ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Steve HOFMANN & José María MARTELL

Uniform rectifiability and harmonic measure I: Uniform rectifiability implies Poisson kernels in L^p

UNIFORM RECTIFIABILITY AND HARMONIC MEASURE I: UNIFORM RECTIFIABILITY IMPLIES POISSON KERNELS IN L^p

BY Steve HOFMANN AND José María MARTELL

ABSTRACT. – We present a higher dimensional, scale-invariant version of a classical theorem of F. and M. Riesz [37]. More precisely, we establish scale invariant absolute continuity of harmonic measure with respect to surface measure, along with higher integrability of the Poisson kernel, for a domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, with a uniformly rectifiable boundary, which satisfies the Harnack chain condition plus an interior (but not exterior) Corkscrew condition. In a companion paper to this one [28], we also establish a converse, in which we deduce uniform rectifiability of the boundary, assuming scale invariant L^q bounds, with $q > 1$, on the Poisson kernel.

RÉSUMÉ. – On présente une version invariante par échelles et en dimension supérieure à 3, d'un théorème classique de F. et M. Riesz [37]. Plus précisément, on établit l'absolue continuité de la mesure harmonique par rapport à la mesure de surface, ainsi qu'un gain d'intégrabilité pour le noyau de Poisson, pour un domaine $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, à bord uniformément rectifiable, vérifiant une condition de chaîne de Harnack et une condition de type « points d'ancrage » ou « Corkscrew » intérieure (mais pas extérieure). L'article associé [28] établit une réciproque, c'est-à-dire l'uniforme rectifiabilité du bord en supposant des estimées invariantes par échelle L^q pour $q > 1$ sur le noyau de Poisson.

1. Introduction

In [37], F. and M. Riesz showed that for a simply connected domain in the complex plane with a rectifiable boundary, harmonic measure is absolutely continuous with respect to arc length measure. A quantitative version of this theorem was obtained by Lavrentiev [35]. More generally, if only a portion of the boundary is rectifiable, Bishop and Jones [8] have shown that harmonic measure is absolutely continuous with respect to arclength on that portion.

They also present a counter-example to show that the result of [37] may fail in the absence of some topological hypothesis (e.g., simple connectedness).

In this paper we extend the results of [37] and [35] to higher dimensions, without imposing extra assumptions on either the exterior domain or the boundary, as has been done previously. Our extension (Theorem 1.26 below) is "scale-invariant", i.e., assuming scale-invariant analogues of the hypotheses of [37], we show that harmonic measure satisfies a scale-invariant version of absolute continuity, namely the weak-A_∞ condition (cf. Definition 1.19 below). More precisely, let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a connected, open set. We establish the weak-A_∞ property of harmonic measure, assuming that $\partial\Omega$ is uniformly rectifiable (cf. (1.13) below), and that Ω satisfies interior (but not necessarily exterior) Corkscrew and Harnack chain conditions (cf. Definitions 1.4 and 1.6 below). Uniform rectifiability is the scale-invariant version of rectifiability, while the Corkscrew and Harnack chain conditions are scale invariant analogues of the topological properties of openness and path connectedness, respectively. We emphasize that in contrast to previous work in this area in dimensions $n + 1 \geq 3$, we impose no restriction on the geometry of the *exterior* domain $\Omega_{\text{ext}} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$, nor any extra condition on the geometry of the boundary, beyond uniform rectifiability. In particular, we do not require that any component of Ω_{ext} satisfy a Corkscrew condition (as in [29], [39, 6]) or even an n-disk condition as in [17]; nor do we assume that $\partial\Omega$ contains "Big Pieces" of the boundaries of Lipschitz sub-domains of Ω, as in [7]. The absence of such assumptions is the main advance in the present paper.

In addition, in a companion paper to this one [28], written jointly with I. Uriarte-Tuero, we establish a converse, Theorem 1.28, in which we deduce uniform rectifiability of the boundary, given a certain scale invariant local L^q estimate, with $q > 1$, for the Poisson kernel (cf. (1.24)). The method of proof in [28] may be of independent interest, as it entails a novel use of "Tb" theory to obtain a free boundary result.

Taken together, the main results of the present paper and of [28], namely Theorems 1.26 and 1.28 below, may be summarized as follows (the terminology and notation used in the statement will be clarified or cross-referenced immediately afterwards):

THEOREM 1.1. – *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a connected open set which satisfies interior Corkscrew and Harnack chain conditions, and whose boundary $\partial\Omega$ is n-dimensional Ahlfors-David regular. Then the following are equivalent:*

1. *$\partial\Omega$ is uniformly rectifiable.*

2. *For every surface ball $\Delta = \Delta(x, r) \subset \partial\Omega$, with radius $r \lesssim \operatorname{diam} \partial\Omega$, the harmonic measure $\omega^{X_\Delta} \in$ weak-$A_\infty(\Delta)$.*

3. *$\omega \ll \sigma$, and there is a $q > 1$ such that the Poisson kernel k^{X_Δ} satisfies the scale invariant L^q bound (1.24), for every $\Delta = \Delta(x, r) \subset \partial\Omega$, with radius $r \lesssim \operatorname{diam} \partial\Omega$.*

REMARK 1.2. – By the counter-example of [8], one would not expect to obtain the implication (1) $\Longrightarrow$ (2), without some sort of connectivity assumption; for us, the interior Harnack chain condition plays this role.

Given a domain $\Omega \subset \mathbb{R}^{n+1}$, a "surface ball" is a set $\Delta = \Delta(x, r) := B(x, r) \cap \partial\Omega$, where $x \in \partial\Omega$, and $B(x, r)$ denotes the standard $(n + 1)$-dimensional Euclidean ball of radius r centered at x. For such a surface ball Δ, we let ω^{X_Δ} denote harmonic measure for Ω, with pole at the

"Corkscrew point" X_Δ (see Definition 1.4). The Corkscrew and Harnack chain conditions, as well as the notions of Ahlfors-David regularity (ADR), uniform rectifiability (UR) and weak-A_∞, are described in Definitions 1.4, 1.6, 1.7, 1.9, and 1.19 below.

The present paper treats the direction (1) implies (2). That (2) implies (3) is well known (see the discussion following Definition 1.19). The main result in [28] is that (3) implies (1). We mention also that we obtain in the present paper an extension of (1) implies (2), in which our hypotheses are assumed to hold only in an "interior big pieces" sense (cf. Definition 1.14 and Theorem 1.27 below).

To place Theorem 1.1 in context, we review previous related work in dimension $n + 1 \geq 3$. We recall that in [29], the authors introduce the notion of a "non-tangentially accessible" (NTA) domain: Ω is said to be NTA if it satisfies the Corkscrew and Harnack chain conditions ("interior Corkscrew and Harnack chain conditions"), and also if the exterior domain, $\Omega_{\text{ext}} := \mathbb{R}^{n+1} \setminus \overline{\Omega}$ (which need not be connected), satisfies the Corkscrew condition ("exterior Corkscrew condition"). The latter was relaxed in [17] to allow a sort of "weak exterior Corkscrew" condition in which the analogue of the exterior Corkscrew point is the center merely of an n-dimensional disk in Ω_{ext}, rather than of a full Euclidean ball. A key observation made in [17] was that the weak exterior Corkscrew condition is still enough to obtain local Hölder continuity at the boundary of harmonic functions which vanish on a surface ball. In [17], the authors prove that, in the presence of Ahlfors-David regularity of the boundary, the NTA condition of [29] or even its relaxed version with "weak exterior Corkscrews", implies that Ω satisfies an "interior big pieces" of Lipschitz sub-domains condition (cf. Definition 1.14 below). By a simple maximum principle argument (plus the deep result of [15]), one then almost immediately obtains a certain lower bound for harmonic measure, to wit that there are constants $\eta \in (0, 1)$ and $c_0 > 0$ such that for each surface ball $\Delta \subset \partial\Omega$, and any Borel subset $A \subset \Delta$, we have

$$(1.3) \qquad \omega^{X_\Delta}(A) \geq c_0, \qquad \text{whenever } \sigma(A) \geq \eta\,\sigma(\Delta).$$

In turn, still given NTA, or at least the relaxed version of [17], the latter bound self-improves to an A_∞ estimate for harmonic measure, via the comparison principle. The same A_∞ conclusion was also obtained by a different argument in [39], under the full NTA condition of [29]. In [7], the authors impose an interior Corkscrew condition, but in lieu of the Harnack chain and exterior (or weak exterior) Corkscrew conditions, the authors assume instead the consequence of these conditions deduced in [17], namely, that Ω satisfies the aforementioned condition concerning "interior big pieces" of Lipschitz sub-domains. The bound (1.3) (suitably interpreted) then holds almost immediately (again by the maximum principle), but the self-improvement argument, in the absence of the Harnack chain and exterior (or weak exterior) Corkscrew conditions, is now more problematic (indeed, the usual proofs of the comparison principle rely on Harnack's inequality and local Hölder continuity at the boundary), and the authors conclude in [7] only that ω is weak-A_∞. On the other hand, they give an example to show that this conclusion is best possible (that is, they construct a domain which satisfies the "interior big pieces" condition, but whose harmonic measure fails to be doubling). We mention also in this context the recent paper [6], in which the geometric conclusion of [17], namely the existence of "interior big pieces" of Lipschitz sub-domains, is shown to hold assuming the full NTA condition (with two-sided Corkscrews), but in

which only the lower (but not the upper) bound is required in the Ahlfors-David condition (cf. (1.8)).

In the present paper, we improve the results of [7] and of [17] by removing the "big pieces of Lipschitz sub-domains" hypothesis, as well as all assumptions regarding the exterior domain. That is, in Theorem 1.26, we assume only that Ω satisfies *interior* Corkscrew and Harnack chain conditions, and that its boundary is uniformly rectifiable. More generally, in Theorem 1.27, we suppose only that these hypotheses hold in an appropriate "interior big pieces" sense (in particular, our results include those of [7] as a special case, since their Lipschitz sub-domains clearly satisfy our hypotheses). The difficulty now, and the heart of the proof, is to establish (1.3); with the latter in hand, the self-improvement to weak A_∞ proceeds as in [7]. We mention that by an unpublished example of Hrycak, UR does not, in general, imply big pieces of Lipschitz graphs[1] (that the opposite implication does hold for ADR sets is easy, and well known). Moreover, in [28] we obtain a converse which shows that the UR hypothesis is optimal. In this connection, we mention also the following observation, which was brought to our attention by M. Badger and T. Toro. Let $F \subset \mathbb{R}^2$ denote the "4 corners Cantor set" of J. Garnett (see, e.g., [19, p. 4]), and let $F^* := F \times \mathbb{R} \subset \mathbb{R}^3$ be the "cylinder" above F. Then $\Omega := \mathbb{R}^3 \setminus F^*$ satisfies the (interior) Corkscrew and Harnack chain conditions, and has a 2-dimensional ADR boundary, but the boundary is not UR, and therefore its harmonic measure is not weak-A_∞.

We conclude this historical survey by providing some additional context for our work here and in [28], namely, that our results may be viewed as a "large constant" analogue of the work of Kenig and Toro [32, 33, 34]. The latter, taken collectively, say that in the presence of a Reifenberg flatness condition and Ahlfors-David regularity, one has that $\log k \in$ VMO if and only if $\nu \in$ VMO, where k is the Poisson kernel with pole at some fixed point, and ν is the unit normal to the boundary. Moreover, given the same background hypotheses, the condition that $\nu \in$ VMO is equivalent to a uniform rectifiability (UR) condition with vanishing trace, thus $\log k \in$ VMO $\iff$ vanishing UR. On the other hand, our large constant version "almost" says "$\log k \in$ BMO $\iff UR$", given interior Corkscrews and Harnack chains. Indeed, it is well known that the A_∞ condition (i.e., weak-A_∞ plus the doubling property) implies that $\log k \in$ BMO, while if $\log k \in$ BMO with small norm, then $k \in A_\infty$.

In order to state our results precisely, we shall first need to discuss some preliminary matters.

1.1. Notation and definitions

– We use the letters c, C to denote harmless positive constants, not necessarily the same at each occurrence, which depend only on dimension and the constants appearing in the hypotheses of the theorems (which we refer to as the "allowable parameters"). We shall also

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[1]   On the other hand, Azzam and Schul [5] have recently shown that every UR set contains "big pieces of big pieces of Lipschitz graphs" (see [19, pp. 15-16] or [5] for a precise formulation). This is a beautiful result, but seems inapplicable to the estimates for harmonic measure considered here: to enable essential use of the maximum principle, one would need "*interior* big pieces (cf. Definition 1.14 below) of *interior* big pieces of Lipschitz subdomains" (say, in the presence of the 1-sided NTA condition), and it is not clear that the methods of [5] would yield such a result . We do expect that the methods of the present paper could be pushed to do so, and we plan to present these arguments, with applications to more general elliptic-harmonic measures, in a forthcoming paper.

sometimes write $a \lesssim b$ and $a \approx b$ to mean, respectively, that $a \le Cb$ and $0 < c \le a/b \le C$, where the constants c and C are as above, unless explicitly noted to the contrary. At times, we shall designate by M a particular constant whose value will remain unchanged throughout the proof of a given lemma or proposition, but which may have a different value during the proof of a different lemma or proposition.

– Given a domain $\Omega \subset \mathbb{R}^{n+1}$, we shall use lower case letters x, y, z, etc., to denote points on $\partial\Omega$, and capital letters X, Y, Z, etc., to denote generic points in $\mathbb{R}^{n+1}$ (especially those in $\mathbb{R}^{n+1} \setminus \partial\Omega$).

– The open $(n + 1)$-dimensional Euclidean ball of radius r will be denoted $B(x, r)$ when the center x lies on $\partial\Omega$, or $B(X, r)$ when the center $X \in \mathbb{R}^{n+1} \setminus \partial\Omega$. A "surface ball" is denoted $\Delta(x, r) := B(x, r) \cap \partial\Omega$.

– Given a Euclidean ball B or surface ball Δ, its radius will be denoted r_B or r_Δ, respectively.

– Given a Euclidean or surface ball $B = B(X, r)$ or $\Delta = \Delta(x, r)$, its concentric dilate by a factor of $\kappa > 0$ will be denoted by $\kappa B := B(X, \kappa r)$ or $\kappa \Delta := \Delta(x, \kappa r)$.

– For $X \in \mathbb{R}^{n+1}$, we set $\delta(X) := \mathrm{dist}(X, \partial\Omega)$.

– We let H^n denote n-dimensional Hausdorff measure, and let $\sigma := H^n\big|_{\partial\Omega}$ denote the "surface measure" on $\partial\Omega$.

– For a Borel set $A \subset \mathbb{R}^{n+1}$, we let 1_A denote the usual indicator function of A, i.e., $1_A(x) = 1$ if $x \in A$, and $1_A(x) = 0$ if $x \notin A$.

– For a Borel set $A \subset \mathbb{R}^{n+1}$, we let $\mathrm{int}(A)$ denote the interior of A. If $A \subset \partial\Omega$, then $\mathrm{int}(A)$ will denote the relative interior, i.e., the largest relatively open set in $\partial\Omega$ contained in A. Thus, for $A \subset \partial\Omega$, the boundary is then well defined by $\partial A := \overline{A} \setminus \mathrm{int}(A)$.

– For a Borel set A, we denote by $\mathscr{C}(A)$ the space of continuous functions on A, by $\mathscr{C}_c(A)$ the subspace of $\mathscr{C}(A)$ with compact support in A, and by $\mathscr{C}_b(A)$ the space of bounded continuous functions on A. If A is unbounded, we denote by $\mathscr{C}_0(A)$ the space of continuous functions on A converging to 0 at infinity.

– For a Borel subset $A \subset \partial\Omega$, we set $\fint_A f d\sigma := \sigma(A)^{-1} \int_A f d\sigma$.

– We shall use the letter I (and sometimes J) to denote a closed $(n + 1)$-dimensional Euclidean cube with sides parallel to the co-ordinate axes, and we let $\ell(I)$ denote the side length of I. We use Q to denote a dyadic "cube" on $\partial\Omega$. The latter exists, given that $\partial\Omega$ is ADR (cf. [18, 13]), and enjoy certain properties which we enumerate in Lemma **??** below.

DEFINITION 1.4 (Corkscrew condition). – Following [29], we say that a domain $\Omega \subset \mathbb{R}^{n+1}$ satisfies the "Corkscrew condition" if for some uniform constant $c > 0$ and for every surface ball $\Delta := \Delta(x, r)$, with $x \in \partial\Omega$ and $0 < r < \mathrm{diam}(\partial\Omega)$, there is a ball $B(X_\Delta, cr) \subset B(x, r) \cap \Omega$. The point $X_\Delta \subset \Omega$ is called a "Corkscrew point" relative to Δ. We note that we may allow $r < C\,\mathrm{diam}(\partial\Omega)$ for any fixed C, simply by adjusting the constant c.

REMARK 1.5. – We note that, on the other hand, every $X \in \Omega$, with $\delta(X) < \mathrm{diam}(\partial\Omega)$, may be viewed as a Corkscrew point, relative to some surface ball $\Delta \subset \partial\Omega$. Indeed, set $r = K\delta(X)$, with $K > 1$, fix $x \in \partial\Omega$ such that $|X - x| = \delta(X)$, and let $\Delta := \Delta(x, r)$.

DEFINITION 1.6 (Harnack chain condition). – Again following [29], we say that Ω satisfies the Harnack chain condition if there is a uniform constant C such that for every $\rho > 0$, $\Lambda \geq 1$, and every pair of points $X, X' \in \Omega$ with $\delta(X), \delta(X') \geq \rho$ and $|X - X'| < \Lambda\,\rho$, there is a chain of open balls $B_1, \ldots, B_N \subset \Omega$, $N \leq C(\Lambda)$, with $X \in B_1$, $X' \in B_N$, $B_k \cap B_{k+1} \neq \varnothing$ and $C^{-1}\,\mathrm{diam}(B_k) \leq \mathrm{dist}(B_k, \partial\Omega) \leq C\,\mathrm{diam}(B_k)$. The chain of balls is called a "Harnack chain".

We remark that the Corkscrew condition is a quantitative, scale invariant version of the fact that Ω is open, and the Harnack chain condition is a scale invariant version of path connectedness.

DEFINITION 1.7 (Ahlfors-David regular). – We say that a closed set $E \subset \mathbb{R}^{n+1}$ is n-dimensional ADR (or simply ADR) ("Ahlfors-David regular") if there is some uniform constant C such that

$$(1.8) \qquad \frac{1}{C}\, r^n \leq H^n(E \cap B(x, r)) \leq C\, r^n, \ \ \forall r \in (0, R_0), x \in E,$$

where R_0 is the diameter of E (which may be infinite). When $E = \partial\Omega$, the boundary of a domain Ω, we shall sometimes for convenience simply say that "Ω has the ADR property" to mean that $\partial\Omega$ is ADR.

DEFINITION 1.9 (Uniform Rectifiability). – Following David and Semmes [18, 19], we say that a closed set $E \subset \mathbb{R}^{n+1}$ is n-dimensional UR (or simply UR) ("Uniformly Rectifiable"), if it satisfies the ADR condition (1.8), and if for some uniform constant C and for every Euclidean ball $B := B(x_0, r)$, $r \leq \mathrm{diam}(E)$, centered at any point $x_0 \in E$, we have the Carleson measure estimate

$$(1.10) \qquad \iint_B |\nabla^2 \mathcal{S}1(X)|^2 \, \mathrm{dist}(X, E)\, dX \leq C r^n,$$

where $\mathcal{S}f$ is the single layer potential of f, i.e.,

$$(1.11) \qquad \mathcal{S}f(X) := c_n \int_E |X - y|^{1-n} f(y)\, dH^n(y).$$

Here, the normalizing constant c_n is chosen so that $\mathcal{E}(X) := c_n |X|^{1-n}$ is the usual fundamental solution for the Laplacian in $\mathbb{R}^{n+1}$. When $E = \partial\Omega$, the boundary of a domain Ω, we shall sometimes for convenience simply say that "Ω has the UR property" to mean that $\partial\Omega$ is UR.

We note that there are numerous characterizations of uniform rectifiability given in [18, 19]; the one stated above will be most useful for our purposes, and appears in [19, Chapter 3, Part III]. We remark that the UR sets are precisely those for which all "sufficiently nice" singular integrals are bounded on L^2 (see [18]).

We recall that "Uniform Rectifiability" is the scale invariant analogue of rectifiability; in particular, using an idea of P. Jones [30], one may derive, for UR sets, a quantitative version

of the fact that rectifiability may be characterized in terms of existence a.e. of approximate tangent planes. For $x \in E$, $t > 0$, we set

$$(1.12) \qquad \beta_2(x,t) \equiv \inf_P \left(\frac{1}{t^n} \int_{B(x,t) \cap E} \left(\frac{\mathrm{dist}(y,P)}{t} \right)^2 dH^n(y) \right)^{1/2},$$

where the infimum runs over all n-planes P. Then a closed, ADR set E is UR if and only if the following Carleson measure estimate holds on $E \times \mathbb{R}_+$:

$$(1.13) \qquad \sup_{x_0 \in E, \, r > 0} r^{-n} \int_0^r \int_{B(x_0,t) \cap E} \beta_2(x,t)^2 dH^n(x) \frac{dt}{t} < \infty.$$

Again see [18] for details.

DEFINITION 1.14 ("Interior Big Pieces"). – Given a domain $\Omega \subset \mathbb{R}^{n+1}$, with ADR boundary, and a collection $\mathscr{S}$ of domains in $\mathbb{R}^{n+1}$, we say that Ω has "interior big pieces of $\mathscr{S}$" (denoted $\Omega \in IBP(\mathscr{S})$) if there are constants $\alpha > 0$, $K > 1$ such that for every $X \in \Omega$, with $\delta(X) < \mathrm{diam}(\partial\Omega)$, there is a point $x \in \partial\Omega$, with $|x - X| = \delta(X)$, and a domain $\Omega' \in \mathscr{S}$ for which, with $r := K\delta(X)$, we have

1. $\Omega' \subset \Omega$.

2. $H^n(\partial\Omega' \cap \Delta(x,r)) \geq \alpha H^n(\Delta(x,r)) \approx \alpha r^n$.

3. X is a Corkscrew point for Ω', relative to $\Delta_*(y, 2r) := B(y, 2r) \cap \partial\Omega'$, for some $y \in \partial\Omega' \cap \Delta$ (we note that X is also a Corkscrew point for Ω, relative to Δ, by construction; cf. Remark 1.5).

LEMMA 1.15 (Existence and properties of the "dyadic grid" [18, 19, 13])

Suppose that $E \subset \mathbb{R}^{n+1}$ satisfies the ADR condition (1.8). *Then there exist constants $a_0 > 0$, $\eta > 0$ and $C_1 < \infty$, depending only on dimension and the ADR constants, such that for each $k \in \mathbb{Z}$, there is a collection of Borel sets ("cubes")*

$$\mathbb{D}_k := \{Q_j^k \subset E : j \in \mathfrak{I}_k\},$$

where $\mathfrak{I}_k$ denotes some (possibly finite) index set depending on k, satisfying

(i) *$E = \bigcup_j Q_j^k$ for each $k \in \mathbb{Z}$.*

(ii) *If $m \geq k$ then either $Q_i^m \subset Q_j^k$ or $Q_i^m \cap Q_j^k = \varnothing$.*

(iii) *For each (j,k) and each $m < k$, there is a unique m such that $Q_j^k \subset Q_i^m$.*

(iv) *Diameter $\left(Q_j^k\right) \leq C_1 2^{-k}$.*

(v) *Each Q_j^k contains some "surface ball" $\Delta\left(x_j^k, a_0 2^{-k}\right) := B\left(x_j^k, a_0 2^{-k}\right) \cap E$.*

(vi) *$H^n\left(\left\{x \in Q_j^k : \mathrm{dist}(x, E \setminus Q_j^k) \leq \tau \, 2^{-k}\right\}\right) \leq C_1 \, \tau^\eta \, H^n\left(Q_j^k\right)$, for all k, j and for all $\tau \in (0, a_0)$.*

A few remarks are in order concerning this lemma.

– In the setting of a general space of homogeneous type, this lemma has been proved by Christ [13]. In that setting, the dyadic parameter $1/2$ should be replaced by some constant $\delta \in (0,1)$. It is a routine matter to verify that one may take $\delta = 1/2$ in the presence of the Ahlfors-David property (1.8) (in this more restrictive context, the result already appears in [18, 19]).

– For our purposes, we may ignore those $k \in \mathbb{Z}$ such that $2^{-k} \gtrsim \operatorname{diam}(E)$, in the case that the latter is finite.

– We shall denote by $\mathbb{D} = \mathbb{D}(E)$ the collection of all relevant Q_j^k, i.e.,

$$\mathbb{D} := \bigcup_k \mathbb{D}_k,$$

where, if $\operatorname{diam}(E)$ is finite, the union runs over those k such that $2^{-k} \lesssim \operatorname{diam}(E)$.

– Properties (iv) and (v) imply that for each cube $Q \in \mathbb{D}_k$, there is a point $x_Q \in E$, a Euclidean ball $B(x_Q, r)$ and a surface ball $\Delta(x_Q, r) := B(x_Q, r) \cap E$ such that $r \approx 2^{-k} \approx \operatorname{diam}(Q)$ and

(1.16) $\Delta(x_Q, r) \subset Q \subset \Delta(x_Q, Cr),$

for some uniform constant C. We shall denote this ball and surface ball by

(1.17) $B_Q := B(x_Q, r), \qquad \Delta_Q := \Delta(x_Q, r),$

and we shall refer to the point x_Q as the "center" of Q.

– Let us now specialize to the case that $E = \partial\Omega$, with Ω satisfying the Corkscrew condition. Given $Q \in \mathbb{D}(\partial\Omega)$, we shall sometimes refer to a "Corkscrew point relative to Q", which we denote by X_Q, and which we define to be the Corkscrew point X_Δ relative to the ball $\Delta := \Delta_Q$ (cf. (1.16), (1.17) and Definition 1.4). We note that

(1.18) $\delta(X_Q) \approx \operatorname{dist}(X_Q, Q) \approx \operatorname{diam}(Q).$

– For a dyadic cube $Q \in \mathbb{D}_k$, we shall set $\ell(Q) = 2^{-k}$, and we shall refer to this quantity as the "length" of Q. Evidently, $\ell(Q) \approx \operatorname{diam}(Q)$.

– For a dyadic cube $Q \in \mathbb{D}$, we let $k(Q)$ denote the "dyadic generation" to which Q belongs, i.e., we set $k = k(Q)$ if $Q \in \mathbb{D}_k$; thus, $\ell(Q) = 2^{-k(Q)}$.

DEFINITION 1.19 (A_∞, A_∞^{dyadic} and weak-A_∞). – Given a surface ball $\Delta = B \cap \partial\Omega$, a Borel measure ω defined on $\partial\Omega$ is said to belong to the class $A_\infty(\Delta)$ if there are positive constants C and θ such that for every $\Delta' = B' \cap \partial\Omega$ with $B' \subseteq B$, and every Borel set $F \subset \Delta'$, we have

(1.20) $\omega(F) \le C \left(\dfrac{\sigma(F)}{\sigma(\Delta')} \right)^\theta \omega(\Delta').$

If we replace the surface balls Δ and Δ' by a dyadic cube Q and its dyadic subcubes Q', with $F \subset Q'$, then we say that $\omega \in A_\infty^{\text{dyadic}}(Q)$:

(1.21) $\omega(F) \le C \left(\dfrac{\sigma(F)}{\sigma(Q')} \right)^\theta \omega(Q').$

Similarly, $\omega \in$ weak-$A_\infty(\Delta)$, with $\Delta = B \cap \partial\Omega$, if for every $\Delta' = B' \cap \partial\Omega$ with $2B' \subseteq B$, we have

$$(1.22) \qquad \omega(F) \le C \left(\frac{\sigma(F)}{\sigma(\Delta')} \right)^\theta \omega(2\Delta').$$

As is well known [14, 21], [38], the A_∞ (resp. weak-A_∞) condition is equivalent to the property that the measure ω is absolutely continuous with respect to σ, and that its density satisfies a reverse Hölder (resp. weak reverse Hölder) condition. In this paper, we are interested in the case that $\omega = \omega^X$, the harmonic measure with pole at X. In that setting, we let $k^X := d\omega^X/d\sigma$ denote the Poisson kernel, so that (1.20) is equivalent to the reverse Hölder estimate

$$(1.23) \qquad \left(\fint_{\Delta'} \left(k^X \right)^q d\sigma \right)^{1/q} \le C \fint_{\Delta'} k^X \, d\sigma,$$

for some $q > 1$ and for some uniform constant C. In particular, when $\Delta' = \Delta$, and $X = X_\Delta$, a Corkscrew point relative to Δ, the latter estimate reduces to

$$(1.24) \qquad \int_\Delta \left(k^{X_\Delta} \right)^q d\sigma \le C \, \sigma(\Delta)^{1-q}.$$

Similarly, (1.22) is equivalent to

$$(1.25) \qquad \left(\fint_{\Delta'} \left(k^X \right)^q d\sigma \right)^{1/q} \le C \fint_{2\Delta'} k^X \, d\sigma .$$

Assuming that the latter bound holds with $\Delta' = \Delta$, and with $X = X_\Delta$, then one again obtains (1.24).

1.2. Statement of the main results

Our main results are as follows. We shall use the terminology that a connected open set $\Omega \subset \mathbb{R}^{n+1}$ is a *1-sided* NTA domain if it satisfies *interior* (but not necessarily exterior) Corkscrew and Harnack chain conditions[2].

THEOREM 1.26. – *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \ge 2$, be a 1-sided NTA domain whose boundary $\partial\Omega$ is n-dimensional UR. Then for each surface ball Δ, the harmonic measure ω^{X_Δ} belongs to weak-$A_\infty(\Delta)$, with uniform weak-A_∞ constants depending only on dimension and on the constants in the ADR, UR, Corkscrew and Harnack chain conditions.*

We emphasize again that we impose no hypothesis (as in [29, 39], [17]) on the geometry of the exterior domain, nor do we assume as in [7] that the boundary has "Big Pieces" of boundaries of Lipschitz subdomains of Ω.

We shall also obtain a certain "self-improvement" of Theorem 1.26, in which the hypotheses are assumed to hold only in an appropriate "big pieces" sense.

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[2] We recall that such domains are sometimes denoted "uniform" domains in the literature, but we prefer the terminology "1-sided NTA", both because it is more descriptive of the actual properties enjoyed by such domains, and to avoid confusion with the completely different notion of "uniform rectifiability".

THEOREM 1.27. – *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a connected open set whose boundary $\partial\Omega$ is n-dimensional ADR. Suppose further that $\Omega \in IBP(\mathcal{S})$ (cf. Definition 1.14), where $\mathcal{S}$ is a collection of 1-sided NTA domains with UR boundaries, with uniform control of all of the relevant Corkscrew, Harnack chain, ADR and UR constants. Then for each surface ball $\Delta = \Delta(x, r)$, and for every $X \in \Omega \setminus B(x, r)$, the harmonic measure ω^X belongs to weak-$A_\infty(\Delta)$, with uniform weak-A_∞ constants that depend only on dimension, on the constants in the ADR and interior big pieces conditions, and on the relevant constants for the subdomains.*

REMARK. – We note that in Theorem 1.27, we have obtained that ω^X belongs to weak-$A_\infty(\Delta(x, r))$, for all $X \in \Omega \setminus B(x, r)$. In the presence of the Harnack chain Condition, as in Theorem 1.26, one may obtain the same conclusion for $X = X_\Delta$, the Corkscrew point relative to Δ. On the other hand, in Theorem 1.26, we of course also obtain that ω^X belongs to weak-$A_\infty(\Delta(x, r))$, for all $X \in \Omega \setminus B(x, r)$.

In a companion paper to this one [28], we shall establish the converse to Theorem 1.26:

THEOREM 1.28. – *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a 1-sided NTA domain, whose boundary is n-dimensional ADR. Suppose also that harmonic measure ω is absolutely continuous with respect to surface measure and that there is a $q > 1$ such that for every surface ball $\Delta = \Delta(x, r)$ with radius $r \lesssim \operatorname{diam} \partial\Omega$, the Poisson kernel satisfies the scale invariant estimate (1.24). Then $\partial\Omega$ is UR.*

We also mention that in [28] we obtain a "big pieces" version of the previous result in the following sense. Let $E \subset \mathbb{R}^{n+1}$ be a closed set and assume that E is n-dimensional ADR. Assume that there exists $q > 1$ such that E has "big pieces of boundaries of $\mathcal{S}$" (i.e., for every surface ball $B(x, r) \cap E$ there is $\Omega' \in \mathcal{S}$ whose boundary has an "ample" contact with $E \cap B(x, r)$), where $\mathcal{S}$ is a collection of domains Ω' each of them satisfying the hypotheses of Theorem 1.28 (with q fixed) and with uniform control on the relevant constants. Then E is UR. See [28] for the precise statement.

Acknowledgements

2. Outline of the strategy of the proof

Let us sketch the strategy of the proofs of Theorems 1.26 and 1.27. We shall do most of our analysis in certain approximating domains which enjoy additional qualitative properties. Given these qualitative properties, we shall prove some a priori estimates for the Green function G and for harmonic measure ω, beginning with Lemma 3.30 in Section 3, whose proofs rely on being able to "hide" certain small quantities, which must therefore be known in advance to be finite. An interesting feature of these a priori estimates is that they permit us to deduce the doubling property for ω, as well as a comparison principle for G, in the absence of an exterior disk or Corkscrew condition (the exterior conditions enable one to

prove boundary Hölder continuity of solutions vanishing on a surface ball). We obtain these properties for G and ω without establishing boundary Hölder continuity. We note that, by the work of Aikawa [1, 2], some of the preliminary estimates that we prove in Section 3, in particular, the "Carleson estimate" Lemma 3.37, and the Comparison Principle (aka "Boundary Harnack Principle") Lemma 3.64, are known, but we include our own relatively short proofs here for the sake of self-containment.

We also establish several geometric preliminaries as follows. In Section 4, we use the Harnack chain property to prove a Poincaré inequality (Lemma 4.8), which we use in turn, in Section 5, to obtain a criterion for the existence of exterior Corkscrew points in the complement of certain "sawtooth" regions (Lemma 5.10). This criterion stipulates that the Carleson measure (cf. (1.10))

$$(2.1) \qquad\qquad |\nabla^2 \mathcal{S}1(X)|^2 \operatorname{dist}(X, \partial\Omega) dX$$

be *sufficiently small* in the relevant sawtooth region. We then present in Section 6 a variant of the "sawtooth lemma" of [16] (Lemma 6.15), which roughly speaking allows for a comparison, in the sense of A_∞, between the respective harmonic measures, ω and $\omega_{\Omega_\mathcal{G}}$, for the original domain and for the sawtooth domain (more precisely, our version of the sawtooth lemma allows us to transfer the dyadic A_∞ property of $\omega_{\Omega_\mathcal{G}}$ to $\mathcal{P}_\mathcal{G}\omega$, where $\mathcal{P}_\mathcal{G}$ is a sort of "conditional expectation" projection operator, with respect to some collection $\mathcal{G}$ of non-overlapping dyadic cubes from which the sawtooth was constructed). The arguments of Section 6 are an extension, to the present context, of our previous work in the Euclidean setting [27].

With these preliminary matters in hand, we proceed to the heart of our proof, which will exploit the technique of "extrapolation (i.e., bootstrapping) of Carleson measures", as it appears in our previous work [27] (see also [26]), but originating in [12] and [36]. We now describe the application of this technique in our setting. By a Corona type stopping time construction delineated in Section 7, plus an induction scheme (formalized in Lemma 8.5), we reduce matters to verifying that $\mathcal{P}_\mathcal{G}\omega$ (that is, the projection of harmonic measure mentioned above) enjoys the dyadic A_∞ property, in sawtooth domains $\Omega_\mathcal{G}$ in which the Carleson measure (2.1) has *sufficiently small* Carleson norm. In turn, we establish this property for $\mathcal{P}_\mathcal{G}\omega$, by using the preliminary facts noted above: by the smallness of (2.1) in the sawtooth, we deduce that the complement of the sawtooth enjoys an exterior Corkscrew condition. Thus, we may apply the results of [17] to the sawtooth, to obtain that $\omega_{\Omega_\mathcal{G}}$, the harmonic measure for the sawtooth domain, belongs to A_∞ with respect to surface measure on the boundary of the sawtooth. Then, invoking our version of the sawtooth lemma, we find that $\mathcal{P}_\mathcal{G}\omega$ belongs to dyadic A_∞, as desired. The "extrapolation" technology (i.e., Lemma 8.5) now allows us to conclude that ω belongs to A_∞ with respect to surface measure, in a local, but scale invariant way. However, at this point, we have only reached this conclusion in our approximating domains Ω_N, albeit with A_∞ constants independent of N. Here $\{\Omega_N\}$ is a nested increasing sequence of sub-domains of Ω, each of which enjoys the qualitative properties mentioned above, such that $\Omega_N \nearrow \Omega$. It is not clear whether the A_∞ property of harmonic measure, or even the doubling property, are transmitted in the limit to harmonic measure on Ω. However, a maximum principle argument (in the case of Theorem 1.27, there are two separate maximum principle arguments) allows us to transfer, at least, the property that there are uniform

constants $c_0, \eta \in (0, 1)$ such that for any Borel subset $A \subset \Delta$,

$$(*)\qquad\qquad\qquad \sigma(A) > \eta\,\sigma(\Delta) \quad\Longrightarrow\quad \omega^{X_\Delta}(A) \geq c_0\,.$$

The fact that $(*)$ holds, in the absence of assumptions on the exterior domain Ω_{ext} or on $\partial\Omega$ (beyond UR), is really the main result of this paper. Given $(*)$, we obtain the conclusion of Theorems 1.26 and 1.27 by invoking the arguments of [7].

3. Some fundamental estimates

In this section we recall or establish certain fundamental estimates for harmonic measure and the Green function. In the sequel, $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, will be a connected, open set, ω^X will denote harmonic measure for Ω, with pole at X, and $G(X, Y)$ will be the Green function. At least in the case that Ω is bounded, we may, as usual, define ω^X via the maximum principle and the Riesz representation theorem, after first using the method of Perron (see, e.g., [22, pp. 24–25]) to construct a harmonic function "associated" to arbitrary continuous boundary data.[3] For unbounded Ω, we may still define harmonic measure via a standard approximation scheme as follows. Given $R > 0$, set $\Omega_R := \Omega \cap B(x_0, 2R)$, where x_0 is a fixed point on $\partial\Omega$. Define a smooth cut-off function $\eta \in \mathscr{C}_0^\infty([-2, 2])$, with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $[-1, 1]$, and η monotone decreasing on $(1, 2)$ and monotone increasing on $(-2, -1)$. Suppose now that $0 \leq f \in \mathscr{C}_b(\partial\Omega)$ and set

$$(3.1)\qquad\qquad\qquad f_R(x) := f(x)\,\eta\left(\frac{|x - x_0|}{R}\right).$$

Extending f_R to be zero outside of its support defines a continuous function on $\partial\Omega_R$, so we may construct the corresponding Perron solution u_R in Ω_R. By the maximum principle,

$$u_R \leq u_{R'} \text{ in } \Omega_R, \text{ if } R' > R, \qquad \text{and} \qquad \sup_{\Omega_R} u_R \leq \sup_{\partial\Omega_R} f_R \leq \sup_{\partial\Omega} f.$$

Consequently, by Harnack's convergence theorem ([22, p. 22]), there is a harmonic function u in Ω such that

$$(3.2)\qquad\qquad\qquad \lim_{R\to\infty} u_R = u,$$

with the convergence being uniform on compacta in Ω. Moreover, u satisfies the maximum principle

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} f.$$

Thus, we may again define harmonic measure ω^X for $X \in \Omega$ via the Riesz representation theorem. We note for future reference that ω^X is a non-negative, finite Borel measure which satisfies the outer regularity property

$$(3.3)\qquad\qquad\qquad \omega^X(A) := \inf_{A \subseteq O} \omega^X(O),$$

for every Borel set $A \subset \partial\Omega$, where the infimum runs over all (relatively) open $O \subset \partial\Omega$ containing A.

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[3] Since we have made no assumption as regards Wiener's regularity criterion, our harmonic function is a generalized solution, which may not be continuous up to the boundary.

