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STRONG KARUSH–KUHN–TUCKER OPTIMALITY CONDITIONS FOR MULTIOBJECTIVE SEMI-INFINITE PROGRAMMING *VIA* TANGENTIAL SUBDIFFERENTIAL[☆]

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Abstract. The main aim of this paper is to study strong Karush–Kuhn–Tucker (KKT) optimality conditions for nonsmooth multiobjective semi-infinite programming (MSIP) problems. By using tangential subdifferential and suitable regularity conditions, we establish some strong necessary optimality conditions for some types of efficient solutions of nonsmooth MSIP problems. In addition to the theoretical results, some examples are provided to illustrate the advantages of our outcomes.

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1. Introduction

A simultaneous minimization of a finite number of objective functions over an infinite number of constraints is called a multiobjective semi-infinite programming (MSIP) problem. Applied and theoretical aspects of semi-infinite programming problems have been considered by many researchers, see *e.g.* [16, 43] and references therein. Recently, Karush–Kuhn–Tucker (KKT) optimality conditions for MSIP have been addressed by many authors. In [4, 5], optimality conditions for some types of efficient solutions of MSIP were investigated in terms of Mordukhovich subdifferential. The Mangasarian–Fromovitz and Farkas–Minkowski constraint qualifications were extended and employed to consider optimality conditions for MSIP in [39]. The papers [17, 18] dealt with the optimality conditions and constraint qualifications in convex vector semi-infinite optimization. KKT optimality conditions for weakly efficient solutions and Pareto efficient solutions were obtained in [27] by using some regularity conditions in the sense of Clarke gradient. The paper [2] considered the necessary optimality conditions for MSIP *via* Michel–Penot subdifferential. Strong KKT optimality conditions give more information than weak KKT optimality conditions since all the multipliers corresponding to the objective functions are positive. In [36], many regularity conditions for differentiable functions were investigated to establish the strong KKT optimality conditions for multiobjective optimization problem (MOP) with inequality constraints. Strong KKT optimality conditions for smooth MOP with mixed constraints were given in [13, 21]. The regularity conditions in the sense of semidifferentiable function in [41] and in the sense of Clarke gradient in [3, 14, 33, 50]

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were utilized to obtain strong KKT optimality conditions for some types of MOPs. Strong KKT for weakly efficient solution of MSIP were given in [26] by using some regularity conditions in the sense of Clarke gradient.

The tangential subdifferential including both convex subdifferential and Gâteaux derivative was proposed and applied to establish optimality conditions for optimization problem in [42]. The papers [15, 22, 23, 41] applied effectively the tangential subdifferential to establish weak KKT optimality conditions for MOP. In line of [36], some regularity conditions in the sense of tangential subdifferential were investigated to obtain the strong KKT optimality conditions for nonsmooth MOP in [14]. In [37], tangential subdifferential was employed to unify the KKT type theorem of convex optimization with the convex feasible set for differentiable functions in [31] and for Clarke regular functions in [8]. By using some suitable generalized constraint qualifications in the sense of tangential subdifferential, we established the necessary and sufficient optimality conditions for some types of efficient solutions of nonsmooth MSIP in [46]. Observe that the tangential subdifferential also includes Clarke/Michel-Penot regular gradient. Hence, the optimality conditions in terms of tangential subdifferential obtain more generalized results than the optimality conditions utilizing Clarke/Michel-Penot regular gradient such as in [8, 29, 45, 47, 48] and references therein.

To the best of our knowledge, there is no paper studying the strong KKT optimality conditions for MSIP by using regularity conditions in the sense of tangential subdifferential. Motivated by the above observations, in this paper, we establish strong KKT optimality conditions for Pareto efficient solutions and weakly efficient solutions of MSIP in terms of tangential subdifferential. The paper is organized as follows. Section 1 recalls basic concepts and some preliminaries. Section 2 is devoted to establishing the KKT optimality conditions for weakly efficient solution and Pareto efficient solution of MSIP. In Section 3, regularity conditions in the sense of tangential subdifferential and their relations are investigated. Our results not only extend the results in [14] from MOP to MSIP but also consider the strong KKT for weakly efficient solution, which was not investigated in [14]. Some examples are provided to illustrate our outcomes.

2. Preliminaries

The following notations and definitions will be used throughout the paper. Let $\mathbb{R}^n$ be a finite-dimensional normed space. The notation $\langle\cdot,\cdot\rangle$ is utilized to denote inner product. For a given $\bar{x}$, $\mathcal{U}(\bar{x})$ is the system of the neighborhoods of $\bar{x}$. For $S \subseteq \mathbb{R}^n$, $\text{int}S$, $\text{cl}S$, $\text{aff}S$, and $\text{co}S$ stand for its interior, closure, affine hull, convex hull of S, respectively (resp). The cone and the convex cone (containing the origin) generated by S are denoted resp by $C(S)$, $\text{cone}S$. We denote by $\text{ri}S$ the relative interior of a convex set S. The negative polar cone and strictly negative polar cone of S are defined resp by

$$S^- = \{x^* \in \mathbb{R}^n | \langle x^*, x\rangle \le 0 \quad \forall x \in S\},$$
$$S^s = \{x^* \in \mathbb{R}^n | \langle x^*, x\rangle < 0 \quad \forall x \in S \setminus \{0\}\}.$$

It is easy to check that $S^s \subset S^-$ and if $S^s \neq \emptyset$ then $\text{cl}S^s = S^-$. Moreover, the bipolar theorem, see *e.g.* [1], states that $S^{--} = \text{cl cone}S$.

Definition 2.1 ([1])**.** Let S be a nonempty subset of $\mathbb{R}^n$.

(i) The contingent (or Bouligand) cone of S at $\bar{x} \in \text{cl}S$ is

$$T(S,\bar{x}) := \{x \in \mathbb{R}^n \mid \exists \tau_k \downarrow 0, \quad \exists x_k \to x,\ \forall k \in \mathbb{N},\ \bar{x} + \tau_k x_k \in S\}.$$

(ii) The adjacent cone of of S at $\bar{x} \in \text{cl}S$ is

$$A(S,\bar{x}) := \{x \in \mathbb{R}^n \mid \forall \tau_k \downarrow 0, \quad \exists x_k \to x,\ \forall k \in \mathbb{N},\ \bar{x} + \tau_k x_k \in S\}.$$

(iii) The cone of the feasible directions of S at $\bar{x}$ is

$$D(S,\bar{x}) := \{x \in \mathbb{R}^n \mid \exists \delta > 0, \quad \bar{x} + \tau x \in S, \forall \tau \in (0,\delta)\}.$$

(iv) The cone of the weak feasible directions of S at $\bar{x}$ is

$$F(S,\bar{x}) := \{x \in \mathbb{R}^n \mid \exists \tau_k \downarrow 0, \quad \forall k \in \mathbb{N},\ \bar{x} + \tau x_k \in S\}$$

Remark 2.2. The following properties can be checked directly.

(i) $D(S,\bar{x}) \subset F(S,\bar{x}) \subset T(S,\bar{x})$.
(ii) $D(S,\bar{x}) \subset A(S,\bar{x}) \subset T(S,\bar{x})$.
(iii) If S is a convex set then $A(S,\bar{x}) = T(S,\bar{x}) = \mathrm{cl}C(S-\bar{x})$.
(iv) If S is a convex set then $D(S,\bar{x}) = F(S,\bar{x}) = C(S-\bar{x})$.

For a nonempty set $S \subseteq \mathbb{R}^n$, the function $\sigma : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, defined by

$$\sigma_S(x) := \sup_{s\in S} \langle s, x\rangle \quad \forall x \in \mathbb{R}^n,$$

is called the support function of S. Notice that σ is sublinear and lower semicontinuous, *i.e.*, $\liminf_{x'\to x} \sigma(x') = \sigma(x)$ for all $x \in \mathbb{R}^n$. Moreover, σ is finite everywhere if and only if S is bounded.

Lemma 2.3 ([19])**.** *If $\sigma : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous and sublinear function then there exists a nonempty closed convex set S_σ such that σ is the support function of S_σ, or, equivalently,*

$$S_\sigma = \{x^* \in \mathbb{R}^n \mid \sigma(d) \le \langle x^*, d\rangle, \quad \forall d \in \mathbb{R}^n\}.$$

Definition 2.4. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x}, d \in \mathbb{R}^n$.

(i) The directional derivative (or Dini derivative) of ϕ at $\bar{x}$ in the direction d is

$$\phi'(\bar{x}, d) := \lim_{\tau\downarrow 0} \frac{\phi(\bar{x}+\tau d) - \phi(\bar{x})}{\tau}.$$

For $d = 0$, define $\phi'(\bar{x}, 0) = 0$. We say that ϕ is directionally differentiable at $\bar{x}$ if its directional derivative exists in all directions d.

(ii) The Hadamard directional derivative of ϕ at $\bar{x}$ in the direction d is $\bar{x}$ in the direction d is

$$\phi^H(\bar{x}, d) := \lim_{\tau\downarrow 0, d'\to d} \frac{\phi(\bar{x}+\tau d') - \phi(\bar{x})}{\tau}.$$

We say that ϕ is Hadamard directionally differentiable at $\bar{x}$ if its Hadamard directional derivative exists in all directions d.

Note that if $\phi^H(\bar{x}, d)$ exists, then $\phi'(\bar{x}, d)$ also exists and they are equal. Conversely, if ϕ is Lipschitzian on a neighborhood U of $\bar{x}$, then ϕ is Hadamard directionally differentiable at $\bar{x}$ in every direction d in which ϕ is directionally differentiable.

Definition 2.5 ([32, 37])**.** A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is called tangentially convex at $\bar{x} \in \mathbb{R}^n$ if for every $d \in \mathbb{R}^n$, $\phi'(\bar{x}, d)$ exists, is finite and the function $\phi'(\bar{x}, .) : \mathbb{R}^n \to \mathbb{R}$ is a convex function of d.

