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SINGULAR CURVES AND THE ÉTALE BRAUER-MANIN OBSTRUCTION FOR SURFACES

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Abstract. – We give an elementary construction of a smooth and projective surface over an arbitrary number field k that is a counterexample to the Hasse principle but has infinite étale Brauer-Manin set. Our surface has a surjective morphism to a curve with exactly one k-point such that the unique k-fibre is geometrically a union of projective lines with an adelic point and the trivial Brauer group, but no k-point.

Résumé. – Nous présentons une construction élémentaire d'une surface lisse et projective sur un corps de nombres quelconque k qui constitue un contre-exemple au principe de Hasse et possède l'ensemble de Brauer-Manin infini. La surface est munie d'un morphisme surjectif vers une courbe avec un seul k-point tel que l'unique fibre rationnelle, qui géométriquement est l'union de droites projectives, a un point adélique et le groupe de Brauer trivial, mais pas de k-points.

Introduction

For a variety X over a number field k one can study the set $X(k)$ of k-points of X by embedding it into the topological space of adelic points $X(\mathbb{A}_k)$. In 1970 Manin [10] suggested to use the pairing

$$X(\mathbb{A}_k) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}$$

provided by local class field theory. The left kernel of this pairing $X(\mathbb{A}_k)^{\mathrm{Br}}$ is a closed subset of $X(\mathbb{A}_k)$, and the reciprocity law of global class field theory implies that $X(k)$ is contained in $X(\mathbb{A}_k)^{\mathrm{Br}}$. The first example of a smooth and projective variety X such that $X(k) = \varnothing$ but $X(\mathbb{A}_k)^{\mathrm{Br}} \neq \varnothing$ was constructed in [18] (see [1] for a similar example; an earlier example conditional on the Bombieri-Lang conjecture was found in [14]). Later, Harari [6] found many varieties X such that $X(k)$ is not dense in $X(\mathbb{A}_k)^{\mathrm{Br}}$. For all of these examples except for that of [14] the failure of the Hasse principle or weak approximation can be explained by the étale Brauer-Manin obstruction (introduced in [18], see also [13]): the closure of $X(k)$ in $X(\mathbb{A}_k)$ is contained in the étale Brauer-Manin set $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} \subset X(\mathbb{A}_k)^{\mathrm{Br}}$ which in these cases is smaller than $X(\mathbb{A}_k)^{\mathrm{Br}}$. Recently Poonen [13] constructed threefolds (fibred into rational surfaces over a curve of genus at least 1) such that $X(k) = \varnothing$ but $X(\mathbb{A}_k)^{\text{ét,Br}} \neq \varnothing$. It is known that $X(\mathbb{A}_k)^{\text{ét,Br}}$ coincides with the set of adelic points surviving the descent obstructions defined by torsors of arbitrary linear algebraic groups (as proved in [3, 17] using [7, 19]).

In 1997 Scharaschkin and the second author independently asked the question whether $X(k) = \varnothing$ if and only if $X(\mathbb{A}_k)^{\text{Br}} = \varnothing$ when X is a smooth and projective curve. They also asked if the embedding of $X(k)$ into $X(\mathbb{A}_k)^{\text{Br}}$ defines a bijection between the closure of $X(k)$ in $X(\mathbb{A}_k)$ and the set of connected components of $X(\mathbb{A}_k)^{\text{Br}}$. Despite some evidence for these conjectures, it may be prudent to consider also their weaker analogues with $X(\mathbb{A}_k)^{\text{ét,Br}}$ in place of $X(\mathbb{A}_k)^{\text{Br}}$.

In this note we give an elementary construction of a smooth and projective surface X over an arbitrary number field k that is a counterexample to the Hasse principle and has infinite étale Brauer-Manin set (Section 3). Even simpler is our counterexample to weak approximation (Section 2). This is a smooth and projective surface X over k with a unique k-point and infinite étale Brauer-Manin set $X(\mathbb{A}_k)^{\text{ét,Br}}$; moreover, infinitely many elements of $X(\mathbb{A}_k)^{\text{ét,Br}}$ have all their local components in the Zariski open set $X \setminus X(k)$. Following Poonen we consider families of curves parameterized by a curve with exactly one k-point. The new idea is to make the unique k-fibre a singular curve, geometrically a union of projective lines, and then use properties of rational and adelic points on singular curves.

The structure of the Picard group of a singular projective curve is well known, see [2, Section 9.2] or [9, Section 7.5]. In Section 1 we give a formula for the Brauer group of a reduced projective curve, see Theorem 1.3. A singular curve over k can have surprising properties that no smooth curve can ever have: it can contain infinitely many adelic points, only finitely many k-points or none at all, and yet have the trivial Brauer group. See Corollary 3.2 for a singular, geometrically connected, projective curve over an arbitrary number field k that is a counterexample to the Hasse principle not explained by the Brauer-Manin obstruction. In [8] the first author proves that every counterexample to the Hasse principle on a curve which geometrically is a union of projective lines, can be explained by finite descent (and hence by the étale Brauer-Manin obstruction). Here we note that geometrically connected and simply connected projective curves over number fields satisfy the Hasse principle, a statement that does not generalize to higher dimension, see Proposition 2.1 and Remark 2.2.

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1. The Brauer group of singular curves

Let k be a field of characteristic 0 with an algebraic closure $\bar{k}$ and the Galois group $\Gamma_k = \text{Gal}(\bar{k}/k)$. For a scheme X over k we write $\overline{X} = X \times_k \bar{k}$. All cohomology groups in this paper are Galois or étale cohomology groups. Let C be a *reduced, geometrically connected, projective* curve over k. We define the *normalization* $\widetilde{C}$ as the disjoint union of normalizations of the irreducible components of C. The normalization morphism $\nu : \widetilde{C} \to C$ factors as

$$\widetilde{C} \xrightarrow{\nu'} C' \xrightarrow{\nu''} C,$$

where C' is a maximal intermediate curve universally homeomorphic to C, see [2, Section 9.2, p. 247] or [9, Section 7.5, p. 308]. The curve C' is obtained from $\widetilde{C}$ by identifying the points which have the same image in C. In particular, there is a canonical bijection $\nu'' : C'(K)\tilde{\longrightarrow}C(K)$ for any field extension K/k. The curve C' has mildest possible singularities: for each singular point $s \in C'(\bar{k})$ the branches of $\overline{C}'$ through s intersect like n coordinate axes at $0 \in \mathbb{A}^n_{\bar{k}}$.

