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ON AN M/G/1 QUEUE IN RANDOM ENVIRONMENT WITH $Min(N, V)$ POLICY

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Abstract. In this paper, we analyze an $M/G/1$ queue operating in multi-phase random environment with $Min(N, V)$ vacation policy. In operative phase i, $i = 1, 2, \dots, n$, customers are served according to the discipline of First Come First Served (FCFS). When the system becomes empty, the server takes a vacation under the $Min(N, V)$ policy, causing the system to move to vacation phase 0. At the end of a vacation, if the server finds no customer waiting, another vacation begins. Otherwise, the system jumps from the phase 0 to some operative phase i with probability q_i, $i = 1, 2, \dots, n$. And whenever the number of the waiting customers in the system reaches N, the server interrupts its vacation immediately and the system jumps from the phase 0 to some operative phase i with probability q_i, $i = 1, 2, \dots, n$, too. Using the method of supplementary variable, we derive the distribution for the stationary system size at arbitrary epoch. We also obtain mean system size, the results of the cycle analysis and the sojourn time distribution. In addition, some special cases and numerical examples are presented.

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1. Introduction

Queueing models with vacations have been well studied in the past three decades because of their applications in modeling many real life situations such as communication networks, computer systems, and manufacturing/service systems (see Fuhrmann and Cooper [8]). For more detail on vacation models readers are referred to the surveys of Doshi [6], the monographs of Takagi [24] and Tian and Zhang [25] as well as the references therein. A brief summary of the most recent research works on vacation queueing models was provided by Ke *et al.* [14].

Yadin and Naor [30] first introduced the N policy, which is characterized by the following conditions: (i) the server is turned off when there are no customers present; (ii) the server is turned on whenever $N (N \geq 1)$ or more customers are present; (iii) After the server is turned off, the server may not be turned on until the number of customers in the system reaches N. The N policy $M/G/1$ queue was first investigated by Heyman [9] and was considered by many researchers such as Bell [2], Tijms [26], Wang and Ke [27], Wang *et al.* [28], and so on. Ke and Wang [13] used the supplementary variable technique to analyze the N policy $G/M/1$ queueing system with finite capacity. Using the embedded Markov chain method, Zhang and Tian [32] analyzed the infinite buffer $GI/M/1$ queueing system with the N policy. The N policy $M/G/1$ queue with vacations was first

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studied by Kella [15]. Lee *et al.* [17] analyzed the batch arrival $M/G/1$ queue with N policy and single vacation using the method of supplementary variables. The N policy $M/G/1$ queueing system with server vacations, startup and breakdowns was investigated by Ke [12]. In [12, 15, 17], the queueing systems operate under the $Max(N,V)$ policy, where V denotes the length of a vacation time. The $Max(N,V)$ policy means that the server re-starts the service only if a vacation ends and the number of customers waiting for service reaches or exceeds a predefined threshold value N. Recently, Wu *et al.* [29] and Jing *et al.* [11] studied an $M/G/1$ vacation queue with $Min(N,V)$ policy. In [11, 29], if either condition of N- or V-policy is satisfied, then the server re-starts the service. Under the $Min(N,V)$ policy a better balance between quick response time and efficient use of resources can be most likely achieved. In fact, in a queueing system with vacations, the server can take some additional work during a vacation, that is, server vacations are helpful for efficient use of resources. And if the queueing system operates under the $Min(N,V)$ vacation policy, once the number of the waiting customers in the system reaches N or the system is not empty after a vacation ends, whichever happens first, the server re-starts the service immediately. So, the $Min(N,V)$ vacation policy is also beneficial to quick response to customer service.

Due to the applications in many fields such as manufacturing systems, communication networks, the queues operating in random environment have increasingly received attention from researchers. Random environment is an external stochastic process on a finite phase (state) space. When the environment resides in some fixed phase, the queue operates as a classic queue of the corresponding type. But, the queueing parameters may change with the change of the phase of the random environment. That is, different from the classic queueing system, the queueing systems operating in random environment are heterogeneous. The first systematic work on queueing systems operating in random environment is due to Yechiali and Naor [31]. They studied an $M/M/1$ queue in a two-phase random environment, in which the arrival and service rates were heterogeneous for different phase. Neuts [21] extended their work to the $M/G/1$ case. Several authors have contributed to the investigation of the $M/G/1$ queue operating in a two-phase random environment, including Sengupta [22], Boxma and Kurkova [4], Huang and Lee [10]. Cordeiro and Kharoufeh [5] considered an $M/M/1$ retrial queue whose operating mechanism is governed by a multi-phase random environment. The $M/M/\infty$ queue operating in a semi-Markovian environment was studied by Falin [7]. Blom *et al.* [3] analyzed the $M/G/\infty$ queue operating in a Markov-modulated environment. Baykal-Gursoy *et al.* [1] investigated the $M/M/C$ queue operating in a two-phase Markovian environment. The $M/M/C$ queue operating in a multi-phase Markovian environment was studied by Liu and Yu [20]. Kim and Kim [16] considered a single server queue with varying service rates and impatient customers, in which the service rate varied according to a Markov random environment. Most recently, an $M/M/1$ queue in a multi-phase random environment with vacations was investigated by Li and Liu [18]. In their model, when vacation phase ends, the system moves to some operative phase with a certain probability. Li *et al.* [19] generalized Li and Liu's work to an $M/G/1$ queue.

To the best of our knowledge, the queueing systems operating in multi-phase random environment with $Min(N,V)$ vacation policy have not been investigated. However, these queueing systems can be utilized to many practical applications like manufacturing system. For example, the operator (server) has a vacation period when there is no raw material (customer) to be processed. When the server returns from vacation and finds raw materials existing, he/she serves them immediately; otherwise, he/she takes another vacation. Moreover, whenever the amount of raw material reaches N, the server interrupts its vacation and re-starts the service. But the new service rate may be different from the previous service rate. The new service rate is affected by the new arrival rate, environmental conditions and operator experience. This motivates us to investigate the $M/G/1$ queue operating in multi-phase random environment with $Min(N,V)$ vacation policy.