Since $\phi'(\bar{x},.)$ is positively homogeneous, if ϕ is tangentially convex at $\bar{x}$ then $\phi'(\bar{x},.)$ is sublinear. Then, by Lemma 2.3, there exists a nonempty compact convex set of $\mathbb{R}^n$ such that $\phi'(\bar{x},.)$ is the support function of that set.

Definition 2.6 ([37, 42])**.** Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be tangentially convex at $\bar{x} \in \mathbb{R}^n$. The nonempty compact convex $\partial^T \phi(\bar{x})$ of $\mathbb{R}^n$ is called the tangential subdifferential of ϕ at $\bar{x}$ if $\phi'(\bar{x}, d) = \max_{x^* \in \partial^T \phi(\bar{x})} \langle x^*, d\rangle$, which is equivalent to

$$\partial^T \phi(\bar{x}) = \{x^* \in \mathbb{R}^n \mid \langle x^*, d\rangle \leq \phi'(\bar{x}, d), \quad \forall d \in \mathbb{R}^n\}.$$

Definition 2.7 ([6])**.** Let $\bar{x} \in \mathbb{R}^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Clarke directional derivative of ϕ at $\bar{x}$ in direction d is defined by

$$\phi^o(\bar{x}, d) := \limsup_{\tau \downarrow 0, x \to \bar{x}} \frac{\phi(x + \tau d) - \phi(x)}{\tau}.$$

The Clarke gradient of ϕ at $\bar{x}$ is

$$\partial^C \phi(\bar{x}) = \{x^* \in \mathbb{R}^n \mid \langle x^*, d\rangle \leq \phi^o(\bar{x}, d), \quad \forall d \in \mathbb{R}^n\}.$$

We say that ϕ is Clarke regular at $\bar{x}$ if $\phi'(\bar{x}, d)$ exists and $\phi^o(\bar{x}, d) = \phi'(\bar{x}, d)$ for all $d \in \mathbb{R}^n$.

Definition 2.8 ([38])**.** Let $\bar{x} \in \mathbb{R}^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Michel-Penot (MP) directional derivative of $\phi : \mathbb{R}^n \to \mathbb{R}$ at $\bar{x}$ in direction u is defined by

$$\phi^\diamond(\bar{x}, u) := \sup_{v \in \mathbb{R}^n} \limsup_{\tau \downarrow 0} \frac{\phi(\bar{x} + \tau(u + v)) - \phi(\bar{x} + \tau v)}{\tau}.$$

The MP subdifferential of ϕ at $\bar{x}$ is

$$\partial^{MP} \phi(\bar{x}) := \{x^* \in \mathbb{R}^n | \langle x^*, d\rangle \leq \phi^\diamond(\bar{x}, d), \quad \forall d \in \mathbb{R}^n\}.$$

We say that ϕ is MP regular at $\bar{x}$ if $\phi'(\bar{x}, d)$ exists and $\phi^\diamond(\bar{x}, d) = \phi'(\bar{x}, d)$ for all $d \in \mathbb{R}^n$.

Remark 2.9. Let ϕ be a function from $\mathbb{R}^n$ to $\mathbb{R}$ and $\bar{x} \in \mathbb{R}^n$. Some important classes of tangentially convex functions are considered as follows.

(i) If ϕ is Gâteaux differentiable at $\bar{x}$ then ϕ is tangentially convex at $\bar{x}$ and $\partial^T \phi(\bar{x}) = \{\nabla \phi(\bar{x})\}$.
(ii) If ϕ is convex then ϕ is tangentially convex at $\bar{x}$ and $\partial^T \phi(\bar{x}) = \partial \phi(\bar{x})$, where ∂ denotes the subdifferential in the sense of convex analysis.
(iii) If ϕ is locally Lipschitz at $\bar{x}$ and Clarke regular at $\bar{x}$, then ϕ is tangentially convex at $\bar{x}$ and $\partial^T \phi(\bar{x}) = \partial^C \phi(\bar{x})$.
(iv) If ϕ is locally Lipschitz at $\bar{x}$ and MP regular at $\bar{x}$, then ϕ is tangentially convex at $\bar{x}$ and $\partial^T \phi(\bar{x}) = \partial^{MP} \phi(\bar{x})$.
(v) If $\phi^H(\bar{x}, d)$ exists, is finite and the function $\phi^H(\bar{x}, .) : \mathbb{R}^n \to \mathbb{R}$ is a convex function of d, then ϕ is tangentially convex at $\bar{x}$ and $\partial^T \phi(\bar{x}) = \partial^H \phi(\bar{x})$, where $\partial^H \phi(\bar{x}) := \{x^* \in \mathbb{R}^n | \langle x^*, d\rangle \leq \phi^H(\bar{x}, d), \ \forall d \in \mathbb{R}^n\}$.

Now, we give some examples to illustrate some advantages of tangential subdifferential in some cases.

Example 2.10. Let $\bar{x} = 0$ and $\phi : \mathbb{R} \to \mathbb{R}$ be defined by $\phi(x) = \max\{x^3, x\} + x$. Then, $\phi'(0, u) = \max\{u, 2u\}$ is a convex function, and hence, $\partial^T(\bar{x}) = [1, 2]$. Note that ϕ is not is Gâteaux differentiable at $\bar{x}$, while it is locally Lipschitz at $\bar{x}$ and Clarke regular at $\bar{x}$.

Example 2.11. Let $\bar{x} = 0$ and $\phi : \mathbb{R} \to \mathbb{R}$ be defined as follows

$$\phi(x) = \begin{cases} x^2 \sin \frac{2}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then, ϕ is locally Lipschitz at $\bar{x}$ and $\phi^\diamond(\bar{x}, u) = \phi'(\bar{x}, u) = \{0\}$ for all $u \in \mathbb{R}^n$. Hence, ϕ is MP regular at $\bar{x}$ and $\partial^T \phi(\bar{x}) = \{0\} \subsetneqq \partial^C \phi(\bar{x}) = [-2, 2]$.

Example 2.12. Let $\bar{x} = (0, 0)$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows

$$\phi(x) = \begin{cases} \frac{x_1^3}{x_2} + x_1, & \text{if } x \neq 0, \\ 0, & \text{if } x_1 = 0 \text{ or } x_2 = 0. \end{cases}$$

Then, ϕ is not continuous at $\bar{x}$. Since $\phi'(\bar{x}, u) = \lim\limits_{\tau \downarrow 0} \frac{\tau u_2^3}{u_1} = 0$, ϕ is Gâteaux differentiable at $\bar{x}$ and $\partial^T \phi(\bar{x}) = \{0\}$.

Definition 2.13 ([14])**.** Let $S \subset \mathbb{R}^n$ be a convex set, $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in S$.

(i) ϕ is quasiconvex at $\bar{x}$ if

$$\forall x \in S, \quad \phi(x) \leq \phi(\bar{x}) \Rightarrow \phi(\lambda x + (1 - \lambda)\bar{x}) \leq \phi(\bar{x}) \quad \forall \lambda \in (0, 1).$$

(ii) ϕ is Dini-convex at $\bar{x}$ if $\forall x \in S, \phi(x) \geq \phi(\bar{x}) + \phi'(\bar{x}, x - \bar{x})$.
(iii) ϕ is strictly Dini-convex at $\bar{x}$ if $\forall x \in S \setminus \{\bar{x}\}, \phi(x) > \phi(\bar{x}) + \phi'(\bar{x}, x - \bar{x})$.
(iv) ϕ is Dini-pseudoconvex at $\bar{x}$ if $\forall x \in S, \phi(x) < \phi(\bar{x}) \Rightarrow \phi'(\bar{x}, x - \bar{x}) < 0$.
(v) ϕ is strictly Dini-pseudoconvex at $\bar{x}$ if

$$\forall x \in S \setminus \{\bar{x}\}, \quad \phi(x) \leq \phi(\bar{x}) \Rightarrow \phi'(\bar{x}, x - \bar{x}) < 0.$$

(vi) ϕ is Dini-quasiconvex at $\bar{x}$ if $\forall x \in S, \phi(x) \leq \phi(\bar{x}) \Rightarrow \phi'(\bar{x}, x - \bar{x}) \leq 0$.
(vii) ϕ is Dini-linearlike at $\bar{x}$ if $\forall x \in S, \phi(x) = \phi(\bar{x}) + \phi'(\bar{x}, x - \bar{x})$.
(viii) ϕ is quasilinear, Dini-pseudolinear or Dini-quasilinear at $\bar{x}$, if ϕ and $-\phi$ are quasiconvex, Dini-pseudoconvex or Dini-quasiconvex at $\bar{x}$, resp.
(ix) ϕ is quasiconvex on S if ϕ is quasiconvex on each point of S. The other concepts here introduced can be defined on a set in a similar way.

Remark 2.14 ([12, 14])**.** Let $S \subset \mathbb{R}^n$ be a convex set, $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in S$. Some properties of generalized convex functions are summarized as follows.

(i) Let ϕ be directional differentiable at $\bar{x}$. If ϕ is quasiconvex at $\bar{x}$ then ϕ is Dini-quasiconvex at $\bar{x}$.
(ii) If ϕ is Dini-pseudoconvex at $\bar{x}$ and continuous on S, then ϕ is quasiconvex at $\bar{x}$.
(iii) If ϕ is Dini-quasiconvex on S and continuous on S, then ϕ is quasiconvex on S.
(iv) If ϕ is Dini-linearlike at $\bar{x}$, then ϕ is Dini-pseudolinear and Dini-quasilinear at $\bar{x}$.
(v) If ϕ is quasilinear and directional differentiable at $\bar{x}$, then ϕ is is Dini-quasilinear at $\bar{x}$.
(vi) If ϕ is Dini-convex at $\bar{x}$ then ϕ is both Dini-pseudoconvex and Dini-quasiconvex at $\bar{x}$.

