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GLOBAL EXISTENCE FOR THE EULER-MAXWELL SYSTEM

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Abstract. – The Euler-Maxwell system describes the evolution of a plasma when the collisions are important enough that each species is in a hydrodynamic equilibrium. In this paper we prove global existence of small solutions to this system set in the whole three-dimensional space, by combining the space-time resonance method (to obtain decay) and energy estimates (to control high frequencies). The non-integrable decay of the solutions makes it necessary to examine resonances within the energy estimate argument.

Résumé. – Le système d'Euler-Maxwell décrit l'évolution d'un plasma quand les collisions sont suffisamment importantes pour que chaque espèce soit dans un état d'équilibre hydrodynamique. On prouve dans cet article l'existence globale de petites solutions à ce système, posé en dimension 3 d'espace, en combinant la méthode des résonances en espace-temps (pour obtenir la décroissance des solutions) et des estimations d'énergie (pour contrôler la régularité des solutions). La décroissance non intégrable des solutions impose de combiner étroitement ces deux arguments en examinant le rôle des résonances au sein des estimations d'énergie.

1. Introduction

1.1. Plasma physics and Euler-Maxwell

There are different models to describe the state of a plasma depending on several parameters such as the Debey length, the plasma frequency, the collision frequencies between the different species… Formal derivation of these models can be found in Plasma Physics textbooks (see for instance Bellan [1], Boyd and Sanderson [4], Dendy [8] and the paper [2]…).

Since the plasma consists of a very large number of interacting particles, it is appropriate to adopt a statistical approach to describe it. In the kinetic description, it is only necessary to evolve the distribution function $f_\alpha(t,x,v)$ for each species in the system. The Vlasov equation is used in this case with the Lorentz force term and a collision term. It is coupled with the Maxwell equations for the electromagnetic fields.

If collisions are important, then each species is in a local equilibrium and the plasma is treated as a fluid. More precisely it is treated as a mixture of two or more interacting fluids.

This is the two-fluid model or the so-called Euler-Maxwell system. We refer to [22] for a discussion about the possible derivation of this system from kinetic models, namely from the two species Vlasov-Boltzmann-Maxwell system. We also refer to [24, 17, 23] for more about hydrodynamic limits of the Boltzmann equation. Another level of approximation consists in treating the plasma as a single fluid either by using the fact that the mass of the electrons is much smaller than the mass of the ions or from the hydrodynamic limit which requires (in a particular limit) that the two species evolve with a common velocity and temperature [22]. This is the model which we are going to consider in this paper.

1.2. The Euler-Maxwell equation

The Cauchy problem for the one-fluid version of the Euler-Maxwell system reads

$$\begin{cases} \rho\left(\partial_t u + u\cdot\nabla u\right) = -\frac{p'(\rho)}{m}\nabla\rho - \frac{e\rho}{m}\left(E + \frac{1}{c}u\times B\right) \\ \partial_t\rho + \nabla\cdot(\rho u) = 0 \\ \partial_t B + c\nabla\times E = 0 \\ \partial_t E - c\nabla\times B = 4\pi e\rho u \\ \nabla\cdot E = 4\pi e(\bar\rho - \rho) \\ \nabla\cdot B = 0 \\ (u,\rho,E,B)(t=0) = (u_0,\rho_0,E_0,B_0). \end{cases} \tag{1.1}$$

The unknown functions are: ρ, the density of electrons; u, the average velocity of the electrons; E, the electric field; B the magnetic field. The physical constants are: c, the speed of light; e, the charge of the electron; m, the mass of the electron. Finally, $\bar\rho$ is the uniform density of ions, and the electron gas is supposed to be barotropic, the pressure being given by $p(\rho)$.

Let us first recall a few results related to (1.1). Global existence of weak solutions was obtained for a related 1d model in [5] using compensated compactness. Also, several asymptotic problems (WKB asymptotics, incompressible limit, non-relativistic limit, quasi-neutral limit…) were studied to derive simplified models starting from the Euler-Maxwell system [33, 35, 34, 30]. We also refer to [27] where the incompressible Navier-Stokes system is studied.

Going back to our system (1.1), we notice that the last two equations above can be removed, as soon as they are satisfied at the initial time, *which we assume from now on*: they are then conserved by the flow given by the first four.

1.3. Vicinity of the trivial equilibrium state

An obvious equilibrium state of the above system is $(\rho,u,E,B) = (\bar\rho,0,0,0)$. In order to study its stability, it is instructive to linearize the above system, and compute evolution equations for its unknowns. It is convenient to introduce at this point the projections P, respectively Q onto divergence-free, respectively curl-free vector fields; they are given by

$$Qu := \left(\frac{\nabla}{\Delta}\right)\nabla\cdot u \quad \text{and} \quad Pu := u - Qu.$$

Split then accordingly u and E: $u = Pu + Qu$ and $E = PE + QE$. The linearized system can be written

$$\begin{cases} \left(\partial_t^2 - c_s^2 \Delta + \omega_p^2\right) \begin{pmatrix} QE \\ \rho - \bar\rho \\ Qu \end{pmatrix} = 0 \\ \left(\partial_t^2 - c^2 \Delta + \omega_p^2\right) \begin{pmatrix} PE \\ \nabla \times B + \frac{4\pi e \bar\rho}{c} Pu \end{pmatrix} = 0 \\ \partial_t \left(B - \frac{cm}{e} \nabla \times u\right) = 0 \end{cases}$$

where the speed of sound c_s and the plasma frequency ω_p are given by

$$c_s = \sqrt{\frac{p'(\bar\rho)}{m}} \quad \text{and} \quad \omega_p = \sqrt{\frac{4\pi e^2 \bar\rho}{m}}.$$

Thus around the equilibrium, and at a linear level, some unknowns are governed by the Klein-Gordon equation (with different speeds), whereas the quantity $B - \frac{cm}{e} \nabla \times u$ is conserved. The Klein-Gordon equations entail decay, which is one of the keys of the global stability result which we will prove; as for the quantity $B - \frac{cm}{e} \nabla \times u$, no decay is to be expected a priori. We will therefore set it to zero, which, as it turns out, is conserved by the nonlinear flow.

1.4. Adimensionalization and reductions

In the following, we set for simplicity the physical constants m, e, c, as well as $\bar\rho$ to 1. We also drop the 4π factors, since they are irrelevant. However $c_s^2 = p'(\bar\rho) = p'(1)$ remains a number less than 1. In order to simplify a little bit the estimates, we assume

$$p(\rho) \stackrel{\text{def}}{=} \frac{c_s^2}{3} \rho^3.$$

Finally, set

$$n \stackrel{\text{def}}{=} \rho - 1.$$

The Cauchy problem becomes

$$(EM) \qquad \begin{cases} \partial_t u + u \cdot \nabla u = -c_s^2 \rho \nabla \rho - E - u \times B \\ \partial_t \rho + \nabla \cdot (\rho u) = 0 \\ \partial_t B + \nabla \times E = 0 \\ \partial_t E - \nabla \times B = \rho u \\ \nabla \cdot E = -n \\ \nabla \cdot B = 0 \\ (u, n, E, B)(t = 0) = (u_0, n_0, E_0, B_0). \end{cases}$$

We shall furthermore assume that, initially,

$$B = \nabla \times u. \tag{1.2}$$

This condition is conserved by the flow of the above system: in order to see this, use the identity $u \cdot \nabla u = -u \times (\nabla \times u) + \nabla \frac{|u|^2}{2}$ to compute

$$
\begin{aligned}
\partial_t (B - \nabla \times u) &= \nabla \times (u \cdot \nabla u + u \times B) \\
&= \nabla \times \left(-u \times (\nabla \times u) + \nabla \frac{|u|^2}{2} \right) - \nabla \times (u \times B) \\
&= \nabla \times (u \times (B - \nabla \times u)) .
\end{aligned}
$$

The linearized system reads now

$$
\begin{cases}
\left(\partial_t^2 - c_s^2 \Delta + 1\right) \begin{pmatrix} Qu \\ n \\ QE \end{pmatrix} = 0 \\
\left(\partial_t^2 - \Delta + 1\right) \begin{pmatrix} Pu \\ PE \\ B \end{pmatrix} = 0.
\end{cases} \tag{1.3}
$$

1.5. Obtained results

Prior to stating our theorem, we need to define the operator $A \overset{\text{def}}{=} \frac{\langle D \rangle}{|D|}$ (see Section 2 for the precise definition of this operator).

THEOREM 1.1. – *Assume that the resonance separation condition* (4.1) *holds; it is the case generically in c_s. Fix $\alpha_0 > 0$. Then there exist $C_0, \varepsilon_0, N_0 > 0$ such that: if $\varepsilon < \varepsilon_0$, $N > N_0$ and*

$$
\left\| \langle x \rangle^{1+\alpha_0} (u_0, An_0, E_0, AB_0) \right\|_{H^N} < \varepsilon,
$$

then there exists a unique global solution of (EM) such that

$$
\sup_t \left[\langle t \rangle^{-C_0 \varepsilon} \|(u, An, E, AB)(t)\|_{H^N} + \sqrt{\langle t \rangle} \|(u, An, E, AB)(t)\|_3 \right] \lesssim \varepsilon
$$

(we refer to Section 2 for the definition of the norms appearing above). Furthermore, it scatters as t goes to infinity in that there exists a solution $(u_\ell, n_\ell, E_\ell, B_\ell)$ of the linear system (1.3) *corresponding to initial data in H^{N-2} such that*

$$
\|(u, n, E, B)(t) - (u_\ell, n_\ell, E_\ell, B_\ell)(t)\|_{H^{N-2}} \to 0 \quad \textit{as} \quad t \to \infty.
$$

REMARK 1.2. – A few observations on the hypotheses on the initial data:

- What is meant by the condition (4.1) being generic? This condition amounts to requiring that a finite number of real analytic functions of the speed of sound c_s do not vanish. The actual system seems to be too complicated to be solved analytically, but a numerical computation in [10] reveals that the condition (4.1) is met for the value $c_s = \frac{1}{5}$. Since non-zero analytic functions have separated zeros, the condition (4.1) holds except at most for a discrete set of c_s.
- The requirements on An_0 and AB_0 imply necessarily that $\int n_0 = 0$ and $\int B_0 = 0$. In particular this is consistent with the electric neutrality. Notice that this electric neutrality assumption could recently be removed for the related Euler-Poisson system, see [12].

– We did not try to optimize the number of derivatives in L^2 required (N), but rather aimed at a proof as simple as possible. On the other hand, the weight appearing above ($\langle x\rangle^{1+\alpha}$) seems nearly optimal; a more precise analysis would maybe allow $\langle x\rangle$ instead of $\langle x\rangle^{1+\alpha}$.

The proof will be essentially split into two parts: controlling the H^N norm of (u, n, E, B); and proving the decay in various norms. The former is achieved by an energy estimate; and the latter by the method of space-time resonances, which was introduced in [13]. It was also used to prove global existence of small data solutions for water waves [14, 15].

One of the main novelties in this paper appears in the energy estimate performed in Section 9 in order to control $\|u\|_{H^N}$. The non-integrable decay of the solution (it decays slightly better than $\frac{1}{\sqrt{t}}$ in L^∞) prevents one from obtaining control (or slow growth) of $\|u\|_{H^N}$ in a straightforward way. Instead, one needs to perform a transformation similar to a normal form transform within the energy estimate. More precisely, one can identify in the term appearing in the right-hand side of the energy estimate, (10.10), a factor oscillating in time, and integrate by parts to take advantage of it. This improves the decay, but leads to a new loss of derivatives, which can be once again absorbed thanks to a manipulation reminiscent of an energy estimate. See Section 9 for the full details.

1.6. Stability of compressible Euler and related models in dimension 3

It is instructive to compare the above results to earlier works on compressible Euler in dimension 3, or couplings of compressible Euler with various fields (electrostatic, electromagnetic, gravitational…). For all these models, a fundamental question is whether given data lead to blow-up or a global solution.

A first class of results gives blow-up for various types of data. The fundamental work is due to Sideris [32], who proved finite time blow-up of the C^1 norm for compressible Euler; he was able to obtain this result for data arbitrarily close to the equilibrium state given by a zero velocity, and a constant density. Many results followed: finite time blow-up was showed for the compressible Euler equation with compactly supported data by Makino, Ukai, and Kawashima [26]; for the attractive Euler-Poisson equation with compactly supported data by Perthame [31]; for the Euler-Poisson equation with compactly supported data by Makino and Perthame [25]; and for the relativistic compressible Euler equation by Guo and Tahvildar-Zadeh [20] (this paper also addressing relativistic Euler-Maxwell) and Pan and Smoller [29].

All of the aforementioned results rely on a non-constructive proof, and do not say much about the nature of the singularity. Recently, Christodoulou [6] was able to describe in a very precise manner the blow-up process for the relativistic compressible Euler equation.

Another line of research gives global existence (and scattering) for data close to the equilibrium state given by constant density, and all the fields (including the velocity) equal to zero. Such a result was first obtained by Guo [18] for the Euler-Poisson equation for the electrons; and by Guo and Pausader [19] for the Euler-Poisson equation for the ions. In both cases, the curl of the data is assumed to be zero, and this condition is conserved by the flow of the equation. Finally, global existence for Euler-Maxwell with relaxation was obtained by Duan [9].

Focusing on the case of small data (i.e., close to an equilibrium), some common features emerge from the results which have been mentioned. Global existence is only known under the assumption that the flow is irrotational: this eliminates a mode which is linearly non-decaying. Under this assumption, a crucial point is then the nature of the linearized equation: roughly speaking, blow-up may occur if it is a wave equation, whereas global existence is expected if it is a Klein-Gordon equation. The relevant difference between these two situations is that the latter gives a decay $\sim \frac{1}{t^{3/2}}$, whereas the former only decays $\sim \frac{1}{t}$.

In the case of Euler-Maxwell, which is treated in this paper, the condition $B = \nabla \times u$ is also meant to restrict the solution to the subspace along which the linearized problem is governed by Klein-Gordon equations. The novelty is that these Klein-Gordon equations have different speeds, making the nonlinear interaction more intricate.

2. Notations

We shall use the following standard notations:

- $A \lesssim B$ if $A \leq CB$ for some implicit constant C. The value of C may change from line to line.
- $A \sim B$ means that both $A \lesssim B$ and $B \lesssim A$.
- For any real number α, the "Japanese brackets" $\langle \cdot \rangle_\alpha$ stand for $\langle x \rangle_\alpha = \sqrt{1 + \alpha^2 x^2}$. We also denote $\langle x \rangle = \langle x \rangle_1$.
- If f is a function over $\mathbb{R}^3$ then its Fourier transform, denoted by $\widehat{f}$, or $\mathscr{F} f$, is given by

$$\widehat{f}(\xi) = \mathscr{F} f(\xi) = \frac{1}{(2\pi)^{3/2}} \int e^{-ix\xi} f(x)\, dx \quad \text{thus} \quad f(x) = \frac{1}{(2\pi)^{3/2}} \int e^{ix\xi} \widehat{f}(\xi)\, d\xi.$$

  In the text, we systematically drop the constants such as $\frac{1}{(2\pi)^{3/2}}$ since they are not relevant.
- The Fourier multiplier with symbol $m(\xi)$ is defined by

$$m(D) f = \mathscr{F}^{-1} \left[m \mathscr{F} f \right].$$

- The bilinear pseudo-product with symbol $m(\xi, \eta)$ is given by its Fourier transform

$$\mathscr{F} \left[T_m(f, g) \right](\xi) = \int m(\xi, \eta) \widehat{f}(\eta) \widehat{g}(\xi - \eta)\, d\eta.$$

  Similarly, the trilinear pseudo-product with symbol $m(\xi, \eta, \nu)$ is given by

$$\mathscr{F} \left[T_m(f, g, h) \right](\xi) = \int m(\xi, \eta, \nu) \widehat{f}(\nu) \widehat{g}(\eta) \widehat{h}(\xi - \eta - \nu)\, d\eta\, d\nu.$$

- The norm of the Lebesgue space L^p is denoted $\| \cdot \|_p$.
- H^N is given by the norm $\|f\|_{H^N} = \| \langle D \rangle^N f \|_2$.
- $W^{s,p}$ is given by the norm $\|f\|_{W^{s,p}} = \| \langle D \rangle^s f \|_p$.