The remainder of this paper is organized as follows. Section 2 provides the model description. In Section 3, the probability generating function (PGF) of the stationary system size distribution at an arbitrary epoch is derived. Section 4 is devoted to obtain various important performance measures of this model. Some special cases and numerical results are given in Sections 5 and 6. Finally, Section 7 concludes the paper.

2. Model description

In this paper, we study an $M/G/1$ queueing system operating in a multi-phase random environment with $Min(N,V)$ vacation policy. In operative phase i, $i=1,2,\dots,n$, the arrivals occur according to a Poisson process with rate λ_i, and the service times of the customers in the system are independent and identically distributed random variables having a general distribution function $B_i(v)$ and a finite mean $1/\mu_i$. Customers are served according to the discipline of FCFS. When the system becomes empty, the server begins a vacation of random length V, causing the system to jump to vacation phase 0. The duration of a vacation is an exponential random variable with rate θ. In phase 0, the arrivals occur according to a Poisson process with rate λ_0, and the server stops processing customers. At the end of a vacation, if the server finds no customer waiting, another vacation is taken. This repeats until there are arrivals during a vacation. Whenever the number of waiting customers in the system reaches N, the server interrupts its vacation immediately and the system jumps from the phase 0 to some operative phase i with probability q_i, $i=1,2,\dots,n$, where $q_i>0$, $\sum_{i=1}^{n} q_i=1$, and the server operates until the system is empty. If there are arrivals during a vacation, but not reach N, after the vacation ends, the system also jumps from the phase 0 to some operative phase i with probability q_i, $i=1,2,\dots,n$, and the server operates until the system is empty. That is, the system operates under the $Min(N,V)$ vacation policy. Moreover, The inter-arrival times, the service times and the vacation times are assumed to be mutually independent in our model.

We describe the system by the process $\{(L(t),J(t),B^-(t)),t\geq 0\}$, where $L(t)$ represents the number of customers in the system at time t, $J(t)$ is the phase in which the system operates at time t, and if $J(t)=i$, $i=1,2,\dots,n$, then $B^-(t)=B_i^-(t)$ denotes the elapsed service time of the customer being served during phase i. It is obvious that the process $\{(L(t),J(t),B^-(t)),t\geq 0\}$ is a Markov process with the state space expressed as

$$\Omega=\{(k,0),0\leq k\leq N-1\}\cup\{(k,i,x),k\geq 1,x\geq 0,i=1,2,\dots,n\}.$$

3. Steady-state distribution

It is readily seen that the system we consider operates as the classic $M/G/1$ queueing system with Poisson arrival rate λ_i and service rate μ_i during phase $i,i=1,2,\dots,n$. And it is well known that the necessary and sufficient condition for the stability of the classic $M/G/1$ queueing system with arrival rate λ and service rate μ is $\rho=\lambda/\mu<1$. So as long as $\rho_i=\lambda_i/\mu_i<1$, $i=1,2,\dots,n$, the system that we consider is stable. That is, $\rho_i=\lambda_i/\mu_i<1$, $i=1,2,\dots,n$, is the sufficient condition for the stability of the system that we consider. Next, we use the reduction to absurdity to prove the necessary condition. Assume that the system is stable. If there is an $i,i=1,2,\dots,n$, $\rho_i=\lambda_i/\mu_i\geq 1$, then, when the system moves to phase i, the system that we consider, which operates as the classic $M/G/1$ queueing system with Poisson arrival rate λ_i and service rate μ_i during phase i, will not be stable. Therefore, $\rho_i=\lambda_i/\mu_i<1$, $i=1,2,\dots,n$, is also the necessary condition for the stability of the system that we consider. Assume that the system is in a stability condition. Now, we analyze the stationary distribution of the system size.

We define the hazard rate functions of service time during phase $i,i=1,2,\dots,n$, as follows

$$\mu_i(x)\mathrm{d}x=\frac{\mathrm{d}B_i(x)}{1-B_i(x)}, \tag{3.1}$$

then,

$$B_i(v)=1-\exp\left\{-\int_0^v \mu_i(x)\mathrm{d}x\right\}. \tag{3.2}$$

And we let

$$P_{k,0} = \lim_{t\to\infty} P\{L(t) = k, J(t) = 0\}, \quad 0 \le k \le N-1,$$

$$P_{k,i}(x)\mathrm{d}x = \lim_{t\to\infty} P\{L(t) = k, J(t) = i, x < B_i^-(t) \le x + \mathrm{d}x\}, \quad k \ge 1, x \ge 0, i = 1, 2, \dots, n.$$

The Kolmogorov equations for the system size distribution can be written by

$$\lambda_0 P_{0,0} = \sum_{i=1}^{n} \int_0^\infty P_{1,i}(x)\mu_i(x)\mathrm{d}x, \tag{3.3}$$

$$(\lambda_0 + \theta)P_{k,0} = \lambda_0 P_{k-1,0}, \;\; 1 \le k \le N-1, \tag{3.4}$$

$$\frac{\mathrm{d}P_{k,i}(x)}{\mathrm{d}x} + [\lambda_i + \mu_i(x)]P_{k,i}(x) = \lambda_i(1-\delta_{k,1})P_{k-1,i}(x), k \ge 1, \tag{3.5}$$

where $\delta_{k,1}$ is the Kronecker symbol.

At $x = 0$, the boundary conditions are

$$P_{k,i}(0) = \int_0^\infty P_{k+1,i}(x)\mu_i(x)\mathrm{d}x + q_i\theta P_{k,0}, \quad 1 \le k \le N-1, i = 1, 2, \dots, n, \tag{3.6}$$

$$P_{N,i}(0) = \int_0^\infty P_{N+1,i}(x)\mu_i(x)\mathrm{d}x + q_i\lambda_0 P_{N-1,0}, \quad k = N, i = 1, 2, \dots, n, \tag{3.7}$$

$$P_{k,i}(0) = \int_0^\infty P_{k+1,i}(x)\mu_i(x)\mathrm{d}x, \quad k \ge N+1, i = 1, 2, \dots, n. \tag{3.8}$$

Finally we have the normalizing condition as follows:

$$P_{0,0} + \sum_{k=1}^{N-1} P_{k,0} + \sum_{i=1}^{n}\sum_{k=1}^{\infty}\int_0^\infty P_{k,i}(x)\mathrm{d}x = 1. \tag{3.9}$$