Remark 2.15. Let $S \subset \mathbb{R}^n$ be a convex set, $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in S$. Suppose that ϕ is tangentially convex at $\bar{x}$.

(i) If ϕ is Dini-convex at $\bar{x}$ and $x \in S$, then

$$\phi(x) \geq \phi(\bar{x}) + \langle \partial^T \phi(\bar{x}), x - \bar{x} \rangle,$$

where $\phi(x) \geq \phi(\bar{x}) + \langle \partial^T \phi(\bar{x}), x - \bar{x} \rangle$ denotes $\phi(x) \geq \phi(\bar{x}) + \langle x^*, x - \bar{x} \rangle$ for all $x^* \in \partial^T \phi(\bar{x})$.

(ii) If ϕ is strictly Dini-convex at $\bar{x}$ and $x \in S \setminus \{\bar{x}\}$, then

$$\phi(x) > \phi(\bar{x}) + \langle \partial^T \phi(\bar{x}), x - \bar{x} \rangle.$$

(iii) If ϕ is Dini-pseudoconvex at $\bar{x}$ and $x \in S, \phi(x) < \phi(\bar{x})$, then

$$\langle \partial^T \phi(\bar{x}), x - \bar{x} \rangle < 0.$$

(iv) If ϕ is strictly Dini-pseudoconvex at $\bar{x}$ and $x \in S \setminus \{\bar{x}\}, \phi(x) \leq \phi(\bar{x})$, then

$$\langle \partial^T \phi(\bar{x}), x - \bar{x} \rangle < 0.$$

(v) If ϕ is Dini-quasiconvex at $\bar{x}$ and $x \in S, \phi(x) \leq \phi(\bar{x})$, then

$$\langle \partial^T \phi(\bar{x}), x - \bar{x} \rangle \leq 0.$$

Lemma 2.16 ([44]). *Let $\{C_i | i = 1, \dots, m\}$ be a collection of nonempty convex sets in $\mathbb{R}^n$ and $K = \mathrm{co}\left(\bigcup_{i=1}^{m} C_i\right)$. Then,*

$$\mathrm{ri}K = \bigcup \left\{ \sum_{i=1}^{m} \lambda_i \mathrm{ri}C_i \mid \sum_{i=1}^{m} \lambda_i = 1, \quad \lambda_i > 0, \ i = 1, \dots, m \right\}.$$

Lemma 2.17 ([44]). *Let C_1 and C_2 be non-empty convex sets in $\mathbb{R}^n$. In order that there exist a hyperplane separating C_1 and C_2 properly, it is necessary and sufficient that $\mathrm{ri}C_1$ and $\mathrm{ri}C_2$ have no point in common.*

Lemma 2.18 ([44]). *Let $\{C_t | t \in \Gamma\}$ be an arbitrary collection of nonempty convex sets in $\mathbb{R}^n$ and $K = \mathrm{cone}\left(\bigcup_{t \in \Gamma} C_t\right)$. Then, every nonzero vector of K can be expressed as a non-negative linear combination of n or fewer linear independent vectors, each belonging to a different C_t.*

Lemma 2.19 ([16]). *Suppose that S, P are arbitrary (possibly infinite) index sets, $a_s = a(s) = (a_1(s), \dots, a_n(s))$ maps S onto $\mathbb{R}^n$, and so does a_p. Suppose that the set $\mathrm{co}\{a_s, s \in S\} + \mathrm{cone}\{a_p, p \in P\}$ is closed. Then the following statements are equivalent:*

$$I: \begin{cases} \langle a_s, x \rangle < 0, & S \neq \emptyset \\ \langle a_p, x \rangle \leq 0, & p \in P \end{cases} \text{has no solution } x \in \mathbb{R}^n;$$
$$II: \ 0 \in \mathrm{co}\{a_s, s \in S\} + \mathrm{cone}\{a_p, p \in P\}.$$

Lemma 2.20 ([19]). *If S is a nonempty compact subset of $\mathbb{R}^n$, then,*

(i) $\mathrm{co}S$ *is a compact set.*
(ii) *If $0 \notin \mathrm{co}S$, then $\mathrm{cone}S$ is a closed cone.*

3. Strong KKT optimality conditions

In this section, we consider the following multiobjective semi-infinite programming

$$\text{(P)} \qquad \begin{aligned} &\min\nolimits_{\mathbb{R}^m_+} f(x) := (f_1(x), \dots, f_m(x)) \\ &\text{s.t.} \quad g_t(x) \leq 0, \quad t \in T, \ x \in \mathbb{R}^n, \end{aligned}$$

where f_i, $i = 1, \ldots, m$, $g_t, t \in T$ are functions from $\mathbb{R}^n$ to $\mathbb{R}$. The index set T is an arbitrary nonempty set, not necessary finite. Set $I := \{1, \ldots, m\}$, $f := (f_1, \ldots, f_m)$ and $g := (g_t)_{t \in T}$. Denote the feasible solution set of (P)

$$\Omega := \{x \in \mathbb{R}^n \mid g_t(x) \leq 0, \quad t \in T\}.$$

Definition 3.1. For problem (P), let $\bar{x} \in \Omega$.

(i) $\bar{x}$ is a locally weakly efficient solution of (P), denoted by $\bar{x} \in \mathrm{LWE(P)}$, if there exists $U \in \mathcal{U}(\bar{x})$ such that

$$f(\bar{x}) - f(x) \notin \mathrm{int}\mathbb{R}^m_+, \quad \forall x \in \Omega \cap U.$$

(ii) $\bar{x}$ is a locally (Pareto) efficient solution of (P), denoted by $\bar{x} \in \mathrm{LE(P)}$, if there exists $U \in \mathcal{U}(\bar{x})$ such that

$$f(\bar{x}) - f(x) \notin \mathbb{R}^m_+ \setminus \{0\}, \quad \forall x \in \Omega \cap U.$$

If $U = \mathbb{R}^n$, the word "locally" is omitted. In this case, the weakly efficient solution sets/the weakly efficient solution sets are denoted by WE(P)/E(P). It is easy to check that $\mathrm{LE(P)} \subset \mathrm{LWE(P)}$; see *e.g.* [35] for more details.

Denote $\mathbb{R}^{|T|}_+$ the collection of all the functions $\lambda : T \to \mathbb{R}$ taking values λ_t's positive only at finitely many points of T, and equal to zero at the other points. For a given $\bar{x} \in \Omega$, denote $T(\bar{x}) := \{t \in T | g_t(\bar{x}) = 0\}$ the index set of all active constraints at $\bar{x}$. The set of active constraint multipliers at $\bar{x} \in \Omega$ is

$$\Lambda(\bar{x}) := \{\lambda \in \mathbb{R}^{|T|}_+ | \lambda_t g_t(\bar{x}) = 0, \quad \forall t \in T\}.$$

Notice that $\lambda \in \Lambda(\bar{x})$ if there exists a finite index set $I \subset T(\bar{x})$ such that $\lambda_t > 0$ for all $t \in I$ and $\lambda_t = 0$ for all $t \in T \setminus I$. For a given $\bar{x} \in \Omega$, define the extension of constraints system of (P)

$$Q^i := \left\{x \in \mathbb{R}^n \mid f_j(x) \leq f_j(\bar{x}), \quad g_t(x) \leq 0, \ j \in I \setminus \{i\}, \ t \in T\right\}, \quad i \in I.$$

Setting $Q := \bigcap\limits_{i \in I} Q^i$, we have $Q = \Omega$.

Definition 3.2. Let $\bar{x} \in \Omega$.

(i) We say that the Assumption (A1) holds at $\bar{x} \in \Omega$ if $f_i, i \in I$, is Hadamard differentiable at $\bar{x}$, the function $f^H_i(\bar{x}, .) : \mathbb{R}^n \to \mathbb{R}$ is a convex function for all $i \in I$ and $g_t, t \in T$, are tangentially convex at $\bar{x}$.
(ii) We say that the Assumption (A2) holds at $\bar{x} \in \Omega$ if $f_i, i \in I$, and $g_t, t \in T$, are tangentially convex at $\bar{x}$.

3.1. Strong KKT optimality conditions for weakly efficient solution

Lemma 3.3. *Let* $\bar{x} \in \mathrm{LWE(P)}$.

(i) *If* (A1) *holds, then* $\left(\bigcup_{i=1}^m \partial^T f_i(\bar{x})\right)^s \cap T(\Omega, \bar{x}) = \emptyset$.
(ii) *If* (A2) *holds, then* $\left(\bigcup_{i=1}^m \partial^T f_i(\bar{x})\right)^s \cap F(\Omega, \bar{x}) = \emptyset$.

Proof. (i) Suppose to the contrary that there exists $d \in \left(\bigcup\limits_{i=1}^m \partial^T f_i(\bar{x})\right)^s \cap T(\Omega, \bar{x})$. It follows from $d \in \left(\bigcup\limits_{i=1}^m \partial^T f_i(\bar{x})\right)^s$ that

$$\langle x^*, d \rangle < 0, \quad \forall x^* \in \partial^T f_i(\bar{x}), \ \forall i \in I.$$

Since the function $\phi : \partial^T f_i(\bar{x}) \subset \mathbb{R}^n \to \mathbb{R}$, defined by $\phi(x^*) = \langle x^*, d\rangle$, is continuous on the compact set $\partial^T f_i(\bar{x})$, there exists a point $\bar{x}^* \in \partial^T f_i(\bar{x})$ such that $\phi(\bar{x}^*) = \max_{x^* \in \partial^T f_i(\bar{x})} \langle x^*, d\rangle$. This implies that

$$f_i'(\bar{x}, d) = \max_{x^* \in \partial^T f_i(\bar{x})} \langle x^*, d\rangle = \langle \bar{x}^*, d\rangle < 0, \quad \forall i \in I. \tag{3.1}$$

By $d \in T(\Omega, \bar{x})$, there exist $\tau_k \downarrow 0$ and $d_k \to d$ such that $\bar{x} + \tau_k d_k \in \Omega$ for all k. Since $\bar{x} \in$ LWE(P), there exists, taking subsequence if necessary, an index $i_0 \in I$ such that

$$f_{i_0}'(\bar{x}, d) = f_{i_0}^H(\bar{x}, d) = \lim_{k\to\infty} \frac{f(\bar{x} + \tau_k d_k) - f(\bar{x})}{\tau_k} \geq 0,$$

which contradicts (3.1).