3. A formulation adapted to energy estimates

Our aim here is to rewrite the equation in such a way that its dispersive properties become more transparent, but energy estimates can also be easily obtained.

Split

$$\begin{pmatrix} u \\ n \\ E \\ B \end{pmatrix} = \begin{pmatrix} Qu \\ n \\ QE \\ 0 \end{pmatrix} + \begin{pmatrix} Pu \\ 0 \\ PE \\ B \end{pmatrix} \overset{\mathrm{def}}{=} V_a + V_p$$

where V_p contains the unknowns which (in the linearization (1.3)) propagate as Klein-Gordon waves at the speed of light, and V_a the unknowns which (still in the linearization (1.3)) propagate as Klein-Gordon waves at the speed of sound.

3.1. The fluid system

We focus here on the evolution of $V_a = (Qu\,,\, n\,,\, QE\,,\, 0)$. It is governed by the system

$$\begin{cases} \partial_t Qu = -QE - \nabla \frac{|u|^2}{2} - c_s^2 \rho \nabla \rho \\ \partial_t n = -\nabla \cdot (\rho u) \\ \nabla \cdot E = -n. \end{cases}$$

In order to diagonalize this system, let us switch to the unknown function

$$\mathcal{A} = \frac{1}{2} \left(\frac{\langle D \rangle_{c_s}}{|D|} n + i \frac{\nabla}{|D|} \cdot u \right)$$

so that

$$Qu = -2 \frac{\nabla}{|D|} \mathfrak{Im}\, \mathcal{A}, \qquad n = 2 \frac{|D|}{\langle D \rangle_{c_s}} \mathfrak{Re}\, \mathcal{A} \quad \text{and} \quad QE = \frac{2\nabla}{|D| \langle D \rangle_{c_s}} \mathfrak{Re}\, \mathcal{A}.$$

Therefore, V_a can be fully described by the complex function $\mathcal{A}$. The evolution of $\mathcal{A}$ is given by

$$2 \partial_t \mathcal{A} = 2i \langle D \rangle_{c_s} \mathcal{A} - \frac{\langle D \rangle_{c_s} \nabla}{|D|} \cdot (nu) + \frac{i|D|}{2} \left(|u|^2 + c_s^2 |n|^2 \right). \tag{3.1}$$

3.2. The electromagnetic system

We focus here on the evolution of $V_p = (Pu\,,\, 0\,,\, PE\,,\, B)$. By (1.2), it suffices to consider PE and B. These fields are governed by the equations

$$\begin{cases} \partial_t B = -\nabla \times E \\ \partial_t PE = \nabla \times B + P(\rho u) \end{cases}$$

which implies

$$\partial_t^2 B - \Delta B + B = -\nabla \times (nu).$$

Setting

$$\mathcal{B} = \frac{\partial_t}{|D|} B + i \frac{\langle D \rangle}{|D|} B, \mathcal{B}$$

it satisfies

$$\partial_t \mathcal{B} - i\langle D\rangle \mathcal{B} = -\frac{\nabla}{|D|} \times (nu),$$

and the original unknown functions Pu, PE and B can be recovered by

$$Pu = \frac{\nabla}{|D|\langle D\rangle} \times \mathfrak{Im}\,\mathcal{B} \quad , \quad PE = -\frac{\nabla}{|D|} \times \mathfrak{Re}\,\mathcal{B} \quad \text{and} \quad B = \frac{|D|}{\langle D\rangle}\mathfrak{Im}\,\mathcal{B}. \tag{3.2}$$

Thus V_b can be fully described by the curl-free function $\mathcal{B}$ valued in $\mathbb{C}^3$.

3.3. Summarizing

The Euler-Maxwell system now reads

$$(EM') \qquad \begin{cases} \partial_t \mathcal{A} - i\langle D\rangle_{c_s} \mathcal{A} = -\frac{1}{2}\frac{\langle D\rangle_{c_s}\nabla}{|D|} \cdot (nu) + \frac{1}{4} i|D| \left(|u|^2 + |n|^2\right) \\ \partial_t \mathcal{B} - i\langle D\rangle \mathcal{B} = -\frac{\nabla}{|D|} \times (nu) \\ (\mathcal{A}, \mathcal{B})(t=0) = (\mathcal{A}_0, \mathcal{B}_0) \end{cases}$$

with

$$\begin{cases} Qu = -2\frac{\nabla}{|D|}\mathfrak{Im}\,\mathcal{A} \\ n = 2\frac{|D|}{\langle D\rangle_{c_s}}\mathfrak{Re}\,\mathcal{A} \\ Pu = \frac{\nabla}{|D|\langle D\rangle} \times \mathfrak{Im}\,\mathcal{B}. \end{cases}$$

It is interesting at this point to count dimensions: we started with (EM), a system whose unknown functions form a 10-vector, with 2 scalar constraints; we added the constraint (1.2), which corresponds to 2 scalar dimensions, since both sides of the equality are curl-free. All in all, the unknown is 6-dimensional. We represent it by $\mathcal{A}$ and $\mathcal{B}$, valued respectively in $\mathbb{C}$ and $\mathbb{C}^3$. Notice however that $\mathcal{B}$ is curl-free, which corresponds to a 2-dimensional constraint: therefore, $\mathcal{A}$ and $\mathcal{B}$ together give 6 degrees of freedom.

The data $(\mathcal{A}_0, \mathcal{B}_0)$ of (EM') are easily expressed in terms of the data (u_0, n_0, E_0, B_0) of (EM):

$$\mathcal{A}_0 \overset{\text{def}}{=} \frac{1}{2}\left(\frac{\langle D\rangle_{c_s}}{|D|} n_0 + i\frac{\nabla}{|D|} \cdot u_0\right) \quad \text{and} \quad \mathcal{B}_0 = -\frac{\nabla}{|D|} \times E_0 + i\frac{\langle D\rangle}{|D|} B_0.$$

Let us finally define the profiles of $\mathcal{A}$ and $\mathcal{B}$

$$a(t) \overset{\text{def}}{=} e^{-it\langle D\rangle_{c_s}} \mathcal{A}(t) \quad \text{and} \quad b(t) \overset{\text{def}}{=} e^{-it\langle D\rangle} \mathcal{B}(t).$$

4. A formulation adapted to decay estimates

As we saw, the system (EM') written above is equivalent to (EM); it will be the correct formulation to perform energy estimates. However, as far as dispersive estimates go, we will not need all the structure of (EM'): only resonances will play an important rôle. It will be convenient to write (EM') in a more compact form.

4.1. Duhamel's formula in Fourier space

Writing Duhamel's formula in terms of a and b gives

$$\begin{cases} a(t) = \mathscr{A}_0 + \int_0^t e^{-is\langle D\rangle_{c_s}} \left[-\frac{1}{2}\frac{\langle D\rangle_{c_s}\nabla}{|D|}\cdot(nu) + \frac{1}{4}i|D|\left(|u|^2+|n|^2\right)\right] ds \\ b(t) = \mathscr{B}_0 - \int_0^t e^{-is\langle D\rangle}\left[\frac{\nabla}{|D|}\times(nu)\right] ds. \end{cases}$$

Taking the Fourier transform gives

$$\begin{cases} \widehat{a}(t,\xi) = \widehat{\mathscr{A}_0}(\xi) + \text{“nonlinear term”} \\ \widehat{b}(t,\xi) = \widehat{\mathscr{B}_0}(\xi) + \text{“nonlinear term”}. \end{cases}$$

In order to make notations lighter and estimates easier, we will now give up some of the structure of the above system.

Convention 1. – We will denote indifferently $\mathscr{C}(t)$ for $\mathscr{A}(t)$ or $\mathscr{B}(t)$, or their complex conjugates, and $c(t)$ for $a(t)$ or $b(t)$, or their complex conjugates. Similarly, we denote $e^{\pm it\langle D\rangle_\ell}$ for any of the groups $e^{it\langle D\rangle}$, $e^{-it\langle D\rangle}$, $e^{it\langle D\rangle_{c_s}}$, or $e^{-it\langle D\rangle_{c_s}}$.

For instance, u or n is a linear combination of terms of the type $e^{\pm it\langle D\rangle_\ell}c$ with coefficients in the class of Fourier multipliers of order zero.

It is always understood that, in an expression of the form $e^{\pm it\langle D\rangle_\ell}c(t)$, the meaning of $e^{\pm it\langle D\rangle_\ell}$ is consistent with that of c. For instance, if c stands for a, then $e^{\pm it\langle D\rangle_\ell}$ stands for $e^{it\langle D\rangle_{c_s}}$.

With this convention, it is easy to see from the above that the “nonlinear terms” can all be written as a linear combination of terms of the following type (which we denote generically by g)

$$\widehat{g}(t,\xi) = \int_0^t \int e^{is\phi(\xi,\eta)} m(\xi,\eta)\widehat{c}(\eta,s)\widehat{c}(\xi-\eta,s)\,d\eta\,ds, \tag{4.1}$$

where m is such that
(4.2)

$$m(\xi,\eta) = m_0(\xi)m_1(\eta)m_2(\xi-\eta) \qquad \text{with} \begin{cases} \left|\partial_\xi^\alpha m_0(\xi)\right| \lesssim \frac{1}{|\xi|^{|\alpha|}} & \text{if } |\xi|\le 1 \\ \left|\partial_\xi^\alpha m_0(\xi)\right| \lesssim \frac{1}{|\xi|^{|\alpha|-1}} & \text{if } |\xi|\ge 1 \\ \left|\partial_\xi^\alpha m_1(\xi)\right|,\ \left|\partial_\xi^\alpha m_2(\xi)\right| \lesssim \frac{1}{|\xi|^{|\alpha|}} & \text{for any } \xi \end{cases}$$

(notice here that m_0 is order 1 whereas m_1 and m_2 are order 0) and ϕ is one of the $\phi^{\varepsilon_1,\varepsilon_2}_{k,\ell,m}$ given by

$$\phi^{\varepsilon_1,\varepsilon_2}_{k,\ell,m}(\xi,\eta) \overset{\text{def}}{=} -\langle\xi\rangle_k + \varepsilon_1\langle\eta\rangle_\ell + \varepsilon_2\langle\xi-\eta\rangle_m \tag{4.3}$$

where $\varepsilon_1,\varepsilon_2 = \pm$ and k,ℓ,m are either 1 or c_s.

4.2. Space-time resonances in the context of Euler-Maxwell

Seeing (4.1) as an oscillatory integral, it becomes clear that the cancellation properties of ϕ and $\partial_\eta \phi$ will provide a key to understanding the large time behavior of our system: this is the idea of space time resonances. See [10] for a general presentation, and [11] for the case of (semilinear) Klein-Gordon equations with different propagation speeds.

Recall that the phase functions were defined in (4.3), and define for each interaction the space, time, and space-time resonant sets

$$\begin{aligned}
\mathcal{S}^{\varepsilon_1,\varepsilon_2}_{k,\ell,m} &\overset{\text{def}}{=} \{(\xi,\eta) \,|\, \phi^{\varepsilon_1,\varepsilon_2}_{k,\ell,m} = 0\} \quad \text{("space resonances")}\\
\mathcal{T}^{\varepsilon_1,\varepsilon_2}_{k,\ell,m} &\overset{\text{def}}{=} \{(\xi,\eta) \,|\, \partial_\eta \phi^{\varepsilon_1,\varepsilon_2}_{k,\ell,m} = 0\} \quad \text{("time resonances")}\\
\mathcal{R}^{\varepsilon_1,\varepsilon_2}_{k,\ell,m} &\overset{\text{def}}{=} \mathcal{S}^{\varepsilon_1,\varepsilon_2}_{k,\ell,m} \cap \mathcal{T}^{\varepsilon_1,\varepsilon_2}_{k,\ell,m} \quad \text{("space-time resonances").}
\end{aligned}$$

The set of all space-time resonances is

$$\mathcal{R} = \cup_{\varepsilon_1,\varepsilon_2,k,\ell,m} \mathcal{R}^{\varepsilon_1,\varepsilon_2}_{k,\ell,m};$$

it is compact (actually, it is a union of sets of the form $\{|\xi| = M, \eta = \lambda\xi\}$ for constants M and λ, see [10]). We denote by $C_{\mathcal{R}} - 1$ the radius of a ball that contains $\mathcal{R}$. Finally, define the outcome and germ, or source frequencies for space-time resonances: these are simply the projections of $\mathcal{R}^{\varepsilon_1,\varepsilon_2}_{k,\ell,m}$ in the ξ variable, and the union of the projections in the η and $\xi - \eta$ variables, respectively. More precisely if $\pi_\xi(\xi',\eta') = \xi'$, $\pi_\eta(\xi',\eta') = \eta'$ and $\pi_{\xi-\eta}(\xi',\eta') = \xi' - \eta'$, we set

$$\begin{aligned}
\mathcal{O} &\overset{\text{def}}{=} \pi_\xi\left(\mathcal{R}\right)\\
\mathcal{G} &\overset{\text{def}}{=} \pi_\eta\left(\mathcal{R}\right) \cup \pi_{\xi-\eta}\left(\mathcal{R}\right).
\end{aligned}$$

Definition 4.1. – *Space-time resonances are said to be separated if no outcome frequency is also a germ frequency. In mathematical terms,* $\mathcal{G} \cap \mathcal{O} = \varnothing$.

5. Some linear and bilinear cutoff Fourier multipliers

We use here some of the cut-off functions defined in [11]; see Proposition 12.1 for results on the boundedness of the associated operators.

5.1. Low or high frequency cutoff: Z_l, Z_h

First pick M_0 large enough (the precise value of M_0 will be fixed in the following, for the moment it is simply $\geq C_{\mathcal{R}}$ defined in Section 4.2).

It will be necessary in the proof to distinguish between high and low frequencies. To this end, we introduce the cut off function θ, which is such that

$$\theta \in \mathcal{C}^\infty_0 \quad , \quad \theta = 1 \text{ on } B(0,1) \quad \text{and} \quad \theta = 0 \text{ on } B(0,2)^c. \tag{5.1}$$

We restrict to low or high frequencies using the operators Z_l and Z_h, respectively. These operators are given by

$$Z_h \overset{\text{def}}{=} 1 - \theta\left(\frac{D}{M_0}\right) \qquad Z_l \overset{\text{def}}{=} \theta\left(\frac{D}{M_0}\right).$$

5.2. Cutoff for $\mathcal{O}$: the operators $Z_{\mathcal{O}}$, $\widetilde{Z}_{\mathcal{O}}$

Recall that $\mathcal{O}$ and $\mathcal{G}$ were defined in Section 4.2.

Under the resonance separation condition (Definition 4.1), it is possible to find δ_0 such that no frequency in $B_{10\delta_0}(\mathcal{O})$ (a $10\delta_0$-neighborhood of $\mathcal{O}$) is a source of a space-time resonance:

$$B_{10\delta_0}(\mathcal{O}) \cap \mathcal{G} = \varnothing.$$

Define $\chi_{\mathcal{O}}$ a smooth cut-off function such that

$$\begin{aligned} \chi_{\mathcal{O}} &= 1 \quad \text{on } B_{\delta_0/2}(\mathcal{O}) \\ \chi_{\mathcal{O}} &= 0 \quad \text{outside of } B_{\delta_0}(\mathcal{O}) \end{aligned}$$

and then let $\widetilde{\chi}_{\mathcal{O}}$ satisfy

$$\chi_{\mathcal{O}} + \widetilde{\chi}_{\mathcal{O}} = 1.$$

The corresponding operators are

$$Z_{\mathcal{O}} \stackrel{\text{def}}{=} \chi_{\mathcal{O}}(D) \quad \text{ and } \quad \widetilde{Z}_{\mathcal{O}} \stackrel{\text{def}}{=} \widetilde{\chi}_{\mathcal{O}}(D).$$

5.3. Cutoff for $\mathcal{S}$ and $\mathcal{T}$: the symbols $\chi_{\mathcal{S}}$ and $\chi_{\mathcal{T}}$

The cut-off functions which we are about to define will, for a given set of indices $\varepsilon_1, \varepsilon_2, k, \ell, m$ separate $\mathcal{T}^{\varepsilon_1,\varepsilon_2}_{k,\ell,m}$ from $\mathcal{S}^{\varepsilon_1,\varepsilon_2}_{k,\ell,m}$; of course this can only be done away from a neighborhood of $\mathcal{R}^{\varepsilon_1,\varepsilon_2}_{k,\ell,m}$, where these two sets intersect. Dropping for simplicity the indices, the function $\chi_{\mathcal{S}}$ localizes away from $\mathcal{T}$, whereas $\chi_{\mathcal{T}}$ localizes away from $\mathcal{S}$. Since $\mathcal{T} = \{\phi = 0\}$ whereas $\mathcal{S} = \{\partial_\eta \phi = 0\}$, this explains the inequalities (5.2).