To solve equations (3.3)–(3.9), we introduce the probability generating functions as follows:

$$P_0(z) = \sum_{k=0}^{N-1} P_{k,0}z^k,$$

$$P_i(x,z) = \sum_{k=1}^{\infty} P_{k,i}(x)z^k, \quad i = 1, 2, \dots, n,$$

$$P_i(z) = \int_0^\infty P_i(x,z)\mathrm{d}x, \quad i = 1,2,\dots,n.$$

From (3.4), we have

$$P_{k,0} = \left(\frac{\lambda_0}{\lambda_0+\theta}\right)^k P_{0,0}, \quad 1 \le k \le N-1. \tag{3.10}$$

Then, we obtain

$$P_0(z) = \sum_{k=0}^{N-1} P_{k,0}z^k = \frac{(\lambda_0+\theta)^N - (\lambda_0 z)^N}{(\lambda_0+\theta)^{N-1}[\lambda_0(1-z)+\theta]}P_{0,0}. \tag{3.11}$$

Multiplying (3.5)–(3.8) by z^k and summing over k, we get

$$\frac{\mathrm{d}P_i(x,z)}{\mathrm{d}x} = -[\lambda_i(1-z)+\mu_i(x)]P_i(x,z), \quad i = 1,2,\dots,n, \tag{3.12}$$

$$zP_i(0,z) = \int_0^\infty P_i(x,z)\mu_i(x)\mathrm{d}x - z\int_0^\infty P_{1,i}(x)\mu_i(x)\mathrm{d}x + q_i\theta z[P_0(z)-P_{0,0}] + q_i\lambda_0 P_{N-1,0}z^{N+1}. \tag{3.13}$$

Solving differential equation (3.12), we have

$$P_i(x,z) = P_i(0,z)\exp\{-\lambda_i(1-z)x\}[1-B_i(x)]. \tag{3.14}$$

Substituting (3.14) into (3.13), it yields

$$[z - B_i^*(\lambda_i(1-z))]P_i(0,z) = q_i\theta z[P_0(z)-P_{0,0}] + q_i\lambda_0 P_{N-1,0}z^{N+1} - z\int_0^\infty P_{1,i}(x)\mu_i(x)\mathrm{d}x. \tag{3.15}$$

Setting $z=1$ in (3.15), we get

$$\int_0^\infty P_{1,i}(x)\mu_i(x)\mathrm{d}x = q_i\theta[P_0(1)-P_{0,0}] + q_i\lambda_0 P_{N-1,0}. \tag{3.16}$$

Substituting (3.16) into (3.15), we obtain

$$P_i(0,z) = \frac{q_i\theta z[P_0(z)-P_0(1)] - q_i\lambda_0 P_{N-1,0}z(1-z^N)}{z - B_i^*(\lambda_i(1-z))}. \tag{3.17}$$

Taking (3.17) into (3.14), we get

$$P_i(x,z) = \frac{q_i\theta z[P_0(z)-P_0(1)] - q_i\lambda_0 P_{N-1,0}z(1-z^N)}{z - B_i^*(\lambda_i(1-z))} exp\{-\lambda_i(1-z)x\}[1-B_i(x)]. \tag{3.18}$$

Let $B_i^*(s)$ be the Laplace-Stieltjes transform (LST) of service time distribution $B_i(v)$. Using the following result

$$\int_0^\infty e^{-sx}[1-B_i(x)]\mathrm{d}x = \frac{1-B_i^*(s)}{s},$$

and integrating (3.18) with respect to x, we obtain

$$P_i(z) = \frac{\{q_i\theta z[P_0(z)-P_0(1)] - q_i\lambda_0 P_{N-1,0} z(1-z^N)\}[1-B_i^*(\lambda_i(1-z))]}{[z-B_i^*(\lambda_i(1-z))]\lambda_i(1-z)}. \tag{3.19}$$

Inserting $z=1$ in (3.11), we get

$$P_0(1) = \frac{(\lambda_0+\theta)^N - \lambda_0{}^N}{(\lambda_0+\theta)^{N-1}\theta} P_{0,0}. \tag{3.20}$$

Using (3.10), (3.11), (3.19), and (3.20), we finally obtain

$$P_i(z) = \frac{q_i\lambda_0 P_{0,0} z\left[(\lambda_0+\theta)^N - (\lambda_0 z)^N\right][1-B_i^*(\lambda_i(1-z))]}{(\lambda_0+\theta)^{N-1}\lambda_i[\lambda_0(z-1)-\theta]\left[z-B_i^*(\lambda_i(1-z))\right]}. \tag{3.21}$$

Putting

$$\mathcal{N}(z) = z[(\lambda_0+\theta)^N - (\lambda_0 z)^N][1-B_i^*(\lambda_i(1-z))],$$

$$\mathcal{D}(z) = [\lambda_0(z-1)-\theta][z-B_i^*(\lambda_i(1-z))].$$

Using L'Hospital's rule, we have

$$P_i(1) = \lim_{z\to 1} P_i(z) = \frac{q_i\lambda_0 P_{0,0}}{(\lambda_0+\theta)^{N-1}\lambda_i}\frac{\mathcal{N}'(1)}{\mathcal{D}'(1)}$$

$$= \frac{q_i\lambda_0[(\lambda_0+\theta)^N - \lambda_0^N]P_{0,0}}{(\lambda_0+\theta)^{N-1}\theta(\mu_i-\lambda_i)}. \tag{3.22}$$

We apply the normalization condition (3.9), *i.e.*,

$$P_0(1) + \sum_{i=1}^n P_i(1) = 1,$$

combining (3.20) and (3.22), then we get

$$P_{0,0} = \frac{1}{\alpha\lambda_0}, \tag{3.23}$$

where

$$\alpha = \left[\frac{\lambda_0+\theta}{\lambda_0\theta} + \sum_{i=1}^n \frac{q_i(\lambda_0+\theta)}{\theta(\mu_i-\lambda_i)}\right]\left[1-\left(\frac{\lambda_0}{\lambda_0+\theta}\right)^N\right]. \tag{3.24}$$

We summarize the above results in the following theorem.

