distortion, and such that $U(z) \cap J(f) \subset X'_m$. It follows that for $w \in J(f)$,

$$\mathscr{P}^*_m(w; h_0) := \sum_{z \in X_m\,:\, f^m(z) = w} |Df^m(z)|^{-h_0} \asymp \sum_{z \in X_m\,:\, f^m(z) = w} \mu(U(z)) \leq \mu(X'_m).$$

Since

$$\mu(A_m) = \int_{f(A)} \mathscr{P}^*_m(w; h_0) d\mu(w),$$

inequality (60) follows.

We have completed the proof of Theorem E.

7.2. Proof of Theorem F

This section is devoted to the proof of Theorem F, which is based on the following proposition; see § 2.2.4 for the definition of $\gamma(f)$.

PROPOSITION 7.3 (Poincaré series). – *Assume that $f \in \mathscr{A}^*$ satisfies $\gamma(f) > 1$. Then $\delta_{\mathrm{Poin}}(f) = \mathrm{HD}(J(f))$. More precisely,*

1. *For every $x_0 \in \mathrm{dom}(f)$ that is not asymptotically exceptional, we have $\mathscr{P}(x_0; \mathrm{HD}(J(f))) = \infty$.*
2. *There is a subset E of $J(f)$ with $\mathrm{HD}(E) < \mathrm{HD}(J(f))$ and a neighborhood U of $J(f)$ such that for every $x_0 \in U \setminus E$, and every $s > \mathrm{HD}(J(f))$, the Poincaré series $\mathscr{P}(x_0; s)$ converges.*

Proof. – By Corollary D, f has a conformal measure μ of exponent $\mathrm{HD}(J(f))$ that is supported on $J_{\mathrm{con}}(f)$.

Part 1 follows from the existence of such a conformal measure, see for example [34, Theorem 5.2].

To prove part 2, take $t_0 \in (\delta_{\mathrm{bad}}(f), \mathrm{HD}(J(f)))$ and $\beta \in (0, \beta_{\max}(f))$ such that $\gamma = (\mathrm{HD}(J(f)) - t_0)\beta > 1$, and let $h = \mathrm{HD}(J(f))/\gamma$. Put

$$E_0 := \bigcap_{n_0 \geq 1} \left(\bigcup_{n \geq n_0} \bigcup_{c \in \mathrm{Crit}'(f)} B(f^n(c), n^{-1/h}) \right), \; E_1 := E_0 \cup \left(\bigcup_{n=1}^{\infty} f^n(\mathrm{Crit}'(f)) \right).$$

Moreover, let $(\widehat{V}, V)$ be a nice couple such that $\delta_{\mathrm{bad}}(\widehat{V}) < t_0$ and put

$$E = (K(V) \cap J(f)) \cup \bigcup_{n=0}^{\infty} f^{-n}(E_1).$$

Then $\mathrm{HD}(E_1) = \mathrm{HD}(E_0) \le h < \mathrm{HD}(J(f))$, so $\mathrm{HD}(E) < \mathrm{HD}(J(f))$.

By the (essentially) topologically exact property of $J(f)$, we have that $\mu(B(x,\delta)) > 0$ for every $x \in \mathrm{Crit}'(f)$ and every $\delta > 0$. Let $\delta_0 > 0$ be the constant given by Proposition 7.2 for the conformal measure μ and $h_0 = \mathrm{HD}(J(f))$. Let U be the δ_0-neighborhood of $J(f)$. Reducing δ_0 if necessary we assume that $U \setminus J(f)$ is disjoint from $\bigcup_{n=1}^{\infty} f^n(\mathrm{Crit}(f))$.

We first prove that for $x \in U \setminus J(f)$, and $s > \mathrm{HD}(J(f))$, we have $\mathscr{P}(x; s) < \infty$. To this end, take $z \in J(f)$ such that $x \in B(z, \delta_0)$, and take $\delta > 0$ small such that $B(x, 2\delta) \subset B(z, \delta_0) \setminus J(f)$. Then by the Koebe principle, we obtain

$$\mathscr{P}(x; s) \asymp \mathscr{L}_s(B(x,\delta)) \le \mathscr{L}_s(B(z,\delta_0)) < \infty.$$