LEMMA 5.1. – *For each set of indices $\varepsilon_1, \varepsilon_2, k, \ell, m$, it is possible to find cut-off functions*

$$\chi_{\mathcal{T}^{\varepsilon_1,\varepsilon_2}_{k,\ell,m}}(\xi,\eta) \quad , \quad \chi_{\mathcal{S}^{\varepsilon_1,\varepsilon_2}_{k,\ell,m}}(\xi,\eta)$$

such that (in the following, we drop the indices $\varepsilon_1, \varepsilon_2, k, \ell, m$ for simplicity)

- *$\chi_{\mathcal{T}}$, $\chi_{\mathcal{S}}$ are smooth;*
- *their sum equals one away from $\mathcal{R}$: $\chi_{\mathcal{T}} + \chi_{\mathcal{S}} = 1$ if $\operatorname{dist}((\xi,\eta),\mathcal{R}) > \delta_0/10$;*
- *the derivatives of $\frac{\chi_{\mathcal{S}}}{\phi}$ and $\frac{\chi_{\mathcal{T}}\partial_\eta\phi}{|\partial_\eta\phi|^2}$ satisfy*

$$\text{if } |\alpha| \leq 20, \text{ then} \qquad \left|\partial^\alpha_{\xi,\eta}\frac{\chi_{\mathcal{S}}}{\phi}\right| \;,\; \left|\partial^\alpha_{\xi,\eta}\frac{\chi_{\mathcal{T}}\partial_\eta\phi}{|\partial_\eta\phi|^2}\right| \lesssim |\xi,\eta|^{n_0} \tag{5.2}$$

for some integer n_0.

5.4. Paraproduct decomposition: the symbols ζ^1 and ζ^2

Following the original idea of Bony [3], we would like to distinguish between regions where $|\eta| \gtrsim |\xi - \eta|$ or $|\xi-\eta| \gtrsim |\eta|$.

We pick two functions $\zeta^1(\xi,\eta)$ and $\zeta^2(\xi,\eta)$ such that

- ζ^2 and ζ^1 are smooth.
- ζ^2 and ζ^1 are homogeneous of degree zero outside of $B(0,1)$.
- $\zeta^2(\xi,\eta) + \zeta^1(\xi,\eta) = 1$ for any (ξ,η).
- If $|(\xi,\eta)| \geq 1$ and $(\xi,\eta) \in \operatorname{Supp}(\zeta^1)$, then $|\xi-\eta| \leq c|\eta|$ for a constant c.
- If $|(\xi,\eta)| \geq 1$ and $(\xi,\eta) \in \operatorname{Supp}(\zeta^2)$, then $|\eta| \leq c|\xi-\eta|$ for a constant c.

6. Plan of the proof

6.1. The a priori estimates

In order to prove Theorem 1.1, we will prove the following a priori estimates, valid if ε is small enough.

Energy estimate:

– $\|(\mathcal{A},\mathcal{B})\|_{H^N} \lesssim \varepsilon \langle t\rangle^{C_0\varepsilon}$ for a constant C_0, and any t (regularity in L^2).

Decay estimates:

– $\|(\mathcal{A},\mathcal{B})\|_{W^{N'',\left(\frac{1}{3}-\delta_1\right)^{-1}}} \lesssim \dfrac{\varepsilon}{\langle t\rangle^{\frac{1}{2}+3\delta_1}}$ (square integrable decay above L^3).
– $\left\|\widetilde{Z}_{\mathcal{O}}\left(\mathcal{A},\mathcal{B}\right)\right\|_{W^{N'',\left(\frac{1}{6}+\delta_1\right)^{-1}}} \lesssim \dfrac{\varepsilon}{\langle t\rangle^{1-3\delta_1}}$ (decay slightly below L^6 for "non-outcome" frequencies).
– $\left\|\widetilde{Z}_{\mathcal{O}}\left(\mathcal{A},\mathcal{B}\right)\right\|_{W^{2,\infty}}, \left\|\widetilde{Z}_{\mathcal{O}}\left(u,n\right)\right\|_{W^{2,\infty}} \lesssim \dfrac{\varepsilon}{\langle t\rangle}$ (decay $\sim \frac{1}{t}$ in L^∞ for "non-outcome" frequencies).

Localization estimates:

– $\||x|(a,b)\|_{H^{N'}} \lesssim \varepsilon\sqrt{\langle t\rangle}$ (localization in $H^{N'}$)
– $\left\||x|^{1/8}\widetilde{Z}_{\mathcal{O}}(a,b)\right\|_2 \lesssim \varepsilon$ (localization in L^2 for "non-outcome" frequencies).

The constants N, N', N'' are chosen such that $N-N_1 > N''-N_1 > N' > N_1$, for a sufficiently big constant N_1; in particular, N is sufficiently big for the necessary arguments in [11] to apply. The constant δ_1 is chosen sufficiently small for the necessary parts of the argument in [11] to apply.

6.2. The continuation argument

We encapsulate the above quantities in the norm $\|\cdot\|_X$ given by

$$\begin{aligned}\|(\mathcal{A},\mathcal{B})\|_X \overset{\text{def}}{=} \sup_t \Bigg[&\langle t\rangle^{-C_0\varepsilon}\|(\mathcal{A},\mathcal{B})\|_{H^N} + \langle t\rangle^{\frac{1}{2}+3\delta_1}\|(\mathcal{A},\mathcal{B})\|_{W^{N'',\left(\frac{1}{3}-\delta_1\right)^{-1}}}\\ &+\langle t\rangle^{1-3\delta_1}\left\|\widetilde{Z}_{\mathcal{O}}(\mathcal{A},\mathcal{B})\right\|_{W^{N'',\left(\frac{1}{6}+\delta_1\right)^{-1}}}\\ &+\langle t\rangle\left\|\widetilde{Z}_{\mathcal{O}}(\mathcal{A},\mathcal{B},u,n)\right\|_{W^{2,\infty}} + \frac{1}{\sqrt{\langle t\rangle}}\||x|(a,b)\|_{H^{N'}} + \left\||x|^{1/8}\widetilde{Z}_{\mathcal{O}}(a,b)\right\|_2\Bigg]\end{aligned}$$

(recall that $a(t) = e^{-it\langle D\rangle_{c_s}}\mathcal{A}(t)$ and $b(t) = e^{-it\langle D\rangle}\mathcal{B}(t)$).

Local well-posedness is classical, since the system is hyperbolic symmetrizable; it is also classical to show that, if the norm $W^{1,\infty}$ remains bounded, so does the H^N norm. Since X controls the norm $L^\infty_t W^{1,\infty}$, it suffices to show the following a priori estimate, which can be turned into an existence proof by a continuation argument.

The crucial a priori estimate, from which the energy, decay and localization estimates in Subsection 6.1 follow, is given by: for data such that

$$\|(e^{it\langle D\rangle_{c_s}}\mathcal{A}_0, e^{it\langle D\rangle}\mathcal{B}_0)\|_X \lesssim \varepsilon,$$

there holds

$$\|(\mathcal{A},\mathcal{B})\|_X \lesssim \|(e^{it\langle D\rangle_{c_s}}\mathcal{A}_0, e^{it\langle D\rangle}\mathcal{B}_0)\|_X + \|(\mathcal{A},\mathcal{B})\|_X^2 + \|(\mathcal{A},\mathcal{B})\|_X^3$$

(the cubic term, which can seem out of place since the nonlinearity is quadratic, follows from normal form transforms). We will proceed by showing that all the quantities appearing in the definition of X can be controlled by the above right-hand side. More precisely, the plan is as follows:

- Decay estimates are proved in Section 7.
- Localization estimates are proved in Section 8.
- The energy estimate for $\mathcal{A}$: $\sup_t \langle t\rangle^{-C_0\varepsilon}\|\mathcal{A}(t)\|_{H^N} \lesssim \varepsilon$ is proved in Section 10.
- The energy estimate for $\mathcal{B}$: $\sup_t \langle t\rangle^{-C_0\varepsilon}\|\mathcal{B}(t)\|_{H^N} \lesssim \varepsilon$ is proved in Section 9.
- Finally, in Section 11 we give a sketch of the proof of the scattering.

7. Decay estimates

We want to prove here that

(7.1)

$$\sup_t \left[\langle t\rangle^{\frac{1}{2}+3\delta_1}\|(\mathcal{A},\mathcal{B})\|_{W^{N'',\left(\frac{1}{3}-\delta_1\right)^{-1}}} + \langle t\rangle^{1-3\delta_1}\left\|\widetilde{Z}_{\mathcal{O}}(\mathcal{A},\mathcal{B})\right\|_{W^{N'',\left(\frac{1}{6}+\delta_1\right)^{-1}}} + \langle t\rangle\left\|\widetilde{Z}_{\mathcal{O}}(\mathcal{A},\mathcal{B},u,n)\right\|_{W^{2,\infty}}\right]$$
$$\lesssim \left\|(e^{it\langle D\rangle_{c_s}}\mathcal{A}_0 + e^{it\langle D\rangle}\mathcal{B}_0)\right\|_X + \|(\mathcal{A},\mathcal{B})\|_X^2 .$$

7.1. Control of the $W^{N'',\left(\frac{1}{6}+\delta_1\right)^{-1}}$ and $W^{N'',\left(\frac{1}{3}-\delta_1\right)^{-1}}$ norms

This can be achieved as in [11]:

- First, the norm $W^{N'',\left(\frac{1}{6}+\delta_1\right)^{-1}}$ is controlled in [11] in Section 5.5. As far as the control of this norm goes, the main difference between the Euler-Maxwell system and the setting of [11] is the quasilinearity of Euler-Maxwell. This induces a further loss of regularity in the nonlinear term, which is however easily absorbed using the H^N norm.
- Second, the norm $W^{N'',\left(\frac{1}{3}-\delta_1\right)^{-1}}$ is controlled in [11] in Section 5.6. This estimate is only problematic on $\mathcal{O}$: away from this set, it can be obtained by interpolating between $W^{N'',\left(\frac{1}{6}+\delta_1\right)^{-1}}$ and the conserved energy, which gives an L^2 bound. Therefore, this is a low frequency question, and the argument of [11] applies identically.

We do not detail these two points, and focus directly on the third norm in (7.1).

7.2. Control of the $W^{2,\infty}$ norm

Proceeding as in Subsection 4.1, we can derive a generic term g corresponding to the nonlinear term in Duhamel's formula for u and n. It turns out, since u and n are given from $\mathcal{A}$ and $\mathcal{B}$ by the action of a Fourier multiplier, that this g would satisfy exactly the properties listed in Subsection 4.1.

Thus all we need to do is to prove that, for g as in Subsection 4.1,

$$\left\|\widetilde{Z}_{\mathcal{O}}e^{it\langle D\rangle_\ell}g(t)\right\|_{W^{2,\infty}} \lesssim \frac{1}{\langle t\rangle}\|(\mathcal{A},\mathcal{B})\|_X^2 .$$

In order to prove this, we shall split $\widetilde{Z}_\emptyset g(t)$ as follows

$$\mathscr{F}(\widetilde{Z}_\emptyset g(t)) = \int_0^1 \int \widetilde{\chi}_\emptyset(\xi) e^{is\phi} m(\xi,\eta)\widehat{c}(s,\eta)\widehat{c}(s,\xi-\eta)\,d\eta\,ds \tag{7.2a}$$

$$+ \int_1^t \int \widetilde{\chi}_\emptyset(\xi)\chi_{\mathscr{S}}(\xi,\eta) e^{is\phi} m(\xi,\eta)\widehat{c}(s,\eta)\widehat{c}(s,\xi-\eta)\,d\eta\,ds \tag{7.2b}$$

$$+ \int_1^t \int \widetilde{\chi}_\emptyset(\xi)\chi_{\mathscr{T}}(\xi,\eta) e^{is\phi} m(\xi,\eta)\widehat{c}(s,\eta)\widehat{c}(s,\xi-\eta)\,d\eta\,ds. \tag{7.2c}$$

In the above, we have used the cut-off functions $\chi_{\mathscr{S}}$ and $\chi_{\mathscr{T}}$. Remember that these were defined in 5.1 depending on the quadratic interaction considered; they were therefore labeled $\chi_{\mathscr{S}^{k,l,m}_{\varepsilon_1,\varepsilon_2}}$ and $\chi_{\mathscr{T}^{k,l,m}_{\varepsilon_1,\varepsilon_2}}$. The above equation is written in generic terms, but it is tacitly understood that the cut-off functions used are the ones corresponding to the quadratic interaction at hand.

7.3. Preliminary estimate on $\partial_s c$

Observe from Subsection 4.1 that $e^{is\langle\xi\rangle_k}\partial_s\widehat{c}(\xi)$ can be written as a sum of terms of the type

$$\int m(\xi,\eta)\widehat{\mathscr{C}}(\eta)\widehat{\mathscr{C}}(\xi-\eta)\,ds,$$

where m satisfies the estimates of that section. Therefore, by Proposition 12.1 (i),

$$\left\| e^{is\langle D\rangle}\partial_s c\right\|_{W^{N''-1,3/2}} \lesssim \|\mathscr{C}\|^2_{W^{N'',3}} \lesssim \frac{1}{t}\|\mathscr{C}\|^2_X . \tag{7.3}$$

7.4. The small time term (7.2a)

Using repeatedly the Sobolev embedding theorem, and the dispersive estimate (12.3) gives (assuming $t>1$, the case $t<1$ being trivial)

$$\begin{aligned}
\left\| e^{it\langle D\rangle_\ell}\mathscr{F}^{-1}(7.2\mathrm{a})\right\|_{W^{2,\infty}} &\lesssim \left\| \int_0^1 e^{i(t-s)\langle D\rangle_\ell} T_{\widetilde{\chi}_\emptyset(\xi)m(\xi,\eta)}(\mathscr{C},\mathscr{C})\,ds\right\|_{W^{3,6}}\\
&\lesssim \frac{1}{t}\int_0^1 \left\| T_{\widetilde{\chi}_\emptyset(\xi)m(\xi,\eta)}(\mathscr{C},\mathscr{C})\right\|_{W^{5,6/5}}\,ds\\
&\lesssim \frac{1}{t}\int_0^1 \|\mathscr{C}\|^2_{W^{6,12/5}}\,ds\\
&\lesssim \frac{1}{t}\int_0^1 \|\mathscr{C}\|^2_{H^7}\,ds \lesssim \frac{1}{t}\|\mathscr{C}\|^2_X .
\end{aligned}$$