Theorem 3.1. *If $\rho_i = \lambda_i/\mu_i < 1$, $i = 1, 2, \ldots, n$, holds, the PGF of the stationary distribution of system size is given by*

$$P(z) = P_0(z) + \sum_{i=1}^{n} P_i(z)$$

$$= \left\{ \frac{(\lambda_0+\theta)^N - (\lambda_0 z)^N}{(\lambda_0+\theta)^{N-1}\left[\lambda_0(1-z)+\theta\right]} + \frac{\left[(\lambda_0+\theta)^N - (\lambda_0 z)^N\right]\lambda_0 z}{(\lambda_0+\theta)^{N-1}\left[\lambda_0(z-1)-\theta\right]} \sum_{i=1}^{n} \frac{q_i\left[1-B_i^*(\lambda_i(1-z))\right]}{\lambda_i\left[z-B_i^*(\lambda_i(1-z))\right]} \right\} P_{0,0}, \quad (3.25)$$

where

$$P_{0,0} = \frac{1}{\alpha\lambda_0}, \alpha = \left[\frac{\lambda_0+\theta}{\lambda_0\theta} + \sum_{i=1}^{n} \frac{q_i(\lambda_0+\theta)}{\theta(\mu_i-\lambda_i)}\right]\left[1-\left(\frac{\lambda_0}{\lambda_0+\theta}\right)^N\right]. \quad (3.26)$$

4. Performance measures

In the following we present some important performance measures.

4.1. Mean system size

Let $E(L)$ denote the mean system size. Differentiating equation (3.25) with respect to z, and then setting $z = 1$, we note that the numerator and denominator are both 0. We apply L'Hospital's rule twice and finally the mean system size is given by

$$E(L) = \frac{dP(z)}{dz}|_{z=1} = \frac{dP_0(z)}{dz}|_{z=1} + \sum_{i=1}^{n} \frac{\mathrm{d}P_i(z)}{dz}|_{z=1}$$

$$= \frac{[(\lambda_0+\theta)^N - \lambda_0^N]\lambda_0 - N\theta\lambda_0^N}{(\lambda_0+\theta)^{N-1}\theta^2} P_{0,0}$$

$$+ \frac{\lambda_0 P_{0,0}}{(\lambda_0+\theta)^{N-1}} \sum_{i=1}^{n} \frac{q_i[\mathcal{N}''(1)\mathcal{D}'(1) - \mathcal{N}'(1)\mathcal{D}''(1)]}{\lambda_i[2(\mathcal{D}'(1))^2]}, \quad (4.1)$$

where

$$\mathcal{N}'(1) = -\frac{\lambda_i}{\mu_i}\left[(\lambda_0+\theta)^N - \lambda_0^N\right],$$

$$\mathcal{N}''(1) = -\left[2\frac{\lambda_i}{\mu_i} + \lambda_i^2\beta_i^{(2)}\right]\left[(\lambda_0+\theta)^N - \lambda_0^N\right] + 2N\frac{\lambda_i}{\mu_i}\lambda_0^N,$$

$$\mathcal{D}'(1) = -\theta\left(1 - \frac{\lambda_i}{\mu_i}\right),$$

$$\mathcal{D}''(1) = 2\lambda_0\left(1 - \frac{\lambda_i}{\mu_i}\right) + \theta\lambda_i^2\beta_i^{(2)},$$

and $\beta_i^{(2)}$ denotes the 2nd moment of the service time distribution. After some algebraic manipulation, the mean system size can be written as

$$E(L) = \frac{[(\lambda_0+\theta)^N - \lambda_0^N] - N\theta\lambda_0^{N-1}}{\alpha(\lambda_0+\theta)^{N-1}\theta^2} + \sum_{i=1}^{n} \frac{q_i\left\{[(\lambda_0+\theta)^N - \lambda_0^N][2(\lambda_0+\theta)(\mu_i-\lambda_i)+\theta\lambda_i\mu_i^2\beta_i^{(2)}] - 2N\theta\lambda_0^N(\mu_i-\lambda_i)\right\}}{2\alpha(\lambda_0+\theta)^{N-1}\theta^2(\mu_i-\lambda_i)^2}. \tag{4.2}$$

4.2. Number of customers in the system at the end of phase 0

Let Q_0 be the number of customers in the system at the end of phase 0. The distribution of Q_0 will be presented.

Denote by $A_l, l = 1, 2, 3, \ldots$, the inter-arrive times during a vacation. Note that $A_l, l = 1, 2, 3, \ldots$, are independent and identically exponentially distributed random variables with a rate of λ_0. Then the probability that at least j customers arrive during a vacation, denoted by $P_j(V)$, is given by

$$\begin{aligned} P_j(V) = P\{\sum_{l=1}^{j} A_j < V\} &= \int_0^\infty e^{-\theta t}\frac{\lambda_0(\lambda_0 t)^{j-1}}{(j-1)!}e^{-\lambda_0 t}\mathrm{d}t \\ &= \int_0^\infty \frac{\lambda_0^j(\lambda_0+\theta)[(\lambda_0+\theta)t)]^{j-1}}{(\lambda_0+\theta)^j(j-1)!}e^{-(\lambda_0+\theta)t}\mathrm{d}t \\ &= \left(\frac{\lambda_0}{\lambda_0+\theta}\right)^j, \quad j \geq 0. \end{aligned} \tag{4.3}$$

In particular, the probability that at least one customer arrives during a vacation is $\lambda_0/(\lambda_0+\theta)$. That is, the probability that no customer arrives during a vacation is $\theta/(\lambda_0+\theta)$.