Let us define the following reduced 0-dimensional schemes:

$$\Lambda = \mathrm{Spec}(H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}})), \quad \Pi = C_{\mathrm{sing}}, \quad \Psi = \big(\Pi \times_C \widetilde{C}\big)_{\mathrm{red}}. \tag{1.1}$$

Here Λ is the k-scheme of geometric irreducible components of C (or the geometric connected components of $\widetilde{C}$); it is the disjoint union of closed points $\lambda = \mathrm{Spec}(k(\lambda))$ such that $k(\lambda)$ is the algebraic closure of k in the function field of the corresponding irreducible component $k(C_\lambda) = k(\widetilde{C}_\lambda)$. Next, Π is the union of singular points of C, and Ψ is the union of fibres of $\nu : \widetilde{C} \to C$ over the singular points of C with their reduced subscheme structure. The morphism ν'' induces an isomorphism $\big(\Pi \times_C C'\big)_{\mathrm{red}}\tilde{\longrightarrow}\Pi$, so we can identify these schemes. Let $i : \Pi \to C$, $i' : \Pi \to C$ and $j : \Psi \to \widetilde{C}$ be the natural closed immersions. We have a commutative diagram

$$\begin{array}{ccccc} \widetilde{C} & \xrightarrow{\nu'} & C' & \xrightarrow{\nu''} & C \\ {\scriptstyle j}\uparrow & & {\scriptstyle i'}\uparrow & \nearrow_{i} & \\ \Psi & \xrightarrow{\nu'} & \Pi. & & \end{array}$$

The restriction of ν to the smooth locus of C induces isomorphisms

$$\widetilde{C} \setminus j(\Psi)\tilde{\longrightarrow}C' \setminus i'(\Pi)\tilde{\longrightarrow}C \setminus i(\Pi).$$

An algebraic group over Π is a product $G = \prod_\pi i_{\pi*}(G_\pi)$, where π ranges over the irreducible components of Π, $i_\pi : \mathrm{Spec}(k(\pi)) \to \Pi$ is the natural closed immersion, and G_π is an algebraic group over the field $k(\pi)$.

Proposition 1.1. – (i) *The canonical maps* $\mathbb{G}_{m,C'} \to \nu'_*\mathbb{G}_{m,\widetilde{C}}$ *and* $\mathbb{G}_{m,C'} \to i'_*\mathbb{G}_{m,\Pi}$ *give rise to the exact sequence of étale sheaves on* C'

$$0 \to \mathbb{G}_{m,C'} \to \nu'_*\mathbb{G}_{m,\widetilde{C}} \oplus i'_*\mathbb{G}_{m,\Pi} \to i'_*\nu'_*\mathbb{G}_{m,\Psi} \to 0, \tag{1.2}$$

where $\nu'_*\mathbb{G}_{m,\Psi}$ *is an algebraic torus over* Π.

(ii) *The canonical map* $\mathbb{G}_{m,C} \to \nu''_*\mathbb{G}_{m,C'}$ *gives rise to the exact sequence of étale sheaves on* C*:*

$$0 \to \mathbb{G}_{m,C} \to \nu''_*\mathbb{G}_{m,C'} \to i_*\mathcal{U} \to 0, \tag{1.3}$$

where $\mathcal{U}$ *is a commutative unipotent group over* Π.

Proof. – This is essentially well known, see [2], the proofs of Propositions 9.2.9 and 9.2.10, or [9, Lemma 7.5.12]. By [11, Thm. II.2.15 (b), (c)] it is enough to check the exactness of (1.2) at each geometric point $\bar{x}$ of C'. If $\bar{x} \notin i'(\Pi)$, this is obvious since locally at $\bar{x}$ the morphism ν' is an isomorphism, and the stalks $(i'_*\mathbb{G}_{m,\Pi})_{\bar{x}}$ and $(i'_*\nu'_*\mathbb{G}_{m,\Psi})_{\bar{x}}$ are zero. Now let $\bar{x} \in i'(\Pi)$, and let $\mathcal{O}_{\bar{x}}$ be the strict Henselisation of the local ring of $\bar{x}$ in C'. Each geometric point $\bar{y}$ of $\widetilde{C}$ belongs to exactly one geometric connected component of $\widetilde{C}$, and we denote by $\mathcal{O}_{\bar{y}}$

the strict Henselisation of the local ring of $\bar{y}$ in its geometric connected component. By the construction of C' we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\bar{x}} \longrightarrow k(\bar{x}) \times \prod_{\nu'(\bar{y})=\bar{x}} \mathcal{O}_{\bar{y}} \longrightarrow \prod_{\nu'(\bar{y})=\bar{x}} k(\bar{y}) \longrightarrow 0,$$

where $\mathcal{O}_{\bar{y}} \to k(\bar{y})$ is the reduction modulo the maximal ideal of $\mathcal{O}_{\bar{y}}$, and $k(\bar{x}) \to k(\bar{y})$ is the multiplication by -1. We obtain an exact sequence of abelian groups

$$1 \longrightarrow \mathcal{O}^*_{\bar{x}} \longrightarrow k(\bar{x})^* \times \prod_{\nu'(\bar{y})=\bar{x}} \mathcal{O}^*_{\bar{y}} \longrightarrow \prod_{\nu'(\bar{y})=\bar{x}} k(\bar{y})^* \longrightarrow 1.$$

Using [11, Cor. II.3.5 (a), (c)] one sees that this is the sequence of stalks of (1.2) at $\bar{x}$, so that (i) is proved.

To prove (ii) consider the exact sequence

$$0 \to \mathbb{G}_{m,C} \to \nu''_* \mathbb{G}_{m,C'} \to \nu''_* \mathbb{G}_{m,C'}/\mathbb{G}_{m,C} \to 0.$$

Since ν'' is an isomorphism away from $i(\Pi)$, the restriction of the sheaf $\nu''_*\mathbb{G}_{m,C'}/\mathbb{G}_{m,C}$ to $C \setminus i(\Pi)$ is zero, hence $\nu''_*\mathbb{G}_{m,C'}/\mathbb{G}_{m,C} = i_*\mathcal{U}$ for some sheaf $\mathcal{U}$ on Π. To see that $\mathcal{U}$ is a unipotent group scheme it is enough to check the stalks at geometric points. Let $\bar{x}$ be a geometric point of $i(\Pi)$, and let $\bar{y}$ be the unique geometric point of C' such that $\nu''(\bar{y}) = \bar{x}$. Let $\mathcal{O}_{\bar{x}}$ and $\mathcal{O}_{\bar{y}}$ be the corresponding strictly Henselian local rings. The stalk $(\nu''_*\mathbb{G}_{m,C'}/\mathbb{G}_{m,C})_{\bar{x}}$ is $\mathcal{O}^*_{\bar{y}}/\mathcal{O}^*_{\bar{x}}$, and according to [9, Lemma 7.5.12 (c)], this is a unipotent group over the field $k(\bar{x})$. This finishes the proof. □