To complete the proof, let us prove that $\mathscr{P}(x; s) < \infty$ for all $x \in J(f) \setminus E$ and $s > \mathrm{HD}(J(f))$. Since $x \notin K(V)$, there exists an integer $n \ge 0$ such that $x_0 := f^n(x) \in V \setminus E_1$. It suffices to prove that $\mathscr{P}(x_0; s) < \infty$. To this end, we first observe that there exists a constant $C(x_0) > 0$ such that $\Delta_m(x_0) \ge C(x_0)(m+1)^{-1/h}$ for all $m = 0, 1, \ldots$. Next, letting $t \in (0, s)$ be such that $\beta(t - t_0) > s/h$, we have that there is a constant $C_0 > 0$ such that for each $\widetilde{Y} \in \mathfrak{B}_m(\widehat{V})$,

$$\mathrm{diam}(\widetilde{Y})^{t-t_0} \Delta_m(x_0)^{-s} \le C_0 C(x_0)^{-s} m^{-\beta(t-t_0)} m^{s/h} \le C_0 C(x_0)^{-s},$$

so

$$\mathscr{L}_m^{\mathrm{bad}}(\widehat{V}; t) \Delta_m(x_0)^{-s} \le C_0 C(x_0)^{-s} \mathscr{L}_m^{\mathrm{bad}}(\widehat{V}; t_0).$$

By part 1 of Proposition 7.1, we obtain that there are constants $C > 0$ and $C_1 > 0$ such that,

$$\begin{aligned} \mathscr{P}(x_0; s) &\le C \sum_{m=0}^{\infty} \mathscr{L}_m^{\mathrm{bad}}(\widehat{V}; t) \xi_m(x_0)^{s-t} \Delta_m(x_0)^{-s} \\ &\le C_1 \sum_{m=0}^{\infty} \mathscr{L}_m^{\mathrm{bad}}(\widehat{V}; t) \Delta_m(x_0)^{-s} \le C_1 C_0 C(x_0)^{-s} \sum_{m=0}^{\infty} \mathscr{L}_m^{\mathrm{bad}}(\widehat{V}; t_0) < \infty. \quad \square \end{aligned}$$

Proof of Theorem F. – By Theorem C and Corollary D, we have $\mathrm{HD}(J(f)) = \mathrm{HD}_{\mathrm{hyp}}(f)$, and there is a conformal measure of exponent $\mathrm{HD}(J(f))$ supported on the conical Julia set of f. On the other hand, by Proposition 7.3 the Poincaré exponent of f is equal to $\mathrm{HD}(J(f))$. So, to complete the proof of the theorem it would be enough to prove $\overline{\mathrm{BD}}(J(f)) \le \delta_{\mathrm{Poin}}(f)$. If $J(f)$ has a nonempty interior, then there is nothing to prove. So let us assume the contrary. Then the Julia set has zero Lebesgue measure by part 1 of Corollary D. In the case $f \in \mathscr{A}_{\mathbb{C}}$, the conclusion then follows from [16, Fact 8.1 and Lemma 8.2], in which $\overline{\mathrm{BD}}(J(f)) = \delta_{\mathrm{Poin}}(f)$ was proved directly. The proof extends to the case of $f \in \mathscr{A}_{\mathbb{R}}$ with the following minor modifications and gives us the desired inequality:

- Instead of taking one point z_j from each cycle of periodic components of $\mathscr{F}$, we may need to take two points, as in the real case, we may only find a "fundamental domain" that is the union of two intervals;
- Instead of the displayed formula (24) in page 392 of [16] derived from [15, Lemma 7], we apply the Koebe principle and obtain a one-sided inequality: $\mathrm{dist}(y, J(f)) \ge C^{-1}|Df^n(y)|^{-1}$ for $y \in f^{-n}(z_j)$, where C is a Koebe constant. $\square$

7.3. Proof of Theorem G

If $\mathrm{Crit}'(f) = \varnothing$ then f is uniformly hyperbolic and the result follows easily from the removability result [21, Theorem 5], see also [16, Fact 9.1]. The latter statement of Theorem G follows from the former one by Theorems A and B, by Fact 2.9 and by [50, Corollary 8.3]. To prove the former statement of Theorem G, assume $\beta_{\max}(f)(2 - \delta_{\mathrm{bad}}(f)) > 1$. By [21, Theorem 5], it suffices to prove that for every $x \in J(f)$ there exists a constant $\delta_0 = \delta_0(x) > 0$ such that $\mathscr{L}(B(x, \delta_0); 2) < \infty$. To do this, take $\beta \in (0, \beta_{\max}(f))$ and $t > \delta_{\mathrm{bad}}(f)$ such that $\beta(2 - t) > 1$. Since the normalized Lebesgue measure μ is a conformal measure of exponent 2 and $\mu(J_{\mathrm{con}}(f)) = 0$, Proposition 7.2 applies and gives us the desired property.