By (4.3), the probability that j customers arrive during a vacation, denoted by a_j, can be written as

$$a_j = P_j(V) - P_{j+1}(V), \qquad j \geq 0. \tag{4.4}$$

With the help of (4.3) and (4.4), we obtain the distribution of Q_0 as follows:

$$\begin{aligned} P\{Q_0 = j\} &= \sum_{k=1}^{\infty}\left(\frac{\theta}{\lambda_0+\theta}\right)^{k-1} a_j = \frac{\lambda_0+\theta}{\lambda_0}[P_j(V) - P_{j+1}(V)] \\ &= \left(\frac{\lambda_0}{\lambda_0+\theta}\right)^{j-1}\frac{\theta}{\lambda_0+\theta}, \quad 1 \leq j \leq N-1; \\ P\{Q_0 = N\} &= \sum_{k=1}^{\infty}\left(\frac{\theta}{\lambda_0+\theta}\right)^{k-1} P_N(V) = \left(\frac{\lambda_0}{\lambda_0+\theta}\right)^{N-1}. \end{aligned} \tag{4.5}$$

Then, the mean number of customers in the system at the end of phase 0 is given by

$$E(Q_0) = \sum_{j=1}^{N-1} j\left(\frac{\lambda_0}{\lambda_0+\theta}\right)^{j-1}\left(\frac{\theta}{\lambda_0+\theta}\right) + N\left(\frac{\lambda_0}{\lambda_0+\theta}\right)^{N-1}$$
$$= \frac{\lambda_0+\theta}{\theta}\left[1-\left(\frac{\lambda_0}{\lambda_0+\theta}\right)^N\right]. \tag{4.6}$$

4.3. Cycle analysis

A cycle is defined as the time interval between two successive instants at which the phase i commences, $i = 0, 1, 2, \ldots, n$. Then there are $(n + 1)$ types of cycles. We denote by C_i the length of type-i cycle.

In the type-0 cycle, the system visits vacation phase 0 and some operative phase i one time, $i = 1, 2, \ldots, n$, respectively. By (4.6), the mean time that the system resides in phase 0 during type-0 cycle, denoted by T_0, is given by

$$T_0 = \frac{1}{\lambda_0}E(Q_0) = \frac{\lambda_0+\theta}{\lambda_0\theta}\left[1-\left(\frac{\lambda_0}{\lambda_0+\theta}\right)^N\right], \tag{4.7}$$

which is the product of the mean number of customers in the system at the end of phase 0 and the mean time of one customer arrival in phase 0.

If phase 0 ends, and the system moves into phase i with probability q_i, $i = 1, 2, \ldots, n$, then the mean time that the system resides in phase i during a cycle, denoted by T_i, is given by

$$T_i = E(Q_0)\frac{1}{\mu_i-\lambda_i} = \frac{\lambda_0+\theta}{\theta(\mu_i-\lambda_i)}\left[1-\left(\frac{\lambda_0}{\lambda_0+\theta}\right)^N\right], \tag{4.8}$$

where $1/(\mu_i - \lambda_i)$ is the mean busy period that one customer induces in phase i.

Combining (4.7) and (4.8), we get

$$E(C_0) = T_0 + \sum_{i=1}^{n} q_iT_i = \alpha. \tag{4.9}$$

Then, the probability that the system resides in state $(0, 0)$ can be also obtained by

$$P_{0,0} = \frac{1/\lambda_0}{E(C_0)} = \frac{1}{\alpha\lambda_0},$$

which is the proportion of time that the system resides in sate $(0, 0)$ during the type-0 cycle. Similarly, the probability that the system resides in phase 0 can be obtained by

$$P_0(1) = \frac{T_0}{E(C_0)} = \frac{(\lambda_0+\theta)^N-\lambda_0^N}{(\lambda_0+\theta)^{N-1}\theta\lambda_0\alpha}.$$

Next, we derive the mean length of the type-i cycle, $i = 1, 2, \ldots, n$.

In the type-i cycle, the times of the system visiting phase 0 is a random variable having a geometric distribution with parameter q_i, and if the system visits phase 0 k times, the mean length of type-i cycle is given by

$$T_i + T_0 + (k-1)\left[T_0 + \sum_{j=1,j\neq i}^{n} \frac{q_j}{1-q_i} T_j\right].$$

Therefore, for $i = 1, 2, \ldots, n$, we obtain

$$\begin{aligned} E(C_i) &= \sum_{k=1}^{\infty} (1-q_i)^{k-1} q_i \left[T_i + T_0 + (k-1)\left(T_0 + \sum_{j=1,j\neq i}^{n} \frac{q_j}{1-q_i} T_j\right)\right] \\ &= T_i + T_0 + \sum_{k=1}^{\infty} (1-q_i)^{k-1} q_i (k-1)\left[T_0 + \frac{1}{1-q_i}(\alpha - T_0 - q_i T_i)\right] \\ &= T_i + T_0 + \frac{1-q_i}{q_i}\left[T_0 + \frac{1}{1-q_i}(\alpha - T_0 - q_i T_i)\right] \\ &= \frac{\alpha}{q_i}. \end{aligned} \tag{4.10}$$

Then, the probability that the system resides in phase i can be obtained by

$$P_i(1) = \frac{T_i}{E(C_i)} = \frac{q_i[(\lambda_0+\theta)^N - \lambda_0^N]}{(\lambda_0+\theta)^{N-1}(\mu_i - \lambda_i)\theta\alpha},$$

which is the proportion of time that the system resides in phase i during the type-i cycle.

4.4. Sojourn time distribution

Let W and $W^*(s)$ respectively denote the stationary sojourn time of an arbitrary customer and its LST, and Let $W_{k,0}$ and $W^*_{k,0}(s)$ respectively denote the stationary sojourn time of a customer that arrives in state $(k, 0)$ and its LST. The stationary sojourn time of a customer that arrives in the state $(k, i, x), k \geq 1, x \geq 0, i = 1, 2, \ldots, n$, is denoted by $W_{k,i,x}$, with its LST $W^*_{k,i,x}(s)$.

When a customer arrives in state $(k, i, x), k = 1, 2, \ldots, x \geq 0, i = 1, 2, \ldots, n$, the sojourn time until departure consists of (i) the remaining service time; (ii) the service time of the k customers. Thus, we have

$$W^*_{k,i,x}(s) = E(e^{-sW_{k,i,x}}) = B_i^{+*}(s)(B_i^*(s))^k. \tag{4.11}$$

where $B_i^{+*}(s) = \frac{\mu_i[1-B_i^*(s)]}{s}$ is the LST of the remaining service time distribution of the customer being served.