REMARK 1.2. – The first part of Proposition 1.1 has an alternative proof which is easier to generalize to higher dimension. Let X be a projective k-variety with normalization morphism $\nu : \widetilde{X} \to X$. Assume that $\widetilde{X}$, X_{sing} and $\widetilde{X}_{\text{crit}}$ are smooth, where X_{sing} is the singular locus of X and $\widetilde{X}_{\text{crit}} = \nu^{-1}(X_{\text{sing}}) \subseteq \widetilde{X}$ is the critical locus of ν. (This assumption is automatically satisfied when X is a curve.) The analogue of C' is the K-variety X' given by the pushout in the square

$$\begin{array}{ccc} \widetilde{X}_{\text{crit}} & \xrightarrow{j} & \widetilde{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \nu'} \\ X_{\text{sing}} & \xrightarrow{i'} & X'. \end{array}$$

This pushout exists in the category of K-varieties since i' is a closed embedding and g is an affine morphism of smooth projective varieties (see [4, Thm. 5.4]). One then proves that the sequence of sheaves

$$0 \longrightarrow \mathbb{G}_{m,X'} \longrightarrow \nu'_*\mathbb{G}_{m,\widetilde{X}} \oplus i'_*\mathbb{G}_{m,X_{\text{sing}}} \longrightarrow \nu'_* j_* \mathbb{G}_{m,\widetilde{X}_{\text{crit}}} \longrightarrow 0$$

is exact, as follows. From the definition of X' we obtain that the square

$$\begin{array}{ccc} \mathcal{O}_{X'} & \longrightarrow & \nu'_*\mathcal{O}_{\widetilde{X}} \\ \downarrow & & \downarrow \\ i'_*\mathcal{O}_{X_{\text{sing}}} & \longrightarrow & \nu'_* j_* \mathcal{O}_{\widetilde{X}_{\text{crit}}} \end{array}$$

is a Cartesian diagram of sheaves of rings on X' (where Cartesian for sheaves means Cartesian on the stalks). The functor $\mathbb{G}_m$ from the category of rings with 1 to the category of abelian groups that associates to a ring its group of units, commutes with limits and filtered colimits (e.g., with taking stalks). This implies that the diagram

$$\begin{array}{ccc} \mathbb{G}_{m,X'} & \longrightarrow & \nu'_*\mathbb{G}_{m,\widetilde{X}} \\ \downarrow & & \downarrow \\ i'_*\mathbb{G}_{m,X_{\mathrm{sing}}} & \longrightarrow & \nu'_*j_*\mathbb{G}_{m,\widetilde{X}_{\mathrm{crit}}} \end{array}$$

is Cartesian. Hence the exactness of the sequence

$$0 \longrightarrow \mathbb{G}_{m,X'} \longrightarrow \nu'_*\mathbb{G}_{m,\widetilde{X}} \oplus i'_*\mathbb{G}_{m,X_{\mathrm{sing}}} \longrightarrow \nu'_*j_*\mathbb{G}_{m,\widetilde{X}_{\mathrm{crit}}}.$$

It remains to check that the last map here is surjective. The morphism ν is finite, hence the functor ν_* is exact [11, Cor. II.3.6]. The map $\mathbb{G}_{m,\widetilde{X}} \to j_*\mathbb{G}_{m,\widetilde{X}_{\mathrm{crit}}}$ is surjective, because $j : \widetilde{X}_{\mathrm{crit}} \to \widetilde{X}$ is a closed embedding, and thus $\nu'_*\mathbb{G}_{m,\widetilde{X}} \to \nu'_*j_*\mathbb{G}_{m,\widetilde{X}_{\mathrm{crit}}}$ is surjective too.

For fields $k_1, \ldots, k_n$, we have $\mathrm{Br}\left(\coprod_{i=1}^n \mathrm{Spec}(k_i)\right) = \oplus_{i=1}^n \mathrm{Br}(k_i)$.

THEOREM 1.3. – *Let C be a reduced, geometrically connected, projective curve, and let Λ, Π and Ψ be the schemes defined in* (1.1). *Let $\Lambda = \coprod_\lambda \mathrm{Spec}(k(\lambda))$ be the decomposition into the disjoint union of connected components, so that $\widetilde{C} = \coprod_\lambda \widetilde{C}_\lambda$, where $\widetilde{C}_\lambda$ is a geometrically integral, smooth, projective curve over the field $k(\lambda)$. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(C) \longrightarrow \mathrm{Br}(\Pi) \oplus \bigoplus_{\lambda\in\Lambda} \mathrm{Br}(\widetilde{C}_\lambda) \longrightarrow \mathrm{Br}(\Psi), \tag{1.4}$$

where the maps are the composition of canonical maps

$$\mathrm{Br}(\widetilde{C}_\lambda) \to \mathrm{Br}(\widetilde{C}_\lambda \cap \Psi) \to \mathrm{Br}(\Psi),$$

and the opposite of the restriction map $\mathrm{Br}(\Pi) \to \mathrm{Br}(\Psi)$.

Proof. – Let π range over the irreducible components of Π, so that $\mathcal{U} = \prod_\pi i_{\pi*}(U_\pi)$, where U_π is a commutative unipotent group over the field $k(\pi)$. Since i_* is an exact functor [11, Cor. II.3.6], we have $H^n(C, i_*\mathcal{U}) = H^n(\Pi, \mathcal{U}) = \prod_\pi H^n(k(\pi), U_\pi)$. The field k has characteristic 0, and it is well known that this implies that any commutative unipotent group has zero cohomology in degree $n > 0$. (Such a group has a composition series with factors $\mathbb{G}_a$, and $H^n(k, \mathbb{G}_a) = 0$ for any $n > 0$, see [15, X, Prop. 1].) Thus the long exact sequence of cohomology groups associated to (1.3) gives rise to an isomorphism $\mathrm{Br}(C) = H^2(C, \mathbb{G}_{m,C}) \xrightarrow{\sim} H^2(C, \nu''_*\mathbb{G}_{m,C'})$. Since ν'' is finite, the functor ν''_* is exact [11, Cor. II.3.6], so we obtain an isomorphism $\mathrm{Br}(C) \xrightarrow{\sim} \mathrm{Br}(C')$. We now apply similar arguments to (1.2). Hilbert's Theorem 90 gives $H^1(\Pi, \nu'_*\mathbb{G}_{m,\Psi}) = H^1(\Psi, \mathbb{G}_{m,\Psi}) = 0$, so we obtain the exact sequence (1.4). □

Recall that a reduced, geometrically connected, projective curve S over a field k is called *semi-stable* if all the singular points of S are ordinary double points [2, Def. 9.2.6].

DEFINITION 1.4. – A semi-stable curve is called *bipartite* if it is a union of two smooth curves without common irreducible components.