(ii) Reasoning by contraposition, assume the existence of $d \in \left(\bigcup_{i=1}^m \partial^T f_i(\bar{x})\right)^s \cap F(\Omega, \bar{x})$. Hence, $\langle x^*, d\rangle < 0, \quad \forall x^* \in \partial^T f_i(\bar{x}), \quad \forall i \in I$. Since the function $\phi : \partial^T f_i(\bar{x}) \subset \mathbb{R}^n \to \mathbb{R}$, defined by $\phi(x^*) = \langle x^*, d\rangle$, is continuous on the compact set $\partial^T f_i(\bar{x})$, there exists a point $\bar{x}^* \in \partial^T f_i(\bar{x})$ such that $\phi(\bar{x}^*) = \max_{x^* \in \partial^T f_i(\bar{x})} \langle x^*, d\rangle$. This shows that

$$f_i'(\bar{x}, d) = \max_{x^* \in \partial^T f_i(\bar{x})} \langle x^*, d\rangle = \langle \bar{x}^*, d\rangle < 0, \quad \forall i \in I. \tag{3.2}$$

By $d \in F(\Omega, \bar{x})$, there exists $\tau_k \downarrow 0$ such that $\bar{x} + \tau_k d \in \Omega$ for all k. Since $\bar{x} \in$ LWE(P), there exists, taking subsequence if necessary, an index $i_0 \in I$ such that

$$f_{i_0}'(\bar{x}, d) = \lim_{k\to\infty} \frac{f(\bar{x} + \tau_k d) - f(\bar{x})}{\tau_k} \geq 0,$$

which contradicts (3.2). □

Now, we establish some strong KKT necessary optimality conditions for locally weakly efficient solutions of MSIP under the following regularity conditions (RCs):

$$(\text{EARC}): \left(\bigcup_{i=1}^m \partial^T f_i(\bar{x})\right)^s \cap \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^- \subseteq T(\Omega, \bar{x}),$$

$$(\text{EFRC}): \left(\bigcup_{i=1}^m \partial^T f_i(\bar{x})\right)^s \cap \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^- \subseteq \text{cl}F(\Omega, \bar{x}),$$

$$(\text{GCRC}): \left(\bigcup_{j\in I\setminus\{i\}} \partial^T f_i(\bar{x})\right)^s \cap \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s = \emptyset, \quad \forall i \in I.$$

Proposition 3.4. *Let* $\bar{x} \in \mathrm{LWE(P)}$*. Suppose that* (A1) *holds at* $\bar{x}$*. If* (EARC) *holds at* $\bar{x}$ *and* $\mathrm{cone}\left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)$ *is closed, then there exist* $\alpha \in \mathbb{R}^m_+$ *with* $\sum_{i=1}^m \alpha_i = 1$ *and* $\lambda \in \Lambda(\bar{x})$ *such that*

$$0 \in \sum_{i=1}^m \alpha_i \partial^T f_i(\bar{x}) + \sum_{t\in T} \lambda_t \partial^T g_t(\bar{x}).$$

Moreover, if (GCRC) *holds, then* $\alpha_i > 0$ *for all* $i \in I$*.*

Proof. It follows from Lemma 3.3 (i) that

$$\left(\bigcup_{i=1}^m \partial^T f_i(\bar{x})\right)^s \cap T(\Omega, \bar{x}) = \emptyset. \tag{3.3}$$

The above equation together with (EARC) implies that

$$\left(\bigcup_{i=1}^m \partial^T f_i(\bar{x})\right)^s \cap \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^- \subseteq \left(\bigcup_{i=1}^m \partial^T f_i(\bar{x})\right)^s \cap T(\Omega, \bar{x}) = \emptyset. \tag{3.4}$$

Hence,

$$\left(\mathrm{co}\bigcup_{i=1}^m \partial^T f_i(\bar{x})\right)^s \cap \left(\mathrm{cone}\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^- = \left(\bigcup_{i=1}^m \partial^T f_i(\bar{x})\right)^s \cap \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^- = \emptyset.$$

Now, we prove that

$$\left(\mathrm{co}\bigcup_{i=1}^m \partial^T f_i(\bar{x})\right) \cap \left(-\mathrm{cone}\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right) \neq \emptyset. \tag{3.5}$$

Suppose to the contrary that (3.5) does not hold. Since $\mathrm{co}\bigcup\limits_{i=1}^m \partial^T f_i(\bar{x})$ is a nonempty compact set and $-\mathrm{cone}\bigcup\limits_{t\in T(\bar{x})} \partial^T g_t(\bar{x})$ is a closed convex cone, by the strong separation theorem, there exists $x \in \mathbb{R}^n$ such that

$$\begin{cases} \langle x^*, x\rangle < 0, & \forall x^* \in \mathrm{co}\bigcup\limits_{i=1}^m \partial^T f_i(\bar{x}), \\ \langle x^*, x\rangle \geq 0, & \forall x^* \in -\mathrm{cone}\bigcup\limits_{t\in T(\bar{x})} \partial^T g_t(\bar{x}). \end{cases}$$

This yields that

$$x \in \left(\mathrm{co}\bigcup_{i=1}^m \partial^T f_i(\bar{x})\right)^s \cap \left(\mathrm{cone}\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^-,$$

which contradicts (3.4). Therefore, (3.5) holds, leading to

$$0 \in \operatorname{co} \bigcup_{i=1}^{m} \partial^T f_i(\bar{x}) + \operatorname{cone} \bigcup_{t \in T(\bar{x})} \partial^T g_t(\bar{x}).$$

It follows from the above inclusion and Lemma 2.18 that there exists $\alpha \in \mathbb{R}^m_+$ with $\sum_{i=1}^m \alpha_i = 1$, $\lambda \in \Lambda(\bar{x})$, $x_i^* \in \partial^T f_i(\bar{x})$ and $y_j^* \in \partial^T g_j(\bar{x})$ with $(i,j) \in I \times J$, where J is a finite subset of $T(\bar{x})$, such that

$$\sum_{i=1}^{m} \alpha_i x_i^* + \sum_{j \in J} \lambda_j y_j^* = 0. \tag{3.6}$$

Suppose to the contrary that $\alpha_i = 0$ for some $i \in I$. As (GCRC) holds, there exists u such that

$$\begin{cases} \langle x_j^*, u \rangle < 0, & j \in I \setminus \{i\}, \\ \langle y_t^*, u \rangle < 0, & t \in T(\bar{x}). \end{cases}$$

The above inequalities together with (3.6) deduces that

$$0 = \sum_{i=1}^{m} \alpha_i \langle x_i^*, u \rangle + \sum_{j \in J} \lambda_j \langle y_j^*, u \rangle < 0,$$

which is absurd. Hence, $\alpha_i > 0, \forall i \in I$ and the conclusion is obtained. □

Proposition 3.5. *Let $\bar{x} \in$ LWE(P). Suppose that* (A2) *holds at $\bar{x}$. If* (EFRC) *holds at $\bar{x}$ and* cone $\left(\bigcup_{t \in T(\bar{x})} \partial^T g_t(\bar{x})\right)$ *is closed, then there exist $\alpha \in \mathbb{R}^m_+$ with $\sum_{i=1}^m \alpha_i = 1$ and $2\lambda \in \Lambda(\bar{x})$ such that*

$$0 \in \sum_{i=1}^{m} \alpha_i \partial^T f_i(\bar{x}) + \sum_{t \in T} \lambda_t \partial^T g_t(\bar{x}).$$

Moreover, if (GCRC) *holds, then $\alpha_i > 0$ for all $i \in I$.*

Proof. It follows from Lemma 3.3 (ii) that

$$\left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x}) \right)^s \cap F(\Omega, \bar{x}) = \emptyset. \tag{3.7}$$

This clearly leads that

$$\operatorname{int} \left(\left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x}) \right)^s \right) \cap \operatorname{cl} F(\Omega, \bar{x}) = \left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x}) \right)^s \cap \operatorname{cl} F(\Omega, \bar{x}) = \emptyset.$$

The above equation together with (EFRC) ensures that

$$\left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x}) \right)^s \cap \left(\bigcup_{t \in T(\bar{x})} \partial^T g_t(\bar{x}) \right)^- \subseteq \left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x}) \right)^s \cap \operatorname{cl} F(\Omega, \bar{x}) = \emptyset.$$

The proof is continued just as in the proof of Proposition 3.4. □

Example 3.6. Let $n = 2, T = [0,1]$ and $f : \mathbb{R}^2 \to \mathbb{R}$, $g_t : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f_1(x) = \begin{cases} \frac{x_1^3}{x_2} - x_1, & \text{if } x \neq 0, \\ 0, & \text{if } x_1 = 0 \text{ or } x_2 = 0, \end{cases} \qquad f_2(x) = -2x_1, g_t(x) = x_2 - t, t \in T.$$

Then, $\Omega = \{x \in \mathbb{R}^2 \mid x_2 \leq 0\}$ and for $\bar{x} = (0,0) \in \Omega$, one has

$$\partial^T f_1(\bar{x}) = \{(-1,0)\}, \partial^T f_2(\bar{x}) = \{(-2,0)\},\ T(\bar{x}) = \{0\},\ \bigcup_{t \in T(\bar{x})} \partial^T g_t(\bar{x}) = \{(0,1)\}.$$