7.4. Proof of Propositions 7.1 and 7.2

Throughout this section we fix a map f in the class $\mathscr{A}^*$ defined in § 2.2.3. Moreover, we fix a nice couple $(\widehat{V}, V)$ for f. For each integer $n \geq 1$, each $W \in \mathscr{M}_n(\widehat{V})$ and each $z \in \widehat{V} \cap J(f)$, let

$$\mathscr{P}_W(z; s) := \sum_{y \in f^{-n}(z) \cap W} |Df^n(y)|^{-s}.$$

Moreover, for $Q \subset \widehat{V}$ let $\mathscr{M}_W(Q) \subset \mathscr{M}_n(Q)$ be the collection of components of $f^{-n}(Q) \cap W$ and let

$$\mathscr{L}_W(Q; s) := \sum_{P \in \mathscr{M}_W(Q)} d_Q(P) \operatorname{diam}(P)^s.$$

Let $\mathfrak{G}_0(\widehat{V}) = \mathscr{M}_0(\widehat{V})$ and for $n \geq 1$, let $\mathfrak{G}_n(\widehat{V})$ be the collection of all diffeomorphic pull-backs of $\widehat{V}$ by f^n. For $Q \subset \widehat{V}$, define

$$\mathscr{M}_n^o(Q) := \bigcup_{W \in \mathfrak{G}_n(\widehat{V})} \mathscr{M}_W(Q), \text{ and } \mathscr{M}^o(Q) := \bigcup_{n=0}^{\infty} \mathscr{M}_n^o(Q).$$

Moreover, let $\mathscr{L}_n^o(Q; s)$, $\mathscr{L}^o(Q; s)$, $\mathscr{P}_n^o(z; s)$ and $\mathscr{P}^o(z; s)$ be defined in a self-evident way. Now let us prove Proposition 7.1. We start with a lemma proving upper bounds for $\mathscr{L}_n^o$ and $\mathscr{L}^o$.

LEMMA 7.4. – *Assume that f has a conformal measure μ of exponent h_0 that charges each open set intersecting* $\mathrm{Crit}'(f)$. *Then the following hold:*

1. *For $s > h_0$, we have $\mathscr{L}^o(V; s) < \infty$;*
2. $\sup_{n=0}^{\infty} \mathscr{L}_n^o(V; h_0) < \infty$*;*
3. *If furthermore $\mu(J_{\mathrm{con}}(f)) = 0$, then $\mathscr{L}^o(V; h_0) < \infty$.*

Proof. – 1. Let us first prove

$$\sum_{W \in \mathscr{M}^o(V) \,:\, W \subset V} \operatorname{diam}(W)^s < \infty. \tag{61}$$

Indeed, each $W \in \mathscr{M}^o(V)$ with $W \subset V$ is a component of $\mathrm{dom}(F^n)$ for some $n \geq 0$, and $F^n|W$ has uniformly bounded distortion. So there is a constant $C_1 > 0$ independent of n, such that

$$\sum_{\text{components } W \text{ of } \mathrm{dom}(F^n)} \operatorname{diam}(W)^{h_0} \leq C_1 \mu(\mathrm{dom}(F^n)).$$

Since $\operatorname{diam}(W)$ is exponentially small in terms of n, the same sum, but with the exponent h_0 replaced by some $s > h_0$, is exponentially small with n. Hence (61) holds.