The Green function may now be constructed by setting

$$(3.4) \qquad G(X,Y) := \mathcal{E}(X - Y) - \int_{\partial\Omega} \mathcal{E}(X - z)\, d\omega^Y(z),$$

where $\mathcal{E}(X) := c_n|X|^{1-n}$ is the usual fundamental solution for the Laplacian in $\mathbb{R}^{n+1}$. We choose the normalization that makes $\mathcal{E}$ positive. Given this normalization, we shall also have that $G \geq 0$ (cf. Lemma 3.11 below.)

Before proceeding further, let us note one more fact for future reference. Assuming that Ω is unbounded, and using the notation above, let ω_R^X and $G_R(X,Y)$ denote, respectively, harmonic measure and Green's function for the approximating domain Ω_R. We then have

$$(3.5) \qquad \lim_{R\to\infty} G_R(X,Y) = G(X,Y),$$

with the convergence being uniform on compacta in Ω, in the Y variable with $X \in \Omega$ fixed. Indeed, fixing X, choosing R so large that $R \gg |X - x_0|$, and setting $f := \mathcal{E}(X - \cdot)$, with f_R defined as in (3.1), we have that

$$\int_{\partial\Omega_R} f\, d\omega_R^Y = \int_{\partial\Omega_R} f_R\, d\omega_R^Y + O(R^{1-n}) := u_R(Y) + O(R^{1-n}).$$

We then obtain (3.5) immediately from (3.2) and the definition of the Green function (3.4).

LEMMA 3.6 (Bourgain [9]). – *Suppose that $\partial\Omega$ is n-dimensional ADR. Then there are uniform constants $c \in (0,1)$ and $C \in (1,\infty)$, such that for every $x \in \partial\Omega$, and every $r \in (0, \operatorname{diam}(\partial\Omega))$, if $Y \in \Omega \cap B(x,cr)$, then*

$$(3.7) \qquad \omega^Y(\Delta(x,r)) \geq 1/C > 0\,.$$

In particular, if Ω satisfies the Corkscrew and Harnack chain conditions, then for every surface ball Δ, we have

$$(3.8) \qquad \omega^{X_\Delta}(\Delta) \geq 1/C > 0\,.$$

We refer the reader to [9, Lemma 1] for the proof.

We next introduce some notation. We say that a domain Ω satisfies the *qualitative exterior Corkscrew* condition if there exists $N \gg 1$ such that Ω has exterior Corkscrew points at all scales smaller than 2^{-N}. That is, there exists a constant c_N such that for every surface ball $\Delta = \Delta(x,r)$, with $x \in \partial\Omega$ and $r \leq 2^{-N}$, there is a ball $B(X_\Delta^{\mathrm{ext}}, c_N r) \subset B(x,r) \cap \Omega_{\mathrm{ext}}$.

Given a ball B_0 centered on $\partial\Omega$, and $X \in \Omega \setminus B_0$, we also introduce the quantity

$$(3.9) \qquad \Upsilon_{B_0}(X) := \sup_{B:2B\subseteq B_0} \frac{r_\Delta^{1-n}\omega^X(\Delta)}{G(X_\Delta, X)},$$

where the sup runs over all the balls B centered at $\partial\Omega$ with $2B \subseteq B_0$ and where as usual $\Delta = B \cap \partial\Omega$. We also set $\|\Upsilon_{B_0}\| = \sup_{X\in\Omega\setminus B_0} \Upsilon_{B_0}(X)$. The quantity Υ_{B_0} will enter in the proof of Lemma 3.30 below.

REMARK 3.10. – Let us observe that if Ω satisfies the qualitative exterior Corkscrew condition, then every point in $\partial\Omega$ is regular in the sense of Wiener. Moreover, for 1-sided NTA domains, the qualitative exterior Corkscrew points allow local Hölder continuity at the boundary (albeit with bounds which may depend badly on N), so that the program of [29] may be followed to prove that $\Upsilon_{B_0}(X)$ is a priori finite (possibly depending on N, X

and B_0). Eventually, we shall apply Lemmas 3.11 and 3.30 below (and several related lemmas and corollaries) to certain approximating domains Ω_N which will inherit the stated quantitative hypotheses from the original domain Ω, but which also satisfy the qualitative exterior Corkscrew conditions for scales $\lesssim 2^{-N}$. We emphasize that all of the *quantitative* bounds that we shall establish will depend only upon dimension and on the parameters in the 1-sided NTA and UR (including ADR) conditions, and thus these bounds will hold uniformly for the entire family of approximating domains.

LEMMA 3.11. – *There are positive, finite constants C, depending only on dimension, and $c(n, \theta)$, depending on dimension and $\theta \in (0, 1)$, such that the Green function satisfies*

$$(3.12) \qquad\qquad G(X, Y) \leq C \, |X - Y|^{1-n}$$

$$(3.13) \qquad c(n, \theta) \, |X - Y|^{1-n} \leq G(X, Y), \quad \text{if } |X - Y| \leq \theta \, \delta(X), \ \theta \in (0, 1) \, .$$

Moreover, if every point on $\partial\Omega$ is regular in the sense of Wiener, then

$$(3.14) \qquad\qquad G(X, Y) \geq 0, \qquad \forall X, Y \in \Omega, \ X \neq Y;$$

$$(3.15) \qquad\qquad G(X, Y) = G(Y, X), \qquad \forall X, Y \in \Omega, \ X \neq Y;$$

and

$$(3.16) \qquad\qquad \int_{\partial\Omega} \Phi \, d\omega^X = -\iint_\Omega \nabla_Y G(Y, X) \cdot \nabla \Phi(Y) \, dY,$$

for every $X \in \Omega$ and $\Phi \in C_0^\infty(\mathbb{R}^{n+1})$ with $\Phi(X) = 0$.

Proof. – Some of these facts are standard, but we include the simple proof here. Recall that we have chosen the normalization $\mathcal{E}(X) := c_n |X|^{1-n}$ with $c_n > 0$. Inequality (3.12) is then trivial, by Definition (3.4), since $\int_{\partial\Omega} \mathcal{E}(X - z) \, d\omega^Y(z) \geq 0$. We now consider (3.13). Suppose that $0 < \theta < 1$, and that $|X - Y| \leq \theta \, \delta(X)$. Then,

$$\int_{\partial\Omega} |X - z|^{1-n} \, d\omega^Y(z) \, \leq \, \delta(X)^{1-n} \, \leq \, \theta^{n-1} \, |X - Y|^{1-n}.$$

Thus, $G(X, Y) \geq c_n(1 - \theta^{n-1}) \, |X - Y|^{1-n}$, as desired.

We now assume that every boundary point is regular in the sense of Wiener. Let us prove (3.14). Suppose first that Ω is bounded. Fix $X \in \Omega$, and observe that by (3.13), it is enough to consider the case that $Y \in \Omega' := \Omega \setminus \overline{B(X, \delta(X)/2)}$. Moreover, by (3.13), we have in particular that $G(X, \cdot) > 0$ on $\partial B(X, \delta(X)/2)$. On the other hand, since every boundary point is regular, we have by Definition (3.4) that $G(X, \cdot) \equiv 0$ on $\partial\Omega$. Applying the maximum principle in Ω', we then obtain (3.14), at least when Ω is bounded. If Ω is unbounded, we may invoke (3.5).

Next, we establish the symmetry condition (3.15), again assuming that every boundary point is regular in the sense of Wiener. By (3.5), it is enough to treat the case that Ω is bounded. Specializing to the case of the Laplacian, the Green function constructed in [23], which we denote temporarily by $\widetilde{G}(X, Y)$, is symmetric (see [23, Theorem 1.3]). Therefore, it is enough to verify that our Green function is the same as the one constructed in [23]. To this

end, we first recall that by [23, Theorem 1.1] $\widetilde{G}$ is unique among all those real valued, non-negative functions defined on $\Omega \times \Omega \setminus \{(X,Y) \in \Omega \times \Omega : X = Y\}$, such that for each $X \in \Omega$ and $r > 0$,

$$(3.17) \qquad \widetilde{G}(X,\cdot) \in W^{1,2}\big(\Omega \setminus B(X,r)\big) \cap W_0^{1,1}(\Omega)$$

$$(3.18) \qquad \iint_\Omega \nabla_Y \widetilde{G}(X,Y) \cdot \nabla\phi(Y)\, dY = \phi(X), \qquad \forall \phi \in C_0^\infty(\Omega).$$

It is clear that (3.18) holds for our Green function $G(X,Y)$, by Definition 3.4. Thus, we need only show that G satisfies (3.17). As in [31, p. 5], for $X \in \Omega$ fixed, we may construct $v(X,\cdot)$, the variational solution to the Dirichlet problem with data $\mathcal{E}(X - \cdot)$. In particular, $v(X,\cdot) \in W^{1,2}(\Omega)$. Since $\mathcal{E}(X - \cdot)$ is Lipschitz on $\partial\Omega$, and since every point on $\partial\Omega$ is Wiener regular, it follows as in [31, p. 5] that $v(X,\cdot) \in C(\overline{\Omega})$, and therefore

$$(3.19) \qquad v(X,Y) := \int_{\partial\Omega} \mathcal{E}\,(X - z)\, d\omega^Y(z)$$

(see, e.g., [22], p. 25). Thus, $G(X,Y) = \mathcal{E}(X-Y) - v(X,Y)$ (cf. (3.4)), and since $v \in W^{1,2}(\Omega)$, we obtain (3.17).

Finally we verify (3.16). We begin by reducing matters to the case that Ω is bounded. Indeed, for the left hand side of (3.16), we may pass immediately from the bounded to the unbounded case by splitting Φ into positive and negative parts, and using (3.2). To pass to the limit on the right hand side is more delicate, and we proceed as follows. As above, given an unbounded domain Ω, let G_R denote the Green function for the domain $\Omega_R := \Omega \cap B(x_0, 2R)$, for some fixed $x_0 \in \partial\Omega$. We claim that

$$(3.20) \qquad \lim_{R\to\infty} \iint_{\Omega_R} \nabla_Y G_R(Y,X) \cdot \mathbf{h}(Y)\, dY = \iint_\Omega \nabla_Y G(Y,X) \cdot \mathbf{h}(Y)\, dY,$$

for all Lipschitz vector-valued $\mathbf{h}$ with compact support in $\mathbb{R}^{n+1}$. Given the claim, and assuming that (3.16) holds for bounded Ω, we may then pass to the unbounded case by setting $\mathbf{h} = \nabla\Phi$.

Thus, to reduce the proof of (3.16) to the case that Ω is bounded, it remains to prove (3.20). To this end, we first recall our previous observation that for bounded domains with Wiener regular boundaries, our Green function is the same as that constructed in [23]. Thus, there is a purely dimensional constant C_n such that for every $R < \infty$, and $X \in \Omega_R$, $\nabla G_R(\cdot, X)$ enjoys the weak-$L^{(n+1)/n}$ estimate

$$\big|\{Y \in \Omega_R : |\nabla_Y\, G_R(Y,X)| > \lambda\}\big| \leq C_n\, \lambda^{-(n+1)/n}\,.$$

Consequently, if $A \subset \Omega_R$, we have that

$$(3.21) \qquad \iint_A |\nabla_Y\, G_R(Y,X)|^p\, dY \leq C(n,p,|A|), \qquad \forall p < (n+1)/n,$$

as may be deduced from the weak-type inequality by arguing as in the proof of Kolmogorov's lemma. We emphasize that the constant in the last inequality depends only upon n, p and $|A|$, but not on R. Let us now fix a ball $B_0 := B(X_0, R_0) \subset \mathbb{R}^{n+1}$, and consider a Lipschitz function $\mathbf{h}$ supported in B_0. We note that

(3.22)

$$\iint_\Omega \nabla_Y G_R(Y,X) \cdot \mathbf{h}(Y)\, dY = \iint_\Omega G_R(Y,X)\, \mathrm{div}\, \mathbf{h}(Y)\, dY \to \iint_\Omega G(Y,X)\, \mathrm{div}\, \mathbf{h}(Y)\, dY,$$

as $R \to \infty$, where we have used first that $G_R \in W_0^{1,1}(\Omega_R)$ (again, because G_R coincides with the [23] Green function), and then (3.5) (in $\Omega \cap B_0 \cap \{\delta(Y) > \varepsilon\}$), along with (3.12) (to control small errors in the "border strip" $\Omega \cap B_0 \cap \{\delta(Y) \leq \varepsilon\}$). Here we may suppose that $R_0 \ll R$ so that $B_0 \cap \Omega \subset \Omega_R$. Let us now extend $G(\cdot, X)$ to be zero in $\mathbb{R}^{n+1} \setminus \overline{\Omega}$, and call this extension $\mathscr{G}$. Then from (3.21) and (3.22) it follows that

$$(3.23) \qquad \left| \iint_{\mathbb{R}^{n+1}} \mathscr{G}(Y, X) \operatorname{div} \mathbf{h}(Y) \, dY \right| \leq C(n, p, |B_0|) \|h\|_{p'}, \quad 1 < p < (n+1)/n.$$

Taking a supremum over all Lipschitz $\mathbf{h}$ supported in B_0, with $\|\mathbf{h}\|_{p'} = 1$, we obtain that for $p \in (1, (n+1)/n)$,

$$(3.24) \qquad \nabla \mathscr{G}(\cdot, X) \in L^p(B_0), \quad \text{with } \|\nabla \mathscr{G}\|_{L^p(B_0 \setminus \{X\})} \leq C(n, p, |B_0|).$$

Now let $\psi \in C^\infty(\mathbb{R})$, with $0 \leq \psi \leq 1$, $\psi(t) \equiv 0$ if $t \leq 1$, $\psi(t) \equiv 1$ if $t \geq 2$. We fix $\mathbf{h}$ as above, let $\varepsilon > 0$, and set $\mathbf{h}_\varepsilon(Y) := \mathbf{h}(Y)\psi(\delta(Y)/\varepsilon)$. Then, by (3.5),

$$(3.25) \qquad \iint_\Omega \nabla_Y G_R(Y, X) \cdot \mathbf{h}_\varepsilon(Y) \, dY = \iint_\Omega G_R(Y, X) \operatorname{div} \mathbf{h}_\varepsilon(Y) \, dY$$

$$\to \iint_\Omega G(Y, X) \operatorname{div} \mathbf{h}_\varepsilon(Y) \, dY = \iint_\Omega \nabla_Y G(Y, X) \cdot \mathbf{h}_\varepsilon(Y) \, dY,$$

as $R \to \infty$. Also, by (3.21), (3.24) and Hölder's inequality, for $1 < p < (n+1)/n$, we have

$$(3.26) \qquad \left| \iint_\Omega \nabla_Y G_R(Y, X) \cdot (\mathbf{h} - \mathbf{h}_\varepsilon)(Y) \, dY \right| + \left| \iint_\Omega \nabla_Y G(Y, X) \cdot (\mathbf{h} - \mathbf{h}_\varepsilon)(Y) \, dY \right|$$

$$\leq C(n, p, |B_0|) \|\mathbf{h}\|_\infty \left| \{ Y \in \Omega \cap B_0 : \delta(Y) < 2\varepsilon \} \right|^{1/p'} \to 0,$$

as $\varepsilon \to 0$, uniformly in R. Then by (3.25)-(3.26), we have that

$$(3.27) \qquad \lim_{R \to \infty} \iint_\Omega \nabla_Y G_R(Y, X) \cdot \mathbf{h}(Y) \, dY$$

$$= \lim_{R \to \infty} \iint_\Omega \nabla_Y G_R(Y, X) \cdot \mathbf{h}_\varepsilon(Y) \, dY + o(1) = \iint_\Omega \nabla_Y G(Y, X) \cdot \mathbf{h}(Y) \, dY + o(1),$$

as $\varepsilon \to 0$. Letting $\varepsilon \to 0$, we obtain (3.20).

We may now assume that Ω is bounded, and proceed to prove (3.16) in that case. As above, Wiener regularity then guarantees that a given Perron solution, with Lipschitz data, coincides with the corresponding variational solution with the same data. This is true for the function $v(X, Y)$ defined in (3.19), as well as for

$$u(X) := \int_{\partial \Omega} \Phi \, d\omega^X.$$

Thus, in particular, $u - \Phi \in W_0^{1,2}(\Omega)$, and we claim that

$$(3.28) \qquad \iint_\Omega \nabla_Y G(Y, X) \cdot \left(\nabla u(Y) - \nabla \Phi(Y) \right) dY = u(X) - \Phi(X) = u(X).$$

If $u - \Phi$ were in $C_0^\infty(\Omega)$, the claim would follow immediately from (3.18). We will pass from $C_0^\infty(\Omega)$ to $W_0^{1,2}(\Omega)$ by a density argument, with a slight complication since the Green function is not in $W^{1,2}$ near the pole. To address this technical issue, we multiply $(u - \Phi)$ by a smooth cut-off function supported in a small neighborhood of the pole X. For the part near

the pole we may invoke (3.18): u is harmonic, therefore smooth in Ω and u times a smooth cut-off is $C_0^\infty(\Omega)$. For the part away from the pole we use (3.17) and (3.18), plus the routine density argument mentioned above. We leave the details, which are standard, to the reader.

At this point, (3.16) follows immediately from (3.28) and the fact that

$$(3.29) \qquad \iint_\Omega \nabla_Y G(Y,X)\,\nabla u(Y)\,dY = 0\,.$$

In turn, we may verify the latter identity as follows. For $0 < \varepsilon \ll \delta(X)$, set $\phi_\varepsilon(Y) := \phi\big((X-Y)/\varepsilon\big)$, where $\phi \in C^\infty(\mathbb{R}^{n+1})$, $\phi \equiv 1$ in $\mathbb{R}^{n+1} \setminus B(0,2)$, $\phi \equiv 0$ in $B(0,1)$. Then

$$\iint_\Omega \nabla_Y G(Y,X) \cdot \nabla u(Y)\,dY \;=\; \iint_\Omega \nabla_Y\Big(\mathcal{E}(Y-X)\,\phi_\varepsilon(Y) - v(Y,X)\Big) \cdot \nabla u(Y)\,dY$$

$$+ \iint_\Omega \nabla_Y\Big(\mathcal{E}(Y-X)\big(1 - \phi_\varepsilon(Y)\big)\Big) \cdot \nabla u(Y)\,dY = 0 + O(\varepsilon)\,,$$

where in the vanishing term we have used the definition of weak solution, since $\mathcal{E}(\cdot - X)\,\phi_\varepsilon(\cdot) - v(\cdot, X) \in W_0^{1,2}(\Omega)$. To obtain the $O(\varepsilon)$ bound, we have used standard estimates for the fundamental solution and its gradient, along with the fact that ∇u is harmonic and therefore locally bounded in Ω. Finally, we obtain (3.29) by letting $\varepsilon \to 0^+$. $\qquad\square$

LEMMA 3.30. – *Let Ω be a 1-sided NTA domain with n-dimensional ADR boundary, and suppose that every $x \in \partial\Omega$ is regular in the sense of Wiener. Fix $B_0 := B(x_0, r_0)$ with $x_0 \in \partial\Omega$, and $\Delta_0 := B_0 \cap \partial\Omega$. Let $B := B(x,r)$, $x \in \partial\Omega$, and $\Delta := B \cap \partial\Omega$, and suppose that $2B \subset B_0$. Then for $X \in \Omega \setminus B_0$ we have*

$$(3.31) \qquad r^{n-1} G(X_\Delta, X) \leq C\omega^X(\Delta).$$

If, in addition, Ω satisfies the qualitative exterior Corkscrew condition, then

$$(3.32) \qquad \omega^X(\Delta) \leq Cr^{n-1}G(X_\Delta, X).$$

The constants in (3.31) and (3.32) depend only *on dimension and on the constants in the ADR and 1-sided NTA conditions.*

REMARK. – Let us emphasize that in several results below we will assume that certain domains satisfy the hypotheses of Lemma 3.30; by this we mean that the domains are 1-sided NTA with n-dimensional ADR boundary, which moreover satisfy the qualitative exterior Corkscrew condition (in particular then, every boundary point is regular in the sense of Wiener, cf. Remark 3.10). Notice that in such a case (3.32) holds with C depending *only* on dimension and on the constants in the ADR and 1-sided NTA conditions. In particular, C does not depend on the parameter N from the qualitative assumption. This will be crucial when applied to approximating domains.

Proof. – The first estimate (3.31) may be obtained by a well known argument (cf. [10] or [31, Lemma 1.3.3]) using (3.8) plus Harnack's inequality, the upper bound for $G(X,Y)$ in (3.12), and the maximum principle in $\Omega \setminus B(X_\Delta, \delta(X_\Delta)/2)$ (the use of the maximum principle is justified even in the case that Ω is unbounded, by virtue of the decay of the Green function at infinity). We omit the details.

The proof of the second estimate (3.32) will require a bit more work. In contrast to the case of previous proofs of this estimate [10, 29], we do not use local Hölder continuity at the boundary for solutions vanishing on a surface ball (since this depends on the parameter N in our qualitative assumption). Instead, we proceed as follows. Fix B_0 centered on $\partial\Omega$, and $X \in \Omega \setminus B_0$, and write $\Upsilon_{B_0} = \Upsilon_{B_0}(X)$ (cf. (3.9)). As observed in Remark 3.10, Υ_{B_0} is a priori finite (possibly depending on N). Thus, it will suffice to show that $\Upsilon_{B_0} \leq C_\varepsilon + C\varepsilon \Upsilon_{B_0}$, for every small $\varepsilon > 0$. Choose now $B = B(x, r)$, with $2B \subseteq B_0$, such that

$$\frac{1}{2}\Upsilon_{B_0} \leq \frac{\omega^X(\Delta)}{r^{n-1}G(X_\Delta, X)},$$

where as usual $\Delta = B \cap \partial\Omega$.

Now set $B := B(x, r)$ and $\widetilde{B} := B(x, 5r/4)$. Taking $\Phi \in C_0^\infty(\widetilde{B})$, with $0 \leq \Phi \leq 1$, $\Phi(Y) \equiv 1$ on $B(x, r)$, and $\|\nabla\Phi\|_\infty \lesssim 1/r$, we deduce from (3.16) that

(3.33)

$$\omega^X(\Delta) \lesssim \frac{1}{r}\iint_{\Omega \cap \widetilde{B}} |\nabla_Y G(Y, X)|\, dY$$

$$= \frac{1}{r}\iint_{\Omega \cap \widetilde{B} \cap \{\delta(Y) > \varepsilon r\}} |\nabla_Y G(Y, X)|\, dY + \frac{1}{r}\iint_{\Omega \cap \widetilde{B} \cap \{\delta(Y) \leq \varepsilon r\}} |\nabla_Y G(Y, X)|\, dY$$

$$\lesssim \frac{1}{r}\iint_{\Omega \cap \widetilde{B} \cap \{\delta(Y) > \varepsilon r\}} \frac{G(Y, X)}{\delta(Y)}\, dY + \frac{1}{r}\iint_{\Omega \cap \widetilde{B} \cap \{\delta(Y) \leq \varepsilon r\}} \frac{G(Y, X)}{\delta(Y)}\, dY$$

$$=: I + II,$$

where $\varepsilon > 0$ is at our disposal, and where in the next to last line we have used standard interior estimates for harmonic functions. By Harnack's inequality and the Harnack chain condition, we have that

$$I \leq C_\varepsilon r^{n-1}G(X_\Delta, X)$$

as desired. To handle term II, choose $y \in \partial\Omega$ such that $|Y - y| = \delta(Y)$, and set $\Delta(Y) := \Delta(y, \delta(Y))$. Then by (3.31) and the Harnack chain condition we have

$$II \leq \frac{1}{r}\iint_{\Omega \cap \widetilde{B} \cap \{\delta(Y) \leq \varepsilon r\}} \frac{\omega^X(\Delta(Y))}{(\delta(Y))^n}\, dY$$

$$\lesssim \frac{1}{r}\sum_{k:2^{-k} \lesssim \varepsilon r}\iint_{\Omega \cap \widetilde{B} \cap \{2^{-k-1} < \delta(Y) \leq 2^{-k}\}} \frac{\omega^X(\Delta(Y))}{(\delta(Y))^n}\, dY$$

$$\lesssim \frac{1}{r}\sum_{k:2^{-k} \lesssim \varepsilon r}\sum_j \iint_{B_j^k \cap \{2^{-k-1} < \delta(Y) \leq 2^{-k}\}} \frac{\omega^X(\Delta(Y))}{(\delta(Y))^n}\, dY,$$

where in the last step $B_j^k := B(x_j^k, 2^{-k+1})$, and for each k in the sum, $\mathscr{B}_k := \{B_j^k\}_j$ is a collection of balls whose doubles have bounded overlaps, such that $x_j^k \in \partial\Omega$,

(3.34) $\left(\Omega \cap \widetilde{B} \cap \{2^{-k-1} < \delta(Y) \leq 2^{-k}\}\right) \subset \bigcup_j B_j^k,$ and $\bigcup_j 2B_j^k \subset B(x, 3r/2).$

We leave it to the reader to verify that such a collection exists, by virtue of the ADR property of $\partial\Omega$, for all sufficiently small ε. We then have that

$$II \lesssim \frac{1}{r} \sum_{k:2^{-k}\lesssim\varepsilon r} 2^{-k} \sum_j \omega^X \left(\Delta(x_j^k, 2^{-k+2})\right) \leq C\varepsilon\, \omega^X \left(\Delta(x, 3r/2)\right)$$

(3.35)

$$\leq C\varepsilon \sum_{\Delta'} \omega^X(\Delta') \leq C\varepsilon\, \Upsilon_{B_0}\, r^{n-1} \sum_{\Delta'} G(X_{\Delta'}, X) \leq C\varepsilon\, \Upsilon_{B_0}\, r^{n-1} G(X_\Delta, X),$$

where in the second, third, fourth and fifth inequalities we have used, respectively, the bounded overlap property of the balls $2B_j^k$; the ADR property to cover $\Delta(x, 3r/2)$ by a collection $\{\Delta'\}$ of bounded cardinality, such that $r_{\Delta'} \approx r$, and $\Delta' = B' \cap \partial\Omega$, with B' centered on $\partial\Omega$ and $2B' \subset B_0$; the definition of Υ_{B_0}; and the Harnack chain condition. We then obtain (3.32) by choosing ε sufficiently small. $\qquad\square$

REMARK. – Let us observe that from the previous proof it follows that we are not really using the full strength of the qualitative exterior Corkscrew condition, but only that every boundary point is regular in the sense of Wiener and that $\Upsilon(B_0)$ is a priori finite. Although these relaxed qualitative hypotheses suffice for our purposes, the qualitative exterior Corkscrew condition is cleaner, easier to check in practice and holds for the approximating domains introduced below.

COROLLARY 3.36. – *Suppose that Ω is a 1-sided NTA domain with n-dimensional ADR boundary and that it also satisfies the qualitative exterior Corkscrew condition. Let $B := B(x, r)$, $x \in \partial\Omega$, $\Delta := B \cap \partial\Omega$ and $X \in \Omega \setminus 4B$. Then there is a uniform constant C such that*

$$\omega^X(2\Delta) \leq C\omega^X(\Delta).$$

Proof. – The conclusion of the corollary follows immediately from the combination of (3.31) and (3.32), and Harnack's inequality. We omit the details. $\qquad\square$

Next, we establish a bound of "Carleson-type" for the Green function. The Carleson estimate is already known for arbitrary non-negative harmonic functions vanishing on a surface ball [1, 2]; however, specializing to the Green function, one may give a fairly simple direct proof, based upon that of the previous lemma.

LEMMA 3.37. – *Suppose that Ω is a 1-sided NTA domain with n-dimensional ADR boundary and that it also satisfies the qualitative exterior Corkscrew condition. Then there is a uniform constant C such that for each $B := B(x, r)$, $x \in \partial\Omega$, $\Delta = B \cap \partial\Omega$, and $X \in \Omega \setminus 2B$, we have*

$$(3.38) \qquad\qquad \sup_{Y \in B \cap \Omega} G(Y, X) \leq C\, G(X_\Delta, X).$$

Proof. – Fix B, Δ and X as in the statement of the lemma, and set $u(Y) := G(Y, X)$. Extending u to be zero in Ω_{ext}, we obtain from (3.16) that u is subharmonic in $B(x, 3r/2)$.

Let $B' := B(x, 5r/4)$. By the sub-mean value inequality, we have that

$$
\sup_{Y \in B} u(Y) \lesssim \frac{1}{|B'|} \iint_{B'} u = \frac{1}{|B'|} \iint_{B' \cap \Omega} u
$$

$$
= \frac{1}{|B'|} \iint_{B' \cap \Omega \cap \{\delta(Y) > \varepsilon r\}} u(Y) \, dY \; + \; \frac{1}{|B'|} \iint_{B' \cap \Omega \cap \{\delta(Y) \le \varepsilon r\}} u(Y) \, dY
$$

$$
=: I^* + II^* \le C_\varepsilon \, u(X_\Delta) + II^*,
$$

where in the last step we have used the Harnack chain condition to estimate term I^*, and we have fixed a small ε as in the proof of Lemma 3.30 so that (3.34) holds. Moreover, by definition of u,

$$
II^* \lesssim \frac{1}{|B'|} \varepsilon r \iint_{B' \cap \Omega \cap \{\delta(Y) \le \varepsilon r\}} \frac{G(Y, X)}{\delta(Y)} \, dY \approx \varepsilon \, r^{1-n} \, II,
$$

where II is exactly the same term as in (3.33). In turn, by (3.35) and the fact that $\Upsilon_{B_0}(X)$ is uniformly bounded (the latter fact is simply a restatement of (3.32)), we find that

$$
II^* \le C\varepsilon^2 G(X_\Delta, X). \qquad \qquad \square
$$

Under the same hypotheses as in the previous two lemmata, we shall obtain a comparison principle for the Green function, again without the use of Hölder continuity at the boundary. In order to state our comparison principle, we shall need to introduce the notion of a Carleson region. Given a "dyadic cube" $Q \in \mathbb{D}(\partial\Omega)$, the *discretized Carleson region* $\mathbb{D}_Q$ is defined to be

$$
(3.39) \qquad \qquad \mathbb{D}_Q := \{Q' \in \mathbb{D} : Q' \subseteq Q\}.
$$

For future reference, we also introduce discretized sawtooth regions as follows. Given a family $\mathcal{F}$ of disjoint cubes $\{Q_j\} \subset \mathbb{D}$, we define the *global discretized sawtooth* relative to $\mathcal{F}$ by

$$
(3.40) \qquad \qquad \mathbb{D}_\mathcal{F} := \mathbb{D} \setminus \bigcup_\mathcal{F} \mathbb{D}_{Q_j},
$$

i.e., $\mathbb{D}_\mathcal{F}$ is the collection of all $Q \in \mathbb{D}$ that are not contained in any $Q_j \in \mathcal{F}$. Given some fixed cube Q, the *local discretized sawtooth* relative to $\mathcal{F}$ by

$$
(3.41) \qquad \qquad \mathbb{D}_{\mathcal{F},Q} := \mathbb{D}_Q \setminus \bigcup_\mathcal{F} \mathbb{D}_{Q_j} = \mathbb{D}_\mathcal{F} \cap \mathbb{D}_Q.
$$

We shall also require "geometric" Carleson regions and sawtooths. Let us first recall that we write $k = k(Q)$ if $Q \in \mathbb{D}_k$ (cf. Lemma **??**), and in that case the "length" of Q is denoted by $\ell(Q) = 2^{-k(Q)}$. We also recall that there is a Corkscrew point X_Q, relative to each $Q \in \mathbb{D}$ (in fact, there are many such, but we just pick one). Given such a Q, we define an associated "Whitney region" as follows. Let $\mathcal{W} = \mathcal{W}(\Omega)$ denote a collection of (closed) dyadic Whitney cubes of Ω, so that the cubes in $\mathcal{W}$ form a pairwise non-overlapping covering of Ω, which satisfy

$$
(3.42) \qquad 4 \operatorname{diam}(I) \le \operatorname{dist}(4I, \partial\Omega) \le \operatorname{dist}(I, \partial\Omega) \le 40 \operatorname{diam}(I), \qquad \forall I \in \mathcal{W}
$$

(just dyadically divide the standard Whitney cubes, as constructed in [40, Chapter VI], into cubes with side length 1/8 as large) and also

$$(1/4)\operatorname{diam}(I_1) \le \operatorname{diam}(I_2) \le 4\operatorname{diam}(I_1),$$

whenever I_1 and I_2 touch. Let $\ell(I)$ denote the side length of I, and write $k = k_I$ if $\ell(I) = 2^{-k}$. We set

$$(3.43) \quad \mathcal{W}_Q := \left\{ I \in \mathcal{W} : k(Q) - m_0 \le k_I \le k(Q) + 1, \text{ and } \operatorname{dist}(I, Q) \le C_0\, 2^{-k(Q)} \right\},$$

where we may (and do) choose the constant C_0 and positive integer m_0, depending only on the constants in the Corkscrew condition and in the dyadic cube construction (cf. Lemma **??**), so that $X_Q \in I$ for some $I \in \mathcal{W}_Q$, and for each dyadic child Q^j of Q, the respective Corkscrew points $X_{Q^j} \in I^j$ for some $I^j \in \mathcal{W}_Q$. In particular, the collection $\mathcal{W}_Q$ is non-empty for every $Q \in \mathbb{D}$. Moreover as long as C_0 is chosen large enough depending on the constant c in the Corkscrew condition, then by the properties of Whitney cubes, we may always find an $I \in \mathcal{W}_Q$ with the slightly more precise property that $k(Q) - 1 \le k_I \le k(Q)$. We may further suppose, by choosing C_0 large enough, that

$$(3.44) \quad \mathcal{W}_{Q_1} \cap \mathcal{W}_{Q_2} \ne \varnothing, \text{ whenever } 1 \le \frac{\ell(Q_2)}{\ell(Q_1)} \le 2, \text{ and } \operatorname{dist}(Q_1, Q_2) \le 1000\ell(Q_2).$$

We omit the details. In the sequel, we shall assume always that C_0 has been so chosen, and further that $C_0 \ge 1000\sqrt{n}$.

We shall need to augment $\mathcal{W}_Q$ in order to exploit the Harnack chain condition. It will be convenient to introduce the following notation: given a subset $A \subset \Omega$, we write

$$X \to_A Y$$

if the interior of A contains all the balls in a Harnack chain (in Ω), connecting X to Y, and if, moreover, for any point Z contained in any ball in the Harnack chain, we have

$$(3.45) \quad\quad\quad \operatorname{dist}(Z, \partial\Omega) \approx \operatorname{dist}(Z, \Omega \setminus A),$$

with uniform control of the implicit constants. We denote by $X(I)$ the center of a cube $I \in \mathbb{R}^{n+1}$, and we recall that X_Q denotes a designated Corkscrew point relative to Q, which we may, from this point on, assume without loss of generality to be the center of some Whitney cube I such that $\ell(I) \approx \ell(Q) \approx \operatorname{dist}(I, Q)$. More precisely, we note the following.

REMARK 3.46. – Having fixed the collection $\mathcal{W}$ (the Whitney cubes of Ω), by taking the Corkscrew constant c to be slightly smaller, if necessary, we may assume that the Corkscrew point X_Q is the center of some $I \in \mathcal{W}$, with $I \subset B_Q \cap \Omega$ and $\ell(I) \approx \ell(Q)$.

We now define the augmented collection $\mathcal{W}_Q^*$ as follows. For each $I \in \mathcal{W}_Q$, we form a Harnack chain, call it $H(I)$, from the center $X(I)$ to the Corkscrew point X_Q. We now denote by $\mathcal{W}(I)$ the collection of all Whitney cubes which meet at least one ball in the chain $H(I)$, and we set

$$\mathcal{W}_Q^* := \bigcup_{I \in \mathcal{W}_Q} \mathcal{W}(I).$$

We also define, for $\lambda \in (0, 1)$ to be chosen momentarily,

$$(3.47) \qquad U_Q := \bigcup_{\mathscr{W}_Q^*}(1 + \lambda)I =: \bigcup_{I \in \mathscr{W}_Q^*} I^* .$$

By construction, we then have that

$$(3.48) \qquad \mathscr{W}_Q \subset \mathscr{W}_Q^* \subset \mathscr{W} \quad \text{and} \quad X_Q \in U_Q, \ X_{Q^j} \in U_Q,$$

for each child Q^j of Q. It is also clear that there are uniform constants k^* and K_0 such that

$$k(Q) - k^* \le k_I \le k(Q) + k^*, \qquad \forall I \in \mathscr{W}_Q^*$$

$$(3.49) \qquad X(I) \to_{U_Q} X_Q, \qquad \forall I \in \mathscr{W}_Q^*$$

$$\operatorname{dist}(I, Q) \le K_0\, 2^{-k(Q)}, \qquad \forall I \in \mathscr{W}_Q^*,$$

where k^*, K_0 and the implicit constants in (3.45) (which pertain to the condition $X(I) \to_{U_Q} X_Q$), depend only on the "allowable parameters" (since m_0 and C_0 also have such dependence) and on λ. Thus, by the addition of a few nearby Whitney cubes of diameter also comparable to that of Q, we can "augment" $\mathscr{W}_Q$ so that the Harnack chain condition holds in U_Q.

We fix the parameter λ so that for any $I, J \in \mathscr{W}$,

$$\operatorname{dist}(I^*, J^*) \approx \operatorname{dist}(I, J)$$

$$(3.50) \qquad \operatorname{int}(I^*) \cap \operatorname{int}(J^*) \ne \varnothing \iff \partial I \cap \partial J \ne \varnothing$$

(the fattening thus ensures overlap of I^* and J^* for any pair $I, J \in \mathscr{W}$ whose boundaries touch, so that the Harnack chain property then holds locally, with constants depending upon λ, in $I^* \cup J^*$). By choosing λ sufficiently small, we may also suppose that there is a $\tau \in (1/2, 1)$ such that for distinct $I, J \in \mathscr{W}$,

$$(3.51) \qquad \tau J \cap I^* = \varnothing .$$

We remark that any sufficiently small choice of λ (say $0 < \lambda \le \lambda_0$) will do for our purposes.

Of course, there may be some flexibility in the choice of additional Whitney cubes which we add to form the augmented collection $\mathscr{W}_Q^*$, but having made such a choice for each $Q \in \mathbb{D}$, we fix it for all time. We may then define the *Carleson box* associated to Q by

$$(3.52) \qquad T_Q := \operatorname{int}\left(\bigcup_{Q' \in \mathbb{D}_Q} U_{Q'}\right) .$$

Similarly, we may define geometric sawtooth regions as follows. As above, give a family $\mathscr{F}$ of disjoint cubes $\{Q_j\} \subset \mathbb{D}$, we define the *global sawtooth* relative to $\mathscr{F}$ by

$$(3.53) \qquad \Omega_{\mathscr{F}} := \operatorname{int}\left(\bigcup_{Q' \in \mathbb{D}_{\mathscr{F}}} U_{Q'}\right) ,$$

and again given some fixed $Q \in \mathbb{D}$, the *local sawtooth* relative to $\mathscr{F}$ by

$$(3.54) \qquad \Omega_{\mathscr{F}, Q} := \operatorname{int}\left(\bigcup_{Q' \in \mathbb{D}_{\mathscr{F}, Q}} U_{Q'}\right) .$$

For future reference, we present the following.

LEMMA 3.55. – *Suppose that Ω is a 1-sided NTA domain with an ADR boundary. Given $Q \in \mathbb{D}$, let $B_Q := B(x_Q, r)$, $r \approx \ell(Q)$, and $\Delta_Q := B_Q \cap \partial\Omega \subset Q$, be as in (1.16) and (1.17). Then for each Q, there is a ball $B_Q' := B(x_Q, s) \subset B_Q$, with $s \approx \ell(Q) \approx r$, such that*

$$(3.56) \qquad\qquad B_Q' \cap \Omega \subset T_Q.$$

Moreover, for a somewhat smaller choice of $s \approx (K_0)^{-1}\ell(Q)$, we have for every pairwise disjoint family $\mathcal{F} \subset \mathbb{D}$, and for each $Q_0 \in \mathbb{D}$ containing Q, that

$$(3.57) \qquad\qquad B_Q' \cap \Omega_{\mathcal{F}, Q_0} = B_Q' \cap \Omega_{\mathcal{F}, Q}.$$

Proof. – We prove (3.56) first. Let $Y \in \Omega$, with $|Y - x_Q| < cr =: s$, where $c > 0$ is to be determined. Then $Y \in I \in \mathcal{W}$, with

$$\ell(I) \approx \delta(Y) \leq |Y - x_Q| < cr.$$

Fix $Q_I \in \mathbb{D}$ such that $\ell(Q_I) = \ell(I)$, and $\mathrm{dist}(Q_I, I) \approx \ell(I)$. In particular, $I \in \mathcal{W}_{Q_I} \subset \mathcal{W}_{Q_I}^*$, so that $I \subset \mathrm{int}(I^*) \subset \mathrm{int}(U_{Q_I})$. By the triangle inequality, for all $x \in Q_I$, we have

$$|x - x_Q| \leq |x - Y| + |Y - x_Q| \leq C\ell(I) + cr \leq Ccr < r,$$

if c is chosen small enough. Hence, $Q_I \subset \Delta_Q \subset Q$, so $Y \in \bigcup_{Q' \in \mathbb{D}_Q} \mathrm{int}(U_{Q'}) \subset T_Q$.