7.5. The term away from $\mathscr{T}$ (7.2b)

In order to deal with this term, integrate by parts in time using the identity $\frac{1}{i\phi}\partial_s e^{is\phi} = e^{is\phi}$. Thus

$$(7.2\mathrm{b}) = \int \widetilde{\chi}_\emptyset(\xi)\chi_{\mathscr{S}}(\xi,\eta) m(\xi,\eta)\frac{e^{-it\langle\xi\rangle_\ell}}{i\phi}\widehat{\mathscr{C}}(t,\eta)\widehat{\mathscr{C}}(t,\xi-\eta)\,d\eta \tag{7.4a}$$

$$- \int_1^t \int \widetilde{\chi}_\emptyset(\xi)\chi_{\mathscr{S}}(\xi,\eta) m(\xi,\eta)\frac{1}{i\phi}e^{is\phi}\partial_s\widehat{c}(s,\eta)\widehat{c}(s,\xi-\eta)\,d\eta\,ds \tag{7.4b}$$

$$+ \{\text{symmetric and easier terms}\}, \tag{7.4c}$$

where the "symmetric and easier terms" correspond to the case where the partial derivative ∂_s hits the other c, and to the boundary term at $s=1$. Using successively the Sobolev embedding Theorem 12.1 and Proposition 12.1 (ii) gives

$$\begin{aligned}\left\|e^{it\langle D\rangle_\ell}\mathcal{F}^{-1}(7.4\mathrm{a})\right\|_{W^{2,\infty}} &= \left\|T_{\widetilde{\chi}_\Theta(\xi)\chi_{\mathscr{I}}(\xi,\eta)m(\xi,\eta)\frac{1}{i\phi}}(\mathscr{C},\mathscr{C})\right\|_{W^{2,\infty}}\\ &\lesssim \left\|T_{\widetilde{\chi}_\Theta(\xi)\chi_{\mathscr{I}}(\xi,\eta)m(\xi,\eta)\frac{1}{i\phi}}(\mathscr{C},\mathscr{C})\right\|_{W^{4,\left(\frac{2}{3}-2\delta_1\right)^{-1}}}\\ &\lesssim \|\mathscr{C}\|_{W^{n+4,\left(\frac{1}{3}-\delta_1\right)^{-1}}}\|\mathscr{C}\|_{W^{n+4,\left(\frac{1}{3}-\delta_1\right)^{-1}}}\\ &\lesssim \frac{1}{t^{1+6\delta_1}}\|\mathscr{C}\|_X^2.\end{aligned}$$

In order to estimate (7.4b), split it as follows

$$\begin{aligned}\mathcal{F}^{-1}(7.4\mathrm{b}) &= \int_1^t\int \widetilde{\chi}_\Theta(\xi)\chi_{\mathscr{I}}(\xi,\eta)m(\xi,\eta)\frac{1}{i\phi}e^{is\phi}\partial_s\widehat{c}(s,\eta)\widehat{c}(s,\xi-\eta)\,d\eta\,ds\\ &= \int_1^{t-1}+\int_{t-1}^t\cdots \overset{\text{def}}{=} I+II.\end{aligned}$$

Use the Sobolev embedding theorem, the dispersive estimate (12.3), Proposition 12.1 and the preliminary estimate (7.3) to get, for $\delta>0$ small enough,

$$\begin{aligned}\left\|e^{it\langle D\rangle_\ell}\mathcal{F}^{-1}I\right\|_{W^{2,\infty}} &= \left\|e^{it\langle D\rangle_\ell}\int_1^{t-1}T_{\widetilde{\chi}_\Theta(\xi)\chi_{\mathscr{I}}(\xi,\eta)m(\xi,\eta)\frac{1}{i\phi}}\left(e^{\pm is\langle D\rangle_\ell}(\partial_s c),\mathscr{C}\right)ds\right\|_{W^{2,\infty}}\\ &\lesssim \int_1^{t-1}\frac{1}{(t-s)^{(3/2-3\delta)}}\left\|T_{\widetilde{\chi}_\Theta(\xi)\chi_{\mathscr{I}}(\xi,\eta)m(\xi,\eta)\frac{1}{i\phi}}\left(e^{\pm is\langle D\rangle_\ell}(\partial_s c),\mathscr{C}\right)\right\|_{W^{5,(1-\delta)^{-1}}}ds\\ &\lesssim \int_1^{t-1}\frac{1}{(t-s)^{3/2-3\delta}}\left\|e^{\pm is\langle D\rangle_\ell}(\partial_s c)\right\|_{W^{5+n,3/2}}\|\mathscr{C}\|_{W^{5+n,(1/3-\delta)^{-1}}}\,ds\\ &\lesssim \int_1^{t-1}\frac{1}{(t-s)^{3/2-3\delta}}\frac{1}{s}\frac{1}{s^{1/2+3\delta}}\|\mathscr{C}\|_X^3\,ds\\ &\lesssim \|\mathscr{C}\|_X^2\frac{1}{t^{3/2-3\delta}}.\end{aligned}$$

As for II, use repeatedly the Sobolev embedding Theorem 12.1, Proposition 12.1 and the preliminary estimate (7.3) to get

$$\begin{aligned}\left\|e^{it\langle D\rangle_\ell}\mathcal{F}^{-1}II\right\|_{W^{2,\infty}} &\lesssim \left\|e^{it\langle D\rangle_\ell}\mathcal{F}^{-1}II\right\|_{W^{4,2}} = \left\|\mathcal{F}^{-1}II\right\|_{W^{4,2}}\\ &\lesssim \int_{t-1}^t\left\|T_{\widetilde{\chi}_\Theta(\xi)\chi_{\mathscr{I}}(\xi,\eta)m(\xi,\eta)\frac{1}{i\phi}}\left(e^{\pm is\langle D\rangle_\ell}(\partial_s c),\mathscr{C}\right)\right\|_{W^{4,2}}ds\\ &\lesssim \int_{t-1}^t\left\|T_{\widetilde{\chi}_\Theta(\xi)\chi_{\mathscr{I}}(\xi,\eta)(\xi,\eta)m(\xi,\eta)\frac{1}{i\phi}}\left(e^{\pm is\langle D\rangle_\ell}(\partial_s c),\mathscr{C}\right)\right\|_{W^{6,(1-\delta)^{-1}}}ds\\ &\lesssim \int_{t-1}^t\left\|e^{\pm is\langle D\rangle_\ell}(\partial_s c)\right\|_{W^{6+n,3/2}}\|\mathscr{C}\|_{W^{6+n,(1/3-\delta)^{-1}}}\,ds\\ &\lesssim \int_{t-1}^t\frac{1}{s}\frac{1}{s^{1/2+3\delta}}\|\mathscr{C}\|_X^2\,ds\\ &\lesssim \|\mathscr{C}\|_X^2\frac{1}{t^{3/2+3\delta}}.\end{aligned}$$

7.6. The term away from $\mathscr{S}$ (7.2c)

First transform this term by an integration by parts using the identity $\frac{\partial_\eta \phi}{is|\partial_\eta\phi|^2} \cdot \partial_\eta e^{is\phi} = e^{is\phi}$. This gives

$$\text{(7.2c)} = -\int_1^t \int \widetilde{\chi}_{\mathscr{O}}(\xi)\chi_{\mathscr{T}}(\xi,\eta)\frac{\partial_\eta\phi}{is|\partial_\eta\phi|^2} m(\xi,\eta)e^{is\phi}\partial_\eta\widehat{c}(\eta)\widehat{c}(\xi-\eta)\,d\eta\,ds \tag{7.5a}$$

$$-\int_1^t \int \widetilde{\chi}_{\mathscr{O}}(\xi)\chi_{\mathscr{T}}(\xi,\eta)\frac{\partial_\eta\phi}{is|\partial_\eta\phi|^2}\partial_\eta m(\xi,\eta)e^{is\phi}\widehat{c}(\eta)\widehat{c}(\xi-\eta)\,d\eta\,ds \tag{7.5b}$$

$$+\{\text{symmetric and easier terms}\}. \tag{7.5c}$$

Let us begin with (7.5a), which we split as follows:

$$\begin{aligned} -\text{(7.5a)} &= \int_1^t \int \widetilde{\chi}_{\mathscr{O}}(\xi)\chi_{\mathscr{T}}(\xi,\eta)\frac{\partial_\eta\phi}{is|\partial_\eta\phi|^2} m(\xi,\eta)e^{is\phi}\partial_\eta\widehat{c}(\eta)\widehat{c}(\xi-\eta)\,d\eta\,ds \\ &= \int_1^{t/2} + \int_{t/2}^t \cdots \overset{\text{def}}{=} I + II. \end{aligned} \tag{7.6}$$

For $\delta < 0$, $|\delta|$ small, apply successively the Sobolev embedding theorem, the dispersive estimate (12.3), and Proposition 12.1 to get

(7.7)

$$\begin{aligned} \left\| e^{it\langle D\rangle_\ell}\mathscr{F}^{-1} I\right\|_{W^{2,\infty}} &\lesssim \int_1^{t/2} \left\| e^{i(t-s)\langle D\rangle}\frac{1}{s} T_{\widetilde{\chi}_{\mathscr{O}}(\xi)\chi_{\mathscr{T}}(\xi,\eta)\frac{\partial_\eta\phi}{i|\partial_\eta\phi|^2}m(\xi,\eta)}\left(e^{\pm is\langle D\rangle_\ell}(xc),\mathscr{C}\right)\right\|_{W^{3,\left(\frac{1}{6}+\delta\right)^{-1}}} ds \\ &\lesssim \int_1^{t/2} \frac{1}{(t-s)^{1-3\delta}}\frac{1}{s}\left\| T_{\widetilde{\chi}_{\mathscr{O}}(\xi)\chi_{\mathscr{T}}(\xi,\eta)\frac{\partial_\eta\phi}{i|\partial_\eta\phi|^2}m(\xi,\eta)}\left(e^{\pm is\langle D\rangle_\ell}(xc),\mathscr{C}\right)\right\|_{W^{5,\left(\frac{5}{6}-\delta\right)^{-1}}} ds \\ &\lesssim \int_1^{t/2} \frac{1}{(t-s)^{1-3\delta}}\frac{1}{s}\|xc\|_{H^{n+5}}\|\mathscr{C}\|_{W^{n+5,\left(\frac{1}{3}-\delta\right)^{-1}}} ds \\ &\lesssim \int_1^{t/2} \frac{1}{(t-s)^{1-3\delta}}\frac{1}{s}\|\mathscr{C}\|_X^2\sqrt{s}\frac{1}{s^{\frac{1}{2}+3\delta}}\,ds \\ &\lesssim \|\mathscr{C}\|_X^2\frac{1}{t}. \end{aligned}$$

Taking this time $\delta > 0$ and small, and retracing the above steps, one gets

$$\left\| e^{it\langle D\rangle_\ell}\mathscr{F}^{-1} II\right\|_{W^{2,\infty}} \lesssim \|\mathscr{C}\|_X^2\frac{1}{t}.$$

The term (7.5b) can be estimated in a very similar way. Indeed, since m satisfies the estimates (4.2), $\partial_\eta m(\xi,\eta)$ yields at worst singularities of the type $\frac{1}{|\eta|}$, $\frac{1}{|\xi-\eta|}$. The above scheme can then be employed since by Hardy's inequality, and Plancherel's equality, $\left\|\frac{1}{|\xi|}\widehat{c}(\xi)\right\|_2 \lesssim \|\partial_\xi\widehat{c}(\xi)\|_2 = \|xc\|_2$.

8. Localization estimates

We want to prove here that

$$\sup_t \left[\frac{1}{\sqrt{t}}\||x|(a,b)\|_{H^{N'}} + \left\||x|^{1/8}\widetilde{Z}_{\mathscr{O}}(a,b)\right\|_2\right] \lesssim \left\|(e^{it\langle D\rangle_{c_s}}\mathscr{A}_0, e^{it\langle D\rangle}\mathscr{B}_0)\right\|_X + \|(\mathscr{A},\mathscr{B})\|_X^2.$$

As above, this reduces to proving that the generic term

$$g(t)=\mathcal{F}^{-1}\int_0^t e^{is\phi(\xi,\eta)}m(\xi,\eta)\widehat{c}(\eta,s)\widehat{c}(\xi-\eta,s)\,ds$$

defined in (4.1) satisfies the localization estimates

$$\sup_t\left[\frac{1}{\sqrt{t}}\,\| |x|g\|_{H^{N'}}+\left\||x|^{1/8}\widetilde{Z}_{\mathcal{O}}g\right\|_2\right]\lesssim\|(\mathcal{A},\mathcal{B})\|_X^2\,. \tag{8.1}$$

By symmetry, it suffices to control

$$g'(t)=\mathcal{F}^{-1}\int_0^t e^{is\phi(\xi,\eta)}m(\xi,\eta)\zeta^1(\xi,\eta)\widehat{c}(\eta,s)\widehat{c}(\xi-\eta,s)\,ds$$

(where the cut-off symbol ζ^1, defined in Section 5, ensures that $|\xi-\eta|\lesssim|\eta|$ for (ξ,η) large). The bound for the second norm in (8.1) was derived in [11] (in Section 5.7), and the same scheme of proof applies here (once again, the novelty compared to [11] is that the Euler-Maxwell system is quasilinear, but the resulting loss of regularity in the nonlinear term is easily absorbed by the H^N norm). Therefore, we focus on the first norm in (8.1), for which some new difficulties arise. It will be helpful to split m as $m=m_0m_1m_2$ (see Subsection 4.1). Multiplying g by the weight x corresponds in Fourier space to differentiating $\widehat{g}$ with respect to ξ. This gives

$$\partial_\xi\widehat{g'}(\xi)=\int_0^t\int e^{is\phi}m(\xi,\eta)\zeta^1(\xi,\eta)\widehat{c}(\eta)\partial_\xi\widehat{c}(\xi-\eta)\,d\eta\,ds \tag{8.2a}$$

$$+\int_0^t\int is\partial_\xi\phi e^{is\phi}m(\xi,\eta)\zeta^1(\xi,\eta)\widehat{c}(\eta)\widehat{c}(\xi-\eta)\,d\eta\,ds \tag{8.2b}$$

$$+\int_0^t\int e^{is\phi}m_0(\xi)m_1(\eta)\partial_\xi m_2(\xi-\eta)\zeta^1(\xi,\eta)\widehat{c}(\eta)\widehat{c}(\xi-\eta)\,d\eta\,ds \tag{8.2c}$$

$$+\int_0^t\int e^{is\phi}m_0(\xi)m_1(\eta)m_2(\xi-\eta)\partial_\xi\zeta^1(\xi,\eta)\widehat{c}(\eta)\widehat{c}(\xi-\eta)\,d\eta\,ds \tag{8.2d}$$

$$+\,\partial_\xi m_0(\xi)\int_0^t\int e^{is\phi}m_1(\eta)m_2(\xi-\eta)\zeta^1(\xi,\eta)\widehat{c}(\eta)\widehat{c}(\xi-\eta)\,d\eta\,ds. \tag{8.2e}$$

8.1. Estimate of (8.2a)

To estimate (8.2a), use the Strichartz estimate (12.4) and Proposition 12.1 (iii) to get

$$\begin{aligned}\left\|\mathcal{F}^{-1}(8.2\text{a})\right\|_{H^{N'}}&\lesssim\left\|\int_1^t e^{is\langle D\rangle}T_{m(\xi,\eta)\zeta^1(\xi,\eta)}(\mathcal{C},e^{is\langle D\rangle}xc)\,ds\right\|_{H^{N'}}\\&\lesssim\left\|T_{m(\xi,\eta)\zeta^1(\xi,\eta)}(\mathcal{C},e^{is\langle D\rangle}xc)\right\|_{L_t^{\left(\frac12+\frac32\delta_1\right)^{-1}}W_x^{N'+1,\left(\frac56-\delta_1\right)^{-1}}}\\&\lesssim\left\|\|\mathcal{C}\|_{W^{n+N'+1,\left(\frac13-\delta_1\right)^{-1}}}\|xc\|_2\right\|_{L_t^{\left(\frac12+\frac32\delta_1\right)^{-1}}}\\&\lesssim\|\mathcal{C}\|_X^2\left\|\sqrt{\langle s\rangle}\langle s\rangle^{-\frac12-3\delta_1}\right\|_{L_t^{\left(\frac12+\frac32\delta_1\right)^{-1}}}\\&\lesssim\|\mathcal{C}\|_X^2\sqrt{\langle t\rangle}.\end{aligned}$$

8.2. Estimate of (8.2b)

To estimate (8.2b), distinguish three types of interactions, by writing $c = Z_\emptyset c + \widetilde{Z}_\emptyset c$. The term (8.2b) can be written as

$$\mathcal{F}^{-1}(8.2\mathrm{b}) = \int_0^t e^{is\langle D\rangle} s T_{m(\xi,\eta)\zeta^1(\xi,\eta)\partial_\xi\phi}(Z_\emptyset \mathcal{C}, Z_\emptyset \mathcal{C})\, ds \tag{8.3a}$$

$$+ \int_0^t e^{is\langle D\rangle} s T_{m(\xi,\eta)\zeta^1(\xi,\eta)\partial_\xi\phi}(\widetilde{Z}_\emptyset \mathcal{C}, \widetilde{Z}_\emptyset \mathcal{C})\, ds \tag{8.3b}$$

$$+ \int_0^t e^{is\langle D\rangle} s T_{m(\xi,\eta)\zeta^1(\xi,\eta)\partial_\xi\phi}(\widetilde{Z}_\emptyset \mathcal{C}, Z_\emptyset \mathcal{C})\, ds \tag{8.3c}$$

$$+ \int_0^t e^{is\langle D\rangle} s T_{m(\xi,\eta)\zeta^1(\xi,\eta)\partial_\xi\phi}(Z_\emptyset \mathcal{C}, \widetilde{Z}_\emptyset \mathcal{C})\, ds. \tag{8.3d}$$

Examining all these terms, only the ones involving high frequencies are harder to treat than in [11], due to the quasilinearity of Euler-Maxwell. For this reason, the term (8.3a) can be treated exactly as in [11], and we skip it. Next we shall bound the term (8.3c). The term (8.3b) is comparatively easier, since the two interacting waves correspond to non-outcome frequencies, thus enjoying better bounds. As for the term (8.3d) it is also easier: indeed for this term, the symbol $\zeta^1(\xi,\eta)$ imposes that $\widetilde{Z}_\emptyset\mathcal{C}$ is lower frequency than $Z_\emptyset\mathcal{C}$; but this is possible only if both are low frequency.