Let $Q_{k,0}$ denote the number of customers in the system at the end of phase 0 conditioned on a customer arriving in state $(k, 0), k = 0, 1, \ldots, N-1$. Next, we present the distribution of $Q_{k,0}$. Note that the remaining vacation time is stochastically equal to a new vacation time due to the memoryless property. Then, for $k =$

$0,1,\ldots,N-2$, we have

$$\begin{aligned}
P(Q_{k,0}=j) &= \frac{P_{j-k-1}(V)-P_{j-k}(V)}{P_{k+1}(V)} \\
&= \frac{\theta}{\lambda_0+\theta}\left(\frac{\lambda_0}{\lambda_0+\theta}\right)^{j-2(k+1)}, \qquad k+1\le j\le N-1; \\
P(Q_{k,0}=N) &= \frac{P_{N-k-1}(V)}{P_{k+1}(V)} = \left(\frac{\lambda_0}{\lambda_0+\theta}\right)^{N-2(k+1)}. \qquad (4.12)
\end{aligned}$$

And for $k=N-1$, we have

$$P(Q_{k,0}=N) \quad = \quad 1. \qquad (4.13)$$

Thus, when a customer arrives in state $(k,0), k=0,1,\ldots,N-2$, the sojourn time until departure is given by

$$\begin{aligned}
W^*_{k,0}(s) &= \sum_{j=k+1}^{N-1} P(Q_{k,0}=j)\sum_{i=1}^{n} q_i[A^*(s)]^{j-k-1}[B_i^*(s)]^j \\
&\quad + P(Q_{k,0}=N)\sum_{i=1}^{n} q_i[A^*(s)]^{N-k-1}[B_i^*(s)]^N \\
&= \sum_{j=k+1}^{N-1}\frac{\theta}{\lambda_0+\theta}\left(\frac{\lambda_0}{\lambda_0+\theta}\right)^{j-2(k+1)}\sum_{i=1}^{n} q_i\left[\frac{\lambda_0}{\lambda_0+s}\right]^{j-k-1}[B_i^*(s)]^j \\
&\quad + \left(\frac{\lambda_0}{\lambda_0+\theta}\right)^{N-2(k+1)}\sum_{i=1}^{n} q_i\left[\frac{\lambda_0}{\lambda_0+s}\right]^{N-k-1}[B_i^*(s)]^N \\
&= \sum_{j=k+1}^{N}\left(\frac{\delta_{j,N}\lambda_0}{\lambda_0+\theta}+\frac{\theta}{\lambda_0+\theta}\right)\left(\frac{\lambda_0}{\lambda_0+\theta}\right)^{j-2(k+1)}\sum_{i=1}^{n} q_i\left[\frac{\lambda_0}{\lambda_0+s}\right]^{j-k-1}[B_i^*(s)]^j, \qquad (4.14)
\end{aligned}$$

where $\delta_{j,N}$ is the Kronecker's delta, and $A^*(s)=\frac{\lambda_0}{\lambda_0+s}$ is the LST of the inter-arrival time during phase 0. When a customer arrives in state $(N-1,0)$, the sojourn time until departure is given by

$$W^*_{N-1,0} = \sum_{i=1}^{n} q_i\left[B_i^*(s)\right]^N. \qquad (4.15)$$

Combining (4.11), (4.14) and (4.15), the LST of the sojourn time distribution of an arbitrary customer is given by

$$W^*(s) = \sum_{k=0}^{N-2} P_{k,0}W^*_{k,0}(s) + P_{N-1,0}W^*_{N-1,0}(s) + \sum_{i=1}^{n}\sum_{k=1}^{\infty}\int_0^\infty P_{k,i}(x)W^*_{k,i,x}(s)\mathrm{d}x$$
$$= \sum_{k=0}^{N-1} P_{k,0}W^*_{k,0}(s) + \sum_{i=1}^{n} B_i^{+*}(s)P_i(B_i^*(s)). \tag{4.16}$$

5. Special cases

In this section, we show that some vacation models are special cases of our model.

5.1. The $M/G/1$ queue with $Min(N,V)$ vacation policy

If the system is homogeneous, that is, $\lambda_i = \lambda$, and $\mu_i = \mu$, the system becomes the $M/G/1$ queue with $Min(N,V)$ vacation policy. Using (3.25) and (3.26), The PGF of the system size distribution in the $M/G/1$ queue with $Min(N,V)$ vacation policy is given by

$$P(z) = \frac{[(\lambda+\theta)^N - (\lambda z)^N]\theta}{[\lambda(1-z)+\theta][(\lambda+\theta)^N - \lambda^N]}\frac{(1-\rho)(1-z)B^*(\lambda(1-z))}{B^*(\lambda(1-z))-z}, \tag{5.1}$$

where $\rho = \lambda/\mu$, $B^*(s)$ is the LST of the service time distribution. From (4.2), the mean system size is

$$E(L) = \rho + \frac{\lambda^2\beta^{(2)}}{2(1-\rho)} + \frac{\lambda}{\theta} - N\left(\frac{\lambda}{\lambda+\theta}\right)^N \Bigg/ \left[1-\left(\frac{\lambda}{\lambda+\theta}\right)^N\right], \tag{5.2}$$

where $\beta^{(2)}$ is the 2nd moment of the service time distribution. All the above results are in agreement with the results of $M/G/1$ queue with $Min(N,V)$ vacation policy reported in [11].

5.2. The $M/G/1$ queue with classical multiple vacations

By letting $N \to \infty$ in (5.1) and (5.2) we obtain

$$P(z) = \frac{\theta}{\lambda(1-z)+\theta}\frac{(1-\rho)(1-z)B^*(\lambda(1-z))}{B^*(\lambda(1-z))-z}, \tag{5.3}$$

and

$$E(L) = \rho + \frac{\lambda}{\theta} + \frac{\lambda^2\beta^{(2)}}{2(1-\rho)}, \tag{5.4}$$

which are the PGF of the system size distribution and mean system size in the $M/G/1$ queue with classical multiple vacations, respectively. The above results coincide with those in Shanthikumar [23].