We now turn to the proof of (3.57). Since $Q \subset Q_0$, the "right to left" containment is trivial, for any choice of B_Q'. We therefore suppose that $Y \in B_Q' \cap \Omega_{\mathcal{F}, Q_0}$, where again $B_Q' := B(x_Q, s)$, and s will be chosen momentarily. It is enough to show that $Y \in \Omega_{\mathcal{F}, Q}$, for some choice of $s \approx (K_0)^{-1}\ell(Q)$. Since $Y \in \Omega_{\mathcal{F}, Q_0}$, by definition there is some $Q' \in \mathbb{D}_{Q_0} \cap \mathbb{D}_{\mathcal{F}}$, for which $Y \in I^* = (1 + \lambda)I$, with $I \in \mathcal{W}_{Q'}^*$. Then

$$\ell(Q') \approx \ell(I) \approx \delta(Y) \leq |Y - x_Q| \leq s,$$

where in the last step we have used that $Y \in B_Q'$. Moreover, for every $y' \in Q'$, we have

$$|y' - Y| \lesssim \mathrm{dist}(Y, Q') \lesssim \mathrm{dist}(I, Q') \lesssim K_0 \ell(Q') \lesssim K_0 s.$$

Thus, by the triangle inequality, for every $y' \in Q'$, we have

$$|y' - x_Q| \lesssim K_0 s < r \approx \ell(Q),$$

by choice of $s = c(K_0)^{-1}\ell(Q)$ with c sufficiently small. Thus, $Q' \subset \Delta_Q \subset Q$, whence $Y \in \bigcup_{Q' \in \mathbb{D}_{\mathcal{F}, Q}} U_{Q'}$, i.e., we have shown that $B_Q' \cap \Omega_{\mathcal{F}, Q_0} \subset \bigcup_{Q' \in \mathbb{D}_{\mathcal{F}, Q}} U_{Q'}$, and therefore $\mathrm{int}(B_Q') \cap \Omega_{\mathcal{F}, Q_0} \subset \Omega_{\mathcal{F}, Q}$, by definition. Choosing s slightly smaller, which amounts to replacing B_Q' by a slightly smaller ball, we obtain (3.57). $\qquad\square$

We also define as follows the "Carleson box" T_Δ associated to a surface ball $\Delta := \Delta(x_\Delta, r)$. Let $k(\Delta)$ denote the unique $k \in \mathbb{Z}$ such that $2^{-k-1} < 200\,r \leq 2^{-k}$, and set

$$(3.58) \qquad\qquad \mathbb{D}^\Delta := \{Q \in \mathbb{D}_{k(\Delta)} : Q \cap 2\Delta \neq \varnothing\}.$$

We then define

$$(3.59) \qquad\qquad T_\Delta := \mathrm{int}\left(\bigcup_{Q \in \mathbb{D}^\Delta} \overline{T_Q}\right).$$

For future reference, we record the following analogue of Lemma 3.55. Set $B_\Delta := B(x_\Delta, r)$, so that $\Delta = B_\Delta \cap \partial\Omega$. Then

$$(3.60) \qquad\qquad \frac{5}{4}B_\Delta \cap \Omega \subset T_\Delta.$$

Indeed, let $X \in \Omega$ with $|X - x_\Delta| < 5r/4$. Then $X \in I \in \mathcal{W}$ with $\ell(I) \approx \delta(X) < 5r/4$, so that $\ell(I) \leq \ell(Q)$, for each $Q \in \mathbb{D}^\Delta$.

Suppose first that $\delta(X) < 3r/4$. There is an $x_1 \in \partial\Omega$ such that $|X - x_1| = \delta(X)$, so that by the triangle inequality, $|x_1 - x_\Delta| < 2r$. Consequently, there is a $Q \in \mathbb{D}^\Delta$ for which $x_1 \in \overline{Q}$, whence there is a $Q' \subset Q$, whose closure contains x_1, such that $\ell(Q') = \ell(I)$, and $\operatorname{dist}(Q', I) \leq \delta(X) \leq 41 \operatorname{diam}(I) \ll C_0 \ell(Q')$ (cf (3.42)-(3.43)). Thus, $I \in \mathcal{W}_{Q'}$, so $X \in \operatorname{int}(I^*) \subset \operatorname{int}(U_{Q'}) \subset T_Q \subset T_\Delta$.

Now suppose that $3r/4 \leq \delta(X) < 5r/4$. Then $X \in I$ with $\ell(I) \approx r$, and $\operatorname{dist}(I, Q') \approx r$ for every Q' contained in any $Q \in \mathbb{D}^\Delta$, with $\ell(Q') \approx \ell(I)$. In that case, we have that $I \in \mathcal{W}_{Q'}$, for each such Q', so that $X \in T_Q, \forall Q \in \mathbb{D}_\Delta$.

LEMMA 3.61. – *Suppose that Ω is a 1-sided NTA domain with an ADR boundary. Then all of its Carleson boxes T_Q and T_Δ, and sawtooth regions $\Omega_{\mathcal{F}}$, and $\Omega_{\mathcal{F},Q}$ are also 1-sided NTA domains with ADR boundaries. If in addition $\partial\Omega$ is also UR, then so is the boundary of each Carleson box T_Q and T_Δ. In all cases, the implicit constants are uniform, and depend only on dimension and on the corresponding constants for Ω.*

We defer the proof until Appendix A.

We remark that it seems likely that one could show that the sawtooth regions also inherit the UR property, but in our case, the only sawtooths that we work with will enjoy an even stronger property, so we shall not bother to explore this issue here.

LEMMA 3.62. – *Suppose that Ω is a 1-sided NTA domain with an ADR boundary and that Ω also satisfies the qualitative exterior Corkscrew condition. Then all of its Carleson boxes T_Q and T_Δ, and sawtooth regions $\Omega_{\mathcal{F}}$, and $\Omega_{\mathcal{F},Q}$ satisfy the qualitative exterior Corkscrew condition. In all cases, the implicit constants are uniform, and depend only on dimension and on the corresponding constants for Ω.*

The proof of this result is almost trivial. Consider for instance the domain $\Omega_{\mathcal{F},Q}$, and let $x \in \partial\Omega_{\mathcal{F},Q}$ and $r \leq 2^{-N}$, with N corresponding to the qualitative exterior Corkscrew condition assumed on Ω. If either $x \in \partial\Omega$ or $x \in \Omega$ with $\delta(x) < r/2$, there exists $y \in \partial\Omega$ such that $B(y, r/2) \subset B(x, r)$. Then the exterior Corkscrew point relative to $\Delta(y, r/2)$ is also a Corkscrew point relative to $B(x, r) \cap \Omega_{\mathcal{F},Q}$. The case $x \in \Omega$ with $\delta(x) \geq r/2$ is as follows. There exists a Whitney box I, with $\ell(I) \approx \delta(X)$, such that $x \in \partial I^*$ and $\operatorname{int}(I^*) \subset \Omega_{\mathcal{F},Q}$. Note that ∂I^* can be covered by Whitney boxes J that meet I by (3.50). Since x is a boundary point of $\Omega_{\mathcal{F},Q}$, there is a $J \ni x$, with $J \notin \mathcal{W}_{Q'}^*$ for any $Q' \in \mathbb{D}_{\mathcal{F},Q}$. Consequently, $B(x, r)$ has an "ample" intersection with $J \setminus \Omega_{\mathcal{F},Q}$, wherein we may find the required Corkscrew point. Further details are left to the reader.

We are now ready to state our comparison principle for the Green function. The result is already known [1], but we include the proof here for the sake of self-containment. Given a

surface ball $\Delta := \Delta(x, r)$, let $B_\Delta := B(x, r)$, so that $\Delta = B_\Delta \cap \partial\Omega$. We fix κ_0 large enough that

$$(3.63) \qquad \overline{T_\Delta} \subset \kappa_0 B_\Delta \cap \overline{\Omega}.$$

LEMMA 3.64. – *Suppose that Ω is a 1-sided NTA domain with n-dimensional ADR boundary and that it also satisfies the qualitative exterior Corkscrew condition. Then there is a uniform constant C such that for each surface ball Δ, and for every $X, Y \in \Omega \backslash 2\kappa_0 B_\Delta$, and $Z \in B_\Delta \cap \Omega$, we have*

$$\frac{1}{C} \frac{G(Z, X)}{G(Z, Y)} \leq \frac{G(X_\Delta, X)}{G(X_\Delta, Y)} \leq C \frac{G(Z, X)}{G(Z, Y)}.$$

REMARK 3.65. – By Lemma 3.61 and Lemma 3.62, every Carleson box $T_\Delta \subset \Omega$ is a 1-sided NTA domain with n-dimensional ADR boundary and also satisfies the qualitative exterior Corkscrew condition.

Proof. – We follow [31, Lemma 1.3.7]. Given a surface ball Δ, fix $X, Y \in \Omega \setminus 2\kappa_0 B_\Delta$, and let ω_Δ^Z denote harmonic measure for the sub-domain T_Δ. Set

$$S_1 := \partial T_\Delta \cap \{Z \in \Omega : \delta(Z) > r/2\},$$

where r is the radius of B_Δ. By Remark 3.65, Corollary 3.36 applies in T_Δ, whence by (3.60),

$$(3.66) \qquad \omega_\Delta^Z(\partial T_\Delta \cap \Omega) \leq C \omega_\Delta^Z(S_1), \qquad \forall Z \in B_\Delta \cap \Omega.$$

Now set $B^* := \kappa_0 B_\Delta$ and $\Delta^* := B^* \cap \partial\Omega$. By Lemma 3.37 and the Harnack chain condition we have

$$G(Z, X) \leq C\, G(X_{\Delta^*}, X) \lesssim G(X_\Delta, X), \qquad \forall Z \in B^* \cap \Omega.$$

By the maximum principle and (3.63), we then have

$$(3.67) \qquad G(Z, X) \leq C\, G(X_\Delta, X)\, \omega_\Delta^Z(\partial T_\Delta \cap \Omega), \qquad \forall Z \in T_\Delta.$$

On the other hand, by the Harnack chain condition,

$$G(Z, Y) \geq C^{-1} G(X_\Delta, Y), \qquad \forall Z \in S_1,$$

and therefore by the maximum principle we have

$$(3.68) \qquad G(Z, Y) \geq C^{-1} G(X_\Delta, Y)\, \omega_\Delta^Z(S_1), \qquad \forall Z \in T_\Delta.$$

Combining (3.66), (3.67) and (3.68), we obtain

$$\frac{G(Z, X)}{G(X_\Delta, X)} \lesssim \frac{G(Z, Y)}{G(X_\Delta, Y)}, \qquad \forall Z \in B_\Delta \cap \Omega.$$

The opposite inequality follows by interchanging the roles of X and Y. $\qquad\qquad\square$

COROLLARY 3.69. – *Given the same hypotheses as in Lemma 3.64, there is a uniform constant C such that for every pair of surface balls $\Delta := B \cap \partial\Omega$, and $\Delta' := B' \cap \partial\Omega$, with $B' \subseteq B$, and for every $X \in \Omega \setminus 2\kappa_0 B$, where κ_0 is the constant in (3.63), we have*

$$\frac{1}{C}\, \omega^{X_\Delta}(\Delta') \leq \frac{\omega^X(\Delta')}{\omega^X(\Delta)} \leq C\, \omega^{X_\Delta}(\Delta').$$

Proof. – We follow [31, Corollary 1.3.8]. Fix Δ', Δ, B', B and X as in the statement of the present corollary. Set $B^{**} = \kappa_1 B$ and $\Delta^{**} := B^{**} \cap \partial\Omega$, where we may choose κ_1 large enough, depending only on κ_0 and on the constants in the Corkscrew condition, such that $X_{\Delta^{**}} \in \Omega \setminus 2\kappa_0 B$. Let r' and r denote the respective radii of B' and B. By Lemma 3.30, we have

$$\omega^X(\Delta') \approx (r')^{n-1} G(X_{\Delta'}, X),$$

$$\omega^X(\Delta) \approx r^{n-1} G(X_\Delta, X),$$

$$\omega^{X_\Delta}(\Delta') \approx \omega^{X_{\Delta^{**}}}(\Delta')(r')^{n-1} G(X_{\Delta'}, X_{\Delta^{**}}),$$

where in the third line we have also used the Harnack chain condition. Moreover, by Lemma 3.11 and the Harnack chain condition, we have

$$r^{n-1} G(X_\Delta, X_{\Delta^{**}}) \approx 1.$$

Note that $X_{\Delta'} \subset B' \subset B$. Thus, by Lemma 3.64, we have

$$\frac{G(X_{\Delta'}, X)}{G(X_{\Delta'}, X_{\Delta^{**}})} \approx \frac{G(X_\Delta, X)}{G(X_\Delta, X_{\Delta^{**}})},$$

and the conclusion of the corollary follows.   □

4. Harnack chains imply a Poincaré inequality

In this section we prove that a certain Poincaré inequality holds in any domain Ω satisfying the ADR, Corkscrew and Harnack chain properties. We therefore impose those three hypotheses throughout this section. It will be convenient to set some additional notation. As above, we let $\mathcal{W}$ denote the collection of Whitney cubes of Ω, and we recall that these have been constructed so that for $I \in \mathcal{W}$, we have $\mathrm{dist}(4I, \partial\Omega) \approx \ell(I)$ (cf. (3.42)). Given a pairwise disjoint family $\mathcal{F} \in \mathbb{D}$, and a constant $\rho > 0$, we derive from $\mathcal{F}$ another family $\mathcal{F}(\rho) \subset \mathbb{D}$, as follows. We augment $\mathcal{F}$ by adjoining to it all those $Q \in \mathbb{D}$ of side length $\ell(Q) \leq \rho$, and we denote this augmented collection by $\mathfrak{C}(\mathcal{F}, \rho)$. We then let $\mathcal{F}(\rho)$ denote the collection of the maximal cubes of $\mathfrak{C}(\mathcal{F}, \rho)$. Thus, the corresponding discrete sawtooth $\mathbb{D}_{\mathcal{F}(\rho)}$ consists precisely of those $Q \in \mathbb{D}_\mathcal{F}$ such that $\ell(Q) > \rho$.

Having constructed the family $\mathcal{F}(\rho)$, and given $Q \in \mathbb{D}$ with $\ell(Q) > \rho$, we may then define local discrete and geometric sawtooth regions $\mathbb{D}_{\mathcal{F}(\rho),Q}$ and $\Omega_{\mathcal{F}(\rho),Q}$ with respect to this family as in (3.41)-(3.47) and (3.54).

We shall also find it useful to consider certain "fattened" versions of the sawtooth regions, as follows. Bearing in mind (3.42), we set

$$(4.1) \qquad U_Q^{\text{fat}} := \bigcup_{\mathcal{W}_Q^*} 4I,$$

$$(4.2) \qquad T_Q^{\text{fat}} := \text{int} \left(\bigcup_{Q' \in \mathbb{D}_Q} U_{Q'}^{\text{fat}} \right),$$

$$(4.3) \qquad \Omega_{\mathcal{F},Q}^{\text{fat}} := \text{int} \left(\bigcup_{Q' \in \mathbb{D}_{\mathcal{F},Q}} U_{Q'}^{\text{fat}} \right)$$

(compare to (3.47), (3.52) and (3.54)). We note that, by construction,

$$(4.4) \qquad \delta(X) \gtrsim \rho, \quad \text{if } X \in \Omega_{\mathcal{F}(\rho),Q}^{\text{fat}}$$

$$(4.5) \qquad \delta(X) \lesssim \rho, \quad \text{if } X \in \Omega_{\mathcal{F},Q} \setminus \Omega_{\mathcal{F}(\rho),Q}.$$

Given a pairwise disjoint family $\mathcal{F} \in \mathbb{D}$, and a cube $Q \in \mathbb{D}_{\mathcal{F}}$, we define

$$(4.6) \qquad \mathcal{W}_{\mathcal{F}} := \bigcup_{Q' \in \mathbb{D}_{\mathcal{F}}} \mathcal{W}_{Q'}^*, \qquad \mathcal{W}_{\mathcal{F},Q} := \bigcup_{Q' \in \mathbb{D}_{\mathcal{F},Q}} \mathcal{W}_{Q'}^*,$$

so that in particular, we may write

$$(4.7) \qquad \Omega_{\mathcal{F},Q} = \text{int} \Big(\bigcup_{I \in \mathcal{W}_{\mathcal{F},Q}} I^* \Big), \qquad \Omega_{\mathcal{F},Q}^{\text{fat}} = \text{int} \Big(\bigcup_{I \in \mathcal{W}_{\mathcal{F},Q}} 4I \Big),$$

where we recall that $I^* := (1 + \lambda)I$.

Suppose now that $Q \in \mathbb{D}_{\mathcal{F}}$, let $r \approx \ell(Q)$ and fix a small $\varepsilon > 0$. Then for $I, J \in \mathcal{W}_{\mathcal{F}(\varepsilon r),Q}$, we have $\ell(I) \approx \ell(J)$ and $\text{dist}(I, J) \lesssim \ell(I)$ (where the implicit constants depend upon ε). By Lemma 3.61 there is a chain $\{I_1, I_2, \ldots, I_N\} \subset \mathcal{W}_{\mathcal{F}(\varepsilon r),Q}$, of bounded cardinality N depending only on dimension, the Harnack chain constants, and ε, such that $I_1 = J$, $I_N = I$, $\ell(I_j) \approx \ell(I)$ for each j (again the implicit constants depend upon ε), and for which $\bigcup_{j=1}^N I_j^*$ contains a Harnack chain which connects the centers of I and J. Moreover, for λ chosen small enough, the chain may be constructed so that for each j, $1 \leq j \leq N - 1$, either $I_j^* \subset 4I_{j+1}$, or $I_{j+1}^* \subset 4I_j$. In the sequel, we shall refer to such a chain $\{I_1, I_2, \ldots, I_N\}$ as a "Harnack chain of Whitney cubes connecting J to I" (we beg the reader's indulgence for this mild abuse of terminology: it is of course really the dilates $\{I_j^*\}$ which form a Harnack chain).

LEMMA 4.8. – *Suppose that Ω is a 1-sided NTA domain with ADR boundary. Fix $Q_0 \in \mathbb{D}$, and a pairwise disjoint family $\mathcal{F} \subset \mathbb{D}_{Q_0}$, and let $Q \in \mathbb{D}_{\mathcal{F},Q_0}$. If $r \approx \ell(Q)$, then for every p, $1 \leq p < \infty$, and for every small $\varepsilon > 0$, there is a constant $C_{\varepsilon,p}$ such that*

$$\iint_{\Omega_{\mathcal{F}(\varepsilon r),Q}} |f - c_{Q,\varepsilon}|^p \leq C_{\varepsilon,p} \, r^p \iint_{\Omega_{\mathcal{F}(\varepsilon r),Q}^{\text{fat}}} |\nabla f|^p,$$

where $c_{Q,\varepsilon} := |\Omega_{\mathcal{F}(\varepsilon r),Q}|^{-1} \iint_{\Omega_{\mathcal{F}(\varepsilon r),Q}} f$.

Proof. – Let $X \in \Omega_{\mathscr{F}(\varepsilon r),Q}$, so that in particular, $X \in I_X^*$ where $I_X \in \mathcal{W}_{\mathscr{F}(\varepsilon r),Q}$, and observe that

$$
\left| f(X) - \frac{1}{|\Omega_{\mathscr{F}(\varepsilon r),Q}|} \iint_{\Omega_{\mathscr{F}(\varepsilon r),Q}} f \right| = \left| \frac{1}{|\Omega_{\mathscr{F}(\varepsilon r),Q}|} \iint_{\Omega_{\mathscr{F}(\varepsilon r),Q}} \Big(f(X) - f(Y) \Big) \, dY \right|
$$

$$
\leq \frac{1}{|\Omega_{\mathscr{F}(\varepsilon r),Q}|} \sum_{I \in \mathcal{W}_{\mathscr{F}(\varepsilon r),Q}} \iint_{I^*} \big| f(X) - f_{I_1^*} + f_{I_1^*} - \cdots + f_{I_N^*} - f(Y) \big| \, dY,
$$

where $I_1, \ldots, I_N$ is a Harnack chain of Whitney cubes connecting I_X to I and $f_{I_j^*} := |I_j^*|^{-1} \iint_{I_j^*} f$. Of course, N depends upon ε. It also depends upon I in the sum, but in a uniformly bounded manner for ε fixed. Consequently, it is enough to consider

$$
\iint_{J^*} \left(\frac{1}{|\Omega_{\mathscr{F}(\varepsilon r),Q}|} \iint_{I^*} \big| f(X) - f_{I_1^*} + f_{I_1^*} - \cdots + f_{I_N^*} - f(Y) \big| \, dY \right)^p dX,
$$

where $J, I \in \mathcal{W}_{\mathscr{F}(\varepsilon r),Q}$ and are connected by the chain of cubes $I_1, I_2, \ldots, I_N$. The desired bound may now be obtained from the standard Poincaré inequality as follows. First,

$$
\iint_{J^*} \left(\frac{1}{|\Omega_{\mathscr{F}(\varepsilon r),Q}|} \iint_{I^*} \big| f(X) - f_{I_1^*} \big| \, dY \right)^p dX
$$

$$
\leq \iint_{J^*} \big| f(X) - f_{I_1^*} \big|^p \, dX \leq C_p \, \ell(J)^p \iint_{J^*} |\nabla f|^p,
$$

since $I_1 = J$. Similarly, the contribution of $f_{I_N^*} - f(Y)$ is bounded by

$$
\iint_{I^*} \big| f_{I_N^*} - f(Y) \big|^p \, dY \leq C_p \, \ell(I)^p \iint_{I^*} |\nabla f|^p,
$$

since $I_N = I$. Finally, to handle the contribution of any term $f_{I_j^*} - f_{I_{j+1}^*}$, we observe that

$$
\left| f_{I_j^*} - f_{I_{j+1}^*} \right| \leq \left| f_{I_j^*} - f_{I_j^{**}} \right| + \left| f_{I_j^{**}} - f_{I_{j+1}^*} \right|,
$$

where $I_j^{**} := 4I_j$ or $4I_{j+1}$, whichever has the larger diameter. Then for example,

$$
\left| f_{I_j^*} - f_{I_j^{**}} \right| \leq \frac{1}{|I_j^*|} \iint_{I_j^*} |f - f_{I_j^{**}}| \lesssim \ell(I_j) \frac{1}{|I_j|} \iint_{I_j^{**}} |\nabla f|,
$$

and similarly for the term $\left| f_{I_j^{**}} - f_{I_{j+1}^*} \right|$ since, as noted above, I_j^{**} contains both I_j^* and I_{j+1}^*. The Poincaré inequality now follows, since each $I \in \mathcal{W}_{\mathscr{F}(\varepsilon r),Q}$, and thus every I_j in any of the chains, has side length proportional to r, depending on ε. $\qquad\square$

5. A criterion for exterior Corkscrew points

We present a criterion for the existence of Corkscrew points in the domain *exterior* to a sawtooth region. We begin with a series of lemmas in which we establish some local estimates for the single layer potential operator $\mathcal{S}$ defined in (1.11), and also prove some geometric properties of sawtooth regions and of domains with ADR boundaries.

LEMMA 5.1. – *Suppose that $E \subset \mathbb{R}^{n+1}$ is n-dimensional ADR, and let $\kappa > 1$. If $1 \le q < (n+1)/n$, there is a constant $C_{q,\kappa}$ depending only on n, q, κ and the ADR constants such that for every $x \in E$, $B := B(x,r)$ and $\kappa\Delta := \kappa B \cap E$, we have*

$$(5.2) \qquad \iint_B |\nabla\!\!\!/\, 1_{\kappa\Delta}(X)|^q \, dX \;\le\; C_{q,\kappa}\, r^{n+1}.$$

Proof. – The left hand side of (5.2) is crudely dominated by a constant times

$$\iint_B \left| \int_{\kappa\Delta} \frac{1}{|X-y|^n}\, dH^n(y) \right|^q dX \le H^n(\kappa\Delta)^{q-1} \iint_B \left(\int_{\kappa\Delta} \frac{1}{|X-y|^{nq}}\, dH^n(y) \right) dX$$

$$\lesssim r^{n(q-1)} \int_{\kappa\Delta} \left(\iint_{|X-y|<(\kappa+1)r} \frac{1}{|X-y|^{nq}}\, dX \right) dH^n(y)$$

$$\approx r^{n(q-1)}\, r^n\, r^{n+1-nq} \;=\; r^{n+1},$$

where of course the implicit constants depend on κ and q. $\square$

LEMMA 5.3. – *Suppose that $E \subset \mathbb{R}^{n+1}$ is n-dimensional ADR. For $\rho > 0$, define the "boundary strip" $\Sigma_\rho := \{X \in \mathbb{R}^{n+1} \setminus E : \mathrm{dist}(X,E) < \rho\}$. Then there is a uniform constant C such that for every ball $B := B(x,r)$ centered on E, and for $\rho \le r$, we have*

$$(5.4) \qquad\qquad |\Sigma_\rho \cap B| \le C\rho r^n.$$

Proof. – Let $\mathcal{W}_E$ denote the collection of cubes in the Whitney decomposition of $\mathbb{R}^{n+1} \setminus E$, and for each $k \in \mathbb{Z}$, set $\mathcal{W}_k := \{I \in \mathcal{W}_E : \ell(I) = 2^{-k}\}$. For each $I \in \mathcal{W}_E$, choose $Q_I \in \mathbb{D}(E)$ such that $\ell(Q_I) = \ell(I)$, and $\mathrm{dist}(I, Q_I) = \mathrm{dist}(I, E) \approx \ell(I)$. By ADR, for each I there are at most a bounded number of $Q \in \mathbb{D}(E)$ having these properties, and we just pick one. We note that if $I \cap B$ is non-empty, and if $\ell(I) \lesssim \rho \le r$, then $Q_I \subset \kappa_1 B$ for some uniform constant κ_1. Moreover, the collection $\{Q_I\}_{I \in \mathcal{W}_k}$ has bounded overlaps for each fixed k. We then have

$$|\Sigma_\rho \cap B| \le \sum_{k: 2^{-k} \lesssim \rho}\ \sum_{I \in \mathcal{W}_k} |I \cap B|$$

$$(5.5) \qquad \approx \sum_{k: 2^{-k} \lesssim \rho}\ \sum_{I \in \mathcal{W}_k} |I \cap B|\, \ell(I)^{-n}\, H^n(Q_I) \qquad \text{(by ADR)}$$

$$\lesssim \sum_{k: 2^{-k} \lesssim \rho} 2^{-k} \sum_{I \in \mathcal{W}_k:\, Q_I \subset \kappa_1 B} H^n(Q_I) \;\lesssim\; \rho r^n,$$

where in the last step we have used the bounded overlap property of the Q_I's. $\square$

COROLLARY 5.6. – *Let $0 < \gamma < 1/(n+1)$, and suppose that $E \subset \mathbb{R}^{n+1}$ is n-dimensional ADR. Then there is a uniform constant $C_{\gamma,\kappa}$ such that for every ball $B := B(x,r)$ centered on E, $\kappa\Delta := \kappa B \cap E$, and every ρ with $0 < \rho \le r$, we have*

$$\iint_{\Sigma_\rho \cap B} |\nabla\!\!\!/\, 1_{\kappa\Delta}(X)|\, dX \le C_{\gamma,\kappa}\, \rho^\gamma r^{n+1-\gamma}.$$

Proof. – The corollary follows immediately from Hölder's inequality and the previous two lemmata. We omit the routine details. $\square$

LEMMA 5.7. – *Suppose that $\partial\Omega$ is ADR, and let $B := B(x,r)$, $\Delta := B \cap \partial\Omega$, with $r \leq \operatorname{diam}(\partial\Omega)$, and $x \in \partial\Omega$. If*

$$(5.8) \qquad\qquad |B \cap (\mathbb{R}^{n+1} \setminus \overline{\Omega})| \geq ar^{n+1},$$

for some $a > 0$, then there is a point $X_\Delta^- \in B \setminus \overline{\Omega}$, and a constant c_1 depending only on a,n and the ADR constants such that

$$B(X_\Delta^-, c_1 r) \subset \mathbb{R}^{n+1} \setminus \overline{\Omega}.$$

Proof. – For notational convenience, we set

$$B^- := B \cap (\mathbb{R}^{n+1} \setminus \overline{\Omega}).$$

We apply Lemma 5.3, with $E = \partial\Omega$, and $\rho = ar/(2C)$ (notice that without loss of generality we may assume that $a \leq 1$, $C \geq 1$), and (5.8) to deduce that

$$\left| B^- \setminus \Sigma_{ar/(2C)} \right| \geq \frac{1}{2} ar^{n+1}.$$

In particular $B^- \setminus \Sigma_{ar/(2C)}$ is non-empty. Moreover, by definition of Σ_ρ, we have that $\operatorname{dist}(X, \partial\Omega) \geq ar/(2C)$, for every $X \in B^- \setminus \Sigma_{ar/(2C)}$. Therefore, any such X may be taken as the point X_Δ^-, with $c_1 := a/(4C)$. $\qquad\square$

REMARK. – Given a domain Ω, we shall henceforth refer to a Corkscrew point for the domain $\mathbb{R}^{n+1} \setminus \overline{\Omega}$, such as the point X_Δ^- in the lemma, as an "exterior Corkscrew point".

LEMMA 5.9. – *Suppose that Ω is a 1-sided NTA domain with ADR boundary. Let $\mathscr{F} \subset \mathbb{D}$ be a pairwise disjoint family. Then for every $Q \subseteq Q_j \in \mathscr{F}$, there is a ball $B' \subset \mathbb{R}^{n+1} \setminus \Omega_\mathscr{F}$, centered at $\partial\Omega$, with radius $r' \approx \ell(Q)/K_0$, and $\Delta' := B' \cap \partial\Omega \subset Q$.*

Proof. – Recall that there exist $B_Q := B(x_Q, r)$ and $\Delta_Q := B_Q \cap \partial\Omega \subset Q$, as defined in (1.16) and (1.17), where $r \approx \ell(Q)$. We now set

$$B' = B\left(x_Q, (MK_0)^{-1} r\right),$$

where M is a sufficiently large number to be chosen momentarily. We need only verify that $B' \cap \Omega_\mathscr{F} = \varnothing$. Suppose not. Then by definition of $\Omega_\mathscr{F}$, there is a Whitney cube $I \in \mathscr{W}_\mathscr{F}$ (cf. (4.6)) such that I^* meets B'. Since I^* meets B', there is a point $Y_I \in I^*$ such that

$$\ell(I) \approx \operatorname{dist}(I^*, \partial\Omega) \leq |Y_I - x_Q| \leq r/(MK_0) \approx \ell(Q)/(MK_0).$$

On the other hand, since $I \in \mathscr{W}_\mathscr{F}$, there is a $Q_I \in \mathbb{D}_\mathscr{F}$ (hence Q_I is not contained in Q_j) with $\ell(I) \approx \ell(Q_I)$, and $\operatorname{dist}(Q_I, Y_I) \approx \operatorname{dist}(Q_I, I) \lesssim K_0 \ell(I) \lesssim \ell(Q)/M$. Then by the triangle inequality,

$$|y - x_Q| \lesssim \ell(Q)/M, \qquad \forall y \in Q_I.$$

Thus, if M is chosen large enough, $Q_I \subset \Delta_Q \subset Q \subset Q_j$, a contradiction. $\qquad\square$

We now come to the main lemma of this section.

LEMMA 5.10. – *Suppose that Ω is a 1-sided NTA domain with ADR boundary. Fix $Q_0 \in \mathbb{D}$, and a pairwise disjoint family $\mathscr{F} \subset \mathbb{D}_{Q_0}$, and let $\Omega_{\mathscr{F},Q_0}$ be the corresponding sawtooth domain. Suppose also that for some $\eta > 0$, we have*

$$(5.11) \qquad \sup_{Q \in \mathbb{D}_{Q_0}} \frac{1}{\sigma(Q)} \iint_{\Omega_{\mathscr{F},Q}^{\mathrm{fat}}} |\nabla^2 \mathscr{J} 1(X)|^2 \delta(X)\, dX \; \le \; \eta.$$

If $\eta \le \eta_0$ with η_0 small enough, depending only on n, K_0, and the Corkscrew, Harnack chain and ADR constants for Ω, then for every $B := B(x,r)$ and $\Delta_\star := B \cap \partial\Omega_{\mathscr{F},Q_0}$, with $x \in \partial\Omega_{\mathscr{F},Q_0}$ and $r \le \mathrm{diam}(Q_0)$, there is an exterior Corkscrew point $X_{\Delta_\star} \in B \cap (\mathbb{R}^{n+1} \setminus \Omega_{\mathscr{F},Q_0})$. Moreover, the exterior Corkscrew constants depend only upon η_0, K_0, and the other parameters stated above.

To avoid confusion, we note that, as usual, $\delta(X) := \mathrm{dist}(X, \partial\Omega)$, and $\Delta = B \cap \Omega$ denotes a surface ball on $\partial\Omega$; we shall use the notation $\delta_\star(X) := \mathrm{dist}(X, \partial\Omega_{\mathscr{F},Q_0})$, and $\Delta_\star := B \cap \partial\Omega_{\mathscr{F},Q_0}$.

Proof. – We fix $x \in \partial\Omega_{\mathscr{F},Q_0}$, and consider two separate cases. Let M be a sufficiently large constant, to be chosen, whose value will remain fixed throughout the proof of the present lemma.

CASE 1: $\mathrm{dist}(x, \partial\Omega) > r/(MK_0)$.

In this case, $x \in \partial I^* \cap J$, where as usual $I^* = (1 + \lambda)I$ and $I \in \mathscr{W}_{\mathscr{F},Q_0}$ (cf. (4.6)), and where $J \in \mathscr{W}$ with $\tau J \subset \mathbb{R}^{n+1} \setminus \Omega_{\mathscr{F},Q_0}$, for some $\tau \in (1/2, 1)$ (cf. (3.51)). By the nature of Whitney cubes, we have $\ell(I) \approx \ell(J) \approx \mathrm{dist}(x, \partial\Omega) > r/(MK_0)$. In this case, it is evident that there is a Corkscrew point in J, with $c \approx (MK_0)^{-1}$.

CASE 2: $\mathrm{dist}(x, \partial\Omega) \le r/(MK_0)$.

In this case, either $x \in \partial\Omega \cap \partial\Omega_{\mathscr{F},Q_0}$, or else x lies on a face of I^* for some Whitney cube $I \in \mathscr{W}_{\mathscr{F},Q_0}$, with $\ell(I) \lesssim r/(MK_0)$. In the former scenario, by Proposition 6.1 below, we may choose $Q \in \mathbb{D}_{Q_0}$, with $x \in \overline{Q} \subset B$, and $\ell(Q) \approx r$. If $Q \subseteq Q_j$, for some $Q_j \in \mathscr{F}$ (which might happen if $x \in \partial Q_j$), then by Lemma 5.9 we immediately obtain the existence of the desired exterior Corkscrew point for $\Omega_{\mathscr{F},Q_0}$, at the scale r. Thus, in this scenario, it is enough to suppose that Q is not contained in any $Q_j \in \mathscr{F}$.

Otherwise, if $x \in \partial I^*$ for some $I \in \mathscr{W}_{\mathscr{F},Q_0}$, with $\ell(I) \lesssim r/(MK_0)$, then there is a $Q_I \in \mathbb{D}_{\mathscr{F},Q_0}$ such that $\ell(Q_I) \approx \ell(I)$, and $\mathrm{dist}(Q_I, I) \lesssim K_0\, \ell(I) \lesssim r/M$. Consequently, we have $\mathrm{dist}(I, Q) \lesssim r/M$ for any $Q \in \mathbb{D}$ with $Q_I \subseteq Q \subseteq Q_0$. Choosing M large enough, we may then fix such a Q with $\ell(Q) \approx r$, and $Q \subset B$. If Q is contained in some $Q_j \in \mathscr{F}$, then by Lemma 5.9, we again obtain the existence of an exterior Corkscrew point exactly as before.

Therefore, in either scenario, we have reduced matters to the following situation: there is a $Q \in \mathbb{D}_{\mathscr{F},Q_0}$ (i.e., not contained in any $Q_j \in \mathscr{F}$), with $\ell(Q) \approx r$, and $Q \subset B$. Having fixed this Q, we recall that, by Lemma 3.55, there is a ball $B'_Q := B(x_Q, s)$, with radius $s \approx (K_0)^{-1}\ell(Q)$, such that (3.57) holds.

By Lemma 3.61, the sawtooth domain $\Omega_{\mathscr{F},Q_0}$ inherits the 1-sided NTA (i.e., interior Corkscrew and Harnack chain) and ADR properties from Ω. Thus, by Lemma 5.7, applied with $\Omega_{\mathscr{F},Q_0}$ in place of Ω, and B'_Q in place of B, it is enough to establish the analogue of (5.8) with a depending only on the allowable parameters.

To this end, we proceed by a variant of the argument in [19, pp. 254–256]. We remind the reader of the definition of the family $\mathscr{F}(\rho)$ (see the discussion at the beginning of Section 4), and we also note that, by construction, there is a purely dimensional constant C_n such that

$$(5.12) \qquad\qquad T_Q^{\mathrm{fat}} \subset B(x_Q, C_n K_0 \, \ell(Q)) =: B_Q^*.$$

Set $\Delta_Q^* := B_Q^* \cap \partial\Omega$ and $\Delta_Q' := B_Q' \cap \partial\Omega$, and let $\Phi \in C_0^\infty(B_Q')$, with $\Phi \equiv 1$ on $B(x_Q, s/2)$, $0 \leq \Phi \leq 1$, and $\|\nabla\Phi\|_\infty \lesssim s^{-1}$. Let $\mathscr{L} := \nabla\cdot\nabla$ denote the usual Laplacian in $\mathbb{R}^{n+1}$. By the ADR property, and the fact that $s \approx \ell(Q) \approx r$, we have

(5.13)

$$
\begin{aligned}
r^{n+1} &\approx r\,\sigma\left(\tfrac{1}{2}\Delta_Q'\right) \leq r\int_{\partial\Omega} \Phi\,d\sigma = r\left\langle -\mathscr{L}\,\mathcal{S}1, \Phi\right\rangle \\
&= r \iint_{\mathbb{R}^{n+1}} \left(\nabla\mathcal{S}1(X) - \nabla\mathcal{S}1_{(2\Delta_Q^*)^c}(x_Q) - \vec{\alpha}\right)\cdot\nabla\Phi(X)\,dX \\
&\lesssim \iint_{B_Q'} \left|\nabla\mathcal{S}1(X) - \nabla\mathcal{S}1_{(2\Delta_Q^*)^c}(x_Q) - \vec{\alpha}\right| dX = \iint_{\Omega\cap B_Q'} + \iint_{\Omega_{\mathrm{ext}}\cap B_Q'} \\
&= \iint_{\Omega_{\mathscr{F}(\varepsilon r),Q}\cap B_Q'} + \iint_{\left(\Omega_{\mathscr{F},Q_0}\setminus\Omega_{\mathscr{F}(\varepsilon r),Q}\right)\cap B_Q'} + \iint_{\left(\Omega\setminus\Omega_{\mathscr{F},Q_0}\right)\cap B_Q'} + \iint_{\Omega_{\mathrm{ext}}\cap B_Q'} \\
&=: I + II + III + IV,
\end{aligned}
$$

where $\vec{\alpha}$ is a constant vector at out disposal, $\varepsilon > 0$ is a small number to be determined, and where as above, $\Omega_{\mathrm{ext}} := \mathbb{R}^{n+1}\setminus\overline{\Omega}$.