Hence, by some calculations, we have $\mathrm{cl}F(\Omega,\bar{x}) = F(\Omega,\bar{x}) = \{x \in \mathbb{R}^2 \mid x_2 \leq 0\}$,

$$\left(\bigcup_{j \in I \setminus \{i\}} \partial^T f_j(\bar{x})\right)^s = \mathrm{int}\mathbb{R}_+ \times \mathbb{R}, \quad i \in I = \{1,2\}, \quad \left(\bigcup_{i=1}^{2} \partial^T f_i(\bar{x})\right)^s = \mathrm{int}\mathbb{R}_+ \times \mathbb{R},$$

$$\left(\bigcup_{t \in T(\bar{x})} \partial^T g_t(\bar{x})\right)^- = \mathbb{R} \times (-\mathbb{R}_+), \left(\bigcup_{t \in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s = \mathbb{R} \times (-\mathrm{int}\mathbb{R}_+).$$

Thus, cone $\bigcup_{t \in T(\bar{x})} \partial^T g_t(\bar{x})$ is closed and (EFRC) holds at $\bar{x}$. Moreover,

$$\left(\bigcup_{j \in I \setminus \{i\}} \partial^T f_i(\bar{x})\right)^s \cap \left(\bigcup_{t \in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s = \emptyset, \quad \forall i \in I,$$

which justifies that (GCRC) holds. It is easy to check that there are no $\alpha \in \mathrm{int}\mathbb{R}_+^2$ (or $\alpha \in \mathbb{R}_+^2$) with $\alpha_1 + \alpha_2 = 1$ and $\lambda \in \Lambda(\bar{x})$ such that

$$(0,0) \in \alpha_1(-2,0) + \alpha_2(-1,0) + \sum_{t \in T} \lambda_t \partial^T g_t(\bar{x}) = \alpha_1(-2,0) + \alpha_2(-1,0) + \lambda_0(0,2).$$

Hence, Proposition 3.5 asserts that $\bar{x}$ is not a locally weakly efficient solution of (P). Since f_1 is not locally Lipschitz at $\bar{x}$, Theorem 3.1 in [4, 5], Theorem 6 in [2], Theorem 3.4 in [27] and Theorem 4 in [26] cannot be used to reject $\bar{x}$.

Remark 3.7. Note that the tangential subdifferential is also an upper regular convexificators, see [7, 20, 25]. The convexificators were used to investigate the optimality conditions for SIP in [28, 40]. Recently, the KKT conditions for (weakly) efficient solutions of a nonsmooth MSIP utilizing convexificators have been established in [24]. However, our approach in this paper is different from that of [24]. Moreover, we also consider the strong KKT conditions for (weakly) efficient solutions of a nonsmooth MSIP, which were not investigated in [24].

3.2. Strong KKT optimality conditions for efficient solution

Lemma 3.8. *Let* $\bar{x} \in \mathrm{LE(P)}$. *Suppose that* (A1) *holds at* $\bar{x}$. *Then,*

$$\left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x})^s\right) \cap \left(\bigcap_{i=1}^{m} T(Q^i, \bar{x})\right) = \emptyset.$$

Proof. It suffices only to prove that

$$(\partial^T f_i(\bar{x}))^s \cap T(Q^i, \bar{x}) = \emptyset, \quad \forall i \in I.$$

Suppose to the contrary that there exist $i_0 \in I$ and a vector d such that

$$d \in (\partial^T f_{i_0}(\bar{x}))^s \cap T(Q^{i_0}, \bar{x}). \tag{3.8}$$

Since $d \in (\partial^T f_{i_0}(\bar{x}))^s$, one has $\langle x^*, d\rangle < 0, \forall x^* \in \partial^T f_{i_0}(\bar{x})$. This gives us the inequality

$$f'_{i_0}(\bar{x}, d) = \max_{x^* \in \partial^T f_{i_0}(\bar{x})} \langle x^*, d\rangle < 0. \tag{3.9}$$

As $d \in T(Q^{i_0}, \bar{x})$, there exist $\tau_k \downarrow 0, d_k \to d$ such that $\bar{x} + \tau_k d_k \in Q^{i_0}$ for all k, *i.e.*,

$$\begin{cases} f_i(\bar{x} + \tau_k d_k) \le f_i(\bar{x}), & \forall i \in I \setminus \{i_0\}, \\ \bar{x} + \tau_k d_k \in \Omega, & \forall k. \end{cases} \tag{3.10}$$

Moreover, it follows from (3.9) that

$$\lim_{k\to\infty} \frac{f_{i_0}(\bar{x} + \tau_k d_k) - f_{i_0}(\bar{x})}{\tau_k} = f^H_{i_0}(\bar{x}, d) = f'_{i_0}(\bar{x}, d) < 0.$$

This derives that, for k large enough,

$$f_{i_0}(\bar{x} + \tau_k d_k) < f_{i_0}(\bar{x}),$$

which together with (3.10) contradicts the efficiency of $\bar{x}$. □

Now, we establish some strong KKT necessary optimality conditions for locally efficient solutions of MSIP under the following regularity conditions:

$$(\mathrm{GARC}) \ : \ \left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x})\right)^- \cap \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^- \subseteq \bigcap_{i=1}^{m} T(Q^i, \bar{x}),$$

$$(\mathrm{GGRC}) \ : \ \left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x})\right)^- \cap \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^- \subseteq \bigcap_{i=1}^{m} \mathrm{cl}\ \mathrm{co}T(Q^i, \bar{x}).$$

Lemma 3.9. *Suppose that* (A1)*and* (GARC) *holds at* $\bar{x}$*. If* $\bar{x} \in \mathrm{LE(P)}$*, then for each* $i \in I$*, the following system has no solution* $d \in \mathbb{R}^n$*:*

$$\begin{cases} \langle \partial^T f_i(\bar{x}), d\rangle < 0, \\ \langle \partial^T f_j(\bar{x}), d\rangle \leq 0, \quad \forall j \in I \setminus \{i\}, \\ \langle \partial^T g_t(\bar{x}), d\rangle \leq 0, \quad \forall t \in T(\bar{x}). \end{cases} \tag{3.11}$$

Proof. Reasoning by contraposition, suppose the existence of $i_0 \in I$ such that (3.11) has a solution $d \in \mathbb{R}^n$. Then, we get from (3.11) that $d \in \left(\bigcup_{i=1}^m \partial^T f_i(\bar{x})\right)^- \cap \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^-$. Since (GARC) holds, for any $i \in I$, one has $d \in T(Q^i, \bar{x})$. Hence, for $i_0 \in I$, there exist $\tau_k \downarrow 0, d_k \to d$ such that $\bar{x} + \tau_k d_k \in Q^{i_0}$ for all k, *i.e.*,

$$\begin{cases} f_i(\bar{x} + \tau_k d_k) \leq f_i(\bar{x}), & \forall i \in I \setminus \{i_0\}, \forall k, \\ \bar{x} + \tau_k d_k \in \Omega, & \forall k. \end{cases} \tag{3.12}$$

It follows from (3.11) with $i = i_0$ that

$$f_i^H(\bar{x}, d) = f'_{i_0}(\bar{x}, d) = \max_{x^* \in \partial^T f_{i_0}(\bar{x})} \langle x^*, d\rangle < 0.$$

Therefore,

$$f_{i_0}^H(\bar{x}, d) = \lim_{k\to\infty} \frac{f(\bar{x} + \tau_k d_k) - f(\bar{x})}{\tau_k} < 0,$$

and thus, for k large enough,

$$f_{i_0}(\bar{x} + \tau_k d_k) < f_{i_0}(\bar{x}).$$

The above inequality together with (3.12) contradicts local efficiency of $\bar{x}$. $\square$

Proposition 3.10. *Let* $\bar{x} \in \mathrm{LE(P)}$*. Suppose that* (A1) *and* (GARC) *hold at* $\bar{x}$*. Let one of the following conditions hold*

(i) (C1) : $\left(\bigcup_{i=1}^m \partial^T f_i(\bar{x})\right)^- \setminus \{0\} \subseteq \bigcup_{i=1}^m (\partial^T f_i(\bar{x}))^s$,

(ii) (C2) : *For each* $i \in I$*, the set*

$$D_i := \partial^T f_i(\bar{x}) + \mathrm{cone}\left(\bigcup_{j\in I\setminus\{i\}} \partial^T f_j(\bar{x}) \cup \bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)$$

are closed.