Coming back to (8.3c), use Proposition 12.1 to get

$$\begin{aligned}\|8.3c\|_{H^{N'}} &\lesssim \int_0^t s \left\| e^{is\langle D\rangle} T_{m(\xi,\eta)\zeta^1(\xi,\eta)\partial_\xi\phi}(\widetilde{Z}_\emptyset \mathcal{C}, Z_\emptyset \mathcal{C})\right\|_{H^{N'}} ds \\ &\lesssim \int_0^t s \, \|Z_\emptyset \mathcal{C}\|_{L^{\left(\frac{1}{3}-\delta_1\right)^{-1}}} \left\|\widetilde{Z}_\emptyset \mathcal{C}\right\|_{W^{N'+n,\left(\frac{1}{6}+\delta_1\right)^{-1}}} ds \\ &\lesssim \|\mathcal{C}\|_X^2 \int_0^t s \frac{1}{s^{\frac{1}{2}+3\delta_1}} \frac{1}{s^{1-3\delta_1}}\, ds \lesssim \|\mathcal{C}\|_X^2 \sqrt{t}.\end{aligned}$$

8.3. Estimate of (8.2c)

By (4.2), $\partial_\xi m_2(\xi-\eta)$ can be bounded by $\frac{C}{|\xi-\eta|}$. Bounding by Hardy's inequality $\frac{1}{|\xi-\eta|}\widehat{c}(\xi-\eta)$ in L^2 by $\partial_\xi \widehat{c}(\xi-\eta)$ in L^2, the estimate for (8.2a) can be easily adapted.

8.4. Estimate of (8.2d)

Since $\partial_\xi\zeta^1(\xi,\eta)$ does not have a singularity, this term is easy and we skip it.

8.5. Estimate of (8.2e)

By (4.2), $\partial_\xi m_0(\xi)$ can be bounded by 1 for high frequencies, and $\frac{1}{|\xi|}$ for small frequencies. Forgetting about high frequencies, which are easily dealt with, we need to bound

$$\mathcal{F}^{-1}\frac{1}{|\xi|}\int_0^t \int e^{is\phi} m_1(\eta) m_2(\xi-\eta)\zeta^1(\xi,\eta)\widehat{c}(\eta)\widehat{c}(\xi-\eta)\, d\eta\, ds$$

in $H^{N'}$. By Hardy's inequality, it suffices to bound

$$\mathcal{F}^{-1}\partial_\xi \int_0^t \int e^{is\phi} m_1(\eta) m_2(\xi-\eta)\zeta^1(\xi,\eta)\widehat{c}(\eta)\widehat{c}(\xi-\eta)\, d\eta\, ds$$

in $H^{N'}$. But expanding the ξ derivative yields terms similar to (8.2a) (8.2b) (8.2c) (8.2d), which we have just seen how to estimate.

9. Sobolev regularity for the electromagnetic part

We shall prove in this section that

$$\|\mathcal{B}\|_{H^N} \lesssim \|(\mathcal{A}_0, \mathcal{B}_0)\|_{H^N} + \|(\mathcal{A}, \mathcal{B})\|_X^2 + \|(\mathcal{A}, \mathcal{B})\|_X \int_0^t \frac{1}{\langle s\rangle} \|(\mathcal{A}, \mathcal{B})\|_{H^N}\, ds. \tag{9.1}$$

Together with (10.1), this will imply that

$$\|(\mathcal{A}, \mathcal{B})\|_{H^N} \lesssim \varepsilon \langle t\rangle^{C_0\varepsilon}.$$

The following observation will be crucial: it follows from their definition that $\mathcal{A}$ and $\mathcal{B}$ control the physical unknowns u and n as follows:

$$\|Qu\|_{H^N} \lesssim \|\mathcal{A}\|_{H^N}, \quad \|Pu\|_{H^{N+1}} \lesssim \|\mathcal{B}\|_{H^N}, \quad \text{and} \quad \|n\|_{H^N} \lesssim \|\mathcal{A}\|_{H^N}. \tag{9.2}$$

9.1. Preliminary estimate: $\partial_s a$

It follows from (EM') that

$$\partial_t a = e^{-it\langle D\rangle_{c_s}} \left[-\frac{1}{2} \frac{\langle D\rangle \nabla}{|D|} \cdot (nu) + \frac{i}{4}|D| \left(|u|^2 + |n|^2 \right) \right].$$

Therefore by the product estimates (12.2),

$$\begin{aligned} \|\partial_t a\|_{H^{N-1}} &\lesssim \|nu\|_{H^N} + \left\|u^2\right\|_{H^N} + \left\|n^2\right\|_{H^N} \\ &\lesssim (\|n\|_{H^N} + \|u\|_{H^N}) (\|n\|_\infty + \|u\|_\infty) \\ &\lesssim \frac{1}{\langle t\rangle^{\frac{1}{2}+3\delta_1 - C_0\varepsilon}} \|(\mathcal{A}, \mathcal{B})\|_X^2. \end{aligned} \tag{9.3}$$

9.2. Distinction between outcome and non-outcome frequencies

Consider the integral equation satisfied by b:

$$b(t) = \mathcal{B}_0 - \int_0^t e^{-is\langle D\rangle} \frac{\nabla}{\langle D\rangle} \times (nu)\, ds.$$

Split n and u into $Z_\emptyset n + \widetilde{Z}_\emptyset n$, respectively $Z_\emptyset u + \widetilde{Z}_\emptyset u$. This gives

$$b(t) = \mathcal{B}_0 + \int_0^t e^{-is\langle D\rangle} \frac{\nabla}{\langle D\rangle} \times (\widetilde{Z}_\emptyset n\, \widetilde{Z}_\emptyset u)\, ds \tag{9.4a}$$

$$+ \int_0^t e^{-is\langle D\rangle} \frac{\nabla}{\langle D\rangle} \times (Z_\emptyset n\, Z_\emptyset u)\, ds \tag{9.4b}$$

$$+ \int_0^t e^{-is\langle D\rangle} \frac{\nabla}{\langle D\rangle} \times (Z_\emptyset n\, \widetilde{Z}_\emptyset u + \widetilde{Z}_\emptyset n Z_\emptyset u). \tag{9.4c}$$

The term (9.4a) is easily estimated: using the product law (12.2), and the Sobolev embedding theorem:

$$\begin{aligned}\|(9.4\mathrm{a})\|_{H^N} &= \left\|\int_0^t e^{-is\langle D\rangle}\frac{\nabla}{\langle D\rangle}\times(\widetilde{Z}_{\emptyset}n\,\widetilde{Z}_{\emptyset}u)\,ds\right\|_{H^N}\\ &\lesssim \int_0^t \left[\left\|\widetilde{Z}_{\emptyset}n\right\|_{H^N}\left\|\widetilde{Z}_{\emptyset}u\right\|_{L^\infty}+\left\|\widetilde{Z}_{\emptyset}n\right\|_{L^\infty}\left\|\widetilde{Z}_{\emptyset}u\right\|_{H^N}\right]ds\\ &\lesssim \|(\mathscr{A},\mathscr{B})\|_X\int_0^t\frac{1}{\langle s\rangle}\|(u,n)\|_{H^N}\,ds\\ &\lesssim \|(\mathscr{A},\mathscr{B})\|_X\int_0^t\frac{1}{\langle s\rangle}\|(\mathscr{A},\mathscr{B})\|_{H^N}\,ds.\end{aligned}$$

For the term (9.4b), we take advantage of the frequency localization of $Z_{\emptyset}n$ and $Z_{\emptyset}u$ to write, with the help of Bernstein's inequality,

$$\begin{aligned}\|(9.4\mathrm{b})\|_{H^N} &= \left\|\int_0^t e^{-is\langle D\rangle}\frac{\nabla}{\langle D\rangle}\times(Z_{\emptyset}n\,Z_{\emptyset}u)\,ds\right\|_{H^N}\\ &\lesssim \int_0^t \|Z_{\emptyset}n\,Z_{\emptyset}u\|_{\left(\frac{2}{3}-2\delta\right)^{-1}}\,ds\\ &\lesssim \int_0^t \|Z_{\emptyset}n\|_{\left(\frac{1}{3}-\delta\right)^{-1}}\|Z_{\emptyset}u\|_{\left(\frac{1}{3}-\delta\right)^{-1}}\,ds\\ &\lesssim \|(\mathscr{A},\mathscr{B})\|_X^2\int_0^t\frac{ds}{\langle s\rangle^{1+6\delta_1}}\,ds\lesssim\|(\mathscr{A},\mathscr{B})\|_X^2\end{aligned}$$

(here we used $\|Z_{\emptyset}(u,n)\|_p\lesssim\|(\mathscr{A},\mathscr{B})\|_p$ for $1<p<\infty$, which follows from the formulas giving u and n appearing in Section 3).

9.3. Interactions between outcome and non-outcome frequencies

Thus we now take a closer look at (9.4c), wich reads

$$(9.4\mathrm{c})=\int_0^t e^{-is\langle D\rangle}\frac{\nabla}{\langle D\rangle}\times(Z_{\emptyset}n\,\widetilde{Z}_{\emptyset}Pu)\,ds \tag{9.5a}$$

$$+\int_0^t e^{-is\langle D\rangle}\frac{\nabla}{\langle D\rangle}\times(Z_{\emptyset}n\,\widetilde{Z}_{\emptyset}Qu)\,ds \tag{9.5b}$$

$$+\int_0^t e^{-is\langle D\rangle}\frac{\nabla}{\langle D\rangle}\times(\widetilde{Z}_{\emptyset}n\,Z_{\emptyset}u)\,ds. \tag{9.5c}$$

The first term, (9.5a), can be estimated with the help of the Strichartz estimate (12.4) and the standard product law 12.2:

$$\begin{aligned}\|(9.5\mathrm{a})\|_{H^N} &= \left\|\int_0^t e^{-is\langle D\rangle}\frac{\nabla}{\langle D\rangle}\times(Z_{\emptyset}n\,\widetilde{Z}_{\emptyset}Pu)\,ds\right\|_{H^N}\\ &\lesssim \left\|Z_{\emptyset}n\,\widetilde{Z}_{\emptyset}Pu\right\|_{L_t^{\left(\frac{1}{2}+\frac{3}{2}\delta_1\right)^{-1}}W_x^{\left(1-\frac{5}{2}\delta_1+N\right),\left(\frac{5}{6}-\delta_1\right)^{-1}}}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left\| \|Z_{\emptyset} n\|_{L_x^{\left(\frac{1}{3}-\delta_1\right)^{-1}}} \left\|\widetilde{Z}_{\emptyset} Pu\right\|_{H_x^{N+1}} + \|Z_{\emptyset} n\|_{H_x^{N+1}} \left\|\widetilde{Z}_{\emptyset} Pu\right\|_{L_x^{\left(\frac{1}{3}-\delta_1\right)^{-1}}} \right\|_{L_t^{\left(\frac{1}{2}+\frac{3}{2}\delta_1\right)^{-1}}} \\
&\lesssim \|(\mathscr{A},\mathscr{B})\|_X^2 \left\| \langle t\rangle^{-\frac{1}{2}-3\delta_1} \right\|_{L_t^{\left(\frac{1}{2}+\frac{3}{2}\delta_1\right)^{-1}}} \\
&\lesssim \|(\mathscr{A},\mathscr{B})\|_X^2 .
\end{aligned}
$$

The estimates for the terms (9.5b) and (9.5c) are exactly the same, changing the roles of n and Qu. We will only treat (9.5b). The term (9.5b) can be decomposed as

$$
\text{(9.6a)} \qquad (9.5\text{b}) = \int_0^t e^{-is\langle D\rangle} \frac{\nabla}{\langle D\rangle} \times \left(Z_{\emptyset} n\, \widetilde{Z}_{\emptyset} Z_l Q u\right) ds
$$

$$
\text{(9.6b)} \qquad + \int_0^t e^{-is\langle D\rangle} \frac{\nabla}{\langle D\rangle} \times \left(Z_{\emptyset} n\, Z_h Q u\right) ds
$$

(recall that Z_l and Z_h have been defined in Section 5, and that M_0 has been chosen so big that $Z_h \widetilde{Z}_{\emptyset} = Z_h$). The term (9.6a) can be estimated exactly as (9.4b); thus we skip it. We are left with (9.6b). Recall now that n and Qu can be written as

$$
n(t) = 2\frac{|D|}{\langle D\rangle_{c_s}} \mathfrak{Re}\, e^{it\langle D\rangle_{c_s}} a(t) \quad \text{ and } \quad Qu(t) = -2\frac{\nabla}{|D|} \mathfrak{Im}\, e^{it\langle D\rangle_{c_s}} a(t).
$$

This implies that the Fourier transform of (9.6b) can be written as a sum of terms of the type

$$
\text{(9.7)} \quad \mathscr{F}(9.6\text{b}) = \int_0^t \int e^{is\phi(\xi,\eta)} \chi_{\emptyset}(\eta) \left(1 - \chi\left(\frac{\xi-\eta}{M_0}\right)\right) \widetilde{m}(\xi,\eta) \widehat{a}(\eta,s) \widehat{a}(\xi-\eta,s)\, d\eta\, ds
$$

where, for simplicity, $\widehat{a}$ stands for $\widehat{a}$ and $\overline{\widehat{a}}$, ϕ has the form

$$
\phi(\xi,\eta) = -\langle \xi\rangle + \varepsilon_1 \langle \eta\rangle_{c_s} + \varepsilon_2 \langle \xi - \eta\rangle_{c_s} \qquad \text{with } \varepsilon_1, \varepsilon_2 \in \{\pm 1\},
$$

and $\widetilde{m}(\xi,\eta)$ is a (matrix-valued) symbol of the form $m_1(\xi) m_2(\eta) m_3(\xi-\eta)$ with

$$
\text{for any } i = 1,2,3, \quad \left|\partial_\xi^\alpha m_i(\xi)\right| \lesssim \frac{1}{|\xi|^{|\alpha|}}.
$$

A crucial point will be that, on the support of $\chi_{\emptyset}(\eta)\left(1 - \theta\left(\frac{\xi-\eta}{M_0}\right)\right)$, since M_0 is chosen big enough, $|\xi| \gg |\eta| \sim 1$, and ϕ satisfies the inequalities

$$
\left| \partial_\xi^\alpha \partial_\eta^\beta \frac{1}{\phi(\eta,\xi)} \right| \lesssim \frac{1}{|\xi|^{|\alpha|+|\beta|+1}}.
$$