We now set

$$\vec{\alpha} := \frac{1}{|\Omega_{\mathscr{F}(\varepsilon r),Q}|} \iint_{\Omega_{\mathscr{F}(\varepsilon r),Q}} \left(\nabla\mathcal{S}1(X) - \nabla\mathcal{S}1_{(2\Delta_Q^*)^c}(x_Q)\right) dX.$$

We note for future reference that by standard Calderón-Zygmund estimates,

$$(5.14) \qquad \left|\nabla\mathcal{S}1_{(2\Delta_Q^*)^c}(X) - \nabla\mathcal{S}1_{(2\Delta_Q^*)^c}(x_Q)\right| \leq C, \qquad \forall X \in B_Q^*.$$

We also note that by Lemma 3.61, the sawtooth domain $\Omega_{\mathscr{F}(\varepsilon r),Q}$, if non-empty, must contain a Corkscrew point at the scale of $\ell(Q) \approx r$, so that, in particular,

$$r^{n+1} \lesssim |\Omega_{\mathscr{F}(\varepsilon r),Q}|.$$

Consequently, by (5.12) and the fact that $\Omega_{\mathscr{F},Q} \subset T_Q \subset T_Q^{\mathrm{fat}}$ for any pairwise disjoint family $\mathscr{F}$ and every $Q \in \mathbb{D}$, we have

$$(5.15) \qquad |\vec{\alpha}| \leq \frac{C_{K_0}}{|B_Q^*|} \iint_{B_Q^*} \left|\nabla\mathcal{S}1_{2\Delta_Q^*}(X)\right| dX + C \leq C_{K_0},$$

where in the last step we have used Lemma 5.1.

By the Poincaré inequality (Lemma 4.8), (5.12), and (4.4) with $\rho = \varepsilon r$, we obtain

$$I \leq C_\varepsilon r \iint_{\Omega_{\mathscr{G}(\varepsilon r), Q}^{\text{fat}}} |\nabla^2 \mathscr{S}1(X)| \, dX$$

$$\leq C_{\varepsilon, K_0} \, r^{(n+3)/2} \left(\iint_{\Omega_{\mathscr{G}(\varepsilon r), Q}^{\text{fat}}} |\nabla^2 \mathscr{S}1(X)|^2 \, dX \right)^{1/2}$$

$$\leq C_{\varepsilon, K_0} \, r^{(n+2)/2} \left(\iint_{\Omega_{\mathscr{G}(\varepsilon r), Q}^{\text{fat}}} |\nabla^2 \mathscr{S}1(X)|^2 \delta(X) \, dX \right)^{1/2}$$

$$\leq C_{\varepsilon, K_0} \sqrt{\eta} \, r^{n+1},$$

by Hypothesis (5.11), since $\ell(Q) \approx r$, and $\Omega_{\mathscr{G}(\varepsilon r), Q}^{\text{fat}} \subset \Omega_{\mathscr{G}, Q}^{\text{fat}}$.

Next, we claim that, for each $\gamma \in (0, 1/(n+1))$, we have

(5.16) $$II \leq C_{\gamma, K_0} \, \varepsilon^\gamma r^{n+1}.$$

We defer the proof of this claim momentarily, and observe that

$$III + IV = \iint_{\left(\mathbb{R}^{n+1} \setminus \overline{\Omega_{\mathscr{G}, Q_0}}\right) \cap B_Q'} \left| \nabla \mathscr{S}1(X) - \nabla \mathscr{S}1_{(2\Delta_Q^*)^c}(x_Q) - \vec{\alpha} \right| dX$$

(to avoid possible confusion, we point out that the boundaries of Ω and all of its sub-domains that we consider here, have $(n+1)$-dimensional Lebesgue measure equal to zero). Then by (5.14), (5.15), Hölder's inequality and Lemma 5.1, we deduce that for any $q \in (1, (n+1)/n)$,

$$III + IV \leq C_{K_0} \left(\left| \left(\mathbb{R}^{n+1} \setminus \overline{\Omega_{\mathscr{G}, Q_0}}\right) \cap B_Q' \right|^{1/q'} r^{(n+1)/q} + \left| \left(\mathbb{R}^{n+1} \setminus \overline{\Omega_{\mathscr{G}, Q_0}}\right) \cap B_Q' \right| \right).$$

Now, choosing first ε, and then η sufficiently small, we can hide $I + II$ on the left hand side of (5.13). Our estimate for $III + IV$ then implies that

$$r^{n+1} \leq C_{K_0} \left| \left(\mathbb{R}^{n+1} \setminus \overline{\Omega_{\mathscr{G}, Q_0}}\right) \cap B_Q' \right|.$$

As noted above, the existence of an exterior Corkscrew point now follows by applying Lemma 5.7, with B_Q' in place of B, and $\Omega_{\mathscr{G}, Q_0}$ in place of Ω.

To complete the proof of Lemma 5.10, it remains only to prove the claimed estimate (5.16). By (3.57), we may replace $\Omega_{\mathscr{G}, Q_0}$ by $\Omega_{\mathscr{G}, Q}$ in the domain of integration which defines II. Consequently, by (4.5) with $\rho = \varepsilon r$, we have that

$$II \leq \int_{\Sigma_{C \varepsilon r} \cap B_Q'} \left| \nabla \mathscr{S}1(X) - \nabla \mathscr{S}1_{(2\Delta_Q^*)^c}(x_Q) - \vec{\alpha} \right| dX$$

where $\Sigma_\rho := \{X \in \mathbb{R}^{n+1} : \delta(X) < \rho\}$. The desired bound now follows readily from (5.14), (5.15), Lemma 5.3 and Corollary 5.6. We omit the routine details. $\qquad\square$

We conclude this section with an estimate for harmonic measure in "good" sawtooth regions (that is, those for which (5.11) holds for sufficiently small η). Given a subdomain $\Omega' \subset \Omega$, we shall use the notational convention that $\omega_\star^X$ denotes harmonic measure for Ω' with pole at X, when there is no chance for confusion.

COROLLARY 5.17. – *Suppose that Ω is a 1-sided NTA domain with ADR boundary. Suppose also that (5.11) holds for some $Q_0 \in \mathbb{D}$, and some pairwise disjoint family $\mathcal{F} \subset \mathbb{D}_{Q_0}$, with $\eta \leq \eta_0$ (cf. Lemma 5.10). Let $\omega_\star^X$ denote harmonic measure for $\Omega_{\mathcal{F},Q_0}$ with pole at X. Then, for every $x \in \partial\Omega_{\mathcal{F},Q_0}$, every $r \leq \mathrm{diam}(Q_0)$, and every surface ball $\Delta_\star = \Delta_\star(x,r)$, the harmonic measure $\omega_\star^{X_{\Delta_\star}}$ belongs to $A_\infty(\Delta_\star)$ (cf. Definition 1.19), with uniform A_∞ constants depending only upon dimension and the ADR, Harnack chain and Corkscrew constants, including K_0.*

Proof. – By Lemma 3.61, $\Omega_{\mathcal{F},Q_0}$ is a 1-sided NTA domain with ADR boundary. Moreover, by Lemma 5.10, it also satisfies an exterior Corkscrew condition. The conclusion of the corollary now follows immediately by [17, Theorem 2]. $\square$

6. $\mathcal{F}$-Projections and a Dahlberg-Jerison-Kenig "sawtooth lemma"

In this section, we present a dyadic version of the main lemma of [16]. Our approach here is modeled on an analogous result in the Euclidean case which appeared in our previous work [27] (see also [26]). As in [27], we shall utilize a certain projection operator adapted to a pairwise disjoint family $\mathcal{F}$.

Consider now such a family $\mathcal{F} = \{Q_j\} \subset \mathbb{D}$. The projection operator $\mathscr{P}_{\mathcal{F}}$ associated to $\mathcal{F}$ (the "$\mathcal{F}$-projection operator") is defined by:

$$\mathscr{P}_{\mathcal{F}} f(x) := f(x)\, 1_{\partial\Omega \setminus (\bigcup_{\mathcal{F}} Q_j)}(x) + \sum_{\mathcal{F}} \left(\fint_{Q_j} f\, d\sigma \right) 1_{Q_j}(x).$$

We may naturally extend $\mathscr{P}_{\mathcal{F}}$ to act on non-negative Borel measures on $\partial\Omega$. Suppose that μ is such a measure, and let $A \subset \partial\Omega$. We then define the measure $\mathscr{P}_{\mathcal{F}}\mu$ as follows:

$$\mathscr{P}_{\mathcal{F}}\mu(A) := \int_{\partial\Omega} \mathscr{P}_{\mathcal{F}}(1_A)\, d\mu = \mu(A \setminus \bigcup_{\mathcal{F}} Q_j) + \sum_{\mathcal{F}} \frac{\sigma(A \cap Q_j)}{\sigma(Q_j)}\, \mu(Q_j).$$

In particular, we have that $\mathscr{P}_{\mathcal{F}}\mu(Q) = \mu(Q)$, for every $Q \in \mathbb{D}_{\mathcal{F}}$ (i.e., for Q not contained in any $Q_j \in \mathcal{F}$), and also that $\mathscr{P}_{\mathcal{F}}\mu(Q_j) = \mu(Q_j)$ for every $Q_j \in \mathcal{F}$.

We shall prove a version of the main lemma in [16] which is valid for $\mathcal{F}$-projections of harmonic measure. Our proof follows the idea of the argument in [16], but is technically simpler (given certain geometric preliminaries), owing to the dyadic setting in which we work here. In more precise detail, we follow our earlier Euclidean version of this lemma, which appears in [27, Lemma A.1].

Let us set a bit of notation: given $Q_0 \in \mathbb{D}$, a pairwise disjoint family $\mathcal{F} \subset \mathbb{D}$, and the corresponding sawtooth domain $\Omega_{\mathcal{F},Q_0}$ (cf. (3.39)-(3.47) and (3.54); also (4.6) and (4.7)), we let $\Delta_\star$, $\delta_\star$, and $\omega_\star^X$ denote, respectively, a surface ball on $\partial\Omega_{\mathcal{F},Q_0}$, the distance to the boundary of $\partial\Omega_{\mathcal{F},Q_0}$, and harmonic measure for the domain $\Omega_{\mathcal{F},Q_0}$ with pole at X; i.e., for $x \in \partial\Omega_{\mathcal{F},Q_0}$, $\Delta_\star(x,r) = B(x,r) \cap \partial\Omega_{\mathcal{F},Q_0}$, and for $X \in \Omega_{\mathcal{F},Q_0}$, $\delta_\star(X) := \mathrm{dist}(X, \partial\Omega_{\mathcal{F},Q_0})$. We continue to use $\Delta = \Delta(x,r), \delta(X)$ and ω^X to denote the analogous objects in reference to the original domain Ω and its boundary.

Before stating our sawtooth lemma, let us record some useful geometric observations. We recall that by Lemma 3.61, the sawtooth domain $\Omega_{\mathcal{F},Q_0}$ inherits the 1-sided NTA (i.e., interior Corkscrew and Harnack chain) and ADR properties from Ω. We begin with the following.

PROPOSITION 6.1. – *Suppose that Ω is a 1-sided NTA domain with ADR boundary. Fix $Q_0 \in \mathbb{D}$, and let $\mathscr{F} \subset \mathbb{D}_{Q_0}$ be a disjoint family. Then*

$$(6.2) \qquad Q_0 \setminus \left(\bigcup_{\mathscr{F}} Q_j \right) \subset \partial\Omega \cap \partial\Omega_{\mathscr{F},Q_0} \subset \overline{Q_0} \setminus \left(\bigcup_{\mathscr{F}} \text{int}(Q_j) \right).$$

Proof. – We first prove the right hand containment. Suppose that $x \in \partial\Omega \cap \partial\Omega_{\mathscr{F},Q_0}$. Then there is a sequence $X^k \in \Omega_{\mathscr{F},Q_0}$, with $X^k \to x$. By definition of $\Omega_{\mathscr{F},Q_0}$, each X^k is contained in I_k^* for some $I_k \in \mathscr{W}_{\mathscr{F},Q_0}$ (cf. (4.6)-(4.7)), so that $\ell(I_k) \approx \delta(X^k) \to 0$. Moreover, again by definition, each I_k belongs to some $\mathscr{W}_{Q^k}^*$, $Q^k \in \mathbb{D}_{\mathscr{F}}$ so that,

$$\text{dist}(Q^k, I_k) \lesssim K_0 \, \ell(Q^k) \approx K_0 \, \ell(I_k) \to 0.$$

Consequently, $\text{dist}(Q^k, x) \to 0$. Since each $Q^k \subset Q_0$, we have $x \in \overline{Q_0}$. On the other hand, if $x \in \text{int}(Q_j)$, for some $Q_j \in \mathscr{F}$, then there is an $\varepsilon > 0$ such that $\text{dist}(x, Q) > \varepsilon$ for every $Q \in \mathbb{D}_{\mathscr{F},Q_0}$ with $\ell(Q) \ll \varepsilon$, because no $Q \in \mathbb{D}_{\mathscr{F},Q_0}$ can be contained in any Q_j. Since this cannot happen if $\ell(Q^k) + \text{dist}(Q^k, x) \to 0$, the right hand containment is established.

Now suppose that $x \in Q_0 \setminus (\bigcup_{\mathscr{F}} Q_j)$. By definition, if $x \in Q \in \mathbb{D}_{Q_0}$, then $Q \in \mathbb{D}_{\mathscr{F},Q_0}$. Therefore, we may choose a sequence $\{Q^k\} \subset \mathbb{D}_{\mathscr{F},Q_0}$ shrinking to x, whence there exist $I_k \in \mathscr{W}_{Q^k}^* \subset \mathscr{W}_{\mathscr{F},Q_0}$ with $\text{dist}(I_k, x) \to 0$. The left hand containment now follows. $\square$

PROPOSITION 6.3. – *Suppose that $\partial\Omega$ is ADR, and that μ is a doubling measure on $\partial\Omega$; i.e., there is a uniform constant M_0 such that $\mu(2\Delta) \leq M_0 \, \mu(\Delta)$ for every surface ball Δ. Then $\partial Q := \overline{Q} \setminus \text{int}(Q)$ has μ-measure 0, for every $Q \in \mathbb{D}$. In particular, the sets in (6.2) have the same μ measure.*

Proof. – The argument is a refinement of that in [21, p. 403], where the Euclidean case was treated. Fix an integer k, a cube $Q \in \mathbb{D}_k$, and a positive integer m to be chosen. We set

$$\{Q_i^1\} := \mathbb{D}^1 := \mathbb{D}_Q \cap \mathbb{D}_{k+m},$$

and make the disjoint decomposition $Q = \cup Q_i^1$. We then split $\mathbb{D}^1 = \mathbb{D}^{1,1} \cup \mathbb{D}^{1,2}$, where $Q_i^1 \in \mathbb{D}^{1,1}$ if ∂Q_i^1 meets ∂Q, and $Q_i^1 \in \mathbb{D}^{1,2}$ otherwise. We then write $\overline{Q} = R^{1,1} \cup R^{1,2}$, where

$$R^{1,1} := \bigcup_{\mathbb{D}^{1,1}} \widehat{Q}_i^1, \qquad R^{1,2} := \bigcup_{\mathbb{D}^{1,2}} Q_i^1,$$

and for each cube $Q_i^1 \in \mathbb{D}^{1,1}$, we construct $\widehat{Q}_i^1$ as follows. We enumerate the elements in $\mathbb{D}^{1,1}$ as $Q_{i_1}^1, Q_{i_2}^1, \ldots, Q_{i_N}^1$, and then set $(Q_i^1)^* = Q_i^1 \cup (\partial Q_i^1 \cap \partial Q)$ and

$$\widehat{Q}_{i_1}^1 := (Q_{i_1}^1)^*, \quad \widehat{Q}_{i_2}^1 := (Q_{i_2}^1)^* \setminus (Q_{i_1}^1)^*, \quad \widehat{Q}_{i_3}^1 := (Q_{i_3}^1)^* \setminus ((Q_{i_1}^1)^* \cup (Q_{i_2}^1)^*), \ldots$$

so that $R^{1,1}$ covers ∂Q and the modified cubes $\widehat{Q}_i^1$ are pairwise disjoint.

We recall the surface ball $\Delta_Q = \Delta(x_Q, r) \subset Q$, with $r \approx \ell(Q)$ as in (1.16)-(1.17). Then

$$\text{dist}\left(\Delta(x_Q, r/2), \partial Q \right) \geq \frac{r}{2} \geq c_0 \, \ell(Q) = c_0 \, 2^{-k},$$

for some uniform constant c_0. By Lemma **??**, there is a uniform constant C_1 such that $\text{diam}(Q') \leq C_1 \ell(Q')$, for every $Q' \in \mathbb{D}$. We may therefore choose m depending only on the ADR constants and dimension so that $2^{-m} < c_0/C_1$, whence

$$\text{diam}(Q_i^1) \leq C_1 2^{-k-m} < c_0 \, 2^{-k}.$$

Consequently, $R^{1,1}$ misses $\Delta(x_Q, r/2)$, so that by the doubling property,

$$\mu(\overline{Q}) \leq C_{M_0} \mu(\Delta(x_Q, r/2) \leq C_{M_0} \mu(R^{1,2}).$$

Since $R^{1,1}$ and $R^{1,2}$ are disjoint, the latter estimate yields

$$\mu(R^{1,1}) \leq \left(1 - \frac{1}{C_{M_0}}\right) \mu(\overline{Q}) =: \theta \, \mu(\overline{Q}),$$

where we note that $\theta < 1$.

Let us now repeat this procedure, decomposing $\widehat{Q}_i^1$ for each $Q_i^1 \in \mathbb{D}^{1,1}$. We set $\mathbb{D}^2(Q_i^1) = \mathbb{D}_{Q_i^1} \cap \mathbb{D}_{k+2m}$ and split it into $\mathbb{D}^{2,1}(Q_i^1)$ and $\mathbb{D}^{2,2}(Q_i^1)$ where $Q' \in \mathbb{D}^{2,1}(Q_i^1)$ if $\partial Q'$ meets $\partial Q \cap \widehat{Q}_i^1$ (this set plays the role of ∂Q in the previous step). Associated to any $Q' \in \mathbb{D}^{2,1}(Q_i^1)$ we set $(Q')^* = (Q' \cap \widehat{Q}_i^1) \cup (\partial Q' \cap (\partial Q \cap \widehat{Q}_i^1))$. Then we make these sets disjoint as before and we have that $R^{2,1}(Q_i^1)$ is defined as the disjoint union of the corresponding $\widehat{Q'}$. Note that $\widehat{Q}_i^1 = R^{2,1}(Q_i^1) \cup R^{2,2}(Q_i^1)$ and this a disjoint union. As before, $R^{2,1}(Q_i^1)$ misses $(1/2)\Delta_{Q_i^1}$ so that by the doubling property

$$\mu(\widehat{Q}_i^1) \leq C_{M_0} \mu\left(\frac{1}{2}\Delta_{Q_i^1}\right) \leq C_{M_0} \mu(R^{2,2}(Q_i^1))$$

and then $\mu(R^{2,1}) \leq \theta \, \mu(\widehat{Q}_i^1)$. Next we set $R^{2,1}$ and $R^{2,2}$ as the union of the corresponding $R^{2,1}(Q_i^1)$ and $R^{2,2}(Q_i^1)$ with $Q_i^1 \in \mathbb{D}^{1,1}$. Then,

$$\mu(R^{2,1}) := \mu\Big(\bigcup_{Q_i^1 \in \mathbb{D}^{1,1}} R^{2,1}(Q_i^1) \Big) = \sum_{Q_i^1 \in \mathbb{D}^{1,1}} \mu\big(R^{2,1}(Q_i^1)\big)$$

$$\leq \theta \sum_{Q_i^1 \in \mathbb{D}^{1,1}} \mu(\widehat{Q}_i^1) = \theta \, \mu(R^{1,1}) \leq \theta^2 \, \mu(\overline{Q}).$$

A straightforward iteration argument now yields that $\mu(\partial Q) = 0$. We omit the details. $\qquad \square$

PROPOSITION 6.4. – *Suppose that Ω is a 1-sided NTA domain with ADR boundary. Fix $Q_0 \in \mathbb{D}$, and let $\mathcal{F} \subset \mathbb{D}_{Q_0}$ be a disjoint family. Then for each $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$, there is a radius $r_Q \approx K_0 \ell(Q)$, and a point $A_Q \in \Omega_{\mathcal{F}, Q_0}$ which serves as a Corkscrew point simultaneously for $\Omega_{\mathcal{F}, Q_0}$, with respect to the surface ball $\Delta_\star(y_Q, r_Q)$, for some $y_Q \in \partial\Omega_{\mathcal{F}, Q_0}$, and for Ω, with respect to each surface ball $\Delta(x, r_Q)$, for every $x \in Q$.*

Proof. – Let $Q \in \mathbb{D}_{\mathcal{F}, Q_0}$. Recall that by construction, $\mathcal{W}_Q$ is non-empty. It follows that there is an I for which $\ell(I) \approx \ell(Q)$ and $\mathrm{dist}(Q, I) \leq C_0 \ell(Q)$. Furthermore, $I \subset \Omega_{\mathcal{F}, Q_0}$, and $\mathrm{dist}(I, \partial\Omega_{\mathcal{F}, Q_0}) \leq \mathrm{dist}(I, \partial\Omega) \approx \ell(I)$. We let A_Q denote the center of this particular I, so that

(6.5) $$\ell(Q) \approx \mathrm{dist}(A_Q, \partial\Omega_{\mathcal{F}, Q_0}) \leq \mathrm{dist}(A_Q, Q) \lesssim C_0 \ell(Q).$$

Fix $y_Q \in \partial\Omega_{\mathcal{F}, Q_0}$ so that $\mathrm{dist}(A_Q, \partial\Omega_{\mathcal{F}, Q_0}) = |A_Q - y_Q|$. Then A_Q is the promised simultaneous Corkscrew point, for $r_Q \approx K_0 \ell(Q) \geq C_0 \ell(Q)$. $\qquad \square$

COROLLARY 6.6. – *The point A_{Q_0} is a Corkscrew point with respect to $\Delta_\star(x, r_{Q_0})$, for all $x \in \partial\Omega_{\mathcal{F}, Q_0}$, and for $\Delta(x, r_{Q_0})$, for all $x \in Q_0$, with $r_{Q_0} \approx K_0 \ell(Q_0)$.*

The proof is almost immediate, since $\mathrm{diam}(\Omega_{\mathcal{F}, Q_0}) \lesssim K_0 \ell(Q_0)$, and we omit it.

PROPOSITION 6.7. – *Suppose that Ω is a 1-sided NTA domain with ADR boundary. Fix $Q_0 \in \mathbb{D}$, and let $\mathscr{F} \subset \mathbb{D}_{Q_0}$ be a disjoint family. Then for each $Q_j \in \mathscr{F}$, there is an n-dimensional cube $P_j \subset \partial\Omega_{\mathscr{F},Q_0}$, which is contained in a face of I^*, for some $I \in \mathcal{W}$, and which satisfies*

$$(6.8) \qquad \ell(P_j) \approx \operatorname{dist}(P_j, Q_j) \approx \operatorname{dist}(P_j, \partial\Omega) \approx \ell(I) \approx \ell(Q_j),$$

where the uniform implicit constants are allowed to depend upon K_0.

Proof. – Fix $Q_j \in \mathscr{F}$. It follows from Lemma 5.9 (with $Q = Q_j$) and the Corkscrew condition that there is an $I_1 \in \mathcal{W}$ with $I_1 \subset \Omega \setminus \Omega_{\mathscr{F},Q_0}$, $\ell(I_1) \approx \ell(Q_j)/K_0$, and $\operatorname{dist}(I_1, Q_j) \lesssim \ell(Q_j)/K_0$. On the other hand, the dyadic parent $\widetilde{Q}_j$ of Q_j belongs to $\mathbb{D}_{\mathscr{F},Q_0}$, so there is an $I_2 \in \mathcal{W}^*_{\widetilde{Q}_j}$ with $I_2 \subset \Omega_{\mathscr{F},Q_0}$, $\ell(I_2) \approx \ell(Q_j)$, and $\operatorname{dist}(Q_j, I_2) \lesssim K_0\ell(Q_j)$. The Harnack chain (in Ω) connecting the centers of I_1 and I_2, then passes through $\partial\Omega_{\mathscr{F},Q_0}$, and maintains a distance to $\partial\Omega$ on the order of $\ell(Q_j)$. Consequently, there is an interface between some pair $I, J \in \mathcal{W}$, with $\operatorname{int}(I^*) \subset \Omega_{\mathscr{F},Q_0}$ and $J \notin \mathcal{W}^*_Q$, for any $Q \in \mathbb{D}_{\mathscr{F},Q_0}$ (so that $\tau J \subset \Omega \setminus \Omega_{\mathscr{F},Q_0}$ for some $\tau \in (1/2,1)$; cf. (3.51)), and

$$\operatorname{dist}(I, Q_j) \approx \operatorname{dist}(J, Q_j) \approx \ell(I) \approx \ell(J) \approx \ell(Q_j)$$

(here, some of the implicit constants may depend upon K_0). Of course, the interface between I and J is precisely one face of the smaller of these two cubes. Therefore, if λ is chosen small enough, then $\partial I^* \cap J$ contains an n-dimensional cube P_j with the stated properties. $\qquad\square$

REMARK 6.9. – It follows from the proof that if $P_j \cap P_k$ then $\ell(Q_j) \approx \ell(Q_k)$ since two adjacent Whitney cubes have comparable side length. Thus, $\operatorname{dist}(Q_j, Q_k) \lesssim \ell(Q_j)$ and therefore we have the bounded overlap property

$$\sum_j 1_{P_j}(x) \leq C,$$

with C depending on the ADR constants.

For future reference, we note that, under the assumptions of Proposition 6.7, if $x_j^\star$ denotes the center of P_j, then for an appropriate choice of $r_j \approx K_0\ell(Q_j)$, we have $P_j \subset \Delta_\star(x_j^\star, r_j)$ and

$$(6.10) \qquad \overline{T_{Q_j}} \subset B(x_j^\star, r_j),$$

since $\operatorname{diam}(T_{Q_j}) \lesssim K_0\ell(Q_j)$. Moreover, given $Q \in \mathbb{D}_{\mathscr{F},Q_0}$ and $r_Q \approx K_0\ell(Q)$ from Proposition 6.4, by choosing $\widehat{r}_Q \approx r_Q$ (with implicit constants depending on K_0) we may suppose that

$$(6.11) \qquad Q \cup \left(\bigcup_{Q_j \in \mathscr{F}: Q_j \subset Q} B(x_j^\star, r_j) \right) \subset B(y_Q, \widehat{r}_Q).$$

Here, y_Q is the center of $\Delta_\star(y_Q, r_Q) \subset \partial\Omega_{\mathscr{F},Q_0}$, appearing in Proposition 6.4. We omit the routine geometric argument.

We conclude this preamble with the following.

PROPOSITION 6.12. – *Suppose that Ω is a 1-sided NTA domain with ADR boundary. Fix $Q_0 \in \mathbb{D}$, and let $\mathscr{F} \subset \mathbb{D}_{Q_0}$ be a disjoint family. For $Q_j \in \mathscr{F}$, let $B(x_j^\star, r_j)$ be the ball, concentric with P_j, satisfying (6.10). Then for each $Q \in \mathbb{D}_{\mathscr{F}, Q_0}$, there is a surface ball*

$$\Delta_\star^Q := \Delta_\star(x_Q^\star, t_Q) \subset (Q \cap \partial\Omega_{\mathscr{F}, Q_0}) \cup \left(\bigcup_{Q_j \in \mathscr{F}: Q_j \subset Q} (B(x_j^\star, r_j) \cap \partial\Omega_{\mathscr{F}, Q_0}) \right),$$

with $t_Q \approx \ell(Q)$, $x_Q^\star \in \partial\Omega_{\mathscr{F}, Q_0}$, and $\operatorname{dist}(Q, \Delta_\star^Q) \lesssim \ell(Q)$, where the implicit constants may depend upon K_0.

Proof. – Suppose first that there is some $Q_{j_0} \subset Q$, for which $\ell(Q_{j_0}) \geq \ell(Q)/M$, where M is a sufficiently large number to be chosen. We then set $\Delta_\star^Q = \Delta_\star(x_{j_0}^\star, \ell(P_{j_0})/2)$, a surface ball contained in the cube P_{j_0} whose existence was established in Proposition 6.7.

Now suppose that $\ell(Q_j) < \ell(Q)/M$, for every $Q_j \subset Q$. By Lemma 3.55, there is a ball $B_Q' = B(x_Q, s)$ with $s \approx \ell(Q)$ and $B_Q' \cap \Omega \subset T_Q \subseteq T_{Q_0}$. In particular, B_Q' misses $\partial T_{Q_0} \setminus Q_0$. Moreover, $\Delta_Q' := B_Q' \cap \partial\Omega \subset Q$. Consider those $Q_j \subset Q$ which meet $\Delta(x_Q, s/(4\sqrt{M}))$. If there are no such Q_j, then we set $\Delta_\star^Q = \Delta(x_Q, s/(4\sqrt{M}))$, which in this case is contained in $Q \cap \partial\Omega_{\mathscr{F}, Q_0}$ by Proposition 6.1. On the other hand, suppose that there is some $Q_{j_0} \subset Q$ which meets $\Delta(x_Q, s/(4\sqrt{M}))$. Then for M large enough, depending on K_0, we have $P_{j_0} \subset B(x_Q, s/(2\sqrt{M}))$, and thus also

(6.13) $\Delta_\star^Q := \Delta_\star(x_{j_0}^\star, s/(2\sqrt{M})) \subset B(x_Q, s/\sqrt{M}) \subset B_Q',$

by the triangle inequality. Consequently, $\Delta_\star^Q$ misses $\partial T_{Q_0} \setminus Q_0$, and $\Delta_\star^Q \cap \partial\Omega \subset Q$, by the properties of B_Q'. Moreover, we claim that

(6.14) $\partial\Omega_{\mathscr{F}, Q_0} \subset (\partial T_{Q_0} \setminus Q_0) \cup (\partial\Omega_{\mathscr{F}, Q_0} \cap Q_0) \cup \left(\bigcup_{Q_j \in \mathscr{F}} (\partial\Omega_{\mathscr{F}, Q_0} \cap \overline{T_{Q_j}}) \right).$

Let us defer for the moment the proof of this claim. Given (6.14), by (6.10) and properties of dyadic cubes, it is enough to verify that if $\Delta_\star^Q$ meets $\overline{T_{Q_j}}$, for some $Q_j \in \mathscr{F}$, then Q_j meets Q. Suppose now that $\Delta_\star^Q$ meets $\overline{T_{Q_j}}$. By (6.13) and the definition of $\overline{T_{Q_j}}$, this means that there is a $Q' \subseteq Q_j$, and an $I \in \mathscr{W}_{Q'}^*$ such that I^* meets $B(x_Q, s/\sqrt{M})$. It follows that

$$\ell(Q') \approx \ell(I) \approx \operatorname{dist}(I^*, \partial\Omega) \leq \operatorname{dist}(I^*, x_Q) \leq s/\sqrt{M}.$$

Since $\operatorname{dist}(I^*, Q') \lesssim K_0 \ell(Q')$, by the triangle inequality we have

$$|y - x_Q| \lesssim K_0 s/\sqrt{M} < s, \qquad \forall y \in Q',$$

if $M \gg (K_0)^2$; i.e., $Q' \subset \Delta_Q' := \Delta(x_Q, s) \subset Q$, whence Q_j meets Q, as desired.

Finally, we establish (6.14). Let $X \in \partial\Omega_{\mathscr{F}, Q_0}$. There are two cases.

CASE 1: $\delta(X) = 0$. If $X \in Q_0$ we are done. Otherwise, since

$$\partial\Omega_{\mathscr{F}, Q_0} \subset \overline{\Omega_{\mathscr{F}, Q_0}} \subset \overline{T_{Q_0}},$$

it suffices to show that $X \notin T_{Q_0}$. But this is trivial, since $T_{Q_0} \subset \Omega$, and for $X \in \Omega$, we have that $\delta(X) > 0$.

CASE 2: $\delta(X) > 0$. If $X \in \overline{T_{Q_j}}$ for some j, we are done, so suppose that this never happens. As in Case 1, it is enough to show that $X \notin T_{Q_0}$, so suppose by way of contradiction that

$X \in T_{Q_0}$. Since T_{Q_0} is open, this means that there is a small number $\varepsilon_0 \ll \delta(X)$ such that the ball $B(X, \varepsilon) \subset T_{Q_0}$, whenever $\varepsilon \leq \varepsilon_0$. By definition of T_{Q_0}, and properties of Whitney cubes, there exist a uniformly bounded number of Whitney cubes, say, $I_1, \ldots, I_M$, such that

$$B(X, \varepsilon_0) \subset \bigcup_{k=1}^{M} I_k^*,$$

and for each $k \in [1, M]$, there is a $Q^k \in \mathbb{D}_{Q_0}$ with $I_k \in \mathcal{W}_{Q^k}^*$. It is possible that for a smaller ε, there may be a smaller collection of I_k's required to cover $B(X, \varepsilon)$, but these I_k's are of course always chosen from the original collection (i.e., the one for ε_0.) Observe that since $B(X, \varepsilon)$ is open, if $B(X, \varepsilon)$ meets I_k^*, then it meets $\text{int}(I_k^*)$. For a given ε, we may assume that the covering collection is "minimal" in the sense that $B(X, \varepsilon)$ meets $\text{int}(I_k^*)$ for each k, i.e., we remove those I_k^* which do not meet $B(X, \varepsilon)$.

We claim that there must be some $\varepsilon > 0$ and a corresponding "minimal" collection, with the property that each $Q^k \in \mathbb{D}_{\mathcal{F}, Q_0}$. Indeed, if not, then there is a sequence $\varepsilon_i \to 0$, and for each i, a $Q_j(i) \in \mathcal{F}$, and a $k(i) \in [1, M]$, such that $Q^{k(i)} \in \mathbb{D}_{Q_j(i)}$. Since there were only a bounded number of I_k's and thus also Q^k's, to start with, there must be some subsequence, again call it ε_i, such that $k(i) = constant$. But this means that there is a fixed $Q_j \in \mathcal{F}$, and a sequence ε_i such that $B(X, \varepsilon_i)$ meets T_{Q_j}, which contradicts our assumption that $X \notin \overline{T_{Q_j}}$ for any $Q_j \in \mathcal{F}$. This proves the claim.

We now choose ε as in the claim, and observe that we then have

$$B(X, \varepsilon) \subset \bigcup_{Q \in \mathbb{D}_{\mathcal{F}, Q_0}} \bigcup_{\mathcal{W}_Q^*} I^*,$$

i.e., that X is an *interior* point for the set $\bigcup_{Q \in \mathbb{D}_{\mathcal{F}, Q_0}} \bigcup_{\mathcal{W}_Q^*} I^*$. But by definition, this means that $X \in \Omega_{\mathcal{F}, Q_0}$, and since the latter set is open, this contradicts that $X \in \partial\Omega_{\mathcal{F}, Q_0}$. $\qquad\square$

LEMMA 6.15 (Dyadic sawtooth lemma for projections). – *Suppose that Ω is a 1-sided NTA domain with ADR boundary and that it also satisfies the qualitative exterior Corkscrew condition. Fix $Q_0 \in \mathbb{D}$, let $\mathcal{F} = \{Q_j\} \subset \mathbb{D}_{Q_0}$ be a family of pairwise disjoint dyadic cubes and let $\mathscr{P}_{\mathcal{F}}$ be the corresponding projection operator. We write $\omega = \omega^{X_0}$ and $\omega_\star = \omega_\star^{X_0}$ to denote the respective harmonic measures for the domains Ω and $\Omega_{\mathcal{F}, Q_0}$, with fixed pole at the Corkscrew point $X_0 := A_{Q_0}$ whose existence was noted in Proposition 6.4 and Corollary 6.6. Let $\nu = \nu^{X_0}$ be the measure defined by*

$$(6.16) \qquad \nu(F) = \omega_\star\left(F \setminus (\bigcup_{\mathcal{F}} Q_j)\right) + \sum_{Q_j \in \mathcal{F}} \frac{\omega(F \cap Q_j)}{\omega(Q_j)} \omega_\star(P_j), \qquad F \subset Q_0,$$

where P_j is the n-dimensional cube produced by Proposition 6.7. Then $\mathscr{P}_{\mathcal{F}}\nu$ depends only on $\omega_\star$ and not on ω. More precisely,

$$(6.17) \qquad \mathscr{P}_{\mathcal{F}}\nu(F) = \omega_\star\left(F \setminus (\bigcup_{\mathcal{F}} Q_j)\right) + \sum_{Q_j \in \mathcal{F}} \frac{\sigma(F \cap Q_j)}{\sigma(Q_j)} \omega_\star(P_j), \qquad F \subset Q_0.$$

Moreover, there exists $\theta > 0$ such that for all $Q \in \mathbb{D}_{Q_0}$ and $F \subset Q$, we have

$$(6.18) \qquad \left(\frac{\mathscr{P}_{\mathcal{F}}\omega(F)}{\mathscr{P}_{\mathcal{F}}\omega(Q)}\right)^\theta \lesssim \frac{\mathscr{P}_{\mathcal{F}}\nu(F)}{\mathscr{P}_{\mathcal{F}}\nu(Q)} \lesssim \frac{\mathscr{P}_{\mathcal{F}}\omega(F)}{\mathscr{P}_{\mathcal{F}}\omega(Q)}.$$

Proof. – We observe that (6.17) follows immediately from the definitions of $\mathscr{P}_{\mathscr{G}}$ and ν, as the reader may readily verify. We omit the details.

Our first main task is to establish the right hand side inequality in (6.18). Let us fix $Q \in \mathbb{D}_{Q_0}, F \subset Q$.

CASE 1: There exists $Q_j \in \mathscr{G}$ such that $Q \subset Q_j$. Note that by (6.17) we have

$$\frac{\mathscr{P}_{\mathscr{G}}\nu(F)}{\mathscr{P}_{\mathscr{G}}\nu(Q)} = \frac{\frac{\sigma(F \cap Q_j)}{\sigma(Q_j)}\,\omega_\star(P_j)}{\frac{\sigma(Q \cap Q_j)}{\sigma(Q_j)}\,\omega_\star(P_j)} = \frac{\sigma(F)}{\sigma(Q)} = \frac{\frac{\sigma(F \cap Q_j)}{\sigma(Q_j)}\,\omega(Q_j)}{\frac{\sigma(Q \cap Q_j)}{\sigma(Q_j)}\,\omega(Q_j)} = \frac{\mathscr{P}_{\mathscr{G}}\omega(F)}{\mathscr{P}_{\mathscr{G}}\omega(Q)}.$$

CASE 2: Q is not contained in any $Q_j \in \mathscr{G}$ (i.e., $Q \in \mathbb{D}_{\mathscr{G},Q_0}$). Notice that if $Q_j \in \mathscr{G}$ with $Q_j \cap Q \neq \varnothing$, then Q_j is *strictly* contained in Q. Let us note also that $\omega_\star$ satisfies the doubling property, by Lemma 3.61, Lemma 3.62 and Corollary 3.36. Set $E_0 = Q_0 \setminus (\bigcup_{\mathscr{G}} Q_j)$. Using (6.17) we observe that

$$\begin{aligned}
(6.19) \qquad \mathscr{P}_{\mathscr{G}}\nu(Q) &= \omega_\star(Q \cap E_0) + \sum_{Q_j \in \mathscr{G}, Q_j \subsetneq Q} \frac{\sigma(Q \cap Q_j)}{\sigma(Q_j)}\,\omega_\star(P_j) \\
&= \omega_\star(Q \cap E_0) + \sum_{Q_j \in \mathscr{G}, Q_j \subsetneq Q} \omega_\star(P_j) \\
&\gtrsim \omega_\star(Q \cap E_0) + \sum_{Q_j \in \mathscr{G}, Q_j \subsetneq Q} \omega_\star\left(B(x_j^\star, r_j) \cap \partial\Omega_{\mathscr{G},Q_0}\right) \\
&\geq \omega_\star(\Delta_\star^Q),
\end{aligned}$$

where in the third line we have used the doubling property of $\omega_\star$ (plus a subdivision and Harnack chain argument if $\ell(Q_j) \approx \ell(Q_0)$), and in the last line we have used Proposition 6.12, along with Propositions 6.1 and 6.3 and the doubling property to ignore the difference between $Q \setminus (\bigcup_{\mathscr{G}} Q_j)$ and $Q \cap \partial\Omega_{\mathscr{G},Q_0}$.