5.3. The $M/G/1$ queue in multi-phase random environment with classical multiple vacations

Letting $N \to \infty$ in our model introduced in Section 2, the system translates into the $M/G/1$ queue operating in multi-phase random environment with classical multiple vacations. Then, from (3.25) and (3.26), the PGF

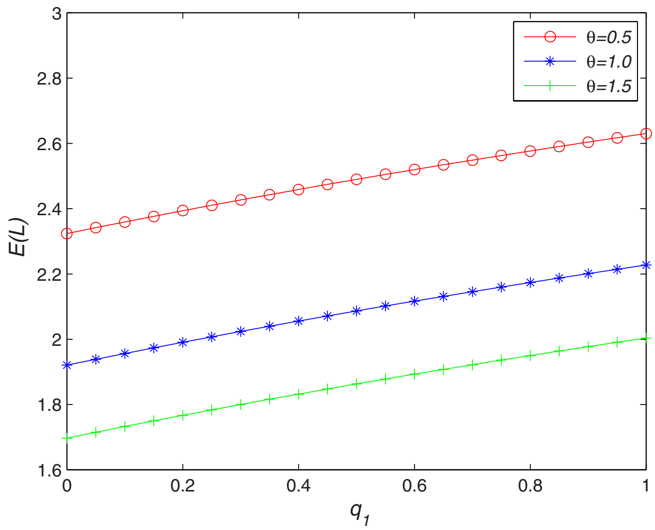

FIGURE 1. Impact of q_1 on $E(L)$.

of the system size in the $M/G/1$ queue operating in multi-phase random environment with classical multiple vacations is given by

$$P(z) = P_0(z) + \sum_{i=1}^{n} P_i(z)$$
$$= \left\{ \frac{\lambda_0 + \theta}{\lambda_0(1-z) + \theta} + \frac{(\lambda_0 + \theta)\lambda_0 z}{\lambda_0(z-1) - \theta} \sum_{i=1}^{n} \frac{q_i[1 - B_i^*(\lambda_i(1-z))]}{\lambda_i[z - B_i^*(\lambda_i(1-z))]} \right\} P_{0,0}, \tag{5.5}$$

where

$$P_{0,0} = \frac{1}{\alpha\lambda_0}, \quad \alpha = \frac{\lambda_0 + \theta}{\lambda_0 \theta} + \sum_{i=1}^{n} \frac{q_i(\lambda_0 + \theta)}{\theta(\mu_i - \lambda_i)}. \tag{5.6}$$

From (4.2), the mean system size in the $M/G/1$ queue operating in multi-phase random environment with classical multiple vacations is given by

$$E(L) = \frac{\lambda_0 + \theta}{\alpha\theta^2} + \frac{\lambda_0 + \theta}{\alpha\theta^2} \sum_{i=1}^{n} \frac{q_i[2(\lambda_0 + \theta)(\mu_i - \lambda_i) + \theta\lambda_i\mu_i^2\beta_i^{(2)}]}{2(\mu_i - \lambda_i)^2}. \tag{5.7}$$

The above results coincide with those in Li *et al.* [19].

6. Numerical illustration

In this section, we present some numerical experiments to explore the effect of the model parameters on the main performance measures. We consider the $M/M/1$ queue operating in 3-phase random environment with $Min(N, V)$ vacation policy, that is, the service time distribution is exponential, and the system has two operative phases and one vacation phase. We choose $\lambda_0 = 1, \lambda_1 = 1.2, \lambda_2 = 1.4, \mu_1 = 2, \mu_2 = 2.5, q_1 = 0.4, q_2 =$

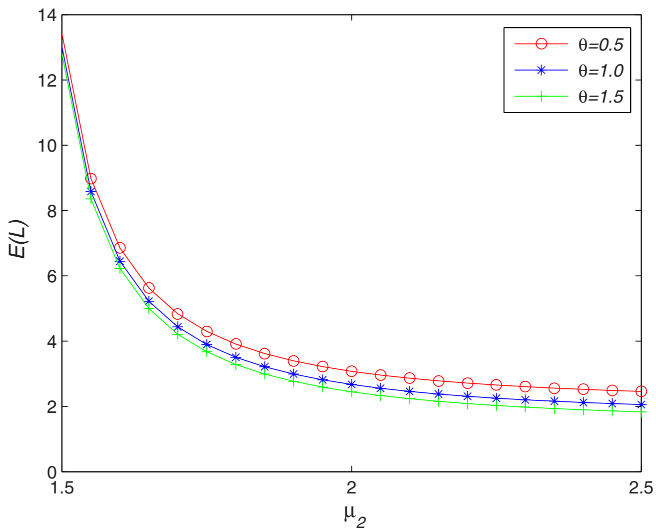

FIGURE 2. Impact of μ_2 on $E(L)$.

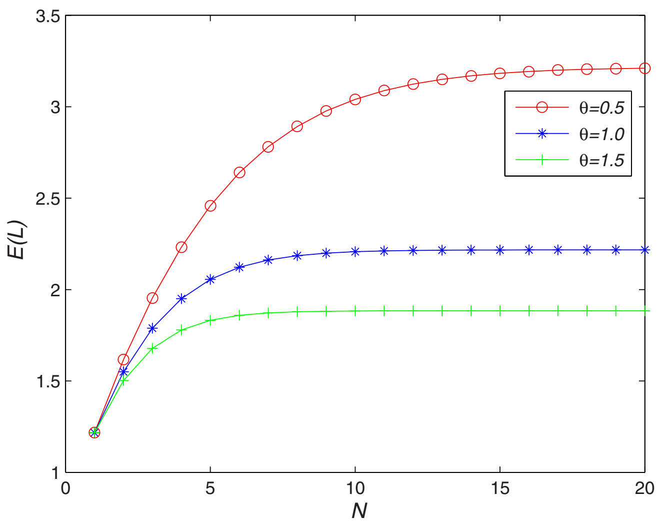

FIGURE 3. Impact of N on $E(L)$.

$0.6, N = 5,$ and $\theta = 0.5, 1.0, 1.5$, unless they are considered as variables or their values are given in the respective figures.