Corollary 1.5. – *Let $S = S^+ \cup S^-$ be a bipartite curve, where S^+ and S^- are smooth curves such that $S^+ \cap S^-$ is finite. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{Br}(S) \longrightarrow \mathrm{Br}(S^+) \oplus \mathrm{Br}(S^-) \longrightarrow \mathrm{Br}(S^+ \cap S^-), \tag{1.5}$$

where $\mathrm{Br}(S) \to \mathrm{Br}(S^+) \oplus \mathrm{Br}(S^-)$ *is the natural map, and* $\mathrm{Br}(S^\pm) \to \mathrm{Br}(S^+ \cap S^-)$ *is the restriction map multiplied by* ± 1.

Proof. – In our previous notation we have $\widetilde{S} = S^+ \coprod S^-$, $\Pi = S^+ \cap S^-$, and Ψ is the disjoint union of two copies of Π, namely $\Psi^+ = \Psi \cap S^+$ and $\Psi^- = \Psi \cap S^-$. In particular, the restriction map $\mathrm{Br}(\Pi) \to \mathrm{Br}(\Psi)$ is injective. Thus taking the quotients by $\mathrm{Br}(\Pi)$ in the middle and last terms of (1.4), we obtain (1.5). □

The constructions in Sections 2 and 3 use singular curves of the following special kind.

Definition 1.6. – A reduced, geometrically connected, projective curve C over k is called *conical* if every irreducible component of $\overline{C}$ is rational.

For *bipartite conical* curves the calculation of the Brauer group can be carried out using only the Brauer groups of fields.

Corollary 1.7. – *Let $C = C^+ \cup C^-$ be a bipartite conical curve, and let $\Lambda = \Lambda^+ \coprod \Lambda^-$ be the corresponding decomposition of Λ. Then $\mathrm{Br}(C)$ is the kernel of the map*

$$\bigoplus_{\lambda \in \Lambda^+} \mathrm{Br}(k(\lambda))/[\widetilde{C}_\lambda] \ \oplus \ \bigoplus_{\lambda \in \Lambda^-} \mathrm{Br}(k(\lambda))/[\widetilde{C}_\lambda] \ \longrightarrow \mathrm{Br}(C^+ \cap C^-),$$

where $[\widetilde{C}_\lambda] \in \mathrm{Br}(k(\lambda))$ is the class of the conic $\widetilde{C}_\lambda$ over the field $k(\lambda)$, and the map $\mathrm{Br}(k(\lambda))/[\widetilde{C}_\lambda] \to \mathrm{Br}(C^+ \cap C^-)$ is the restriction followed by multiplication by ± 1 when $\lambda \in \Lambda^\pm$.

Proof. – This follows directly from Proposition 1.5 and the well known fact that the Brauer group of a conic over k is the quotient of $\mathrm{Br}(k)$ by the cyclic subgroup generated by the class of this conic. □

In some cases one can compute $\mathrm{Br}(C)$ using the Hochschild-Serre spectral sequence $H^p(k, H^q(\overline{C}, \mathbb{G}_m)) \Rightarrow H^{p+q}(C, \mathbb{G}_m)$. In [5, III, Cor. 1.2] Grothendieck proved that $\mathrm{Br}(\overline{C}) = 0$ for any curve C. Hence the spectral sequence identifies the cokernel of the natural map $\mathrm{Br}(k) \to \mathrm{Br}(C)$ with a subgroup of $H^1(k, \mathrm{Pic}(\overline{C}))$.

The structure of Γ_k-module $\mathrm{Pic}(\overline{C})$ is well known, at least up to its maximal unipotent subgroup. It is convenient to describe this structure in combinatorial terms. Recall that $\overline{\Lambda}$, $\overline{\Pi}$, $\overline{\Psi}$ are the $\bar{k}$-schemes obtained from Λ, Π, Ψ by extending the ground field to $\bar{k}$. We associate to C the *incidence graph* $X(C)$ defined as the directed graph whose vertices are $X(C)_0 = \overline{\Lambda} \cup \overline{\Pi}$ and the edges are $X(C)_1 = \overline{\Psi}$. The edge $Q \in \overline{\Psi}$ goes from $L \in \overline{\Lambda}$ to $P \in \overline{\Pi}$ when $\nu(Q) = P$ and Q is contained in the irreducible component L of $\widetilde{C}$. The source and target maps $X(C)_1 \to X(C)_0$ can be described as a morphism of k-schemes

$$(s,t) : \Psi \longrightarrow \Lambda \coprod \Pi,$$

where $t : \Psi \to \Pi$ is induced by ν', and s is the composition of the closed immersion $j : \Psi \to \widetilde{C}$ and the canonical morphism $\widetilde{C} \to \Lambda$. By construction $X(C)$ is a connected bipartite graph with a natural action of the Galois group Γ_k.

For a reduced 0-dimensional k-scheme $p : \Sigma \to \mathrm{Spec}(k)$ of finite type, the k-group scheme $p_*\mathbb{G}_m$ is an algebraic torus over k. If we write $\Sigma = \coprod_{i=1}^n \mathrm{Spec}(k_i)$, where $k_1, \dots, k_n$ are finite field extensions of k, then $p_*\mathbb{G}_m$ is the product of Weil restrictions $\prod_{i=1}^n R_{k_i/k}(\mathbb{G}_m)$. For a reduced 0-dimensional scheme $p' : \Sigma' \to \mathrm{Spec}(k)$ of finite type a morphism of k-schemes $f : \Sigma' \to \Sigma$ gives rise to a canonical morphism $\mathbb{G}_{m,\Sigma} \to f_*\mathbb{G}_{m,\Sigma'}$, and hence to a canonical homomorphism of k-tori $p_*\mathbb{G}_m \to p'_*\mathbb{G}_m$, which we denote by f^*. Let us denote the structure morphism $\Lambda \to \mathrm{Spec}(k)$ by p_Λ, and use the same convention for Ψ and Π. Let T be the algebraic k-torus defined by the exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow p_{\Lambda *}\mathbb{G}_m \oplus p_{\Pi *}\mathbb{G}_m \longrightarrow p_{\Psi *}\mathbb{G}_m \longrightarrow T \longrightarrow 1,$$

where the middle arrow is $s^*(t^*)^{-1}$.

Remark 1.8. – The character group $\hat{T}$ with its natural action of the Galois group Γ_k, is canonically isomorphic to $H_1(X(C), \mathbb{Z})$, the *first homology group* of the graph $X(C)$. Since $X(C)$ is connected, we have $T = \{1\}$ if and only if $X(C)$ is a tree.