Let A_Q be as in Proposition 6.4. Then by Corollary 3.69 plus the doubling property and Harnack chain condition, and a differentiation argument, we have that for any Borel set $H \subset Q$,

$$(6.20) \qquad\qquad \omega^{A_Q}(H) \approx \frac{\omega^{X_0}(H)}{\omega^{X_0}(Q)} = \frac{\omega(H)}{\omega(Q)}.$$

The same occurs for $\omega_\star$ and $\omega_\star^{A_Q}$ and for any $H_\star \subset \Delta_\star(y_Q, \hat{r}_Q)$, (see (6.11) and Proposition 6.4). More precisely,

$$(6.21) \qquad \omega_\star^{A_Q}(H_\star) \approx \frac{\omega_\star^{X_0}(H_\star)}{\omega_\star^{X_0}(\Delta_\star(y_Q, \widehat{r}_Q))} = \frac{\omega_\star(H_\star)}{\omega_\star(\Delta_\star(y_Q, \widehat{r}_Q))} \approx \frac{\omega_\star(H_\star)}{\omega_\star\left(\Delta_\star^Q\right)},$$

where $\Delta_\star^Q$ is the surface ball in Proposition 6.12, and where the last step follows by the doubling property of $\omega_\star$, since $\mathrm{dist}(\Delta_\star^Q, \Delta_\star(y_Q, \widehat{r}_Q)) \lesssim \ell(Q)$, and the radius of each surface ball is comparable to $\ell(Q)$.

Using (6.19) and (6.21) (and (6.11)), we obtain

$$(6.22) \qquad \frac{\mathscr{P}_{\mathscr{G}}\nu(F)}{\mathscr{P}_{\mathscr{G}}\nu(Q)} \lesssim \frac{\omega_{\star}(F \cap E_0)}{\omega_{\star}(\Delta_{\star}^{Q})} + \sum_{Q_j \in \mathscr{F}, Q_j \subsetneq Q} \frac{\sigma(F \cap Q_j)}{\sigma(Q_j)} \frac{\omega_{\star}(P_j)}{\omega_{\star}(\Delta_{\star}^{Q})}$$

$$\approx \omega_{\star}^{A_Q}(F \cap E_0) + \sum_{Q_j \in \mathscr{F}, Q_j \subsetneq Q} \frac{\sigma(F \cap Q_j)}{\sigma(Q_j)} \omega_{\star}^{A_Q}(P_j).$$

We claim that the following estimates hold:

$$(6.23) \qquad \omega_{\star}^{A_Q}(F \cap E_0) \lesssim \omega^{A_Q}(F \cap E_0), \qquad \omega_{\star}^{A_Q}(P_j) \lesssim \omega^{A_Q}(Q_j).$$

Indeed, the first estimate follows immediately from the maximum principle, since $\Omega_{\mathscr{F},Q_0} \subset \Omega$, and $E_0 \subset \partial\Omega \cap \partial\Omega_{\mathscr{F},Q_0}$, by Proposition 6.1. To prove the second estimate, we observe that, again by the maximum principle, it suffices to show that $\omega^X(Q_j) \gtrsim 1$, for $X \in P_j$. But the latter bound follows immediately from (3.8) with $\Delta = \Delta_{Q_j}$ (cf. (1.16)-(1.17)), the Harnack chain condition and (6.8).

The bounds in (6.22), (6.23) and (6.20) imply

$$\frac{\mathscr{P}_{\mathscr{G}}\nu(F)}{\mathscr{P}_{\mathscr{G}}\nu(Q)} \lesssim \omega^{A_Q}(F \cap E_0) + \sum_{Q_j \in \mathscr{F}, Q_j \subsetneq Q} \frac{\sigma(F \cap Q_j)}{\sigma(Q_j)} \omega^{A_Q}(Q_j)$$

$$\approx \frac{\omega(F \cap E_0)}{\omega(Q)} + \sum_{Q_j \in \mathscr{F}, Q_j \subsetneq Q} \frac{\sigma(F \cap Q_j)}{\sigma(Q_j)} \frac{\omega(Q_j)}{\omega(Q)} = \frac{\mathscr{P}_{\mathscr{G}}\omega(F)}{\omega(Q)} = \frac{\mathscr{P}_{\mathscr{G}}\omega(F)}{\mathscr{P}_{\mathscr{G}}\omega(Q)},$$

where in the last equality we have used that $\mathscr{P}_{\mathscr{G}}\omega(Q) = \omega(Q)$. Thus, we have established the right hand inequality in (6.18). We may now obtain the left hand side of (6.18) by a direct application of Lemma B.7 (see Appendix B below), using the fact that $\mathscr{P}_{\mathscr{G}}\omega$ and $\mathscr{P}_{\mathscr{G}}\nu$ are dyadically doubling by Lemmas B.1 and B.2. $\qquad\square$

7. A discrete Corona decomposition

In this section we present a discretized version of the stopping time decomposition of a Carleson region appearing in [12, 3], [4, 27] and [26] (cf. [11, 36], [25]). We suppose that $\{\alpha_Q\}_{Q \in \mathbb{D}}$ is a sequence of non-negative numbers indexed on the dyadic "cubes", and for any collection $\mathbb{D}' \subset \mathbb{D}$, we define

$$\mathfrak{m}(\mathbb{D}') := \sum_{Q \in \mathbb{D}'} \alpha_Q.$$

For a fixed $Q_0 \in \mathbb{D}$, we say that $\mathfrak{m}$ is a "Carleson measure" on $\mathbb{D}_{Q_0}$ (with respect to σ), and we write $\mathfrak{m} \in \mathscr{C}(Q_0)$, if

$$\|\mathfrak{m}\|_{\mathscr{C}(Q_0)} := \sup_{Q \in \mathbb{D}_{Q_0}} \frac{\mathfrak{m}(\mathbb{D}_Q)}{\sigma(Q)} < \infty.$$

We also write

$$(7.1) \qquad \|\mathfrak{m}\|_{\mathscr{C}} := \sup_{Q \in \mathbb{D}} \frac{\mathfrak{m}(\mathbb{D}_Q)}{\sigma(Q)} < \infty$$

Q

$P(Q, Q') = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4}$

$\frac{1}{2}$
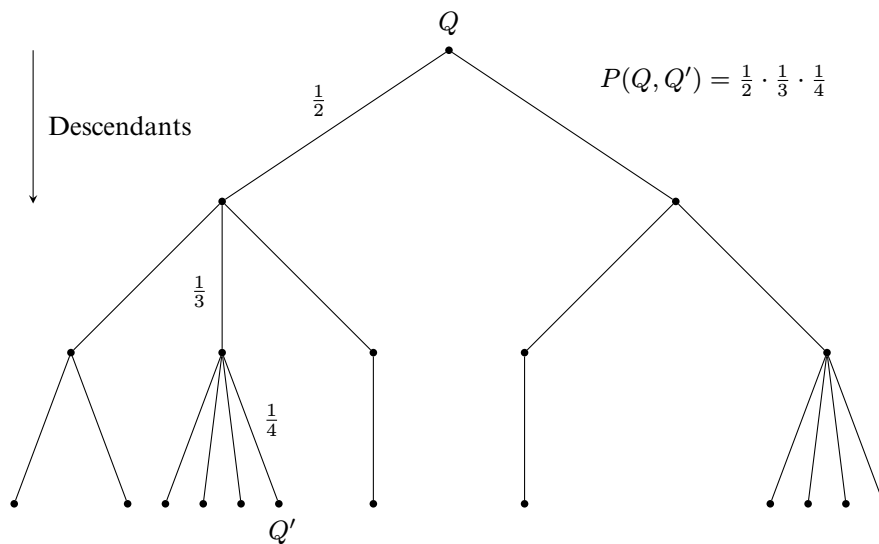
Descendants

$\frac{1}{3}$

$\frac{1}{4}$

Q'

FIGURE 1. "Tree-graph" with vertex Q and its random walk

to denote the "global" Carleson norm on $\mathbb{D}$. We furthermore set $\mathbb{D}_Q^{\text{short}} := \mathbb{D}_Q \setminus \{Q\}$, and given a family $\mathscr{F} \subset \mathbb{D}$ of pairwise disjoint cubes, we define the "restriction of $\mathfrak{m}$ to the sawtooth $\mathbb{D}_{\mathscr{F}}$" by

$$\mathfrak{m}_{\mathscr{F}}(\mathbb{D}') := \mathfrak{m}(\mathbb{D}' \cap \mathbb{D}_{\mathscr{F}}) = \sum_{Q \in \mathbb{D}' \setminus (\bigcup_{\mathscr{F}} \mathbb{D}_{Q_j})} \alpha_Q.$$

We fix $Q_0 \in \mathbb{D}$, and construct a "tree-graph" with a vertex for each $Q \in \mathbb{D}_{Q_0}$, and with edges connecting a given Q to each of its dyadic "children" (these are the subcubes of Q which lie in the very next dyadic generation $\mathbb{D}_{k(Q)+1}$). We consider a random walk along the graph, in which it is permitted to move only to the descendant generation, but not to the ancestral generation (nor to any other cube in the same generation), and we suppose that from a given $Q \in \mathbb{D}_{Q_0}$, there is an equal probability of arriving at any of its children. We set $P(Q, Q) = 1$, and in general for $Q' \subseteq Q \in \mathbb{D}_{Q_0}$, we denote by $P(Q, Q')$ the probability that such a random walk beginning at Q arrives at Q' (thus also if Q is strictly contained in Q', or if Q and Q' are disjoint, we have $P(Q, Q') = 0$). See Figure 1.

LEMMA 7.2. – *Suppose that $\partial\Omega$ is ADR. Fix $Q_0 \in \mathbb{D}$ and $\mathfrak{m}$ as above. Let $a \geq 0$ and $b > 0$, and suppose that $\mathfrak{m}(\mathbb{D}_{Q_0}) \leq (a+b)\,\sigma(Q_0)$. Then there is a family $\mathscr{F} = \{Q_j\} \subset \mathbb{D}_{Q_0}$ of pairwise disjoint cubes, and a constant C depending only on dimension and the ADR constants such that*

$$(7.3) \qquad\qquad\qquad \|\mathfrak{m}_{\mathscr{F}}\|_{\mathscr{C}(Q_0)} \leq Cb,$$

$$(7.4) \qquad\qquad\qquad \sigma(B) \leq \frac{a+b}{a+2b}\,\sigma(Q_0),$$

where B is the union of those $Q_j \in \mathscr{F}$ such that $\mathfrak{m}(\mathbb{D}_{Q_j}^{\text{short}}) > a\,\sigma(Q_j)$.

REMARK 7.5. – In the proof of this result, the only feature of σ that we shall use, is that it is a non-negative Borel measure satisfying the "dyadically doubling property on Q_0" (by this

we mean that there is a uniform constant c_σ such that $\sigma(\widetilde{Q}) \leq c_\sigma \, \sigma(Q)$, whenever $\widetilde{Q} \in \mathbb{D}_{Q_0}$ is the dyadic parent of Q). Notice that this property follows at once for our measure $\sigma = H^n\big|_{\partial\Omega}$ by the ADR property. Therefore, Lemma 7.2 admits an extension in which σ can be any non-negative dyadically doubling Borel measure on Q_0.

Proof. – We note that $\mathfrak{m}(\mathbb{D}_{Q_0}) = \mathfrak{m}(\mathbb{D}_{Q_0}^{\text{short}}) + \alpha_{Q_0}$. Thus, if $\alpha_{Q_0} > b\sigma(Q_0)$, the result is trivial: in this case $\mathfrak{m}(\mathbb{D}_{Q_0}^{\text{short}}) \leq a\sigma(Q_0)$, so we may set $\mathscr{F} = \{Q_0\}$, and $B = \varnothing$.

Suppose now that $\alpha_{Q_0} \leq b\,\sigma(Q_0)$. For $Q' \in \mathbb{D}_{Q_0}$, we set

$$\beta(Q') := \sum_{Q:\, Q' \subseteq Q \subseteq Q_0} P(Q, Q')\,\alpha_Q.$$

In particular, $\beta(Q_0) = \alpha_{Q_0} \leq b\,\sigma(Q_0)$. We now perform a standard stopping time argument to select the collection $\mathscr{F} = \{Q_j\}$, comprised of the subcubes of Q_0 which are maximal with respect to the property that

$$(7.6) \qquad\qquad\qquad \beta(Q_j) > 2b\sigma(Q_j).$$

If $\mathscr{F}$ is empty, we simply have that $\mathbb{D}_{\mathscr{F}} = \mathbb{D}$, $\mathfrak{m}_{\mathscr{F}} = \mathfrak{m}$ and $B = \varnothing$.

We now verify that $\mathscr{F}$ satisfies the desired properties. We start by proving (7.4). To this end let us record some useful facts. We first note that, given a fixed $Q \in \mathbb{D}_{Q_0}$,

$$(7.7) \qquad\qquad \sum_{Q_j \in \mathscr{F}} P(Q, Q_j) = \sum_{Q_j \in \mathscr{F}:\, Q_j \subseteq Q} P(Q, Q_j) \leq 1,$$

since the cubes in $\mathscr{F}$, and therefore also the events in the sum, are disjoint. Next, we note that since $P(Q_j, Q_j) = 1$,

$$(7.8) \qquad \mathfrak{m}(\mathbb{D}_{Q_j}^{\text{short}}) + \beta(Q_j) = \mathfrak{m}(\mathbb{D}_{Q_j}) + \sum_{Q:\, Q_j \subsetneq Q \subseteq Q_0} P(Q, Q_j)\,\alpha_Q,$$

where the last sum runs over those $Q \in \mathbb{D}_{Q_0}$ that *strictly* contain Q_j. Consequently,

$$(7.9) \quad \sum_{Q_j \in \mathscr{F}} \left(\mathfrak{m}(\mathbb{D}_{Q_j}^{\text{short}}) + \beta(Q_j) \right)$$

$$= \sum_{Q_j \in \mathscr{F}} \mathfrak{m}(\mathbb{D}_{Q_j}) + \sum_{Q \in \mathbb{D}_{\mathscr{F}, Q_0}} \alpha_Q \sum_{Q_j \in \mathscr{F}:\, Q_j \subsetneq Q} P(Q, Q_j) \leq \mathfrak{m}(\mathbb{D}_{Q_0}),$$

by (7.7) and the definition of $\mathbb{D}_{\mathscr{F}, Q_0}$ (cf. (3.41)).

We now set $\mathscr{F}_{\text{bad}} := \{Q_j \in \mathscr{F} : \mathfrak{m}(\mathbb{D}_{Q_j}^{\text{short}}) > a\,\sigma(Q_j)\}$. Then by definition of B and the stopping time construction,

$$(a + 2b)\,\sigma(B) = (a + 2b) \sum_{Q_j \in \mathscr{F}_{\text{bad}}} \sigma(Q_j)$$

$$\leq \sum_{Q_j \in \mathscr{F}_{\text{bad}}} \left(\mathfrak{m}(\mathbb{D}_{Q_j}^{\text{short}}) + \beta(Q_j) \right) \leq \mathfrak{m}(\mathbb{D}_{Q_0}) \leq (a + b)\,\sigma(Q_0),$$

where in the last line we have used (7.9) and our hypothesis. Estimate (7.4) follows.

We now turn to the proof of (7.3). Let us fix $Q \in \mathbb{D}_{Q_0}$. We consider

$$\mathfrak{m}_{\mathscr{F}}(\mathbb{D}_Q) = \mathfrak{m}(\mathbb{D}_{\mathscr{F}, Q}) = \sum_{Q' \in \mathbb{D}_{\mathscr{F}, Q}} \alpha_{Q'} = \lim_{N \to \infty} \sum_{Q' \in \mathbb{D}_{\mathscr{F}_N, Q}} \alpha_{Q'} = \lim_{N \to \infty} \mathfrak{m}(\mathbb{D}_{\mathscr{F}_N, Q}),$$

where $\mathscr{F}_N := \mathscr{F}(2^{-N-1})$ is derived from $\mathscr{F}$ as in the discussion at the beginning of Section 4; i.e., $\mathscr{F}_N = \{Q_i^N\}$ is the collection of maximal cubes of

$$\mathscr{F} \cup \{Q' \in \mathbb{D}_Q : \ell(Q') \leq 2^{-N-1}\}.$$

Thus,

$$\mathbb{D}_{\mathscr{F}_N,Q} = \{Q' \in \mathbb{D}_{\mathscr{F},Q} : \ell(Q') \geq 2^{-N}\}, \qquad N \geq k(Q).$$

It is therefore enough to establish the bound

(7.10) $$\mathfrak{m}(\mathbb{D}_{\mathscr{F}_N,Q}) \leq Cb\,\sigma(Q)$$

uniformly in N. To this end, we observe that equality holds in (7.7), for a given cube Q and pairwise disjoint family $\mathscr{F}$, if Q is covered by a union of cubes in $\mathscr{F}$. Since this is the case for the family $\mathscr{F}_N$ and for every $Q' \in \mathbb{D}_{\mathscr{F}_N,Q}$, we have

$$\mathfrak{m}(\mathbb{D}_{\mathscr{F}_N,Q}) = \sum_{Q' \in \mathbb{D}_{\mathscr{F}_N,Q}} \alpha_{Q'} \sum_{Q_i^N \in \mathscr{F}_N : Q_i^N \subseteq Q'} P(Q',Q_i^N)$$

$$= \sum_{Q_i^N \in \mathscr{F}_N \cap \mathbb{D}_Q} \sum_{Q' : Q_i^N \subset Q' \in \mathbb{D}_{\mathscr{F}_N,Q}} P(Q',Q_i^N)\,\alpha_{Q'} = \Sigma_1 + \Sigma_2,$$

where in Σ_1 the first sum runs over those $Q_i^N \in \mathscr{F}_N \cap \mathbb{D}_Q$ which are equal to some $Q_j \in \mathscr{F}$ (i.e., $Q_i^N \in \mathscr{F}_N \cap \mathscr{F} \cap \mathbb{D}_Q$), while in Σ_2 the first sum runs over the remaining cubes in $\mathscr{F}_N \cap \mathbb{D}_Q$ (i.e., over $Q_i^N \in (\mathscr{F}_N \setminus \mathscr{F}) \cap \mathbb{D}_Q$, equivalently those Q_i^N which are not contained in any $Q_j \in \mathscr{F}$). We then have

$$\Sigma_2 = \sum_{Q_i^N \in (\mathscr{F}_N \setminus \mathscr{F}) \cap \mathbb{D}_Q} \left(\sum_{Q' : Q_i^N \subset Q' \in \mathbb{D}_{\mathscr{F}_N,Q}} P(Q',Q_i^N)\,\alpha_{Q'} \right)$$

$$\leq \sum_{Q_i^N \in (\mathscr{F}_N \setminus \mathscr{F}) \cap \mathbb{D}_Q} \beta(Q_i^N) \leq 2b \sum_{Q_i^N \in (\mathscr{F}_N \setminus \mathscr{F}) \cap \mathbb{D}_Q} \sigma(Q_i^N) \leq 2b\,\sigma(Q),$$

by the stopping time construction of $\mathscr{F}$, since Q_i^N is not contained in any $Q_j \in \mathscr{F}$, and $Q_i^N \in \mathbb{D}_Q$.

We now consider Σ_1. We first note that no Q' appearing in the sum can be contained in any $Q_j \in \mathscr{F}$, since $\mathbb{D}_{\mathscr{F}_N,Q} \subset \mathbb{D}_{\mathscr{F},Q}$. Therefore, if some $Q_j \in \mathscr{F}$ is contained in any such Q', then so is its dyadic parent $\widetilde{Q}_j$. Moreover,

$$P(Q',Q_j) \leq P(Q',\widetilde{Q}_j), \qquad \forall Q_j \in \mathscr{F}.$$

We then have that, by definition,

$$\Sigma_1 \leq \sum_{Q_j \in \mathscr{F}} \sum_{Q' : \widetilde{Q}_j \subseteq Q' \subseteq Q} P(Q',\widetilde{Q}_j)\,\alpha_{Q'} \leq \sum_{Q_j \in \mathscr{F} : Q_j \subset Q} \beta(\widetilde{Q}_j)$$

$$\leq 2b \sum_{Q_j \in \mathscr{F} : Q_j \subset Q} \sigma(\widetilde{Q}_j) \leq Cb\,\sigma(Q),$$

where the next-to-last inequality holds because the cubes Q_j are maximal with respect to the property (7.6), and the last one holds by the dyadic doubling property of σ (see Remark 7.5), and the pairwise disjointness of the cubes in $\mathscr{F}$. $\square$

8. Proofs of Theorems 1.26 and 1.27

8.1. Relating geometric and discrete Carleson measures

We recall that the UR property may be characterized in terms of the Carleson measure estimate (1.10), which we shall invoke with $E = \partial\Omega$. We also remind the reader that we may assume that for every Whitney cube $I \in \mathcal{W}$, we have $\mathrm{dist}(4I, \partial\Omega) \approx \ell(I)$ (cf. (3.42)). In this case, the "fattened" Whitney cubes $4I$ have bounded overlaps. From this fact, properties of Whitney cubes, and the ADR property, it follows that the fattened Whitney regions $U_Q^{\mathrm{fat}} = \bigcup_{\mathcal{W}_Q^*} 4I$ (cf. (3.43)-(3.49) and (4.1)) also have the bounded overlap property:

$$(8.1) \qquad \sum_{Q\in\mathbb{D}} 1_{U_Q^{\mathrm{fat}}}(X) \le C.$$

We now set

$$(8.2) \qquad \alpha_Q := \iint_{U_Q^{\mathrm{fat}}} |\nabla^2 \mathcal{S}1(X)|^2 \, \delta(X) \, dX,$$

and for any sub-collection $\mathbb{D}' \subset \mathbb{D}$, we define

$$(8.3) \qquad \mathfrak{m}(\mathbb{D}') := \sum_{Q\in\mathbb{D}'} \alpha_Q,$$

as in the Section 7. By (8.1), for every pairwise disjoint family $\mathcal{F} \subset \mathbb{D}$, and every $Q \in \mathbb{D}_{\mathcal{F}}$, we have

$$(8.4) \qquad \mathfrak{m}_{\mathcal{F}}(\mathbb{D}_Q) = \mathfrak{m}(\mathbb{D}_{\mathcal{F},Q}) \approx \iint_{\Omega_{\mathcal{F},Q}^{\mathrm{fat}}} |\nabla^2 \mathcal{S}1(X)|^2 \, \delta(X) \, dX$$

where $\Omega_{\mathcal{F},Q}^{\mathrm{fat}}$ is defined in (4.3) (we have used in (8.4) the rather trivial fact that $\partial\Omega_{\mathcal{F},Q}^{\mathrm{fat}}$ has $(n+1)$-dimensional Lebesgue measure 0). In particular, taking $\mathcal{F} = \varnothing$, in which case $\mathbb{D}_{\mathcal{F},Q} = \mathbb{D}_Q$, and $\Omega_{\mathcal{F},Q}^{\mathrm{fat}} = T_Q^{\mathrm{fat}}$ (cf. (4.2)), we obtain from (5.12) that $\mathfrak{m}$ inherits the Carleson measure property (7.1) from (1.10), and that the Carleson norm $\|\mathfrak{m}\|_{\mathcal{C}}$ depends only on dimension and the various ADR, UR, Corkscrew and Harnack chain constants for Ω (including K_0).

8.2. Proof of Theorem 1.26 with "qualitative assumptions"

In this subsection, we present the proof of Theorem 1.26, in the special case that the qualitative exterior Corkscrew condition holds in Ω (and therefore also in its sawtooths and Carleson boxes). As we observed in Section 3, this qualitative hypothesis (along with our standard quantitative assumptions), were enough to imply the doubling condition for the harmonic measure for Ω and the sawtooth regions, and also to allow us to obtain the "Dyadic Sawtooth" Lemma 6.15. We shall remove the qualitative assumptions, and also give the proof of Theorem 1.27, in Subsection 8.3.

We shall use the method of "extrapolation of Carleson measures", based on ideas originating in [12] and [36] (cf. [25, 3], [4, 27]). In more precise detail, we follow our related work in the Euclidean setting [27]. In the sequel, we say that a measure μ is "dyadically doubling on Q_0" if there is a uniform constant c_μ such that $\mu(\widetilde{Q}) \le c_\mu \, \mu(Q)$, whenever $\widetilde{Q} \in \mathbb{D}_{Q_0}$ is the dyadic parent of Q.

Lemma 8.5. – *We fix $Q_0 \in \mathbb{D}$. Let σ and ω be a pair of non-negative, dyadically doubling Borel measures on Q_0, and let $\mathfrak{m}$ be a discrete Carleson measure with respect to σ (cf. Section 7) with*

$$\|\mathfrak{m}\|_{\mathscr{C}(Q_0)} \leq M_0.$$

Suppose that there is a $\gamma > 0$ such that for every $Q \in \mathbb{D}_{Q_0}$ and every family of pairwise disjoint dyadic subcubes $\mathscr{F} = \{Q_j\} \subset \mathbb{D}_Q$ verifying

$$(8.6) \qquad\qquad\qquad \|\mathfrak{m}_{\mathscr{F}}\|_{\mathscr{C}(Q)} \leq \gamma,$$

we have that $\mathscr{P}_{\mathscr{F}} \omega$ satisfies the following property:

$$(8.7) \quad \forall \varepsilon \in (0,1), \ \exists C_\varepsilon > 1 \text{ such that } \left(F \subset Q, \ \frac{\sigma(F)}{\sigma(Q)} \geq \varepsilon \ \implies \ \frac{\mathscr{P}_{\mathscr{F}}\omega(F)}{\mathscr{P}_{\mathscr{F}}\omega(Q)} \geq \frac{1}{C_\varepsilon} \right).$$

Then, there exist $\eta_0 \in (0,1)$ and $C_0 < \infty$ such that, for every $Q \in \mathbb{D}_{Q_0}$,

$$(8.8) \qquad\qquad F \subset Q, \ \frac{\sigma(F)}{\sigma(Q)} \geq 1 - \eta_0 \ \implies \ \frac{\omega(F)}{\omega(Q)} \geq \frac{1}{C_0}.$$

I.e., $\omega \in A_\infty^{\mathrm{dyadic}}(Q_0)$.

Remark 8.9. – Notice that in the statement of the lemma, σ and ω are allowed to be any pair of non-negative, dyadically doubling Borel measures on Q_0, and that σ plays the role of underlying measure. Therefore, σ appears implicitly in the Carleson conditions, $\mathscr{P}_{\mathscr{F}}$ and in the definition of the class $A_\infty^{\mathrm{dyadic}}(Q_0)$. In the present paper, we shall apply this result in the special case that $\sigma = H^n\big|_{\partial\Omega}$, and $\omega = \omega^{X_{Q_0}}$, the harmonic measure with pole at the Corkscrew point X_{Q_0}.

Remark 8.10. – It is known that (8.8), for every $Q \in \mathbb{D}_{Q_0}$, self-improves to (1.21), but this fact may also be gleaned from Remark B.10 below.

Remark 8.11. – The key hypothesis of the lemma, and the main point that must be verified in applications, is that (8.6) implies (8.7), for sufficiently small γ.

In the remainder of this subsection, we shall use Lemma 8.5 to prove Theorem 1.26, assuming the extra qualitative exterior Corkscrew condition. The qualitative hypothesis will then be removed in the next subsection. We defer the proof of the lemma to Section 9.

Proof of Theorem 1.26 with qualitative hypothesis. – To begin, we let $\sigma = H^n\big|_{\partial\Omega}$, which is dyadically doubling by the ADR property. Let us fix $Q_0 \in \mathbb{D}$, and set $\omega := \omega^{X_{Q_0}}$, where as usual X_{Q_0} is a Corkscrew point relative to Q_0. Given the qualitative hypothesis, it holds in particular that ω is a doubling measure (cf. Corollary 3.36), and therefore also dyadically doubling, on Q_0 (to obtain dyadic doubling when $\ell(Q) \approx \ell(Q_0)$, we may need to invoke the Harnack chain condition); moreover, the doubling constants depend only upon the constants in the *quantitative* hypotheses of Theorem 1.26 (i.e., dimension, UR, ADR, Harnack chain and Corkscrew, including the constant K_0 which ultimately depended only upon the other stated parameters). We define $\mathfrak{m}$ as in (8.3), with α_Q as in (8.2). As observed in the previous subsection, this $\mathfrak{m}$ inherits the discrete Carleson measure property (7.1) from (1.10). Therefore, once we have verified that (8.6), with γ small enough, implies (8.7), we may then conclude from Lemma 8.5 that $\omega = \omega^{X_{Q_0}} \in A_\infty^{\mathrm{dyadic}}(Q_0)$, for every $Q_0 \in \mathbb{D}$, and thus by the Harnack chain condition that $\omega^{X_{Q_0}} \in A_\infty^{\mathrm{dyadic}}(Q_1)$, for every Q_1 of the same generation

as Q_0 such that $\operatorname{dist}(Q_0, Q_1) \leq 100 \operatorname{diam}(Q_0)$. Since this is true for every $Q_0 \in \mathbb{D}$, and since $\omega^{X_{Q_0}}$ is *concentrically* doubling, we may conclude that $\omega^{X_\Delta} \in A_\infty(\Delta)$, for every surface ball $\Delta = \Delta(x, r)$, $x \in \partial\Omega$ and $r \leq \operatorname{diam}(\partial\Omega)$, with A_∞ constants uniformly controlled, and depending only upon dimension and the UR, ADR, Harnack chain and Corkscrew constants. We reached this conclusion by imposing the extra qualitative exterior Corkscrew condition, but as our estimates do not depend quantitatively on that hypothesis, we shall be able, in Subsection 8.3, to remove it by an approximation argument, but at the the loss of the doubling property of ω.

To complete our task in this subsection, it now remains only to verify that (8.6), with γ small enough, implies (8.7). To this end, we fix $Q \in \mathbb{D}_{Q_0}$ and a pairwise disjoint family $\mathcal{F} \subset \mathbb{D}_Q$, and we suppose that (8.6) holds for some small γ to be chosen momentarily. By (8.4), we deduce that

$$(8.12) \qquad \sup_{Q' \in \mathbb{D}_Q} \frac{1}{\sigma(Q')} \iint_{\Omega_{\mathcal{F},Q'}^{\mathrm{fat}}} |\nabla^2 \mathscr{S}1(X)|^2 \, \delta(X) \, dX \ \leq \ C\gamma.$$

Consequently, if γ is small enough, depending only upon the allowable quantitative parameters, we may apply Corollary 5.17, with Q in place of Q_0, to obtain, for every surface ball $\Delta_\star = B \cap \partial\Omega_{\mathcal{F},Q}$, with $B = B(x, r)$, $x \in \partial\Omega_{\mathcal{F},Q}$, and $r \leq \operatorname{diam}(Q)$, that $\omega_\star^{X_{\Delta_\star}} \in A_\infty(\Delta_\star)$, where $\omega_\star^X$ denotes harmonic measure for $\Omega_{\mathcal{F},Q}$. Moreover, the A_∞ constants are uniformly controlled by the stated parameters. By the Harnack chain condition, we obtain that $\omega_\star^{A_Q} \in A_\infty(\partial\Omega_{\mathcal{F},Q})$ (meaning that we view $\partial\Omega_{\mathcal{F},Q}$ itself as a surface ball $\Delta_\star^Q$ of radius $r(\Delta_\star^Q) \approx K_0 \ell(Q)$, and that $\omega_\star^{A_Q} \in A_\infty(\Delta_\star^Q)$) where A_Q is the simultaneous Corkscrew point produced in Corollary 6.6, applied with Q in place of Q_0.

Let $P_{\mathcal{F}}\nu$ be defined as in Lemma 6.15, but again with Q in place of Q_0. We shall prove in Lemma B.6 (Appendix B below) that $P_{\mathcal{F}}\nu \in A_\infty^{\mathrm{dyadic}}(Q)$. Thus, by Lemma 6.15, with Q in place of Q_0, we obtain that $P_{\mathcal{F}}\omega^{A_Q} \in A_\infty^{\mathrm{dyadic}}(Q)$. We may then use Corollary 3.69 (here we only consider the case that Q is not contained in any Q_j, otherwise $\mathscr{P}_{\mathcal{F}}\omega^{X_{Q_0}} \in A_\infty^{\mathrm{dyadic}}(Q)$ trivially) along with the Harnack chain condition and a differentiation argument, to replace the pole A_Q by X_{Q_0}, the Corkscrew point for the ambient cube Q_0, and to conclude that $\mathscr{P}_{\mathcal{F}}\omega = \mathscr{P}_{\mathcal{F}}\omega^{X_{Q_0}} \in A_\infty^{\mathrm{dyadic}}(Q)$. In particular, (8.7) holds, by Lemmas B.7 and B.1 in Appendix B. $\qquad\square$

8.3. Removing the qualitative hypothesis, and conclusion of the proofs of Theorems 1.26 and 1.27

In this subsection, we first complete the proof of Theorem 1.26 (modulo the proof of Lemma 8.5 and the technical lemmata that we have deferred to Appendices), by removing the qualitative exterior Corkscrew condition. We then conclude by giving the proof of Theorem 1.27.

We define approximating domains as follows. For each large integer N, set $\mathcal{F}_N := \mathbb{D}_N$. We then let $\Omega_N := \Omega_{\mathcal{F}_N}$ denote the usual (global) sawtooth with respect to the family $\mathcal{F}_N$ (cf. (3.49), (3.47) and (3.53).) Thus,

$$(8.13) \qquad \Omega_N = \operatorname{int}\left(\bigcup_{Q \in \mathbb{D}:\, \ell(Q) \geq 2^{-N+1}} U_Q\right),$$

so that $\overline{\Omega_N}$ is the union of fattened Whitney cubes $I^* = (1 + \lambda)I$, with $\ell(I) \gtrsim 2^{-N}$, and the boundary of Ω_N consists of portions of faces of I^* with $\ell(I) \approx 2^{-N}$. By virtue of Lemma 3.61, each Ω_N satisfies the ADR, Corkscrew and Harnack chain properties. Moreover, $\partial\Omega_N$ is UR. We defer the proof of the UR property to Appendix C. We note that, for each of these properties, the constants are uniform in N, and depend only on dimension and on the corresponding constants for Ω. In addition, by construction, Ω_N has exterior Corkscrew points at all scales $\lesssim 2^{-N}$. By Lemma 3.62, the same statement applies to the Carleson boxes T_Q and T_Δ, and to the sawtooth domains $\Omega_{\mathcal{F}}$ and $\Omega_{\mathcal{F},Q}$ (all of them relative to Ω_N) and even to Carleson boxes within saw tooths.

Consequently, by the arguments in the previous subsection, we conclude that for every surface ball $\Delta_\star = \Delta_\star^N \subset \partial\Omega_N$, the harmonic measure $\omega_N^{X_{\Delta_\star}} \in A_\infty(\Delta_\star)$, uniformly in N. We now consider the limiting case. Fix a surface ball $\Delta := \Delta(x,r) \subset \partial\Omega$, and a Borel subset $A \subset \Delta$. Assuming the hypotheses of Theorem 1.26, we claim that

$$(8.14) \qquad \sigma(A) \geq \eta\,\sigma(\Delta) \implies \omega^{X_\Delta}(A) \geq c_0\,\eta^\theta, \qquad \forall\eta \in (0,1),$$

for some uniform positive constants c_0 and θ, where as usual X_Δ denotes a Corkscrew point relative to Δ. By the outer regularity property of ω^X (cf. (3.3)), we may assume that A is (relatively) open. It then follows that we may write $A = \bigcup_k Q_k$, where $\{Q_k\} \subset \mathbb{D}$ is a pairwise disjoint collection. We set $\Delta_k = \Delta(x_k, r_k) := \Delta_{Q_k}$, where $\Delta_{Q_k} := B_{Q_k} \cap \partial\Omega \subset Q_k$ is the surface ball defined in (1.16)-(1.17), so that $r_k \approx \ell(Q_k)$ and $\sigma(Q_k) \approx \sigma(\Delta_k)$. Then

$$\eta\,\sigma(\Delta) \leq \sigma(A) = \sum_k \sigma(Q_k) \approx \sum_k \sigma(\Delta_k).$$

We now set $A' := \bigcup_{k=1}^M \Delta_k$, where M is chosen large enough (depending on A) so that

$$(8.15) \qquad \sigma(A') = \sum_{k=1}^M \sigma(\Delta_k) \geq \frac{1}{C}\,\eta\,\sigma(\Delta).$$

By the ADR property and a covering lemma argument, we may further suppose that the Euclidean balls $B_k := B_{Q_k}, 1 \leq k \leq M$, are pairwise disjoint. We now fix N so large that $2^{-N} \ll \min_{1 \leq k \leq M} r_k$, and $2^N \gg \operatorname{diam}(\Delta)$. Fix also a point $\hat{x} \in \partial\Omega_N$, with $|x - \hat{x}| \approx 2^{-N}$ (such a point exists, with implicit constants possibly depending on K_0, since Ω satisfies the Corkscrew condition). We shall approximate Ω by a domain $\widehat{\Omega}_N$, which is defined as follows, and whose harmonic measure we denote by $\hat{\omega}^X$. If Ω is bounded, we set $\widehat{\Omega}_N = \Omega_N$. Otherwise, we define $\widehat{\Omega}_N := T_{\Delta_\star^N}$, where $T_{\Delta_\star^N} \subset \Omega_N$ denotes the Carleson box corresponding to $\Delta_\star^N = B(\hat{x}, 2^N) \cap \partial\Omega_N$ for the domain Ω_N. Then by the arguments in Subsection 8.2, for every surface ball $\Delta_\star = \Delta_{\star,N} \subset \partial\widehat{\Omega}_N$, the harmonic measure $\hat{\omega}_N^{X_{\Delta_\star}} \in A_\infty(\Delta_\star)$, uniformly in N, since $\widehat{\Omega}_N$ is either equal to Ω_N, or else is a subdomain of Ω_N which inherits all of the requisite properties as observed above.

We set $B_k' := cB_k$, where $c \in (0,1)$ is the constant in Lemma 3.6. As noted above, the collection $\{B_k\}_{1 \leq k \leq M}$, hence also $\{B_k'\}_{1 \leq k \leq M}$, may be taken to be pairwise disjoint. Let us also note that, since $2^{-N} \ll \min_{1 \leq k \leq M} r_k$, by the ADR properties of $\partial\widehat{\Omega}_N$ and $\partial\Omega$, we have

$$(8.16) \qquad H^n\left(\bigcup_{k=1}^M B_k' \cap \partial\widehat{\Omega}_N\right) \gtrsim \sum_{k=1}^M r_k^n \gtrsim \eta\,\sigma(\Delta) \gtrsim \eta\,H^n\big(\Delta_\star(\hat{x}, Cr)\big),$$

where in the last pair of inequalities we have used (8.15) and ADR (for both $\partial\Omega$ and $\partial\Omega_N$). Moreover, since $2^{-N} \ll \text{diam}(\Delta) \ll 2^N$, we have that $X_\Delta \in \widehat{\Omega}_N$ is also a Corkscrew point for $\widehat{\Omega}_N$ with respect to the surface ball $\Delta_\star(\hat{x}, Cr)$, where $\hat{x}$ is as above, and where C is chosen large enough that $\bigcup_{k=1}^{M} B'_k \cap \partial\widehat{\Omega}_N \subset \Delta_\star(\hat{x}, Cr)$.

We observe that $u(X) := \omega^X(A')$ is harmonic in Ω, and thus also in the bounded subdomain $\widehat{\Omega}_N$. Since in bounded domains we have uniqueness by the maximum principle, we obtain

$$(8.17) \qquad \omega^{X_\Delta}(A') = \int_{\partial\widehat{\Omega}_N} \omega^Y(A') \, d\hat{\omega}_N^{X_\Delta}(Y) \geq \sum_{k=1}^{M} \int_{B'_k \cap \partial\widehat{\Omega}_N} \omega^Y(\Delta_k) \, d\hat{\omega}_N^{X_\Delta}(Y)$$

$$\gtrsim \sum_{k=1}^{M} \int_{B'_k \cap \partial\widehat{\Omega}_N} d\hat{\omega}_N^{X_\Delta}(Y) = \hat{\omega}_N^{X_\Delta}\Big(\bigcup_{k=1}^{M} B'_k \cap \partial\widehat{\Omega}_N\Big) \gtrsim \eta^\theta,$$

where in the last line we have used Lemma 3.6 and then (8.16) and the A_∞ property of $\hat{\omega}^{X_\Delta}$ (recall that X_Δ serves as a Corkscrew point for $\Delta_\star(\hat{x}, Cr)$, as we have noted above). Since $A' \subset A$, we then obtain (8.14).