Figure 1 demonstrates the effect of q_1 on the mean system size $E(L)$ for different vacation rate θ. As to be expected, the mean system size $E(L)$ increases with increasing the value of q_1 for any θ. Further, the mean system size $E(L)$ decreases with the increase of θ for a fixed q_1. This is because the mean time of vacation decreases as θ increases, and there are fewer customers in the system at the end of phase 0.

The effect of μ_2 on $E(L)$ for different values of θ is shown in Figure 2. It can be observed that the mean system size $E(L)$ decreases with the increase of μ_2 for any θ, which agrees with intuitive expectations. On the other hand, if μ_2 is fixed, the mean system size $E(L)$ decreases as θ increases.

Figure 3 describes the behavior of the mean system size with varying values of N. As it can be seen from Figure 3, for any θ, the mean system size $E(L)$ increases with the increase of N, and when N is relatively

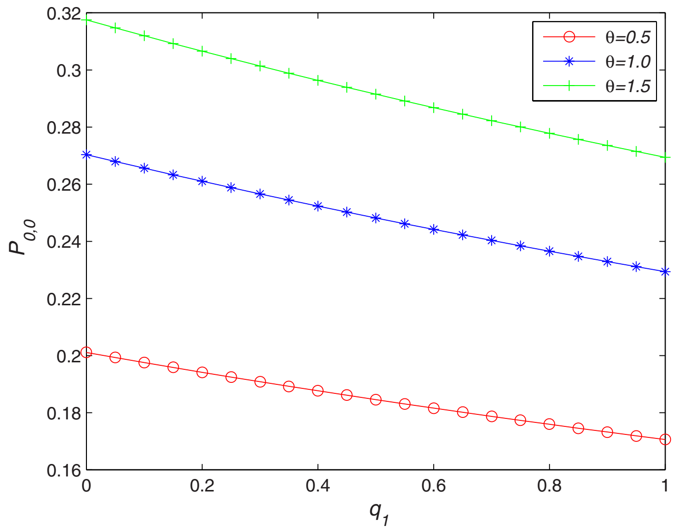

FIGURE 4. Impact of q_1 on $P_{0,0}$.

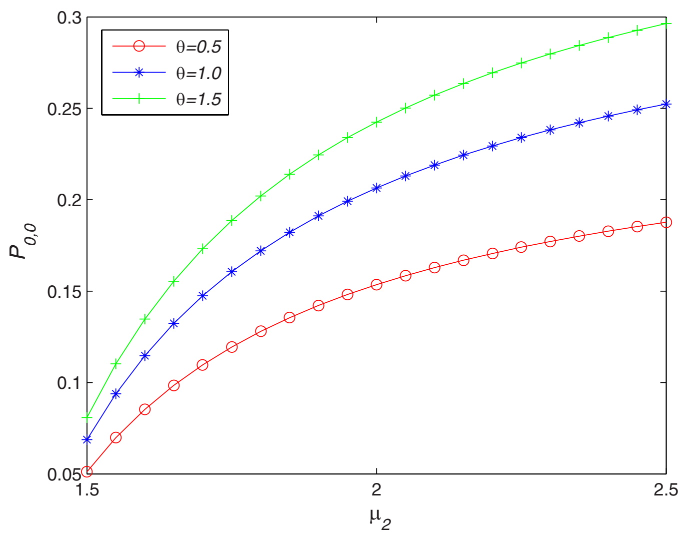

FIGURE 5. Impact of μ_2 on $P_{0,0}$.

small, the change of $E(L)$ is large, however, when N is relatively large, the change of $E(L)$ is small. That is to say, N hardly affects the mean system size $E(L)$ when N is relatively large, as we expected. Moreover, for a fixed N, as intuition tells us, the mean system size $E(L)$ decreases with the increase of θ.

The effect of q_1 on the probability that the system is empty $P_{0,0}$ is illustrated in Figure 4. It is observed that $P_{0,0}$ decreases as q_1 increases, which also coincides with the intuitive expectations. In addition, for a fixed q_1, $P_{0,0}$ increases with the increase of θ. Figure 5 describes the effect of μ_2 on $P_{0,0}$. $P_{0,0}$ increases with the increase of μ_2, and for a fixed μ_2, $P_{0,0}$ increases with the increase of θ. In Figure 6, we plot the $P_{0,0}$ versus N. As it can be observed from Figure 6, $P_{0,0}$ decreases as N increases for any θ, and N hardly affects $P_{0,0}$ when N is relatively large. Moreover, for a fixed N, $P_{0,0}$ increases with the increase of θ.

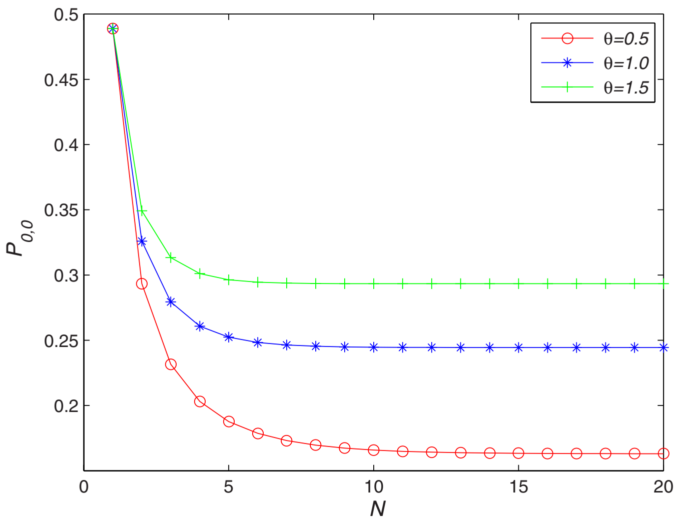

FIGURE 6. Impact of N on $P_{0,0}$.

7. Conclusion

In this paper, we consider an $M/G/1$ queue operating in multi-phase random environment with $Min(N,V)$ vacation policy. For this model, we obtain the probability generating function of the system size at arbitrary epoch. Some important performance characteristics such as the mean system size, the distribution of system size at the end of phase 0, the mean length of type-i cycle, and the sojourn time distribution of an arbitrary customer are obtained. We present some special cases of our model. Finally, the effect of various parameters on the performance characteristics is demonstrated through some numerical experiments.

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