Proposition 1.9. – *Let C be a reduced, geometrically connected, projective curve over k. We have the following exact sequences of Γ_k-modules:*

$$0 \longrightarrow T(\bar{k}) \longrightarrow \mathrm{Pic}(\overline{C'}) \longrightarrow \mathrm{Pic}(\widetilde{C} \times_k \bar{k}) \longrightarrow 0, \tag{1.6}$$

$$0 \longrightarrow U(\bar{k}) \longrightarrow \mathrm{Pic}(\overline{C}) \longrightarrow \mathrm{Pic}(\overline{C'}) \longrightarrow 0, \tag{1.7}$$

where U is a commutative unipotent algebraic group over k.

Proof. – To obtain (1.6) we apply the direct image functor with respect to the structure morphism $C' \to \mathrm{Spec}(k)$ to the exact sequence (1.2). The sequence (1.7) is obtained from (1.3) in a similar way. □

Corollary 1.10. – *If C is a conical curve such that $X(C)$ is a tree, then $H^1(k, \mathrm{Pic}(\overline{C})) = 0$, so that the natural map* $\mathrm{Br}(k) \to \mathrm{Br}(C)$ *is surjective.*

Proof. – Since k has characteristic 0, we have $H^n(k, U) = 0$ for $n > 0$. Thus (1.7) gives an isomorphism $H^1(k, \mathrm{Pic}(\overline{C})) = H^1(k, \mathrm{Pic}(\overline{C'}))$. We have $T = \{1\}$, hence $\mathrm{Pic}(\overline{C'}) = \mathrm{Pic}(\widetilde{C} \times_k \bar{k})$. The curve $\widetilde{C}$ is the disjoint union of conics defined over finite extensions of k, thus the free abelian group $\mathrm{Pic}(\widetilde{C} \times_k \bar{k})$ has a natural Γ_k-stable $\mathbb{Z}$-basis. Hence $H^1(k, \mathrm{Pic}(\widetilde{C} \times_k \bar{k})) = 0$. □

2. Weak approximation

Let k be a number field. Recall that the étale Brauer-Manin set $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}}$ is the set of adelic points $(M_v) \in X(\mathbb{A}_k)$ satisfying the following property: for any torsor $f : Y \to X$ of a finite k-group scheme G there exists a k-torsor Z of G such that (M_v) is the image of an adelic point in the Brauer-Manin set of $(Y \times_k Z)/G$. Here G acts simultaneously on both factors, and the morphism $(Y \times_k Z)/G \to X$ is induced by $Y \to X$. It is clear that the étale Brauer-Manin obstruction is a functor from the category of varieties over k to the category of sets. Note that $(Y \times_k Z)/G \to X$ is a torsor of an inner form of G, called the twist of Y/X by Z, see [16, pp. 20–21] for details.

In this section we construct a simple example of a smooth projective surface X over k such that $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}}$ is infinite but $X(k)$ contains only one point. Thus $X(k)$ is far from dense in $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}}$; in fact, infinitely many points of $X(\mathbb{A}_k)^{\text{ét},\mathrm{Br}}$ have all their local components in the complement to $X(k)$ in X.

We start with the following statement which shows that on an everywhere locally soluble conical curve C such that $X(C)$ is a tree all the adelic points survive the étale Brauer-Manin obstruction.

Proposition 2.1. – *Let k be a number field, and let C be a conical curve over k such that $X(C)$ is a tree and $C(\mathbb{A}_k) \neq \varnothing$. Then*

1. $C(k) \neq \varnothing$*;*
2. *the natural map* $\mathrm{Br}(k) \to \mathrm{Br}(C)$ *is an isomorphism;*
3. $C(\mathbb{A}_k)^{\text{ét},\mathrm{Br}} = C(\mathbb{A}_k)$.

Remark 2.2. – Proposition 2.1 (1) implies that *geometrically connected and simply connected projective curves over number fields satisfy the Hasse principle*. It is easy to see that this statement does not generalize to higher dimension. Consider a conic $C \subset \mathbb{P}^2_k$ without a k-point, and choose a quadratic extension K/k so that all the places v with $C(k_v) = \varnothing$ are split in K. Then the union of two planes conjugate over K and intersecting at C is a geometrically connected and simply connected projective surface that is a counterexample to the Hasse principle.

Proof of Proposition 2.1. – Let us prove (1). It is well known that any group acting on a finite connected tree fixes a vertex or an edge. (This is easily proved by induction on the diameter of a tree, that is, on the length of a longest path contained in it.) We apply this to the action of the Galois group Γ_k on $X(C)$. If Γ_k fixes a point of $\overline{\Pi}$ or $\overline{\Psi}$, then $C(k) \neq \varnothing$. If Γ_k fixes a point of $\overline{\Lambda}$, then C has an irreducible component C_0 which is a geometrically integral geometrically rational curve. Let $\widetilde{C}_0$ be the normalization of C_0. Since $X(C)$ is a tree, the morphism $\widetilde{C}_0 \to C_0$ is a bijection on points. Thus if we can prove that $C_0(\mathbb{A}_k) \neq \varnothing$, then $\widetilde{C}_0(\mathbb{A}_k) \neq \varnothing$, and by the Hasse-Minkowski theorem $\widetilde{C}_0(k) \neq \varnothing$, so that finally $C_0(k) \neq \varnothing$. Since $X(C)$ is a connected tree, each connected component of $\overline{C} \setminus \overline{C}_0$ meets $\overline{C}_0$ in exactly one point. Let k_v be a completion of k. Suppose that $C_0(k_v) = \varnothing$. Since $C(k_v) \neq \varnothing$, at least one of the connected components of $\overline{C} \setminus \overline{C}_0$ is fixed by the Galois group Γ_{k_v}, and hence it intersects C_0 in a k_v-point. This contradiction proves (1).

By (1) the natural map $\mathrm{Br}(k) \to \mathrm{Br}(C)$ has a retraction, and so is injective, but it is also surjective by Corollary 1.10. This proves (2).

Let us prove (3). Let G be a finite k-group scheme, and let $\mathcal{T} \to C$ be a torsor of G. Fix a k-point P in C. The fibre $\mathcal{T}_P$ is then a k-torsor. The twist of $\mathcal{T}$ by $\mathcal{T}_P$ is the quotient of $\mathcal{T} \times_k \mathcal{T}_P$ by the diagonal action of G. This is a C-torsor of an inner form of G such that the fibre at P has a k-point, namely the quotient by G of the diagonal in $\mathcal{T}_P \times_k \mathcal{T}_P$. Thus, twisting $\mathcal{T}$ by a k-torsor of G, and replacing G by an inner form we can assume that $\mathcal{T}_P$ contains a k-point Q. Since all irreducible components of $\overline{C}$ are homeomorphic to $\mathbb{P}^1_{\bar{k}}$, the torsor $\overline{\mathcal{T}} \to \overline{C}$ trivializes over each component of $\overline{C}$. But $X(C)$ is a tree, and this implies that the torsor $\overline{\mathcal{T}} \to \overline{C}$ is trivial, that is, $\overline{\mathcal{T}} \simeq (C \times_k G) \times_k \bar{k}$. The connected component of $\overline{\mathcal{T}}$ that contains Q is thus defined over k, and hence it gives a section s of $\mathcal{T} \to C$ such that $s(P) = Q$. Hence any adelic point on C lifts to an adelic point on $s(C) \subset \mathcal{T}$. Since $\mathrm{Br}(C) = \mathrm{Br}(k)$ we conclude that $C(\mathbb{A}_k)$ is contained in, and hence is equal to the étale Brauer-Manin set $C(\mathbb{A}_k)^{\text{ét,Br}}$. □