We now note that (8.14) trivially implies the following weak version of itself: for bounded Ω (the unbounded case is treated below) satisfying the hypotheses of Theorem 1.26, there exist uniform constants $\eta \in (0,1)$ and $c_0 > 0$ such that

$$(8.18) \qquad\qquad \sigma(A) \geq \eta\,\sigma(\Delta) \implies \omega^{X_\Delta}(A) \geq c_0.$$

We remark here that to establish (8.18) has really been our main goal. Indeed, given (8.18), the remainder of the proof of Theorem 1.26 will follow the arguments in [7]. We further remark that in [7], (8.18) is essentially taken as a starting point: by the maximum principle, and the result of [15], an appropriate version of (8.18) (cf. (8.26) below) follows immediately from the main hypothesis in [7], that Ω has "interior big pieces" (in the sense of Definition 1.14) of Lipschitz sub-domains of Ω, with uniform constants. Eventually, we shall see that (8.18), suitably interpreted, continues to hold under the hypotheses of Theorem 1.27.

We now proceed to describe the remaining steps needed to deduce the weak-A_∞ property of harmonic measure. By [7, Lemma 3.1], it suffices to show that for each $\varepsilon \in (0, 1/1000)$, there are uniform constants $\eta_\varepsilon \in (0,1)$ and $C_\varepsilon \in (1,\infty)$, such that given balls B, B', centered on $\partial\Omega$, with $2B' \subset B$, and corresponding surface balls $\Delta := B \cap \partial\Omega$ and $\Delta' := B' \cap \partial\Omega$, and a Borel subset $A \subset 2\Delta'$ with $\sigma(A) \geq \eta_\varepsilon\,\sigma(2\Delta')$, we have

$$(8.19) \qquad\qquad \omega^{X_\Delta}(\Delta') \leq \varepsilon\,\omega^{X_\Delta}(2\Delta') + C_\varepsilon\,\omega^{X_\Delta}(A).$$

In fact, [7, Lemma 3.1] is a purely real variable result which says that any positive Borel measure μ on $\partial\Omega$ satisfying (8.19) belongs to weak-$A_\infty(\Delta)$ (equivalently, satisfies the weak reverse Hölder estimate (1.25) for some $q > 1$), assuming only that $\partial\Omega$ is ADR. Under the hypotheses of Theorem 1.26, we shall establish (8.19) with $\eta_\varepsilon := \eta$, the constant in (8.18) (independently of ε).

Let us now give the proof of (8.19). We prove the desired bound first in the case that Ω is bounded. This restriction will be removed at the end of the proof. We follow the argument in [7, Lemma 2.2] almost verbatim, with some small simplifications permitted by our hypothesis that the Harnack chain condition holds in Theorem 1.26. Let $B' = B(z, s)$,

$\Delta' := B' \cap \partial\Omega$, and suppose $2B' \subseteq B := B(x, r)$. We cover $\frac{3}{2}\Delta' \setminus \frac{5}{4}\Delta'$ by annuli of thickness $\approx \varepsilon s$. More precisely, we set

$$(8.20) \qquad U_k := \big\{y \in \partial\Omega : (5/4 + \varepsilon k)\, s \le |y - z| < \big(5/4 + \varepsilon(k+1)\big)s\big\},$$

$$S_k := \big\{X \in \Omega : |X - z| = \big(5/4 + \varepsilon(k + 1/2)\big)\, s\big\},$$

where $0 \le k \lesssim 1/(4\varepsilon)$. Suppose now $A \subset 2\Delta'$, with $\sigma(A) \ge \eta\, \sigma(2\Delta')$, for η as in (8.18). Let $c \in (0, 1)$ be the constant in Lemma 3.6. By the Harnack chain condition and (8.18), applied to $2\Delta'$ in place of Δ, we have

$$(8.21) \qquad \omega^X(A) \ge c_\varepsilon\, c_0, \qquad \forall X \in S_k \cap \{X : \delta(X) \ge c\varepsilon s/100\},$$

uniformly in k. On the other hand, if $X \in S_k \cap \{X : \delta(X) < c\varepsilon s/100\}$, then for a suitable uniform constant C, we have

$$(8.22) \qquad C\omega^X(U_k) \ge C\omega^X\big(\Delta(\hat{x}, \varepsilon s/10)\big) \ge 1,$$

by Lemma 3.6, where $\hat{x} \in \partial\Omega$ is chosen so that $|X - \hat{x}| = \delta(X)$. Thus,

$$(8.23) \qquad \omega^X(\Delta') \le 1 \le C\omega^X(U_k) + C_\varepsilon\, \omega^X(A), \qquad \forall X \in S_k,$$

where $C_\varepsilon = 1/(c_\varepsilon c_0)$. By the maximum principle, this implies in particular that (8.23) continues to hold for $X \in \Omega \setminus \frac{7}{4}B'$, since $S_k \subset \frac{7}{4}B'$, if ε is small, for every relevant k (i.e., those for which U_k meets $\frac{3}{2}B' \setminus \frac{5}{4}B'$). Since this set of k's has cardinality $\approx 1/\varepsilon$, summing in k we obtain

$$(8.24) \qquad \frac{1}{\varepsilon}\omega^X(\Delta') \le C\omega^X(2\Delta') + C_\varepsilon\omega^X(A), \qquad \forall X \in \Omega \setminus \frac{7}{4}B',$$

since the U_k's are pairwise disjoint and contained in $2\Delta'$. The desired bound (8.19) now follows, at least in the case that Ω is bounded.

Now suppose that Ω is unbounded. Given a surface ball $\Delta = \Delta(x, r)$, we choose $R \gg r$, set $\Delta_R = \Delta(x, R)$, and consider the domain $\Omega_R := T_{\Delta_R}$, the Carleson Box associated to Δ_R. For each such R, the argument above may be applied, to obtain (8.19) for each of the corresponding harmonic measures $\omega_R^{X_\Delta}$. For any fixed Borel subset $F \subset \Delta$, we have that the solutions $u_R(X) := \omega_R^X(F)$ are monotone increasing on any fixed Ω_{R_0}, as $R_0 \le R \to \infty$, by the maximum principle. We then obtain that $u_R(X) \to u(X) := \omega^X(F)$, uniformly on compacta, by Harnack's convergence theorem (as in the discussion at the beginning of Section 3), whence (8.19) follows. The proof of Theorem 1.26 is now complete, modulo the deferred arguments.

Proof of Theorem 1.27. – Finally, we discuss the modifications needed to prove Theorem 1.27. By [7, Lemma 3.1] (and a limiting process to treat the case of an unbounded domain), it again suffices to establish, for bounded Ω now satisfying the hypotheses of Theorem 1.27, an appropriate version of (8.19). That is, we seek to show that for each $\varepsilon \in (0, 1/1000)$, there are uniform constants $\eta_\varepsilon \in (0, 1)$ and $C_\varepsilon \in (1, \infty)$, such that given balls $B = B(x, r)$ and $B' = B(z, s)$, with $2B' \subset B$, and corresponding surface balls $\Delta := B \cap \partial\Omega$ and $\Delta' := B' \cap \partial\Omega$, if $A \subset 2\Delta'$ with $\sigma(A) \ge \eta_\varepsilon\, \sigma(2\Delta')$, then

$$(8.25) \qquad \omega^X(\Delta') \le \varepsilon\, \omega^X(2\Delta') + C_\varepsilon\, \omega^X(A), \qquad \forall X \in \Omega \setminus B.$$

To this end, we first establish a suitable variant of (8.18). Given $X \in \Omega$, under the hypotheses of Theorem 1.27, there is a point $x \in \partial\Omega$, with $|X - x| = \delta(X)$, and a subdomain $\Omega' \subset \Omega$ satisfying the hypotheses of Theorem 1.26, with the property that for some constants $K > 1$ and $\alpha > 0$, we have

$$\sigma(\partial\Omega' \cap \Delta_X) \geq \alpha\,\sigma(\Delta_X),$$

where $\Delta_X := \Delta(x, K\delta(X))$. We may further suppose that X serves as a Corkscrew point for Ω' relative to a surface ball $\Delta_\star := B(y, 2K\delta(X)) \cap \partial\Omega'$, with $y \in \Delta_X \cap \partial\Omega'$. That Ω' exists, with uniform control of the various constants involved, is simply a re-statement of the "big pieces" hypothesis of Theorem 1.27 (cf. Definition 1.14). We claim that there exist uniform constants $\eta \in (0, 1)$ and $c_0 > 0$, such that for any Borel subset $A \subset \Delta_X$,

$$(8.26) \qquad\qquad \sigma(A) \geq \eta\,\sigma(\Delta_X) \implies \omega^X(A) \geq c_0\,.$$

Let us now prove this claim. Suppose that $A \subset \Delta_X$, with $\sigma(A) \geq (1 - \alpha/2)\sigma(\Delta_X)$. Then

$$\sigma'(\partial\Omega' \cap A) \geq \frac{\alpha}{2}\,\sigma(\Delta_X) \approx \alpha\,\sigma'(\Delta_\star)\,,$$

where $\sigma' := H^n\big|_{\partial\Omega'}$ denotes surface measure on $\partial\Omega'$ (so $\sigma = \sigma'$ on $\partial\Omega \cap \partial\Omega'$), and where we have used that $\partial\Omega$ and $\partial\Omega'$ are both ADR. Since the hypotheses of Theorem 1.26 apply in Ω', we deduce from (8.14) and a formal application of the maximum principle that

$$(8.27) \qquad\qquad \alpha^\theta \lesssim \omega^X_{\Omega'}(\partial\Omega' \cap A) \leq \omega^X(A),$$

where $\omega_{\Omega'}$ is harmonic measure for Ω'. Thus, we obtain (8.26), with $\eta = (1 - \alpha/2)$. We caution the reader that our use of the maximum principle to obtain the second inequality in (8.27) is not routine, since we are working in a regime where the Wiener test may fail, and our solutions $X \to \omega^X(A)$ and $X \to \omega^X_{\Omega'}(\partial\Omega' \cap A)$ are not Perron solutions for the same domain, nor are they continuous on the closures of the respective domains under consideration. We shall give a rigorous justification of the essential inequality in (8.27) (namely, that $\alpha^\theta \lesssim \omega^X(A)$), at the end of this section.

It remains to establish (8.25). To this end, we again follow the argument in [7, Lemma 2.2]. Fix B and B' as above, and define U_k and S_k as in (8.20). In fact, we proceed as we did under the hypotheses of Theorem 1.26, except that the proof of (8.21) will now be somewhat more delicate, as we may no longer simply invoke the Harnack chain condition. Instead, we return to the original approach of [7]. It is enough to verify (8.23), as the remainder of the proof is unchanged. In particular, we obtain (8.24), which in turn yields (8.25), since $2B' \subset B$.

As before, (8.23) is a direct consequence of (8.21) and (8.22). The latter always holds, by Lemma 3.6, so we consider (8.21). Again we follow [7] essentially verbatim. We suppose first that there exists $Y \in S_k$ with $\delta(Y) = c\varepsilon s/(100K)$, where c is the constant in Lemma 3.6. For each such Y, we fix $y \in \partial\Omega$, with $|Y - y| = \delta(Y)$, and set $\Delta_Y := \Delta(y, K\delta(Y))$. If $\eta_\varepsilon \in (0, 1)$ is chosen close enough to 1, depending on ε and the ADR constants of $\partial\Omega$, and if $A \subset 2\Delta'$ with $\sigma(A) \geq \eta_\varepsilon\,\sigma(2\Delta')$, then

$$\sigma(A \cap \Delta_Y) \geq \eta\,\sigma(\Delta_Y),$$

for η as in (8.26), so that $\omega^Y(A) \geq \omega^Y(A \cap \Delta_Y) \geq c_0$. Thus, (8.21) holds in this case (with $c\varepsilon s/100$ now multiplied by $1/K$), by Harnack's inequality, because even in the absence of the

Harnack chain condition, there is a Harnack path from any

$$X \in S_k \cap \{X : \delta(X) \geq c\varepsilon s/(100K)\}$$

to a point Y in S_k with $\delta(Y) = c\varepsilon s/(100K)$, if the latter exists (just follow a geodesic path on S_k from X to the nearest such Y).

On the other hand, suppose that there is no such Y. Then either

$$S_k \subset \{X \in \Omega : \delta(X) > c\varepsilon s/(100K)\},$$

or $S_k \subset \{X \in \Omega : \delta(X) < c\varepsilon s/(100K)\}$. In the latter case, (8.22) holds now for all $X \in S_k$, so (8.23) follows trivially. Otherwise, by continuity of δ, there is a number $\rho > 0$ such that

$$(8.28) \quad \{X \in \Omega : \rho \leq |X - z| \leq (5/4 + \varepsilon(k + 1/2))s\}$$
$$\subset \{X \in \Omega : \delta(X) \geq c\varepsilon s/(100K)\},$$

and $\delta(Y) = c\varepsilon s/(100K)$ for some $Y \in S(\rho) := \{X \in \Omega : |X - z| = \rho\}$. In this case, we may repeat the analysis above, in which there was such a Y on S_k. In the present scenario, we have that (8.21) holds for all $X \in S(\rho) \cap \{X \in \Omega : \delta(X) \geq c\varepsilon s/(100K)\}$, which in fact is all of $S(\rho)$ by (8.28). But then by Harnack's inequality we obtain (8.21) (for all $X \in S_k$), because the containment in (8.28) allows us to form a radial Harnack path between any $X \in S_k$, and its projection onto $S(\rho)$. We conclude that (8.23) holds under all circumstances.

To finish the proof of Theorem 1.27, it remains only to provide a rigorous justification of (8.27). We shall make up for the lack of continuity of the solutions by proceeding as in the removal of the qualitative hypothesis in the proof of Theorem 1.26, with a few minor modifications. We fix $\varepsilon_1 > 0$ to be chosen momentarily, and set $F := A \cap \partial\Omega'$. We recall that $H^n(F) \geq (\alpha/2)H^n(\Delta_X)$. By outer regularity of Hausdorff measure and ω, there is a set $\mathcal{O}$, relatively open in $\partial\Omega$, such that $F \subset \mathcal{O} \subset \Delta_X \subset \partial\Omega$, and

$$H^n(\mathcal{O} \setminus F) + \omega^X(\mathcal{O} \setminus F) < \varepsilon_1.$$

We let $\mathcal{F} \subset \mathbb{D}(\partial\Omega)$ be a family of non-overlapping dyadic cubes whose union equals $\mathcal{O}$, so that $H^n(\mathcal{O}) = \sum_{\mathcal{F}} H^n(Q_k)$, and we set

$$\mathcal{F}' := \left\{Q_k \in \mathcal{F} : H^n(Q_k \cap F) \geq \frac{1}{4}H^n(Q_k)\right\}.$$

We claim that

$$(8.29) \qquad \alpha H^n(\Delta_X) \lesssim \sum_{\mathcal{F}'} H^n(Q_k).$$

Indeed, we have that

$$H^n(F) = \sum_{\mathcal{F}\setminus\mathcal{F}'} H^n(Q_k \cap F) + \sum_{\mathcal{F}'} H^n(Q_k \cap F) \leq \frac{1}{4}H^n(\mathcal{O}) + \sum_{\mathcal{F}'} H^n(Q_k),$$

whence (8.29) follows, if we choose $\varepsilon_1 \ll H^n(F)$.

Since each $Q_k \in \mathcal{F}'$ has an ample intersection with F, by Lemma **??** (vi), we may choose a point $x_k \in Q_k \cap F \subset \partial\Omega' \cap \partial\Omega$, and a radius $r_k \approx \ell(Q_k)$, such that $Q_k \supset \partial\Omega \cap B_k$, where $B_k := B(x_k, r_k)$. We emphasize that, in particular, each B_k is centered on $\partial\Omega' \cap \partial\Omega$. Set $B'_k := cB_k$, where $c \in (0, 1)$ is the constant in Lemma 3.6. Set $\mathcal{F}'_N := \mathbb{D}_N(\partial\Omega')$, and let $\Omega'_N := \Omega'_{\mathcal{F}_N}$ be the corresponding approximating domain relative to Ω'. By the ADR

property and a covering lemma argument, and by choice of N sufficiently large, we can select a finite, pairwise disjoint sub-collection $\{B_k\}_{1 \leq k \leq M}$, such that

$$\alpha\, H^n(\Delta_\star^N) \approx \alpha\, H^n(\Delta_X) \lesssim \sum_{k=1}^{M} H^n(B_k' \cap \partial\Omega_N')$$

where $\Delta_\star^N$ is a surface ball on $\partial\Omega_N'$ of radius $\approx \delta(X)$, such that X is a Corkscrew point for $\Delta_\star^N$ in Ω_N', and $\bigcup_{k=1}^{M}(B_k' \cap \partial\Omega_N') \subset \Delta_\star^N$. We may apply Theorem 1.26 in Ω_N' (see the discussion immediately following (8.13) above), to obtain that ω_N^X, the harmonic measure for the approximating domain Ω_N', belongs to $A_\infty(\Delta_\star^N)$ with bounds that are independent of N.

We now set $A' := \left(\bigcup_{k=1}^{M} B_k\right) \cap \partial\Omega$, and observe that $A' \subset \mathcal{O}$. Since $X \to \omega^X(A')$ is continuous on $\overline{\Omega_N'}$, we may repeat the argument in (8.17), *mutatis mutandis*, to obtain that

$$\omega^X(A) + \varepsilon_1 \geq \omega^X(F) + \varepsilon_1 \geq \omega^X(\mathcal{O}) \geq \omega^X(A') \gtrsim \alpha^\theta.$$

We choose $\varepsilon_1 \ll \alpha^\theta$, and it follows that $\alpha^\theta \lesssim \omega^X(A)$, as desired. $\qquad\square$

9. Proof of the extrapolation lemma

To finish the proofs of Theorems 1.26 and 1.27, it remains to prove Lemma 8.5.

Proof of Lemma 8.5. – The proof follows the strategy introduced in [36], and developed further in [25, 3] and [4]. In more precise detail, the argument is based on the systematic treatment given in [27] in the Euclidean setting.

The proof uses an induction argument with continuous parameter. The induction hypothesis is the following: given $a \geq 0$,

$$\boxed{H(a)} \quad \boxed{\begin{array}{l} \text{There exist } \eta_a \in (0,1) \text{ and } C_a < \infty \text{ such that for every } Q \in \mathbb{D}_{Q_0} \\ \text{satisfying } \mathfrak{m}(\mathbb{D}_Q) \leq a\, \sigma(Q), \text{ it follows that} \\[2mm] \qquad F \subset Q, \quad \dfrac{\sigma(F)}{\sigma(Q)} \geq 1 - \eta_a \quad \Longrightarrow \quad \dfrac{\omega(F)}{\omega(Q)} \geq \dfrac{1}{C_a}. \end{array}}$$

The induction argument is split in two steps.

STEP 1. – Show that $H(0)$ holds.

STEP 2. – Show that there exists b depending on γ, dimension, and the ADR property such that for all $0 \leq a \leq M_0$, $H(a)$ implies $H(a + b)$.

Once these steps have been carried out, the proof follows easily: pick $k \geq 1$ such that $(k-1)\,b < M_0 \leq k\,b$ (note that k only depends on b and M_0). By STEP 1 and STEP 2, it follows that $H(k\,b)$ holds. Observe that $\|\mathfrak{m}\|_{\mathcal{C}(Q_0)} \leq M_0 \leq k\,b$ implies $\mathfrak{m}(\mathbb{D}_Q) \leq k\,b\,\sigma(Q)$ for all $Q \in \mathbb{D}_{Q_0}$, and by $H(k\,b)$ we conclude (8.8).

Step 1. $H(0)$ holds

If $\mathfrak{m}(\mathbb{D}_Q) = 0$ then we take $\mathcal{F}$ to be empty, so that $\mathbb{D}_Q \cap \mathbb{D}_{\mathcal{F}} = \mathbb{D}_Q$, and $\mathcal{P}_{\mathcal{F}}\,\omega = \omega$. Then (8.6) holds (since $0 \leq \gamma$) and therefore we can use (8.7) with ω in place of $\mathcal{P}_{\mathcal{F}}\,\omega$, which is the desired property.

Step 2. $H(a)$ **implies** $H(a + b)$

Fix $0 \leq a \leq M_0$ and $Q \in \mathbb{D}_{Q_0}$ such that $\mathfrak{m}(\mathbb{D}_Q) \leq (a + b)\,\sigma(Q)$, where we choose b so that $C\,b := \gamma$ and C is the constant in the right hand side of (7.3). We also fix $F \subset Q$ with $\sigma(F) \geq (1 - \eta)\,\sigma(Q)$, where $0 < \eta \leq \eta_{a,b}$ and $\eta_{a,b}$ is to be chosen. We may now apply Lemma 7.2 and Remark 7.5 to the cube Q, to construct the non-overlapping family of cubes $\mathscr{F} = \{Q_j\} \subset \mathbb{D}_Q$ with the stated properties. Set

$$E_0 = Q \setminus \bigcup_{Q_j \in \mathscr{F}} Q_j, \qquad G = \bigcup_{Q_j \in \mathscr{F}_{\mathrm{good}}} Q_j, \qquad B = \bigcup_{Q_j \in \mathscr{F} \setminus \mathscr{F}_{\mathrm{good}}} Q_j,$$

where $\mathscr{F}_{\mathrm{good}} = \{Q_j \in \mathscr{F} : \mathfrak{m}(\mathbb{D}_{Q_j}^{\mathrm{short}}) \leq a\,\sigma(Q_j)\}$. We recall that by (7.4), we have $\sigma(B)/\sigma(Q) \leq (a + b)/(a + 2b)$.

We shall also require the following "pigeon hole" lemma, which says that "most" of the cubes Q_j have an ample overlap with F.

LEMMA 9.1. – *Given $0 < \tilde{\eta} < 1$, we set*

$$\mathscr{F}_1 = \{Q_j \in \mathscr{F}_{\mathrm{good}} : \sigma(F \cap Q_j) \geq (1 - \tilde{\eta})\,\sigma(Q_j)\}, \qquad G_1 = \bigcup_{Q_j \in \mathscr{F}_1} Q_j.$$

If $0 < \eta \leq \eta_1 := \tilde{\eta}\,\frac{1}{2}\left(1 - \frac{M_0 + b}{M_0 + 2b}\right)$, then $\sigma(E_0 \cup G_1) \geq \eta_1\,\sigma(Q)$.

Proof. – Take θ such that $\sigma(B) = \theta\,\sigma(Q)$, and $\theta_0 = (M_0 + b)/(M_0 + 2b)$. By (7.4) and since $a \leq M_0$ we obtain that $\theta \leq \theta_0$:

$$\theta\,\sigma(Q) = \sigma(B) \leq \frac{a + b}{a + 2b}\,\sigma(Q) \leq \theta_0\,\sigma(Q).$$

We set $B_1 = \bigcup_{Q_j \in \mathscr{F}_{\mathrm{good}} \setminus \mathscr{F}_1} Q_j$ and observe that $B_1 \subset G \subset Q \setminus B$. Hence,

$$\sigma(F \cap B_1) = \sum_{Q_j \in \mathscr{F}_{\mathrm{good}} \setminus \mathscr{F}_1} \sigma(F \cap Q_j) < (1 - \tilde{\eta}) \sum_{Q_j \in \mathscr{F}_{\mathrm{good}} \setminus \mathscr{F}_1} \sigma(Q_j)$$

$$= (1 - \tilde{\eta})\,\sigma(B_1) \leq (1 - \tilde{\eta})\,\sigma(Q \setminus B) = (1 - \tilde{\eta})\,(1 - \theta)\,\sigma(Q).$$

Thus, using that $\theta \leq \theta_0$, we have

$$(1 - \eta)\,\sigma(Q) \leq \sigma(F) = \sigma(F \cap E_0) + \sigma(F \cap B) + \sigma(F \cap G_1) + \sigma(F \cap B_1)$$

$$\leq \sigma(E_0) + \sigma(B) + \sigma(G_1) + (1 - \tilde{\eta})\,(1 - \theta)\,\sigma(Q)$$

$$= \sigma(E_0) + \sigma(G_1) + \left[\theta + (1 - \tilde{\eta})\,(1 - \theta)\right]\,\sigma(Q)$$

$$\leq \sigma(E_0) + \sigma(G_1) + \left[1 - \tilde{\eta}\,(1 - \theta_0)\right]\,\sigma(Q)$$

and therefore

$$\sigma(E_0 \cup G_1) = \sigma(E_0) + \sigma(G_1) \geq \left[\tilde{\eta}\,(1 - \theta_0) - \eta\right]\,\sigma(Q) \geq \frac{1}{2}\,\tilde{\eta}\,(1 - \theta_0)\,\sigma(Q) = \eta_1\,\sigma(Q),$$

where we have used that $\eta \leq \tilde{\eta}\,(1 - \theta_0)/2 = \eta_1$. $\qquad \square$

We now return to the proof of Step 2. We apply Lemma 9.1, with $\tilde{\eta} \in (0, 1)$ to be chosen. Given $Q_j \in \mathscr{F}_1 \subset \mathscr{F}_{\mathrm{good}}$ we have that $\mathfrak{m}(\mathbb{D}_{Q_j}^{\mathrm{short}}) \leq a\,\sigma(Q_j)$. Moreover,

$$\mathbb{D}_{Q_j}^{\mathrm{short}} = \mathbb{D}_{Q_j} \setminus \{Q_j\} = \bigcup_i \mathbb{D}_{Q_i^j},$$

where $\{Q_i^j\}_i$ is the family of dyadic "children" of Q_j (these are the subcubes of Q_j which lie in the very next dyadic generation $\mathbb{D}_{k(Q_j)+1}$). Then by pigeon-holing, there exists at least one i_0 such that $Q_{i_0}^j =: Q_j'$ satisfies

$$(9.2) \qquad\qquad \mathfrak{m}(\mathbb{D}_{Q_j'}) \leq a\,\sigma(Q_j')$$

(there could be more than one i_0 with this property, but we just pick one). We define $\widetilde{\mathscr{F}}_1$ to be the collection of those selected "children" Q_j', with $Q_j \in \mathscr{F}_1$. Let C_0 be the dyadically doubling constant of σ, i.e., $\sigma(Q) \leq C_0\sigma(Q')$ for every $Q \in \mathbb{D}_{Q_0}$, and for every "child" Q' of Q. Then, for each such Q_j', using the definition of $\mathscr{F}_1$, and taking $0 < \tilde{\eta} = \eta_a/C_0$ (where $0 < \eta_a < 1$ is provided by $H(a)$), we have

$$\sigma(Q_j' \setminus F) \leq \sigma(Q_j \setminus F) \leq \tilde{\eta}\,\sigma(Q_j) \leq \tilde{\eta}\,C_0\sigma(Q_j') = \eta_a\,\sigma(Q_j'),$$

which yields $\sigma(Q_j' \cap F) \geq (1 - \eta_a)\,\sigma(Q_j')$. With this estimate and (9.2) in hand, we can use the induction hypothesis $H(a)$ to deduce:

$$(9.3) \qquad\qquad \omega(Q_j' \cap F) \geq \frac{1}{C_a}\,\omega(Q_j'), \qquad \forall\, Q_j' \in \widetilde{\mathscr{F}}_1.$$

On the other hand, if we set $\widetilde{G}_1 = \bigcup_{Q_j' \in \widetilde{\mathscr{F}}_1} Q_j'$, then

$$\sigma(\widetilde{G}_1) = \sum_{Q_j' \in \widetilde{\mathscr{F}}_1} \sigma(Q_j') \geq C_0^{-1} \sum_{Q_j \in \mathscr{F}_1} \sigma(Q_j) = C_0^{-1}\sigma(G_1).$$

Thus, by Lemma 9.1, having now fixed $\tilde{\eta}$ above, we have that

$$\sigma(E_0 \cup \widetilde{G}_1) = \sigma(E_0) + \sigma(\widetilde{G}_1) \geq C_0^{-1}\,\sigma(E_0 \cup G_1) \geq C_0^{-1}\,\eta_1\sigma(Q) =: \eta_2\,\sigma(Q),$$

if $\eta \leq \eta_1$, from which it follows that

$$\sigma(F \cap (E_0 \cup \widetilde{G}_1)) \geq \frac{1}{2}\eta_2\sigma(Q) =: \eta_{a,b}\,\sigma(Q),$$

if $\eta \leq \eta_2/2$, since $\sigma(Q \setminus F) \leq \eta\,\sigma(Q)$.

We recall that the family $\mathscr{F}$ was constructed using Lemma 7.2 with $C\,b := \gamma$. Consequently, by (7.3), we may deduce that (8.6) holds, so in turn, by hypothesis, we can apply (8.7) to the set $F \cap (E_0 \cup \widetilde{G}_1)$, obtaining

$$\frac{\mathscr{P}_{\mathscr{F}}\,\omega(F \cap (E_0 \cup \widetilde{G}_1))}{\mathscr{P}_{\mathscr{F}}\,\omega(Q)} \geq \frac{1}{C_{\eta_{a,b}}}.$$

As observed before, $\mathscr{P}_{\mathscr{F}}\,\omega(Q) = \omega(Q)$. Thus, in order to establish the conclusion of $H(a+b)$, and consequently to complete the proof of Lemma 8.5, it remains only to show that

$$\mathscr{P}_{\mathscr{F}}\,\omega(F \cap (E_0 \cup \widetilde{G}_1)) \leq C\,\omega(F).$$

To this end, we use first the definition of $\mathscr{P}_{\mathscr{G}}$, then that ω is dyadically doubling and finally (9.3) to obtain

$$\mathscr{P}_{\mathscr{G}}\,\omega(F \cap (E_0 \cup \widetilde{G}_1)) = \mathscr{P}_{\mathscr{G}}\,\omega(F \cap E_0) + \mathscr{P}_{\mathscr{G}}\,\omega(F \cap \widetilde{G}_1)$$

$$= \omega(F \cap E_0) + \sum_{Q_j \in \mathscr{G}_1} \frac{\sigma(Q_j' \cap F)}{\sigma(Q_j)}\,\omega(Q_j)$$

$$\leq \omega(F \cap E_0) + C_\omega \sum_{Q_j' \in \widetilde{\mathscr{G}}_1} \omega(Q_j')$$

$$\leq \omega(F \cap E_0) + C_\omega C_a \sum_{Q_j' \in \widetilde{\mathscr{G}}_1} \omega(Q_j' \cap F) \leq C\,\omega(F).$$

This concludes the proof of Lemma 8.5. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Appendix A

Inheritance of properties by Carleson and sawtooth regions

This section is devoted to the proof of Lemma 3.61, which states that Carleson and Sawtooth regions inherit the Corkscrew, Harnack chain and ADR properties from the original domain Ω. Moreover, in the presence of the Corkscrew, Harnack chain and ADR properties, the UR property is transmitted to the Carleson boxes T_Q and T_Δ. We discuss these properties one at a time. We shall find it convenient for our purposes in this section to continue to let Δ denote a surface ball on $\partial\Omega$, while $\Delta_\star$ will denote a surface ball on the boundary of the sub-domain under consideration. Similarly $\delta(X)$ will continue to denote the distance from X to $\partial\Omega$, while $\delta_\star(X)$ will denote the distance from X to the boundary of the sub-domain under consideration.

In order to avoid possible confusion, let us emphasize that the construction of our sawtooth and Carleson sub-domains is always based on the Whitney decomposition of the domain under consideration at that moment, even if that domain happens to be, say, an approximating domain Ω_N which had been constructed in the first place from Whitney cubes of the original domain Ω.

A.1. Corkscrew

For the sake of specificity, we treat only the case of a local sawtooth region $\Omega_{\mathscr{G},Q}$. The proof for the global sawtooth $\Omega_{\mathscr{G}}$ is almost identical. Moreover, specializing to the case that $\mathscr{G} = \varnothing$, we see that the result for a sawtooth $\Omega_{\mathscr{G},Q}$ applies immediately to the Carleson box T_Q, and therefore also almost immediately to any box T_Δ, since the latter is a union of a bounded number of T_Q's.

We fix $Q_0 \in \mathbb{D}$, and a pairwise disjoint family $\{Q_j\} = \mathscr{G} \subset \mathbb{D}_{Q_0}$, and let $\Omega_{\mathscr{G},Q_0}$ denote the associated local sawtooth region (cf. (3.39)-(3.54)). Set

$$\Delta_\star := \Delta_\star(x,r) := B(x,r) \cap \partial\Omega_{\mathscr{G},Q_0},$$

with $r \lesssim \ell(Q_0)$ and $x \in \partial\Omega_{\mathcal{F},Q_0}$. We suppose first that $x \in \partial\Omega_{\mathcal{F},Q_0} \cap \partial\Omega$. Then by construction of $\Omega_{\mathcal{F},Q_0}$, there is a $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, with $x \in \overline{Q}$, and $r \approx 100K_0\,\ell(Q)$ (see Proposition 6.1). Consequently, by (3.47)-(3.49), we have

$$X_Q \in U_Q \subset B(x,r) \cap \Omega_{\mathcal{F},Q_0},$$

where X_Q is a Corkscrew point for Ω, relative to Q, and which we have assumed (without loss of generality) to be the center of some $I \in \mathcal{W}_Q^*$. This same X_Q then serves as a Corkscrew point for $\Omega_{\mathcal{F},Q_0}$, relative to $\Delta_\star(x,r)$, with Corkscrew constant $c \approx 1/(100K_0)$.

Next, we suppose that $x \in \partial\Omega_{\mathcal{F},Q_0} \setminus \partial\Omega$, where as above $\Delta_\star := \Delta_\star(x,r)$. Then by definition of the sawtooth region, x lies on a face of a fattened Whitney cube $I^* = (1+\lambda)I$, with $I \in \mathcal{W}_Q^*$, for some $Q \in \mathbb{D}_{\mathcal{F},Q_0}$. If $r \lesssim \ell(I)$, then trivially there is a point $X^\star \in I^*$ such that $B(X^\star,cr) \subset B(x,r) \cap \text{int}(I^*) \subset B(x,r) \cap \Omega_{\mathcal{F},Q_0}$. This $X^\star$ is then a Corkscrew point for $\Delta_\star$. On the other hand, if $\ell(I) < r/(MK_0)$, with M sufficiently large to be chosen momentarily, then there is a $Q' \in \mathbb{D}_{\mathcal{F},Q_0}$, with $\ell(Q') \approx r/(MK_0)$, and $Q \subseteq Q'$. Now fix $I' \in \mathcal{W}_{Q'} \subset \mathcal{W}_{Q'}^*$, and observe that

$$|x - X(I')| \lesssim \text{dist}(I,Q) + \text{dist}(Q',I') \lesssim K_0\,\ell(I) + K_0\,\ell(I') \lesssim r/M.$$

Note that $B(X(I'),cr) \subset \text{int}(I')$, for $c \approx (MK_0)^{-1}$. Moreover, for M large enough we have that $B(X(I'),cr) \subset B(x,r) \cap \text{int}(I') \subset B(x,r) \cap \Omega_{\mathcal{F},Q_0}$, so that $X(I')$ is a Corkscrew point for $\Delta_\star$.

A.2. Harnack chain

We establish the Harnack chain condition for a local sawtooth $\Omega_{\mathcal{F},Q}$, of which, as noted above, the Carleson box T_Q is a special case (with $\mathcal{F} = \varnothing$). The proof for a global sawtooth is almost the same, and we omit it. We shall discuss the Carleson boxes T_Δ at the end of this subsection.

Fix $Q \in \mathbb{D}$, and a pairwise disjoint family $\mathcal{F} \subset \mathbb{D}_Q$, and let $\Omega_{\mathcal{F},Q}$ be the corresponding local sawtooth region. Let $X_1, X_2 \in \Omega_{\mathcal{F},Q}$. By definition of the sawtooth regions, there exist $Q_1, Q_2 \in \mathbb{D}_{\mathcal{F},Q}$, with $X_i \in (1+\lambda)I_i = I_i^*$ where $I_i \in \mathcal{W}_{Q_i}^*$, $i = 1,2$. Without loss of generality we may suppose that $\ell(Q_1) \leq \ell(Q_2)$. We first observe that the desired result is clear if $I_1 = I_2$, or more generally, if I_1^* and I_2^* overlap. Therefore, we may suppose that

$$(\text{A.1}) \qquad\qquad \text{dist}(I_1^*, I_2^*) \gtrsim \ell(I_2) \gtrsim \ell(I_1)$$

(cf. (3.50).) In order to construct a Harnack chain under these circumstances, relative to $\Omega_{\mathcal{F},Q}$, from X_1 to X_2, it is convenient to make a few simple reductions and observations, as follows.

(1) It is enough to treat the case that X_i is the center of I_i. If X_i is near the boundary of the sawtooth (and hence also near the boundary of I_i^*), then $\text{dist}(X_i,\partial I_i^*) \approx \text{dist}(X_i,\partial\Omega_{\mathcal{F},Q})$, so that the Harnack chain within I_i^*, that connects X_i to the center $X(I_i)$, is also a Harnack chain for the sawtooth. On the other hand, if X_i is not near the boundary of the sawtooth, then we can easily join X_i with $X(I_i)$ by a bounded number of balls of radius $\approx \ell(I_i)$ with distance to the boundary of $\partial\Omega_{\mathcal{F},Q}$ comparable to $\delta_\star(X_i)$.

(2) By construction (cf. (3.47)-(3.49)), we may then further suppose that $X_i = X_{Q_i}$, the designated Corkscrew point (for the ambient domain Ω), relative to Q_i.

(3) Recall that by construction, if $Q' \subset Q''$ belong to consecutive generations in $\mathbb{D}$ (i.e., $k(Q'') = k(Q') - 1$), then $U_{Q'} \cap U_{Q''}$ contains the Corkscrew point $X_{Q'}$ (cf. (3.48)) and is therefore non-empty. Thus, by (3.47)-(3.49) there is a Harnack chain joining the respective Corkscrew points $X_{Q'}$ and $X_{Q''}$.

(4) We note that by definition, if $Q_i \in \mathbb{D}_{\mathcal{F},Q}$, then also $Q' \in \mathbb{D}_{\mathcal{F},Q}$ for every Q' such that $Q_i \subseteq Q' \subseteq Q$.

(5) If $X(I)$ denotes the center of a Whitney cube I, then $\delta(X(I)) \approx \delta_\star(X(I)) \approx \ell(I)$.

With these observations in mind, we consider three cases. Set $R := |X_1 - X_2|$.

CASE 1. – $Q_1 \subseteq Q_2$. In this case, $R \lesssim \ell(Q_2)$ (with $R \approx \ell(Q_2)$ if $\ell(Q_2) \gg \ell(Q_1)$), and $\min(\delta_\star(X_1), \delta_\star(X_2)) \gtrsim \ell(Q_1)$. Consequently, we may form a Harnack chain of cardinality $\approx k(Q_1) - k(Q_2) + 1$ that connects the Corkscrew points of every Q', with $Q_1 \subseteq Q' \subseteq Q_2$.

Before proceeding to the remaining cases, we observe that if Case 1 does not hold, then Q_1 and Q_2 are disjoint, whence it follows from (A.1) and observations (1) and (5) above that

$$(\text{A.2}) \qquad\qquad R \gtrsim \delta_\star(X_2) \approx \ell(Q_2) \geq \ell(Q_1) \approx \delta_\star(X_1).$$

Of course, we also have $R \lesssim \ell(Q)$.

CASE 2. – $Q_1 \cap Q_2 = \varnothing$, but have a common ancestor $Q^* \subseteq Q$, with $\ell(Q^*) \approx R$. We may then proceed as in Case 1, to construct respective Harnack chains, connecting each of X_1 and X_2, to X_{Q^*}. The union of these two chains connects X_1 to X_2.

CASE 3. – Q_1 and Q_2 have no common ancestor of length $\approx R$. In this case, we may suppose that $R < \ell(Q)/(MK_0)$, where M is a sufficiently large number to be chosen momentarily. Indeed, if not, then $\ell(Q)/(MK_0) \leq R \lesssim \ell(Q)$, in which case Q would be a common ancestor with $\ell(Q) \approx R$.