Let us spread the estimate to all $W \in \mathscr{M}^o(V)$. Let $L : \operatorname{dom}(f) \setminus K(V) \to V$ denote the first landing map onto V. Since the distortion of L on each component of its domain is uniformly bounded, as above we obtain that

$$\sum_{U \in \mathfrak{L}_V} \operatorname{diam}(U)^{h_0} < \infty,$$

where $\mathfrak{L}_V$ denotes the collection of connected components of $\operatorname{dom}(f) \setminus K(V)$. Hence

$$\sum_{U \in \mathfrak{L}_V} \operatorname{diam}(U)^s < \infty. \tag{62}$$

Note that each $W \in \mathscr{M}^o(V)$ is contained in some $U \in \mathfrak{L}_V$, and $L(W) \in \mathscr{M}^o(V)$. By the bounded distortion property of L, there is a constant $C_2 > 0$ that only depends on $(\widehat{V}, V)$, such that

$$\sum_{W \in \mathscr{M}^o(V) \,:\, W \subset U} \operatorname{diam}(W)^s \le C_2 \frac{\operatorname{diam}(U)^s}{\operatorname{diam}(L(U))^s} \sum_{W \in \mathscr{M}^o(V) \,:\, W \subset L(U)} \operatorname{diam}(W)^s.$$

Since $L(U)$ is a component of V, we obtain for some constant $C_3 > 0$,

$$\mathscr{L}^o(V; s) \le C_3 \sum_{U \in \mathfrak{L}_V} \operatorname{diam}(U)^s \sum_{W \in \mathscr{M}^o(V) \,:\, W \subset V} \operatorname{diam}(W)^s < \infty.$$

2. For each $W \in \mathscr{M}^o_n(V)$, f^n maps W diffeomorphically onto a component of V and the map has uniformly bounded distortion, so

$$C^{-1} \operatorname{diam}(W)^{h_0} \le \mu(W) \le C \operatorname{diam}(W)^{h_0},$$

where C is a constant independent of W. It follows that

$$\mathscr{L}^o_n(V; h_0) = \sum_{W \in \mathscr{M}^o_n(V)} \operatorname{diam}(W)^{h_0} \le C\mu(\operatorname{dom}(f)) \le C.$$

3. Arguing as in the proof of part 1, it suffices to prove that $\mu(\operatorname{dom}(F^n))$ is exponentially small in terms of n. But the assumption that $\mu(J_{\mathrm{con}}(f)) = 0$ implies that for some $n_0 \ge 1$ and each component V_c of V, $\mu(V_c \setminus \operatorname{dom}(F^{n_0})) > 0$, hence by the Koebe principle, $\mu(\operatorname{dom}(F^n))$ decreases exponentially fast. □

Lemma 7.5. – *For each $m \ge 0$, $W \in \mathscr{M}_m(\widehat{V})$, each $s \ge t > 0$ and each connected $Q \subset \widehat{V}$,*

$$\mathscr{L}_W(Q; s) \le d_{\widehat{V}}(W) \operatorname{diam}(W)^t \theta_m(Q)^{s-t}. \tag{63}$$

Moreover, there exists a constant $C > 0$ such that for every $z \in J(f) \cap V$,

$$\mathscr{P}_W(z; s) \le C d_{\widehat{V}}(W) \operatorname{diam}(W)^t \xi_m(z)^{s-t} \Delta_m(z)^{-s}. \tag{64}$$

Proof. – For each $P \in \mathscr{M}_W(Q)$,

$$\operatorname{diam}(P) \le \operatorname{diam}(W) \text{ and } \operatorname{diam}(P) \le \theta_m(Q).$$

Then inequality (63) follows from

$$\mathscr{L}_W(Q; s) \le \sum_{P \in \mathscr{M}_W(Q)} d_Q(P) \sup_{P \in \mathscr{M}_W(Q)} \operatorname{diam}(P)^s \le d_{\widehat{V}}(W) \sup_{P \in \mathscr{M}_W(Q)} \operatorname{diam}(P)^s.$$

To obtain (64), observe first that each pull-back of $B(z,\Delta_m(z))$ by f^m is diffeomorphic so, by the definition of ε_0, for each $P \in \mathscr{M}_m(B(z, 2\varepsilon_0\Delta_m(z)))$,

$$|Df^m((f^m|P)^{-1}(z))|^{-1} \leq (2\varepsilon_0)^{-1}\operatorname{diam}(P)\Delta_m(z)^{-1}.$$

So inequality (64) follows from (63) with $Q = B(z, 2\varepsilon_0\Delta_m(z))$. □