Then, there exist $\alpha \in \mathrm{int}\mathbb{R}^m_+$ *and* $\lambda \in \Lambda(\bar{x})$ *such that*

$$0 \in \sum_{i=1}^m \alpha_i \partial^T f_i(\bar{x}) + \sum_{t\in T} \lambda_t \partial^T g_t(\bar{x}).$$

Proof. (i) Let (C1) hold. We first prove that

$$0 \in \operatorname{ri}\left(\operatorname{co}\bigcup_{i=1}^{m}\partial^{MP}f_i(\bar{x})\right)+\operatorname{cone}\bigcup_{t\in T(\bar{x})}\partial^{T}g_t(\bar{x}). \tag{3.13}$$

Suppose to the contrary that (3.13) does not hold. Then,

$$\operatorname{ri}\left(\operatorname{co}\bigcup_{i=1}^{m}\partial^{T}f_i(\bar{x})\right)\cap\left(-\operatorname{cone}\bigcup_{t\in T(\bar{x})}\partial^{T}g_t(\bar{x})\right)=\emptyset.$$

Thus, by Lemma 2.17, there exists $d\in\mathbb{R}^n\setminus\{0\}$ such that

$$d\in\left(\operatorname{co}\bigcup_{i=1}^{m}\partial^{T}f_i(\bar{x})\right)^{-}\cap\left(\operatorname{cone}\bigcup_{t\in T(\bar{x})}\partial^{T}g_t(\bar{x})\right)^{-}=\left(\bigcup_{i=1}^{m}\partial^{T}f_i(\bar{x})\right)^{-}\cap\left(\bigcup_{t\in T(\bar{x})}\partial^{T}g_t(\bar{x})\right)^{-},$$

which together with (C1) and (GARC) contradicts Lemma 3.8. Hence, the validity of (3.13) yields. Then, it follows from (3.13), Lemmas 2.16 and 2.18 that there exist $\alpha\in\operatorname{int}\mathbb{R}^m_+$ with $\sum_{i=1}^m\alpha_i=1$ and $\lambda\in\mathit{\Lambda}(\bar{x})$ such that

$$0\in\sum_{i=1}^{m}\alpha_i\partial^{T}f_i(\bar{x})+\sum_{t\in T}\lambda_t\partial^{T}g_t(\bar{x}).$$

(ii) Let (C2) hold. Employing Lemma 3.9, one gets that, for each $i\in I$, the following system has no solution

$$\begin{cases}\langle\partial^{T}f_i(\bar{x}),d\rangle<0,\\ \langle\partial^{T}f_j(\bar{x}),d\rangle\le 0, & \forall j\in I\setminus\{i\},\\ \langle\partial^{T}g_t(\bar{x}),d\rangle\le 0, & \forall t\in T(\bar{x}).\end{cases}$$

Hence, by applying Lemma 2.19 with the sets $S=\partial^{T}f_i(\bar{x})=\operatorname{co}\partial^{T}f_i(\bar{x})$ and $P=\bigcup_{j\in I\setminus\{i\}}\partial^{T}f_j(\bar{x})\cup\bigcup_{t\in T(\bar{x})}\partial^{T}g_t(\bar{x})$, one has, for each $i\in I$,

$$0\in\operatorname{co}\partial^{T}f_i(\bar{x})+\operatorname{cone}\left(\bigcup_{j\in I\setminus\{i\}}\partial^{T}f_j(\bar{x})\cup\bigcup_{t\in T(\bar{x})}\partial^{T}g_t(\bar{x})\right).$$

Now, according to Lemma 2.18, there exist $\alpha^i_j\ge 0, j\in I\setminus\{i\}$ and $\lambda^i\in\mathit{\Lambda}(\bar{x})$ such that

$$0\in\partial^{T}f_i(\bar{x})+\sum_{j\in I\setminus\{i\}}\alpha^i_j\partial^{T}f_j(\bar{x})+\sum_{t\in T}\lambda^i_t\partial^{T}g_t(\bar{x}),\quad\forall i\in I.$$

Consequently,

$$0\in\sum_{i=1}^{m}\left(\partial^{T}f_i(\bar{x})+\sum_{j\in I\setminus\{i\}}\alpha^i_j\partial^{T}f_j(\bar{x})+\sum_{t\in T}\lambda^i_t\partial^{T}g_t(\bar{x})\right).$$

Setting $\alpha_i := 1 + \sum_{j \in I \setminus \{i\}} \alpha_j^i, i = 1, \ldots, m$ and $\lambda_t := \sum_{i=1}^{m} \lambda_t^i, t \in T$, we arrive at the existence of $\alpha_i > 0$ and $\lambda \in \Lambda(\bar{x})$ such that

$$0 \in \sum_{i=1}^{m} \alpha_i \partial^T f_i(\bar{x}) + \sum_{t \in T} \lambda_t \partial^T g_t(\bar{x}).$$

This completes the proof. □

The following examples show that the condition (C1) or (C2) is essential.

Example 3.11. Let $T = [0,1], D = \{a \in \mathbb{R}^2 \mid -1 \leq a_1 \leq -a_2^2, -1 \leq a_2 \leq 1\}$ and $f_i : \mathbb{R}^2 \to \mathbb{R}, i = 1, 2, g_t : \mathbb{R}^2 \to \mathbb{R}, t \in T$ be defined as follows

$$f_1(x) = x_2,\ f_2(x) = \sigma_D(x) = \sup_{a \in D} \langle a, x \rangle,\ g_t(x) = -x_2 - t, \quad t \in T.$$

Then, $\Omega = \{x \in \mathbb{R}^2 \mid x_2 \geq 0\}$ and, for $\bar{x} = (0,0) \in \Omega$, one has

$$\begin{aligned} &Q^1 = \mathbb{R}_+ \times \{0\},\ Q^2 = \mathbb{R} \times \mathbb{R}_+,\ \partial^T f_1(x) = \{(0,-2)\},\ \partial^T f_2(x) = D, \\ &T(\bar{x}) = \{0\},\ \bigcup_{t \in T(\bar{x})} \partial^T g_t(x) = \{(0,-1)\}. \end{aligned}$$

Hence, by some calculations, we get that $T(Q^1, \bar{x}) = Q^1, T(Q^2, \bar{x}) = Q^2$ and

$$\left(\bigcup_{i=1}^{2} \partial^T f_i(x) \right)^- = \mathbb{R}_+ \times \{0\},\ (\partial^T f_1(\bar{x}))^s = \mathbb{R} \times \mathrm{int}\mathbb{R}_+,$$

$$(\partial^T f_1(\bar{x}))^s = \emptyset,\ \left(\bigcup_{t \in T(\bar{x})} \partial^T g_t(x) \right)^- = \mathbb{R} \times \mathbb{R}_+.$$

Thus, (GARC) holds at $\bar{x}$. However, (C1) does not hold at $\bar{x}$ since

$$(1,0) \in \left(\bigcup_{i=1}^{2} \partial^T f_i(\bar{x}) \right)^- \setminus \{0\},\ (1,0) \notin \bigcup_{i=1}^{2} (\partial^T f_i(\bar{x}))^s.$$

It is easy to check that there are no $\alpha \in \mathrm{int}\mathbb{R}^2_+$ with $\alpha_1 + \alpha_2 = 1$ and $\lambda \in \Lambda(\bar{x})$ such that

$$(0,0) \in \alpha_1(0,-2) + \alpha_2 D + \sum_{t \in T} \lambda_t \partial^T g_t(\bar{x}) = \alpha_1(0,-2) + \alpha_2 D + \lambda_0(0,-1).$$

Example 3.12. Let $T = \mathbb{N} = \{1, 2, \ldots\}$, $f_i : \mathbb{R}^2 \to \mathbb{R}, i = 1, 2$, and $g_t : \mathbb{R}^2 \to \mathbb{R}, t \in T$, be defined as follows

$$f_1(x_1, x_2) = 2x_1, f_2(x_1, x_2) = x_1, g_t(x_1, x_2) = \sup_{(a_1, a_2) \in D_t} a_1 x_1 + a_2 x_2,$$

where $D_t = \mathrm{co}\{(-\xi, -\xi^{t+1}) \mid 0 \leq \xi \leq 1\}$.

Then, $\Omega = \{x \in \mathbb{R}^2 \mid x_1 \geq 0, x_1 + x_2 \geq 0\}$ and, for $\bar{x} = (0,0) \in \Omega$, we have

$$Q^1 = Q^2 = \{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \geq 0\},\ f_1'(\bar{x}, d) = 2d,\ f_2'(\bar{x}, d) = d,\ \partial^T f_1(\bar{x}) = \{(2,0)\},$$

$$\partial^T f_2(\bar{x}) = \{(1,0)\},\ T(\bar{x}) = \mathbb{N},\ g_t'(\bar{x}, d) = g_t(d),\ \partial^T g_t(\bar{x}) = D_t,\ \forall t \in T(\bar{x}).$$

Hence,

$$\left(\bigcup_{i=1}^{2} \partial^T f_i(\bar{x})\right)^- \cap \left(\bigcup_{t \in T(\bar{x})} \partial^T g_t(\bar{x})\right)^- = \{x \in \mathbb{R}^2 \mid x_1 = 0,\ x_2 \geq 0\},$$

$$T(Q^1, \bar{x}) = T(Q^1, \bar{x}) = \{x \in \mathbb{R}^2 \mid x_1 = 0,\ x_2 \geq 0\},$$

which confirm (GARC). However,

$$D_1 = \mathbb{R} \times (-\mathbb{R}_+) \setminus \{(x_1, 0), x_1 < 2\}, \quad D_2 = \mathbb{R} \times (-\mathbb{R}_+) \setminus \{(x_1, 0), x_1 < 1\},$$

and hence, D_1, D_2 are not closed. We can check that $\bar{x}$ is a locally efficient solution of (P) and there is no $\alpha \in \text{int}\mathbb{R}^2_+$ with $\alpha_1 + \alpha_2 = 1$ and $\lambda \in \Lambda(\bar{x})$ such that

$$(0,0) \in \alpha_1(2,0) + \alpha_2(1,0) + \sum_{t \in T} \lambda_t D_t.$$

Lemma 3.13. *Suppose that $\bar{x} \in \text{LE(P)}$, (A2) and (GGRC) hold at $\bar{x}$. Assume further that, for each $i \in I$,*

(i) *$f_j, j \in I \setminus \{i\}, g_t, t \in T(\bar{x})$ are quasiconvex at $\bar{x}$ and $-f_i$ is quasiconvex at $\bar{x}$.*
(ii) *$f_i'(\bar{x}, .)$ is a linear function on $\mathbb{R}^n$.*

Then, for each $i \in I$, the following system has no solution $d \in \mathbb{R}^n$:

$$\begin{cases} \langle \partial^T f_i(\bar{x}), d \rangle < 0, \\ \langle \partial^T f_j(\bar{x}), d \rangle \leq 0, \quad \forall j \in I \setminus \{i\}, \\ \langle \partial^T g_t(\bar{x}), d \rangle \leq 0, \quad \forall t \in T(\bar{x}). \end{cases} \tag{3.14}$$