Integrating by parts in (9.7) using the identity $\frac{1}{i\phi}\partial_s e^{is\phi} = e^{is\phi}$, and denoting

$$
\mu(\xi,\eta) \stackrel{\text{def}}{=} \frac{\chi_{\emptyset}(\eta)\left(1-\theta\left(\frac{\xi-\eta}{M_0}\right)\right) \widetilde{m}(\eta,\xi)}{\phi(\xi,\eta)}
$$

gives

$$\mathscr{F}(9.6\text{b}) = -\int_0^t \int e^{is\phi(\xi,\eta)}\mu(\xi,\eta)\partial_s\widehat{a}(\eta,s)\widehat{a}(\xi-\eta,s)\,d\eta\,ds \tag{9.8a}$$

$$-\int_0^t \int e^{is\phi(\xi,\eta)}\mu(\xi,\eta)\widehat{a}(\eta,s)\partial_s\widehat{a}(\xi-\eta,s)\,d\eta\,ds \tag{9.8b}$$

$$+\int e^{it\phi(\xi,\eta)}\mu(\xi,\eta)\widehat{a}(\eta,t)\widehat{a}(\xi-\eta,t)\,d\eta\,ds \tag{9.8c}$$

$$-\int \mu(\xi,\eta)\widehat{a}(\eta,0)\widehat{a}(\xi-\eta,0)\,d\eta\,ds. \tag{9.8d}$$

We only estimate (9.8b): indeed, (9.8a) is symmetrical, and (9.8c) as well as (9.8d) are easier since they do not involve a time integral. Proposition 12.1 (v) gives that

$$\begin{aligned}
\left\|\mathscr{F}^{-1}(9.8\text{b})\right\|_{H^N} &= \left\|\int_0^t e^{is\langle D\rangle}T_\mu(\mathscr{A}, e^{it\langle D\rangle}\partial_s a(s))\,ds\right\|_{H^N}\\
&\lesssim \int_0^t \left\|T_\mu\left(\mathscr{A},\, e^{it\langle D\rangle}\partial_s a(s)\right)\right\|_{H^N}\,ds\\
&\lesssim \int_0^t \|\mathscr{A}\|_\infty \,\|\partial_s a(s)\|_{H^{N-1}}\,ds\\
&\lesssim \|(\mathscr{A},\mathscr{B})\|_X^3 \int_0^t \frac{1}{\langle s\rangle}\frac{1}{\langle s\rangle^{1/2+3\delta_1-C_0\varepsilon}}\,ds \lesssim \|(\mathscr{A},\mathscr{B})\|_X^3\,.
\end{aligned}$$

10. Sobolev regularity for the fluid part

We shall prove in this section that

$$\|\mathscr{A}\|_{H^N} \lesssim \|(\mathscr{A}_0,\mathscr{B}_0)\|_{H^N} + \|(\mathscr{A},\mathscr{B})\|_X^2 + \int_0^t \frac{1}{\langle s\rangle}\|(\mathscr{A},\mathscr{B})\|_{H^N}\,ds. \tag{10.1}$$

Together with (9.1), this will imply that

$$\|(\mathscr{A},\mathscr{B})\|_{H^N} \lesssim \varepsilon t^{C_0\varepsilon}.$$

10.1. The Equation (3.1)

First we rewrite the evolution Equation (3.1) satisfied by $\mathscr{A}$. We will use the notation $\tilde{D} = \frac{\langle D\rangle_{c_s}\nabla}{|D|}$. We start by expanding the first nonlinear terms appearing in (3.1).

We start by taking N derivatives of (3.1). We get

$$\begin{aligned}
\partial_t\partial^N\mathscr{A} &= i\langle D\rangle_{c_s}\partial^N\mathscr{A} - \frac{1}{2}\tilde{D}\cdot(\partial^N nu + n\partial^N u)\\
&\quad + \frac{i|D|}{2}\left(u\partial^N u + c_s^2 n\partial^N n\right) + R_1^N\\
&= i\langle D\rangle_{c_s}\partial^N\mathscr{A} - \frac{1}{2}u\cdot\tilde{D}\partial^N n - \frac{1}{2}n\tilde{D}\cdot\partial^N u\\
&\quad + iu\cdot\frac{|D|}{2}\partial^N u + ic_s^2 n\frac{|D|}{2}\partial^N n + R_2^N\\
&= i\langle D\rangle_{c_s}\partial^N\mathscr{A} - u\cdot\nabla\partial^N\mathscr{A} + in\langle D\rangle_{c_s}\partial^N\mathscr{A} + R_3^N
\end{aligned} \tag{10.2}$$

where the rest terms R_i^N consist of quadratic lower order terms. In particular, we have

$$\begin{aligned}
R_1^N &= \tilde{D} \cdot \Big(\partial^N(nu) - \partial^N nu + n\partial^N u\Big) \\
&\quad + \frac{i|D|}{4}\left(\partial^N(|u|^2) - 2u\partial^N u + c_s^2\partial^N(|n|^2) - 2c_s^2 n\partial^N n\right) \\
2R_2^N &= 2R_1^N - [\tilde{D}, u]\partial^N n - [\tilde{D}\cdot, n]\partial^N u \\
&\quad + i[|D|, u]\partial^N u + ic_s^2[|D|, n]\partial^N n \\
R_3^N &= R_2^N - in\frac{1}{2|D|}\partial^N n + iu\frac{|D|}{2}\partial^N Pu := R^N.
\end{aligned}$$

Notice that R^N is quadratic in (n, u) and that there are no terms where $N+1$ derivatives hit n or u, except for Pu which has already a gain of 1 derivative on $\mathcal{B}$ (see (3.2)). Summarizing, R is of the form

$$R^N = \sum_{k,l \leq N, k+l \leq N+1} Q_{k,l}(\partial^k(\mathcal{A}, \mathcal{B}), \partial^l(\mathcal{A}, \mathcal{B})) \tag{10.3}$$

where the $Q_{k,l}$ are pseudo-product operators

$$Q_{k,l}(u, v) = \int m_{k,l}(\xi, \eta)\hat{u}(\eta)\hat{v}(\xi - \eta)\, d\eta \tag{10.4}$$

of order 0. Finally, the following estimate holds

$$\|R^N\|_{L^2} \leq \|(n, u)\|_{W^{2,\infty}} \|(\mathcal{A}, \mathcal{B})\|_{H^N}. \tag{10.5}$$

Hence R^N does not lose derivatives, and an energy estimate with a Gronwall inequality will give the desired control, as long as (u, n) decays sufficiently fast. This is the case for non-outcome frequencies; for outcome frequencies, another argument is needed, as we will see in the following.

10.2. Non resonant phase

Due to the slow decay of the $Z_\theta u$ and $Z_\theta n$, we have to use non resonant properties of the second and third terms on the right-hand side of (3.1).

LEMMA 10.1. – *There exist a positive number $\kappa_0 > 0$ and a constant $C_0 > 0$ such that for $|\xi| \geq C_0$ and $|\eta| \leq C_{\mathcal{R}}$, we have*

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \frac{1}{\phi_{c_s,k,\ell}^{\varepsilon_1,\varepsilon_2}(\xi, \eta)}\right| \lesssim \frac{1}{|\xi|^{|\alpha|}} \tag{10.6}$$

for $\varepsilon_1, \varepsilon_2 = \pm$ and $k, \ell = 1, c_s$.

Proof. – We will only consider the phase $\phi_{c_s,c_s,c_s}^{-,+} = -\langle\xi\rangle_{c_s} - \langle\eta\rangle_{c_s} + \langle\xi - \eta\rangle_{c_s}$ since the other phases are easier. Furthermore, we only prove the estimate on $\frac{1}{\phi}$, not its derivatives. We have

$$\begin{aligned}
\langle\xi - \eta\rangle_{c_s} &= c|\xi|\sqrt{1 + \frac{1}{c_s^2|\xi|^2} - 2\frac{\xi.\eta}{|\xi|^2} + \frac{|\eta|^2}{|\xi|^2}} \\
&= c_s|\xi|\left[1 + \frac{1}{2c_s^2|\xi|^2} - 2\frac{\xi.\eta}{2|\xi|^2} + \frac{|\eta|^2}{2|\xi|^2} + O(\frac{1}{|\xi|^2})\right].
\end{aligned}$$

Hence, we see that $-\langle \xi-\eta\rangle_{c_s}+c_s|\xi|+c_s|\eta|\geq -\frac{C}{|\xi|}$, from which we deduce that

$$\phi^{-,+}_{c_s,c_s,c_s}\leq -\langle\xi\rangle_{c_s}-\langle\eta\rangle_{c_s}+c_s|\xi|+c_s|\eta|+\frac{C}{|\xi|}.$$

Hence, if C_0 is big enough then, $\phi^{-,+}_{c_s,c_s,c_s}\leq -\frac{\sqrt{1+(c_sC_{\mathscr{R}})^2}-c_sC_{\mathscr{R}}}{2}<0.$ □

10.3. Sobolev regularity

The Sobolev estimates for the electromagnetic part were performed using simply Strichartz estimates and integration by parts in time depending on the cases. Due to the further loss of a derivative, this method does not apply here. Instead we will perform an iterated energy estimate that we find interesting in its own right.

Using that u and n are both real, we deduce that

$$\partial_t\frac{\|\partial^N\mathscr{A}\|^2_{L^2}}{2}=\mathfrak{Re}\int\nabla.u|\partial^N\mathscr{A}|^2+i[\langle D\rangle_{c_s},n]\partial^N\overline{\mathscr{A}}\partial^N\mathscr{A}+R^N\partial^N\overline{\mathscr{A}}. \tag{10.7}$$

Hence

$$\frac{\|\partial^N\mathscr{A}(t)\|^2_{L^2}}{2}-\frac{\|\partial^N\mathscr{A}_0\|^2_{L^2}}{2}=\int_0^t\mathfrak{Re}\int\left(\nabla.u|\partial^N\mathscr{A}|^2+i[\langle D\rangle_{c_s},n]\partial^N\overline{\mathscr{A}}\partial^N\mathscr{A}+R^N\partial^N\overline{\mathscr{A}}\right)ds. \tag{10.8}$$

We would like now to explain how to control the three terms on the right-hand side of (10.8). For the first term, we split u into the outcome and non-outcome parts $u=Z_{\mathscr{O}}u+\widetilde{Z}_{\mathscr{O}}u$. The non-outcome part has enough decay to apply directly the Gronwall argument. Hence, we will only concentrate on the outcome part. We recall that the profile $a(t)$ associated to $\mathscr{A}$ is defined by $\mathscr{A}(t)=e^{i\langle D\rangle_{c_s}t}a(t)$. Also, we have

$$Z_{\mathscr{O}}u=Z_{\mathscr{O}}\frac{\nabla}{|D|\langle D\rangle}\times\left(\frac{e^{it\langle D\rangle}b-e^{-it\langle D\rangle}\overline{b}}{2i}\right)+iZ_{\mathscr{O}}\frac{\nabla}{|D|}\left(e^{i\langle D\rangle_{c_s}t}a-e^{i\langle D\rangle_{c_s}t}\overline{a}\right). \tag{10.9}$$

We denote by $e^{\pm it\langle D\rangle_l}d(t)=\mathscr{D}(t)$ the divergence of any one of the four terms appearing in (10.9). To control the first term in the right-hand side of (10.8), it is enough to rewrite it in Fourier space. Hence, it is enough to consider

$$\int_0^t\iint e^{is\phi(\xi,\eta)}\widetilde{m}(\xi,\eta)\widehat{d}(s,\eta)\widehat{\partial^N a}(s,\xi-\eta)\overline{\widehat{\partial^N a}}(s,\xi)d\eta\,d\xi\,ds \tag{10.10}$$

where the phase ϕ is given by $\phi(\xi,\eta)=\langle\xi-\eta\rangle_{c_s}-\langle\xi\rangle_{c_s}\pm\langle\eta\rangle_l$and $\widetilde{m}(\xi,\eta)\overset{\text{def}}{=}\chi_{\mathscr{O}}(\eta)$. Split $\widetilde{m}(\xi,\eta)=\theta\left(\frac{\xi}{M_0}\right)\widetilde{m}(\xi,\eta)+\left[1-\theta\left(\frac{\xi}{M_0}\right)\right]\widetilde{m}(\xi,\eta)$. The first term corresponds to low frequencies of $\partial^N\mathscr{A}$, which are easily estimated; thus, we shall consider in the following that

$$\widetilde{m}(\xi,\eta)=\left[1-\theta\left(\frac{\xi}{M_0}\right)\right]\chi_{\mathscr{O}}(\eta).$$

From Lemma 10.1, we know that ϕ is always bounded away from zero in the support of $\widetilde{m}$. Hence, we can integrate by parts in time (using the identity $\frac{1}{i\phi}\partial_se^{is\phi}=e^{is\phi}$) in (10.10) and get

$$i(10.9) = -\int_0^t \iint \widetilde{m}(\xi,\eta)\frac{e^{is\phi}}{\phi}\widehat{d}(s,\eta)\partial_s\Big(\widehat{\partial^N a}(s,\xi-\eta)\overline{\widehat{\partial^N a}}(s,\xi)\Big)d\eta\, d\xi\, ds \tag{10.11a}$$

$$-\int_0^t \iint \widetilde{m}(\xi,\eta)\frac{e^{is\phi}}{\phi}\Big(\partial_s\widehat{d}(s,\eta)\widehat{\partial^N a}(s,\xi-\eta)\overline{\widehat{\partial^N a}}(s,\xi)\Big)d\eta\, d\xi\, ds \tag{10.11b}$$

$$+\iint \frac{e^{it\phi}}{\phi}\widetilde{m}(\xi,\eta)\widehat{d}(t,\eta)\widehat{\partial^N a}(t,\xi-\eta)\overline{\widehat{\partial^N a}}(t,\xi)d\eta\, d\xi \tag{10.11c}$$

$$-\iint \frac{1}{\phi}\widetilde{m}(\xi,\eta)\widehat{d}(0,\eta)\widehat{\partial^N a}(0,\xi-\eta)\overline{\widehat{\partial^N a}}(0,\xi)d\eta\, d\xi. \tag{10.11d}$$

We rewrite the time derivative in (10.11a) as

$$\Big(\partial_s\widehat{\partial^N a}(\xi-\eta)\overline{\widehat{\partial^N a}}(\xi) + \widehat{\partial^N a}(\xi-\eta)\partial_s\overline{\widehat{\partial^N a}}(\xi)\Big). \tag{10.12}$$

From (10.2), we deduce that

$$e^{it\langle D\rangle_{c_s}}\partial_t\partial^N a = -u\cdot\nabla\partial^N \mathcal{A} + in\langle D\rangle_{c_s}\partial^N\mathcal{A} + R^N. \tag{10.13}$$

Hence, (10.11a) can be expanded as

(10.14a)

$$\begin{aligned}(10.11a) = -\int_0^t \iint \widetilde{m}(\xi,\eta)\frac{1}{\phi}\widehat{\mathcal{D}}(\eta)\Big[&[\widehat{u\cdot\nabla\partial^N\mathcal{A}}](\xi-\eta)\overline{\widehat{\partial^N\mathcal{A}}}(\xi)\\ &+[\widehat{\partial^N\mathcal{A}}](\xi-\eta)\overline{[\widehat{u\cdot\nabla\partial^N\mathcal{A}}]}(\xi)\Big]d\eta\, d\xi\, ds\end{aligned}$$

$$\begin{aligned}-\int_0^t \iint \widetilde{m}(\xi,\eta)\frac{i}{\phi}\widehat{\mathcal{D}}(\eta)\Big[&[\widehat{n\langle D\rangle_{c_s}\partial^N\mathcal{A}}](\xi-\eta)\overline{\widehat{\partial^N\mathcal{A}}}(\xi)\\ &-\widehat{\partial^N\mathcal{A}}(\xi-\eta)\overline{[\widehat{n\langle D\rangle_{c_s}\partial^N\mathcal{A}}]}(\xi)\Big]d\eta\, d\xi\, ds\end{aligned} \tag{10.14b}$$

$$-\int_0^t \iint \widetilde{m}(\xi,\eta)\frac{1}{\phi}\widehat{\mathcal{D}}(\eta)\Big[\widehat{R^N}(\xi-\eta)\overline{\widehat{\partial^N\mathcal{A}}}(\xi) - \widehat{\partial^N\mathcal{A}}(\xi-\eta)\overline{\widehat{R^N}}(\xi)\Big]d\eta\, d\xi\, ds. \tag{10.14c}$$