Let k be a number field, and let $f(x, y)$ be a separable homogeneous polynomial such that its zero locus $Z^f \subseteq \mathbb{P}^1_k$ is a 0-dimensional scheme violating the Hasse principle. It is easy to see that such a polynomial exists for any number field k. For example, for $k = \mathbb{Q}$ one can take

$$f(x, y) = (x^2 - 2y^2)(x^2 - 17y^2)(x^2 - 34y^2). \tag{2.1}$$

For an arbitrary number field k the following polynomial will do:

$$f(x, y) = (x^2 - ay^2)(x^2 - by^2)(x^2 - aby^2)(x^2 - cy^2),$$

where $a, b, c \in k^* \setminus k^{*2}$ are such that $ab \notin k^{*2}$, whereas $c \in k_v^{*2}$ for all places v such that $\mathrm{val}_v(a) \neq 0$ or $\mathrm{val}_v(b) \neq 0$, and also for the Archimedean places, and the places with residual characteristic 2. (For fixed a and b one finds c using weak approximation.) Let $d = \deg(f)$.

Let $C^f \subseteq \mathbb{P}^2_k$ be the curve with the equation $f(x, y) = 0$. It is geometrically connected and has a unique singular point $P = (0 : 0 : 1) \in C^f \subset \mathbb{P}^2_k$ which is contained in all irreducible components of C^f. Since $Z^f(k) = \varnothing$ we see that P is the only k-point of C^f. The intersection of C^f with any line in $\mathbb{P}^2_k$ that does not pass through P is isomorphic to Z^f, hence *the smooth locus of C^f contains an infinite subset of $C^f(\mathbb{A}_k) = C^f(\mathbb{A}_k)^{\text{ét,Br}}$*, where the equality is due to Proposition 2.1 (3).

Now let $g(x, y, z)$ be a homogeneous polynomial over k of the same degree as $f(x, y)$ such that the subset of $\mathbb{P}^2_k$ given by $g(x, y, z) = f(x, y) = 0$ consists of d^2 distinct $\bar{k}$-points. Consider the projective surface $Y \subseteq \mathbb{P}^2_k \times \mathbb{P}^1_k$ given by the equation

$$\lambda f(x, y) + \mu g(x, y, z) = 0,$$

where $(\lambda : \mu)$ are homogeneous coordinates on $\mathbb{P}^1_k$. One immediately checks that Y is smooth. The fibre of the projection $Y \to \mathbb{P}^1_k$ over $\infty = (1 : 0)$ is C^f.

Let E be a smooth, geometrically integral, projective curve over k containing exactly one k-rational point M. By [12] such curves exist over any global field k. Choose a dominant morphism $\varphi : E \to \mathbb{P}^1_k$ such that $\varphi(M) = \infty$, and φ is not ramified over all the points of $\mathbb{P}^1_k$ where Y has a singular fibre (including ∞). Define

$$X = E \times_{\mathbb{P}^1_k} Y. \tag{2.2}$$

PROPOSITION 2.3. – *The surface X is smooth, projective and geometrically integral. The set $X(k)$ has exactly one point, whereas the set $X(\mathbb{A}_k)^{\text{ét,Br}}$ is infinite. Furthermore, infinitely many elements of $X(\mathbb{A}_k)^{\text{ét,Br}}$ have all their local components in the Zariski open set $X \setminus X(k)$.*

Proof. – The inverse image of the unique k-point of E in X is C^f, hence X has exactly one k-point. Since the étale Brauer-Manin obstruction is a functor from the category of k-varieties to the category of sets, the inclusion $\iota : C^f \hookrightarrow X$ induces an inclusion

$$\iota_* : C^f(\mathbb{A}_k) = C^f(\mathbb{A}_k)^{\text{ét,Br}} \hookrightarrow X(\mathbb{A}_k)^{\text{ét,Br}}.$$

We conclude that $X(\mathbb{A}_k)^{\text{ét,Br}}$ contains infinitely many points all of whose local components belong to the Zariski open set $X \setminus X(k)$. □

REMARK 2.4. – Following a suggestion of Ambrus Pál made in response to the first version of this paper we now sketch a more elementary construction of a counterexample to weak approximation not explained by the étale Brauer-Manin obstruction. Consider any irreducible binary quadratic form $f(x, y)$ over k. Then our method produces a smooth, projective and geometrically integral surface X fibred into conics over E. The fibre over the unique k-point of E is the irreducible singular conic C^f, hence the set $X(k)$ consists of the singular point of C^f. Let D be the discriminant of $f(x, y)$. If v is a finite place of k that splits in $k(\sqrt{D})$, then $C^f(k_v)$ is the union of two projective lines over k_v. Thus $C^f(\mathbb{A}_k) = C^f(\mathbb{A}_k)^{\text{ét,Br}}$ is infinite, and hence so is $X(\mathbb{A}_k)^{\text{ét,Br}}$. This construction gives a surface of simpler geometric structure than the surface in Proposition 2.3, but it does not possess the stronger property of the previous example: here no element of $X(\mathbb{A}_k)^{\text{ét,Br}}$ is contained in the Zariski open set $X \setminus X(k)$. For this argument we assume that the Jacobian of E has rank 0 and a finite Shafarevich-Tate group, e.g., E is an elliptic curve over $\mathbb{Q}$ of analytic rank 0. Let (M_v) be an adelic point in $X(\mathbb{A}_k)^{\text{ét,Br}}$. Its image (N_v) in E is contained in $E(\mathbb{A}_k)^{\text{Br}}$, but this set is just the connected component of $0 = E(k)$ in $E(\mathbb{A}_k)$, see [16, Prop. 6.2.4]. Thus for all finite places v we have $N_v = 0$. For any place v that does not split in $k(\sqrt{D})$, this implies $M_v \in X(k)$.

3. The Hasse principle

In this section we construct a smooth projective surface X over an arbitrary number field k such that $X(\mathbb{A}_k)^{\text{ét,Br}}$ is infinite and $X(k)$ is empty. This means that X does not satisfy the Hasse principle and this failure is not explained by the étale Brauer-Manin obstruction.