Proof. Reasoning ad absurdum, suppose the existence of $i_0 \in I$ such that (3.15) has a solution $d \in \mathbb{R}^n$. This implies that

$$\begin{cases} f_{i_0}'(\bar{x}, d) = \max\limits_{x^* \in \partial^T f_{i_0}(\bar{x})} \langle x^*, d \rangle < 0, \\ f_j'(\bar{x}, d) = \max\limits_{x^* \in \partial^T f_j(\bar{x})} \langle x^*, d \rangle \leq 0, \quad \forall j \in I \setminus \{i\}, \\ g_T'(\bar{x}, d) = \max\limits_{x^* \in \partial^T g_t(\bar{x})} \langle x^*, d \rangle \leq 0, \quad \forall t \in T(\bar{x}). \end{cases} \tag{3.15}$$

Moreover, (3.15) deduces that $d \in \left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x})\right)^- \cap \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^-$. Since (GGRC) holds, for any $i \in I$, one has $d \in \text{cl co}T(Q^i, \bar{x})$. Then, there exists a sequence $\{d_p\}$ in $\text{co}T(Q^i, \bar{x})$ such that

$$\lim_{p\to\infty} d_p = d, \tag{3.16}$$

and, for any $d_p, p = 1, 2, \ldots$, there exist numbers $k_p, \lambda_{pl} \geq 0$ and $d_{pl} \in T(Q^i, \bar{x})$, where $l = 1, \ldots, k_p$, such that

$$\sum_{l=1}^{k_p} \lambda_{pl} = 1, \ \sum_{l=1}^{k_p} \lambda_{pl} d_{pl} = d_p.$$

We get from $d_{pl} \in T(Q^i, \bar{x})$ the existence of $\tau_{pl}^k \downarrow 0$ and $x_{pl}^k \subset Q^i$ such that

$$\lim_{k\to\infty} x_{pl}^k = \bar{x} \text{ and } \lim_{k\to\infty} \frac{x_{pl}^k - \bar{x}}{\tau_{pl}^k} = d_{pl}. \tag{3.17}$$

Setting $d_{pl}^k := \frac{x_{pl}^k - \bar{x}}{\tau_{pl}^k}$, we deduce from $x_{pl}^k \subset Q^i$ that, for any k,

$$f_j(x_{pl}^k) = f_j(\bar{x} + \tau_{pl}^k d_{pl}^k) \leq f_j(\bar{x}), \quad \forall j \in I \setminus \{i\}, \tag{3.18}$$

$$g_t(x_{pl}^k) = g_t(\bar{x} + \tau_{pl}^k d_{pl}^k) \leq g_t(\bar{x}), \quad \forall t \in T. \tag{3.19}$$

Let $i_0 \in I$. As $x \in \text{LP(E)}$, for any k, one gets

$$f_{i_0}(x_{pl}^k) = f_{i_0}(\bar{x} + +\tau_{pl}^k d_{pl}^k) \geq f_{i_0}(\bar{x}). \tag{3.20}$$

Combining (3.18)–(3.20) and the assumption (i), we obtain

$$f'_{i_0}(\bar{x}, d_{pl}^k) \geq 0, \tag{3.21}$$

$$f'_j(\bar{x}, d_{pl}^k) \leq 0, \quad \forall k \in I \setminus \{i_0\}, \tag{3.22}$$

$$g'_t(\bar{x}, d_{pl}^k) \leq 0, \quad \forall t \in T(\bar{x}). \tag{3.23}$$

Hence, we deduce from (3.16)–(3.18) and the assumption (ii) that

$$f'_{i_0}(\bar{x}, d) \geq 0. \tag{3.24}$$

Also by (3.16)–(3.18), (3.22), (3.23) and (A2), it follows that

$$f'_j(\bar{x}, d) \leq 0, \quad \forall j \in I \setminus \{i_0\}, \tag{3.25}$$

$$g'_t(\bar{x}, d) \leq 0, \quad \forall t \in T(\bar{x}). \tag{3.26}$$

Thus, (3.24)–(3.26) contradicts (3.14). Hence, the conclusion is verified. $\square$

Proposition 3.14. *Let $\bar{x} \in \text{LP(E)}$. Suppose that* (A2) *and* (GGRC) *hold at $\bar{x}$. Assume further that, for each $i \in I$,*

(i) *$f_j, j \in I \setminus \{i\}, g_t, t \in T(\bar{x})$ are quasiconvex at $\bar{x}$ and $-f_i$ is quasiconvex at $\bar{x}$.*

(ii) $f_i'(\bar{x}, .)$ *is a linear function on* $\mathbb{R}^n$,

and, for each $i \in I$, *the sets*

$$D_i := \partial^T f_i(\bar{x}) + \operatorname{cone}\left(\bigcup_{j \in I \setminus \{i\}} \partial^T f_j(\bar{x}) \cup \bigcup_{t \in T(\bar{x})} \partial^T g_t(\bar{x})\right)$$

are closed. Then, there exist $\alpha \in \operatorname{int}\mathbb{R}^m_+$ *and* $\lambda \in \Lambda(\bar{x})$ *such that*

$$0 \in \sum_{i=1}^{m} \alpha_i \partial^T f_i(\bar{x}) + \sum_{t \in T} \lambda_t \partial^T g_t(\bar{x}).$$

Proof. The proof is similar to the proof of Proposition 3.10 (ii). □

The following example shows that the closedness of $D_i, i \in I$, can not be dropped.

Example 3.15. Let $T = \mathbb{N} = \{1, 2, \ldots\}$, $f_i : \mathbb{R}^2 \to \mathbb{R}$ and $g_t : \mathbb{R}^2 \to \mathbb{R}, t \in T$ be defined as follows

$$f_1(x_1, x_2) = x_1, f_2(x_1, x_2) = 2x_1, g_t(x_1, x_2) = \sup_{(a_1, a_2) \in D_t} a_1 x_1 + a_2 x_2,$$

where $D_t = \{a \in \mathbb{R}^2 \mid a_1^2 + a_2^2 + 2ta_2 \leq 0, a_1 \leq 0\}$.
Then, $\Omega = \{x \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\}$ and, for $\bar{x} = (0, 0) \in \Omega$, we have

$$\begin{aligned} &Q^1 = Q^2 = \{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \geq 0\},\ f_1'(\bar{x}, d) = 2d, f_2'(\bar{x}, d) = d, \\ &\partial^T f_1(\bar{x}) = \{(1, 0)\}, \partial^T f_2(\bar{x}) = \{(2, 0)\},\ T(\bar{x}) = \mathbb{N}, \\ &g_t'(\bar{x}, d) = g_t(d),\ \partial^T g_t(\bar{x}) = D_t, \quad \forall t \in T(\bar{x}). \end{aligned}$$

Hence,

$$\left(\bigcup_{i=1}^{2} \partial^T f_i(\bar{x})\right)^- \cap \left(\bigcup_{t \in T(\bar{x})} \partial^T g_t(\bar{x})\right)^- = \{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 \geq 0\},$$

$$T(Q^1, \bar{x}) = T(Q^1, \bar{x}) = \{x \in \mathbb{R}^2 \mid x_1 = 0, \quad x_2 \geq 0\}.$$

Thus, (GGRC) holds. Moreover, we can justify that assumptions (i), (ii) in Proposition 3.14 are fulfilled. However,

$$D_1 = \mathbb{R} \times (-\mathbb{R}_+) \setminus \{(x_1, 0), x_1 < 1\}, D_2 = \mathbb{R} \times (-\mathbb{R}_+) \setminus \{(x_1, 0), x_1 < 2\},$$

and hence, D_1, D_2 are not closed. We can check that $\bar{x}$ is a locally efficient solution of (P) and there is no $\alpha \in \operatorname{int}\mathbb{R}^2_+$ with $\alpha_1 + \alpha_2 = 1$ and $\lambda \in \Lambda(\bar{x})$ such that

$$(0, 0) \in \alpha_1(2, 0) + \alpha_2(1, 0) + \sum_{t \in T} \lambda_t D_t.$$

4. Regularity conditions

In this section, we investigate some sufficient conditions for regularity conditions in Section 2.

Definition 4.1. Suppose that (A2) holds and $G(x) = \sup_{t\in T} g_t(x)$ for all $x \in \Omega$. The Pshenichnyi–Levin–Valadier (PLV) condition holds at $\bar{x} \in \Omega$ if $G(.)$ is tangential convex at $\bar{x}$ and

$$\partial^T G(\bar{x}) \subset \operatorname{co}\left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right).$$

Note that (PLV) conditions were proposed in [34] for the convex semi-infinite system and in [26] for the Lipschitz semi-infinite system. The following example illustrates that (PLV) condition does not hold in general.

Example 4.2. Let $T = \mathbb{N}$, $\bar{x} = 0$ and $g_t : \mathbb{R} \to \mathbb{R}, t \in T$, be defined by

$$g_t(x) = \begin{cases} 4x, & t = 0, \\ 3x - \frac{1}{k+1}, & t = 2k-1, k = 1, 2, \dots \\ 5x - \frac{1}{k+2}, & t = 2k, k = 1, 2, \dots \end{cases}$$

Then, $\Omega = (-\infty, 0], T(\bar{x}) = \{0\}$ and $G(x) = \max\{3x, 5x\}$. Hence G is tangential convex at $\bar{x} \in \Omega$. However, (PLV) condition does not hold at $\bar{x}$ since

$$\partial^T G(\bar{x}) = [3, 5] \not\subset \operatorname{co}\left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right) = \{4\}.$$

Proposition 4.3 ([42])**.** *Let $\bar{x} \in \Omega$. If the following conditions satisfy:*

(i) *T is a compact set,*
(ii) *the function $g_t(x)$ is continuous jointly in both variables in $\mathbb{R}^n \times T$,*
(iii) *for all d, $g_t(\bar{x} + \lambda d) = g_t(\bar{x}) + \lambda g'_t(\bar{x}, d) + \gamma_t(\lambda)$, where $\gamma_t(\lambda) \to 0$ uniformly in t when $\lambda \downarrow 0$,*
(iv) $\operatorname{co}\left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)$ *is closed,*

then, (PLV) *holds at $\bar{x}$.*

Now, we present some regularity conditions in terms of tangential subdifferential.