The difficulty in bounding (10.14a) is that $\mathcal{A}$ appears with $N+1$ derivatives. The main idea here is to use once again a cancellation coming from the energy structure to perform an integration by part so that the extra derivative can be moved on a term with fewer derivatives. Keeping in mind that u is real-valued, we can rewrite (10.14a) as

$$\begin{aligned}(10.14a) &= -\int_0^t \iiint \widetilde{m}(\xi,\eta)\frac{i}{\phi}\widehat{\mathcal{D}}(\eta)\Big[\widehat{u}(\nu)\cdot(\xi-\eta-\nu)\widehat{\partial^N\mathcal{A}}(\xi-\eta-\nu)\widehat{\partial^N\overline{\mathcal{A}}}(-\xi)\\ &\qquad + \widehat{\partial^N\mathcal{A}}(\xi-\eta)\widehat{u}(\nu)\cdot(-\xi-\nu)\widehat{\partial^N\overline{\mathcal{A}}}(-\xi-\nu)\Big]d\eta\, d\xi\, d\nu\, ds\\ &= -i\int_0^t \iiint \widehat{\mathcal{D}}(\eta)\widehat{u}(\nu)\cdot\Big[\mu(\xi,\eta)(\xi-\eta-\nu)\widehat{\partial^N\mathcal{A}}(\xi-\eta-\nu)\widehat{\partial^N\overline{\mathcal{A}}}(-\xi)\\ &\qquad + \mu(\xi-\nu,\eta)\widehat{\partial^N\mathcal{A}}(\xi-\eta-\nu)(-\xi)\widehat{\partial^N\overline{\mathcal{A}}}(-\xi)\Big]d\eta\, d\xi\, d\nu\, ds\end{aligned}$$

where $\mu(\xi,\eta) = \frac{\widetilde{m}(\xi,\eta)}{\phi(\xi,\eta)}$ and we made the change of variable $\xi \to \xi - \nu$ in the last line. The integrand of the term appearing in the last two lines can be rewritten as

$$\widehat{\mathcal{D}}(\eta)\widehat{u}(\nu)\widehat{\partial^N \mathcal{A}}(\xi-\eta-\nu)\widehat{\partial^N \overline{\mathcal{A}}}(-\xi)\cdot[\mu(\xi,\eta)(\xi-\eta-\nu)+\mu(\xi-\nu,\eta)\xi]$$

and the term between brackets is equal to

$$M(\xi,\eta,\nu) = -\mu(\xi,\eta)(\eta+\nu) + [\mu(\xi,\eta)-\mu(\xi-\nu,\eta)]\,\xi. \tag{10.15}$$

Proposition 12.3 gives the desired conclusion, namely that

(10.16)

$$\begin{aligned}\|(10.10a)\|_2 &\lesssim \int_0^t \left| \iiint M(\xi,\eta,\nu)\widehat{\mathcal{D}}(\eta)\widehat{u}(\nu)\widehat{\partial^N \mathcal{A}}(\xi-\eta-\nu)\widehat{\partial^N \overline{\mathcal{A}}}(-\xi)d\eta\, d\xi\, d\nu\right| ds\\ &\lesssim \int_0^t \|u\|_{W^{2,\infty}} \|\mathcal{C}\|_\infty \left\|\partial^N \mathcal{A}\right\|_2^2 ds\\ &\lesssim \|(\mathcal{A},\mathcal{B})\|_X^4 \int_0^t \frac{1}{\langle s\rangle^{1+6\delta_1}}\, ds \lesssim \|(\mathcal{A},\mathcal{B})\|_X^4 .\end{aligned}$$

The treatement of (10.14b) is very similar and we do not detail it here. Finally, (10.14a) is much simpler since it involves at most N derivatives hitting $\mathcal{A}$.

To control the second term on the right-hand side of (10.8), we rewrite it in Fourier space. Hence, it is enough to consider

$$\int_0^t \iint e^{is\phi}\widehat{d}(\eta)(\langle \xi-\eta\rangle_{c_s} - \langle\xi\rangle_{c_s})\widehat{\partial^N a}(\xi-\eta)\overline{\widehat{\partial^N a}}(\xi)d\eta d\xi ds \tag{10.17}$$

where the phase ϕ is given by $\phi(\xi,\eta) = \langle\xi-\eta\rangle_{c_s} - \langle\xi\rangle_{c_s} \pm \langle\eta\rangle_{c_s}$ and $e^{\pm i\langle D\rangle_{c_s} t}d$ is one of the two terms appearing in the decomposition of n as $n = e^{i\langle D\rangle_{c_s} t}N + e^{-i\langle D\rangle_{c_s} t}\overline{N}$. The estimate of (10.17) is exactly the same as the estimate of (10.10) and we do not detail it again.

Now, it remains to control the last term on the right-hand side of (10.8), namely the term involving the rest term. As expressed in ((10.3)), it is a sum of trilinear expressions in $(\mathcal{A},\mathcal{B})$, each of the functions $(\mathcal{A},\mathcal{B})$ carrying a certain number of derivatives. Again, if the frequency of the function with the fewer number of derivatives is non-outcome then we can estimate the term directly using (10.5) and the almost integrable decay of the non-outcome part.

Hence, the only difficult terms are those for which the term that has the lower number of derivatives in (10.3) is outcome. The most difficult terms are very similar to those we treated above by integration by parts in time, namely (10.10) and (10.17). This is the case for instance when $k=1$ and $l=N$ in (10.3) and the phase is similar to the one in (10.10). In addition we have better terms of the type

$$\int_0^t \iint e^{is\phi}\widetilde{m}(\xi,\eta)\widehat{d}(\eta)\overline{\widehat{\partial^N a}}(\xi-\eta)\widehat{\partial^N \overline{a}}(\xi)d\eta d\xi ds \tag{10.18}$$

where the phase ϕ is given by $\phi(\xi,\eta) = -\langle\xi-\eta\rangle_{c_s} - \langle\xi\rangle_{c_s} \pm \langle\eta\rangle_{c_s}$ and terms obtained by taking the complex conjugate. These two types of terms are better than (10.10) due to the presence of the same sign in front of $\langle\xi-\eta\rangle_{c_s}$ and $\langle\xi\rangle_{c_s}$ and hence integration by parts in time gains a factor $\frac{1}{|\phi|}$ that behaves like $\frac{1}{|\xi|}$ in the dangerous region $|\eta| \ll |\xi|$.

We also have terms of the form

$$\int_0^t \iint \widetilde{m}(\xi,\eta) e^{is\phi} \widehat{d}(\eta) \widehat{\partial^{N-k} a}(\xi-\eta) \overline{\widehat{\partial^N a}}(\xi) d\eta d\xi ds \tag{10.19}$$

where $1 \leq k \leq N/2$ and the phase ϕ is given by $\phi(\xi,\eta) = \pm\langle \xi-\eta\rangle_{c_s} \pm \langle\xi\rangle_{c_s} \pm \langle\eta\rangle_l$ and $e^{\pm it\langle D\rangle_l} d(t) = \mathcal{D}(t)$ denote outcome (low frequency) terms and we denote undistinctly $\widehat{a}$ for $\widehat{a}$ or $\widehat{\overline{a}}$ or their complex conjugate. Note here that compared to the definition of d and $\mathcal{D}$ after Equation (10.9), we have k more derivatives on $Z_{\mathcal{Q}}(\mathcal{A},\mathcal{B})$. Besides, the multiplier $\widetilde{m}(\xi,\eta)$ has the same properties as the multiplies $m_{k,l}$ defined in (10.4). From Lemma 10.1, we know that ϕ is always bounded away from zero in the region we are interested in, namely ξ large and $|\eta| \leq C_{\mathcal{R}}$. Integration by parts in time yields terms that are easier to control than above. In particular the corresponding term to (10.11a) can be expressed as

(10.20)

$$= -\int_0^t \iint \widetilde{m}(\xi,\eta) \frac{1}{\phi} \widehat{\mathcal{D}}(\eta) \left[\widehat{u \cdot \nabla \partial^{N-k} \mathcal{A}}(\xi-\eta) \overline{\widehat{\partial^N \mathcal{A}}}(\xi) + \widehat{\partial^{N-k} \mathcal{A}}(\xi-\eta) \overline{\widehat{u \cdot \nabla \partial^N \mathcal{A}}}(\xi) \right]$$

(10.21)

$$-\int_0^t \iint \widetilde{m}(\xi,\eta) \frac{i}{\phi} \widehat{\mathcal{D}}(\eta) \left[\widehat{n \langle D\rangle_{c_s} \partial^{N-k} \mathcal{A}}(\xi-\eta) \overline{\widehat{\partial^N \mathcal{A}}}(\xi) - \widehat{\partial^{N-k} \mathcal{A}}(\xi-\eta) \overline{\widehat{n\langle D\rangle_{c_s} \partial^N \mathcal{A}}}(\xi) \right]$$

(10.22)

$$-\int_0^t \iint \widetilde{m}(\xi,\eta) \frac{1}{\phi} \widehat{\mathcal{D}}(\eta) \left[\widehat{R^{N-k}}(\xi-\eta) \overline{\widehat{\partial^N \mathcal{A}}}(\xi) - \widehat{\partial^N \mathcal{A}}(\xi-\eta) \overline{\widehat{R^N}}(\xi) \right]$$

which can be easily estimated.

Finally, we also have terms for which Pu carries the greatest number of derivatives. For these terms, we cannot use the cancellation coming from the energy estimate. To gain the two factors of $|\xi|$, we take advantage of the fact that Pu is more regular, namely it is in H^{N+1} and the fact that the phase ϕ involved in this case is bounded below by $|\xi|/C$. The term corresponding to (10.10) is of the form

$$\int_0^t \iint \widetilde{m}(\xi,\eta) e^{is\phi} \widehat{d}(\eta) \widehat{\partial^{N+1} b}(\xi-\eta) \overline{\widehat{\partial^N a}}(\xi) d\eta d\xi ds \tag{10.23}$$

where the phase ϕ is given by $\phi(\xi,\eta) = \pm\langle\xi-\eta\rangle_1 - \langle\xi\rangle_{c_s} \pm \langle\eta\rangle_l$ and $e^{\pm it\langle D\rangle_l} d(t) = \mathcal{D}(t)$ denote outcome (low frequency) terms. It is clear that $|\phi| \geq |\xi|/C$ in the region we are interested in, namely ξ large and $|\eta| \leq C_{\mathcal{R}}$. Hence, we can perform an integration by parts in time and conclude as before.

11. Scattering

Let us prove for instance that $\mathcal{A}$ scatters. We write symbolically the equation (3.1) on $\mathcal{A}$ as

$$\partial_t \mathcal{A} - i\langle D\rangle_{c_s} \mathcal{A} = \partial \mathcal{C}\mathcal{C}.$$

By definition, $\mathcal{A}$ will scatter in H^{N-2}, say at $+\infty$, if and only if

$$\int_0^t e^{is\langle D\rangle_{c_s}} \partial\mathcal{C}(s)\mathcal{C}(s) ds$$

converges as $t \to \infty$. By the Strichartz estimates (12.4), it suffices that the right-hand side $\partial\mathscr{C}\mathscr{C}$ belongs to $L_t^{\left(\frac{1}{2}+\frac{3}{2}\delta_1\right)^{-1}}\left([0,\infty), L_x^{\left(\frac{1}{3}-\delta_1\right)^{-1}}\right)$. This is the case since

$$\|\partial\mathscr{C}\mathscr{C}\|_{L_t^{\left(\frac{1}{2}+\frac{3}{2}\delta_1\right)^{-1}} L_x^{\left(\frac{1}{3}-\delta_1\right)^{-1}}} \lesssim \|\mathscr{C}\|_X^2 \left\| \langle t\rangle^{C_0\varepsilon} \langle t\rangle^{-\frac{1}{2}-\frac{3}{2}\delta_1} \right\|_{L^{\left(\frac{1}{2}+\frac{3}{2}\delta_1\right)^{-1}}} < \infty,$$

where the last inequality follows since ε is small enough.

12. Appendix: analytical tools

12.1. Sobolev embedding theorem

If $1 \le p \le q \le \infty$ and

$$k > \frac{3}{p} - \frac{3}{q},$$

then

$$\|f\|_p \lesssim \|f\|_{W^{k,q}}. \tag{12.1}$$

12.2. Product laws

If $1 < p, r < \infty$, $1 \le q \le \infty$, $k \ge 0$ and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r},$$

then

$$\|fg\|_{W^{k,r}} \lesssim \|f\|_{W^{k,p}} \|g\|_q + \|f\|_q \|g\|_{W^{k,p}}. \tag{12.2}$$

12.3. Dispersive and Strichartz estimates

The standard dispersive estimates for Klein-Gordon can be found in Ginibre and Velo [16]. We only state them for $e^{it\langle D\rangle}$, but the same estimates (with different implicit constants) hold for $e^{it\langle D\rangle_{c_s}}$:

$$\left\| e^{it\langle D\rangle} f \right\|_p \lesssim t^{\frac{3}{p}-\frac{3}{2}} \|f\|_{W^{5\left(\frac{1}{2}-\frac{1}{p}\right)+\varepsilon,p'}} \qquad \text{if } 2 \le p \le \infty \text{ and } \varepsilon > 0. \tag{12.3}$$

We need the following Strichartz estimate for the Klein-Gordon equation (see for instance Ibrahim, Masmoudi and Nakanishi [21]): if $\varepsilon > 0$ and $0 \le \delta \le \frac{1}{3}$,

$$\left\| \int_0^t e^{is\langle D\rangle} F(s)\, ds \right\|_2 \lesssim \|F\|_{L^{\left(\frac{1}{2}+\frac{3}{2}\delta\right)^{-1}} W^{\left(\frac{5}{6}-\frac{5}{2}\delta+\varepsilon\right),\left(\frac{5}{6}-\delta\right)^{-1}}}. \tag{12.4}$$

For the reader familiar with Besov spaces, this estimate follows from the interpolation between

$$\left\| \int_0^t e^{is\langle D\rangle} F(s)\, ds \right\|_2 \lesssim \|F\|_{L^1 L^2}$$

and

$$\left\| \int_0^t e^{is\langle D\rangle} F(s)\, ds \right\|_2 \lesssim \|F\|_{L^2 B^{5/6}_{6/5,2}}.$$

12.4. Boundedness of multilinear Fourier multipliers

After cutting off with the help of the functions defined in Section 5, the manipulations which we perform lead to various pseudo product operators. Their boundedness properties are stated in the following proposition; it is very far from optimal, but sufficient for our purposes.

This proposition uses classical harmonic analysis tools, and does not rely on the structure of Euler-Maxwell, except for the order of the multipliers, or the estimates on ϕ.

PROPOSITION 12.1. – *Assume that m satisfies the estimates* (4.2).