Thus, since $R < \ell(Q)/(MK_0)$, there exist $Q_1^*, Q_2^* \in \mathbb{D}_Q$ such that, for $i = 1, 2$, Q_i^* is an ancestor of Q_i, with $\ell(Q_1^*) = \ell(Q_2^*) \approx MK_0R$. Since $\text{dist}(X_i, Q_i) \lesssim K_0 \ell(Q_i)$ by construction (cf. (3.49)), we then have that $\text{dist}(Q_1, Q_2) \lesssim K_0R$ (by (A.2) and the triangle inequality), and therefore also that

$$\text{dist}(Q_1^*, Q_2^*) \leq C\ell(Q_1^*)/M \leq \ell(Q_1^*) = \ell(Q_2^*),$$

by choice of M large enough. Consequently, by (3.44), $\mathcal{W}_{Q_1^*}^* \cap \mathcal{W}_{Q_2^*}^*$ is non-empty, whence there is a Harnack chain connecting the respective Corkscrew points $X_{Q_1^*}$ and $X_{Q_2^*}$. We may then proceed as above to construct a Harnack chain from X_i to $X_{Q_i^*}$, $i = 1, 2$, and the proof of the Harnack chain condition for the sawtooth $\Omega_{\mathcal{F},Q}$ is now complete.

We finish this subsection by verifying the Harnack chain property for a Carleson box T_Δ. Let $X_1, X_2 \in T_\Delta$, with $a := \delta_\star(X_1) \leq \delta_\star(X_2) =: b$. As above, we may suppose that I_1^* and I_2^* are separated, and thus as in observation (1), that each X_i, $i = 1, 2$ is the respective center of the Whitney cube I_i whose dilate contains it.

By definition of T_Δ (cf. (3.58)-(3.59)), and since X_i is the center of I_i, we have $X_i \in T_{Q^i}$, where $Q^i \in \mathbb{D}^\Delta$, $i = 1, 2$. By (3.44), $\mathcal{W}_{Q^1} \cap \mathcal{W}_{Q^2}$ is non-empty. Consequently, there is a Harnack chain connecting the respective Corkscrew points X_{Q^1} and X_{Q^2}, so in the case that $|X_1 - X_2| =: \Lambda a \approx r_\Delta$, we may connect X_1 to X_{Q^1} to X_{Q^2} to X_2.

Therefore, we may now suppose that $|X_1 - X_2| = \Lambda a \leq r_\Delta/(MK_0)$, for some sufficiently large M to be chosen momentarily. We note that there is a uniform constant $c > 0$ such that $\Lambda \geq c$, since X_1 and X_2 are the respective centers of non-overlapping Whitney cubes

(cf. observation (5) above). We now claim that we also have $b \lesssim \Lambda a$. Indeed, if $b \gg \Lambda a$, then by the triangle inequality, $a \geq b - \Lambda a \gg \Lambda a$, which contradicts the uniform lower bound for Λ. Therefore, by observation (5) above, and by construction of each T_{Q^i}, there exist $\widetilde{Q}^1 \subset Q^1$, $\widetilde{Q}^2 \subset Q^2$ such that $X_i \in T_{\widetilde{Q}^i}$, $\ell(\widetilde{Q}^1) = \ell(\widetilde{Q}^2) \approx MK_0 \Lambda a$, and

$$\operatorname{dist}(X_1, \widetilde{Q}^1) \lesssim K_0 a, \qquad \operatorname{dist}(X_2, \widetilde{Q}^2) \lesssim K_0 b \lesssim K_0 \Lambda a.$$

By the triangle inequality, we then have

$$\operatorname{dist}(\widetilde{Q}^1, \widetilde{Q}^2) \leq C\ell(\widetilde{Q}^1)/M \leq \ell(\widetilde{Q}^1) = \ell(\widetilde{Q}^2),$$

by choice of M large enough. By (3.44), $\mathcal{W}_{\widetilde{Q}^1} \cap \mathcal{W}_{\widetilde{Q}^2}$ is non-empty, so that we may construct a Harnack chain from X_1 to X_2, by a now familiar argument, via the Corkscrew points $X_{\widetilde{Q}^1}$ and $X_{\widetilde{Q}^2}$.

A.3. ADR

Suppose that $\partial\Omega$ is ADR, and we show first that for each $Q \in \mathbb{D}(\partial\Omega)$, the boundary of the "Carleson box" T_Q is also ADR. We begin with the upper bound. Let $x \in \partial T_Q$, and let $\Delta_\star := \Delta_\star(x, r) := B(x, r) \cap \partial T_Q$, with $r \lesssim \operatorname{diam} Q$. If $B(x, r)$ meets $\partial\Omega$, then there is a point $x' \in \partial\Omega$ such that $B(x, r) \subset B(x', 2r)$. Consequently,

$$H^n\left(B(x, r) \cap \partial\Omega \cap \partial T_Q\right) \leq H^n\left(\Delta(x', 2r)\right) \lesssim r^n,$$

since $\partial\Omega$ is ADR.

Now consider $\Delta_\star \setminus \partial\Omega$. This portion of $\Delta_\star$ is contained in a union of faces (or partial faces) of fattened Whitney cubes $I^* = (1 + \lambda)I$. Let $\mathscr{I}_Q$ denote the collection of Whitney cubes I for which ∂I^* meets ∂T_Q, and $\operatorname{int}(I^*) \subset T_Q$. Suppose that $I \in \mathscr{I}_Q$ is a Whitney cube such that ∂I^* meets $\Delta_\star$. Then

$$(A.3) \qquad H^n\left(\Delta_\star \cap \partial I^*\right) \leq H^n\left(B(x, r) \cap \partial I^*\right) \lesssim \min(\ell(I)^n, r^n).$$

Therefore,

$$\sum_{I \in \mathscr{I}_Q : \ell(I) \geq r/(MK_0)} H^n\left(\Delta_\star \cap \partial I^*\right) \lesssim r^n,$$

because only a bounded number of terms can appear in this sum. Here, M is a sufficiently large number to be chosen, and K_0 is the same constant appearing in (3.49). It remains to consider

$$\sum_{I \in \mathscr{I}_Q : \ell(I) < r/(MK_0)} H^n\left(\Delta_\star \cap \partial I^*\right) = \sum_{k : 2^{-k} < r/(MK_0)} \sum_{I \in \mathscr{I}_Q^k} H^n\left(\Delta_\star \cap \partial I^*\right),$$

where $\mathscr{I}_Q^k := \{I \in \mathscr{I}_Q : \ell(I) = 2^{-k}\}$. It is then enough to show that there is an $\varepsilon > 0$ such that for each k with $2^{-k} < r/(MK_0)$, we have

$$(A.4) \qquad \sum_{I \in \mathscr{I}_Q^k} H^n\left(\Delta_\star \cap \partial I^*\right) \lesssim 2^{-k\varepsilon} r^{n-\varepsilon}.$$

It follows from (A.3) that the latter bound will hold if the cardinality of the set of I's which make a non-trivial contribution to the sum is no larger than

$$(A.5) \qquad C(2^k r)^{n-\varepsilon}.$$

We recall that by the definition of T_Q (cf. (3.43)-(3.52)), for each $I \in \mathscr{I}_Q^k$, there is a $Q_I \in \mathbb{D}_Q$ such that $\ell(Q_I) \approx \ell(I) = 2^{-k}$, and $\text{dist}(I, Q_I) \lesssim K_0 \ell(I)$. Since $2^{-k} < r$, there is a uniform constant C such that $B(x, Cr)$ contains each such Q_I, for every I such that ∂I^* meets $\Delta_\star$. We may then cover $B(x, Cr) \cap Q$ by a bounded number of subcubes $Q' \in \mathbb{D}_Q$, with $\ell(Q') \approx r$, so that each relevant Q_I is contained in some Q'. It is enough to consider those Q_I contained in one such Q'. We therefore now fix Q' and k, and distinguish two types of $Q_I \subset Q'$:

$$\text{Type 1}: \qquad \text{dist}(Q_I, (Q')^c) > 2^{-\gamma k} r^{1-\gamma}$$

$$\text{Type 2}: \qquad \text{dist}(Q_I, (Q')^c) \leq 2^{-\gamma k} r^{1-\gamma}$$

where we have fixed $\gamma \in (0, 1)$. We note that there are at most a bounded number of I's corresponding to each Q_I. Thus, since $2^{-k} < r/(MK_0) \ll r$, by Lemma **??** (vi) we have that the cardinality of the set of I's for which Q_I is of Type 2 is no larger than $C(2^k r)^{-\gamma \eta} r^n / (2^{-kn}) \approx (2^k r)^{n-\gamma \eta}$, which is (A.5), with $\varepsilon = \gamma \eta$.

We now claim that for M chosen large enough, depending on γ and K_0, the collection of I such that ∂I^* meets $\Delta_\star$, and for which Q_I is of Type 1, is empty. Indeed, if Q_I is of Type 1, and if M is sufficiently large, we then have

$$\text{dist}(Q_I, (Q')^c) > 2^{-\gamma k} r^{1-\gamma} > (MK_0)^{(1-\gamma)} 2^{-k} \gg K_0 \ell(I) \gtrsim \text{dist}(I, Q_I).$$

Consequently, if $y \in \partial \Omega$ satisfies $\text{dist}(I, y) \lesssim K_0 \ell(I)$, then

$$\text{dist}(y, Q_I) \lesssim K_0 \ell(I) \ll \text{dist}(Q_I, (Q')^c),$$

so that $y \in Q'$, and $\text{dist}(y, (Q')^c) \gg K_0 \ell(I)$. Now consider any Whitney cube $J \in \mathcal{W}$ that touches I. Then $\text{dist}(J, \partial \Omega) \approx \ell(J) \approx \ell(I) \approx \text{dist}(I, \partial \Omega)$, so that for some $y_J \in \partial \Omega$, we have $\text{dist}(y_J, J) \approx \text{dist}(y_J, I) \ll C_0 \ell(I) \leq K_0 \ell(I)$ (cf. (3.43) and (3.49).) Thus, $y_J \in Q'$, and $\text{dist}(y_J, (Q')^c) \gg K_0 \ell(I) \approx K_0 \ell(J)$. It follows that there is a $Q_J \in \mathbb{D}(\partial \Omega)$, with $Q_J \subset Q' \subset Q, y_J \in \overline{Q}_J, \ell(Q_J) = \ell(J)$, and $\text{dist}(Q_J, J) \leq \text{dist}(y_J, J) \leq C_0 \ell(J)$. Therefore, $J \in \mathcal{W}_{Q_J}$. Since this is true for all Whitney cubes J that touch I, we have in particular that every point on ∂I^* is an interior point of T_Q, hence $\Delta_\star \cap \partial I^* = \varnothing$. We have now established the upper bound $H^n(\Delta_\star(x, r)) \lesssim r^n$.

The lower bound is easy. Consider $B := B(x, r)$, $r \lesssim \text{diam} \, Q$, with $x \in \partial T_Q$. If $B \cap Q$ contains a surface ball $\Delta \subset \partial \Omega$, with radius $r_\Delta \gtrsim r$, then we are done, by the ADR property of $\partial \Omega$. Otherwise, if $B \cap Q$ contains no such surface ball, then $\text{dist}(x, Q) \gtrsim r$, whence it follows that $x \in \partial I^*$, where I is a Whitney cube with $\ell(I) \gtrsim r/K_0$ (cf. (3.49)), and where x lies in a subset F of a (closed) face of I^*, with $F \subset B \cap \partial T_Q$, and $H^n(F \cap B) = H^n(F) \gtrsim (r/K_0)^n$, as desired.

Next, we discuss the ADR property of a Carleson region T_Δ. By definition (cf. (3.59)), T_Δ is a union of a bounded number of regions T_Q. The upper bound in the ADR condition is then an immediate consequence of the corresponding bound for T_Q. The lower bound is proved in the same way as it was for T_Q depending on whether or not the ball B has an ample intersection with some $Q \in \mathbb{D}^\Delta$. We omit the routine details.

Finally, we establish the ADR property for the global (3.53) and local (3.54) sawtooth regions. The proofs are similar, so for the sake of specificity, we treat the global sawtooth $\Omega_\mathscr{G}$. We first prove the upper bound in the ADR condition. Fix $B := B(x, r)$. The desired bound for $H^n(B \cap \partial \Omega \cap \partial \Omega_\mathscr{G})$ is an immediate consequence of the fact that $\partial \Omega$ is ADR.

Now consider $\Sigma := \partial\Omega_{\mathscr{F}} \setminus \partial\Omega$. We observe that this portion of the boundary consists of (portions of) faces of certain fattened Whitney cubes $J^* = (1 + \lambda)J$, with $\text{int}(J^*) \subset \Omega_{\mathscr{F}}$, which meet some $I \in \mathcal{W}$ for which $I \notin \mathcal{W}_Q^*$, for any $Q \in \mathbb{D}_{\mathscr{F}}$ (so that $\tau I \subset \Omega \setminus \Omega_{\mathscr{F}}$ for some $\tau \in (1/2, 1)$; cf. (3.51).) Necessarily, $I \in \mathcal{W}_{Q'}^*$, where $Q' \in \mathbb{D}_{Q_j}$ for some $Q_j \in \mathscr{F}$. For each $Q_j \in \mathscr{F}$, we set

$$\mathscr{R}_{Q_j} := \bigcup_{Q' \in \mathbb{D}_{Q_j}} \mathcal{W}_{Q'}^*,$$

and denote by $\mathscr{F}_B$ the sub-collection of those $Q_j \in \mathscr{F}$ such that there is an $I \in \mathscr{R}_{Q_j}$ for which $B \cap \Sigma$ meets I. We then split the latter collection into $\mathscr{F}_B = \mathscr{F}_1 \cup \mathscr{F}_2$, where $Q_j \in \mathscr{F}_1$ if $\ell(Q_j) < r$, and $Q_j \in \mathscr{F}_2$ if $\ell(Q_j) \geq r$. We consider the contribution of the latter first. Suppose that Q_j and Q_k are both in $\mathscr{F}_2$, and without loss of generality that $r \leq \ell(Q_j) \leq \ell(Q_k)$. Since B meets some $I \in \mathscr{R}_{Q_j}$, we obtain in particular that $\text{dist}(y, \partial\Omega) \lesssim \ell(Q_j), \forall y \in B$. Thus, $B \cap \Sigma$ lies within $C\ell(Q_j)$ of $\partial\Omega$, and therefore meets no Whitney cubes of side length greater than $C\ell(Q_j)$. Consequently, any such Whitney cube $I' \in \mathscr{R}_{Q_k}$, which meets B, must lie within $CK_0\ell(Q_j)$ of Q_k. Therefore, for any pair $Q_j, Q_k \in \mathscr{F}_2$, we have that $\text{dist}(Q_j, Q_k) \lesssim \min(\ell(Q_j), \ell(Q_k))$ (with implicit constants depending on K_0.) Since the cubes in $\mathscr{F}$ are pairwise disjoint, it follows that the cardinality of $\mathscr{F}_2$ is uniformly bounded, hence

$$H^n\left(B \cap \Sigma \cap \left(\bigcup_{Q_j \in \mathscr{F}_2} \cup_{I \in \mathscr{R}_{Q_j}} I\right)\right) \lesssim \sup_{Q_j \in \mathscr{F}} H^n(B \cap \Sigma_j),$$

where $\Sigma_j := \Sigma \cap (\bigcup_{I \in \mathscr{R}_{Q_j}} I)$. The desired bound for the contribution of $\mathscr{F}_2$ is an immediate consequence of following estimate, which holds for every $Q_j \in \mathscr{F}$:

$$(A.6) \qquad H^n(B \cap \Sigma_j) \lesssim (\min(r, \ell(Q_j)))^n.$$

Let us take the latter bound for granted momentarily, and consider the contribution of $\mathscr{F}_1$. If $Q_j \in \mathscr{F}_1$, then $Q_j \subset B^* := CK_0 B$ for some uniform constant C. By the case $r > \ell(Q_j)$ of (A.6), we have that $H^n(\Sigma_j \cap B) \lesssim H^n(Q_j)$. Therefore,

$$H^n\left(B \cap \left(\bigcup_{Q_j \in \mathscr{F}_1} \Sigma_j\right)\right) \lesssim \sum_{\mathscr{F}_1} H^n(Q_j) \leq H^n(B^* \cap \partial\Omega) \approx (K_0 r)^n,$$

since the Q_j's are pairwise disjoint.

Thus, to finish proving the upper ADR bound for the sawtooth regions, it remains only to establish (A.6). Suppose first that $\ell(Q_j) \lesssim r$. We write

$$\Sigma_j := \bigcup_{k:2^{-k} \lesssim \ell(Q_j)} \Sigma_j^k,$$

where $\Sigma_j^k = \Sigma \cap (\bigcup_{\{I \in \mathscr{R}_{Q_j}: \ell(I) = 2^{-k}\}} I) = \Sigma_j \cap (\bigcup_{\{I \in \mathscr{R}_{Q_j}: \ell(I) = 2^{-k}\}} I)$. We observe that for any $I \in \mathcal{W}$,

$$(A.7) \qquad H^n(\Sigma \cap I) \lesssim \ell(I)^n.$$

Moreover, there are at most a bounded number of $I \in \mathcal{R}_{Q_j}$ for which $\ell(I) \approx \ell(Q_j)$, so that

$$H^n \left(\sum_{k : 2^{-k} \approx \ell(Q_j)} \Sigma_j^k \right) \lesssim \ell(Q_j)^n,$$

as desired. On the other hand, suppose $I \in \mathcal{R}_{Q_j}$, with $\ell(I) = 2^{-k} \ll \ell(Q_j)$. Then there is a $Q_I \in \mathbb{D}_{Q_j}$, with $I \in \mathcal{W}_{Q_I}^*$. In addition, if I meets Σ_j, then I meets J^*, for some $J \in \mathcal{W}_{Q'}^*$, with $Q' \in \mathbb{D}_{\mathcal{F}}$, and $\ell(Q') \approx \ell(J) \approx \ell(I) \ll \ell(Q_j)$. We note that $Q' \cap Q_j = \varnothing$, by definition of $\mathbb{D}_{\mathcal{F}}$, and the fact that $\ell(Q') < \ell(Q_j)$. Consequently,

$$\operatorname{dist}(Q_I, (Q_j)^c) \leq \operatorname{dist}(Q_I, Q') \lesssim K_0 \, 2^{-k}.$$

Notice that for each such Q_I, there are at most a bounded number of $I' \in \mathcal{W}_{Q_I}^*$ (indeed, by definition of $\mathcal{W}_{Q_I}^*$, all such I' satisfy $\ell(I') \approx \ell(Q_I) \approx \operatorname{dist}(I', Q_I)$). By Lemma **??** (vi) we therefore have that

$$\# \left\{ I \in \mathcal{R}_{Q_j} : \ell(I) = 2^{-k}, \, I \cap \Sigma_j^k \neq \varnothing \right\} \lesssim \left(2^k \ell(Q_j) \right)^{n-\eta}$$

(where the implicit constant depends upon K_0), whence it follows that

$$H^n \left(\sum_{k : 2^{-k} \ll \ell(Q_j)} \Sigma_j^k \right) \lesssim \ell(Q_j)^n .$$

Thus (A.6) holds in the case $r \gtrsim \ell(Q_j)$.

Now suppose that $r \ll \ell(Q_j)$. If $x \notin \partial\Omega$, then $B = B(x, r)$ is centered on a face of some J_x^*, with $\operatorname{int}(J_x^*) \subset \Omega_{\mathcal{F}}$. If $\ell(J_x) \gg r$, we are done, by the nature of Whitney cubes. On the other hand, if $\ell(J_x) \lesssim r$, or if $x \in \partial\Omega$, then for each $I \in \mathcal{R}_{Q_j}$ which meets B, we have that $\ell(I) \lesssim r$, and also that $B(x, Cr)$ meets Q_I, for some uniform constant C, where $Q_I \in \mathbb{D}_{Q_j}$ is defined as in the previous paragraph. We may then cover $B(x, Cr) \cap Q_j$ by a bounded number of subcubes $Q^i \subset Q_j$, with $\ell(Q^i) \approx Mr$, so that each relevant Q_I is contained in some Q^i. Here, M is a sufficiently large number, to be fixed momentarily. Now suppose that I meets Σ_j. We may then proceed as in the previous paragraph, except that in this case we consider $\operatorname{dist}(Q_I, (Q^i)^c)$, and $\ell(Q_j)$ is replaced by $\ell(Q^i) \approx Mr$. As above, we find that I meets some J^*, with $J \in \mathcal{W}_{Q'}^*$ and $Q' \in \mathbb{D}_{\mathcal{F}}$, so that $\ell(Q') \approx \ell(J) \approx \ell(I)$. In the present scenario, we have $\ell(I) \lesssim r$, therefore $\ell(Q') < \ell(Q^i)$, for M chosen large enough and consequently $Q' \cap Q^i = \varnothing$. The rest of the argument follows as before. We omit the details.

Finally, to complete our discussion of the ADR property, it remains only to prove the lower ADR bound for the sawtooth regions. For the sake of specificity, we treat only the case of a local sawtooth, as the proof in the global case is similar.

Fix now $Q_0 \in \mathbb{D}$, $r \lesssim \operatorname{diam} Q_0$ and $x \in \partial\Omega_{\mathcal{F}, Q_0}$, where $\mathcal{F} \subset \mathbb{D}$ is a disjoint family, and set $B := B(x, r)$ and $\Delta_\star = \Delta_\star(x, r) := B \cap \partial\Omega_{\mathcal{F}, Q_0}$. We consider two main cases. As usual, M denotes a sufficiently large number to be chosen.

CASE 1: $\delta(x) \geq r/(MK_0)$. In this case, for some J with $\operatorname{int}(J^*) \subset \Omega_{\mathcal{F}, Q_0}$, we have that x lies on a subset F of a (closed) face of J^*, satisfying $H^n(F) \gtrsim (r/(MK_0))^n$, and $F \subset \partial\Omega_{\mathcal{F}, Q_0}$. Thus, $H^n(B \cap \partial\Omega_{\mathcal{F}, Q_0}) \geq H^n(B \cap F) \gtrsim (r/(MK_0))^n$, as desired.

CASE 2: $\delta(x) < r/(MK_0)$. In this case, we have that $\operatorname{dist}(x, Q_0) \lesssim r/M$. Indeed, if $x \in \partial\Omega \cap \partial\Omega_{\mathcal{F}, Q_0}$, then by Proposition 6.1, $x \in \overline{Q_0}$, so that $\operatorname{dist}(x, Q_0) = 0$. Otherwise,

there is some cube $Q \in \mathbb{D}_{\mathscr{F}, Q_0}$ such that x lies on the face of a fattened Whitney cube I^*, with $I \in \mathcal{W}_Q^*$, and $\ell(Q) \approx \ell(I) \approx \delta(x) < r/(MK_0)$. Thus,

$$\operatorname{dist}(x, Q_0) \lesssim \operatorname{dist}(I, Q) \lesssim K_0 \, \ell(Q) \lesssim r/M.$$

Consequently, we may choose $\hat{x} \in Q_0$ such that $|x - \hat{x}| \lesssim r/M$. Fix now $\widehat{Q} \in \mathbb{D}_{Q_0}$ with $\hat{x} \in \widehat{Q}$ and $\ell(\widehat{Q}) \approx r/M$. Then for M chosen large enough we have that $\widehat{Q} \subset B(\hat{x}, r/\sqrt{M}) \subset B(x, r)$. We now consider two sub-cases.

SUB-CASE 2A: $B(\hat{x}, r/\sqrt{M})$ meets a $Q_j \in \mathscr{F}$ with $\ell(Q_j) \geq r/M$. Then in particular, there is a $Q \subseteq Q_j$, with $\ell(Q) \approx r/M$, and $Q \subset B(\hat{x}, 2r/\sqrt{M})$. By Lemma 5.9, there is a ball $B' \subset \mathbb{R}^{n+1} \setminus \overline{\Omega_{\mathscr{F}, Q_0}}$, with radius $r' \approx \ell(Q)/K_0 \approx r/(K_0 M)$, such that $B' \cap \partial\Omega \subset Q$, and thus also $B' \subset B$ (for M large enough). On the other hand, we have already established above that $\Omega_{\mathscr{F}, Q_0}$ satisfies the (interior) Corkscrew condition, so there is another ball $B'' \subset B \cap \Omega_{\mathscr{F}, Q_0}$, with radius $r'' \approx r$. Therefore, by the isoperimetric inequality and the structure theorem for sets of locally finite perimeter (cf. [20], pp. 190 and 205, resp.) we have $H^n(\Delta_\star) \gtrsim c_{K_0} r^n$.

SUB-CASE 2B: There is no Q_j as in sub-case 2a. Thus, if $Q_j \in \mathscr{F}$ meets $B(\hat{x}, r/\sqrt{M})$, then $\ell(Q_j) \leq r/M$. Since $\hat{x} \in Q_0$, there is a surface ball

$$\Delta_1 := \Delta(x_1, cr/\sqrt{M}) \subset Q_0 \cap B(\hat{x}, r/\sqrt{M}) \subset Q_0 \cap B.$$

Let $\mathscr{F}_1$ denote the collection of those $Q_j \in \mathscr{F}$ which meet Δ_1. We then have the covering

$$\Delta_1 \subset \left(\bigcup_{\mathscr{F}_1} Q_j \right) \cup \left(\Delta_1 \setminus \left(\bigcup_{\mathscr{F}_1} Q_j \right) \right).$$

If

$$(A.8) \qquad \sigma\left(\frac{1}{2}\Delta_1 \setminus \left(\bigcup_{\mathscr{F}_1} Q_j \right) \right) \geq \frac{1}{2} \sigma\left(\frac{1}{2}\Delta_1 \right) \approx r^n,$$

then we are done, since $\Delta_1 \setminus (\bigcup_{\mathscr{F}_1} Q_j) \subset (Q_0 \setminus (\bigcup_{\mathscr{F}} Q_j)) \cap B \subset \Delta_\star$, by Proposition 6.1.

Otherwise, if (A.8) fails, then

$$(A.9) \qquad \sum_{Q_j \in \mathscr{F}_1'} \sigma(Q_j) \gtrsim r^n,$$

where $\mathscr{F}_1'$ denotes those $Q_j \in \mathscr{F}_1$ which meet $\frac{1}{2}\Delta_1$. Let us remind the reader that $x_j^\star$ is the center of the n-dimensional cube P_j constructed in Proposition 6.7, and we recall (6.10) and the related discussion. We claim that there is a uniform constant C such that for each such Q_j, the ball $B_{Q_j}^* := B(x_j^\star, CK_0 \ell(Q_j))$ contains both an interior and an exterior Corkscrew point for $\Omega_{\mathscr{F}, Q_0}$, with respect to the surface ball $B_{Q_j}^* \cap \partial\Omega_{\mathscr{F}, Q_0}$ (with Corkscrew constants that may depend upon K_0).

Indeed, the exterior point exists by virtue of Lemma 5.9, while the interior point may be taken to be the center of some $I \in \mathcal{W}_{\widetilde{Q}_j}^*$ with $\ell(I) \approx \ell(\widetilde{Q}_j)$, where $\widetilde{Q}_j$ is the dyadic parent of Q_j, so that $\widetilde{Q}_j \in \mathbb{D}_{\mathscr{F}, Q_0}$ and therefore $I \subset \operatorname{int}(I^*) \subset \Omega_{\mathscr{F}, Q_0}$. Consequently, by the isoperimetric inequality and the structure theorem for sets of locally finite perimeter, we have

$$(A.10) \qquad H^n(B_{Q_j}^* \cap \partial\Omega_{\mathscr{F}, Q_0}) \gtrsim \ell(Q_j)^n \approx \sigma(Q_j).$$

Now, by the ADR property and a covering lemma argument, and (A.9), there is a sub-collection $\mathscr{T}_1'' \subset \mathscr{T}_1'$ such that the balls in $\{B_{Q_j}^*\}_{Q_j \in \mathscr{T}_1''}$ are pairwise disjoint and

$$(A.11) \qquad \sum_{Q_j \in \mathscr{T}_1''} \sigma(Q_j) \gtrsim r^n.$$

Combining (A.10) and (A.11), we obtain that $H^n(\Delta_\star) \gtrsim r^n$, since for M large enough, each $B_{Q_j}^* \subset B$, by construction.

A.4. UR

In this subsection, we show that the Carleson box T_Q inherits the UR property from Ω. This fact extends routinely to any T_Δ, and we omit the details.

Let us note that, since $\partial\Omega$ is UR, we have the global L^2 bound

$$(A.12) \qquad \iint_{\mathbb{R}^{n+1}} |\nabla^2 \mathcal{S} f(X)|^2 \, \delta(X) \, dX \le C \, \|f\|_{L^2(\partial\Omega)}^2,$$

which is equivalent to the Carleson measure condition (1.10) by "$T1$ reasoning".

Fix now $Q \in \mathbb{D}(\partial\Omega)$, and as usual let $\delta_\star(X) := \operatorname{dist}(X, \partial T_Q)$ (in the present context, X need not belong to T_Q, but of course $\delta_\star(X)$ is still well-defined). By "local Tb" theory (see [24] in the present context), it is enough to verify that for every $\Delta_\star = \Delta_\star(x, r) := B(x, r) \cap \partial T_Q$, with $x \in \partial T_Q$ and $r \lesssim \operatorname{diam}(Q)$, there is a function $b_{\Delta_\star}$, supported in $\Delta_\star$, and satisfying

$$(A.13) \qquad \left| \fint_{\Delta_\star} b_{\Delta_\star} \, dH^n \right| \ge \frac{1}{C},$$

$$(A.14) \qquad \fint_{\Delta_\star} |b_{\Delta_\star}|^2 \, dH^n \le C,$$

$$(A.15) \qquad \iint_{B(x,2r)} |\nabla^2 \mathcal{S} b_{\Delta_\star}(X)|^2 \, \delta_\star(X) \, dX \le C r^n,$$

where C is a uniform constant, independent of Q. We fix a large constant M to be chosen. There are two cases:

CASE 1: $\operatorname{dist}(x, \partial\Omega) \le r/(M K_0)$.

In this case, either $x \in \overline{Q}$, or x lies on a face of some $I^* = (1 + \lambda)I$, with $I \in \mathcal{W}_{Q_I}^*$, for some $Q_I \in \mathbb{D}_Q$, where $\ell(Q_I) \approx \ell(I) \lesssim r/(M K_0)$, and $\operatorname{dist}(Q_I, I) \lesssim K_0 \, \ell(I) \lesssim r/M$.

We claim that there is a surface ball $\Delta' = B' \cap \partial\Omega \subset \Delta_\star \cap \partial\Omega$, with $r_{\Delta'} \approx r/M$, and with $B' \cap \Omega \subset T_Q$. Indeed, if $x \in \overline{Q}$, then there is a $Q' \in \mathbb{D}_Q$ such that $x \in \overline{Q'}$ and $\ell(Q') \approx r/M$, and we may then set $\Delta' := B' \cap \partial\Omega$, where $B' := B_{Q'}'$ is the ball promised by Lemma 3.55, applied with Q' in place of Q, so that $B' \cap \Omega \subset T_{Q'} \subset T_Q$. On the other hand, if $x \in \partial I^*$, with $I \in \mathcal{W}_{Q_I}^*$ as above, then there is a $Q' \in \mathbb{D}_Q$ with $Q_I \subseteq Q'$, and $\ell(Q') \approx r/M$. Moreover, by the triangle inequality, $|x - y| \lesssim r/M$, for every $y \in Q'$, so that for M large enough we have $Q' \subset B(x, r) \cap \partial T_Q = \Delta_\star$ by Proposition 6.1. Thus, we may again set $B' := B_{Q'}'$, as in Lemma 3.55, and the claim is established.

We fix $\Delta' = \Delta(x_{Q'}, r_{\Delta'})$ as in the previous paragraph, and then set $b_{\Delta_\star} := 1_{\Delta''}$, where $\Delta'' := \Delta(x_{Q'}, r_{\Delta'}/4)$. Then (A.14) is trivial, and (A.13) holds by the ADR properties

of Ω and T_Q. It remains to establish (A.15). To this end, we claim that $\delta_\star(X) = \delta(X)$, for $X \in 2B'' = \frac{1}{2}B'$. Momentarily taking this claim for granted, we obtain that

$$\iint_{2B''} |\nabla^2 \mathcal{S} b_{\Delta_\star}(X)|^2 \, \delta_\star(X) \, dX \le Cr^n,$$

by (A.12), since $b_{\Delta_\star} = 1_{\Delta''}$ is supported in $\partial\Omega$. Otherwise, for $X \in B(x, 2r) \setminus 2B''$, we have

$$(\text{A.16}) \qquad |\nabla^2 \mathcal{S} b_{\Delta_\star}(X)| \lesssim \left| \int_{|X-y| \gtrsim r_{\Delta'}} |X-y|^{-n-1} 1_{\Delta''}(y) \, d\sigma(y) \right| \lesssim 1/r_{\Delta'} \approx M/r,$$

by the ADR property, and (A.15) follows.

Let us now verify the claim. Fix $X \in \frac{1}{2}B'$. We note that

$$\text{dist}(X, \partial\Omega \cap \partial T_Q) \le \frac{1}{2}r_{\Delta'},$$

since B' is centered on $\partial\Omega \cap \partial T_Q$. On the other hand, $\partial T_Q \setminus \partial\Omega \subset \Omega$, and therefore lies outside of B', since, by construction, $B' \cap \Omega \subset T_Q$. Thus,

$$\text{dist}(X, \partial T_Q \setminus \partial\Omega) \ge \frac{1}{2}r_{\Delta'}.$$

Consequently, $\delta_\star(X) = \text{dist}(X, \partial\Omega \cap \partial T_Q)$. Similarly, we shall have that $\delta(X) = \text{dist}(X, \partial\Omega \cap \partial T_Q)$, and thus $\delta(X) = \delta_\star(X)$ as claimed, once we show that $\partial\Omega \setminus \partial T_Q$ lies outside B', or equivalently, that $\partial\Omega \cap B' \subset \partial T_Q$. So, fix $y \in \partial\Omega \cap B'$. Since B' is open, we have that $B(y, \varepsilon_0) \subset B'$ for ε_0 small enough. Note that $B(y, \varepsilon_k)$ meets Ω for a sequence $\varepsilon_k \to 0$, with $\varepsilon_k < \varepsilon_0$. Thus, there exists a sequence $\{y_k\}_k \subset B' \cap \Omega \subset T_Q$, with $y_k \to y$, whence $y \in \partial T_Q$.

CASE 2: $\text{dist}(x, \partial\Omega) > r/(MK_0)$.

In this case, we can find a Whitney cube I with $x \in \partial I^*$ and $\text{int}(I^*) \subset T_Q$, and a ball $B' = B(x', r')$, with $r' \approx r/(MK_0)$, such that some face F of I^* contains the surface ball $\Delta'_\star := B' \cap \partial T_Q$. We define $b_{\Delta_\star} := 1_{\Delta''_\star}$, where $\Delta''_\star = B'' \cap \partial T_Q$ and $B'' := \frac{1}{4}B'$. We may now proceed as in Case 1, using that of course (A.12) holds when $\partial\Omega$ is replaced by the hyper-plane $\mathcal{H}$ that contains F, and $\delta(X) = \text{dist}(X, \mathcal{H})$. We omit the routine details.

Appendix B

Dyadically doubling and Muckenhoupt weights

Recall that, for a fixed cube $Q_0 \in \mathbb{D}$, we say that ω is dyadically doubling on Q_0 if there exists C_ω such that $\omega(Q) \le C_\omega \omega(Q') < \infty$ for every $Q \in \mathbb{D}_{Q_0}$, and for every dyadic "child" Q' of Q. We write C_σ for the dyadic doubling constant of σ (which depends on the ADR property). Throughout Appendix B, Q_0 will denote a fixed cube in $\mathbb{D}$. Let us also recall that the projection operators $\mathcal{P}_{\mathcal{G}}$ have been introduced in Section 6.

LEMMA B.1. – *Fix Q_0. Let ω be a dyadically doubling measure on Q_0 with constant C_ω. Then for every family $\mathcal{G} \subset \mathbb{D}_{Q_0}$ of pairwise disjoint dyadic cubes, $\mathcal{P}_{\mathcal{G}}\omega$ is dyadically doubling on Q_0, indeed $\mathcal{P}_{\mathcal{G}}\omega(Q) \le \max(C_\omega, C_\sigma) \mathcal{P}_{\mathcal{G}}\omega(Q')$ for every $Q \in \mathbb{D}_{Q_0}$, and for every dyadic "child" Q' of Q.*

Proof. – We follow the proof of [27, Lemma B.1]. Let us fix $Q \in \mathbb{D}_{Q_0}$ and one of its dyadic "children" Q'. We consider several cases.

CASE 1: There exists $Q_k \in \mathscr{F}$ with $Q \subset Q_k$. The estimate is trivial in this case:

$$\mathscr{P}_{\mathscr{F}}\omega(Q) = \frac{\sigma(Q)}{\sigma(Q_k)}\,\omega(Q_k) \leq C_\sigma\,\frac{\sigma(Q')}{\sigma(Q_k)}\,\omega(Q_k) = C_\sigma\,\mathscr{P}_{\mathscr{F}}\omega(Q') < \infty.$$

CASE 2: $Q' \in \mathscr{F}$. Notice that $\mathscr{P}_{\mathscr{F}}\omega(Q') = \omega(Q')$. Let $\mathscr{F}_1$ be the family of cubes $Q_k \in \mathscr{F}$ with $Q_k \cap Q \neq \varnothing$ and observe that if $Q_k \in \mathscr{F}_1$ then $Q_k \subsetneq Q$. Thus,

$$\begin{aligned}
\mathscr{P}_{\mathscr{F}}\omega(Q) &= \omega\big(Q \setminus \big(\bigcup_{Q_k \in \mathscr{F}} Q_k\big)\big) + \sum_{Q_k \in \mathscr{F}_1} \frac{\sigma(Q_k \cap Q)}{\sigma(Q_k)}\omega(Q_k) \\
&= \omega\big(Q \setminus \big(\bigcup_{Q_k \in \mathscr{F}} Q_k\big)\big) + \sum_{Q_k \in \mathscr{F}_1} \omega(Q_k) \\
&= \omega(Q) \leq C_\omega\,\omega(Q') = C_\omega\,\mathscr{P}_{\mathscr{F}}\omega(Q') < \infty.
\end{aligned}$$

CASE 3: None of the conditions in the previous cases occur. We take the same set $\mathscr{F}_1$ and observe that if $Q_k \in \mathscr{F}_1$ then $Q_k \subsetneq Q$ (otherwise we are driven to Case 1). Let $\mathscr{F}_2$ be the family of cubes $Q_k \in \mathscr{F}$ with $Q_k \cap Q' \neq \varnothing$. Notice that if $Q_k \in \mathscr{F}_2$ then $Q_k \subsetneq Q'$: otherwise, either $Q_k = Q'$ which leads us to Case 2, or $Q' \subsetneq Q_k$ which implies $Q \subset Q_k$ and this is Case 1. Then proceeding as in the previous case one obtains that $\mathscr{P}_{\mathscr{F}}\omega(Q) = \omega(Q)$ and $\mathscr{P}_{\mathscr{F}}\omega(Q') = \omega(Q')$ which in turn imply

$$\mathscr{P}_{\mathscr{F}}\omega(Q) = \omega(Q) \leq C_\omega\,\omega(Q') = C_\omega\,\mathscr{P}_{\mathscr{F}}\omega(Q') < \infty. \qquad \square$$

LEMMA B.2. – *Under the hypotheses of Lemma 6.15, ν and $\mathscr{P}_{\mathscr{F}}\nu$ are dyadically doubling on Q_0.*

Proof. – We proceed as in [27, Lemma B.2]. Let us first consider ν. Fix $Q \in \mathbb{D}_{Q_0}$, and one of its dyadic "children" Q'. We recall Proposition 6.7, which for each $Q_k \in \mathscr{F}$ promises the existence of an n-dimensional cube $P_k \subset \partial\Omega_{\mathscr{F},Q_0}$, with $\ell(P_k) \approx \ell(Q_k) \approx \mathrm{dist}(P_k, Q_k) \approx \mathrm{dist}(P_k, \partial\Omega)$.