Definition 4.4. Suppose that (A2) holds at $\bar{x}$. We consider the following regularity conditions (with the convention $\cup_{\alpha\in\emptyset} X_\alpha = \emptyset$).

(i) (CRC) : $\left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x})\right)^s \cap \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s = \emptyset.$
(ii) (GCRC) : $\left(\bigcup_{j\in I\setminus\{i\}} \partial^T f_j(\bar{x})\right)^s \cap \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s = \emptyset, \quad \forall i \in I.$
(iii) (SRC) : $f_i, i \in I, g_t, t \in T$, are Dini-pseudoconvex at $\bar{x}$ and there exists $\tilde{x} \in \mathbb{R}^n$ such that

$$f_i(\tilde{x}) < f_i(\bar{x}) \quad \forall i \in I,\ g_t(\tilde{x}) < 0\ \forall t \in T(\bar{x}).$$

(iv) (GSRC) : $f_i, i \in I, g_t, t \in T$, are Dini-pseudoconvex at $\bar{x}$ and for each $i \in I$, there exists $\tilde{x}_i \in \mathbb{R}^n$ such that

$$f_j(\tilde{x}) < f_j(\bar{x}) \quad \forall j \in I \setminus \{i\},\ g_t(\tilde{x}_i) < 0 \forall t \in T(\bar{x}).$$

(v) (EZRC) : $\left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x})\right)^s \cap \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s \subseteq \mathrm{cl}D(\Omega, \bar{x})$.

(vi) (GZRC) : $\left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x})\right)^- \cap \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s \subseteq \bigcap_{i=1}^{m} D(Q^i, \bar{x})$.

(vii) (EKTRC) : $\left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x})\right)^- \cap \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s \subseteq A(\Omega, \bar{x})$.

(viii) (GFRC) : $\left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x})\right)^- \cap \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s \subseteq \bigcap_{i=1}^{m} \mathrm{cl}F(Q^i, \bar{x})$.

Proposition 4.5. *Suppose that* (A2) *holds at* $\bar{x} \in \Omega$. *The following implications are verified.*

(i) (SRC) $\Rightarrow$ (GSRC) $\Rightarrow$ (GCRC) *and* (SRC) $\Rightarrow$ (CRC) $\Rightarrow$ (GCRC).
(ii) (CRC) + (PLV) $\Rightarrow$ (EZRC) *and* (GCRC) + (PLV) $\Rightarrow$ (GZRC).
(iii) (EZRC) $\Rightarrow$ (EFRC) $\Rightarrow$ (EARC) *and* (EZRC) $\Rightarrow$ (EKTRC) $\Rightarrow$ (EARC).
(iv) (GZRC) $\Rightarrow$ (GFRC) $\Rightarrow$ (GARC) $\Rightarrow$ (GGRC).

Proof. (i) (SRC) $\Rightarrow$ (GSRC). The proof is trivival.

(GSRC) $\Rightarrow$ (GCRC). Suppose that (GSRC) holds. Then, for each $i \in I$, there exists ?$\tilde{x}_i \in \mathbb{R}^n$ such that $f_j(\tilde{x}) < f_j(\bar{x})\ \forall j \in I \setminus \{i\}, g_t(\tilde{x}_i) < 0\ \forall t \in T(\bar{x})$. Since $f_i, i \in I, g_t, t \in T$, are Dini-pseudoconvex at $\bar{x}$, we deduce from the above inequalities that

$$\langle \partial^T f_j(\bar{x}), \tilde{x}_i - \bar{x}\rangle < 0 \quad \forall j \in I \setminus \{i\},\ \langle \partial_t^T g(\bar{x}), \tilde{x}_i - \bar{x}\rangle < 0 \quad \forall t \in T(\bar{x}).$$

Setting $d_i := \tilde{x}_i - \bar{x}$, then for each $i \in I$,

$$d_i \in \left(\bigcup_{j\in I\setminus\{i\}} \partial^T f_j(\bar{x})\right)^s \cap \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s.$$

(SRC) $\Rightarrow$ (CRC). The proof is similar to the proof of (GSRC) $\Rightarrow$ (GCRC).

(CRC) $\Rightarrow$ (GCRC). Since $\left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x})\right)^s \subset \left(\bigcup_{j\in I\setminus\{i\}} \partial^T f_j(\bar{x})\right)^s$, the conclusion is obtained.

(ii) (CRC) + (PLV) $\Rightarrow$ (EZRC). Suppose that (CRC) holds. Then,

$$\left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s \subset \left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x})\right)^s \cap \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s \neq \emptyset.$$

Combining the above relation with (PLV) give us the existence of d such that

$$d \in \left(\bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s = \left(\mathrm{co} \bigcup_{t\in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s \subset (\partial^T G(\bar{x}))^s.$$

Hence, $G'(\bar{x}, d) < 0$, and consequently, there exists $\delta > 0$ such that

$$G(\bar{x} + \tau d) < G(\bar{x}), \quad \forall \tau \in (0, \delta).$$

This leads that, for all $t \in T$ and for all $\tau \in (0, \delta)$, one has $g_t(\bar{x} + \tau d) < 0$. Hence, $\bar{x} + \tau d \in \Omega$ for all $\tau \in (0, \delta)$, *i.e.*, $\bar{x} + \tau d \in D(\Omega, \bar{x})$. This implies that

$$\left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x})\right)^s \cap \left(\bigcup_{t \in T(\bar{x})} \partial^T g_t(\bar{x})\right)^- = \left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x})\right)^s \cap \mathrm{cl} \left(\bigcup_{t \in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s \subset \mathrm{cl} D(\Omega, \bar{x}).$$

(GCRC) + (PLV) $\Rightarrow$ (GZRC). Suppose that (GCRC) holds. Then,

$$\left(\bigcup_{t \in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s \subset \left(\bigcup_{j \in I \setminus \{i\}} \partial^T f_j(\bar{x})\right)^s \cap \left(\bigcup_{t \in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s \neq \emptyset.$$

It follows from (PLV) and the above relation that there exists d such that

$$d \in \left(\bigcup_{j \in I \setminus \{i\}} \partial^T f_j(\bar{x})\right)^s, \tag{4.1}$$

$$d \in \left(\bigcup_{t \in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s = \left(\mathrm{co} \bigcup_{t \in T(\bar{x})} \partial^T g_t(\bar{x})\right)^s \subset (\partial^T G(\bar{x}))^s.$$

Hence, $G'(\bar{x}, d) < 0$, and consequently, there exists $\delta > 0$ such that

$$G(\bar{x} + \tau d) < G(\bar{x}), \quad \forall \tau \in (0, \delta).$$

This ensures that, for all $t \in T$ and for all $\tau \in (0, \delta)$, one has $g_t(\bar{x} + \tau d) < 0$. Moreover, it follows from (4.1) that $\langle \partial^T f_j(\bar{x}), d \rangle < 0, \quad \forall j \in I \setminus \{i\}$, and hence, $f'_j(\bar{x}, d) < 0, \forall j \in I \setminus \{i\}$. This allows us to justify the existence of $\delta' > 0$ such that

$$f_j(\bar{x} + \tau d) < f_j(\bar{x}), \quad \forall \tau \in (0, \delta'),\ \forall j \in I \setminus \{i\}.$$

Setting $\bar{\delta} := \min\{\delta, \delta'\}$, one has, for all $\tau \in (0, \bar{\delta})$,

$$\begin{cases} g_t(\bar{x} + \tau d) < 0, & \forall t \in T, \\ f_j(\bar{x} + \tau d) < f_j(\bar{x}), & \forall j \in I \setminus \{i\}, \end{cases}$$

i.e., $\bar{x} + \tau d \in Q^i$ for all $\tau \in (0, \bar{\delta})$, or equivalently, $d \in D(Q^i, \bar{x})$. This verifies that, for all $i \in I$,

$$\left(\bigcup_{i=1}^{m} \partial^T f_i(\bar{x})\right)^- \subset \left(\bigcup_{j \in I \setminus \{i\}} \partial^T f_j(\bar{x})\right)^- = \mathrm{cl} \left(\bigcup_{j \in I \setminus \{i\}} \partial^T f_j(\bar{x})\right)^s \subset \mathrm{cl} D(Q^i, \bar{x}),$$

$$\left(\bigcup_{t\in T(\bar{x})}\partial^T g_t(\bar{x})\right)^- = \mathrm{cl}\left(\bigcup_{t\in T(\bar{x})}\partial^T g_t(\bar{x})\right)^s \subset \mathrm{cl}D(Q^i,\bar{x}).$$

So, (GZRC) holds.
(iv) Since $\mathrm{cl}D(Q^i,\bar{x}) \subset \mathrm{cl}F(Q^i,\bar{x}) \subset T(Q^i,\bar{x}) \subset \mathrm{cl}\ \mathrm{co}T(Q^i,\bar{x}), \forall \in I$, the conclusion is obtained. □

Remark 4.6. Characterizing properly efficient solutions and robust solutions, see *e.g.* [9–11, 30, 49], in terms of tangential subdifferential can be worth studying, as a future research direction. Notice that proper efficiency and robust efficiency are very important practically, because the former is concerning the boundedness of the trade-offs and the later refers to stability of the solution against changes in the problem data.

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