Let $f(x, y)$ and Z^f be as in the previous section. The scheme Z^f is the disjoint union of $\operatorname{Spec}(K_i)$ for $i = 1, \ldots, n$, where K_i is a finite extension of k. We assume $d = \deg(f) \geq 5$. (For $k = \mathbb{Q}$ one can take $f(x, y)$ of degree 6 as in (2.1), and in general degree 8 will suffice.) We choose field extensions L/k and F/k such that $L \otimes_k K_i$ and $F \otimes_k K_i$ are fields for all $i = 1, \ldots, n$, and, moreover, $[L : k] = d/2 - 1$, $[F : k] = d/2$ if d is even, and $[L : k] = (d-1)/2$, $[F : k] = (d+1)/2$ if d is odd. Fix an embedding

$$\operatorname{Spec}(L) \coprod \operatorname{Spec}(F) \hookrightarrow \mathbb{P}^1_k$$

and let D be the following curve in $\mathbb{P}^1_k \times \mathbb{P}^1_k$:

$$D = \left(Z^f \times \mathbb{P}^1_k\right) \cup \left(\mathbb{P}^1_k \times \operatorname{Spec}(L)\right) \cup \left(\mathbb{P}^1_k \times \operatorname{Spec}(F)\right).$$

This is a bipartite conical curve without k-points, see Definition 1.4. The class of D in $\operatorname{Pic}(\mathbb{P}^1_k \times \mathbb{P}^1_k)$ is $(d, d-1)$ or (d, d) depending on the parity of d.

PROPOSITION 3.1. – *The natural map* $\operatorname{Br}(k) \to \operatorname{Br}(D)$ *is an isomorphism.*

Proof. – Since $D(L) \neq \emptyset$, the natural map $\operatorname{Br}(L) \to \operatorname{Br}(D \times_k L)$ has a retraction, and so is injective. The composition of the restriction $\operatorname{Br}(k) \to \operatorname{Br}(L)$ and the corestriction $\operatorname{Br}(L) \to \operatorname{Br}(k)$ is the multiplication by $[L : k]$, hence for any x in the kernel of the natural map $\operatorname{Br}(k) \to \operatorname{Br}(D)$ we have $[L : k]x = 0$. Similarly, $D(F) \neq \emptyset$ implies that $[F : k]x = 0$.

But $[F:k]=[L:k]+1$, therefore the natural map $\mathrm{Br}(k)\to\mathrm{Br}(D)$ is injective. By Corollary 1.7 we need to prove that $\mathrm{Br}(k)$ is the kernel of the map

$$\mathrm{Br}(L)\oplus\mathrm{Br}(F)\oplus\bigoplus_{i=1}^{n}\mathrm{Br}(K_i)\longrightarrow\bigoplus_{i=1}^{n}\mathrm{Br}(LK_i)\oplus\bigoplus_{i=1}^{n}\mathrm{Br}(FK_i).$$

Recall that the maps $\mathrm{Br}(L)\to\mathrm{Br}(LK_i)$ are the restriction maps, whereas $\mathrm{Br}(K_i)\to\mathrm{Br}(LK_i)$ are opposites of the restriction maps. The same convention applies with F in place of L.

Suppose that we have $\alpha_i\in\mathrm{Br}(K_i)$, $i=1,\dots,n$, $\beta\in\mathrm{Br}(L)$ and $\gamma\in\mathrm{Br}(F)$ such that (α_i,β,γ) goes to zero. Let v be a place of k, and let k_v be a completion of k at v. Since $Z^f(k_v)\neq\varnothing$, there is an index i such that v splits in K_i. Let w be a degree 1 place of K_i over v, so that the natural map $k_v\to K_{i,w}$ is an isomorphism. Let $a_v\in\mathrm{Br}(k_v)=\mathrm{Br}(K_{i,w})$ be the image of α_i under the restriction map $\mathrm{Br}(K_i)\to\mathrm{Br}(K_{i,w})$. This defines $a_v\in\mathrm{Br}(k_v)$ for any place v of k, moreover, we have $a_v=0$ for almost all v since each α_i is unramified away from a finite set of places (and the Brauer group of the ring of integers of k_ν is trivial). We have a commutative diagram of restriction maps

$$\begin{array}{ccccc}\mathrm{Br}(L) & \longrightarrow & \mathrm{Br}(L\otimes_k K_i) & \longleftarrow & \mathrm{Br}(K_i)\\ \downarrow & & \downarrow & & \downarrow\\ \mathrm{Br}(L\otimes_k k_v) & \xrightarrow{\sim} & \mathrm{Br}(L\otimes_k K_{i,w}) & \longleftarrow & \mathrm{Br}(K_{i,w}).\end{array}$$

Here for a family of fields $\{F_i\}$ we write $\mathrm{Br}(\oplus F_i)=\oplus\mathrm{Br}(F_i)$. Since (α_i,β,γ) goes to zero, the restrictions of α_i and β to $\mathrm{Br}(L\otimes_k K_i)$ coincide. Hence the image β_v of β in $\mathrm{Br}(L\otimes_k k_v)$ is the image of $a_v\in\mathrm{Br}(k_v)$. From the global reciprocity law applied to $\beta\in\mathrm{Br}(L)$ we deduce $[L:k]\sum_v\mathrm{inv}_v(a_v)=0$. The same argument with γ instead of β gives $[F:k]\sum_v\mathrm{inv}_v(a_v)=0$. Since $[L:k]$ and $[F:k]$ are coprime we obtain $\sum_v\mathrm{inv}_v(a_v)=0$. By global class field theory there exists $\alpha\in\mathrm{Br}(k)$ such that a_v is the image of α in $\mathrm{Br}(k_v)$. Since the map $\mathrm{Br}(L)\to\oplus_v\mathrm{Br}(L\otimes_k k_v)$ is injective it follows that α goes to β under the restriction map $\mathrm{Br}(k)\to\mathrm{Br}(L)$, and similarly for γ. Modifying α_i, β and γ by the image of α we can now assume that $\beta=0$ and $\gamma=0$. Since α_i goes to zero in $\mathrm{Br}(LK_i)$, a standard restriction-corestriction argument gives $[L:k]\alpha=0$. Similarly, α_i goes to zero in $\mathrm{Br}(FK_i)$, and hence $[F:k]\alpha=0$. Therefore, $\alpha=0$. □

Corollary 3.2. – *We have* $D(\mathbb{A}_k)^{\mathrm{Br}}=D(\mathbb{A}_k)$.