LEMMA 7.6. – *For any $s \geq t > 0$, there exists a constant $C' > 0$ such that for each connected $Q \subset V$ and $z \in V \cap J(f)$,*

$$\mathscr{L}_n(Q;s) \leq C' \sum_{m=0}^{n} \mathscr{L}^o_{n-m}(V;s)\mathscr{L}^{\mathrm{bad}}_m(\widehat{V};t)\theta_m(Q)^{s-t}, \tag{65}$$

$$\mathscr{P}_n(z;s) \leq C' \sum_{m=0}^{n} \mathscr{L}^o_{n-m}(V;s)\mathscr{L}^{\mathrm{bad}}_m(\widehat{V};t)\xi_m(z)^{s-t}\Delta_m(z)^{-s}. \tag{66}$$

Proof. – For each $W \in \mathscr{M}_n(\widehat{V})$, let $k(W)$ be the minimal integer in $\{0,1,\ldots,n\}$ such that $f^{n-k(W)}$ maps a neighborhood of W diffeomorphically onto a component of $\widehat{V}$ and let $\widetilde{Y}_W$ be the component of $f^{-k(W)}(\widehat{V})$ that contains $f^{n-k(W)}(W)$. Then $\widetilde{Y}_W \in \mathfrak{B}_{k(W)}(\widehat{V})$. Note that for each $P \in \mathscr{M}_W(Q)$, we have $P' := f^{n-k(W)}(P) \subset V$. Indeed, if $k(W) = 0$, then $P' = Q \subset V$; otherwise, $\widetilde{Y}_W$ is a bad pull-back of $\widehat{V}$ and hence $P' \subset \widetilde{Y}_W \subset V$. Given $k \in \{0,1,\ldots,n\}$ and $U \in \mathscr{M}^o_{n-k}(\widehat{V})$, by Koebe principle and (63) there are constants $C_0 > 0$ and $C_1 > 0$ such that,

$$\begin{aligned}\sum_{\substack{W\in\mathscr{M}_n(\widehat{V}):\\ W\subset U, k(W)=k}} \mathscr{L}_W(Q;s) &\leq C_0 \operatorname{diam}(U)^s \sum_{\widetilde{Y}\in\mathfrak{B}^{\mathrm{bad}}_k(\widehat{V})} \mathscr{L}_{\widetilde{Y}}(Q;s)\\ &\leq C_1 \operatorname{diam}(U)^s \mathscr{L}^{\mathrm{bad}}_k(\widehat{V};t)\theta_k(Q)^{s-t},\end{aligned}$$

where in the second inequality, we used (63). Summing over all $U \in \mathscr{M}^o_{n-k}(\widehat{V})$ and then over all $k = 0,1,\ldots,n$, we obtain (65).

Repeating the argument, using (64) instead of (63), we obtain (66). □

Proof of Proposition 7.1. – Inequality (54) follows from (66) with $s = h_0$, and from part 2 of Lemma 7.4. Again by (66), when $s > h_0$,

$$\begin{aligned}\mathscr{P}(z;s) &= \sum_{n=0}^{\infty}\mathscr{P}_n(z;s)\\ &\leq C'\sum_{n=0}^{\infty}\sum_{m=0}^{n}\mathscr{L}^o_{n-m}(V;s)\mathscr{L}^{\mathrm{bad}}_m(\widehat{V};t)\xi_m(z)^{s-t}\Delta_m(z)^{-s}\\ &= C'\sum_{n=0}^{\infty}\mathscr{L}^o_n(V;s)\sum_{m=0}^{\infty}\mathscr{L}^{\mathrm{bad}}_m(\widehat{V};t)\xi_m(z)^{s-t}\Delta_m(z)^{-s},\end{aligned}$$

which implies (53) by part 1 of Lemma 7.4. □

Proof of Proposition 7.2. – Take $0 < \beta < \beta_{\max}(f)$ and $t > \delta_{\mathrm{bad}}(f)$ such that $\beta(h_0 - t) > 1$. Let $(\widehat{V}, V)$ be so small such that $\delta_{\mathrm{bad}}(\widehat{V}) < t$ and $\theta(V) \leq 1$. Fix a constant $s > h_0$ if $\mu(J_{\mathrm{con}}(f)) > 0$ and $s \geq h_0$ otherwise. We first prove that there exists a constant $C > 0$ such that for any $Q \subset V$,