(i) *Then for any p, q, r in $(1,\infty)$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and $k \geq 0$,*

$$\|T_m(f,g)\|_{W^{k,r}} \lesssim \|f\|_{W^{k+1,p}} \|g\|_{W^{k+1,q}}.$$

Furthermore, if $pk > 3$,

$$\|T_m(f,g)\|_{W^{k,p}} \lesssim \|f\|_{W^{k+1,p}} \|g\|_{W^{k+1,p}}.$$

(ii) *Assume*

$$\mu(\xi,\eta) = \widetilde{\chi}_{\mathscr{O}}(\xi)\chi_{\mathscr{S}}(\xi,\eta)m(\xi,\eta)\frac{1}{\phi} \quad or \quad \widetilde{\chi}_{\mathscr{O}}(\xi)\chi_{\mathscr{T}}(\xi,\eta)m(\xi,\eta)\frac{\partial_\eta\phi}{|\partial_\eta\phi|^2}.$$

Then there exists a constant, which we denote $n \geq 0$, such that for any p, q, r in $(1,\infty)$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and $k \geq 0$,

$$\|T_\mu(f,g)\|_{W^{k,r}} \lesssim \|f\|_{W^{k+n,p}} \|g\|_{W^{k+n,q}}.$$

(iii) *Assume*

$$\mu(\xi,\eta) = m(\xi,\eta)\zeta^1(\xi,\eta) \quad or \quad m(\xi,\eta)\zeta^1(\xi,\eta)\partial_\xi\phi(\xi,\eta).$$

Then there exists a constant, which we stil denote $n \geq 0$, such that for any p, q, r in $(1,\infty)$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and $k \geq 0$,

$$\|T_\mu(f,g)\|_{W^{k,r}} \lesssim \|f\|_{W^{k+n,q}} \|g\|_{p}.$$

(iv) *Assume*

$$\mu(\xi,\eta) = m(\xi,\eta)\zeta^2(\xi,\eta) \quad or \quad m(\xi,\eta)\zeta^2(\xi,\eta)\partial_\xi\phi(\xi,\eta).$$

Then there exists a constant, which we stil denote $n \geq 0$, such that for any p, q, r in $(1,\infty)$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and $k \geq 0$,

$$\|T_\mu(f,g)\|_{W^{k,r}} \lesssim \|f\|_{p} \|g\|_{W^{k+n,p}}.$$

(v) *Assume*

$$\mu(\xi,\eta) \overset{\text{def}}{=} \frac{\chi_{\mathscr{O}}(\eta)\left(1-\theta\left(\frac{\xi-\eta}{M_0}\right)\right)\widetilde{m}(\eta,\xi)}{\phi(\xi,\eta)}$$

where $\widetilde{m}(\xi,\eta) = m_1(\xi)m_2(\eta)m_3(\xi-\eta)$ and $\frac{1}{\phi}$ satisfy the estimates

$$\text{for any } i = 1,2,3, \quad \left|\partial_\xi^\alpha m_i(\xi)\right| \lesssim \frac{1}{|\xi|^{|\alpha|}} \quad \text{and} \quad \left|\partial_\xi^\alpha \partial_\eta^\beta \frac{1}{\phi}\right| \lesssim (|\xi|+|\eta|)^{-|\alpha|-|\beta|-1}.$$

Then

$$\|T_\mu(f,g)\|_{H^k} \lesssim \|f\|_\infty \|g\|_{H^{k+1}}.$$

Proof. – Estimates similar to the first four points above were proved in [11]. It essentially suffices to use the basic estimate

$$\|T_\mu(f,g)\|_r \lesssim \|\mu\|_{H^{3/2+\varepsilon}}\|f\|_p\|g\|_q$$

if $\varepsilon > 0$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, the estimates given in Section 5 on the various symbols, and a paraproduct decomposition to handle large frequencies.

The fifth point follows from the classical Coifman-Meyer theorem [7], see the recent book of Muscalu and Schlag [28], Section 2.13, for a more modern presentation. ☐

Next, we want to study a particular kind of symbol, which will not satisfy standard Coifman-Meyer bounds, but still admit Hölder-like bounds (in the bilinear case for instance, we only focus on the case $L^\infty \times L^2 \to L^2$ bound, but it should be clear from the proof that more general $L^p \times L^q \to L^r$ bounds, with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, also hold).

LEMMA 12.2. – *Let R be a fixed constant.*

(i) *Let $\mu(\xi,\eta)$ be a smooth symbol such that*

$$\operatorname{Supp}\mu \subset \{|\eta|\leq R\} \quad \textit{and} \quad \left|\partial_\xi^\alpha\partial_\eta^\beta\mu(\xi,\eta)\right| \lesssim \frac{1}{|\xi|^{|\alpha|}} \quad \textit{for any } \xi,\eta.$$

Then $\|T_\mu(f,g)\|_2 \lesssim \|f\|_\infty\|g\|_2$.

(ii) *Let $\mu(\xi,\eta,\nu)$ be a smooth symbol such that*

$$\operatorname{Supp}\mu \subset \{|\eta|\leq R\,,\,|\nu|\leq \frac{1}{200}|\xi|\} \quad \textit{and} \quad \left|\partial_\xi^\alpha\partial_\eta^\beta\partial_\nu^\gamma\mu(\xi,\eta,\nu)\right| \lesssim \frac{1}{|\xi|^{|\alpha|+|\gamma|}} \quad \textit{for any } \xi,\eta,\nu.$$

Then $\|T_\mu(f,g,h)\|_2 \lesssim \|f\|_\infty\|g\|_\infty\|h\|_2$.

Proof. – We take for simplicity $R=1$, and first define standard Fourier space decompositions:

- Let ζ be a non-negative function, equal to 1 on $B(0,.9)$, zero outside of $B(0,2)$, and such that $\sum_{j\in\mathbb{Z}^3}\zeta(\xi-j)=1$ for any ξ. Denote

$$Q_j \overset{\text{def}}{=} \sum_{j-3}^{j+3}\zeta(D-j).$$

- Let ψ be a non-negative function, equal to 1 on $B(1,1.5)$, zero outside of $B(.5,4)$, and such that $\sum_{j\in\mathbb{Z}^3}\psi\left(\frac{\xi}{2^j}\right)=1$ for any $\xi\neq 0$. Further denote

$$\widetilde{\psi}(\xi)=\sum_{j=-1}^{+1}\psi\left(\frac{\xi}{2^j}\right) \quad \text{and} \quad \chi(\xi)=\sum_{j=-\infty}^{+1}\psi\left(\frac{\xi}{2^j}\right)$$

and the associated Fourier multipliers

$$P_j=\psi\left(\frac{D}{2^j}\right) \quad , \quad \widetilde{P}_j=\widetilde{\psi}\left(\frac{D}{2^j}\right) \quad \text{and} \quad S_j=\chi\left(\frac{D}{2^j}\right).$$

Proof of (i). – Split μ as follows

$$\mu(\xi,\eta) = \sum_{j\in\mathbb{Z}^3} \zeta(\xi - j)\mu(\xi,\eta) \overset{\text{def}}{=} \sum_j \mu_j(\xi,\eta).$$

The symbols μ_j are uniformly controlled in $\mathscr{C}^k$ for any k. Thus they define operators which are uniformly bounded $L^\infty \times L^2 \to L^2$. Observe furthermore that, due to frequency localization properties, $T_{\mu_j}(f,g) = T_{\mu_j}(f, Q_j g)$; and that for the same reason, the families $(T_{\mu_j}(f,g))_j$ and $(Q_j g)_j$ are almost orthogonal in L^2. These arguments lead to the following inequalities

$$\|T_\mu(f,g)\|_2^2 \lesssim \sum_j \left\|T_{\mu_j}(f,g)\right\|_2^2 \lesssim \sum_j \|f\|_\infty^2 \|Q_j g\|_2^2 \lesssim \|f\|_\infty^2 \|g\|_2^2,$$

proving (i).

Proof of (ii). – We will essentially run the original argument of Coifman and Meyer [7]. First set $\mu(\xi,\eta,\nu) = \widetilde{\mu}(\xi-\eta-\nu,\eta,\nu)$, and observe that the bounds on μ translate into

$$\left|\partial_\xi^\alpha \partial_\eta^\beta \partial_\nu^\gamma \widetilde{\mu}(\xi,\eta,\nu)\right| \lesssim \frac{1}{|\xi|^{|\alpha|+|\gamma|}}. \tag{12.5}$$

Next split $\widetilde{\mu}$ as follows

$$\widetilde{\mu}(\xi,\eta,\nu) = \sum_j \psi\left(\frac{\xi}{2^j}\right)\widetilde{\mu}(\xi,\eta,\nu) \overset{\text{def}}{=} \sum_{j\geq 1} \widetilde{\mu}_j(\xi,\eta,\nu) \quad \text{up to a remainder}$$

(where the remainder is smooth and compactly supported, thus we can forget about it and focus on the sum over j). The support of $\widetilde{\mu}_j(\xi,\eta,\nu)$ is contained in a box $\{|\eta|\leq 1, |\xi|\leq 2^{j+1}, |\nu| \leq \frac{1}{200}2^{j+1}\}$. It can be expanded in (periodic) Fourier series adapted to the larger box $\{|\eta|\leq 2\,,\, |\xi|\leq 2^{j+2}\,,\, |\nu|\leq 2^{j-7}$, and then recovered by restriction. This gives

$$\widetilde{\mu}_j(\xi,\eta,\nu) = \widetilde{\psi}\left(\frac{\xi}{2^j}\right)\chi(2\eta)\chi\left(\frac{\nu}{2^j-5}\right)\sum_{k,\ell,m\in\mathbb{Z}^3} \alpha^j_{k,\ell,m} e^{i2\pi m\xi 2^{-j-2}} e^{i2\pi k\eta} e^{i2\pi\ell\nu 2^{-j+7}},$$

or, coming back to μ,

$$\mu_j(\xi,\eta,\nu) = \widetilde{\psi}\left(\frac{\xi-\eta-\nu}{2^j}\right)\chi(2\eta)\chi\left(\frac{\nu}{2^j-5}\right)\sum_{k,\ell,m} \alpha^j_{k,\ell,m} e^{i2\pi m(\xi-\eta-\nu)2^{-j-2}} e^{i2\pi k\eta} e^{i2\pi\ell\nu 2^{-j+7}}.$$

Next notice that the $\alpha^j_{k,\ell}$ are uniformly bounded in j, with arbitrarily quickly decaying (inverse) polynomial bounds:

$$\text{for any } N, \quad \sup_j |\alpha^j_{k,\ell,m}| = \alpha_{k,\ell,m} \lesssim \frac{1}{|(k,\ell,m)|^N}. \tag{12.6}$$

This can be seen by simply coming back to their definition:

$$\begin{aligned}
\alpha^j_{k,\ell,m} &= C 2^{-6j} \int_{|\xi|\leq 2^{j+2}} \int_{|\eta|\leq 1} \int_{|\nu|\leq 2^{j-7}} \widetilde{\mu}_j(\xi,\eta,\nu) e^{-i2\pi m\xi 2^{-j-2}} e^{-i2\pi k\eta} e^{-i2\pi\ell\nu 2^{-j+7}}\, d\nu\, d\eta\, d\xi \\
&= C \int_{|\xi|\leq 1} \int_{|\eta|\leq 1} \int_{|\nu|\leq 1} \widetilde{\mu}_j(2^{j+2}\xi, \eta, 2^{j-7}\nu) e^{-i2\pi m\xi} e^{-i2\pi k\eta} e^{-i2\pi\ell\nu}\, d\nu\, d\eta\, d\xi,
\end{aligned}$$

and the conclusion follows since the bounds (12.5) imply a uniform control (in j) of the symbols $\widetilde{\mu}_j(2^{j+2}\xi,\eta,2^{j-7}\nu)$.

Coming back to physical space, we have achieved the following decomposition for T_μ:

$$T_\mu(f,g,h) = \sum_{j,k,\ell,m} \alpha^j_{k,\ell,m} S_{j-5}f(\cdot+k)S_0 g(\cdot+\ell 2^{-j+7})\widetilde{P}_j h(\cdot+m2^{-j-2}).$$

The desired estimates follow easily by almost orthogonality between the j-summands

$$\begin{aligned}
\|T_\mu(f,g,h)\|_2 &\lesssim \sum_{k,\ell,m}\left[\sum_j \alpha^j_{k,\ell,m}\left\|S_{j-5}f(\cdot+k)S_0 g(\cdot+\ell 2^{-j+7})\widetilde{P}_j h(\cdot+m2^{-j-2})\right\|_2^2\right]^{1/2}\\
&\lesssim \sum_{k,\ell,m}\left[\alpha^j_{k,\ell,m}\|f\|_\infty^2\|g\|_\infty^2\sum_j\|\widetilde{P}_j h(\cdot+m2^{-j-2})\|_2^2\right]^{1/2}\\
&\lesssim \sum_{k,\ell,m}\alpha_{k,\ell,m}\left[\|f\|_\infty^2\|g\|_\infty^2\|h\|_2^2\right]^{1/2}\\
&\lesssim \|f\|_\infty\|g\|_\infty\|h\|_2,
\end{aligned}$$

where we used in the last inequality the bound (12.6). □

Equipped with the previous lemma, we can prove the following proposition.

PROPOSITION 12.3. – *Let M be as in* (10.15)*, and fix $\alpha>0$. Then the following estimate holds:*

$$\|T_M(f,g,h)\|_2 \lesssim \|f\|_{W^{1+\alpha,\infty}}\|g\|_\infty\|h\|_2.$$

Proof. – Recall that

$$M(\xi,\eta,\nu) = -\mu(\xi,\eta)(\eta+\nu) + [\mu(\xi,\eta)-\mu(\xi-\nu,\eta)]\,\xi \overset{\text{def}}{=} M_1(\xi,\eta,\nu)+M_2(\xi,\eta,\nu),$$

where the operator μ is supported in a strip $\{|\eta|\lesssim 1\,,\ |\xi|\gg 1\}$ and satisfies the bounds

$$\left|\partial_\xi^\alpha\partial_\eta^\beta\right| \lesssim \frac{1}{|\xi|^{|\alpha|}}.$$

We will treat separately the operators T_{M_1} and T_{M_2}, further distinguishing for the latter between the regions where $|\xi-\eta-\nu|\lesssim\nu$, and those where $|\xi-\eta-\nu|\gg|\nu|$, by writing

$$T_{M_2}(f,g,h) = \sum_{j\geq 0} T_{M_2}(P_j f, g, S_{j+10}h) + \sum_{j\geq 0} T_{M_2}(S_{j-10}f,g,P_j h) \quad \text{up to a remainder.}$$

Since the remainder is smooth and compactly supported, it is easily estimated, and we forget about it in the following in order to concentrate on the sum over j. Notice that we overtook the Littlewood-Paley operators P_j and S_j defined in Lemma 12.2.

The operator T_{M_1}. – Simply observe that

$$T_{M_1}(f,g,h) = T_\mu(\nabla f, gh) + T_\mu(f,(\nabla g)h).$$

Thus Lemma 12.2 gives the conclusion.

The operator T_{M_2} in the case $|\xi-\eta-\nu| \lesssim \nu$. – Recall that M_2 is given by $[\mu(\xi,\eta) - \mu(\xi-\nu,\eta)]\,\xi$. There is no cancellation between the two summands in the range we consider, so $\mu(\xi,\eta)\xi$ and $\mu(\xi-\nu,\eta)\xi$ can be considered separately. Since they are estimated in similar ways, we focus on the first one. Notice that $T_{\mu(\xi,\eta)\xi} = \nabla T_\mu$. Using Lemma 12.2 and proceeding in a straightforward way, we get the desired estimates:

$$\left\| \sum_j \nabla T_\mu(P_j f, g S_{j+10} h) \right\|_2 \lesssim \sum_j 2^j \, \|T_\mu(P_j f, g S_{j+10} h)\|_2 \lesssim \sum_j 2^j \|P_j f\|_\infty \|g\|_\infty \|S_j h\|_2$$
$$\lesssim \|f\|_{W^{1+\alpha,\infty}} \|g\|_2 \|h\|_2$$

(notice that $\alpha > 0$ enables to sum over the dyadic blocks).

The operator T_{M_2} in the case $|\xi-\eta-\nu| \gg \nu$. – In this case, we observe that the operator

$$(f,g,h) \mapsto \sum_j T_{M_2}(S_{j-10} f, g, P_j h)$$

has a symbol

$$M_2'(\xi,\eta,\nu) = M_2(\xi,\eta,\nu) \sum_j \psi\left(\frac{\xi-\eta-\nu}{2^j}\right) \chi\left(\frac{\nu}{2^{j-10}}\right)$$

which can be written

$$M_2'(\xi,\eta,\nu) = \widetilde{M_2}(\xi,\eta,\nu) \cdot \nu$$

with

$$\widetilde{M_2}(\xi,\eta,\nu) \overset{\mathrm{def}}{=} \sum_j \psi\left(\frac{\xi-\eta-\nu}{2^{j-10}}\right) \chi\left(\frac{\nu}{2^j}\right) \xi \int_0^1 \partial_\xi \mu(\xi - t\nu, \eta)\, dt.$$

The key observation is that, due to the hypotheses on μ, $\widetilde{M_2}$ satisfies the conditions of Lemma 12.2. The estimate follows easily:

$$\|T_{M_2}(S_j f, g, P_j h)\|_2 = \left\| T_{\widetilde{M_2}}(\nabla f, g, h) \right\|_2 \lesssim \|\nabla f\|_\infty \|g\|_\infty \|h\|_2 . \qquad \square$$

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