We now construct a conical curve $C\subset\mathbb{P}^1_k\times\mathbb{P}^1_k$ with one singular point such that C and D are linearly equivalent. Let $P=(P_1,P_2)$ be a k-point in $\mathbb{P}^1_k\times\mathbb{P}^1_k$. In the tangent plane to $\mathbb{P}^1_k\times\mathbb{P}^1_k$ at P choose a line ℓ through P such that ℓ is not one of the two tangent directions. Assume first that d is odd, so that $\mathcal{O}(D)=\mathcal{O}(d,d)$. For $i=1,\dots,d$ let $C_i\subset\mathbb{P}^1_k\times\mathbb{P}^1_k$ be pairwise different geometrically irreducible curves through P tangent to ℓ such that $\mathcal{O}(C_i)=\mathcal{O}(1,1)$. (If one embeds $\mathbb{P}^1_k\times\mathbb{P}^1_k$ as a quadric $Q\subset\mathbb{P}^3_k$, then C_i are smooth conic sections of Q by pairwise different hyperplanes passing through ℓ.) Define the curve C as the union of the conics C_i, for $i=1,\dots,d$. Since $(C_i^2)=2$ and all the curves C_i are tangent to each other, we have $C_i\cap C_j=P$ if $i\neq j$. Thus P is the unique singular point of C. If d is even, we define C as the union of C_i, for $i=1,\dots,d-1$, and $L=P_1\times\mathbb{P}^1_k$. We have $L\cap C_i=P$ for $i=1,\dots,d-1$, so P is a unique singular point of C. Therefore, for any d

the curve C is conical and $X(C)$ is a tree, and C and D have the same class in $\mathrm{Pic}(\mathbb{P}^1_k \times \mathbb{P}^1_k)$, and so are linearly equivalent.

By choosing P outside D we can arrange that C does not meet D_{sing}. Then each $\bar{k}$-point s of $C \cap D$ belongs to exactly one geometric irreducible component of each C and D, and these components meet transversally at s.

Let $r(x, y; u, v)$ and $s(x, y; u, v)$ be the bi-homogeneous polynomials of bi-degree (d, d) if d is odd, and $(d, d-1)$ if d is even, such that their zero sets are the curves D and C, respectively. Let $Y \subset (\mathbb{P}^1_k)^3$ be the surface given by

$$\lambda r(x, y; u, v) + \mu s(x, y; u, v) = 0,$$

where $(\lambda : \mu)$ are homogeneous coordinates on the third copy of $\mathbb{P}^1_k$. It is easy to check that Y is smooth. The projection to the third factor $(\mathbb{P}^1_k)^3 \to \mathbb{P}^1_k$ defines a surjective morphism $Y \to \mathbb{P}^1_k$ with fibres $Y_0 = C$ and $Y_\infty = D$. The generic fibre of $Y \to \mathbb{P}^1_k$ is geometrically integral.

As in the previous section, we pick a smooth, geometrically integral, projective curve E with a unique k-point M, and a dominant morphism $\varphi : E \to \mathbb{P}^1_k$ such that $\varphi(M) = \infty$, and φ is not ramified over all points of $\mathbb{P}^1_k$ where Y has a singular fibre (including 0 and ∞). We define X by (2.2); this is a smooth, geometrically integral and projective surface. Let $p : X \to E$ be the natural projection. Since $E(k) = \{M\}$ and $D = p^{-1}(M)$ has no k-points, we see that $X(k) = \varnothing$.

To study the étale Brauer-Manin set of X we need to understand X-torsors of an arbitrary finite k-group scheme G. In the following general proposition the word 'torsor' means 'torsor with structure group G'.

PROPOSITION 3.3. – *Let k be a field of characteristic zero, and let X and B be varieties over k. Let $p : X \to B$ be a proper morphism with geometrically connected fibres. Assume that p has a simply connected geometric fibre. Then for any torsor $f : X' \to X$ there exists a torsor $B' \to B$ such that torsors $X' \to X$ and $X \times_B B' \to X$ are isomorphic.*

Proof. – Let $\delta = |G(\bar{k})|$. Let $X' \xrightarrow{g} B' \xrightarrow{h} B$ be the Stein factorization of the composed morphism $X' \xrightarrow{f} X \xrightarrow{p} B$ (EGA III.4.3.1). Thus we have a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{h} & B \end{array}$$

where g is proper with geometrically connected fibres, and h is a finite morphism. (The variety B' can be defined as the relative spectrum of $(pf)_*\mathcal{O}_{X'}$.) The Stein factorization is a functor from the category of proper schemes over B to the category of finite schemes over B. Thus we obtain an induced action of G on B' such that g is G-equivariant.

Let δ' be the degree of h. We claim that $\delta' \geq \delta$. Indeed, let $\bar{x}$ be a geometric point of B such that $p^{-1}(\bar{x})$ is simply connected. Thus $f^{-1}(p^{-1}(\bar{x}))$ is isomorphic to a disjoint union of δ copies of $p^{-1}(\bar{x})$. Hence $h^{-1}(\bar{x})$ has cardinality δ, and so $\delta' \geq \delta$.

The projection $\pi_X : X \times_B B' \to X$ is a finite morphism of degree δ'. The composition of the natural map $X' \to X \times_B B'$ with π_X is $f : X' \to X$, and this is a finite étale morphism of degree δ. Since $\delta' \geq \delta$ it follows that $X' \to X \times_B B'$ is finite and étale of degree 1, and hence is an isomorphism. We also see that $\delta' = \delta$ and that π_X is an étale morphism of degree δ. It follows that h is also étale of degree δ. Now the action of G equips B' with the structure of a B-torsor. □

Finally, we can prove the main result of this section.

THEOREM 3.4. – *The set $X(k)$ is empty, whereas the set $X(\mathbb{A}_k)^{\text{ét},\text{Br}}$ contains $D(\mathbb{A}_k)$ and so is infinite.*

Proof. – Since $\overline{C}$ is a connected and simply connected variety over $\bar{k}$ which is a geometric fibre of $X \to E$, we can use Proposition 3.3. Thus any X-torsor of a finite k-group scheme has the form $\mathcal{T}_X = \mathcal{T} \times_E X$ for some torsor $\mathcal{T} \to E$. After twisting we can assume that the fibre of $\mathcal{T}$ over the unique k-point of E has a k-point. Thus the restriction of the torsor $\mathcal{T}_X \to X$ to the curve $D \subset X$ has a section σ. Therefore every adelic point on $D \subset X$ is the image of an adelic point on $\sigma(D) \subset \mathcal{T}_X$. By Corollary 3.2 and the functoriality of the Brauer-Manin set we have

$$\sigma(D)(\mathbb{A}_k) = \sigma(D)(\mathbb{A}_k)^{\text{Br}} \subset \mathcal{T}_X(\mathbb{A}_k)^{\text{Br}}.$$

Thus, by the definition of the étale Brauer-Manin set, $X(\mathbb{A}_k)^{\text{ét},\text{Br}}$ contains the infinite set $D(\mathbb{A}_k)$, and hence is infinite. □

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