$$\mathscr{L}(Q; s) \leq C\theta(Q)^{s-t}. \tag{67}$$

Indeed, in case $s > h_0$ by part 1 of Lemma 7.4 and in case $\mu(J_{\mathrm{con}}(f)) = 0$ by part 3 of that lemma, we have $\mathscr{L}^o(V; s) < \infty$. So by (65),

$$\begin{aligned}\mathscr{L}(Q; s) &= \sum_{n=0}^{\infty} \mathscr{L}_n(Q; s)\\ &\leq C \sum_{n=0}^{\infty} \sum_{m=0}^{n} \mathscr{L}^o_{n-m}(V; s) \mathscr{L}^{\mathrm{bad}}_m(\widehat{V}; t) \theta_m(Q)^{s-t}\\ &= C \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \mathscr{L}^o_k(V; s) \mathscr{L}^{\mathrm{bad}}_m(\widehat{V}; t) \theta_m(Q)^{s-t}\\ &\leq C \mathscr{L}^o(V; s) \mathscr{L}^{\mathrm{bad}}(\widehat{V}; t) \theta(Q)^{s-t},\end{aligned}$$

thus (67) holds.

To prove the proposition, it suffices to show that each $x \in J(f)$ has a neighborhood B_x such that $\mathscr{L}(B_x; s) < \infty$ since $J(f)$ is compact.

By (67), $\mathscr{L}(V; s) < \infty$, so for $x \in V \cap J(f)$, we may take $B_x = V$. For $x \in J(f) \setminus K(V)$, letting $n \geq 0$ be such that $f^n(x) \in V$ and taking $B_x \ni x$ such that $f^n(B_x) \subset V$, we have $\mathscr{L}(B_x; s) < \infty$. So let us assume that $x \in K(V) \cap J(f)$.

Let $\delta_0 > 0$ be sufficiently small so that every pull-back of $B(x, 2\delta_0)$ intersecting $\mathrm{Crit}'(f)$ is contained in V, and such that for every $y \in K(V)$ and every $n \geq 1$ the pull-back of $B(f^n(y), 2\delta_0)$ by f^n containing y is diffeomorphic. Let us prove that $\mathscr{L}(B(x, \delta_0); s) < \infty$. Let $\mathfrak{H}^o$ be the collection of all those pull-backs of $B(x, \delta_0)$ such that the corresponding pull-back of $B(x, 2\delta_0)$ is diffeomorphic and let $\mathfrak{H}'$ (resp. $\mathfrak{H}''$) be the collection of all pull-backs of $B(x, 2\delta_0)$ that are non-diffeomorphic (resp. that intersect $\mathrm{Crit}'(f)$). Let W_* be a pull-back of V contained in $B(x, \delta_0)$. Then there is a distortion constant $C_0 > 1$ such that,

$$\sum_{W \in \mathfrak{H}^o} d_{B(x,\delta_0)}(W) \operatorname{diam}(W)^s \leq C_0 \mathscr{L}(W_*; s) < \infty.$$

It is thus enough to prove that

$$\sum_{W \in \mathfrak{H}'} d_{B(x,\delta_0)}(W) \operatorname{diam}(W)^s < \infty.$$

Note that for each integer $n \geq 1$ there is at most one pull-back of $B(x, 2\delta_0)$ by f^n containing a given element of $\mathrm{Crit}'(f)$. On the other hand, since f satisfies the Polynomial Shrinking Condition with exponent β, there is a constant $C_1 > 0$ such that for each integer $n \geq 1$ and each pull-back Q of $B(x, 2\delta_0)$ by f^n, we have $\theta(Q) \leq C_1 n^{-\beta}$. Thus

$$\sum_{Q \in \mathfrak{H}''} \theta(Q)^{s-t} \leq \# \mathrm{Crit}'(f) C_1^{s-t} \sum_{n=1}^{\infty} n^{-\beta(s-t)} < \infty.$$

Therefore,

$$\begin{aligned}\sum_{W\in\mathfrak{H}'} d_{B(x,\delta_0)}(W)\,\mathrm{diam}(W)^s &\le \sum_{Q\in\mathfrak{H}''} 2\ell_{\max}(f)\mathscr{L}(Q;s)\\ &\le 2\ell_{\max}(f)C\sum_{Q\in\mathfrak{H}''}\theta(Q)^{s-t}<\infty.\end{aligned}$$ □

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