CASE 1: There exists $Q_k \in \mathscr{F}$ with $Q \subset Q_k$. The estimate is trivial in this case since ω is dyadically doubling:

$$\nu(Q) = \frac{\omega(Q)}{\omega(Q_k)}\,\omega_\star(P_k) \leq C_\omega\,\frac{\omega(Q')}{\omega(Q_k)}\,\omega_\star(P_k) = C_\omega\,\nu(Q') < \infty.$$

CASE 2: $Q' \in \mathscr{F}$. Write $Q' = Q_1 \in \mathscr{F}$. Notice that $\nu(Q') = \omega_\star(P_1)$. Let $\mathscr{F}_1$ be the family of cubes $Q_k \in \mathscr{F}$ with $Q_k \cap Q \neq \varnothing$ and observe that if $Q_k \in \mathscr{F}_1$ then $Q_k \subsetneq Q$. Thus,

Remark 6.9 implies

$$\nu(Q) = \omega_\star \left(Q \setminus (\bigcup_{Q_k \in \mathscr{T}} Q_k) \right) + \sum_{Q_k \in \mathscr{T}_1} \frac{\omega(Q_k \cap Q)}{\omega(Q_k)} \omega_\star(P_k)$$

$$= \omega_\star \left(Q \setminus (\bigcup_{Q_k \in \mathscr{T}} Q_k) \right) + \sum_{Q_k \in \mathscr{T}_1} \omega_\star(P_k)$$

$$\lesssim \omega_\star \left((Q \setminus (\bigcup_{Q_k \in \mathscr{T}} Q_k)) \cup (\bigcup_{Q_k \in \mathscr{T}_1} P_k) \right).$$

We note that there are uniform positive constants c and C such that

(B.3)
$$(Q \setminus (\bigcup_{Q_k \in \mathscr{T}} Q_k)) \cup (\bigcup_{Q_k \in \mathscr{T}_1} P_k) \subset \Delta_\star(x_1^\star, C\ell(P_1))$$

and

(B.4)
$$\Delta_\star(x_1^\star, c\ell(P_1)) \subset P_1,$$

where as usual $x_1^\star$ denotes the center of the n-dimensional cube P_1. Indeed, (B.4) is trivial, since by construction (cf. Proposition 6.7), $P_1 \subset \partial\Omega_{\mathscr{T}, Q_0}$. To verify (B.3), it is enough to observe that, by Proposition 6.7, and the fact that Q is the dyadic parent of Q_1, for $Q_k \in \mathscr{T}_1$ we have

$$\ell(P_k) \approx \mathrm{dist}(P_k, Q_k) \approx \mathrm{dist}(P_k, Q) \lesssim \ell(P_1) \approx \ell(Q) \approx \mathrm{dist}(Q, P_1).$$

Consequently, since $\omega_\star$ is doubling, we have

$$\nu(Q) \lesssim \omega_\star(\Delta_\star(x_1^\star, C\ell(P_1))) \lesssim \omega_\star(\Delta_\star(x_1^\star, c\ell(P_1))) \leq \omega_\star(P_1) = \nu(Q').$$

CASE 3: None of the conditions in the previous cases occur. We take the same set $\mathscr{T}_1$ and observe that if $Q_k \in \mathscr{T}_1$ then $Q_k \subsetneq Q$ (otherwise we are driven to Case 1). Let $\mathscr{T}_2$ be the family of cubes $Q_k \in \mathscr{T}$ with $Q_k \cap Q' \neq \varnothing$. Notice that if $Q_k \in \mathscr{T}_2$ then $Q_k \subsetneq Q'$: otherwise, either $Q_k = Q'$ which leads us to Case 2, or $Q' \subsetneq Q_k$ which implies $Q \subset Q_k$ and this is Case 1. We claim that for some uniform constant C, we have

(B.5)
$$(Q \setminus (\bigcup_{Q_k \in \mathscr{T}} Q_k)) \cup (\bigcup_{Q_k \in \mathscr{T}_1} P_k) \subset \Delta_\star(x_{Q'}^\star, Ct_{Q'})$$

(see Proposition 6.12 for the notation). Indeed, since Q is the dyadic parent of Q', by the construction in Proposition 6.12 (applied to Q'), we have that

$$\mathrm{dist}(x_{Q'}^\star, Q) \leq \mathrm{dist}(x_{Q'}^\star, Q') \lesssim \ell(Q') \approx \ell(Q) \approx t_{Q'},$$

whence (B.5) follows immediately.

Then we proceed as in the previous case and obtain that

$$\nu(Q) \lesssim \omega_\star \left((Q \setminus (\bigcup_{Q_k \in \mathscr{T}} Q_k)) \cup (\bigcup_{Q_k \in \mathscr{T}_1} P_k) \right)$$

$$\leq \omega_\star(\Delta_\star(x_{Q'}^\star, Ct_{Q'})) \lesssim \omega_\star(\Delta_\star^{Q'}) \lesssim \nu(Q')$$

where we have used that $\omega_\star$ is doubling and where the last inequality follows as in (6.19):

$$\nu(Q') = \omega_\star(Q' \cap E_0) + \sum_{Q_k \in \mathscr{T}_2} \omega_\star(P_k) \gtrsim \omega_\star(\Delta_\star^{Q'}).$$

One might show that $\mathscr{P}_{\mathscr{I}}\nu$ is dyadically doubling by invoking Lemma B.1, but then the doubling constant would depend on ω and $\omega_\star$. This is not the right approach as we have already observed that $\mathscr{P}_{\mathscr{I}}\nu$ does not depend on ω. On the other hand, following the previous argument for ν we can see that the doubling constant does not depend on ω. In Cases 2 and 3, we have that $\mathscr{P}_{\mathscr{I}}\nu(Q) = \nu(Q)$ and $\mathscr{P}_{\mathscr{I}}\nu(Q') = \nu(Q')$ so the doubling condition follows at once from the previous computations, and depends quantitatively only upon the doubling constant for $\omega_\star$, but not on ω. In Case 1 we obtain

$$\mathscr{P}_{\mathscr{I}}\nu(Q) = \frac{\sigma(Q)}{\sigma(Q_k)}\,\omega_\star(P_k) \le C_\sigma \frac{\sigma(Q')}{\sigma(Q_k)}\,\omega_\star(P_k) = C_\sigma \mathscr{P}_{\mathscr{I}}\nu(Q'). \qquad \square$$

Let us remind the reader that, as explained above, we may view $\partial\Omega_{\mathscr{I},Q_0}$ itself as a surface ball $\Delta_\star^{Q_0}$ of radius $r(\Delta_\star^{Q_0}) \approx K_0\,\ell(Q_0)$, and then $A_\infty(\partial\Omega_{\mathscr{I},Q_0})$ is identified with $A_\infty(\Delta_\star^{Q_0})$.

LEMMA B.6. – *Under the hypotheses of Lemma 6.15, if $\omega_\star \in A_\infty(\partial\Omega_{\mathscr{I},Q_0})$, then $\mathscr{P}_{\mathscr{I}}\nu \in A_\infty^{\mathrm{dyadic}}(Q_0)$.*

Proof. – Fix $0 < \eta < 1/2$ and $F \subset Q \in \mathbb{D}_{Q_0}$ with $\sigma(F) \ge (1-\eta)\,\sigma(Q)$.

CASE 1: There exists $Q_k \in \mathscr{I}$ with $Q \subset Q_k$. The estimate is trivial in this case:

$$\frac{\mathscr{P}_{\mathscr{I}}\nu(F)}{\mathscr{P}_{\mathscr{I}}\nu(Q)} = \frac{\frac{\sigma(F)}{\sigma(Q_k)}\omega_\star(P_k)}{\frac{\sigma(Q)}{\sigma(Q_k)}\omega_\star(P_k)} = \frac{\sigma(F)}{\sigma(Q)} \ge 1 - \eta.$$

CASE 2: Q is not contained in any $Q_k \in \mathscr{I}$ (i.e., $Q \in \mathbb{D}_{\mathscr{I},Q_0}$). Let $\mathscr{I}_1$ be the family of cubes $Q_k \in \mathscr{I}$ with $Q_k \cap Q \ne \varnothing$ and observe that if $Q_k \in \mathscr{I}_1$ then $Q_k \subsetneq Q$. We set

$$\widetilde{\mathscr{I}} = \{Q_k \in \mathscr{I}_1 : \sigma(F \cap Q_k) \ge (1-2\eta)\,\sigma(Q_k)\},$$

and

$$E_0 = Q \setminus \bigcup_{Q_k \in \mathscr{I}} Q_k, \qquad G = \bigcup_{Q_k \in \widetilde{\mathscr{I}}} Q_k, \qquad B = \bigcup_{Q_k \in \mathscr{I}_1 \setminus \widetilde{\mathscr{I}}} Q_k.$$

Note that

$$\sigma(F \cap B) = \sum_{Q_k \in \mathscr{I}_1 \setminus \widetilde{\mathscr{I}}} \sigma(F \cap Q_k) \le (1-2\eta) \sum_{Q_k \in \mathscr{I}_1 \setminus \widetilde{\mathscr{I}}} \sigma(Q_k) \le (1-2\eta)\,\sigma(Q).$$

Thus,

$$(1-\eta)\,\sigma(Q) \le \sigma(F) \le \sigma(F \cap E_0) + \sigma(F \cap B) + \sigma(F \cap G)$$
$$\le \sigma\big((F \cap E_0) \cup G\big) + (1-2\eta)\,\sigma(Q),$$

and therefore $\sigma\big((F \cap E_0) \cup G\big) \ge \eta\,\sigma(Q)$.

Note that the ADR property of $\partial\Omega$ and $\partial\Omega_{\mathscr{I},Q_0}$ implies

$$\sigma(Q) \approx \ell(Q)^n \approx (\widehat{r}_Q)^n \approx \sigma_\star(\Delta_\star(y_Q, \widehat{r}_Q))$$

with $\Delta_\star(y_Q, \widehat{r}_Q)$ given in (6.11) (see also Proposition 6.4), where as usual we write $\sigma_\star$ to denote the "surface measure" on $\partial\Omega_{\mathscr{F},Q_0}$, i.e., $\sigma_\star = H^n\big|_{\partial\Omega_{\mathscr{F},Q_0}}$. If we set $G_\star = \bigcup_{Q_k \in \widetilde{\mathscr{F}}} P_k$ we have that $\sigma(G) \approx \sigma_\star(G_\star)$. Indeed, since $\Omega_{\mathscr{F},Q_0}$ is ADR we have that

$$\sigma_\star(P_k) \approx \ell(P_k)^n \approx \ell(Q_k)^n \approx \sigma(Q_k),$$

by Proposition 6.7; thus Remark 6.9 yields

$$\sigma(G) = \sum_{Q_k \in \widetilde{\mathscr{F}}} \sigma(Q_k) \approx \sum_{Q_k \in \widetilde{\mathscr{F}}} \sigma_\star(P_k) \approx \sigma_\star(G_\star).$$

On the other hand, Proposition 6.1 gives $\sigma(F \cap E_0) = \sigma_\star(F \cap E_0)$ and therefore

$$\sigma_\star\big((F \cap E_0) \cup G_\star\big) \approx \sigma\big((F \cap E_0) \cup G\big) \geq \eta\,\sigma(Q) \approx \eta\,\sigma_\star(\Delta_\star(y_Q, \widehat{r}_Q)).$$

Next we use that $\omega_\star \in A_\infty(\partial\Omega_{\mathscr{F},Q_0})$ to obtain

$$\frac{\omega_\star\big((F \cap E_0) \cup G_\star\big)}{\omega_\star(\Delta_\star(y_Q, \widehat{r}_Q))} \gtrsim \left(\frac{\sigma_\star\big((F \cap E_0) \cup G_\star\big)}{\sigma_\star(\Delta_\star(y_Q, \widehat{r}_Q))}\right)^\theta \gtrsim \eta^\theta,$$

where we have used that $(F \cap E_0) \cup G_\star \subset \Delta_\star(y_Q, \widehat{r}_Q)$ by (6.11). Then,

$$\mathscr{P}_{\mathscr{G}}\nu(F) \geq \omega_\star(F \cap E_0) + \sum_{Q_k \in \widetilde{\mathscr{F}}} \frac{\sigma(F \cap Q_k)}{\sigma(Q_k)}\omega_\star(P_k)$$

$$\geq \omega_\star(F \cap E_0) + (1 - 2\,\eta) \sum_{Q_k \in \widetilde{\mathscr{F}}} \omega_\star(P_k)$$

$$\geq (1 - 2\,\eta)\omega_\star\big((F \cap E_0) \cup G_\star\big)$$

$$\gtrsim (1 - 2\,\eta)\eta^\theta \omega_\star(\Delta_\star(y_Q, \widehat{r}_Q))$$

$$\gtrsim (1 - 2\,\eta)\eta^\theta \omega_\star\left(\left(Q \cup \left(\bigcup_{Q_k \in \mathscr{G}:\, Q_k \subset Q} B(x_k^\star, r_k)\right)\right) \cap \partial\Omega_{\mathscr{F},Q_0}\right)$$

where the last inequality follows from (6.11). Next we observe that, by Proposition 6.1,

$$(Q \cap E_0) \cup \left(\bigcup_{Q_k \in \mathscr{G}_1} P_k\right) \subset (Q \cap E_0) \cup \left(\bigcup_{Q_k \in \mathscr{G}_1} \Delta_\star(x_k^\star, r_k)\right)$$

$$\subset \left(Q \cup \left(\bigcup_{Q_k \in \mathscr{G}:\, Q_k \subset Q} B(x_k^\star, r_j)\right)\right) \cap \partial\Omega_{\mathscr{F},Q_0}.$$

Consequently,

$$\mathscr{P}_{\mathscr{G}}\nu(F) \gtrsim (1 - 2\,\eta)\eta^\theta \omega_\star\left((Q \cap E_0) \cup \left(\bigcup_{Q_k \in \mathscr{G}_1} P_k\right)\right)$$

$$\gtrsim (1 - 2\,\eta)\eta^\theta \left(\omega_\star(Q \cap E_0) + \sum_{Q_k \in \mathscr{G}_1} \omega_\star(P_k)\right) = (1 - 2\,\eta)\eta^\theta \mathscr{P}_{\mathscr{G}}\nu(Q).$$

Thus, in both cases we have shown as desired that $\mathscr{P}_{\mathscr{G}}\nu(F)/\mathscr{P}_{\mathscr{G}}\nu(Q) \geq C_\eta$. $\qquad\square$

Next we give a version of the classical result in [14] valid in our situation. The proof of this result follows the standard arguments in [21] although one has to adapt the ideas to the dyadic and local setting considered here. We give the proof for completeness.

LEMMA B.7. – *Let Q_0 be a fixed cube and let ω_1, ω_2 be two dyadically doubling measures on Q_0. Assume that there exist positive constants C_0, θ_0 such that for all $Q \in \mathbb{D}_{Q_0}$ and $F \subset Q$,*

$$\text{(B.8)} \qquad \frac{\omega_2(F)}{\omega_2(Q)} \le C_0 \left(\frac{\omega_1(F)}{\omega_1(Q)} \right)^{\theta_0}.$$

Then, there exist positive constants C_1, θ_1 such that for all $Q \in \mathbb{D}_{Q_0}$ and $F \subset Q$,

$$\text{(B.9)} \qquad \frac{\omega_1(F)}{\omega_1(Q)} \le C_1 \left(\frac{\omega_2(F)}{\omega_2(Q)} \right)^{\theta_1}.$$

REMARK B.10. – The proof shows that the desired estimate can be obtained from the following (apparently) weaker condition: there exist $0 < \alpha, \beta < 1$ such that for every cube $Q \in \mathbb{D}_{Q_0}$,

$$\text{(B.11)} \qquad F \subset Q, \quad \frac{\omega_2(F)}{\omega_2(Q)} < \alpha \quad \implies \quad \frac{\omega_1(F)}{\omega_1(Q)} < \beta.$$

To prove this result we need a local Calderón-Zygmund decomposition for dyadically doubling weights. The proof is standard and we leave it to the interested reader.

LEMMA B.12. – *Given Q_0 and ω a dyadically doubling measure on Q_0 with constant C_ω, we consider the local dyadic Hardy-Littlewood maximal function with respect to ω:*

$$\mathcal{M}_\omega f(x) = \sup_{x \in Q \in \mathbb{D}_{Q_0}} \frac{1}{\omega(Q)} \int_Q |f(y)| \, d\omega(y).$$

For any $0 \le f \in L^1(Q_0, \omega)$ and $\lambda \ge \frac{1}{\omega(Q_0)} \int_{Q_0} |f(y)| \, d\omega(y)$, there exists a collection of maximal and therefore disjoint dyadic cubes $\{Q_j\} \subset \mathbb{D}_{Q_0}$ such that

$$\text{(B.13)} \qquad E_\lambda = \{x \in Q_0 : \mathcal{M}_\omega f(x) > \lambda\} = \bigcup_j Q_j,$$

$$\text{(B.14)} \qquad f(x) \le \lambda, \quad \text{for } \omega\text{-a.e. } x \notin E_\lambda,$$

$$\text{(B.15)} \qquad \lambda < \frac{1}{\omega(Q_j)} \int_{Q_j} f(y) \, d\omega(y) \le C_\omega \, \lambda.$$

Proof of Lemma B.7. – We proceed as in [27, Lemma B.4]. Pick $0 < \alpha < 1$ and $\beta = 1 - \left(\frac{1-\alpha}{C_0} \right)^{1/\theta_0}$, and notice that $0 < \beta < 1$ since $C_0 \ge 1$. Then for any $F \subset Q$, $Q \in \mathbb{D}_{Q_0}$ we apply (B.8) to $Q \setminus F$ and we conclude (B.11). Next we see that this (apparently) weaker condition implies the desired conclusion. Assume momentarily that $\omega_1 \ll \omega_2$. Then the Radon-Nikodym derivative $h = d\omega_1/d\omega_2$ satisfies that $h \in L^1(Q_0, \omega_2)$ and $0 \le h(x) < \infty$ for ω_2-a.e. $x \in Q_0$.

Fixed $Q \in \mathbb{D}_{Q_0}$ we write $\tau = C_{\omega_2}/\alpha$,

$$\lambda_0 = \frac{1}{\omega_2(Q)} \int_Q h(x) \, d\omega_2(x) = \frac{\omega_1(Q)}{\omega_2(Q)}$$

and $\lambda_k = \tau^k \lambda_0$. Notice that $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ since $\tau > C_{\omega_2} \ge 1$. For every $k \ge 0$ we apply Lemma B.12 in Q to h with dyadically doubling measure ω_2: let $\{Q_j^k\}_j \subset \mathbb{D}_Q \subset \mathbb{D}_{Q_0}$

be the corresponding collection of cubes such that $E_k = E_{\lambda_k} = \bigcup_j Q_j^k$. Fix $Q_{j_0}^k$ and observe that if $Q_{j_0}^k \cap Q_j^{k+1} \neq \varnothing$ then $Q_j^{k+1} \subset Q_{j_0}^k$: otherwise we would have $Q_{j_0}^k \subsetneq Q_j^{k+1}$, by (B.15) we observe that $\frac{1}{\omega_2(Q_j^{k+1})} \int_{Q_j^{k+1}} h \, d\omega_2 > \lambda_{k+1} > \lambda_k$ and then $Q_{j_0}^k$ would not be maximal. Then using (B.13) and (B.15) we obtain

$$\omega_2(Q_{j_0}^k \cap E_{k+1}) = \sum_{j: Q_j^{k+1} \subset Q_{j_0}^k} \omega_2(Q_j^{k+1}) < \frac{1}{\lambda_{k+1}} \sum_{j: Q_j^{k+1} \subset Q_{j_0}^k} \int_{Q_j^{k+1}} h \, d\omega_2$$

$$\leq \frac{1}{\lambda_{k+1}} \int_{Q_{j_0}^k} h \, d\omega_2 \leq \frac{C_{\omega_2} \lambda_k}{\lambda_{k+1}} \, \omega_2(Q_{j_0}^k) = \alpha \, \omega_2(Q_{j_0}^k).$$

This estimate allows us to use (B.11) which in turn gives that $\omega_1(Q_{j_0}^k \cap E_{k+1}) < \beta \, \omega_1(Q_{j_0}^k)$. Next we sum on j_0 and conclude that $\omega_1(E_{k+1}) < \beta \, \omega_1(E_k)$ since $E_{k+1} \subset E_k$. By iterating this expression we obtain $\omega_1(E_k) < \beta^k \omega_1(E_0)$. Similarly, $\omega_2(E_k) < \alpha^k \omega_1(E_0)$, which implies

$$\omega_2(\cap_k E_k) = \lim_{k \to \infty} \omega_2(E_k) = 0.$$

Let $0 < \varepsilon < -\log \beta / \log \tau$. Then $0 < \tau^\varepsilon \beta < 1$ and by (B.14)

$$(\text{B.16}) \quad \frac{1}{\omega_2(Q)} \int_Q h(x)^{1+\varepsilon} \, d\omega_2(x)$$

$$= \frac{1}{\omega_2(Q)} \int_{Q \setminus E_0} h(x)^{1+\varepsilon} \, d\omega_2(x) + \frac{1}{\omega_2(Q)} \sum_{k=0}^\infty \int_{E_k \setminus E_{k+1}} h(x)^{1+\varepsilon} \, d\omega_2(x)$$

$$\leq \lambda_0^\varepsilon \frac{1}{\omega_2(Q)} \int_Q h(x) \, d\omega_2(x) + \frac{1}{\omega_2(Q)} \sum_{k=0}^\infty \lambda_{k+1}^\varepsilon \int_{E_k} h(x) \, d\omega_2(x)$$

$$= \lambda_0^\varepsilon \frac{\omega_1(Q)}{\omega_2(Q)} + \frac{1}{\omega_2(Q)} \sum_{k=0}^\infty \lambda_{k+1}^\varepsilon \, \omega_1(E_k)$$

$$\leq \lambda_0^\varepsilon \frac{\omega_1(Q)}{\omega_2(Q)} + \lambda_0^\varepsilon \frac{\omega_1(E_0)}{\omega_2(Q)} \sum_{k=0}^\infty \tau^{(k+1)\varepsilon} \beta^k$$

$$\leq \lambda_0^\varepsilon \frac{\omega_1(Q)}{\omega_2(Q)} (1 + \tau^\varepsilon (1 - \tau^\varepsilon \beta)^{-1}) = \left(\frac{\omega_1(Q)}{\omega_2(Q)} \right)^{1+\varepsilon} C_1^{1+\varepsilon}.$$

This estimate implies that for all $F \subset Q$,

$$\frac{\omega_1(F)}{\omega_2(Q)} = \frac{1}{\omega_2(Q)} \int_Q \chi_F \, h \, d\omega_2 \leq \left(\frac{1}{\omega_2(Q)} \int_Q h^{1+\varepsilon} \, d\omega_2 \right)^{\frac{1}{1+\varepsilon}} \left(\frac{\omega_2(F)}{\omega_2(Q)} \right)^{\frac{1}{(1+\varepsilon)'}}$$

$$\leq \frac{\omega_1(Q)}{\omega_2(Q)} C_1 \left(\frac{\omega_2(F)}{\omega_2(Q)} \right)^{\frac{1}{(1+\varepsilon)'}},$$

which is (B.9) with $\theta_1 = 1/(1+\varepsilon)'$. Notice that ε and C_1 depend only on α, β and C_{ω_2}.

Next we see how to proceed in the general case starting from (B.11). We define a new measure $\tilde{\omega}_2 = \omega_2 + \delta \, \omega_1$ with $\delta > 0$. It is clear that $\omega_1 \ll \tilde{\omega}_2$ and also that $\tilde{\omega}_2$ is dyadically doubling on Q_0 with constant $C_{\tilde{\omega}_2} = C_{\omega_1} + C_{\omega_2}$. We claim that setting

$\tilde{\beta} = 1 - \min\{1 - \beta, \alpha/2\}$, $\tilde{\alpha} = \alpha/2$ we have for every $Q \in \mathbb{D}_{Q_0}$,

(B.17) $$F \subset Q, \quad \frac{\tilde{\omega}_2(F)}{\tilde{\omega}_2(Q)} < \tilde{\alpha} \quad \Longrightarrow \quad \frac{\omega_1(F)}{\omega_1(Q)} < \tilde{\beta}.$$

Assuming this, (B.11) holds for $\omega_1, \tilde{\omega}_2$. By the previous case, since $\omega_1 \ll \tilde{\omega}_2$, there exist $\tilde{\varepsilon}, \tilde{C}_1$ such that for every $Q \in \mathbb{D}_{Q_0}$, $F \subset Q$ we have

$$\frac{\omega_1(F)}{\omega_1(Q)} \leq \tilde{C}_1 \left(\frac{\tilde{\omega}_2(F)}{\tilde{\omega}_2(Q)} \right)^{\frac{1}{(1+\tilde{\varepsilon})'}}.$$

As mentioned above $\tilde{\varepsilon}, \tilde{C}_1$ depend only on $\tilde{\alpha}, \tilde{\beta}, C_{\tilde{\omega}_2}$ and these are ultimately given in terms of $\alpha, \beta, C_{\omega_1}, C_{\omega_2}$. Next we see that $\omega_1 \ll \omega_2$: given $F \subset Q_0$ with $\omega_2(F) = 0$, the previous inequality applied to $Q = Q_0$ gives as desired

$$0 \leq \frac{\omega_1(F)}{\omega_1(Q)} \leq \tilde{C}_1 \left(\frac{\delta \, \omega_1(F)}{\tilde{\omega}_2(Q_0)} \right)^{\frac{1}{(1+\tilde{\varepsilon})'}} \leq \tilde{C}_1 \left(\delta \, \frac{\omega_1(F)}{\omega_2(Q_0)} \right)^{\frac{1}{(1+\tilde{\varepsilon})'}} \longrightarrow 0, \quad \text{as } \delta \to 0^+.$$

Thus, we get back to the first case and obtain (B.16) which eventually leads to (B.9) with C_1 and θ_1 as stated above.

To complete the proof we obtain (B.17). Given F as there, it follows that $\tilde{\omega}_2(Q \setminus F)/\tilde{\omega}_2(Q) > 1 - \alpha/2$. We see that $\omega_1(Q \setminus F)/\omega_1(Q) > \min\{1 - \beta, \alpha/2\}$, which yields as desired $\omega_1(F)/\omega_1(Q) < \tilde{\beta}$. If this were not the case then we would have $\omega_1(Q \setminus F)/\omega_1(Q) \leq \alpha/2$ and also that $\omega_1(F)/\omega_1(Q) \geq \beta$. By (B.11), the latter gives $\omega_2(F)/\omega_2(Q) \geq \alpha$ and therefore $\omega_2(Q \setminus F)/\omega_2(Q) \leq 1 - \alpha$. Gathering these estimates we get a contradiction

$$\frac{\tilde{\omega}_2(Q \setminus F)}{\tilde{\omega}_2(Q)} = \frac{\omega_2(Q \setminus F)}{\tilde{\omega}_2(Q)} + \delta \frac{\omega_1(Q \setminus F)}{\tilde{\omega}_2(Q)} \leq \frac{\omega_2(Q \setminus F)}{\omega_2(Q)} + \frac{\omega_1(Q \setminus F)}{\omega_1(Q)} \leq 1 - \alpha/2. \quad \square$$

REMARK B.18. – Let us observe that (B.16) can be equivalently written as

$$\left(\frac{1}{\omega_2(Q)} \int_Q h(x)^{1+\varepsilon} \, d\omega_2(x) \right)^{\frac{1}{1+\varepsilon}} \leq C_1 \frac{1}{\omega_2(Q)} \int_Q h(x) \, d\omega_2(x),$$

and this shows that $h \in RH_{1+\varepsilon}^{\mathrm{dyadic}}(Q_0, \omega_2)$.

Appendix C

The UR property for approximating domains

We establish the UR property (with uniform constants) for the approximating domains Ω_N defined by (8.13). Recall that we have already observed that Ω_N inherits the ADR, Corkscrew and Harnack chain conditions from Ω.

The proof is based on ideas of Guy David, and uses the following singular integral characterization of UR sets, established in [18]. Suppose that $E \subset \mathbb{R}^{n+1}$ is n-dimensional ADR. The singular integral operators that we shall consider are those of the form

$$T_{E,\varepsilon} f(x) = T_\varepsilon f(x) := \int_E K_\varepsilon(x - y) \, f(y) \, dH^n(y),$$

where $K_\varepsilon(x) := K(x)\,\Phi(|x|/\varepsilon)$, with $0 \leq \Phi \leq 1$, $\Phi(\rho) \equiv 1$ if $\rho \geq 2$, $\Phi(\rho) \equiv 0$ if $\rho \leq 1$, and $\Phi \in C^\infty(\mathbb{R})$, and where the singular kernel K is an odd function, smooth on $\mathbb{R}^{n+1} \setminus \{0\}$, and satisfying

$$(C.1) \qquad |K(x)| \leq C\,|x|^{-n}$$

$$(C.2) \qquad |\nabla^m K(x)| \leq C_m\,|x|^{-n-m}, \qquad \forall m \in \{1, 2, 3, \ldots\}.$$

Then E is UR if and only if for every such kernel K, we have that

$$(C.3) \qquad \sup_{\varepsilon > 0} \int_E |T_\varepsilon f|^2\, dH^n \leq C_K \int_E |f|^2\, dH^n.$$

We refer the reader to [18] for the proof. We shall also require "non-tangential" estimates for an extension of T_ε defined as follows. For K as above, set

$$(C.4) \qquad \mathcal{T}_E f(X) := \int_E K(X - y)\, f(y)\, dH^n(y), \qquad X \in \mathbb{R}^{n+1} \setminus E.$$

We define non-tangential approach regions $\Gamma_\tau(x)$ as follows. Let $\mathcal{W}_E$ denote the collection of cubes in the Whitney decomposition of $\mathbb{R}^{n+1} \setminus E$, and set

$$\mathcal{W}_\tau(x) := \{I \in \mathcal{W}_E : \operatorname{dist}(I, x) < \tau \ell(I)\}.$$

Then we define

$$\Gamma_\tau(x) := \bigcup_{I \in \mathcal{W}_\tau(x)} I^*$$

(thus, roughly speaking, τ is the "aperture" of $\Gamma_\tau(x)$). For $F \in \mathcal{C}(\mathbb{R}^{n+1} \setminus E)$ we may then also define the non-tangential maximal function

$$N_{*,\tau}(F)(x) := \sup_{Y \in \Gamma_\tau(x)} |F(Y)|.$$

We shall sometimes write simply N_* when there is no chance of confusion in leaving implicit the dependence on the aperture τ.

LEMMA C.5. – *Suppose that $E \subset \mathbb{R}^{n+1}$ is n-dimensional UR, and let $\mathcal{T}_E$ be defined as in (C.4). Then for each $\tau \in (0, \infty)$, there is a constant $C_{\tau,K}$ depending only on n, τ, K and the UR constants such that*

$$(C.6) \qquad \int_E (N_{*,\tau}(\mathcal{T}_E f))^2\, dH^n \leq C_{\tau,K} \int_E |f|^2 dH^n.$$

Given (C.3), Lemma C.5 is a variant of the standard "Cotlar inequality" for maximal singular integrals, and we omit the proof.

We are now ready to prove that $\partial\Omega_N$ is UR, uniformly in N. It is enough to establish the estimate (C.3), for all K as above, with E replaced by $\partial\Omega_N$. On the other hand, we are given that $\partial\Omega$ is UR, whence (C.6) holds with $E = \partial\Omega$. Since $\partial\Omega_N$ is ADR, it enjoys the dyadic grid structure promised by Lemma **??**. We then make a partition $\partial\Omega_N = \cup Q_j(N)$, where $Q_j(N) \in \mathbb{D}_N(\partial\Omega_N) =: \mathbb{D}_N(N)$, the dyadic grid on $\partial\Omega_N$ at scale 2^{-N}. We observe that, by the construction of Ω_N, for each $Q_j(N) \in \mathbb{D}_N(N)$, we may choose a $Q_j \in \mathbb{D}_N(\partial\Omega) =: \mathbb{D}_N$ with $\operatorname{dist}(Q_j(N), Q_j) \approx 2^{-N}$. By the ADR property of $\partial\Omega$, a given $Q \in \mathbb{D}_N$ can serve in this

way for at most a bounded number of $Q_j(N) \in \mathbb{D}_N(N)$. Therefore, we have the bounded overlap condition

$$(C.7) \qquad \sum_{Q_j(N) \in \mathbb{D}_N(N)} 1_{Q_j}(x) \le C, \qquad \forall x \in \partial\Omega.$$

As usual, we set $\sigma := H^n|_{\partial\Omega}$, and we now also let $\sigma_N := H^n|_{\partial\Omega_N}$. We then have that for τ large enough,

$$\int_{\partial\Omega_N} |\mathscr{T}_{\partial\Omega} f|^2 \, d\sigma_N = \sum_{Q_j(N) \in \mathbb{D}_N(N)} \int_{Q_j(N)} |\mathscr{T}_{\partial\Omega} f|^2 \, d\sigma_N$$

$$= \sum_{Q_j(N) \in \mathbb{D}_N(N)} \frac{1}{\sigma(Q_j)} \int_{Q_j} \int_{Q_j(N)} |\mathscr{T}_{\partial\Omega} f(x)|^2 \, d\sigma_N(x) \, d\sigma(x')$$

$$\lesssim \sum_{Q_j(N) \in \mathbb{D}_N(N)} \int_{Q_j} \left(N_{*,\tau} \left(\mathscr{T}_{\partial\Omega} f \right) \right)^2 \, d\sigma \le C_{\tau,K} \int_{\partial\Omega} |f|^2 d\sigma,$$

where in the last line we have used first the ADR properties of $\partial\Omega$ and $\partial\Omega_N$, and then (C.7) and (C.6) with $E = \partial\Omega$.

We have thus established that $\mathscr{T}_{\partial\Omega} : L^2(\partial\Omega) \to L^2(\partial\Omega_N)$. Since the kernel K is odd, we therefore obtain by duality that

$$(C.8) \qquad \mathscr{T}_{\partial\Omega_N} : L^2(\partial\Omega_N) \to L^2(\partial\Omega).$$

Now fix $\varepsilon > 0$, and N large enough that $2^{-N} \ll \operatorname{diam} \partial\Omega$. Set $\varepsilon_N = 2^{-N}$. We consider two cases.

CASE 1: $\varepsilon < \varepsilon_N$. In this case,

$$\int_{\partial\Omega_N} |T_{\partial\Omega_N,\varepsilon} f|^2 \, d\sigma_N$$

$$\lesssim \int_{\partial\Omega_N} |T_{\partial\Omega_N,\varepsilon} f - T_{\partial\Omega_N,\varepsilon_N} f|^2 \, d\sigma_N \; + \int_{\partial\Omega_N} |T_{\partial\Omega_N,\varepsilon_N} f|^2 \, d\sigma_N \; =: I + II.$$

Let $Q_j \in \mathbb{D}_N$ denote the cube chosen relative to $Q_j(N) \in \mathbb{D}_N(N)$ as above. Then for $x \in Q_j(N)$, and $x' \in Q_j$, we have by standard Calderón-Zygmund estimates using (C.1) and (C.2), and the ADR property of $\partial\Omega_N$, that

$$|T_{\partial\Omega_N,\varepsilon_N} f(x) - \mathscr{T}_{\partial\Omega_N} f(x')| \lesssim M^N f(x),$$

where M^N denotes the Hardy-Littlewood maximal function on $\partial\Omega_N$. Consequently,

$$II = \sum_{Q_j(N) \in \mathbb{D}_N(N)} \frac{1}{\sigma(Q_j)} \int_{Q_j} \int_{Q_j(N)} |T_{\partial\Omega_N,\varepsilon_N} f(x)|^2 \, d\sigma_N(x) \, d\sigma(x')$$

$$\lesssim \sum_{Q_j(N) \in \mathbb{D}_N(N)} \int_{Q_j(N)} \left(M^N f(x) \right)^2 \, d\sigma_N(x)$$

$$+ \sum_{Q_j(N) \in \mathbb{D}_N(N)} \int_{Q_j} |\mathscr{T}_{\partial\Omega_N} f(x')|^2 \, d\sigma(x') \; =: II' + II''.$$

The desired bound for II' follows immediately, and the bound for II'' follows directly from (C.7) and (C.8).

We turn now to term I. Let us note that since $\varepsilon_N \approx \operatorname{diam}(Q_j(N))$, for $x \in Q_j(N)$ we have

$$T_{\partial \Omega_N, \varepsilon} f(x) - T_{\partial \Omega_N, \varepsilon_N} f(x)$$

$$= \int_{\partial \Omega_N} K(x-y) \left(\Phi\left(\frac{|x-y|}{\varepsilon} \right) - \Phi\left(\frac{|x-y|}{\varepsilon_N} \right) \right) f(y) 1_{\Delta_{N,j}}(y) \, d\sigma_N(y)$$

$$=: T_{\partial \Omega_N, \varepsilon, \varepsilon_N} \left(f \, 1_{\Delta_{N,j}} \right)(x),$$

a doubly truncated singular integral on $\partial \Omega_N$, where $\Delta_{N,j} := B_{N,j} \cap \partial \Omega_N$, and $B_{N,j}$ is a ball centered at some point in $Q_j(N)$, with radius $C \operatorname{diam}(Q_j(N)) \approx 2^{-N}$. By choosing C large enough, we may assume that $Q_j(N) \subset \Delta_{N,j}$. We recall that by definition, $\partial \Omega_N$ is a union of portions of faces of fattened Whitney cubes I^*, of side length $\approx 2^{-N}$. Since only a bounded number of these can meet $B_{N,j}$, we have

$$\Delta_{N,j} \subset \bigcup_{m=1}^{M_0} F_m^j,$$

where M_0 is a uniform constant and each F_m^j is either a portion of a face of some I^*, or else $F_m^j = \varnothing$ (since M_0 is not necessarily equal to the number of faces, but is rather an upper bound for the number of faces). Thus,

$$I \lesssim \sum_{Q_j(N) \in \mathbb{D}_N(N)} \sum_{1 \leq m, m' \leq M_0} \int_{F_m^j} |T_{\partial \Omega_N, \varepsilon, \varepsilon_N} \left(f \, 1_{F_{m'}^j} \right)|^2 \, d\sigma_N .$$

The faces $F_{m'}^j$ have bounded overlaps as we sum in j. Therefore, the case $m = m'$ reduces to the classical case that $\partial \Omega_N$ is a hyperplane. For $m \neq m'$, there are two cases as follows. If $\operatorname{dist}(F_m^j, F_{m'}^j) \approx 2^{-N}$, then using (C.1), we may crudely dominate $T_{\partial \Omega_N, \varepsilon, \varepsilon_N}$ by the Hardy-Littlewood maximal operator. Otherwise, $\operatorname{dist}(F_m^j, F_{m'}^j) \ll 2^{-N}$, in which case F_m^j and $F_{m'}^j$ are contained in respective faces which either lie in the same hyperplane, or else meet at an angle of $\pi/2$. In the latter scenario, after a possible rotation of co-ordinates, we may view $F_m^j \cup F_{m'}^j$ as lying in a Lipschitz graph with Lipschitz constant 1, so that we may estimate $T_{\partial \Omega_N, \varepsilon, \varepsilon_N}$ using an extension of the Coifman-McIntosh-Meyer theorem.

CASE 2: $\varepsilon \geq \varepsilon_N$. We observe that (C.8) also applies to the modified operator $\mathscr{T}^{\varepsilon}_{\partial \Omega_N}$, obtained by replacing the kernel K by the kernel K_ε, since the latter is still odd and still satisfies the Calderón-Zygmund estimates (C.1) and (C.2) (uniformly in ε). The present case may then be handled just like term II above, by writing

$$T_{\partial \Omega_N, \varepsilon} f(x) = \left(T_{\partial \Omega_N, \varepsilon} f(x) - \mathscr{T}^{\varepsilon}_{\partial \Omega_N} f(x') \right) + \mathscr{T}^{\varepsilon}_{\partial \Omega_N} f(x') .$$

There is no term I. We leave the details to the reader.

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Steve Hofmann
Department of Mathematics
University of Missouri
Columbia, MO 65211, USA
E-mail: `hofmanns@missouri.edu`

José María Martell
Instituto de Ciencias Matemáticas
CSIC-UAM-UC3M-UCM
Consejo Superior de Investigaciones Científicas
C/ Nicolás Cabrera, 13-15
E-28049 Madrid, Spain
E-mail: `chema.martell@icmat.es`